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Minimal semiinjective resolutions in the Q-shaped derived category

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Abstract. Semiinjective resolutions of chain complexes are used for the computation of Hom spaces in the derived category $\mathscr{D}(A)$ of a ring A. Minimal semiinjective resolutions have the additional property of being unique. The Q-shaped derived category $\mathscr{D}_Q(A)$ consists of Q-shaped diagrams for a suitable preadditive category Q, and it generalises $\mathscr{D}(A)$. Some special cases of $\mathscr{D}_Q(A)$ are the derived categories of differential modules, m-periodic chain complexes, and N-complexes, and there are many other possibilities. The category $\mathscr{D}_Q(A)$ shares some key properties of $\mathscr{D}(A)$; for instance, it is triangulated and compactly generated. This paper establishes a theory of minimal semiinjective resolutions in $\mathscr{D}_Q(A)$. As a sample application, it generalises a theorem by Ringel and Zhang on differential modules.

1. Introduction

This paper generalises the theory of minimal semiinjective resolutions in $\mathscr{D}(A)$, the classic derived category of a ring A, to $\mathscr{D}_Q(A)$, the Q-shaped derived category.

The Q-shaped derived category was defined in [18] and [20]; see [19] for a quick introduction. The objects of $\mathscr{D}_Q(A)$ are Q-shaped diagrams of A-modules, where Q is a suitable preadditive category. For example, Q could be given by Figure 1 or Figure 2 with the relations that N consecutive arrows compose to zero for some fixed $N \geqslant 2$, and then $\mathscr{D}_Q(A)$ would be the derived category of N-complexes or m-periodic N-complexes. The case $N = 2$ shows that $\mathscr{D}_Q(A)$ can be specialised to $\mathscr{D}(A)$, but there is a range of other choices of Q enabling the construction of bespoke categories $\mathscr{D}_Q(A)$, which are compactly generated triangulated categories like $\mathscr{D}(A)$.

The theory of minimal semiinjective resolutions in $\mathscr{D}(A)$ was developed by Avramov–Foxby–Halperin [1], Christensen–Foxby–Holm (Appendix B of [5]), Foxby (Section 10 of [10]), García Rozas (Sections 2.3 and 2.4 of [12]), Krause (Appendix B of [28]), and, in an abstract version, by Roig [31]. Minimal injective resolutions of modules are a special case, and minimal semiinjective resolutions in $\mathscr{D}(A)$ have a range of applications. Chen–Iyengar and Foxby used them to investigate the small support (see Proposition 2.1 in [4] and Remark 2.9 in [9]), Christensen–Iyengar–Marley used them to prove results on

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$$\cdots \longrightarrow 2 \longrightarrow 1 \longrightarrow 0 \longrightarrow -1 \longrightarrow -2 \longrightarrow \cdots$$

Figure 1. A chain complex is a diagram of this form.

Ext rigidity (see Proposition 3.2, Proposition 3.4 and Theorem 5.1 in [6]), Enochs–Jenda–Xu linked them to relative homological algebra (see Theorem 3.18 in [8]), and Iacob–Iyengar used them to characterise regular rings (see Proposition 2.10 in [23]).

Motivated by this, we will develop a theory of minimal semiinjective resolutions in $\mathscr{D}_Q(A)$. We will provide different characterisations of minimal semiinjective objects in Theorem 3.1 and use them to establish the existence and uniqueness of minimal semiinjective resolutions in Theorems B, C and D. As a sample application, we will generalise Ringel and Zhang's result (Theorem 2 in [30]) on differential modules; see Theorem E.

Background

Let us explain the notation and definitions which will be used in the rest of the paper. First, we fix the following.

- $\Bbbk$ is a hereditary noetherian commutative ring.
- A is a $\Bbbk$-algebra.
- $\mathrm{Mod}(A)$ is the category of A-left modules, and $\mathrm{Inj}(A)$ is the full subcategory of injective modules.
- Q is a small $\Bbbk$-preadditive category where Q_0 denotes the class of objects and $Q(-,-)$ the Hom functor. Using the terminology of Setup 2.1 in [19], we assume that Q satisfies the following conditions.
  - *Hom finiteness*: Each $Q(p,q)$ is a finitely generated free $\Bbbk$-module.
  - *Local boundedness*: For each $q \in Q_0$, the sets
  $$\{p \in Q_0 \mid Q(q,p) \neq 0\} \quad \text{and} \quad \{p \in Q_0 \mid Q(p,q) \neq 0\}$$
  are finite.
  - *Serre functor*: There is a Serre functor $Q \xrightarrow{S} Q$ for which there are isomorphisms $Q(p,q) \cong \mathrm{Hom}_{\Bbbk}\big(Q(q,Sp),\Bbbk\big)$, natural in p and q.
  - *Strong retraction*: There are fixed decompositions of $\Bbbk$-modules
  $$Q(q,q) = (\Bbbk \cdot \mathrm{id}_q) \oplus \mathfrak{r}_q$$
  for q in Q_0 which satisfy
    (i) $\mathfrak{r}_q \circ \mathfrak{r}_q \subseteq \mathfrak{r}_q$,
    (ii) $Q(p,q) \circ Q(q,p) \subseteq \mathfrak{r}_q$ for $p \neq q$.
  The ideal in Q defined by
  $$\mathfrak{r}(p,q) = \begin{cases} \mathfrak{r}_q & \text{if } p = q, \\ Q(p,q) & \text{if } p \neq q, \end{cases}$$
  for $p,q \in Q_0$ is called the pseudoradical.
  - *Nilpotence:* The pseudoradical satisfies $\mathfrak{r}^N = 0$ for some integer $N \geqslant 1$.

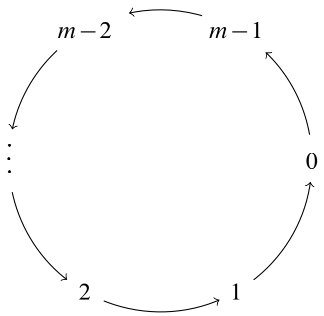


Figure 2. An m-periodic chain complex is a diagram of this form.

Secondly, we recall some items from [18] and [20].

- The category of Q-shaped diagrams with values in $\mathrm{Mod}(A)$ is

$$_{Q,A}\mathrm{Mod} = \{\, \Bbbk\text{-linear functors } Q \to \mathrm{Mod}(A)\,\}.$$

  It is a Grothendieck abelian category which generalises the abelian category of chain complexes of A-modules; see 2.5 in [19]. In $_{Q,A}\mathrm{Mod}$, the Hom functor is $\mathrm{Hom}_{Q,A}$, the ith Ext functor is $\mathrm{Ext}^i_{Q,A}$, and the full subcategory of injective objects is $_{Q,A}\mathrm{Inj}$. For f and g in $\mathrm{Hom}_{Q,A}(X,Y)$, we write $f \sim g$ if $f-g$ factors through an object of $_{Q,A}\mathrm{Inj}$.

- If $A = \Bbbk$, then A will be omitted from the notation in the previous bullet point. For instance, we have

$$_{Q}\mathrm{Mod} = \{\, \Bbbk\text{-linear functors } Q \to \mathrm{Mod}(\Bbbk)\,\}.$$

- The class of exact objects in $_{Q,A}\mathrm{Mod}$ is

$$\mathscr{E} = \{\, X \in {}_{Q,A}\mathrm{Mod} \mid \mathrm{pd}_Q(X) < \infty \,\} = \{\, X \in {}_{Q,A}\mathrm{Mod} \mid \mathrm{id}_Q(X) < \infty \,\}. \tag{1.1}$$

  It generalises the class of exact chain complexes; see 3.2 and 3.4 in [19]. In the formula, $\mathrm{pd}_Q(X)$ and $\mathrm{id}_Q(X)$ are the projective and injective dimensions of X viewed as an object of $_Q\mathrm{Mod}$ by forgetting the A-structure.

- The class of weak equivalences in $_{Q,A}$ Mod is

$$\mathrm{weq} = \left\{ X \xrightarrow{x} Y \text{ in } {}_{Q,A}\mathrm{Mod} \;\middle|\; \begin{array}{l} x = pj \text{ where } j \text{ is monic with cokernel} \\ \text{in } \mathscr{E}, \text{ and } p \text{ is epic with kernel in } \mathscr{E} \end{array} \right\}. \tag{1.2}$$

  It generalises the class of quasi-isomorphisms of chain complexes; see 3.3 and 3.4 in [19].

- The Q-shaped derived category, obtained by inverting the morphisms in weq, is

$$\mathscr{D}_Q(A) = \mathrm{weq}^{-1}\, {}_{Q,A}\mathrm{Mod}.$$

  It generalises the classic derived category of A; see 3.10 in [19].

- For each q in Q_0, there are stalk functors

$$S\langle q\rangle = Q(q,-)/\mathfrak{r}(q,-) \quad \text{and} \quad S\{q\} = Q(-,q)/\mathfrak{r}(-,q)$$

which take the value $\Bbbk$ at q and zero elsewhere; they are objects of ${}_Q\mathrm{Mod}$, respectively ${}_{Q^{\mathrm{op}}}\mathrm{Mod}$. Given i in $\mathbb{Z}$, they permit the definition of the (co)homology functors

$$\text{(1.3)} \qquad \mathbb{H}^i_{[q]}(-) = \mathrm{Ext}^i_Q(S\langle q\rangle, -) \quad \text{and} \quad \mathbb{H}^{[q]}_i(-) = \mathrm{Tor}^Q_i(S\{q\}, -).$$

These are $\Bbbk$-linear functors ${}_{Q,A}\mathrm{Mod} \to \mathrm{Mod}(A)$ which generalise the classic (co)homology functors on chain complexes; see 3.4 in [19].

- The classes $\mathscr{E}$ and weq satisfy

$$\text{(1.4)} \qquad \begin{aligned} \mathscr{E} &= \{\, X \in {}_{Q,A}\mathrm{Mod} \mid \mathbb{H}^1_{[q]}(X) = 0 \text{ for } q \text{ in } Q_0 \,\} \\ &= \{\, X \in {}_{Q,A}\mathrm{Mod} \mid \mathbb{H}^{[q]}_1(X) = 0 \text{ for } q \text{ in } Q_0 \,\} \end{aligned}$$

and

$$\text{(1.5)} \qquad \begin{aligned} \mathrm{weq} &= \{\, X \xrightarrow{x} Y \text{ in } {}_{Q,A}\mathrm{Mod} \mid \mathbb{H}^1_{[q]}(x) \text{ and } \mathbb{H}^2_{[q]}(x) \text{ are isomorphisms} \,\} \\ &= \{\, X \xrightarrow{x} Y \text{ in } {}_{Q,A}\mathrm{Mod} \mid \mathbb{H}^{[q]}_1(x) \text{ and } \mathbb{H}^{[q]}_2(x) \text{ are isomorphisms} \,\}. \end{aligned}$$

This generalises the characterisation by (co)homology of exact chain complexes and quasi-isomorphisms; see 3.4 in [19].

