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Sharp Fourier extension for functions with localized support on the circle

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Abstract. A well-known conjecture states that constant functions are extremizers of the $L^2 \to L^6$ Tomas–Stein extension inequality for the circle. We prove that functions supported in a $\sqrt{6}/80$-neighborhood of a pair of antipodal points on S^1 satisfy the conjectured sharp inequality. In the process, we make progress on a program formulated by Carneiro, Foschi, Oliveira e Silva and Thiele to prove the sharp inequality for all functions.

1. Introduction

We are interested in the conjecture that constant functions are extremizers for the Tomas–Stein Fourier extension inequality for the circle

$$\|\widehat{f\sigma}\|_{L^6(\mathbb{R}^2)} \leq C\|f\|_{L^2(\sigma)}. \tag{1.1}$$

Here σ is the arc length measure on the unit circle $S^1 \subset \mathbb{R}^2$ and $\hat{\mu}(\xi) = \int e^{-ix\xi}\,\mathrm{d}\mu(\xi)$ is the Fourier transform.

The corresponding conjecture for S^2 was proven by Foschi [9], and in [3], Foschi's argument is adapted to S^1, and the conjecture of interest is reduced to the following.

Conjecture 1.1. *The quadratic form*

$$Q(f) := \int_{(S^1)^6} (|\omega_1+\omega_2+\omega_3|^2-1)(f(\omega_1,\omega_2,\omega_3)^2 - f(\omega_1,\omega_2,\omega_3)f(\omega_4,\omega_5,\omega_6))\,\mathrm{d}\Sigma$$

is positive semi-definite on the subspace V of all antipodal functions in $L^2((S^1)^3,\mathbb{R})$. Here we denote

$$\mathrm{d}\Sigma = \mathrm{d}\Sigma(\omega) = \delta\Big(\sum_{j=1}^{6}\omega_k\Big)\prod_{j=1}^{6}\mathrm{d}\sigma(\omega_j),$$

and a function f is antipodal if $f(\pm\omega_1,\pm\omega_2,\pm\omega_3)$ does not depend on the choice of signs.

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Conjecture 1.1 has been verified for all functions with Fourier modes up to degree 120 in [15] and [1], via a numerical computation of the eigenvalues of Q on the finite dimensional space of such functions. Further, using different methods, in [7] the conjectured sharp form of inequality (1.1) has been established for certain infinite dimensional subspaces of $L^2(\sigma)$ with constrained Fourier support. Our main result establishes Conjecture 1.1 for functions with localized spatial support.

Let C_ε be the cylinder of radius ε centered at the line $\mathbb{R}(1,1,1)$, and define

$$V_\varepsilon := \Big\{ f \in V : \operatorname{supp} f(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \subset \bigcup_{k \in \pi\mathbb{Z}^3} k + C_\varepsilon \Big\}.$$

Theorem 1.2. *Let $\varepsilon = 1/20$. Then for all $f \in V_\varepsilon$, it holds that $Q(f) \geq 0$.*

Note that since constant functions are in the kernel of Q, the same result holds for $V_\varepsilon \oplus \langle \mathbf{1} \rangle$, where $\mathbf{1}$ is the constant 1 function.

As a corollary, functions with support sufficiently close to a pair of antipodal points satisfy (1.1) with the conjectured sharp constant. Define

$$\Phi(g) := \frac{\|\widehat{g\sigma}\|_{L^6(\mathbb{R}^2)}}{\|g\|_{L^2(\sigma)}}.$$

Corollary 1.3. *Let $\varepsilon' = \sqrt{3/8}\varepsilon$. Suppose that $g \in L^2(\sigma)$ is such that $g(e^{i\theta})$ is supported in $(-\varepsilon', \varepsilon') + \pi\mathbb{Z}$. Then $\Phi(g) \leq \Phi(\mathbf{1})$, where $\mathbf{1}$ is the constant 1 function on S^1.*

Note that by rotation symmetry, the same holds when $g(e^{i\theta})$ is supported in $I + \pi\mathbb{Z}$ for any interval I of length $2\varepsilon'$.

The constants ε and ε' in Theorem 1.2 and Corollary 1.3 are not optimal. Numerical computations suggest that with our method ε can be improved up to about 0.104, and ε' up to about 0.063, see Section7.

The numerical results in [1] suggest that eigenfunctions of Q on the subspace of functions with Fourier modes up to degree N corresponding to small eigenvalues concentrate in space. Theorem 1.2 shows that Q is positive on all such sufficiently concentrated functions, thus it should be a useful partial result in establishing positive semi-definiteness of Q on the full space of antipodal functions. A more precise observation by Jiaxi Cheng, a graduate student in Bonn, is that the smallest eigenvalue is of size $\sim N^{-2}\log(N)$, see Section 2 of [13]. The existence of such an eigenvalue is also explained by the asymptotic formula for the multiplier m in Lemma 4.1, which looks like $c|\log|x||\,|x|^2$ near 0. Unfortunately, we cannot prove that this is the smallest eigenvalue.

More generally, the topic of sharp Fourier extension inequalities has attracted a lot of interest in recent years. In the following, we consider general dimensions $d \geq 2$. Then the Tomas–Stein extension inequality states that for every

$$q \geq q_d := \frac{2(d+1)}{d-1},$$

there exists $C(d,q) > 0$ such that, for all $f \in L^2(S^{d-1}, \sigma^{d-1})$,

$$\|\widehat{f\sigma}\|_{L^q(\mathbb{R}^d)} \leq C(d,q)\|f\|_{L^2(\sigma)}. \tag{1.2}$$

Here σ^{d-1} denotes the $d-1$-dimensional Hausdorff measure on S^{d-1}.

It is known that extremizers for (1.2) exist when $q > q_d$, for all d, see [8]. At the endpoint $q = q_d$, existence and smoothness of extremizers have been shown for $d = 3$ in [5, 6] and for $d = 2$ in [17, 18]. For higher dimensional spheres $d \geq 4$, existence of extremizers for $q = q_d$ is known conditional on the conjecture that Gaussians maximize the corresponding extension inequality for the paraboloid, see [11].

For certain specific choices of (d, q), a full characterization of the extremizers of (1.2) is known. Most such results grew out of the work of Foschi [9], who showed that constant functions maximize (1.2) for $(d, q) = (2, 4)$, and gave a full characterization of all complex valued maximizers. His method can be adapted for some non-endpoint extension inequalities on higher dimensional spheres, see [4]. Using different methods, maximizers of (1.2) for some choices of (d, q) with even $q > 4$ are characterized in [14]. In some further cases, it is known that constant functions are local maximizers. This was shown in [3] for $(d, q) = (2, 6)$, and in [12] for (d, q_d) with $3 \leq d \leq 60$. For further background and references on sharp Fourier extension inequalities, we refer to [10] and [13].

2. Proof of Corollary 1.3

Corollary 1.3 is a direct consequence of Theorem 1.2 and the program formulated in [3]. We give a brief sketch of the implication here; for the details of the program and proofs, we refer the reader to [3].

Proof of Corollary 1.3. Let $g \in L^2(\sigma)$ be such that $g(e^{i\theta})$ is supported in $(-\varepsilon', \varepsilon') + \pi\mathbb{Z}$. Define $\tilde{g}(x) = g(-x)$ and

$$g_{\#} = \sqrt{\frac{|g|^2 + |\tilde{g}|^2}{2}}.$$

As shown in [3], Step 1 and 2, it holds that $\Phi(g) \leq \Phi(g_{\#})$, and $g_{\#}$ is antipodal and $g_{\#}(e^{i\theta})$ is supported in $(-\varepsilon', \varepsilon') + \pi\mathbb{Z}$. Define $f(\omega_1, \omega_2, \omega_3) := g_{\#}(\omega_1)\, g_{\#}(\omega_2)\, g_{\#}(\omega_3)$. Then the function $f(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3})$ is supported in $\bigcup_{k \in \pi\mathbb{Z}^3} k + (-\varepsilon', \varepsilon')^3$. Since $(-\varepsilon', \varepsilon')^3$ is a subset of the cylinder $C_{\sqrt{8/3}\,\varepsilon'}$, it follows that $f \in V_{\sqrt{8/3}\varepsilon'} = V_\varepsilon$, hence $Q(f) \geq 0$, by Theorem 1.2. This verifies Conjecture 1.4 in [3] for $g_{\#}$. Using Step 3, 4 and 5 in [3], we conclude that $\Phi(g) \leq \Phi(1)$. ■

3. Proof of Theorem 1.2

3.1. Orthogonal decomposition

We consider the sesquilinear form

$$\begin{aligned} B(f, g) = \int_{(S^1)^6} &(|\omega_1 + \omega_2 + \omega_3|^2 - 1) \\ &\cdot \left(f(\omega_1, \omega_2, \omega_3)\, \overline{g(\omega_1, \omega_2, \omega_3)} - f(\omega_1, \omega_2, \omega_3)\, \overline{g(\omega_4, \omega_5, \omega_6)}\right) \mathrm{d}\Sigma(\omega). \end{aligned}$$

