Doc. Math. (Online first)
DOI 10.4171/DM/1042

Tensor products of Drinfeld modules and convolutions of Goss L-series

Wei-Cheng Huang

Abstract. Following the same framework of the special value results (by Papanikolas and the author) of convolutions of Goss and Pellarin L-series attached to Drinfeld modules that take values in Tate algebras, we establish special value results of convolutions of two Goss L-series attached to Drinfeld modules that take values in $\mathbb{F}_q(\!(\frac{1}{\theta})\!)$. Applying the class module formula of Fang to tensor products of two Drinfeld modules, we provide special value formulas for their L-functions. By way of the theory of Schur polynomials these identities take the form of specializations of convolutions of Rankin–Selberg type. Finally, we show an explicit computation of the regulators appearing in Fang's class module formula for tensor products as well as symmetric and alternating squares of Drinfeld modules.

Contents

1. Introduction

1.1. Motivation

Given an elliptic curve E over $\mathbb{Q}$, its L-function has the form

$$L(E,s) := \prod_p Q_p(p^{-s})^{-1},$$

where $Q_p(X) \in \mathbb{Z}[X]$ depends on the reduction of E at p. Recall that if E has a good reduction at p and $\ell \neq p$ is a prime, the polynomial $Q_p(X) \in \mathbb{Z}[X]$ is the characteristic polynomial of the Frobenius element of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the ℓ-adic Tate module of E and it is independent of ℓ. (e.g., see [47]).

Mathematics Subject Classification 2020: 11M38 (primary); 11G09, 11M32 (secondary).
Keywords: Goss L-series, Drinfeld modules, Anderson t-modules, Tate algebras, Poonen pairings, class module formulas, Schur polynomials.

It makes sense to consider the tensor products of Tate modules and study the corresponding L-functions (see [17,23]), but it is still an open problem what the corresponding geometric objects are. However, this works out well in the function field. By Anderson [2], the category of abelian t-modules is anti-equivalent to the category of t-motives, which are analogous to abelian varieties and pure motives in the function field setting. We are interested in the L-function of a tensor product of Drinfeld modules, which is defined to be the corresponding t-module of the tensor product of the corresponding t-motives.

Guided by a series of articles by Anglès, Demeslay, Gezmiş, Pellarin, Taelmann, Tavares Ribeiro [7, 9, 18, 19, 25, 45, 49–51], Papanikolas and the author [37] defined a t-module $\mathbb{E}(\phi \times \psi)$ by some kind of twisting of two Drinfeld modules ϕ and ψ. Then its associated L-function includes a Rankin–Selberg type convolution of a Goss L-series and a Pellarin L-series (see [37, Thms. 6.2.3 and 6.3.5]) and can be evaluated using Demeslay's class module identity (see [37, Cors. 6.2.4 and 6.3.6]).

Inspired by these convolutions, it is natural to consider the Rankin–Selberg type convolution of two Goss L-series and ask to what extent the regulators are related to the special values of logarithms. This leads us to study Goss L-series of tensor products, symmetric squares and alternating squares of Drinfeld modules (see Theorem A and Corollary B). We also provide explicit expressions of their regulators for the rank 2 case (see Corollary C and Theorem D) in terms of these L-values and logarithms. We now summarize these results.

1.2. Convolution Goss L-functions and special values

Let $\mathbb{F}_q$ be a field with $q = p^m$ elements for p a prime. For a variable θ we let $A := \mathbb{F}_q[\theta]$ be the polynomial ring in θ over $\mathbb{F}_q$, and let $K := \mathbb{F}_q(\theta)$ be its fraction field. We take $K_\infty := \mathbb{F}_q((\theta^{-1}))$ for the completion of K at ∞, and let $\mathbb{C}_\infty$ be the completion of an algebraic closure of K_∞. We normalize the ∞-adic norm $|\cdot|_\infty$ on $\mathbb{C}_\infty$ so that $|\theta|_\infty = q$, and letting $\deg := -\operatorname{ord}_\infty = \log_q |\cdot|_\infty$, we see that $\deg a = \deg_\theta a$ for any $a \in A$. We let A_+ denote the subset of monic elements of A. Finally, we let $A[\tau]$ be the ring of twisted polynomials in τ with coefficients in A, subject to the relation $\tau a = a^q \tau$ for $a \in A$.

Let $\mathsf{A} = \mathbb{F}_q[t]$, and let $\phi, \psi \colon \mathsf{A} \to A[\tau]$ be Drinfeld modules defined over A by

$$\phi_t = \theta + \kappa_1 \tau + \cdots + \kappa_r \tau^r, \quad \psi_t = \theta + \eta_1 \tau + \cdots + \eta_\ell \tau^\ell, \quad \kappa_r, \eta_\ell \in \mathbb{F}_q^\times. \tag{1.2.1}$$

Thus ϕ has rank r and ψ has rank ℓ, and moreover because their leading coefficients are in $\mathbb{F}_q^\times$, both ϕ and ψ have everywhere good reduction.

1.2.2. Characteristic polynomials of Frobenius. For our Drinfeld module ϕ in (1.2.1), if we fix $f \in A_+$ irreducible of degree d and let $\lambda \in \mathsf{A}_+$ be irreducible with $\lambda(\theta) \neq f$, then by work of Gekeler, Hsia, Takahashi, and Yu [24, 36, 52], the characteristic polynomial $P_{\phi,f}(X) = \operatorname{Char}(\tau^d, T_\lambda(\overline{\phi}), X) = X^r + c_{r-1}X^{r-1} + \cdots + c_0 \in A[X]$ of τ^d acting on $T_\lambda(\overline{\phi})$, the λ-adic Tate module of the reduction of ϕ modulo f, satisfies $c_0 = (-1)^r \overline{\chi}_\phi(f) f$, where $\chi_\phi(a) := ((-1)^{r+1}\kappa_r)^{\deg a}$ and $\overline{\chi}_\phi = \chi_\phi^{-1}$. We further let $P^\vee_{\phi,f}(X) \in K[X]$ be the characteristic polynomial of τ^d acting on the dual space of $T_\lambda(\overline{\phi})$. See Section 2.4 for more details.

1.2.3. L-functions of tensor products. For the Drinfeld module ϕ from (1.2.1), Goss defined the L-function,

$$L(\phi^\vee, s) = \prod_f Q^\vee_{\phi,f}(f^{-s})^{-1} = \sum_{a \in A_+} \frac{\mu_\phi(a)}{a^{s+1}},$$

where $Q^\vee_{\phi,f}(X)$ is the reciprocal polynomial of $P^\vee_{\phi,f}(X)$. The multiplicative function $\mu_\phi\colon A_+ \to A$ is defined by the generating series,

$$\sum_{m=1}^{\infty} \mu_\phi(f^m) X^m = Q^\vee_f(fX)^{-1}.$$

One of our main goals is to express Dirichlet series for L-function of tensor products in a similar explicit fashion. If $E = \phi \otimes \psi$, Cauchy's identity (see Theorem 2.5.12) implies that the L-function $L(E^\vee, s)$ has a convolution interpretation, following the situation for Maass forms on GL_n (see [12, 28]). When $E = \mathrm{Sym}^2 \phi$ or $\bigwedge^2 \phi$, Littlewood's identities (see Theorem 2.5.15) imply that the L-function $L(E^\vee, s)$ can be factored into a twisted Carlitz zeta function and intriguing L-functions involving $\boldsymbol{\mu}_\phi : (A_+)^{r-1} \to A$ defined in [37]. See Section 2.6 for details.