- For each q in Q_0, there are adjoint pairs (F_q, E_q) and (E_q, G_q) as follows:

$$\text{(1.6)} \qquad {}_{Q,A}\mathrm{Mod} \underset{\overset{\longleftarrow}{G_q}}{\overset{\overset{F_q}{\longleftarrow}}{\xrightarrow{E_q}}} \mathrm{Mod}(A) \quad \text{given by} \quad \begin{aligned} F_q(M) &= Q(q,-)\underset{\Bbbk}{\otimes} M, \\ E_q(X) &= X(q), \\ G_q(M) &= \mathrm{Hom}_{\Bbbk}(Q(-,q), M). \end{aligned}$$

The functor E_q generalises the functor sending a chain complex to its qth component. The functors F_q and G_q generalise the indecomposable projective and injective representations of Q at q known from quiver representations; see Definition 5.3 in [32].

- For each q in Q_0, there are adjoint pairs (C_q, S_q) and (S_q, K_q) as follows:

$$\text{(1.7)} \qquad \mathrm{Mod}(A) \underset{\overset{\longleftarrow}{K_q}}{\overset{\overset{C_q}{\longleftarrow}}{\xrightarrow{S_q}}} {}_{Q,A}\mathrm{Mod} \quad \text{given by} \quad \begin{aligned} C_q(X) &= S\{q\}\underset{Q}{\otimes} X, \\ S_q(M) &= S\langle q\rangle\underset{\Bbbk}{\otimes} M, \\ K_q(X) &= \mathrm{Hom}_Q(S\langle q\rangle, -). \end{aligned}$$

The functor S_q generalises the simple representation of Q at q known from quiver representations; see Definition 2.2 in [32]. The functors C_q and K_q generalise the functors sending a chain complex to the cokernel, respectively kernel, of the differentials which have q as the target, respectively source; see Proposition 7.18 in [18].

The conditions on Q are mainly due to Dell'Ambrogio–Stevenson–Šťovíček (Theorem 1.6 in [7]). They are satisfied in the examples where Q is given by Figure 1 or Figure 2 with the relations that N consecutive arrows compose to zero for some fixed $N \geqslant 2$; see 2.5 in [19]. The definition of $\mathscr{D}_Q(A)$ is based on the insight of Iyama–Minamoto that the key property of Q which makes $\mathscr{D}_Q(A)$ well behaved is the existence of a Serre functor on Q; see [26] and Section 2 of [25].

Semiinjective objects

The following is the key definition of this paper. Note that part (i) appeared in 3.6 of [19], and that the class $\mathscr{E}^{\perp}$ was used intensively in [18, 20].

Definition A. (i) A *semiinjective object* in ${}_{Q,A}\mathrm{Mod}$ is an object in the class

$$\mathscr{E}^{\perp} = \{ I \in {}_{Q,A}\mathrm{Mod} \mid \mathrm{Ext}^1_{Q,A}(\mathscr{E}, I) = 0 \}.$$

(ii) A *minimal semiinjective object* in ${}_{Q,A}\mathrm{Mod}$ is a semiinjective object whose only subobject in ${}_{Q,A}\mathrm{Inj}$ is 0.

(iii) A *semiinjective resolution* of X in ${}_{Q,A}\mathrm{Mod}$ is a weak equivalence $X \to I$ with I semiinjective.

(iv) A *minimal semiinjective resolution* of X in ${}_{Q,A}\mathrm{Mod}$ is a weak equivalence $X \to I$ with I minimal semiinjective.

These concepts generalise (minimal) semiinjective chain complexes and (minimal) semiinjective resolutions of chain complexes; see Proposition 2.3.14 and Section 2.4 of [12], and 3.7 in [19]. Semiinjective chain complexes are due to Bökstedt–Neeman, who used the term "special complexes of injectives" (see Section 2 of [2]), and García-Rozas, who used the term "DG-injective complexes" (see Proposition 2.3.4 in [12]).

One reason for the interest in semiinjective objects is that they can be used to compute Hom spaces in $\mathscr{D}_Q(A)$, which are otherwise hard to access. Each object Y in ${}_{Q,A}\mathrm{Mod}$ has a semiinjective resolution $Y \to I$ by Theorem B (i). If X is also an object in ${}_{Q,A}\mathrm{Mod}$, then

$$\mathrm{Hom}_{\mathscr{D}_Q(A)}(X, Y) \cong \mathrm{Hom}_{Q,A}(X, I)/\sim \tag{1.8}$$

by Proposition 2.1 (ii). Equation (1.8) generalises the computation of Hom spaces in $\mathscr{D}(A)$ using semiinjective resolutions of chain complexes; see Corollary 7.3.22 in [5].

Minimal semiinjective objects and resolutions

Our main results establish the existence and uniqueness of minimal semiinjective resolutions as follows.

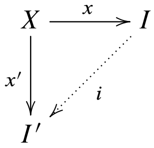
Theorem B. (i) *Each X in ${}_{Q,A}\mathrm{Mod}$ has a minimal semiinjective resolution.*

(ii) *Each semiinjective object I in ${}_{Q,A}\mathrm{Mod}$ has the form $I = I' \oplus J'$ in ${}_{Q,A}\mathrm{Mod}$, with I' a minimal semiinjective object and J' in ${}_{Q,A}\mathrm{Inj}$.*

Theorem C. *If $I \xrightarrow{i} I'$ in ${}_{Q,A}\mathrm{Mod}$ is a weak equivalence between minimal semiinjective objects, then i is an isomorphism in ${}_{Q,A}\mathrm{Mod}$.*

Theorem D. *If* $X \xrightarrow{x} I$ *and* $X \xrightarrow{x'} I'$ *are minimal semiinjective resolutions in* ${}_{Q,A}\mathrm{Mod}$*, then the following holds.*

- *The diagram of solid arrows*



can be completed with a morphism i *such that* $ix \sim x'$ *in* ${}_{Q,A}\mathrm{Mod}$.

- *The morphism* i *is unique up to equivalence under "*$\sim$*".*
- *Each completing morphism* i *is an isomorphism in* ${}_{Q,A}\mathrm{Mod}$.

Note that the first bullet in Theorem D cannot be improved to say $ix = x'$ instead of $ix \sim x'$.

Another main result is Theorem 3.1, which provides different characterisations of minimal semiinjective objects. Finally, Appendix A explains how the results can be specialised to the theory of minimal semiinjective resolutions in $\mathscr{D}(A)$.

The proof of Theorem B uses the full force of the results on the Q-shaped derived category established in [18] and [20] as well as most of the machinery developed in this paper. The easier Theorems C and D could have been obtained as consequences of Corollaries 1 and 2 in [31] because Theorem 3.1 (iii) implies that our notion of minimal semiinjective resolutions is an instance of the right minimal models of Section 1 in [31]. However, we provide short, self contained proofs for the benefit of the reader.

Differential modules

As a sample application of our theory, we will generalise Ringel and Zhang's result (Theorem 2 in [30]) on differential modules.

A differential module over the ring A is a pair (M, ∂) with M in $\mathrm{Mod}(A)$ and $M \xrightarrow{\partial} M$ an endomorphism with $\partial^2 = 0$. This notion was defined by Cartan–Eilenberg under the name "modules with differentiation", see [3], p. 53. There is a Grothendieck abelian category $\mathrm{Diff}(A)$ of differential modules over A in which the notions of injective and Gorenstein injective objects make sense, see Section 7 of [28]. The homology functor $\mathrm{Diff}(A) \xrightarrow{\mathrm{H}} \mathrm{Mod}(A)$ is defined on objects by $\mathrm{H}(M, \partial) = \mathrm{Z}(M, \partial)/\,\mathrm{B}(M, \partial)$, where $\mathrm{Z}(M, \partial) = \operatorname{Ker} \partial$ is the cycles, and $\mathrm{B}(M, \partial) = \operatorname{Im} \partial$ the boundaries.

A notable result on differential modules was proved by Ringel and Zhang in Theorem 2 of [30]. They worked with finite dimensional differential modules over the path algebra of a finite, acyclic quiver. We provide the following generalisation to arbitrary differential modules over a hereditary ring.

Theorem E. *Assume that* A *is a left hereditary ring. Then the homology functor*

$$\mathrm{Diff}(A) \xrightarrow{\mathrm{H}} \mathrm{Mod}(A)$$

induces a bijection

$$\left\{\begin{array}{l}\textit{Isomorphism classes of Gorenstein}\\ \textit{injective objects without non-zero}\\ \textit{injective summands in } \operatorname{Diff}(A)\end{array}\right\} \xrightarrow{\mathrm{H}} \left\{\begin{array}{l}\textit{Isomorphism classes}\\ \textit{in } \operatorname{Mod}(A)\end{array}\right\}. \tag{1.9}$$

To place this result in a wider context, recall from Section 7 of [28] that if $\mathscr{A}$ is a Grothendieck abelian category, then $\operatorname{GInj}\mathscr{A}$, the full subcategory of Gorenstein injective objects of $\mathscr{A}$, is a Frobenius category with projective-injective objects given by $\operatorname{Inj}\mathscr{A}$, the injective objects of $\mathscr{A}$. The naïve quotient category $\operatorname{GInj}\mathscr{A}/\operatorname{Inj}\mathscr{A}$ is triangulated, by Theorem I.2.6 in [15], and is important in the context of Gorenstein approximations and Tate cohomology. Understanding the objects of $\operatorname{GInj}\mathscr{A}/\operatorname{Inj}\mathscr{A}$ amounts to understanding the objects of $\operatorname{GInj}\mathscr{A}$ up to injective summands, see Theorem 13.7 in [16], and this is accomplished by Theorem E for $\mathscr{A} = \operatorname{Diff}(A)$.

Theorem E will be proved by translating the left-hand set of equation (1.9) to the set of isomorphism classes of minimal semiinjective differential modules. The inverse bijection to (1.9) is induced by sending M to a minimal semiinjective resolution of $(M, 0)$. Theorem E is an injective analogue of Corollary 1.4 in [34].