By a change of variables, it holds that $B(f, g) = B(Rf, Rg)$, where $Rf(\omega_1, \omega_2, \omega_3) = f(e^i\omega_1, e^i\omega_2, e^i\omega_3)$. Define

$$Z_d = \{(k_1, k_2, k_3) \in (2\mathbb{Z})^3 : k_1 + k_2 + k_3 = d\}$$

and

$$X_d = \Big\{ \sum_{k\in Z_d} a_k \omega_1^{k_1}\omega_2^{k_2}\omega_3^{k_3} \ : \ (a_k) \in \ell^2(Z_d)\Big\} \subset L^2((S^1)^3).$$

For $d \neq d'$, the spaces X_d and $X_{d'}$ are eigenspaces of R with different eigenvalues e^{id} and $e^{id'}$, and hence are orthogonal with respect to B. Note that the orthogonal projection π_d onto X_d can be expressed as

$$\pi_d(f)(\omega_1,\omega_2,\omega_3) = \int_0^1 e^{-2\pi i d t} f(e^{2\pi i t}\omega_1, e^{2\pi i t}\omega_2, e^{2\pi i t}\omega_3)\, \mathrm{d}t,$$

which implies that $\pi_d(V_\varepsilon) \subset V_\varepsilon$. Therefore, we have that

$$V_\varepsilon = \overline{\bigoplus_{d\in\mathbb{Z}} \pi_d(V_\varepsilon)} = \overline{\bigoplus_{d\in\mathbb{Z}} (V_\varepsilon \cap X_d)}.$$

Hence, it suffices to show positive semi-definiteness of B on each of the spaces

$$X_{d,\varepsilon} := V_\varepsilon \cap X_d.$$

3.2. Reducing the dimension

From now on, we use the convention that

$$\omega_i = (\cos(\theta_i), \sin(\theta_i)), \tag{3.1}$$

and abuse notation by writing $f(\omega(\theta)) = f(\theta)$. We also define

$$\begin{aligned} a(\theta_1,\theta_2,\theta_3) &:= (\cos(\theta_1)+\cos(\theta_2)+\cos(\theta_3))^2 + (\sin(\theta_1)+\sin(\theta_2)+\sin(\theta_3))^2 \\ &= |\omega_1+\omega_2+\omega_3|^2, \end{aligned}$$

so that the weight in the bilinear form B is given by $a-1$, and record the useful identity

$$a(\theta_1,\theta_2,\theta_3) = 3 + 2\cos(\theta_1-\theta_2) + 2\cos(\theta_2-\theta_3) + 2\cos(\theta_3-\theta_1). \tag{3.2}$$

The domain of integration $\omega \in (S^1)^6$ in the bilinear form B becomes $\theta \in \mathbb{R}^6/(2\pi\mathbb{Z})^6$. As we assume that $f \in X_d$ for some d, we fully understand how f transforms under simultaneous rotations of $\omega_1,\omega_2,\omega_3$ by the same angle. We will use this to integrate out such simultaneous rotations of $\omega_1,\omega_2,\omega_3$ and of $\omega_4,\omega_5,\omega_6$. These rotations correspond to shifts of $(\theta_1,\theta_2,\theta_3)$ and $(\theta_4,\theta_5,\theta_6)$ in direction $(1,1,1)$, which makes it natural to choose the following fundamental domain of $\mathbb{R}^3/(2\pi\mathbb{Z})^3$ as our domain of integration in θ.

Lemma 3.1. *Let C be the rhombus with corners*

$$(\pi,-\pi,0), \quad \Big(-\frac{\pi}{3},-\frac{\pi}{3},\frac{2\pi}{3}\Big), \quad (-\pi,\pi,0) \quad \textit{and} \quad \Big(\frac{\pi}{3},\frac{\pi}{3},-\frac{2\pi}{3}\Big).$$

Then the prism $P := C + \{(t,t,t) : t \in [0,2\pi)\}$ over C of height $2\pi\sqrt{3}$ is a fundamental domain for $\mathbb{R}^3/(2\pi\mathbb{Z})^3$.

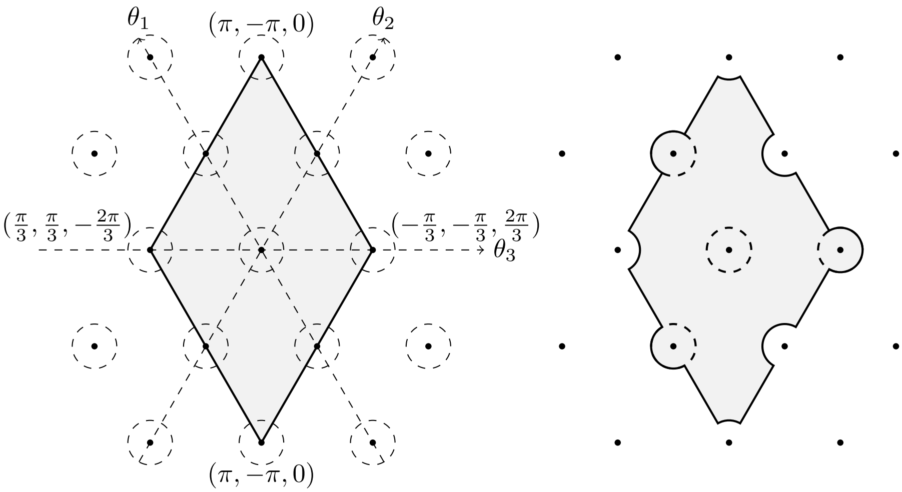

Figure 1. Left: The lattice $\frac{1}{2}\Lambda$ in the hyperplane H and the fundamental domain C (gray) of Λ. The restriction $|f||_H$ is supported in the union of the dashed balls and periodic with respect to $\frac{1}{2}\Lambda$. Right: One possible choice of a fundamental domain C' such that $f|_{C'}$ is supported in the union of the balls (dashed) B_1, B_2, B_3 and B_4.

Proof. Denote by p the orthogonal projection onto the hyperplane

$$H := \{(\theta_1, \theta_2, \theta_3) : \theta_1 + \theta_2 + \theta_3 = 0\}.$$

The image of $(2\pi\mathbb{Z})^3$ under p is the hexagonal lattice

$$\Lambda := \mathbb{Z}v_1 \oplus \mathbb{Z}v_2 \subset H,$$

where

$$v_1 := \Big(\frac{4\pi}{3}, -\frac{2\pi}{3}, -\frac{2\pi}{3}\Big) \quad \text{and} \quad v_2 := \Big(-\frac{2\pi}{3}, \frac{4\pi}{3}, -\frac{2\pi}{3}\Big).$$

It is easy to see that the rhombus C is a fundamental domain of H modulo the lattice Λ. Thus for every x, there exists y with $x - y \in (2\pi\mathbb{Z})^3$ and $p(y) \in C$. Then for an appropriate choice of $k \in \mathbb{Z}$, the point $z = y + 2\pi k(1,1,1)$ lies in P, and $x - z \in (2\pi\mathbb{Z})^3$.

Conversely, let $z, z' \in P$ be such that $z - z' \in (2\pi\mathbb{Z})^3$. Then $p(z) - p(z')$ lies in $p((2\pi\mathbb{Z})^3) = \Lambda$, and $p(z), p(z') \in C$. It follows that $p(z) = p(z')$. Thus $z - z' \in 2\pi\mathbb{Z} \cdot (1,1,1)$, and from $z, z' \in P$, it follows that $z = z'$. ∎

In the next lemma, we perform integrations in direction $(1,1,1)$ in $(\theta_1, \theta_2, \theta_3)$ and $(\theta_4, \theta_5, \theta_6)$, thereby reducing to a quadratic form depending only on the restriction $f|_C$. We define the function $\lambda_d : C \times C \to S^1$ by

$$\lambda_d(\theta_1', \theta_2', \theta_3', \theta_1, \theta_2, \theta_3) = \exp(id \cdot (\arg(e^{i\theta_1'} + e^{i\theta_2'} + e^{i\theta_3'}) - \arg(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3}))).$$

The only property of λ_d that will be used in the proof below is that $|\lambda_d| = 1$.

Lemma 3.2. *For all $d \in \mathbb{Z}$ and all $f \in X_d$, we have that $B(f, f)$ equals*

$$12\pi \int_{C^2} \delta(a(\theta) - a(\theta'))\,(a(\theta) - 1)\,(|f(\theta)|^2 - \lambda_d(\theta', \theta)\, f(\theta)\,\overline{f(\theta')})\,\mathrm{d}\mathcal{H}^2_C(\theta)\,\mathrm{d}\mathcal{H}^2_C(\theta').$$

Here $\mathcal{H}^2_C$ denotes the 2-dimensional Hausdorff measure on C.