When $r, \ell \geqslant 2$, we define an L-function $L(\boldsymbol{\mu}_\phi \times \boldsymbol{\mu}_\psi, s)$ as follows. If $r = \ell$

$$L(\boldsymbol{\mu}_\phi \times \boldsymbol{\mu}_\psi, s) := \sum_{a_1 \in A_+} \cdots \sum_{a_{r-1} \in A_+} \frac{\boldsymbol{\mu}_\phi(a_1, \dots, a_{r-1}) \boldsymbol{\mu}_\psi(a_1, \dots, a_{r-1})}{(a_1 \cdots a_{r-1})^2 (a_1 a_2^2 \cdots a_{r-1}^{r-1})^s}.$$

If $r < \ell$, then

$$L(\boldsymbol{\mu}_\phi \times \boldsymbol{\mu}_\psi, s) := \sum_{a_1, \dots, a_r \in A_+} \frac{\chi_\phi(a_r) \boldsymbol{\mu}_\phi(a_1, \dots, a_{r-1}) \boldsymbol{\mu}_\psi(a_1, \dots, a_r, 1, \dots, 1)}{(a_1 \cdots a_r)^2 (a_1 a_2^2 \cdots a_r^r)^s}.$$

In this way, we interpret $L(\boldsymbol{\mu}_\phi \times \boldsymbol{\mu}_\psi, s)$ as a convolution of two Goss L-series. Since $\phi \otimes \psi \cong \psi \otimes \phi$ (see [34, Prop. 2.5], Remark 3.1.5), the $r > \ell$ case is the same as the $r < \ell$ case in the sense of switching the rolls of ϕ and ψ. We further define two L-functions as follows.

$$L(\widetilde{\boldsymbol{\mu}}_\phi, s) := \sum_{a_1 \in A_+} \cdots \sum_{a_{r-1} \in A_+} \frac{\boldsymbol{\mu}_\phi(a_1^2, \dots, a_{r-1}^2)}{(a_1 \cdots a_{r-1})^2 (a_1 a_2^2 \cdots a_{r-1}^{r-1})^s},$$

and

$$L(\widehat{\boldsymbol{\mu}}_\phi, s) := \sum_{\substack{a_1, \dots, a_{r-1} \in A_+ \\ a_i = 1 \text{ if } 2 \nmid i}} \frac{\boldsymbol{\mu}_\phi(a_1, \dots, a_{r-1})}{a_1 \cdots a_{r-1} (a_1 a_2^2 \cdots a_{r-1}^{r-1})^s}.$$

For the cases $E = \phi^{\otimes 2}$, $\mathrm{Sym}^2 \phi$, $\bigwedge^2 \phi$, the L-series above are related to $L(E^\vee, s)$ by the following result (stated as Theorems 4.3.14, 4.3.17, 4.4.10 and 4.4.13). We let $L(A, \chi, s) = \sum_{a \in A_+} \chi(a) \cdot a^{-s}$ be the twist of the Carlitz zeta function $L(A, s) = \sum_{a \in A_+} a^{-s}$ by a completely mutiplicative function $\chi\colon A_+ \to \mathbb{F}_q^\times$.

Theorem A. *Let ϕ, $\psi\colon \mathsf{A} \to A[\tau]$ be Drinfeld modules of ranks r and ℓ respectively with everywhere good reduction, as defined in (1.2.1). Assume that $r, \ell \geqslant 2$.*

(a) *If $r = \ell$, then*

$$L\big((\phi \otimes \psi)^\vee, s\big) = L(A, \chi_\phi \chi_\psi, rs + 2) \cdot L(\boldsymbol{\mu}_\phi \times \boldsymbol{\mu}_\psi, s).$$

(b) *If $r < \ell$, then*

$$L\big((\phi \otimes \psi)^\vee, s\big) = L(\boldsymbol{\mu}_\phi \times \boldsymbol{\mu}_\psi, s).$$

(c) *Assume further that $p \neq 2$, then*

$$L\big((\operatorname{Sym}^2 \phi)^\vee, s\big) = L(A, \chi_\phi^2, rs + 2) \cdot L(\widetilde{\boldsymbol{\mu}}_\phi, s).$$

and

$$L\Big(\Big(\bigwedge^2 \phi\Big)^\vee, s\Big) = L\Big(A, \chi_\phi, \frac{rs}{2} + 1\Big)^{\frac{(-1)^r+1}{2}} \cdot L(\widehat{\boldsymbol{\mu}}_\phi, s).$$

Substituting $s = 0$ in Theorem A provides special value identities for $L(\boldsymbol{\mu}_\phi \times \boldsymbol{\mu}_\psi, 0)$, $L(\widetilde{\boldsymbol{\mu}}_\phi, 0)$ and $L(\widehat{\boldsymbol{\mu}}_\phi, 0)$. Fang's class module identity (Theorem 4.2.2), which relates the special L-values of an abelian t-module E to its regulator Reg_E and class module $H(E)$, implies the following corollary (stated as Corollaries 4.3.15, 4.3.18, 4.4.11 and 4.4.14).

Corollary B. *Let ϕ, $\psi\colon \mathsf{A} \to A[\tau]$ be Drinfeld modules of ranks r and ℓ respectively with everywhere good reduction, as defined in (1.2.1). Assume that $r, \ell \geqslant 2$.*

(a) *If $r = \ell$, then*

$$\begin{aligned} L(\boldsymbol{\mu}_\phi \times \boldsymbol{\mu}_\psi, 0) &= \sum_{a_1 \in A_+} \cdots \sum_{a_{r-1} \in A_+} \frac{\boldsymbol{\mu}_\phi(a_1, \ldots, a_{r-1}) \boldsymbol{\mu}_\psi(a_1, \ldots, a_{r-1})}{(a_1 \cdots a_{r-1})^2} \\ &= \frac{\operatorname{Reg}_{\phi \otimes \psi} \cdot \big[\mathrm{H}(\phi \otimes \psi)\big]_A}{L(A, \chi_\phi \chi_\psi, 2)}. \end{aligned}$$

(b) *If $r < \ell$, then*

$$\begin{aligned} L(\boldsymbol{\mu}_\phi \times \boldsymbol{\mu}_\psi, 0) &= \sum_{a_1, \ldots, a_r \in A_+} \frac{\chi_\phi(a_r) \boldsymbol{\mu}_\phi(a_1, \ldots, a_{r-1}) \boldsymbol{\mu}_\psi(a_1, \ldots, a_r, 1, \ldots, 1)}{(a_1 \cdots a_r)^2} \\ &= \operatorname{Reg}_{\phi \otimes \psi} \cdot \big[\mathrm{H}(\phi \otimes \psi)\big]_A. \end{aligned}$$