Structure of the paper

Section 2 proves some preliminary results. Section 3 provides different characterisations of minimal semiinjective objects in Theorem 3.1 and uses them to prove Theorems B, C, and D. Section 4 proves Theorem E. Appendix A shows how our theory specialises to the theory of minimal semiinjective resolutions in $\mathscr{D}(A)$.

2. Preliminary results

This section proves some preliminary results required to establish the theorems stated in the introduction. Notation and definitions from the "Background" part of the introduction will be used freely.

Proposition 2.1.

(i) *There are isomorphisms*

$$\operatorname{Hom}_{Q,A}(X, I)/\sim \longrightarrow \operatorname{Hom}_{\mathscr{D}_Q(A)}(X, I),$$

natural with respect to X *in* ${}_{Q,A}\operatorname{Mod}$ *and* I *in* $\mathscr{E}^{\perp}$.

(ii) *If* X *and* Y *are in* ${}_{Q,A}\operatorname{Mod}$ *and* $Y \to I$ *is a semiinjective resolution, then there is an isomorphism*

$$\operatorname{Hom}_{Q,A}(X, I)/\sim \longrightarrow \operatorname{Hom}_{\mathscr{D}_Q(A)}(X, Y).$$

(iii) *If* $X \to Y$ *is a weak equivalence in* ${}_{Q,A}\operatorname{Mod}$ *and* I *is in* $\mathscr{E}^{\perp}$, *then the induced map*

$$\operatorname{Hom}_{Q,A}(Y, I)/\sim \longrightarrow \operatorname{Hom}_{Q,A}(X, I)/\sim$$

is an isomorphism.

Proof. (i) Use Theorem 6.1 (b) in [18], and its proof, to get a "Hovey triple"

$$({}_{Q,A}\mathrm{Mod}, \mathscr{E}, \mathscr{E}^{\perp}).$$

Then apply Theorem 2.6 in [13], where the Hovey triple is called "abelian model category", noting that the "core" ${}_{Q,A}\mathrm{Mod} \cap \mathscr{E} \cap \mathscr{E}^{\perp}$ is ${}_{Q,A}\mathrm{Inj}$ by Theorem 4.4 (b) in [18].

(ii) Compose the isomorphism from part (i) with the inverse of the isomorphism

$$\mathrm{Hom}_{\mathscr{D}_Q(A)}(X, Y) \longrightarrow \mathrm{Hom}_{\mathscr{D}_Q(A)}(X, I)$$

which results from the weak equivalence $Y \to I$ inducing an isomorphism in $\mathscr{D}_Q(A)$.

(iii) By part (i), the morphism $X \to Y$ induces a commutative square

$$\begin{array}{ccc} \mathrm{Hom}_{Q,A}(Y, I)/\sim & \longrightarrow & \mathrm{Hom}_{\mathscr{D}_Q(A)}(Y, I) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{Q,A}(X, I)/\sim & \longrightarrow & \mathrm{Hom}_{\mathscr{D}_Q(A)}(X, I), \end{array}$$

where the horizontal maps are isomorphisms. The right-hand vertical map is an isomorphism because the weak equivalence $X \to Y$ induces an isomorphism in $\mathscr{D}_Q(A)$, so the left-hand vertical map is also an isomorphism. ∎

Proposition 2.2. *Let* $0 \to X' \xrightarrow{x'} X \xrightarrow{x} X'' \to 0$ *be a short exact sequence in* ${}_{Q,A}\mathrm{Mod}$.

(i) x' *is a weak equivalence* $\iff$ X'' *is in* $\mathscr{E}$.

(ii) x *is a weak equivalence* $\iff$ X' *is in* $\mathscr{E}$.

Proof. For each q in Q_0, there are long exact sequences

$$\cdots \to \mathbb{H}^{[q]}_{i+1}(X'') \to \mathbb{H}^{[q]}_{i}(X') \xrightarrow{x'_*} \mathbb{H}^{[q]}_{i}(X) \xrightarrow{x_*} \mathbb{H}^{[q]}_{i}(X'') \to \mathbb{H}^{[q]}_{i-1}(X') \to \cdots$$

and

$$\cdots \to \mathbb{H}^{i-1}_{[q]}(X'') \to \mathbb{H}^{i}_{[q]}(X') \xrightarrow{x'_*} \mathbb{H}^{i}_{[q]}(X) \xrightarrow{x_*} \mathbb{H}^{i}_{[q]}(X'') \to \mathbb{H}^{i+1}_{[q]}(X') \to \cdots .$$

Combining these with Theorems 7.1 and 7.2 in [18] proves the lemma. ∎

The following lemma, and later parts of the paper, use the notions of (special) preenvelopes, left minimal morphisms, and envelopes, see Definitions 2.1.1 and 2.1.12 in [14].

Lemma 2.3. *Let* E *be in* $\mathscr{E}$.

(i) *Each* ${}_{Q,A}\mathrm{Inj}$*-preenvelope* $E \xrightarrow{e} J$ *is a special* $\mathscr{E}^{\perp}$*-preenvelope.*

(ii) *Each* ${}_{Q,A}\mathrm{Inj}$*-envelope* $E \xrightarrow{e} J$ *is an* $\mathscr{E}^{\perp}$*-envelope.*

Proof. (i) Since e is a ${}_{Q,A}\mathrm{Inj}$-preenvelope in the abelian Grothendieck category ${}_{Q,A}\mathrm{Mod}$, it is a monomorphism; see Corollary X.4.3 in [33]. It defines a short exact sequence

$$0 \to E \xrightarrow{e} J \to E' \to 0,$$

which induces an exact sequence

$$\mathrm{Hom}_{Q,A}(J, I) \xrightarrow{e^*} \mathrm{Hom}_{Q,A}(E, I) \to \mathrm{Ext}^1_{Q,A}(E', I)$$

for each I. Since J is in $\mathscr{E}$ by Theorem 4.4(b) in [18], we have E' in $\mathscr{E}$ by the last part of Theorem 4.4 in [18].

If I is in $\mathscr{E}^\perp$, then $\mathrm{Ext}^1_{Q,A}(E', I) = 0$, whence e^* is an epimorphism. But J is in $\mathscr{E}^\perp$ by Theorem 4.4(b) in [18], so e is an $\mathscr{E}^\perp$-preenvelope. It is special because E' is in $\mathscr{E}$, which is equal to ${}^\perp(\mathscr{E}^\perp)$ by Theorem 4.4(b) in [18].

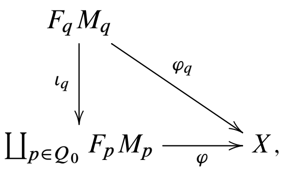
(ii) Since e is a ${}_{Q,A}\mathrm{Inj}$-envelope, it is an $\mathscr{E}^\perp$-preenvelope by part (i). Since it is an envelope, it is a left minimal morphism. Hence it is an $\mathscr{E}^\perp$-envelope. ■

Lemma 2.4. *Let* $\{F_q M_q \xrightarrow{\varphi_q} X\}_{q \in Q_0}$ *be a family of monomorphisms in* ${}_{Q,A}\mathrm{Mod}$. *Then the induced morphism*

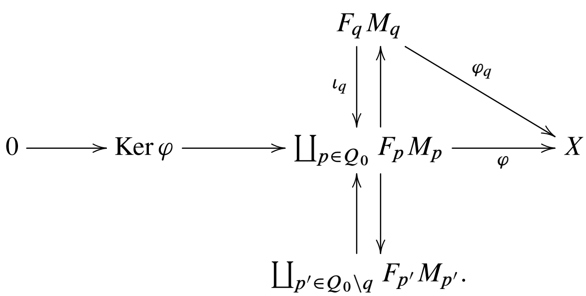
$$\coprod_{p \in Q_0} F_p M_p \xrightarrow{\varphi} X$$

is a monomorphism.

Proof. By definition, φ is the unique morphism such that the following diagram is commutative for each q in Q_0:



where ι_q denotes the coproduct inclusion. The diagram can be extended as follows:



Here the column is a biproduct diagram, φ_q is a monomorphism by assumption, and the row is left exact. Applying the Serre functor S to q gives an object Sq in Q_0, and applying the functor K_{Sq} to the diagram gives the following:

$$
\begin{array}{ccccccc}
 & & & & K_{Sq}(F_q M_q) & & \\
 & & & & {\scriptstyle K_{Sq}(\iota_q)}\downarrow\uparrow & \searrow^{K_{Sq}(\varphi_q)} & \\
0 & \longrightarrow & K_{Sq}(\operatorname{Ker}\varphi) & \longrightarrow & K_{Sq}\big(\coprod_{p\in Q_0} F_p M_p\big) & \xrightarrow[K_{Sq}(\varphi)]{} & K_{Sq}(X) \\
 & & & & \uparrow\downarrow & & \\
 & & & & K_{Sq}\big(\coprod_{p'\in Q_0\setminus q} F_{p'} M_{p'}\big). & &
\end{array}
$$

Since K_{Sq} is additive, the column is a biproduct diagram. Since K_{Sq} is a right adjoint by equation (1.7), hence left exact, $K_{Sq}(\varphi_q)$ is a monomorphism and the row is left exact.

We now compute as follows:

$$
\begin{aligned}
K_{Sq}\Big(\coprod_{p'\in Q_0\setminus q} F_{p'} M_{p'}\Big) &\overset{(a)}{\cong} K_{Sq}\Big(\prod_{p'\in Q_0\setminus q} F_{p'} M_{p'}\Big) \overset{(b)}{\cong} \prod_{p'\in Q_0\setminus q} K_{Sq} F_{p'} M_{p'} \\
&\overset{(c)}{\cong} \prod_{p'\in Q_0\setminus q} K_{Sq} G_{Sp'} M_{p'} \overset{(d)}{\cong} 0.
\end{aligned}
\tag{2.1}
$$

We explain the isomorphisms:

(a) holds because $\coprod_{p'\in Q_0\setminus q} F_{p'} M_{p'} \cong \prod_{p'\in Q_0\setminus q} F_{p'} M_{p'}$ by Proposition 3.7 in [20] since Q satisfies condition *Local boundedness* from the introduction;

(b) holds because K_{Sq} respects products since it is a right adjoint by equation (1.7):

(c) holds because $F_{p'} M_{p'} \cong G_{Sp'} M_{p'}$ by Lemma 3.4 in [20] since Q satisfies conditions *Hom finiteness* and *Serre functor* from the introduction;

(d) holds because $K_{Sq} G_{Sp'} = 0$ for $Sq \neq Sp'$ by Lemma 7.28 (b) in [18].

