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Addendum to "Amenability and acyclicity in bounded cohomology"

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Abstract. We show that a surjective homomorphism $\varphi\colon \Gamma \to K$ of (discrete) groups induces an isomorphism $H_b^\bullet(K;V) \to H_b^\bullet(\Gamma;\varphi^{-1}V)$ in bounded cohomology for all dual normed K-modules V if and only if the kernel of φ is boundedly acyclic. This complements a previous result by the authors that characterized this class of group homomorphisms as bounded cohomology equivalences with respect to $\mathbb{R}$-generated Banach K-modules. We deduce a characterization of the class of maps between path-connected spaces that induce isomorphisms in bounded cohomology with respect to coefficients in all dual normed modules, complementing the corresponding result shown previously in terms of $\mathbb{R}$-generated Banach modules. The main new input is the proof of the fact that every boundedly acyclic group Γ has trivial bounded cohomology with respect to all dual normed *trivial* Γ-modules.

This note concerns improvements and complementary results concerning the following characterization of boundedly n-acyclic maps shown in our previous work [9]. Following the conventions of [9], we restrict to path-connected topological spaces that admit a universal covering whose group of deck transformations is identified with the fundamental group.

Theorem 1.1 (see Theorem C in [9]). *Let $f\colon X \to Y$ be a map between based path-connected spaces, let F denote its homotopy fiber, and let $n \geq 0$ be an integer or $n = \infty$. We denote by $f_*\colon \pi_1(X) \to \pi_1(Y)$ the induced homomorphism between the fundamental groups. Then the following are equivalent*:

(1) *f is boundedly n-acyclic, that is, the induced restriction map*
$$H_b^i(f;V) : H_b^i(Y;V) \to H_b^i(X;f_*^{-1}V)$$
is an isomorphism for $i \leq n$ and injective for $i = n+1$ for every $\mathbb{R}$-generated Banach $\pi_1(Y)$-module V.

(2) *The induced restriction map*
$$H_b^i(f;V) : H_b^i(Y;V) \to H_b^i(X;f_*^{-1}V)$$
is surjective for every $\mathbb{R}$-generated Banach $\pi_1(Y)$-module V and $0 \leq i \leq n$.

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(3) *F is path-connected and $H_b^i(X; f_*^{-1}V) = 0$ for every relatively injective $\mathbb{R}$-generated Banach $\pi_1(Y)$-module V and $1 \le i \le n$.*

(4) *F is boundedly n-acyclic, that is, $H_b^0(F;\mathbb{R}) \cong \mathbb{R}$ and $H_b^i(F;\mathbb{R}) = 0$ for $1 \le i \le n$.*

For a set S and a Banach space V, we write $\ell^\infty(S, V)$ for the Banach space of bounded functions $S \to V$ (in the case of $V = \mathbb{R}$, we simply write $\ell^\infty(S)$). For a group Γ, the *bounded complex* of Γ with respect to trivial coefficients in V (i.e., V regarded as a Γ-module equipped with the trivial action) is the cochain complex

$$0 \to V \xrightarrow{\delta^0} \ell^\infty(\Gamma, V) \xrightarrow{\delta^1} \ell^\infty(\Gamma^2, V) \to \cdots \to \ell^\infty(\Gamma^n, V) \xrightarrow{\delta^n} \cdots$$

with the following coboundary operator, see Section 1.7 of [2]: δ^0 is the zero map, and for all $n \ge 1$, we have

$$\begin{aligned}\delta^n(f)(\gamma_1, \dots, \gamma_{n+1}) &\coloneqq f(\gamma_2, \dots, \gamma_{n+1}) \\ &\quad + \sum_{i=1}^{n} (-1)^i f(\gamma_1, \dots, \gamma_i\gamma_{i+1}, \dots, \gamma_{n+1}) + (-1)^{n+1} f(\gamma_1, \dots, \gamma_n),\end{aligned} \tag{1.1}$$

where $f \in \ell^\infty(\Gamma^n, V)$ and $\gamma_1, \dots, \gamma_n, \gamma_{n+1} \in \Gamma$. We recall that Γ is *boundedly n-acyclic* if the bounded cochain complex $(\ell^\infty(\Gamma^\bullet;\mathbb{R})\,\delta^\bullet)$ has trivial cohomology in degrees $1 \le i \le n$.