(c) *Assume further that $p \neq 2$, then*

$$L(\widetilde{\boldsymbol{\mu}}_\phi, 0) = \sum_{a_1 \in A_+} \cdots \sum_{a_{r-1} \in A_+} \frac{\boldsymbol{\mu}_\phi(a_1^2, \ldots, a_{r-1}^2)}{(a_1 \cdots a_{r-1})^2} = \frac{\operatorname{Reg}_{\operatorname{Sym}^2 \phi} \cdot \big[\mathrm{H}(\operatorname{Sym}^2 \phi)\big]_A}{L(A, \chi_\phi^2, 2)},$$

and

$$L(\widehat{\boldsymbol{\mu}}_\phi, 0) = \sum_{\substack{a_1, \ldots, a_{r-1} \in A_+ \\ a_i = 1 \text{ if } 2 \nmid i}} \frac{\boldsymbol{\mu}_\phi(a_1, \ldots, a_{r-1})}{a_1 \cdots a_{r-1}} = \frac{\operatorname{Reg}_{\bigwedge^2 \phi} \cdot \big[\mathrm{H}(\bigwedge^2 \phi)\big]_A}{L(A, \chi_\phi, 1)^{\frac{(-1)^r+1}{2}}}.$$

We now assume $r=2$ and suppose $\phi_t=\theta+\kappa_1\tau+\kappa_2\tau^2$ and $\mathrm{Log}_\phi(z)=\sum_{m\geq 0}\beta_m z^{q^m}$. In Section 5.2 we define $\mathbb{F}_q$-linear power series $L_i(z)$, $L_i'(z)$, $\widetilde{L}_i(z)$, e.g., see (5.2.24)–(5.2.30) as well as matrices $\mathbf{L}_m\in\mathrm{Mat}_4(K)$ for $m\geqslant 1$, see Corollary 5.2.31, which combine to make the coordinate functions of the logarithm series of $\phi\otimes\psi$, $\mathrm{Sym}^2\phi$ and $\bigwedge^2\phi$. The main result in this part of the paper (Corollary 5.2.31) is the following.

Corollary C. *Let $\phi\colon \mathsf{A}\to A[\tau]$ be a Drinfeld module given by $\phi_t=\theta+\kappa_1\tau+\kappa_2\tau^2$ with $\kappa_2\in\mathbb{F}_q^\times$, then*

(a) $$\mathrm{Log}_{\phi^{\otimes 2}}\begin{pmatrix} z_1\\ z_2\\ z_3\\ z_4\end{pmatrix}=\begin{pmatrix} z_1\\ z_2\\ z_3\\ z_4\end{pmatrix}+\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&-\frac{\kappa_1}{\kappa_2}&1\end{pmatrix}\sum_{m\geq 1}\mathbf{L}_m\begin{pmatrix} z_1^{q^m}\\ z_2^{q^m}\\ z_3^{q^m}\\ z_4^{q^m}\end{pmatrix},$$

(b) $$\mathrm{Log}_{\mathrm{Sym}^2\phi}\begin{pmatrix} z_1\\ z_2\\ z_3\end{pmatrix}=\begin{pmatrix} z_1\\ z_2\\ z_3\end{pmatrix}+\begin{pmatrix}1&&&\\&\frac{1}{2}&\frac{1}{2}&\\&-\frac{\kappa_1}{2\kappa_2}&-\frac{\kappa_1}{2\kappa_2}&1\end{pmatrix}\sum_{m\geq 1}\mathbf{L}_m\begin{pmatrix} z_1^{q^m}\\ z_2^{q^m}\\ z_2^{q^m}\\ z_3^{q^m}\end{pmatrix},$$

(c) $\mathrm{Log}_{\bigwedge^2\phi}(z)=z+\widetilde{L}_0(z)$.

We further define the dilogarithm series $\mathrm{Log}_{\phi,2}(z)=\sum_{m\geq 0}\beta_m^2 z^{q^m}$ and $\widehat{L}_i(z)$ in (5.3.4)–(5.3.6), and obtain the following result on regulators (stated as Theorem 5.3.7). Note that $\widehat{L}_i(z)$ is related to $L_i'(z)$ by the chain rule [42, Lem. 2.4.6] (see also (5.3.10)).

Theorem D. *Let $\phi\colon \mathsf{A}\to A[\tau]$ be a Drinfeld module given by $\phi_t=\theta+\kappa_1\tau+\kappa_2\tau^2$ with $\kappa_2\in\mathbb{F}_q^\times$.*

(a) *Assume that* $\deg(\kappa_1)\leqslant(q+1)/2$. *Then* $\mathrm{Reg}_{\bigwedge^2\phi}=\mathrm{Log}_{\bigwedge^2\phi}(1)$.

(b) *Assume that* $\deg(\kappa_1)\leqslant 1$. *Then*

  (i) *We have an explicit formula for* $\mathrm{Reg}_{\mathrm{Sym}^2\phi}$ *involving values of dilogarithm series* $\mathrm{Log}_{\phi,2}(z)$ *and values of power series* $L_i(z)$, $\widetilde{L}_i(z)$ *and* $\widehat{L}_i(z)$.

  (ii) $\mathrm{Reg}_{\phi^{\otimes 2}}=\mathrm{Reg}_{\mathrm{Sym}^2\phi}\cdot\mathrm{Reg}_{\bigwedge^2\phi}$.

Remark 1.2.4. From Theorem D we see the regulators can be explicitly expressed in terms of coordinate functions of logarithms.

1.3. Outline

After summarizing preliminary material in Section 2, we define tensor products, symmetric and alternating squares of Drinfeld modules from the aspect of t-motives and explore their properties in Section 3. In Section 4 we review the theories of Goss L-series, as well as Fang's class module formula. We consider the L-function of $\phi\otimes\psi$, $\mathrm{Sym}^2\phi$ and $\bigwedge^2\phi$. Then we introduce the convolution L-series $L(\boldsymbol{\mu}_\phi\times\boldsymbol{\mu}_\psi,s)$, $L(\widetilde{\boldsymbol{\mu}}_\phi,s)$ and $L(\widehat{\boldsymbol{\mu}}_\phi,s)$, relate them to $L(\phi\otimes\psi,s)$, $L(\mathrm{Sym}^2\phi,s)$, $L(\bigwedge^2\phi,s)$ and twisted Carlitz L-series, and investigate special value identities using Fang's class module formula. We provide explicit expressions of regulators $\mathrm{Reg}_{\phi\otimes\psi}$, $\mathrm{Reg}_{\mathrm{Sym}^2\phi}$ and $\mathrm{Reg}_{\bigwedge^2\phi}$ for rank 2 case in Section 5.

2. Preliminaries

2.1. Notation

We will use the following notation throughout.