The column in the last diagram is a biproduct diagram, so equation (2.1) implies that $K_{Sq}(\iota_q)$ is an isomorphism. But $K_{Sq}(\varphi_q)$ is a monomorphism, so the commutative triangle in the diagram implies that $K_{Sq}(\varphi)$ is a monomorphism, whence left exactness of the row implies $K_{Sq}(\operatorname{Ker}\varphi) = 0$. This holds for each q in Q_0, so Proposition 7.19 in [18] implies $\operatorname{Ker}\varphi = 0$, since the pseudoradical $\mathfrak{r}$ of Q satisfies condition *Nilpotence* from the introduction. Hence φ is a monomorphism, as claimed. ■

Lemma 2.5. *Let* $M \xrightarrow{m} N$ *be an essential extension in* $\operatorname{Mod}(A)$ (*that is, a monomorphism with essential image*). *Then* $F_q M \xrightarrow{F_q m} F_q N$ *is an essential extension in* ${}_{Q,A}\operatorname{Mod}$ *for each* q *in* Q_0.

Proof. The functor F_q is exact by Corollary 3.9 (a) in [18], so $F_q M \xrightarrow{F_q m} F_q N$ is a monomorphism. Up to isomorphism, it can be written $G_p M \xrightarrow{G_p m} G_p N$ by Lemma 3.4

in [20], where $p = Sq$. We must prove that if $X \subseteq G_p N$ has zero intersection with the image of $G_p m$, then X is zero. So let $X \xrightarrow{x} G_p N$ denote the inclusion and assume that

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle x} \\ G_p M & \xrightarrow[G_p m]{} & G_p N \end{array}$$

is a pullback diagram; we must prove that X is zero.

For r in Q_0, the functor K_r is a right adjoint by equation (1.7), hence left exact. It follows that there is a pullback diagram

$$\begin{array}{ccc} 0 & \longrightarrow & K_r(X) \\ \big\downarrow & & \big\downarrow{\scriptstyle K_r(x)} \\ K_r(G_p M) & \xrightarrow[K_r(G_p m)]{} & K_r(G_p N). \end{array}$$

If $r \neq p$, then $K_r(G_p N) = 0$ by Lemma 7.28(b) in [18], whence the diagram implies $K_r(X) = 0$. If $r = p$, then $K_r G_p \cong \mathrm{id}$ by Lemma 7.28(b) in [18]; in particular, the diagram is isomorphic to a pullback diagram

$$\begin{array}{ccc} 0 & \longrightarrow & K_r(X) \\ \big\downarrow & & \big\downarrow \\ M & \xrightarrow[m]{} & N. \end{array}$$

Since $M \xrightarrow{m} N$ is an essential extension, this implies $K_r(X) = 0$.

Hence $K_r(X) = 0$ for each r in Q_0, so Proposition 7.19 in [18] implies $X = 0$ since the pseudoradical $\mathfrak{r}$ of Q satisfies condition *Nilpotence* from the introduction. ■

Remark 2.6. We recall two properties of adjoint functors.

(i) The adjunction isomorphism $\mathrm{Hom}_A(M, E_q X) \to \mathrm{Hom}_{Q,A}(F_q M, X)$ maps a morphism

$$M \xrightarrow{\mu} E_q X$$

to the *adjoint morphism*

$$F_q M \xrightarrow{\varphi} X$$

defined as the composition of the morphisms

$$F_q M \xrightarrow{F_q \mu} F_q E_q X \xrightarrow{\varepsilon_X} X, \tag{2.2}$$

where ε is the counit of the adjoint pair (F_q, E_q).

(ii) If the diagram

(2.3)
$$\begin{array}{ccc} M & \xrightarrow{m} & N \\ {\scriptstyle\mu}\downarrow & \swarrow{\scriptstyle\nu} & \\ E_q X & & \end{array}$$

is commutative, then so is the diagram

(2.4)
$$\begin{array}{ccc} F_q M & \xrightarrow{F_q m} & F_q N \\ {\scriptstyle\varphi}\downarrow & \swarrow{\scriptstyle\psi} & \\ X, & & \end{array}$$

where the adjoint morphisms of μ and ν are φ and ψ.

Lemma 2.7. *Let q in Q_0 and X in ${}_{Q,A}\mathrm{Mod}$ be given. Consider a morphism $M \xrightarrow{\mu} E_q X$ with adjoint morphism $F_q M \xrightarrow{\varphi} X$. If φ is a monomorphism, then μ is a monomorphism.*

Proof. Assume that φ is a monomorphism. Then $F_q \mu$ is a monomorphism since φ is the composition of the morphisms in equation (2.2). There is an exact sequence

$$0 \to \operatorname{Ker} \mu \to M \xrightarrow{\mu} E_q X,$$

hence an exact sequence

$$0 \to F_q \operatorname{Ker} \mu \to F_q M \xrightarrow{F_q \mu} F_q E_q X$$

because F_q is exact by Corollary 3.9 (a) in [18]. Since $F_q \mu$ is a monomorphism, this shows $F_q \operatorname{Ker} \mu = 0$, whence $\operatorname{Ker} \mu \cong C_q F_q \operatorname{Ker} \mu \cong 0$ by Lemma 7.28 (a) in [18]. So μ is a monomorphism. ∎

Lemma 2.8. *Let q in Q_0 and X in ${}_{Q,A}\mathrm{Mod}$ be given.*

(i) *The following set of A-left submodules of $E_q X$ is non-empty and has a maximal element with respect to inclusion:*

$$\mathscr{M} = \left\{ M \subseteq E_q X \;\middle|\; \begin{array}{l} \text{the inclusion morphism } M \to E_q X \text{ has an adjoint} \\ \text{morphism } F_q M \to X \text{ which is a monomorphism} \end{array} \right\}.$$

(ii) *Suppose that $E_q X$ is in $\mathrm{Inj}(A)$. Then so is each maximal element of $\mathscr{M}$.*

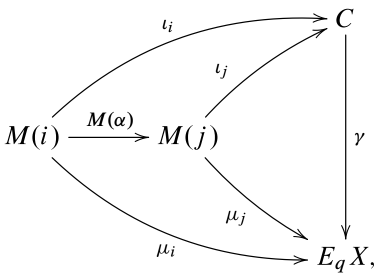
Proof. (i) The set $\mathscr{M}$ is non-empty because it contains $M = 0$. We will use Zorn's lemma to prove that $\mathscr{M}$ has a maximal element, so suppose that $\mathscr{I}$ is a totally ordered subset of $\mathscr{M}$; we must prove that $\mathscr{I}$ has an upper bound in $\mathscr{M}$.

There is a small filtered category I whose objects are the modules in $\mathscr{I}$ and whose morphisms are the inclusions between modules in $\mathscr{I}$. There is a functor $I \xrightarrow{M} \operatorname{Mod}(A)$ acting as the identity on objects and morphisms, and the colimit of M is

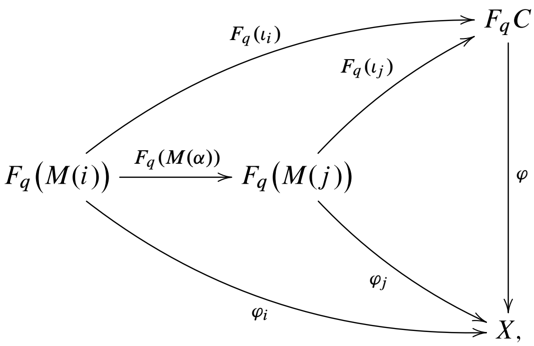
$$C = \bigcup_{i \in \mathscr{I}} M(i).$$

We will prove that C is in $\mathscr{M}$ whence it is clearly an upper bound for $\mathscr{I}$ in $\mathscr{M}$. That is, we will prove that the inclusion morphism $C \xrightarrow{\gamma} E_q X$ has an adjoint morphism $F_q C \xrightarrow{\varphi} X$ which is a monomorphism.

For each morphism $i \xrightarrow{\alpha} j$ in I, there is a commutative diagram



where all arrows are inclusions. The universal cone to C is $\{M(i) \xrightarrow{\iota_i} C\}_{i \in \mathscr{I}}$, and the cone $\{M(i) \xrightarrow{\mu_i} E_q X\}_{i \in \mathscr{I}}$ induces the inclusion morphism $C \xrightarrow{\gamma} E_q X$. Remark 2.6(ii) gives an induced commutative diagram



where φ_i, φ_j and φ are the adjoint morphisms of μ_i, μ_j and γ. The functor F_q is a left adjoint hence preserves colimits, so $\{F_q\big(M(i)\big) \xrightarrow{F_q(\iota_i)} F_q C\}_{i \in \mathscr{I}}$ is the universal cone to the colimit of $F_q \circ M$. The last diagram shows that $\{F_q\big(M(i)\big) \xrightarrow{\varphi_i} X\}_{i \in \mathscr{I}}$ is a cone inducing the adjoint morphism $F_q C \xrightarrow{\varphi} X$. Since the $M(i)$ are in $\mathscr{M}$, the φ_i are monomorphisms. Since ${}_{Q,A}\operatorname{Mod}$ is a Grothendieck abelian category, filtered colimits preserve monomorphisms, so φ is a monomorphism as desired.

(ii) Let $M \subseteq E_q X$ be a maximal element of $\mathscr{M}$. Since $E_q X$ is in $\operatorname{Inj}(A)$, to prove that M is in $\operatorname{Inj}(A)$ we will assume $M \subseteq N \subseteq E_q X$ with M essential in N and prove $M = N$; this is sufficient by Lemma V.2.2 and Proposition V.2.4 in [33].

Let $M \xrightarrow{m} N$ be the inclusion and consider Remark 2.6(ii). There is a commutative diagram (2.3) where μ and ν are the inclusions into $E_q X$, and the remark gives the commutative diagram (2.4) where φ and ψ are the adjoint morphisms of μ and ν. Assume $M \subsetneq N$. Since M is maximal in $\mathscr{M}$, the morphism φ is a monomorphism but the morphism ψ is not. But then diagram (2.4) contradicts that $F_q M \xrightarrow{F_q m} F_q N$ is an essential extension by Lemma 2.5. ∎

3. Main theorems

This section provides different characterisations of minimal semiinjective objects in Theorem 3.1 and uses them to prove Theorems B, C and D, which were stated in the introduction. Not all parts of Theorem 3.1 are required for the subsequent proofs, but we consider them worthwhile in their own right. Notation and definitions from the "Background" part of the introduction will still be used freely.