An $\mathbb{R}$-generated Banach Γ-module is a Banach Γ-module of the form $\ell^\infty(S)$ for some Γ-set S (the Γ-action on $\ell^\infty(S)$ is given by $\gamma \cdot f(s) = f(\gamma^{-1}s)$ for all $\gamma \in \Gamma$ and $s \in S$). As we remarked in our previous work, see Remark 2.41 in [9], the notion of an $\mathbb{R}$-generated Banach Γ-module was motivated by the fact that every boundedly n-acyclic group Γ satisfies $H_b^i(\Gamma; V) = 0$ for $1 \le i \le n$ and every $\mathbb{R}$-generated *trivial* Banach Γ-module V, see Proposition 2.39 in [9]. It turns out that boundedly n-acyclic groups have trivial bounded cohomology with respect to a larger class of coefficient modules (this also answers our question in Remark 2.41 of [9]).

Proposition 1.2. *Let Γ be a boundedly n-acyclic group, $n \ge 1$, and let V be a dual normed trivial Γ-module. Then $H_b^i(\Gamma; V) = 0$ for every $1 \le i \le n$.*

The following lemma is the key observation for the proof of Proposition 1.2. We owe the inspiration for this lemma to the recent work of Glebsky, Lubotzky, Monod and Rangarajan, see Section 4.2 of [3].

We recall that a Banach space X is *injective* if for every Banach space W with a subspace $V \subseteq W$ and a bounded linear map $f : V \to X$, there is an extension of f to a bounded linear map $\tilde{f} : W \to X$, see [5]; equivalently, using the uniform continuity of f and completeness of X, we may restrict to the case where V is a closed subspace, that is, V is again a Banach space. The Banach space $\ell^\infty(S)$ is injective for any set S (using the Hahn–Banach theorem, Proposition 2.5.2 in [1]). Moreover, by using an embedding of X (i.e., an isomorphism onto its image) into some $\ell^\infty(S)$, it follows that X is an injective Banach space if and only if every embedding of X into a Banach space W has a complement in W, see Section 3, pp. 92–93, in [4] (see also Theorem 2.7 and Corollary 2.8 in [5]).

Lemma 1.3. *Let Γ be a boundedly n-acyclic group, $n \ge 1$. For every $1 \le i \le n$, $\mathrm{im}(\delta^{i-1})$ is a closed complemented subspace of $\ell^\infty(\Gamma^i)$. As a consequence, the short exact sequence*

$$0 \to \ker(\delta^i) \to \ell^\infty(\Gamma^i) \to \mathrm{im}(\delta^i) \to 0$$

splits (as Banach spaces) for every $1 \le i \le n - 1$.

Proof. Since Γ is boundedly n-acyclic, we have $\operatorname{im}(\delta^{i-1}) = \ker(\delta^i)$ for every $1 \le i \le n$. Thus, $\operatorname{im}(\delta^{i-1})$ is a closed subspace of $\ell^\infty(\Gamma^i)$. In particular, $\operatorname{im}(\delta^{i-1})$ is a Banach space for every $1 \le i \le n$ ($\operatorname{im}(\delta^0) = \{0\}$). Then, using the bounded n-acyclicity of Γ and the open mapping theorem, the canonical bounded linear map

$$\ell^\infty(\Gamma^{i-1}) \,/\, \operatorname{im}(\delta^{i-2}) \to \operatorname{im}(\delta^{i-1})$$

is an isomorphism for $2 \le i \le n$. Note that the quotient of an inclusion $V \subseteq W$ of injective Banach spaces is again injective, since it is isomorphic to the complement of V in W and so the quotient map $W \to W/V$ admits a section. Hence we conclude inductively that $\operatorname{im}(\delta^{i-1})$ is injective for $1 \le i \le n$, as required. It follows that $\operatorname{im}(\delta^{i-1}) = \ker(\delta^i)$ is a complemented subspace of $\ell^\infty(\Gamma^i)$, and so the inclusion $\ker(\delta^i) \subseteq \ell^\infty(\Gamma^i)$ admits a splitting. ■

Proof of Proposition 1.2. Let V be a dual normed space with the trivial Γ-action. We need to show that the bounded complex of Γ with trivial coefficients in V,

$$0 \to V \xrightarrow{\delta_V^0} \ell^\infty(\Gamma, V) \xrightarrow{\delta_V^1} \ell^\infty(\Gamma^2, V) \xrightarrow{\delta_V^2} \ell^\infty(\Gamma^3, V) \xrightarrow{\delta_V^3} \cdots,$$

has vanishing cohomology in degrees $1 \le i \le n$. By assumption, this holds for $V = \mathbb{R}$. For the general case, suppose that V is the topological dual $\mathcal{B}(W, \mathbb{R})$ of the normed space W with the trivial Γ-action. Then we have a natural identification of Γ-modules

$$\ell^\infty(\Gamma^\bullet, V) \cong \mathcal{B}(W, \ell^\infty(\Gamma^\bullet))$$

induced by

$$f \mapsto \big(w \mapsto ((\gamma_1, \dots, \gamma_\bullet) \mapsto f(\gamma_1, \dots, \gamma_\bullet)(w))\big).$$