Proof. By Lemma 3.1, we have

$$\begin{aligned}
B(f, f) &= \int_{P\times P} \delta(\omega_1 + \omega_2 + \omega_3 - \omega_4 - \omega_5 - \omega_6)\,(|\omega_1 + \omega_2 + \omega_3|^2 - 1) \\
&\qquad \times \big(|f(\omega_1, \omega_2, \omega_3)|^2 - f(\omega_1, \omega_2, \omega_3)\,\overline{f(\omega_4, \omega_5, \omega_6)}\big) \prod_{j=1}^{6} \mathrm{d}\theta_j \\
&= 2\pi\sqrt{3} \int_{C\times P} \delta(\omega_1 + \omega_2 + \omega_3 - \omega_4 - \omega_5 - \omega_6)\,(|\omega_1 + \omega_2 + \omega_3|^2 - 1) \\
&\qquad \times \big(|f(\omega_1, \omega_2, \omega_3)|^2 - f(\omega_1, \omega_2, \omega_3)\,\overline{f(\omega_4, \omega_5, \omega_6)}\big)\,\mathrm{d}\mathcal{H}^2_C(\theta_1, \theta_2, \theta_3) \prod_{j=4}^{6} \mathrm{d}\theta_j.
\end{aligned}$$

Here we have used that $f \in X_d$, to integrate out simultaneous rotations of all 6 points ω_j by the same angle. For $x, y \in \mathbb{R}^2$, it holds that

$$\delta(x - y) = 2\delta(|x|^2 - |y|^2)\,\delta(\arg(x) - \arg(y)).$$

Hence, we can rewrite the last expression as

$$\begin{aligned}
&= 4\pi\sqrt{3} \int_{C\times P} \delta(|\omega_1 + \omega_2 + \omega_3|^2 - |\omega_4 + \omega_5 + \omega_6|^2) \\
&\qquad \times \delta(\arg(\omega_1 + \omega_2 + \omega_3) - \arg(\omega_4 + \omega_5 + \omega_6))\,(|\omega_1 + \omega_2 + \omega_3|^2 - 1) \\
&\qquad \times \big(|f(\omega_1, \omega_2, \omega_3)|^2 - f(\omega_1, \omega_2, \omega_3)\,\overline{f(\omega_4, \omega_5, \omega_6)}\big)\,\mathrm{d}\mathcal{H}^2_C(\theta_1, \theta_2, \theta_3) \prod_{j=4}^{6} \mathrm{d}\theta_j \\
&= 12\pi \int_{C\times C} \int_0^{2\pi} \delta(a(\theta_1, \theta_2, \theta_3) - a(\theta_4, \theta_5, \theta_6)) \\
&\qquad \times \delta(\arg(\omega_1 + \omega_2 + \omega_3) - \arg(\omega_4 + \omega_5 + \omega_6) - t)\,(a(\theta_1, \theta_2, \theta_3) - 1) \\
&\qquad \times \big(|f(\omega_1, \omega_2, \omega_3)|^2 - f(\omega_1, \omega_2, \omega_3)\,\overline{f(e^{it}\omega_4, e^{it}\omega_5, e^{it}\omega_6)}\big)\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^4_{C\times C}(\theta).
\end{aligned}$$

Since $f \in X_d$, we have

$$f(e^{it}\omega_4, e^{it}\omega_5, e^{it}\omega_6) = e^{itd} f(\omega_4, \omega_5, \omega_6).$$

Thus, we can integrate out t and obtain the claimed identity. ∎

3.3. Completing the proof

By Lemma 3.2, we have for all d and all $f \in X_d$,

$$\begin{aligned}B(f,f) \geq 12\pi \int_C |f(\theta)|^2 (a(\theta)-1) \int_C \delta(a(\theta)-a(\theta'))\, \mathrm{d}\mathcal{H}^2_C(\theta')\, \mathrm{d}\mathcal{H}^2_C(\theta) \\ - 12\pi \int_{C^2} \delta(a(\theta)-a(\theta'))\, |a(\theta)-1|\,|f(\theta)|\,|f(\theta')|\, \mathrm{d}\mathcal{H}^2_C(\theta)\, \mathrm{d}\mathcal{H}^2_C(\theta') \\ =: 12\pi(\mathrm{I}-\mathrm{II}). \qquad (3.3)\end{aligned}$$

If $f \in X_{d,\varepsilon}$, then the restriction of f onto the hyperplane $H = \{(\theta_1,\theta_2,\theta_3) : \theta_1+\theta_2+\theta_3 = 0\}$ is supported in $\frac{1}{2}\Lambda + B_\varepsilon(0)$. Furthermore, the function $|f|$ is periodic with respect to $\frac{1}{2}\Lambda$, since it is periodic with respect to $\pi\mathbb{Z}^3$ and invariant under all translations in direction $(1,1,1)$. Thus it suffices to show the following.

Lemma 3.3. *Suppose that $\varepsilon \leq 1/20$. Then for all functions $f : H \to [0,\infty)$ that are periodic with respect to $\frac{1}{2}\Lambda$ and supported in $\frac{1}{2}\Lambda + B_\varepsilon(0)$, it holds that* $\mathrm{I} \geq \mathrm{II}$.

Proof. Recall that C is a fundamental domain of the lattice Λ. The expressions in the integrals for the terms I and II are Λ periodic, so we may replace C by any other fundamental domain C'. Since f is supported in $\frac{1}{2}\Lambda + B(0,\varepsilon)$, there exists a fundamental domain C' such that $f|_{C'}$ is supported in

$$\begin{aligned}B_\varepsilon(0,0,0) \cup B_\varepsilon\Big(\frac{2\pi}{3},-\frac{\pi}{3},-\frac{\pi}{3}\Big) \cup B_\varepsilon\Big(-\frac{\pi}{3},\frac{2\pi}{3},-\frac{\pi}{3}\Big) \cup B_\varepsilon\Big(-\frac{\pi}{3},-\frac{\pi}{3},\frac{2\pi}{3}\Big) \\ =: B_1 \cup B_2 \cup B_3 \cup B_4.\end{aligned}$$

We decompose

$$\mathrm{I} = \sum_{i=1}^4 \int_{B_i} |f(\theta)|^2 (a(\theta)-1) \int_C \delta(a(\theta)-a(\theta'))\, \mathrm{d}\mathcal{H}^2_C(\theta')\, \mathrm{d}\mathcal{H}^2_C(\theta) =: \sum_{i=1}^4 \mathrm{I}_i, \qquad (3.4)$$

$$\begin{aligned}\mathrm{II} = \sum_{1\leq i,j\leq 4} \int_{B_i\times B_j} \delta(a(\theta)-a(\theta'))|a(\theta)-1|\,|f(\theta)|\,|f(\theta')|\, \mathrm{d}\mathcal{H}^2_C(\theta)\, \mathrm{d}\mathcal{H}^2_C(\theta') \\ =: \sum_{1\leq i,j\leq 4} \mathrm{II}_{ij}. \qquad (3.5)\end{aligned}$$

Note that $|\theta| < \pi/6$ implies, by (3.2), that $a(\theta) \geq 3 + 6\cos(\pi/3) = 6$, and that similarly $|\theta - (2\pi/3, -\pi/3, -\pi/3)| < \pi/6$ implies that $a(\theta) \leq 3$. Therefore, for $j = 2,3,4$ the measure $\delta(a(\theta)-a(\theta'))$ vanishes on $B_1 \times B_j$, thus $\mathrm{I}_{1j} = \mathrm{I}_{j1} = 0$.