Theorem 3.1. *Let I be a semiinjective object in ${}_{Q,A}\mathrm{Mod}$. The following conditions are equivalent.*

(i) *If $J \subseteq I$ with J in ${}_{Q,A}\mathrm{Inj}$, then $J = 0$. That is, I is minimal in the sense of Definition A(ii).*

(ii) *If $E \subseteq I$ with E in $\mathscr{E}$, then $E = 0$.*

(iii) *Each weak equivalence $I \to X$ in ${}_{Q,A}\mathrm{Mod}$ is a split monomorphism.*

(iv) *If an endomorphism $I \xrightarrow{f} I$ in ${}_{Q,A}\mathrm{Mod}$ induces an automorphism in $\mathscr{D}_Q(A)$, then f is already an automorphism.*

(v) *For q in Q_0 and D in $\mathrm{Inj}(A)$, if $F_q D \to I$ is a monomorphism, then $D = 0$.*

(vi) *For q in Q_0 and M in $\mathrm{Mod}(A)$, if $F_q M \to I$ is a monomorphism, then $M = 0$.*

(vii) *For q in Q_0 and M in $\mathrm{Mod}(A)$, if a monomorphism $M \xrightarrow{\mu} E_q I$ satisfies*

$$\mathrm{Im}\left(F_q M \xrightarrow{F_q \mu} F_q E_q I\right) \cap Z_q I = 0, \tag{3.1}$$

then $M = 0$. Here we write

$$Z_q X = \mathrm{Ker}(F_q E_q X \xrightarrow{\varepsilon_X} X)$$

for X in ${}_{Q,A}\mathrm{Mod}$, where ε is the counit of the adjoint pair (F_q, E_q).

Proof. Before starting the proof proper, we recall from Theorem 6.5 in [18] that $\mathscr{E}^{\perp}$ is a Frobenius category with projective-injective objects ${}_{Q,A}\mathrm{Inj}$, and that there is an equivalence

$$\mathscr{D}_Q(A) \simeq \frac{\mathscr{E}^{\perp}}{{}_{Q,A}\mathrm{Inj}}.$$

The right-hand side is the naïve quotient category, which has the same objects as $\mathscr{E}^{\perp}$ and Hom spaces obtained by dividing by the subspaces of morphisms factoring through an object of ${}_{Q,A}\mathrm{Inj}$. Equivalently, the Hom spaces are obtained by dividing by the equivalence relation "$\sim$". Hence condition (iv) can be replaced by (iv') If an endomorphism $I \xrightarrow{f} I$ in ${}_{Q,A}\mathrm{Mod}$ induces an automorphism in $\mathscr{E}^{\perp}/{}_{Q,A}\mathrm{Inj}$, then f is already an automorphism.

See also Proposition 2.1 (i).

(i) $\Rightarrow$ (ii). Let $E \subseteq I$ with E in $\mathscr{E}$ be given. Since ${}_{Q,A}\mathrm{Mod}$ is a Grothendieck abelian category, there is a ${}_{Q,A}\mathrm{Inj}$-envelope $E \to J$, see Proposition V.2.5 and Corollary X.4.3 in [33]. It is an $\mathscr{E}^{\perp}$-envelope by Lemma 2.3 (ii), so we can factorise as follows, where the vertical arrow is the inclusion:

$$\begin{array}{ccc} E & \longrightarrow & J \\ \downarrow & \swarrow_{j} & \\ I. & & \end{array}$$

Since $E \to J$ is an essential extension, j is a monomorphism. Identifying J with its image under j, we have $E \subseteq J \subseteq I$. But then $J = 0$ by (i) and $E = 0$ follows.

(ii) $\Rightarrow$ (vi). Let $F_q M \to I$ be a monomorphism. Since $F_q M$ is in $\mathscr{E}$ by Lemma 7.14 and Theorem 7.1 in [18], we have $F_q M = 0$ by (ii). But then $M \cong C_q F_q M = 0$ by Lemma 7.28 (a) in [18].

(vi) $\Rightarrow$ (v) is clear.

(v) $\Rightarrow$ (i) Let $J \subseteq I$ with J in ${}_{Q,A}\mathrm{Inj}$ be given. Combining the proof of Lemma 7.29 in [18] and Lemma 3.4 and Proposition 3.7 in [20], we can write J up to isomorphism as $\coprod_{p \in Q_0} F_p D_p$, where each D_p is in $\mathrm{Inj}(A)$. If q is in Q_0, then there is a commutative diagram

$$\begin{array}{ccc} F_q D_q & \xrightarrow{\iota_q} & \coprod_{p \in Q_0} F_p D_p, \\ {\scriptstyle \varphi_q}\downarrow & \swarrow_{j} & \\ I, & & \end{array}$$

where ι_q denotes the coproduct inclusion, and j the inclusion of J into I. Since ι_q and j are monomorphisms, so is $F_q D_q \xrightarrow{\varphi_q} I$. But then $D_q = 0$, by (v). This holds for each q in Q_0, so $J = 0$.

(i) $\Rightarrow$ (iv'): This part of the proof is divided into three steps.

Step 1. Assume that an endomorphism $I \xrightarrow{f} I$ in ${}_{Q,A}\mathrm{Mod}$ induces the identity morphism in $\mathscr{E}^{\perp}/{}_{Q,A}\mathrm{Inj}$. We will prove that f is a monomorphism.

The assumption means that there are morphisms $I \xrightarrow{a} J \xrightarrow{b} I$ with J in ${}_{Q,A}\mathrm{Inj}$ such that $\mathrm{id}_I - f = ba$. Composing with the inclusion $\mathrm{Ker}\, f \xrightarrow{k} I$ gives $(\mathrm{id}_I - f)k = bak$, that is, $k = bak$. Since k is a monomorphism, so is bak, and hence so is ak. By the proof of Proposition 2.5 in [33], there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Ker}\, f & \xrightarrow{\kappa} & J' \\ {\scriptstyle ak}\downarrow & \swarrow_{j'} & \\ J, & & \end{array}$$

where κ is a ${}_{Q,A}$Inj-envelope and j' the inclusion of a subobject, and this gives $bj'\kappa = bak = k$. Since k is a monomorphism, so is $bj'\kappa$, and hence so is bj', since κ is an essential extension. So bj' lets us view J' as a subobject of I, whence $J' = 0$ by (i). Hence $\operatorname{Ker} f = 0$ and f is a monomorphism, as claimed.

Step 2. Assume that an endomorphism $I \xrightarrow{f} I$ in ${}_{Q,A}$Mod induces an automorphism in $\mathscr{E}^{\perp}/{}_{Q,A}$Inj. We will prove that f is a monomorphism.

Pick $I \xrightarrow{g} I$ such that g induces an inverse of f in $\mathscr{E}^{\perp}/{}_{Q,A}$Inj. Then gf induces the identity morphism in $\mathscr{E}^{\perp}/{}_{Q,A}$Inj, whence gf is a monomorphism by Step 1. Hence f is a monomorphism.

Step 3. Assume that an endomorphism $I \xrightarrow{f} I$ in ${}_{Q,A}$Mod induces an automorphism in $\mathscr{E}^{\perp}/{}_{Q,A}$Inj. We will prove that f is an automorphism.

By Step 2, we know that f is a monomorphism, so there is a short exact sequence

$$0 \to I \xrightarrow{f} I \to J \to 0, \tag{3.2}$$

which induces an exact sequence

$$\operatorname{Ext}^1_{Q,A}(E, I) \to \operatorname{Ext}^1_{Q,A}(E, J) \to \operatorname{Ext}^2_{Q,A}(E, I)$$

for each E. If E is in $\mathscr{E}$, then the outer terms are zero. This is true for the first term because I is in $\mathscr{E}^{\perp}$. For the third term, it is true because I is in $\mathscr{E}^{\perp}$, while $(\mathscr{E}, \mathscr{E}^{\perp})$ is a hereditary cotorsion pair by Theorem 4.4(b) in [18]. Hence the middle term is zero, so J is in $\mathscr{E}^{\perp}$. Thus, (3.2) is a short exact sequence with terms in $\mathscr{E}^{\perp}$, so induces a triangle in the triangulated category $\mathscr{E}^{\perp}/{}_{Q,A}$Inj by Lemma I.2.7 in [15]. Since f induces an automorphism, J must induce the zero object whence J is in ${}_{Q,A}$Inj. But then J is projective-injective in the Frobenius category $\mathscr{E}^{\perp}$, so (3.2) is split exact. Up to isomorphism, J is hence a subobject of I so J is zero by (i). So (3.2) proves that f is an automorphism.

(iv') $\Rightarrow$ (iii): Let $I \xrightarrow{i} X$ be a weak equivalence. It follows from Proposition 2.1(iii) that there is a morphism $X \xrightarrow{x} I$ such that xi induces the identity morphism in $\mathscr{E}^{\perp}/{}_{Q,A}$Inj. Hence xi is an automorphism by (iv'). If θ is the inverse, then $\theta x i = \mathrm{id}_I$, which shows that i is a split monomorphism.

(iii) $\Rightarrow$ (i): Let $J \subseteq I$ with J in ${}_{Q,A}$Inj be given. There is an induced short exact sequence $0 \to J \to I \xrightarrow{f} X \to 0$, and J is in $\mathscr{E}$ by Theorem 4.4(b) in [18], so f is a weak equivalence by Proposition 2.2(ii). But then f is a split monomorphism by (iii), whence $J = 0$.

(vi) $\Leftrightarrow$ (vii): Each of conditions (vi) and (vii) requires that $M = 0$ under certain circumstances. It is hence enough to prove that the two sets of circumstances are the same. We will do so by proving that if $F_q M \xrightarrow{\varphi} I$ is the adjoint morphism of $M \xrightarrow{\mu} E_q I$, then

$$F_q M \xrightarrow{\varphi} I \text{ is a monomorphism} \iff \begin{cases} M \xrightarrow{\mu} E_q I \text{ is a monomorphism} \\ \text{which satisfies equation (3.1).} \end{cases}$$

For the implication "$\Rightarrow$", assume that $F_q M \xrightarrow{\varphi} I$ is a monomorphism. Then $M \xrightarrow{\mu} E_q I$ is a monomorphism by Lemma 2.7, and $M \xrightarrow{\mu} E_q I$ satisfies equation (3.1) because the composition of the morphisms $F_q M \xrightarrow{F_q \mu} F_q E_q X \xrightarrow{\varepsilon_X} X$ is a monomorphism, since it equals $F_q M \xrightarrow{\varphi} I$ by equation (2.2). For the implication "$\Leftarrow$", assume that $M \xrightarrow{\mu} E_q I$ is a monomorphism which satisfies equation (3.1). Then $F_q M \xrightarrow{\varphi} I$ is a monomorphism because it equals the composition of the morphisms $F_q M \xrightarrow{F_q \mu} F_q E_q X \xrightarrow{\varepsilon_X} X$ by equation (2.2), and this composition is a monomorphism since $F_q M \xrightarrow{F_q \mu} F_q E_q I$ is a monomorphism by Corollary 3.9 (a) in [18], while $M \xrightarrow{\mu} E_q I$ satisfies equation (3.1). ■

Proof of Theorem B. (ii) For each q in Q_0, Lemma 2.8 (i) says there is a submodule $D_q \subseteq E_q I$ maximal with respect to the property that the inclusion $D_q \xrightarrow{\delta_q} E_q I$ has an adjoint morphism $F_q D_q \xrightarrow{\varphi_q} I$ which is a monomorphism. The module D_q is in $\operatorname{Inj}(A)$ by Lemma 2.8 (ii) because $E_q I$ is in $\operatorname{Inj}(A)$ by Theorem E in [20].