A	$=$	$\mathbb{F}_q[\theta]$, polynomial ring in variable θ over $\mathbb{F}_q$.
A_+	$=$	the monic elements of A.
K	$=$	$\mathbb{F}_q(\theta)$, the fraction field of A.
K_∞	$=$	$\mathbb{F}_q((\theta^{-1}))$, the completion of K at ∞.
$\mathbb{C}_\infty$	$=$	the completion of an algebraic closure $\overline{K}_\infty$ of K_∞.
$\lvert\cdot\rvert_\infty$; deg	$=$	∞-adic norm on $\mathbb{C}_\infty$, extended to the sup norm on a finite-dimensional $\mathbb{C}_\infty$-vector space; $\deg = -\operatorname{ord}_\infty = \log_q\lvert\cdot\rvert_\infty$.
$\mathbb{F}_f$	$=$	A/fA for $f \in A_+$ irreducible.
A	$=$	$\mathbb{F}_q[t]$, for a variable t independent from θ.
$\mathbb{T}_t$	$=$	Tate algebra in $t = \{\sum a_i t^i \in \mathbb{C}_\infty[\![t]\!] \mid \lvert a_i\rvert_\infty \to 0\}$ = completion of $\mathbb{C}_\infty[t]$ with respect to Gauss norm.

2.1.1. Rings of operators. For a variable t independent from θ we let $\mathsf{A} := \mathbb{F}_q[t]$. We let $\mathbb{T}_t$ denote the standard *Tate algebra*, $\mathbb{T}_t \subseteq \mathbb{C}_\infty[\![t]\!]$, consisting of power series that converge on the closed unit disk of $\mathbb{C}_\infty$, and we define

$$\mathbb{T}_t(K_\infty) := \mathbb{T}_t \cap K_\infty[\![t]\!] = \mathbb{F}_q[t]((\theta^{-1})),$$

where the latter set consists of Laurent series in θ^{-1} with coefficients in the polynomial ring $\mathbb{F}_q[t]$. We let $\|\cdot\|$ denote the Gauss norm on $\mathbb{T}_t$, such that $\|\sum_{i=0}^\infty a_i t^i\| = \max_i\{\lvert a_i\rvert_\infty\}$, under which $\mathbb{T}_t$ is a complete normed $\mathbb{C}_\infty$-vector space, and likewise $\mathbb{T}_t(K_\infty)$ is a complete normed K_∞-vector space. We extend the degree map on $\mathbb{C}_\infty$ to $\mathbb{T}_t$ by taking $\deg = \log_q\|\cdot\|$. We further let $\mathbb{T}_\theta$ denote the Tate algebra, $\mathbb{T}_\theta \subseteq \mathbb{C}_\infty[\![t]\!]$, consisting of power series that converge on the closed disk of radius $\lvert\theta\rvert_\infty$, and $\|\cdot\|_\theta$ denote the norm on $\mathbb{T}_\theta$ such that $\|\sum_{i=0}^\infty a_i t^i\|_\theta = \max_i\{q^i \cdot \lvert a_i\rvert_\infty\}$.

2.1.2. Frobenius operators. We take $\tau\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ for the *q-th power Frobenius automorphism*, which we extend to $\mathbb{C}_\infty((t))$ by requiring it to commute with t. For $g = \sum c_i t^i \in \mathbb{C}_\infty((t))$, we define the *$n$-th Frobenius twist*,

$$g^{(n)} := \tau^n(g) = \sum c_i^{q^n} t^i, \quad \forall n \in \mathbb{Z}.$$

Then τ induces an $\mathbb{F}_q(t)$-linear automorphism of $\mathbb{T}_t$, and the fixed ring of τ is $\mathbb{T}_t^\tau = \mathbb{F}_q[t]$.

2.1.3. Twisted polynomials. Let R be any commutative $\mathbb{F}_q$-algebra, and let $\tau\colon R \to R$ be an injective $\mathbb{F}_q$-algebra endomorphism. Let R^τ be the $\mathbb{F}_q$-subalgebra of R of elements fixed by τ. For $n \in \mathbb{Z}$ for which τ^n is defined on R and a matrix $B = (b_{ij})$ with entries in R, we let $B^{(n)}$ be defined by twisting each entry. That is, $(b_{ij})^{(n)} = (b_{ij}^{(n)})$. For $\ell \geqslant 1$

we let $\mathrm{Mat}_\ell(R)[\tau] = \mathrm{Mat}_\ell(R[\tau])$ be the ring of *twisted polynomials* in τ with coefficients in $\mathrm{Mat}_\ell(R)$, subject to the relation $\tau B = B^{(1)}\tau$ for $B \in \mathrm{Mat}_\ell(R)$. In this way, R^ℓ is a left $\mathrm{Mat}_\ell(R)[\tau]$-module, where if $\beta = B_0 + B_1\tau + \cdots + B_m\tau^m \in \mathrm{Mat}_\ell(R)[\tau]$ and $\mathbf{x} \in R^\ell$, then

$$\beta(\mathbf{x}) = B_0\mathbf{x} + B_1\mathbf{x}^{(1)} + \cdots + B_m\mathbf{x}^{(m)}. \tag{2.1.4}$$

If furthermore τ is an automorphism of R, then we set $\sigma := \tau^{-1}$ and form the twisted polynomial ring $\mathrm{Mat}_\ell(R)[\sigma]$, subject to $\sigma B = B^{(-1)}\sigma$ for $B \in \mathrm{Mat}_\ell(R)$. Then R^ℓ is a left $\mathrm{Mat}_\ell(R)[\sigma]$-module, where for $\gamma = C_0 + C_1\sigma + \cdots + C_m\sigma^m \in \mathrm{Mat}_\ell(R)[\sigma]$ and $\mathbf{x} \in R^\ell$,

$$\gamma(\mathbf{x}) = C_0\mathbf{x} + C_1\mathbf{x}^{(-1)} + \cdots + C_m\mathbf{x}^{(-m)}.$$

For $\beta \in \mathrm{Mat}_\ell(R)[\tau]$ (or $\gamma \in \mathrm{Mat}_\ell(R)[\sigma]$), we write $\partial\beta$ (or $\partial\gamma$) for the constant term with respect to τ (or σ). We have natural inclusions of $\mathbb{F}_q$-algebras,

$$\mathrm{Mat}_\ell(R)[\tau] \subseteq \mathrm{Mat}_\ell(R)[[\tau]], \quad \mathrm{Mat}_\ell(R)[\sigma] \subseteq \mathrm{Mat}_\ell(R)[[\sigma]],$$

into *twisted power series* rings, where the latter holds when τ is an automorphism.

2.1.5. Ore anti-involution. We assume that $\tau\colon R \to R$ is an automorphism, and recall the anti-isomorphism $*\colon R[\tau] \to R[\sigma]$ of $\mathbb{F}_q$-algebras originally defined by Ore [43] (see also [33, §1.7], [42, §2.3], [46]), given by

$$\left(\sum_{i=0}^{\ell} b_i\tau^i\right)^* = \sum_{i=0}^{\ell} b_i^{(-i)}\sigma^i.$$

One verifies that $(\alpha\beta)^* = \beta^*\alpha^*$ for $\alpha, \beta \in R[\tau]$. For $B = (\beta_{ij}) \in \mathrm{Mat}_{k\times\ell}(R[\tau])$, we set

$$B^* := (\beta_{ij}^*)^{\mathsf{T}} \in \mathrm{Mat}_{\ell\times k}\bigl(R[\sigma]\bigr),$$

which then satisfies

$$(BC)^* = C^*B^* \in \mathrm{Mat}_{m\times k}\bigl(R[\sigma]\bigr), \quad B \in \mathrm{Mat}_{k\times\ell}\bigl(R[\tau]\bigr),\ C \in \mathrm{Mat}_{\ell\times m}\bigl(R[\tau]\bigr). \tag{2.1.6}$$

The inverse of $*\colon \mathrm{Mat}_{k\times\ell}(R[\tau]) \to \mathrm{Mat}_{\ell\times k}(R[\sigma])$ is also denoted by "$*$."