Next, we record that $\mathrm{II}_{11} \leq \mathrm{I}_1$, by Cauchy–Schwarz, and since $a(\theta) \geq 6$ on B_1,

$$\begin{aligned}\mathrm{II}_{11} &= \int_{B_1^2} \delta(a(\theta)-a(\theta'))|a(\theta)-1|\,|f(\theta)|\,|f(\theta')|\, \mathrm{d}\mathcal{H}^2_C(\theta)\, \mathrm{d}\mathcal{H}^2_C(\theta') \\ &\leq \frac{1}{2} \int_{B_1^2} \delta(\tilde{a}(\theta)-\tilde{a}(\theta'))\, (\tilde{a}(\theta)-1)\, (|f(\theta)|^2 + |f(\theta')|^2)\, \mathrm{d}\mathcal{H}^2_C(\theta)\, \mathrm{d}\mathcal{H}^2_C(\theta') \\ &\leq \int_{B_1} |f(\theta)|^2 (\tilde{a}(\theta)-1) \int_C \delta(\tilde{a}(\theta)-\tilde{a}(\theta'))\, \mathrm{d}\mathcal{H}^2_C(\theta)\, \mathrm{d}\mathcal{H}^2_C(\theta') = \mathrm{I}_1.\end{aligned}$$

The remaining terms are estimated in the next two sections. By Lemmas 4.1 and 5.1, we have

$$\begin{aligned}\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4 &\geq 30\int_{B_1}|\theta|^2\,|f(\theta)|^2\,\mathrm{d}\mathcal{H}^2_H(\theta) > 9\,\frac{101}{100}\,\pi\int_{B_1}|\theta|^2\,|f(\theta)|^2\,\mathrm{d}\mathcal{H}^2_H(\theta)\\ &\geq \sum_{2\leq i,j\leq 4}\mathrm{II}_{ij},\end{aligned}$$

which completes the proof. ∎

4. Estimating term I

Lemma 4.1. *It holds that*

$$\mathrm{I}_2+\mathrm{I}_3+\mathrm{I}_4 = \int_{B_1} m(\theta)\,|f(\theta)|^2\,\mathrm{d}\mathcal{H}^2_H(\theta), \tag{4.1}$$

where I_j *is defined in* (3.4), *and* $m(\theta)\geq 30|\theta|^2$.

Proof. By definition of the I_j, equation (4.1) holds with

$$m(\theta) = \sum_{j=2}^{4}(a(\theta+c_j)-1)\int_C \delta(a(\theta+c_j)-a(\theta'))\,\mathrm{d}\mathcal{H}^2_C(\theta'),$$

where c_j is the center of the ball B_j. Reversing the argument in the proof of Lemma 3.2, it follows that for $x\in\mathbb{R}^2$,

$$\begin{aligned}&\int_C \delta(|x|^2-a(\theta'))\,\mathrm{d}\mathcal{H}^2_C(\theta')\\ &\quad= \frac{1}{\sqrt{3}}\int_P \delta(|x|^2-|\omega_1+\omega_2+\omega_3|^2)\,\delta(\arg(x)-\arg(\omega_1+\omega_2+\omega_3))\prod_{j=1}^{3}\mathrm{d}\theta_j'\\ &\quad= \frac{1}{2\sqrt{3}}\int_{(S^1)^3}\delta(x-(\omega_1+\omega_2+\omega_3))\prod_{j=1}^{3}\mathrm{d}\sigma(\omega_j) = \frac{1}{2\sqrt{3}}\,\sigma*\sigma*\sigma(x).\end{aligned}$$

The convolution $\sigma*\sigma*\sigma$ is radial. We set $\sigma*\sigma*\sigma(x)=\rho(|x|)$, giving

$$m(\theta) = \frac{1}{2\sqrt{3}}\sum_{j=2}^{4}(a(\theta+c_j)-1)\,\rho\big(\sqrt{a(\theta+c_j)}\big). \tag{4.2}$$

In polar coordinates

$$\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\end{pmatrix} = s\cos(\alpha)\,\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\\ 0\end{pmatrix} + s\sin(\alpha)\frac{1}{\sqrt{6}}\begin{pmatrix}1\\ 1\\ -2\end{pmatrix}, \tag{4.3}$$

we compute in Lemma 6.6 the asymptotic expansion

$$
\begin{aligned}
&(a(\theta + c_4) - 1)\,\rho\big(\sqrt{a(\theta + c_4)}\big) \\
&\quad = -12s^2(3\sin^2(\alpha) - \cos^2(\alpha))\log(s) && (4.4)\\
&\qquad - 6s^2(3\sin^2(\alpha) - \cos^2(\alpha))\log|3\sin^2(\alpha) - \cos^2(\alpha)| && (4.5)\\
&\qquad + 18\log 2\, s^2(3\sin^2(\alpha) - \cos^2(\alpha)) && (4.6)\\
&\qquad + E,
\end{aligned}
$$

with

$$|E| \le -180s^4\log s + 71s^4 \quad \text{when } s \le 1/20.$$

As the function a is invariant under permutation of its arguments and constant in direction $(1,1,1)$, it is invariant under the rotation T by $2\pi/3$ about the line $\mathbb{R}(1,1,1)$. Since

$$c_2 + \theta(\alpha, s) = T(c_4 + \theta(\alpha + 4\pi/3, s)) \quad \text{and} \quad c_3 + \theta(\alpha, s) = T^2(c_4 + \theta(\alpha + 2\pi/3, s)),$$

we obtain the same asymptotic expansion for $a(\theta + c_j)\,\rho(\sqrt{a(\theta + c_j)})$, $j = 2, 3$, but with α replaced by $\alpha + 4\pi/3$ and $\alpha + 2\pi/3$.

We now consider (4.2). The term (4.4) contributes $-6\sqrt{3}s^2\log(s)$ to m, and the term (4.6) contributes $9\sqrt{3}\log(2)\,s^2$, since for all α,

$$\sum_{j=1}^{3}\left(3\sin^2\left(\alpha + \frac{2\pi j}{3}\right) - \cos^2\left(\alpha + \frac{2\pi j}{3}\right)\right) = 3.$$

For term (4.5), we use the sharp estimate

$$
\begin{aligned}
&\sum_{j=1}^{3}\left(3\sin^2\left(\alpha+\frac{2\pi j}{3}\right) - \cos^2\left(\alpha+\frac{2\pi j}{3}\right)\right)\log\left|3\sin^2\left(\alpha+\frac{2\pi j}{3}\right) - \cos^2\left(\alpha+\frac{2\pi j}{3}\right)\right| \\
&\quad \le 3\log(3),
\end{aligned}
$$

which we prove in Lemma 6.7. Hence, for $s \le 1/20$,

$$
\begin{aligned}
m(\theta) &\ge -6\sqrt{3}s^2\log(s) + \big(9\sqrt{3}\log(2) - 3\sqrt{3}\log(3)\big)s^2 + 90\sqrt{3}s^4\log s - 62s^4 \\
&\ge \left(6\sqrt{3}\log(20) + 9\sqrt{3}\log(2) - 3\sqrt{3}\log(3) - \frac{90\sqrt{3}}{400}\log(20) - \frac{62}{400}\right)s^2 \\
&\approx 34.906\, s^2,
\end{aligned}
$$

as claimed. ∎

5. Estimating term II

Lemma 5.1. *For all $2 \le i, j \le 4$ and all f, it holds that*

$$\mathrm{II}_{ij} \le \frac{101}{100}\pi\int_{B_1}|\theta|^2|f(\theta)|^2\,\mathrm{d}\mathcal{H}^2_H(\theta).$$

Proof. We first treat the term II_{44}, and later explain the changes for the other terms. We have

$$\begin{aligned}\mathrm{II}_{44} &= \int_{B_4\times B_4} \delta\Big(1-\frac{1-a(\theta')}{1-a(\theta)}\Big)\,|f(\theta)|\,|f(\theta')|\,\mathrm{d}\mathcal{H}^2_H(\theta)\,\mathrm{d}\mathcal{H}^2_H(\theta')\\ &= \int_{B_1\times B_1} \delta\Big(1-\frac{1-a(c_4+\theta')}{1-a(c_4+\theta)}\Big)\,|f(\theta)|\,|f(\theta')|\,\mathrm{d}\mathcal{H}^2_H(\theta)\,\mathrm{d}\mathcal{H}^2_H(\theta').\end{aligned}$$

We introduce polar coordinates $\theta=\theta(s,\alpha)$ as in (4.3) and write also $\theta'=\theta(t,\beta)$. With the definitions

$$h(s,t,\alpha,\beta) := \frac{1-a(c_4+\theta')}{1-a(c_4+\theta)} \quad \text{and} \quad g(s,\alpha) := |\theta|^2\,|f(\theta)|,$$

we obtain, by changing variables,

$$\mathrm{II}_{44} = \int_0^{2\pi}\int_0^{2\pi}\int_0^{\varepsilon}\int_0^{\varepsilon} \delta(1-h(s,t,\alpha,\beta))\,g(s,\alpha)\,g(t,\beta)\,\frac{\mathrm{d}s}{s}\,\frac{\mathrm{d}t}{t}\,\mathrm{d}\alpha\,\mathrm{d}\beta. \tag{5.1}$$

Doing a Taylor expansion of $1-a(c_4+\theta)$ at 0 yields (see Lemma 6.5)

$$h(s,t,\alpha,\beta) = \frac{t^2}{s^2}\,\frac{3\sin^2(\beta)-\cos^2(\beta)}{3\sin^2(\alpha)-\cos^2(\alpha)}\,\frac{1+\psi(t,\beta)}{1+\psi(s,\alpha)}, \tag{5.2}$$

where $\psi(s,\alpha)$ is a smooth function of s and α, and $\psi(s,\alpha)=O(s^2)$. If the last factor in (5.2) were equal to 1, then the inner two integrals in (5.1) would simplify to

$$\int_0^{\infty} g(s,\alpha)\,g(c(\alpha,\beta)s,\beta)\,\frac{\mathrm{d}s}{s},$$

for some constant $c(\alpha,\beta)$, which is easily estimated using Cauchy–Schwarz. The following is a perturbed version of this argument.