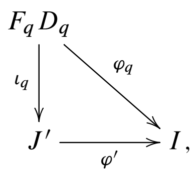
We claim that the object

$$J' = \coprod_{q \in Q_0} F_q D_q$$

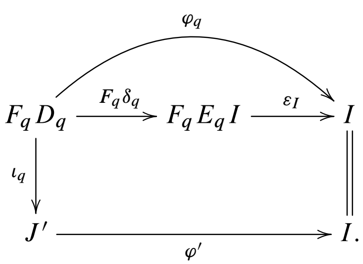
is in ${}_{Q,A}\operatorname{Inj}$. To see so, observe that J' can be written as

$$\prod_{q \in Q_0} G_{S_q} D_q$$

by Lemma 3.4 and Proposition 3.7 in [20], and that $G_{S_q} D_q$ is in ${}_{Q,A}\operatorname{Inj}$ by Lemma 3.11 in [18], since D_q is in $\operatorname{Inj}(A)$. There is a unique morphism φ' such that the following diagram is commutative for each q in Q_0:



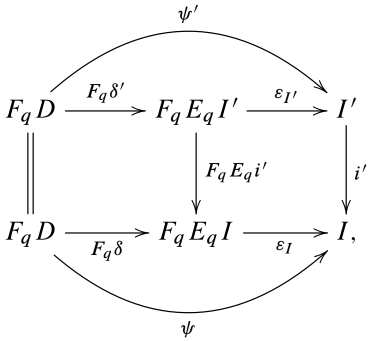
where ι_q denotes the coproduct inclusion. Combining with Remark 2.6 (i) provides the following commutative diagram:



The morphism φ' is a monomorphism by Lemma 2.4. Since J' is in ${}_{Q,A}\operatorname{Inj}$, the morphism φ' is a split monomorphism which can be viewed as the inclusion of a direct summand. The diagram shows that

(3.3) the image of the monomorphism φ_q is contained in the direct summand J' for each q in Q_0.

Consider the complement I' of J' in I. Then $I \cong I' \oplus J'$, so it is clear that I' is semiinjective. To complete the proof, we will prove that I' is minimal semiinjective by proving that it satisfies the condition in Theorem 3.1(v). So let q in Q_0 and D in $\operatorname{Inj}(A)$ be given and assume that $F_q D \xrightarrow{\psi'} I'$ is a monomorphism. Then ψ' is the adjoint morphism of a morphism $D \xrightarrow{\delta'} E_q I'$, which is a monomorphism by Lemma 2.7. The inclusion $I' \xrightarrow{i'} I$ is a split monomorphism, hence so is $E_q I' \xrightarrow{E_q i'} E_q I$. The composition δ of the morphisms $D \xrightarrow{\delta'} E_q I' \xrightarrow{E_q i'} E_q I$ is a monomorphism, and there is a commutative diagram



where ψ' and ψ are the adjoint morphisms of δ' and δ, see Remark 2.6(i). Since ψ' and i' are monomorphisms, so is $\psi = i'\psi'$. The diagram shows that

(3.4) the image of the monomorphism ψ is contained in the direct summand I', which is the complement of J' in I.

Now consider the morphism $D_q \oplus D \xrightarrow{(\delta_q,\delta)} E_q I$. Its adjoint morphism is the composition of the morphisms

$$F_q D_q \oplus F_q D \xrightarrow{(F_q\delta_q, F_q\delta)} F_q E_q I \xrightarrow{\varepsilon_I} I,$$

so its adjoint morphism is (φ_q, ψ) which is a monomorphism by equations (3.3) and (3.4). Hence (δ_q, δ) is a monomorphism by Lemma 2.7. However, by the maximality of D_q, this implies $\delta = 0$, and since δ is a monomorphism, it follows that $D = 0$. Hence we have proved that I' satisfies the condition in Theorem 3.1(v).

(i) By Theorem 5.9 in [18], there is a complete cotorsion pair $(\mathscr{E}, \mathscr{E}^{\perp})$ in the sense of Definition 2.2.1 and Lemma 2.2.6 in [14]. Hence there is a short exact sequence $0 \to X \xrightarrow{x} I \to E \to 0$ with I in $\mathscr{E}^{\perp}$ and E in $\mathscr{E}$. By part (ii) of the theorem, we have $I = I' \oplus J'$ with I' a minimal semiinjective object and J' in ${}_{Q,A}\operatorname{Inj}$. Hence there is a short exact sequence $0 \to J' \to I \xrightarrow{i} I' \to 0$. Note that J' is in $\mathscr{E}$ by Theorem 4.4(b) in [18]. The morphisms $X \xrightarrow{x} I$ and $I \xrightarrow{i} I'$ are weak equivalences by Proposition 2.2, so the composition $X \xrightarrow{ix} I'$ is a weak equivalence by Proposition 5.12 in [22], hence a minimal semiinjective resolution. ■

Proof of Theorem C. Let $I \xrightarrow{i} I'$ be a weak equivalence between minimal semiinjective objects. Theorem 3.1(iii) says that i is a split monomorphism, so there exists a split epimorphism $I' \xrightarrow{i'} I$ such that $i'i = \mathrm{id}_I$. But then Proposition 5.12 in [22] implies that i' is a weak equivalence, so Theorem 3.1(iii) implies that i' is a split monomorphism. In particular, i' is an epimorphism and a monomorphism, hence an isomorphism. ■

Proof of Theorem D. By Proposition 2.1(iii), the morphism $X \xrightarrow{x} I$ induces a bijection

$$\mathrm{Hom}_{Q,A}(I, I')/\sim \longrightarrow \mathrm{Hom}_{Q,A}(X, I')/\sim .$$

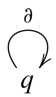
This implies the two first bullet points. To prove the third bullet point, observe that the relation $ix \sim x'$ in ${}_{Q,A}\mathrm{Mod}$ induces an equality in $\mathscr{D}_Q(A)$. This implies that i induces an isomorphism in $\mathscr{D}_Q(A)$ because the weak equivalences x and x' induce isomorphisms in $\mathscr{D}_Q(A)$. But then i is a weak equivalence in ${}_{Q,A}\mathrm{Mod}$ by Theorem 1.2.10(iv) in [21], and then i is an isomorphism in ${}_{Q,A}\mathrm{Mod}$ by Theorem C. ■

4. Differential modules

This section proves Theorem E, which was stated in the introduction. Our theory can be specialised to the theory of differential modules by setting

$$\Bbbk = \mathbb{Z}$$

and setting Q equal to the $\Bbbk$-preadditive category given by



with $\partial^2 = 0$, and we will do so in this section. Then a Q-shaped diagram is a differential module as defined in the introduction, so ${}_{Q,A}\mathrm{Mod}$ is equal to $\mathrm{Diff}(A)$, the category of differential modules over A. Note that this Q satisfies conditions *Hom finiteness* through *Nilpotence* of the introduction with pseudoradical given by $\mathfrak{r}_q = \Bbbk \cdot \partial$. We now explain how some concepts from the theory of ${}_{Q,A}\mathrm{Mod}$ specialise to $\mathrm{Diff}(A)$; see also A.2 in [20].

4.1. Cohomology functors (defined in equation (1.3)) specialise as

$$\mathbb{H}^i_{[q]} \rightsquigarrow \mathrm{H}$$

for each $i \geqslant 1$, where H is the homology functor on differential modules defined in the introduction. This can be proved by computing $\mathbb{H}^i_{[q]}(-) = \mathrm{Ext}^i_Q(S\langle q\rangle, -)$ using the projective resolution $\cdots \to Q(q,-) \to Q(q,-)$ of $S\langle q\rangle$.

4.2. The class $\mathscr{E}$ of exact objects (defined in equation (1.1)) specialises as

$$\mathscr{E} \rightsquigarrow \{\, (M,\partial) \in \operatorname{Diff}(A) \mid (M,\partial) \text{ is exact} \,\},$$

where a differential module (M,∂) is *exact* if $\mathrm{H}(M,\partial) = 0$. This follows from paragraph 4.1 and equation (1.4).

4.3. Weak equivalences (defined in equation (1.2)) specialise as

$$\mathrm{weq} \rightsquigarrow \{\, \mu \mid \mu \text{ is a quasi-isomorphism} \,\},$$

where a morphism μ of differential modules is a *quasi-isomorphism* if $\mathrm{H}(\mu)$ is an isomorphism. This follows from paragraph 4.1 and equation (1.5).

4.4. The class $\mathscr{E}^{\perp}$ of semiinjective objects will only be specialised when the left global dimension of A is finite. Then

$$\mathscr{E}^{\perp} \rightsquigarrow \{\, (J,\partial) \in \operatorname{Diff}(A) \mid J \text{ is in } \operatorname{Inj}(A) \,\}$$

by Theorem E in [20], and the right-hand class can be written as

$$\{\, (J,\partial) \in \operatorname{Diff}(A) \mid J \text{ is Gorenstein injective in } \operatorname{Mod}(A) \,\}$$

by the dual of Proposition 2.27 in [17]. By Theorem 1.1 in [34], the last class can be written as

$$\{\, (J,\partial) \in \operatorname{Diff}(A) \mid (J,\partial) \text{ is Gorenstein injective in } \operatorname{Diff}(A) \,\}.$$

4.5. The class of minimal semiinjective objects will only be specialised when the left global dimension of A is finite. Then

$$\left\{ I \;\middle|\; \begin{array}{l} I \text{ is minimal} \\ \text{semiinjective} \end{array} \right\} \rightsquigarrow \left\{ (J,\partial) \in \operatorname{Diff}(A) \;\middle|\; \begin{array}{l} (J,\partial) \text{ is Gorenstein injective without} \\ \text{non-zero injective summands in } \operatorname{Diff}(A) \end{array} \right\}$$

by paragraph 4.4 and Definition A (ii). Here, the right-hand class modulo isomorphism is the left-hand class in Theorem E.