2.1.7. Orders of finite $F[x]$-modules. For $F[x]$ a polynomial ring in one variable over a field F, we say that an $F[x]$-module is *finite* if it is finitely generated and torsion. Now fix a finite $F[x]$-module M. Then there are monic polynomials $f_1, \dots, f_\ell \in F[x]$ so that

$$M \cong F[x]/(f_1) \oplus \cdots \oplus F[x]/(f_\ell).$$

We set $[M]_{F[x]} := f_1 \cdots f_\ell \in F[x]$, which is a generator of the Fitting ideal of M, and we call $[M]_{F[x]}$ the $F[x]$-*order* of M. If $m_x\colon M \to M$ is left-multiplication by x, then

$$[M]_{F[x]} = \mathrm{Char}(m_x, M, X)|_{X=x},$$

where $\mathrm{Char}(m_x, M, X) \in F[X]$ is the characteristic polynomial of m_x as an F-linear map.

For a variable y independent from x, but M still an $F[x]$-module, we will write

$$[M]_{F[y]} := [M]_{F[x]}|_{x=y} = \mathrm{Char}(m_x, M, y).$$

This will be of particular use for us when M is an A-module (or $\mathbf{A}$-module), where

$$[M]_A = [M]_{\mathsf{A}}|_{t=\theta} = \mathrm{Char}(m_t, M, \theta) \in A, \big(\text{or } [M]_{\mathbb{A}} = [M]_{\mathbf{A}}|_{t=\theta} = \mathrm{Char}(m_t, M, \theta) \in \mathbb{A}\big),$$

coercing A-orders and $\mathbb{A}$-orders to be elements of our scalar fields.

2.2. Drinfeld modules, Anderson t-modules, and their adjoints

Given a field $F \supseteq \mathbb{F}_q$ and an $\mathbb{F}_q$-algebra map $\iota\colon \mathsf{A} \to F$, we call F an A-*field*. The kernel of ι is the *characteristic* of F, and if ι is injective then the characteristic is *generic*. If $F \subseteq \mathbb{C}_\infty$ has generic characteristic, then we always assume that $\iota(t) = \theta$. Otherwise, $\iota(t) =: \overline{\theta} \in F$.

2.2.1. Drinfeld modules and Anderson t-modules. A *Drinfeld module* over F is defined by an $\mathbb{F}_q$-algebra homomorphism $\phi\colon \mathsf{A} \to F[\tau]$ such that

$$\phi_t = \overline{\theta} + \kappa_1 \tau + \cdots + \kappa_r \tau^r, \quad \kappa_r \neq 0. \tag{2.2.2}$$

We say that ϕ has *rank* r. We then make F into an A-module by setting

$$t \cdot x := \phi_t(x) = \overline{\theta} x + \kappa_1 x^q + \cdots + \kappa_r x^{q^r}, \quad x \in F.$$

Similarly an *Anderson t-module of dimension ℓ* over F is defined by an $\mathbb{F}_q$-algebra homomorphism $\psi\colon \mathsf{A} \to \mathrm{Mat}_\ell(F)[\tau]$ such that

$$\psi_t = \partial\psi_t + E_1 \tau + \cdots + E_w \tau^w, \quad E_i \in \mathrm{Mat}_\ell(F), \tag{2.2.3}$$

where $\partial\psi_t - \overline{\theta} \cdot \mathrm{I}_\ell$ is nilpotent. A Drinfeld module is then a t-module of dimension 1. We write $\psi(F)$ for F^ℓ with the A-module structure given by $a \cdot \mathbf{x} := \psi_a(\mathbf{x})$ through (2.1.4). Similarly, we write $\mathrm{Lie}(\psi)(F)$ for F^ℓ with $F[t]$-module structure defined by $\partial\psi_a$ for $a \in \mathsf{A}$. For $a \in \mathsf{A}$, the a-torsion submodule of $\psi(\overline{F})$ is denoted

$$\psi[a] := \big\{\mathbf{x} \in \overline{F}^\ell \mid \psi_a(\mathbf{x}) = 0\big\}.$$

Given t-modules $\phi\colon \mathsf{A} \to \mathrm{Mat}_k(F)[\tau]$, $\psi\colon \mathsf{A} \to \mathrm{Mat}_\ell(F)[\tau]$, a morphism $\eta\colon \phi \to \psi$ is a matrix $\eta \in \mathrm{Mat}_{\ell\times k}(F[\tau])$ such that $\eta\phi_a = \psi_a\eta$ for all $a \in \mathsf{A}$. Moreover, η induces an A-module homomorphism $\eta\colon \phi(F) \to \psi(F)$, and we have a functor $\psi \mapsto \psi(F)$ from the category of t-modules to A-modules. We also have an induced map of $F[t]$-modules, $\partial\psi\colon \mathrm{Lie}(\phi)(F) \to \mathrm{Lie}(\psi)(F)$.

Anderson defined t-modules in [2], and following his language we sometimes abbreviate “Anderson t-module” by “t-module.” For more information about Drinfeld modules and t-modules see [33, 53].

2.2.4. Exponential and logarithm series. Suppose now that $F \subseteq \mathbb{C}_\infty$ has generic characteristic and that ψ is defined over F. Then there is a twisted power series $\mathrm{Exp}_\psi \in \mathrm{Mat}_\ell(F)[\![\tau]\!]$, called the *exponential series* of ψ, such that

$$\mathrm{Exp}_\psi = \sum_{i=0}^{\infty} B_i \tau^i, \quad B_0 = \mathrm{I}_\ell,\ B_i \in \mathrm{Mat}_\ell(F),$$

and for all $a \in \mathsf{A}$, $\mathrm{Exp}_\psi \cdot \partial\psi_a = \psi_a \cdot \mathrm{Exp}_\psi$. This functional identity for $a = t$ induces a recursive relation that uniquely determines Exp_ψ. That the coefficient matrices have entries in F is due to Anderson [2, Prop. 2.1.4, Lem. 2.1.6]. The exponential series induces an $\mathbb{F}_q$-linear and entire function,

$$\mathrm{Exp}_\psi \colon \mathbb{C}_\infty^\ell \to \mathbb{C}_\infty^\ell, \quad \mathrm{Exp}_\psi(\mathbf{z}) = \sum_{i=0}^{\infty} B_i \mathbf{z}^{(i)},\ \mathbf{z} := (z_1, \dots, z_\ell)^{\mathsf{T}},$$

called the *exponential function* of ψ. That Exp_ψ converges everywhere is equivalent to

$$\lim_{i\to\infty} |B_i|_\infty^{1/q^i} = 0 \iff \lim_{i\to\infty} \deg(B_i)/q^i = -\infty.$$