Fix α,β and write $h(s,t)=h(s,t,\alpha,\beta)$. Let $s(t)$ be defined implicitly by $h(s(t),t)=1$ (note that s also depends on α and β). Then

$$\begin{aligned}\int_0^{\varepsilon}\int_0^{\varepsilon} \delta(1-h(s,t))\,g(s,\alpha)\,g(t,\beta)\,\frac{\mathrm{d}s}{s}\,\frac{\mathrm{d}t}{t} &= \int_0^{\varepsilon} g(t,\beta)\,g(s(t),\alpha)\,\frac{1}{|\partial_s h(s(t),t)|}\,\frac{1}{s(t)\,t}\,\mathrm{d}t\\ &= \int_0^{\varepsilon} g(t,\beta)\,g(s(t),\alpha)\,\frac{1}{2+s(t)\,\frac{\psi'(s(t),\alpha)}{1+\psi(s(t),\alpha)}}\,\frac{1}{t}\,\mathrm{d}t.\end{aligned} \tag{5.3}$$

Here we used that

$$\partial_s h(s,t) = -h(s,t)\Big(\frac{2}{s}+\frac{\psi'(s,\alpha)}{1+\psi(s,\alpha)}\Big),$$

and hence

$$-\partial_s h(s(t),t) = \frac{2}{s(t)}+\frac{\psi'(s(t),\alpha)}{1+\psi(s(t),\alpha)}.$$

Applying Cauchy–Schwarz, we obtain that (5.3) is bounded by

$$\begin{aligned}(5.4)\qquad &\left(\int_0^{\varepsilon} g(t,\beta)^2 \,\frac{1}{2+s(t)\frac{\psi'(s(t),\alpha)}{1+\psi(s(t),\alpha)}}\,\frac{1}{t}\,\mathrm{d}t\right)^{1/2}\\ &\qquad\cdot\left(\int_0^{\varepsilon} g(s(t),\alpha)^2 \,\frac{1}{2+s(t)\frac{\psi'(s(t),\alpha)}{1+\psi(s(t),\alpha)}}\,\frac{1}{t}\,\mathrm{d}t\right)^{1/2}.\end{aligned}$$

After substituting $s = s(t)$ in the second integral, its integrand becomes the same as in the first one, but with the roles of (s,α) and (t,β) interchanged. By Lemma 6.5, it holds for $s \le 1/20$ that

$$|\psi(s,\alpha)| < \frac{1}{100} \quad\text{and}\quad |\psi'(s,\alpha)| \le \frac{1}{10},$$

giving

$$\left|s\frac{\psi'(s,\alpha)}{1+\psi(s,\alpha)}\right| \le \frac{1}{198}.$$

Thus, the factor in the integrals in (5.4) is bounded above by $198/395 < 101/200$. It follows that

$$\begin{aligned}\mathrm{II}_{44} &\le \frac{101}{200}\int_0^{2\pi}\int_0^{2\pi}\left(\int_0^{\varepsilon} g(t,\beta)^2\,\frac{\mathrm{d}t}{t}\right)^{1/2}\left(\int_0^{\varepsilon} g(s,\alpha)^2\,\frac{\mathrm{d}s}{s}\right)^{1/2}\mathrm{d}\alpha\,\mathrm{d}\beta\\ &\le \frac{101}{100}\,\pi\int_0^{2\pi}\int_0^{\varepsilon}|g(s,\alpha)|^2\,\frac{\mathrm{d}t}{t}\,\mathrm{d}\alpha = \frac{101}{100}\,\pi\int_{B_1}|\theta|^2\,|f(\theta)|^2\,\mathrm{d}\mathcal{H}^2_H(\theta).\end{aligned}$$

For the other eight integrals, the same estimate holds: by the argument in the proof of Lemma 4.1, changing c_4 to some other c_j only changes the expansion in (5.2) by a translation in α and β. Then the rest of the argument goes through exactly as for II_{44}. ∎

6. Technical estimates

Here we prove the computational lemmas that were used in the main argument.

We have the following explicit formula for ρ (see [3], Lemma 8):

$$(6.1)\qquad \rho(r) = \frac{4}{r}\int_{A(r)}^{1}\frac{\mathrm{d}u}{\sqrt{1-u^2}\sqrt{\frac{(1-r)^2}{2r}+1-u}\sqrt{\frac{(3+r)(1-r)}{2r}+1+u}},$$

with

$$A(r) = -1 + \max\left\{0, \frac{(3+r)(r-1)}{2r}\right\}.$$

From this, we obtain the following asymptotic formula.

Lemma 6.1. *Let ρ be defined by $\rho(|x|) = \sigma * \sigma * \sigma(x)$. Then we have, for all r with $|r-1| \le 1/10$,*

$$|\rho(r) + 6\log|1-r| - 12\log 2| \le -22\,|r-1|\log|r-1| + 23\,|r-1|.$$

We have not tried to optimize the error in this estimate. We give an elementary, self-contained proof below. For an alternative proof, one can use the identity (see p. 12 of [16] or equation (1.2) in [2])

$$\rho(x) = \begin{cases} \frac{16}{\sqrt{(x+1)^3(3-x)}}\, K\Big(\sqrt{\frac{16x}{(x+1)^3(3-x)}}\Big) & \text{if } 0 \le x < 1, \\ \frac{4}{\sqrt{x}}\, K\Big(\sqrt{\frac{(x+1)^3(3-x)}{16x}}\Big) & \text{if } 1 < x \le 3, \\ 0 & \text{if } x > 3, \end{cases} \tag{6.2}$$

where

$$K(k) = \int_0^1 \frac{1}{\sqrt{1-x^2}\,\sqrt{1-k^2x^2}}\,\mathrm{d}x$$

is the complete elliptic integral of the first kind, together with known asymptotics for $K(k)$ as $k \nearrow 1$.

We first prove some auxiliary lemmas.

Lemma 6.2. *For all $\delta > 0$, it holds that*

$$0 \le \int_0^1 \frac{1}{\sqrt{u}\,\sqrt{u+\delta}}\,\mathrm{d}u - \log\Big(\frac{4}{\delta}\Big) \le \frac{1}{2}\,\delta.$$

Proof. We have

$$\int_0^1 \frac{1}{\sqrt{u}\,\sqrt{u+\delta}}\,\mathrm{d}u = -\log(\delta) + 2\log\big(1+\sqrt{1+\delta}\,\big).$$

Furthermore, by the mean value theorem, there exists $0 < \delta' < \delta$ such that

$$\log\big(1+\sqrt{1+\delta}\,\big) = \log(2) + \delta g(\delta'),$$

where

$$0 < g(\delta) = \frac{1}{2(1+\sqrt{1+\delta})\sqrt{1+\delta}} \le \frac{1}{4}$$

is the derivative of $\log(1+\sqrt{1+\delta}\,)$. ∎

Lemma 6.3. *For all $0 < a, b < 1$, we have*

$$\Big|\int_0^1 \frac{1}{\sqrt{1-x^2}\sqrt{a+1-x}\sqrt{b+1+x}}\,\mathrm{d}x - \int_0^1 \frac{1}{\sqrt{1-x^2}\sqrt{a+1-x}\sqrt{1+x}}\,\mathrm{d}x\Big| \le \frac{b}{2}\Big(\log\Big(\frac{4}{a}\Big) + \frac{a}{2}\Big).$$

Proof. By the mean value theorem, we have for all $x \ge 0$,

$$|(b+1+x)^{-1/2} - (1+x)^{-1/2}| \le \frac{1}{2}\,b.$$

Hence the left-hand side of the claimed inequality is estimated by

$$\frac{b}{2}\int_0^1 \frac{1}{\sqrt{1-x}\sqrt{a+1-x}}\,\mathrm{d}x \le \frac{b}{2}\Big(\log\Big(\frac{4}{a}\Big) + \frac{a}{2}\Big),$$

where we applied Lemma 6.2. ∎

Lemma 6.4. *For all* $1 > a > 0$, *we have*

$$\left|\int_0^1 \frac{1}{(1+x)\sqrt{1-x}\sqrt{a+1-x}}\,\mathrm{d}x - \frac{1}{2}\log\left(\frac{8}{a}\right)\right| \le \frac{1}{2}a\log\left(1+\frac{1}{a}\right).$$