4.6. Minimal semiinjective resolutions will only be specialised when the left global dimension of A is finite. Then they become quasi-isomorphisms $(M,\partial_M) \to (J,\partial_J)$, where (J,∂_J) is Gorenstein injective without non-zero injective summands in $\operatorname{Diff}(A)$. This follows from paragraphs 4.3 and 4.5 and Definition A (iv).

Recall from the introduction the functors B, Z and H from $\operatorname{Diff}(A)$ to $\operatorname{Mod}(A)$, which send a differential module to its boundaries, cycles and homology. There is a short exact sequence in $\operatorname{Mod}(A)$,

$$0 \to \mathrm{B}(M,\partial) \to \mathrm{Z}(M,\partial) \xrightarrow{\zeta} \mathrm{H}(M,\partial) \to 0, \tag{4.1}$$

natural with respect to (M,∂) in $\operatorname{Diff}(A)$.

The modules $\mathrm{B}(M,\partial)$, $\mathrm{Z}(M,\partial)$ and $\mathrm{H}(M,\partial)$ can be viewed as differential modules $(\mathrm{B}(M,\partial),0)$, $(\mathrm{Z}(M,\partial),0)$ and $(\mathrm{H}(M,\partial),0)$ with zero differential. There is a canonical short exact sequence in $\mathrm{Diff}(A)$,

$$0 \to (\mathrm{Z}(M,\partial),0) \xrightarrow{j} (M,\partial) \to (\mathrm{B}(M,\partial),0) \to 0, \tag{4.2}$$

natural with respect to (M,∂) in $\mathrm{Diff}(A)$, and $\mathrm{H}(j)$ can be identified with the morphism ζ in the sequence (4.1).

Lemma 4.7. *For (M,∂_M) in $\mathrm{Diff}(A)$, consider the differential module $(\mathrm{H}(M,\partial_M),0)$ with zero differential.*

Assume that the sequence (4.1) in $\mathrm{Mod}(A)$ is split exact. Then there exists a monic quasi-isomorphism $(\mathrm{H}(M,\partial_M),0) \xrightarrow{\eta} (M,\partial_M)$ in $\mathrm{Diff}(A)$.

Proof. Since the sequence (4.1) is split exact, there is a splitting morphism p giving the following diagram:

$$0 \longrightarrow \mathrm{B}(M,\partial_M) \longrightarrow \mathrm{Z}(M,\partial_M) \xrightarrow{\zeta} \mathrm{H}(M,\partial_M) \longrightarrow 0, \qquad p : \mathrm{Z}(M,\partial_M) \to \mathrm{B}(M,\partial_M) \tag{4.3}$$

We can also view p as a morphism $(\mathrm{Z}(M,\partial_M),0) \xrightarrow{p} (\mathrm{B}(M,\partial_M),0)$ and use it to construct the following diagram in $\mathrm{Diff}(A)$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathrm{Z}(M,\partial_M),0) & \xrightarrow{j} & (M,\partial_M) & \longrightarrow & (\mathrm{B}(M,\partial_M),0) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p} & \text{pushout} & \downarrow{\scriptstyle m} & & \| & & \\ 0 & \longrightarrow & (\mathrm{B}(M,\partial_M),0) & \longrightarrow & (V,\partial_V) & \longrightarrow & (\mathrm{B}(M,\partial_M),0) & \longrightarrow & 0, \end{array} \tag{4.4}$$

where the first row is the short exact sequence (4.2), the first square is a pushout square, and the second row is short exact; see Proposition VIII.4.2 in [29]. The snake lemma implies $\operatorname{Ker} p \cong \operatorname{Ker} m$ and $\operatorname{Coker} p \cong \operatorname{Coker} m$. Since p is the splitting morphism from diagram (4.3), we have $\operatorname{Ker} p \cong (\mathrm{H}(M,\partial_M),0)$ and $\operatorname{Coker} p = 0$. Combining this information provides a short exact sequence in $\mathrm{Diff}(A)$,

$$0 \to (\mathrm{H}(M,\partial_M),0) \xrightarrow{\eta} (M,\partial_M) \xrightarrow{m} (V,\partial_V) \to 0.$$

To prove that η is a quasi-isomorphism, it is enough to prove $\mathrm{H}(V,\partial_V) = 0$, by Proposition 2.2(i) and paragraphs 4.3 and 4.2.

The pushout square in (4.4) induces a short exact sequence

$$0 \to (\mathrm{Z}(M,\partial_M),0) \xrightarrow{\binom{j}{-p}} (M,\partial_M) \oplus (\mathrm{B}(M,\partial_M),0) \to (V,\partial_V) \to 0 \tag{4.5}$$

in $\mathrm{Diff}(A)$ by Proposition 2.53 in [11]. Note that $\binom{j}{-p}$ is indeed a monomorphism since j is a monomorphism. As remarked before the proposition, $\mathrm{H}(j)$ can be identified with ζ from diagrams (4.1) and (4.3), and $\mathrm{H}(-p)$ can clearly be identified with

$$\mathrm{Z}(M,\partial_M) \xrightarrow{-p} \mathrm{B}(M,\partial_M),$$

so $\mathrm{H}\binom{j}{-p} = \binom{\mathrm{H}(j)}{\mathrm{H}(-p)}$ can be identified with $\binom{\zeta}{-p}$. This is an isomorphism since p is a splitting morphism, see diagram (4.3). Hence the long exact homology sequence induced by (4.5) implies $\mathrm{H}(V, \partial_V) = 0$ as desired. ∎

Proof of Theorem E. Since A is left hereditary, it has finite left global dimension, so paragraphs 4.1 through 4.6 apply.

It is clear that the homology functor $\mathrm{Diff}(A) \xrightarrow{\mathrm{H}} \mathrm{Mod}(A)$ induces a map H as shown in equation (1.9). We will prove that an inverse map, K, is given by mapping the isomorphism class of M in $\mathrm{Mod}(A)$ to the isomorphism class of (J, ∂_J) in $\mathrm{Diff}(A)$, where $(M, 0) \xrightarrow{\mu} (J, \partial_J)$ is a quasi-isomorphism and (J, ∂_J) is Gorenstein injective without non-zero injective summands in $\mathrm{Diff}(A)$.

Such a μ is a minimal semiinjective resolution by paragraph 4.6, so it exists by Theorem B(i), and (J, ∂_J) is determined up to isomorphism by Theorem D.

The map K takes values in the left-hand set of equation (1.9) by construction.

(a) HK = id.

The quasi-isomorphism $(M, 0) \xrightarrow{\mu} (J, \partial_J)$ provides the second equality in the following computation up to isomorphism:

$$\mathrm{HK}(M) = \mathrm{H}(J, \partial_J) = \mathrm{H}(M, 0) = M$$

(b) KH = id.

Let (J, ∂_J) be Gorenstein injective without non-zero injective summands in $\mathrm{Diff}(A)$. By paragraph 4.4, we have J in $\mathrm{Inj}(A)$ whence the quotient $\mathrm{B}(J, \partial_J)$ of J is also in $\mathrm{Inj}(A)$ because A is left hereditary. This implies that the sequence (4.1) is split exact, so Lemma 4.7 gives a quasi-isomorphism $(\mathrm{H}(J, \partial_J), 0) \xrightarrow{\eta} (J, \partial_J)$, and by the definition of K, this shows

$$\mathrm{KH}(J, \partial_J) = (J, \partial_J).$$ ∎

A. Minimal semiinjective resolutions in the classic derived category

Our theory can be specialised to the theory of minimal semiinjective resolutions in $\mathscr{D}(A)$ by setting Q equal to the $\Bbbk$-preadditive category given by Figure 1 modulo the relations that consecutive arrows compose to zero, and we will do so in this appendix. Then a Q-shaped diagram is a chain complex, so ${}_{Q,A}\mathrm{Mod}$ is equal to $\mathrm{Ch}(A)$, the category of chain complexes and chain maps over A, and $\mathscr{D}_Q(A)$ is equal to $\mathscr{D}(A)$. Theorems B, C, D and 3.1 specialise to the following results due to [1], Appendix B of [5], Section 10 of [10], Sections 2.3 and 2.4 of [12], and Appendix B of [28].

Theorem B for complexes.

(i) *Each X in* $\mathrm{Ch}(A)$ *has a minimal semiinjective resolution.*

(ii) *Each semiinjective complex I in* $\mathrm{Ch}(A)$ *has the form $I = I' \oplus J'$ in* $\mathrm{Ch}(A)$, *with I' a minimal semiinjective complex and J' a null homotopic complex of injective modules.*

Theorem C for complexes. *If $I \xrightarrow{i} I'$ in $\mathrm{Ch}(A)$ is a quasi-isomorphism between minimal semiinjective complexes, then i is an isomorphism in $\mathrm{Ch}(A)$.*

Theorem D for complexes. *If $X \xrightarrow{x} I$ and $X \xrightarrow{x'} I'$ are minimal semiinjective resolutions in $\mathrm{Ch}(A)$, then the following holds.*

- *The diagram of solid arrows*

$$\begin{array}{ccc} X & \xrightarrow{x} & I \\ {\scriptstyle x'}\big\downarrow & \swarrow{\scriptstyle i} & \\ I' & & \end{array}$$

  can be completed with a chain map i such that ix is chain homotopic to x' in $\mathrm{Ch}(A)$.
- *The chain map i is unique up to chain homotopy.*
- *Each completing chain map i is an isomorphism in $\mathrm{Ch}(A)$.*

Theorem 3.1 for complexes. *Let*

$$I = \cdots \to I_2 \xrightarrow{\partial_2} I_1 \xrightarrow{\partial_1} I_0 \xrightarrow{\partial_0} I_{-1} \xrightarrow{\partial_{-1}} I_{-2} \to \cdots$$

be a semiinjective complex in $\mathrm{Ch}(A)$. The following conditions are equivalent.