We also identify the exponential function with the $\mathbb{F}_q$-linear formal power series $\mathrm{Exp}_\psi(\mathbf{z}) \in \mathbb{C}_\infty[\![\mathbf{z}]\!]^\ell$. The functional equation for Exp_ψ induces the identities,

$$\mathrm{Exp}_\psi(\partial\psi_a \mathbf{z}) = \psi_a\big(\mathrm{Exp}_\psi(\mathbf{z})\big), \quad \forall\, a \in \mathsf{A}.$$

The exponential function of ψ is always surjective for Drinfeld modules, but it may not be surjective when $\ell \geqslant 2$. We say that ψ is *uniformizable* if $\mathrm{Exp}_\psi \colon \mathbb{C}_\infty^\ell \to \mathbb{C}_\infty^\ell$ is surjective. The kernel of $\mathrm{Exp}_\psi \subseteq \mathbb{C}_\infty^\ell$,

$$\Lambda_\psi := \ker \mathrm{Exp}_\psi,$$

is a finitely generated and discrete $\partial\psi(\mathsf{A})$-submodule of $\mathbb{C}_\infty^\ell$ called the *period lattice* of ψ. Thus if ψ is uniformizable, then we obtain an exact sequence of A-modules,

$$0 \to \Lambda_\psi \to \mathbb{C}_\infty^\ell \xrightarrow{\mathrm{Exp}_\psi} \psi(\mathbb{C}_\infty) \to 0.$$

As an element of $\mathrm{Mat}_\ell(F)[\![\tau]\!]$ the series Exp_ψ is invertible, and we let

$$\mathrm{Log}_\psi := \mathrm{Exp}_\psi^{-1} \in \mathrm{Mat}_\ell(F)[\![\tau]\!]$$

be the *logarithm series* of ψ, satisfying

$$\mathrm{Log}_\psi = \sum_{i=0}^{\infty} C_i \tau^i, \quad C_0 = \mathrm{I}_\ell,\ C_i \in \mathrm{Mat}_\ell(F).$$

Together with the *logarithm function*, $\mathrm{Log}_\psi(\mathbf{z}) = \sum_{i\geqslant 0} C_i \mathbf{z}^{(i)} \in \mathbb{C}_\infty[\![\mathbf{z}]\!]^\ell$, we have $\partial\psi_a \cdot \mathrm{Log}_\psi = \mathrm{Log}_\psi \cdot \psi_a$ and $\partial\psi_a(\mathrm{Log}_\psi(\mathbf{z})) = \mathrm{Log}_\psi(\psi_a(\mathbf{z}))$, for all $a \in \mathsf{A}$. In general $\mathrm{Log}_\psi(\mathbf{z})$

converges only on an open polydisc in $\mathbb{C}_\infty^\ell$. For example, if $\phi\colon \mathsf{A} \to \mathbb{C}_\infty[\tau]$ is a Drinfeld module as in (2.2.2), then $\operatorname{Log}_\phi(z)$ converges on the open disk of radius R_ϕ, where

$$R_\phi = |\theta|_\infty^{-\max\{(\deg \kappa_i - q^i)/(q^i-1) \mid 1 \leqslant i \leqslant r, \kappa_i \neq 0\}} \tag{2.2.5}$$

(see [20, Rem. 6.11] and [39, Cor. 4.5]).

2.2.6. Adjoints of t-modules. Assume now that F is a perfect A-field and that $\psi\colon \mathsf{A} \to \operatorname{Mat}_\ell(F)[\tau]$ is an Anderson t-module over F defined as in (2.2.3). The *adjoint* of ψ is defined to be the $\mathbb{F}_q$-algebra homomorphism $\psi^*\colon \mathsf{A} \to \operatorname{Mat}_\ell(F)[\sigma]$ defined by

$$\psi_a^* := (\psi_a)^*, \quad \forall a \in \mathsf{A}.$$

Since for $a, b \in \mathsf{A}$ we have $\psi_{ab} = \psi_a\psi_b = \psi_b\psi_a$, (2.1.6) implies that ψ^* respects multiplication, which is the nontrivial part of checking that ψ^* is an $\mathbb{F}_q$-algebra homomorphism. From (2.2.3), we have

$$\psi_t^* = (\psi_t)^* = (\partial\psi_t)^{\mathsf{T}} + (E_1^{(-1)})^{\mathsf{T}}\sigma + \cdots + (E_w^{(-w)})^{\mathsf{T}}\sigma^w,$$

and so for any $\mathbf{x} \in F^\ell$, we have $\psi_t^*(\mathbf{x}) = (\partial\psi_t)^{\mathsf{T}}\mathbf{x} + (E_1^{(-1)})^{\mathsf{T}}\mathbf{x}^{(-1)} + \cdots + (E_w^{(-w)})^{\mathsf{T}}\mathbf{x}^{(-w)}$. In this way, the map ψ^* induces an A-module structure on F^ℓ, which we denote $\psi^*(F)$. Similarly we denote $\operatorname{Lie}(\psi^*)(F) = F^\ell$ with an $F[t]$-module structure induced by $\partial\psi_a^{\mathsf{T}}$ for $a \in \mathsf{A}$. For $a \in \mathsf{A}$, the a-torsion submodule of $\psi^*(\overline{F})$ is denoted

$$\psi^*[a] := \{\mathbf{x} \in \overline{F}^\ell \mid \psi_a^*(\mathbf{x}) = 0\}.$$

If $\eta\colon \phi \to \psi$ is a morphism of t-modules as above, then $\eta^* \in \operatorname{Mat}_{k\times\ell}(F)[\sigma]$ provides a morphism $\eta^*\colon \psi^* \to \phi^*$ such that $\eta^*\psi_a^* = \phi_a^*\eta^*$ for all $a \in \mathsf{A}$ (and vice versa). Furthermore, $\partial\eta^*\colon \operatorname{Lie}(\psi^*)(F) \to \operatorname{Lie}(\phi^*)(F)$ is an $F[t]$-module homomorphism. Adjoints of Drinfeld modules were investigated extensively by Goss [33, §4.14] and Poonen [46].

2.3. t-motives and dual t-motives

For this subsection we fix a perfect A-field F and t-module $\psi\colon \mathsf{A} \to \operatorname{Mat}_\ell(F)[\tau]$ as in (2.2.3). Recall that $\overline{\theta} = \iota(t) \in F$.