Proof. We have, with $v = 1-x$,

$$\int_0^1 \frac{1}{(1+x)\sqrt{1-x}\sqrt{a+1-x}}\,\mathrm{d}x = \int_0^1 \frac{1}{(2-v)\sqrt{v}\sqrt{a+v}}\,\mathrm{d}v,$$

which can be expanded to equal

$$\frac{1}{2}\int_0^1 \frac{1}{\sqrt{v}\sqrt{a+v}}\,\mathrm{d}v + \frac{1}{2}\int_0^1 \frac{1}{2-v}\,\mathrm{d}v - \frac{a}{2}\int_0^1 \frac{1}{(2-v)\sqrt{a+v}(\sqrt{v}+\sqrt{a+v})}\,\mathrm{d}v.$$

Computing the second integral and using Lemma 6.2 for the first one yields the main term $\log(8/a)/2$. For the error estimate, we combine Lemma 6.2 and the bound

$$\int_0^1 \frac{1}{(2-v)\sqrt{a+v}(\sqrt{v}+\sqrt{a+v})}\,\mathrm{d}v \le \int_0^1 \frac{1}{v+a}\,\mathrm{d}v = \log\left(1+\frac{1}{a}\right),$$

and note that the errors have opposite signs. ∎

Proof of Lemma 6.1. We start with the case $r = 1-\varepsilon < 1$. By (6.1), we have

$$\frac{1-\varepsilon}{4}\rho(1-\varepsilon) = \int_{-1}^1 \frac{1}{\sqrt{1-u^2}\sqrt{\frac{\varepsilon^2}{2-2\varepsilon}+1-u}\sqrt{\frac{(4-\varepsilon)\varepsilon}{2-2\varepsilon}+1+u}}\,\mathrm{d}u.$$

Combining Lemmas 6.3 and 6.4 with $a = \varepsilon^2/(2-2\varepsilon)$ and $b = (4-\varepsilon)\varepsilon/(2-2\varepsilon)$, we obtain that this integral equals

$$\frac{1}{2}\left(\log\left(\frac{8}{a}\right) + \log\left(\frac{8}{b}\right)\right) + E = 3\log(2) - \frac{3}{2}\log(\varepsilon) - \log(2-2\varepsilon) + \frac{1}{2}\log(4-\varepsilon) + E,$$

with

$$|E| \le \frac{1}{2}\left(b\log\left(\frac{4}{a}\right) + a\log\left(\frac{4}{b}\right) + ab + a\log\left(1+\frac{1}{a}\right) + b\log\left(1+\frac{1}{b}\right)\right). \tag{6.3}$$

It is easy to see that

$$\left|\frac{1}{2}\log(4-\varepsilon) - \log(2-2\varepsilon)\right| \le \frac{\varepsilon}{2}.$$

Further, one verifies that, when $0 < \varepsilon \le 1/10$,

$$a \le \frac{1}{18}\varepsilon, \quad b \le \frac{19}{9}\varepsilon, \quad \log\left(\frac{4}{a}\right) \le 3\log(2) - 2\log(\varepsilon), \quad \log\left(\frac{4}{b}\right) \le \log(2) - \log(\varepsilon)$$

and

$$\log\left(1+\frac{1}{a}\right) \le \log(2) - 2\log(\varepsilon), \quad \log\left(1+\frac{1}{b}\right) \le -\log(\varepsilon).$$

Using this, one can check that

$$|E| \leq \frac{13}{4}\,\varepsilon \log\left(\frac{1}{\varepsilon}\right) + \frac{5}{2}\,\varepsilon.$$

To summarize, we have shown that

$$\left|\frac{1-\varepsilon}{4}\,\rho(1-\varepsilon) - 3\log(2) + \frac{3}{2}\log(\varepsilon)\right| \leq \frac{13}{4}\,\varepsilon \log\left(\frac{1}{\varepsilon}\right) + 3\varepsilon.$$

We multiply by $4/(1-\varepsilon)$, and use that $|4/(1-\varepsilon) - 4| \leq 40\varepsilon/9$ to obtain

$$|\rho(1-\varepsilon) - 12\log(2) + 6\log(\varepsilon)| \leq 22\varepsilon \log\left(\frac{1}{\varepsilon}\right) + 23\varepsilon.$$

Now we turn to the case $r = 1 + \varepsilon > 1$. There we have

$$\begin{aligned}\rho(1+\varepsilon) &= \frac{4}{1+\varepsilon}\int_{-1+\frac{(4+\varepsilon)\varepsilon}{2+2\varepsilon}}^{1} \frac{1}{\sqrt{1-u^2}\sqrt{\frac{\varepsilon^2}{2+2\varepsilon}+1-u}\sqrt{-\frac{(4+\varepsilon)\varepsilon}{2+2\varepsilon}+1+u}}\,\mathrm{d}u \\ &= \frac{16}{4-\varepsilon^2}\int_{-1}^{1} \frac{1}{\sqrt{1-v^2}\sqrt{\frac{2\varepsilon^2}{4-\varepsilon^2}+1-v}\sqrt{\frac{8\varepsilon}{4-\varepsilon^2}+1+v}}\,\mathrm{d}v.\end{aligned}$$

We first approximate the integral. We can argue as in the case $r < 1$, now with $a = 2\varepsilon^2/(4-\varepsilon^2)$ and $b = 8\varepsilon/(4-\varepsilon^2)$. The main term is easily seen to be the same as in the case $r < 1$, and the error is bounded by

$$-\log\left(1 - \frac{\varepsilon^2}{4}\right) + E \leq \frac{\varepsilon}{40} + E,$$

with E satisfying (6.3). Now we have

$$a \leq \frac{1}{15}\,\varepsilon, \quad b \leq \frac{800}{399}\,\varepsilon, \quad \log\left(\frac{4}{a}\right) \leq 3\log(2) - 2\log(\varepsilon), \quad \log\left(\frac{4}{b}\right) \leq \log(2) - \log(\varepsilon),$$

and

$$\log\left(1 + \frac{1}{a}\right) \leq \log\left(\frac{201}{100}\right) - 2\log(\varepsilon), \quad \log\left(1 + \frac{1}{b}\right) \leq -\log(\varepsilon).$$

Using this, we obtain

$$|E| + \frac{\varepsilon}{40} \leq \frac{13}{4}\,\varepsilon \log\left(\frac{1}{\varepsilon}\right) + \frac{9}{4}\,\varepsilon.$$

In other words, it holds that

$$\left|\frac{4-\varepsilon^2}{16}\rho(1+\varepsilon) - 3\log(2) + \frac{3}{2}\log(\varepsilon)\right| \leq \frac{13}{4}\,\varepsilon \log\left(\frac{1}{\varepsilon}\right) + \frac{9}{4}\,\varepsilon.$$

We multiply by $16/(4-\varepsilon^2)$ and use that $|16/(4-\varepsilon^2) - 4| \leq 40\varepsilon/399$ to obtain

$$|\rho(1+\varepsilon) - 12\log(2) + 6\log(\varepsilon)| \leq 14\varepsilon \log\left(\frac{1}{\varepsilon}\right) + 9\varepsilon.$$

This completes the proof. ∎

Lemma 6.5. *Let θ be given by (4.3). Then it holds that*

$$a(c_4+\theta)-1 = s^2(3\sin^2(\alpha)-\cos^2(\alpha))\,(1+\psi(s,\alpha)),$$

where $\psi(s,\alpha)$ is a smooth function satisfying the following estimates:

$$|\psi(s,\alpha)| \le \frac{7}{24}s^2+\frac{17}{720}s^4+s^6e^{\sqrt{2}s},$$
$$|\psi'(s,\alpha)| \le \frac{14}{24}s+\frac{17}{180}s^3+2s^5e^{\sqrt{2}s}.$$

Proof. By the definition of h, the trigonometric identities and the Taylor expansion of cos, we have

$$\begin{aligned} a(c_4+\theta)-1 &= a((0,0,\pi)+\theta)-1 \\ &= (\cos(\theta_1)+\cos(\theta_2)-\cos(\theta_3))^2+(\sin(\theta_1)+\sin(\theta_2)-\sin(\theta_3))^2-1 \\ &= 2+2\cos(\theta_1-\theta_2)-2\cos(\theta_1-\theta_3)-2\cos(\theta_2-\theta_3) \\ &= 2\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k)!}\left((\theta_1-\theta_2)^{2k}-(\theta_1-\theta_3)^{2k}-(\theta_2-\theta_3)^{2k}\right) \qquad (6.4) \\ &=: \sum_{k=1}^{\infty}s^{2k}\frac{(-1)^k}{(2k)!}P_{2k}(\sin(\alpha),\cos(\alpha)). \end{aligned}$$