(i) *If $J \subseteq I$ with J a null homotopic complex of injective modules, then $J = 0$. That is, I is minimal.*

(ii) *If $E \subseteq I$ with E an exact complex, then $E = 0$.*

(iii) *Each quasi-isomorphism $I \to X$ in $\mathrm{Ch}(A)$ is a split monomorphism.*

(iv) *If an endomorphism $I \xrightarrow{f} I$ in $\mathrm{Ch}(A)$ induces an automorphism in $\mathscr{D}(A)$, then f is already an automorphism.*

(v) $\mathrm{Ker}\,\partial_q$ *is an essential submodule of I_q for each q.*

The specialisations are obtained by applying Figure 3, which explains how some concepts from ${}_{Q,A}\mathrm{Mod}$ specialise to $\mathrm{Ch}(A)$. Note that items (i)-(iv) in Theorem 3.1 specialise to items (i)-(iv) in Theorem 3.1 for complexes, while items (v)-(vii) in Theorem 3.1 all specialise to item (v) in Theorem 3.1 for complexes.

For instance, consider Theorem 3.1(vi). The functor F_q from equation (1.6) specialises to

$$\mathrm{Ch}(A) \xleftarrow{\;F_q\;} \mathrm{Mod}(A),$$

given on objects by

$$F_q M = \cdots \to 0 \to M \xrightarrow{\mathrm{id}} M \to 0 \to \cdots$$

with M is placed in homological degrees q and $q-1$.

	${}_{Q,A}\mathrm{Mod}$	$\mathrm{Ch}(A)$
(a)	$\mathscr{D}_Q(A)$	$\mathscr{D}(A)$
(b)	$\mathscr{E}$	exact complexes
(c)	weq	quasi-isomorphisms
(d)	semiinjective object	semiinjective complex
(e)	semiinjective resolution	semiinjective resolution
(f)	minimal semiinjective resolution	minimal semiinjective resolution
(g)	${}_{Q,A}\mathrm{Inj}$	null homotopic complexes of injective modules
(h)	$\sim$ in the category $\mathscr{E}^{\perp}$	chain homotopy of chain maps

Figure 3. Set Q equal to the $\Bbbk$-preadditive category given by Figure 1 modulo the relations that consecutive arrows compose to zero. Then ${}_{Q,A}\mathrm{Mod}$ is equal to $\mathrm{Ch}(A)$. This table explains how some concepts now specialise. Items (a)-(e) are given by 3.4, 3.7 and 3.10 in [19], item (g) is Exercise 14.8 in [27], and item (h) follows from item (g). Item (f) holds because Theorem 3.1 (i), characterising minimal semiinjective objects in ${}_{Q,A}\mathrm{Mod}$, specialises to Proposition 2.3.14 (b) in [12], characterising minimal semiinjective complexes.

A morphism $F_q M \xrightarrow{\varphi} I$ is a chain map of the following form:

$$
\begin{array}{ccccccccccccc}
F_q M & = & \cdots & \longrightarrow & 0 & \longrightarrow & M & \xrightarrow{\mathrm{id}} & M & \longrightarrow & 0 & \longrightarrow & \cdots \\
\Big\downarrow{\scriptstyle\varphi} & & & & \Big\downarrow & & \Big\downarrow{\scriptstyle\mu} & & \Big\downarrow{\scriptstyle\partial_q\mu} & & \Big\downarrow & & \\
I & = & \cdots & \longrightarrow & I_{q+1} & \longrightarrow & I_q & \xrightarrow[\partial_q]{} & I_{q-1} & \longrightarrow & I_{q-2} & \longrightarrow & \cdots
\end{array}
$$

Hence Theorem 3.1(vi) specialises to the statement that if $M \xrightarrow{\mu} I_q$ and $M \xrightarrow{\partial_q\mu} I_{q-1}$ are both monomorphisms, then $M = 0$. That is, if $M \xrightarrow{\mu} I_q$ is a monomorphism for which $\mathrm{Im}\,\mu \cap \mathrm{Ker}\,\partial_q = 0$, then $M = 0$. This is equivalent to item (v) in Theorem 3.1 for complexes.

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References

[1] Avramov, L. L., Foxby, H.-B., and Halperin, S.: *Differential graded homological algebra*. Unpublished notes.

[2] Bökstedt, M. and Neeman, A.: Homotopy limits in triangulated categories. *Compositio Math.* **86** (1993), no. 2, 209–234. Zbl 0802.18008 MR 1214458

[3] Cartan, H. and Eilenberg, S.: *Homological algebra*. Princeton University Press, Princeton, NJ, 1956. Zbl 0075.24305 MR 0077480

[4] Chen, X.-W. and Iyengar, S. B.: Support and injective resolutions of complexes over commutative rings. *Homology Homotopy Appl.* **12** (2010), no. 1, 39–44. Zbl 1185.13015 MR 2594681

[5] Christensen, L. W., Foxby, H.-B. and Holm, H.: *Derived category methods in commutative algebra*. Springer Monogr. Math., Springer, Cham, 2024. Zbl 07954534 MR 4890472

[6] Christensen, L. W., Iyengar, S. B. and Marley, T.: Rigidity of Ext and Tor with coefficients in residue fields of a commutative Noetherian ring. *Proc. Edinb. Math. Soc. (2)* **62** (2019), no. 2, 305–321. Zbl 1412.13020 MR 3935802

[7] Dell'Ambrogio, I., Stevenson, G. and Šťovíček, J.: Gorenstein homological algebra and universal coefficient theorems. *Math. Z.* **287** (2017), no. 3-4, 1109–1155. Zbl 1405.16014 MR 3719530

[8] Enochs, E. E., Jenda, O. M. G. and Xu, J.: Orthogonality in the category of complexes. *Math. J. Okayama Univ.* **38** (1996), 25–46. Zbl 0940.18006 MR 1644453

[9] Foxby, H.-B.: Bounded complexes of flat modules. *J. Pure Appl. Algebra* **15** (1979), no. 2, 149–172. Zbl 0411.13006 MR 0535182

[10] Foxby, H.-B.: A homological theory of complexes of modules. Preprint, 1981.

[11] Freyd, P.: *Abelian categories. An introduction to the theory of functors*. Harper's Ser. Modern Math., Harper & Row, New York, 1964. Zbl 0121.02103 MR 0166240

[12] García Rozas, J. R.: *Covers and envelopes in the category of complexes of modules*. Res. Notes Math. 407, Chapman & Hall/CRC, Boca Raton, FL, 1999. Zbl 0922.16001 MR 1693036

[13] Gillespie, J.: Hereditary abelian model categories. *Bull. Lond. Math. Soc.* **48** (2016), no. 6, 895–922. Zbl 1372.18001 MR 3608936

[14] Göbel, R. and Trlifaj, J.: *Approximations and endomorphism algebras of modules*. De Gruyter Exp. Math. 41, Walter de Gruyter, Berlin, 2006. Zbl 1121.16002 MR 2251271

[15] Happel, D.: *Triangulated categories in the representation theory of finite-dimensional algebras*. London Math. Soc. Lecture Note Ser. 119, Cambridge University Press, Cambridge, 1988. Zbl 0635.16017 MR 0935124

[16] Hilton, P.: *Homotopy theory and duality*. Gordon and Breach, New York-London-Paris, 1965. Zbl 0155.50801 MR 0198466

[17] Holm, H.: Gorenstein homological dimensions. *J. Pure Appl. Algebra* **189** (2004), no. 1-3, 167–193. Zbl 1050.16003 MR 2038564

[18] Holm, H. and Jørgensen, P.: The Q-shaped derived category of a ring. *J. Lond. Math. Soc. (2)* **106** (2022), no. 4, 3263–3316. Zbl 08027258 MR 4524199

[19] Holm, H. and Jørgensen, P.: A brief introduction to the $\mathcal{Q}$-shaped derived category. In *Triangulated categories in representation theory and beyond – the Abel Symposium 2022*, pp. 141–167. Abel Symp. 17, Springer, Cham, 2024. Zbl 07950526 MR 4786505

[20] Holm, H. and Jørgensen, P.: The Q-shaped derived category of a ring – compact and perfect objects. *Trans. Amer. Math. Soc.* **377** (2024), no. 5, 3095–3128. Zbl 07876042 MR 4744776

[21] Hovey, M.: *Model categories*. Math. Surveys Monogr. 63, American Mathematical Society, Providence, RI, 1999. Zbl 0909.55001 MR 1650134

[22] Hovey, M.: Cotorsion pairs, model category structures, and representation theory. *Math. Z.* **241** (2002), no. 3, 553–592. Zbl 1016.55010 MR 1938704

[23] Iacob, A. and Iyengar, S. B.: Homological dimensions and regular rings. *J. Algebra* **322** (2009), no. 10, 3451–3458. Zbl 1186.13012 MR 2568347

[24] Iyama, O., Kato, K. and Miyachi, J.-i.: Derived categories of N-complexes. *J. Lond. Math. Soc. (2)* **96** (2017), no. 3, 687–716. Zbl 1409.18013 MR 3742439

[25] Iyama, O. and Minamoto, H.: On a generalization of complexes and their derived categories. In *Proceedings of the 47th Symposium on ring theory and representation theory*, pp. 63–68. Symp. Ring Theory Represent. Theory Organ. Comm., Okayama, 2015. MR 3363946

[26] Iyama, O. and Minamoto, H.: $\mathscr{A}$-derived categories. In preparation.

[27] Kashiwara, M. and Schapira, P.: *Categories and sheaves*. Grundlehren Math. Wiss. 332, Springer, Berlin, 2006. Zbl 1118.18001 MR 2182076

[28] Krause, H.: The stable derived category of a Noetherian scheme. *Compos. Math.* **141** (2005), no. 5, 1128–1162. Zbl 1090.18006 MR 2157133

[29] Mac Lane, S.: *Categories for the working mathematician*. Second edition. Grad. Texts in Math. 5, Springer, New York, 1998. Zbl 0906.18001 MR 1712872

[30] Ringel, C. M. and Zhang, P.: Representations of quivers over the algebra of dual numbers. *J. Algebra* **475** (2017), 327–360. Zbl 1406.16010 MR 3612474

[31] Roig, A.: Minimal resolutions and other minimal models. *Publ. Mat.* **37** (1993), no. 2, 285–303. Zbl 0814.55005 MR 1249232

[32] Schiffler, R.: *Quiver representations*. CMS Books Math., Springer, Cham, 2014. Zbl 1310.16015 MR 3308668

[33] Stenström, B.: *Rings of quotients. An introduction to methods of ring theory*. Grundlehren Math. Wiss. 217, Springer, New York-Heidelberg, 1975. Zbl 0296.16001 MR 0389953

[34] Wei, J.: Gorenstein homological theory for differential modules. *Proc. Roy. Soc. Edinburgh Sect. A* **145** (2015), no. 3, 639–655. Zbl 1352.16006 MR 3371571

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