2.3.1. t-motive of ψ. We let $\mathcal{M}_\psi := \operatorname{Mat}_{1\times\ell}(F[\tau])$, and make $\mathcal{M}_\psi$ into a left $F[t,\tau]$-module by using the inherent structure as a left $F[\tau]$-module and setting

$$a \cdot m := m\psi_a, \quad m \in \mathcal{M}_\psi,\ a \in \mathsf{A}.$$

Then $\mathcal{M}_\psi$ is called the *t-motive* of ψ. We note that for any $m \in \mathcal{M}_\psi$,

$$(t - \overline{\theta})^\ell \cdot m \in \tau\mathcal{M}_\psi,$$

since $\partial\psi_t - \overline{\theta}\mathrm{I}_\ell$ is nilpotent (and F is perfect). If we need to emphasize the dependence on the base field F, we write

$$\mathcal{M}_\psi(F) := \mathcal{M}_\psi = \operatorname{Mat}_{1\times\ell}\big(F[\tau]\big).$$

A morphism $\eta\colon \phi \to \psi$ of t-modules over F of dimensions k and ℓ, defined as in Section 2.2, induces a morphism of left $F[t,\tau]$-modules $\eta^{\dagger}\colon \mathcal{M}_{\psi} \to \mathcal{M}_{\phi}$, given by $\eta^{\dagger}(m) := m\eta$ for $m \in \mathcal{M}_{\psi}$. The functor from t-modules over F to t-motives over F is fully faithful, and so every left $F[t,\tau]$-module homomorphism $\mathcal{M}_{\psi} \to \mathcal{M}_{\phi}$ arises in this way.

By construction $\mathcal{M}_{\psi}$ is free of rank ℓ as a left $F[\tau]$-module, and we say ℓ is the *dimension* of $\mathcal{M}_{\psi}$. If $\mathcal{M}_{\psi}$ is further free of finite rank over $F[t]$, then $\mathcal{M}_{\psi}$ is said to be *abelian* and $r = \operatorname{rank}_{F[t]} \mathcal{M}_{\psi}$ is the *rank* of $\mathcal{M}_{\psi}$. We will say that ψ is abelian or has rank r if $\mathcal{M}_{\psi}$ possesses the corresponding properties. The t-motives in Anderson's original definition in [2] are abelian, as will be most of the t-motives in this paper, but for example, see [11], [33, Ch. 5], [42, Chs. 2–4], [35] for t-motives in this wider context.

2.3.2. Dual t-motive of ψ. We let $\mathcal{N}_{\psi} := \operatorname{Mat}_{1\times\ell}(F[\sigma])$, and similar to the case of t-motives, we define a left $F[t,\sigma]$-module structure on $\mathcal{N}_{\psi}$ by setting

$$a \cdot n := n\psi_a^*, \quad n \in \mathcal{N}_{\psi},\ a \in \mathsf{A}.$$

The module $\mathcal{N}_{\psi}$ is the *dual t-motive* of ψ. As in the case of t-motives, for any $n \in \mathcal{N}_{\psi}$ we have $(t-\overline{\theta})^{\ell} \cdot n \in \sigma\mathcal{N}_{\psi}$. Also if we need to emphasize the dependence on F, we write

$$\mathcal{N}_{\psi}(F) := \mathcal{N}_{\psi} = \operatorname{Mat}_{1\times\ell}\bigl(F[\sigma]\bigr).$$

Again for a morphism $\eta\colon \phi \to \psi$ of t-modules of dimensions k and ℓ, we obtain a morphism of left $F[t,\sigma]$-modules, $\eta^{\ddagger}\colon \mathcal{N}_{\phi} \to \mathcal{N}_{\psi}$, given by $\eta^{\ddagger}(n) := n\eta^*$ for $n \in \mathcal{N}_{\phi}$. Also, every morphism of left $F[t,\sigma]$-modules $\mathcal{N}_{\phi} \to \mathcal{N}_{\psi}$ arises in this way.

The dual t-motive $\mathcal{N}_{\psi}$ is free of rank ℓ as a left $F[\sigma]$-module, and ℓ is the *dimension* of $\mathcal{N}_{\psi}$. If $\mathcal{N}_{\psi}$ is free of finite rank over $F[t]$, then we say $\mathcal{N}_{\psi}$ is A-*finite*, and we call $r = \operatorname{rank}_{F[t]}(\mathcal{N}_{\psi})$ the *rank* of $\mathcal{N}_{\psi}$. It has been shown by Maurischat [41] that for a t-module ψ, the t-motive $\mathcal{M}_{\psi}$ is abelian if and only if the dual t-motive $\mathcal{N}_{\psi}$ is A-finite. In this case the rank of $\mathcal{M}_{\psi}$ is the same as the rank of $\mathcal{N}_{\psi}$. We will say that ψ is A-finite or has rank r if $\mathcal{N}_{\psi}$ has those properties. Dual t-motives were initially introduced in [3] over fields of generic characteristic. See [11, 35, 41], [42, Chs. 2–4], for more information.

We call $\mathbf{m} = (m_1,\dots,m_r)^{\mathsf{T}} \in \operatorname{Mat}_{r\times 1}(\mathcal{M}_{\psi}(F))$ a *basis* of $\mathcal{M}_{\psi}(F)$ if $m_1,\dots,m_r$ form an $F[t]$-basis of $\mathcal{M}_{\psi}(F)$. Likewise $\mathbf{n} = (n_1,\dots,n_r)^{\mathsf{T}} \in \operatorname{Mat}_{r\times 1}(\mathcal{N}_{\psi}(F))$ is a *basis* of $\mathcal{N}_{\psi}(F)$ if $n_1,\dots,n_r$ form an $F[t]$-basis of $\mathcal{N}_{\psi}(F)$. We then define $\Gamma, \Phi \in \operatorname{Mat}_r(F[t])$ so that

$$\tau\mathbf{m} = \Gamma\mathbf{m}, \quad \sigma\mathbf{n} = \Phi\mathbf{n}.$$

It follows that $\det\Gamma = c(t-\theta)^{\ell}$, $\det\Phi = c'(t-\theta)^{\ell}$, where $c, c' \in F^{\times}$ (e.g., see [42, §3.2]). Then Γ *represents multiplication by* τ on $\mathcal{M}_{\psi}$ and Φ *represents multiplication by* σ on $\mathcal{N}_{\psi}$.

Example 2.3.3. *Carlitz module.* The Carlitz module $\mathsf{C}\colon \mathsf{A} \to F[\tau]$ over F is defined by

$$\mathsf{C}_t = \overline{\theta} + \tau,$$

and it has dimension 1 and rank 1. Then $\mathbf{m} = \{1\}$ is an $F[t]$-basis for $\mathcal{M}_{\mathsf{C}} = F[\tau]$, and $\mathbf{n} = \{1\}$ is an $F[t]$-basis for $\mathcal{N}_{\mathsf{C}} = F[\sigma]$. One finds that $\tau \cdot 1 = (t-\theta)\cdot 1$ in $\mathcal{M}_{\mathsf{C}}$ and $\sigma \cdot 1 = (t-\theta)\cdot 1$ in $\mathcal{N}_{\mathsf{C}}$, so $\Gamma = \Phi = t - \theta$.

Example 2.3.4 (Drinfeld modules). Let $\phi\colon \mathsf{A} \to F[\tau]$ be a Drinfeld module over F of rank r defined as in (2.2.2). Then $\mathbf{m} = (1, \tau, \dots, \tau^{r-1})^{\mathsf{T}}$ is a basis for $\mathcal{M}_\psi$ and $\mathbf{n} = (1, \sigma, \dots, \sigma^{r-1})^{\mathsf{T}}$ is a basis for $\mathcal{N}_\psi$. Furthermore, $\tau\mathbf{m} = \Gamma\mathbf{m}$ and $\sigma\mathbf{n} = \Phi\mathbf{n}$, where

$$\Gamma = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ (t-\overline{\theta})/\kappa_r & -\kappa_1/\kappa_r & \cdots & -\kappa_{r-1}/\kappa_r \end{pmatrix}, \tag{2.3.5}$$

and Φ occurs similarly. See [15, §3.3–3.4], [42, Exs. 3.5.14, 4.6.7], [44, §4.2] for details.