It follows from (6.4) that each P_{2k} vanishes when $\theta_1=\theta_3$ and when $\theta_2=\theta_3$, which is equivalent to $\alpha=\pm\pi/6$, or to $\cos(\alpha)=\pm\sqrt{3}\sin(\alpha)$. Hence, the homogeneous polynomial $P_{2k}(X,Y)$ vanishes on the lines $\sqrt{3}X+Y=0$ and $\sqrt{3}X-Y=0$. We conclude that for all k, the factor $3X^2-Y^2$ divides $P_{2k}(X,Y)$. Define Q_{2k} by

$$Q_{2k}(X,Y)(3X^2-Y^2)=(-1)^kP_{2k}(X,Y).$$

Then we have, using that $Q_2=1$,

$$a(c_4+\theta)-1=s^2(3\sin^2(\alpha)-\cos^2(\alpha))(1+\psi(s,\alpha)),$$

where ψ is defined by

$$\psi(s,\alpha)=\sum_{k=2}^{\infty}s^{2k-2}\frac{1}{(2k)!}Q_{2k}(\cos(\alpha),\sin(\alpha)).$$

Now we fix k and estimate

$$p(\alpha):=P_{2k}(\sin(\alpha),\cos(\alpha)) \quad \text{and} \quad q(\alpha):=Q_{2k}(\sin(\alpha),\cos(\alpha)).$$

By (4.3), we have that

$$\begin{aligned} \theta_1-\theta_2 &= \sqrt{2}\cos(\alpha), \\ \theta_3-\theta_1 &= -\frac{1}{\sqrt{2}}\cos(\alpha)-\frac{\sqrt{3}}{\sqrt{2}}\sin(\alpha)=\sqrt{2}\cos\left(\alpha+\frac{2\pi}{3}\right), \\ \theta_2-\theta_3 &= -\frac{1}{\sqrt{2}}\cos(\alpha)+\frac{\sqrt{3}}{\sqrt{2}}\sin(\alpha)=\sqrt{2}\cos\left(\alpha-\frac{2\pi}{3}\right). \end{aligned}$$

k	$P_{2k}(X,Y)$	$Q_{2k}(X,Y)$
1	$-3X^2+Y^2$	1
2	$-9X^4-18X^2Y^2+7Y^4$	$-3X^2-7Y^2$
3	$-\frac{1}{2}(27X^6+135X^4Y^2+45X^2Y^4-31Y^6)$	$\frac{1}{2}(9X^4+48X^2Y^2+31Y^4)$

Table 1. The polynomials P_{2k} and Q_{2k} for small values of k.

Thus, by (6.4),

$$p(\alpha)=2^{k+1}(-1)^k\Big(\cos(\alpha)^{2k}-\cos\Big(\alpha+\frac{2\pi}{3}\Big)^{2k}-\cos\Big(\alpha-\frac{2\pi}{3}\Big)^{2k}\Big).$$

Taking derivatives, and noting that the terms inside the brackets are each at most 1, we obtain

$$|p(\alpha)|\le 6\cdot 2^k,\quad |p'(\alpha)|\le 12k2^k \quad\text{and}\quad |p''(\alpha)|\le 24k^2 2^k.$$

Denote

$$q(\alpha):=\frac{p(\alpha)}{3\sin^2(\alpha)-\cos^2(\alpha)}=\frac{p(\alpha)}{(\sqrt{3}\sin(\alpha)-\cos(\alpha))\,(\sqrt{3}\sin(\alpha)+\cos(\alpha))}.$$

If both factors $|\sqrt{3}\sin(\alpha)\pm\cos(\alpha)|$ are at least $1/2$, we have that

$$q(\alpha)\le 24\cdot 2^k.$$

If not, then $|\alpha-\pi/6|<1/5$ or $|\alpha+\pi/6|<1/5$. Without loss of generality, we are in the first case. Then, by Taylor's formula,

$$\Big|\frac{p(\alpha)}{\alpha-\pi/6}-p'(\pi/6)\Big|\le\frac{1}{2}\Big|\alpha-\frac{\pi}{6}\Big|\sup|p''|\le\frac{1}{10}24k^2 2^k,$$

hence

$$\Big|\frac{p(\alpha)}{\alpha-\pi/6}\Big|\le 15k^2 2^k.$$

Furthermore, since $|\alpha-\pi/6|\le 1/5$,

$$\Big|\frac{\alpha-\pi/6}{(\sqrt{3}\sin(\alpha)-\cos(\alpha))(\sqrt{3}\sin(\alpha)+\cos(\alpha))}\Big|\le 2\Big|\frac{\alpha-\pi/6}{\sqrt{3}\sin(\alpha)-\cos(\alpha)}\Big|\le\frac{1/5}{\sin(1/5)}<2.$$

Multiplying the last two estimates, we conclude that $|q|\le 30k^2 2^k$. We also directly compute, for small k,

$$|Q_4(\sin(\alpha),\cos(\alpha))|=|-7\cos^2(\alpha)-3\sin^2(\alpha)|\le 7$$

and

$$|Q_6(\sin(\alpha),\cos(\alpha))|=\frac{1}{2}|9\sin^4(\alpha)+48\sin^2(\alpha)\cos^2(\alpha)+31\cos^2(\alpha)|\le\frac{5125}{312}<17.$$

Plugging in these estimates, we obtain

$$|\psi(s,\alpha)| \le \frac{7}{24}s^2 + \frac{17}{720}s^4 + \sum_{k=4}^{\infty}\frac{60k^2}{(2k)!}(\sqrt{2}s)^{2k-2} \le \frac{7}{24}s^2 + \frac{17}{720}s^4 + s^6 e^{\sqrt{2}s}$$

and

$$|\psi'(s,\alpha)| \le \frac{14}{24}s + \frac{17}{180}s^3 + \sqrt{2}\sum_{k=4}^{\infty}\frac{60k^2}{(2k-1)!}(\sqrt{2}s)^{2k-3} \le \frac{14}{24}s + \frac{17}{180}s^4 + 2s^5 e^{\sqrt{2}s},$$

as claimed. ∎

Lemma 6.6. *Let θ be given by (4.3). Then for all $0 \le s \le 1/20$, we have*

$$\begin{aligned}(a(c_4+\theta)-1)\rho(\sqrt{a(c_4+\theta)}) = {} & -12s^2(3\sin^2(\alpha)-\cos^2(\alpha))\log(s) \\ & - 6s^2(3\sin^2(\alpha)-\cos^2(\alpha))\log|3\sin^2(\alpha)-\cos^2(\alpha)| \\ & + 18\log 2\, s^2(3\sin^2(\alpha)-\cos^2(\alpha)) + E,\end{aligned}$$

with

$$|E| \le -180s^4\log s + 71s^4.$$

Proof. By Lemma 6.1, it holds that

$$\begin{aligned}(x^2-1)\rho(x) &= -6(x^2-1)\log|x-1| + 12\log(2)(x^2-1) + (x^2-1)E_1 \\ &= -6(x^2-1)\log|x^2-1| + 18\log(2)(x^2-1) + (x^2-1)E_1 \\ &\quad + 6(x^2-1)\log\Big(1+\frac{1}{2}(x-1)\Big),\end{aligned} \tag{6.5}$$

where

$$|E_1| \le -22|x-1|\log|x-1| + 23|x-1|.$$

Denote also the last term in (6.5) by E_2. We set

$$x = \sqrt{a(c_4+\theta)}.$$

Lemma 6.5 implies that

$$|x-1| \le |x^2-1| \le 2s^2.$$

Using this and monotonicity of $r\log r$, we obtain

$$\begin{aligned}|(x^2-1)E_1| &\le |x^2-1|(-22|x-1|\log|x-1| + 23|x-1|) \\ &\le -176s^4\log(s) + 32s^4\end{aligned} \tag{6.6}$$

and

$$|E_2| \le 6|x^2-1|\left|\log\Big(1+\frac{1}{2}(x-1)\Big)\right| \le 24s^4. \tag{6.7}$$

By Lemma 6.5, it holds that

$$
\begin{aligned}
&-6(x^2-1)\log|x^2-1| \\
&\quad = -6s^2(3\sin^2(\alpha)-\cos^2(\alpha))(1+\psi(s,\alpha))(2\log(s)+\log(3\sin^2(\alpha)-\cos^2(\alpha)) \\
&\qquad + \log(1+\psi(s,\alpha))) \\
&\quad = -12s^2\log(s)(3\sin^2(\alpha)-\cos^2(\alpha)) \qquad (6.8)\\
&\qquad - 6s^2(3\sin^2(\alpha)-\cos^2(\alpha))\log|3\sin^2(\alpha)-\cos^2(\alpha)| \qquad (6.9)\\
&\qquad - 6s^2(3\sin^2(\alpha)-\cos^2(\alpha))\log(1+\psi(s,\alpha))) \qquad (6.10)\\
&\qquad - 6s^2\psi(s,\alpha)(3\sin^2(\alpha)-\cos^2(\alpha)) \\
&\qquad\quad \times (2\log(s)+\log|3\sin^2(\alpha)-\cos^2(\alpha)|+\log(1+\psi(s,\alpha))). \qquad (6.11)
\end{aligned}
$$