Moreover, these identifications yield isomorphic cochain complexes

$$(V, \ell^\infty(\Gamma^\bullet, V); \delta_V^\bullet) \cong \big(\mathcal{B}(W, \mathbb{R}), \mathcal{B}(W, \ell^\infty(\Gamma^\bullet)); \mathcal{B}(W, \delta^\bullet)\big).$$

By Lemma 1.3, the inclusion $\operatorname{im}(\delta^{n-1}) = \ker(\delta^n) \subseteq \ell^\infty(\Gamma^n)$ has a complement Q and we denote by $q\colon \ell^\infty(\Gamma^n) \to Q$ the canonical projection. Note that there is a bijective bounded linear map $Q \to \operatorname{im}(\delta^n)$. Further, by Lemma 1.3, the “modified truncation” of the cochain complex $(\mathbb{R}, \ell^\infty(\Gamma^\bullet); \delta^\bullet)$,

$$0 \to \mathbb{R} \xrightarrow{\delta^0 = 0} \ell^\infty(\Gamma) \to \cdots \to \ell^\infty(\Gamma^{n-1}) \xrightarrow{\delta^{n-1}} \ell^\infty(\Gamma^n) \xrightarrow{q} Q \to 0 \to \cdots,$$

is a contractible cochain complex of Banach spaces (it decomposes into a direct sum of split short exact sequences), and applying the functor $\mathcal{B}(W, -)$ (that preserves split short exact sequences) yields again a contractible cochain complex. The cohomology of the latter cochain complex agrees with the cohomology of the original cochain complex $(V, \ell^\infty(\Gamma^\bullet); \delta_V^\bullet)$ in degrees $\le n$, and the result follows. ■

Proposition 1.2 essentially makes it possible to replace $\mathbb{R}$-generated Banach Γ-modules in [9] with dual normed Γ-modules (as we had actually done originally in an old preprint version [8] of our paper!). At the same time, the restriction to $\mathbb{R}$-generated Banach

Γ-modules and Theorem 1.1 could still be useful as it provides a smaller test set of coefficient modules for the detection of boundedly acyclic maps. On the other hand, Remark 1.7 will show that a similar result cannot hold if we only restrict to $\mathbb{R}$ as coefficients.

The proofs of the following results are the same as those of Theorem 4.1 and Theorem C in [9], and together they provide a more complete characterization of boundedly n-acyclic maps that also includes dual normed modules.

Theorem 1.4. *Let $\varphi\colon \Gamma \to K$ be a homomorphism of discrete groups and let H denote its kernel. Let $n \geq 0$ be an integer or $n = \infty$. Then the following are equivalent:*

(1) *The induced restriction map*
$$H_b^i(\varphi; V) : H_b^i(K; V) \to H_b^i(\Gamma; \varphi^{-1} V)$$
is an isomorphism for $i \leq n$ and injective for $i = n+1$ for every dual normed K-module V.

(2) *The induced restriction map*
$$H_b^i(\varphi; V) : H_b^i(K; V) \to H_b^i(\Gamma; \varphi^{-1} V)$$
is surjective for $0 \leq i \leq n$ and every dual normed K-module V.

(3) *φ is surjective and $H_b^i(\Gamma; \varphi^{-1} V) = 0$ for $1 \leq i \leq n$ and every relatively injective dual normed K-module V.*

(4) *φ is surjective and H is a boundedly n-acyclic group.*

Theorem 1.5. *Let $f\colon X \to Y$ be a map between based path-connected spaces and let F denote its homotopy fiber. Let $n \geq 0$ be an integer or $n = \infty$. We denote by $f_*\colon \pi_1(X) \to \pi_1(Y)$ the induced homomorphism between the fundamental groups. Then the following are equivalent:*

(1) *The induced restriction map*
$$H_b^i(f; V) : H_b^i(Y; V) \to H_b^i(X; f_*^{-1} V)$$
is an isomorphism for $i \leq n$ and injective for $i = n+1$ for every dual normed $\pi_1(Y)$-module V.

(2) *The induced restriction map*
$$H_b^i(f; V) : H_b^i(Y; V) \to H_b^i(X; f_*^{-1} V)$$
is surjective for $0 \leq i \leq n$ and every dual normed $\pi_1(Y)$-module V.