The term (6.10) bounded by $18s^2|\psi(s,\alpha)| \leq 6s^4$. The term (6.11) is bounded by

$$-18s^2\log(s)|\psi(s,\alpha)| + 4s^2|\psi(s,\alpha)| + 18s^2\psi(s,\alpha)^2 \leq -6s^4\log(s) + 2s^4.$$

For the second term in (6.5), we have

$$
\begin{aligned}
&18\log(2)(x^2-1) \\
&= 18\log(2)s^2(3\sin^2(\alpha)-\cos^2(\alpha)) + 18\log(2)s^2(3\sin^2(\alpha)-\cos^2(\alpha))\psi(s,\alpha), \qquad (6.12)
\end{aligned}
$$

with the second term bounded by

$$27\log(2)s^2|\psi(s,\alpha)| \leq 9\log(2)s^4.$$

Putting together the main terms (6.8), (6.9) and (6.12), and the estimates for the error terms in (6.6), (6.7), (6.10), (6.11) and in (6.12), one obtains the lemma. ∎

Lemma 6.7. *For all α, it holds that*

$$
\begin{aligned}
&\sum_{j=1}^{3}\left(3\sin^2\left(\alpha+\frac{2\pi j}{3}\right)-\cos^2\left(\alpha+\frac{2\pi j}{3}\right)\right)\log\left|3\sin^2\left(\alpha+\frac{2\pi j}{3}\right)-\cos^2\left(\alpha+\frac{2\pi j}{3}\right)\right| \\
&\qquad \leq 3\log(3).
\end{aligned}
$$

Proof. Let

$$a_j = \sin^2\left(\alpha+\frac{2\pi j}{3}\right) - \frac{1}{3}\cos^2\left(\alpha+\frac{2\pi j}{3}\right) = \frac{1}{3} - \frac{2}{3}\cos\left(2\alpha+\frac{4\pi j}{3}\right).$$

It is easy to check that

$$a_1 + a_2 + a_3 = 1 \quad \text{and} \quad a_1^2 + a_2^2 + a_3^2 = 1. \tag{6.13}$$

Defining

$$b_j = \frac{a_j + a_{j-1}}{2}$$

(note that $a_{j+3} = a_j$), it follows that

$$b_1 + b_2 + b_3 = 1 \quad \text{and} \quad b_1^2 + b_2^2 + b_3^2 = 1/2,$$

hence $b_1, b_2, b_3 \geq 0$. Using Jensen's inequality, we deduce

$$\sum_{j=1}^{3} a_j \log(|a_j|) = \sum_{j=1}^{3} b_j \log\Big(\frac{|a_j||a_{j-1}|}{|a_{j-2}|}\Big) \leq \log\Big(\sum_{j=1}^{3} b_j \frac{|a_j||a_{j-1}|}{|a_{j-2}|}\Big).$$

By (6.13), we have that

$$2a_j a_{j-1} = (a_j + a_{j-1})^2 - (a_j^2 + a_{j-1}^2) = (1 - a_{j-2})^2 - (1 - a_{j-2}^2) = 2a_{j-2}(a_{j-2} - 1).$$

Thus, using again (6.13)

$$\sum_{j=1}^{3} b_j \frac{|a_j||a_{j-1}|}{|a_{j-2}|} = \sum_{j=1}^{3} b_j (1 - a_{j-2}) = 1.$$

We conclude that

$$\sum_{j=1}^{3} 3a_j \log(|3a_j|) = 3\log(3) + 3\sum_{j=1}^{3} a_j \log|a_j| \leq 3\log 3. \qquad \blacksquare$$

7. Discussion

7.1. Optimal value of ε

An inspection of the above argument shows that $Q(f) \geq 0$ for all $f \in V_\varepsilon$ as long as

$$\inf_{\theta \in H, |\theta| \leq \varepsilon} \frac{1}{2} \sum_{j=2}^{4} (a(\theta + c_j) - 1)\, \rho\big(\sqrt{a(\theta + c_j)}\big) \geq 18\pi \sup_{s \leq \varepsilon, \alpha \in [0,2\pi]} \frac{1}{2 + s\frac{\psi'(s,\alpha)}{1+\psi(s,\alpha)}}. \tag{7.1}$$

(Non-rigorous) numerical computations suggest that this inequality holds up to $\varepsilon = 0.104$. The constant ε' in Corollary 1.3 could then be increased to 0.063.

7.2. Fourier coefficients of Q

In [1], some numerical observations on the Fourier coefficients

$$\hat{B}(k, l) := B(\omega_1^{k_1} \omega_2^{k_2} \omega_3^{k_3}, \omega_4^{l_1} \omega_5^{l_2} \omega_6^{l_3})$$

of B with $k_1 + k_2 + k_3 = l_1 + l_2 + l_3 = 0$ are discussed. Namely, they are very large only when k is very close to l and when $k_1^2 + k_2^2 + k_3^2 \approx l_1^2 + l_2^2 + l_3^2$. We can explain this using Lemma 3.2 as follows.

By Lemma 3.2 and since $\lambda_0 = 1$, for all $f \in X_0$, the form $B(f, f)$ can be expressed as

$$\int_C m(\theta)\, |f(\theta)|^2 \,\mathrm{d}\mathcal{H}_C^2(\theta) + \int_{C^2} n(\theta)\, \delta(a(\theta) - a(\theta'))\, f(\theta)\, \overline{f(\theta')}\, \mathrm{d}\mathcal{H}_C^2(\theta)\, \mathrm{d}\mathcal{H}_C^2(\theta')$$

for certain functions m and n.

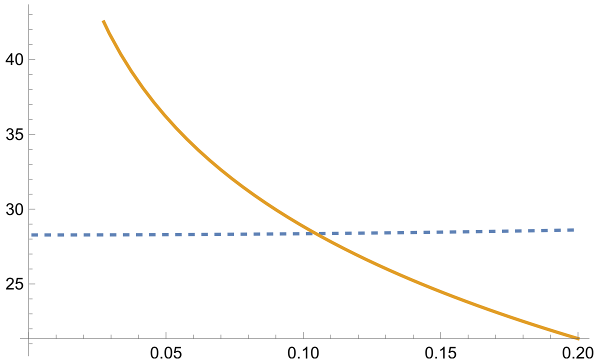

Figure 2. The left-hand side (solid) and the right-hand side (dashed) of (7.1).

The first term is a multiplier, hence it acts on the Fourier side by convolution with a fixed bump function. This bump function decays at least like $|k-l|^{-3}$, because the third derivative of m is still integrable. This explains the large coefficients when k is close to l.

The Fourier coefficients of the second term are the Fourier coefficients of the measure

$$\mu := n(\theta)\,\delta(a(\theta)-a(\theta'))$$

supported on the 3-manifold

$$M := \{(x,y) \in C^2 : a(x) = a(y)\} \subset \mathbb{R}^6.$$

The measure μ has a smooth, bounded density with respect to the Hausdorff measure on this manifold, except in the critical points of a. The Fourier transform of the parts where the measure has a smooth, bounded density can be estimated using the method of stationary phase, and are of lower order than the contribution of the critical points. To explain what happens at a critical point (where $\det D^2 a \neq 0$), we choose coordinates x_1, x_2, y_1, y_2 for C^2, such that the critical point of a is at 0. After a scaling in a and a linear change of variables, either

$$a(x) = x_1^2 + x_2^2 + O(|x|^3) \quad \text{or} \quad a(x) = x_1^2 - x_2^2 + O(|x|^3). \tag{7.2}$$

Thus, ignoring higher order terms,

$$\delta(a(x)-a(y)) \approx \delta(|x|^2-|y|^2) \quad \text{or} \quad \delta(a(x)-a(y)) \approx \delta(x_1^2-x_2^2-y_1^2+y_2^2).$$

The Fourier transforms of these measures can be explicitly computed, in fact, they are up to a constant factor their own Fourier transform. Now, a has one local maximum and two local minima, which together with the above discussion explain why $\hat{B}(k,l)$ is very large on the cone $|k|^2 = |l|^2$. The contribution of all other critical points is of smaller order, since the weight n vanishes there.

This discussion can be turned into a rigorous proof that the Fourier coefficients of μ concentrate near the cone $|k|^2 = |l|^2$. However, we can only show that they concentrate in, e.g.,

$$\{(k,l) : ||k|-|l|| \leq C|k|^{1/2}\},$$

and not in an $O(1)$ neighborhood of the cone, because of the higher order terms in (7.2).

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