(3) *F is path-connected and $H_b^i(X; f_*^{-1} V) = 0$ for $1 \leq i \leq n$ and every relatively injective dual normed $\pi_1(Y)$-module V.*

(4) *F is boundedly n-acyclic, that is, $H_b^0(F; \mathbb{R}) \cong \mathbb{R}$ and $H_b^i(F; \mathbb{R}) = 0$ for $1 \leq i \leq n$.*

Proofs of Theorem 1.4 and Theorem 1.5. Concerning Theorem 1.4, all the implications (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4) are the same as for the corresponding implications of Theorem 4.1 in [9]. The last implication (4)$\Rightarrow$(1) is also shown similarly by applying Proposition 1.2 instead of Proposition 2.39 in [9]. Theorem 1.5 follows from Theorem 1.4 using the mapping theorem, similarly to the proof of Theorem C in [9] (see also the proof of Proposition 2.11 in [9]). ∎

Note that the last item in the previous theorems is independent of the choice of coefficient modules. Thus, combining these with Theorem 1.1, we conclude that bounded acyclicity of maps (as defined in [9]) can also be defined in terms of dual normed modules.

Corollary 1.6. *Let $f\colon X \to Y$ be a map between based path-connected spaces, and let $n \geq 0$ be an integer or $n = \infty$. We denote by $f_*\colon \pi_1(X) \to \pi_1(Y)$ the induced homomorphism between the fundamental groups. Then the following are equivalent*:

(a) *The induced restriction map $H^i_b(f;V)\colon H^i_b(Y;V) \to H^i_b(X; f_*^{-1}V)$ is an isomorphism for $i \leq n$ and injective for $i = n+1$ for every dual normed $\pi_1(Y)$-module V.*

(b) *The induced restriction map $H^i_b(f;V)\colon H^i_b(Y;V) \to H^i_b(X; f_*^{-1}V)$ is an isomorphism for $i \leq n$ and injective for $i = n+1$ for every $\mathbb{R}$-generated Banach $\pi_1(Y)$-module V.*

Remark 1.7. We emphasize that the previous result is specific to the respective classes of coefficient modules. Indeed, if we consider just $\mathbb{R}$ as coefficients or even (semi-)separable coefficients in the sense of Monod (Section 3.C of [6]), then the resulting notion of bounded acyclicity is strictly weaker.

More precisely, Monod proved that the restricted wreath product

$$F_2 \wr \mathbb{Z} := \Big(\bigoplus_{\mathbb{Z}} F_2\Big) \rtimes \mathbb{Z}$$

is boundedly acyclic with respect to all dual (semi-)separable coefficients, see [7]. Also, $\mathbb{Z}$ has trivial bounded cohomology for all dual normed coefficient modules. But the group $\bigoplus_{\mathbb{Z}} F_2$ is not boundedly acyclic (this already fails in degree 2), so the canonical homomorphism $F_2 \wr \mathbb{Z} \to \mathbb{Z}$ does not satisfy the conditions of Theorem 1.4. In fact, one can show (by using cohomological induction) that if we consider the $\mathbb{R}$-generated Banach $(F_2 \wr \mathbb{Z})$-module $\ell^\infty(\mathbb{Z})$, then $H^2_b(F_2 \wr \mathbb{Z}; \ell^\infty(\mathbb{Z})) \neq 0$, see Section 4.3 in [7].

In this specific example, we also see the different nature of the two classes of coefficient modules. Indeed, given a group Γ, the basic example of a semi-separable coefficient module is the space of essentially bounded functions $L^\infty(\Omega)$, where Ω is a standard Borel probability space with a measurable measure-preserving Γ-action, see Section 3.C in [6]. However, $\mathbb{R}$-generated modules are defined through bounded functions over *arbitrary discrete Γ-sets*.

The fact that surjective group homomorphisms induce an injective map in degree 2 in bounded cohomology (Example 4.2 in [9]) extends to the class of dual normed coefficient modules.

We also have the following version of Corollary 4.3 in [9].

Corollary 1.8. *Let $n \geq 1$ be an integer or $n = \infty$. Let Γ be a boundedly n-acyclic group, let $H \trianglelefteq \Gamma$ be a normal subgroup, and let $\varphi\colon \Gamma \to \Gamma/H$ be the quotient homomorphism. Then H is boundedly n-acyclic if and only if*

$$H^i_b(\Gamma; \varphi^{-1}V) = 0$$

for every relatively injective dual normed Γ/H-module V and $1 \leq i \leq n$.

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