2.4. Tate modules and characteristic polynomials for Drinfeld modules

We fix a Drinfeld module $\phi\colon \mathsf{A} \to A[\tau]$ of rank r in generic characteristic, given by

$$\phi_t = \theta + \kappa_1\tau + \cdots + \kappa_r\tau^r, \quad \kappa_i \in A,\ \kappa_r \neq 0.$$

Letting $f \in A_+$ be irreducible of degree d, the reduction of ϕ modulo f is a Drinfeld module $\overline{\phi}\colon \mathsf{A} \to \mathbb{F}_f[\tau]$ of rank $r_0 \leqslant r$, where $\mathbb{F}_f = A/fA$. Then ϕ has *good reduction* modulo f if $r_0 = r$ or equivalently if $f \nmid \kappa_r$.

For $\lambda \in \mathsf{A}_+$ irreducible, we form the λ-adic *Tate modules*,

$$T_\lambda(\phi) := \varprojlim \phi[\lambda^m], \quad T_\lambda(\overline{\phi}) := \varprojlim \overline{\phi}[\lambda^m].$$

As an A_λ-module, $T_\lambda(\phi) \cong \mathsf{A}_\lambda^r$, and if $\lambda(\theta) \neq f$, then likewise $T_\lambda(\overline{\phi}) \cong \mathsf{A}_\lambda^{r_0}$. Fixing henceforth that $\lambda(\theta) \neq f$, we set $P_f(X) := \mathrm{Char}(\tau^d, T_\lambda(\overline{\phi}), X)|_{t=\theta}$ to be the characteristic polynomial of the q^d-th power Frobenius acting on $T_\lambda(\overline{\phi})$ but, for convenience, with coefficients forced into A (rather than A). Thus we have

$$P_f(X) = X^{r_0} + c_{r_0-1}X^{r_0-1} + \cdots + c_0 \in A[X]. \tag{2.4.1}$$

Takahashi [52, Prop. 3] showed that the coefficients are in A and are independent of the choice of λ (see also Gekeler [24, Cor. 3.4]). We note that if ϕ has good reduction modulo λ and if $\alpha_f \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ is a Frobenius element, then (e.g., see [31, §3], [33, §8.6])

$$\mathrm{Char}\left(\tau^d, T_\lambda(\overline{\phi}), X\right) = \mathrm{Char}\left(\alpha_f, T_\lambda(\phi), X\right) \in \mathsf{A}[X].$$

2.4.2. Properties of $P_f(X)$. The following results are due to Gekeler [24, Thm. 5.1] and Takahashi [52, Lem. 2, Prop. 3].

- We have $c_0 = c_f^{-1} f$ for some $c_f \in \mathbb{F}_q^\times$.
- The ideal $(P_f(1)) \subseteq A$ is an Euler–Poincaré characteristic for $\overline{\phi}(\mathbb{F}_f)$.
- The roots $\gamma_1, \dots, \gamma_{r_0}$ of $P_f(x)$ in $\overline{K}$ satisfy $\deg_\theta \gamma_i = d/r_0$.

Extending these a little further, for $1 \leqslant j \leqslant r_0$, we have $\deg_\theta c_{r_0-j} \leqslant jd/r_0$. Additionally,

$$\left[\overline{\phi}(\mathbb{F}_f)\right]_A = c_f P_f(1)$$

by [14, Cor. 3.2]. Here we use the convention from Section 2.1.7 that

$$[\overline{\phi}(\mathbb{F}_f)]_A = [\overline{\phi}(\mathbb{F}_f)]_{\mathsf{A}}\big|_{t=\theta}.$$

Following the exposition in [14, §3], we let $P_f^\vee(X) := \mathrm{Char}(\tau^d, T_\lambda(\overline{\phi})^\vee, X)|_{t=\theta}$ be the characteristic polynomial in $K[X]$ of τ^d acting on the dual space of $T_\lambda(\overline{\phi})$. We let $Q_f(X) = X^{r_0} P_f(1/X)$ and $Q_f^\vee(X) = X^{r_0} P_f^\vee(1/X)$ be the reciprocal polynomials of $P_f(X)$ and $P_f^\vee(X)$, and consider

$$Q_f^\vee(fX) = 1 + c_f c_1 X + c_f c_2 f X^2 + \cdots + c_f c_{r_0-1} f^{r_0-2} X^{r_0-1} + c_f f^{r_0-1} X^{r_0}.$$

To denote the dependence on ϕ, we write $P_{\phi,f}(X)$, $Q_{\phi,f}(X)$, etc.

We use $Q_f^\vee(fX)$ and $Q_f(X)$ to define the multiplicative functions μ_ϕ and $\nu_\phi\colon A_+ \to A$, which satisfy the following relations on the powers of a given f:

$$\sum_{m=1}^{\infty} \mu_\phi(f^m) X^m := \frac{1}{Q_f^\vee(fX)}, \quad \sum_{m=1}^{\infty} \nu_\phi(f^m) X^m := \frac{1}{Q_f(X)}. \tag{2.4.3}$$

2.4.4. Everywhere good reduction. Hsia and Yu [36] have determined precise formulas for c_f in terms of the $(q-1)$-st power residue symbol. Of particular interest presently is the case that ϕ has everywhere good reduction, i.e., when $\kappa_r \in \mathbb{F}_q^\times$. In this case, Hsia and Yu [36, Thm. 3.2, Eqs. (2) and (8)] showed that $c_f = (-1)^{r+d(r+1)}\kappa_r^d$. This prompts the definition of a completely multiplicative function $\chi_\phi\colon A_+ \to \mathbb{F}_q^\times$,

$$\chi_\phi(a) := \bigl((-1)^{r+1}\kappa_r\bigr)^{\deg_\theta a}, \tag{2.4.5}$$

for which we see that $c_f = (-1)^r \chi_\phi(f)$. Letting $\overline{\chi}_\phi\colon A_+ \to \mathbb{F}_q^\times$ be the multiplicative inverse of χ_ϕ, we see that

$$\begin{aligned} P_f(X) &= X^r + c_{r-1}X^{r-1} + \cdots + c_1 X + (-1)^r \overline{\chi}_\phi(f)\cdot f, \\ P_f^\vee(X) &= X^r + \frac{(-1)^r\chi_\phi(f)c_1}{f} X^{r-1} + \cdots + \frac{(-1)^r\chi_\phi(f)c_{r-1}}{f} X \\ &\quad + \frac{(-1)^r\chi_\phi(f)}{f}, \end{aligned} \tag{2.4.6}$$

and likewise

$$\begin{aligned} Q_f(X) &= 1 + c_{r-1}X + \cdots + c_1 X^{r-1} + (-1)^r \overline{\chi}_\phi(f)\cdot f X^r, \\ Q_f^\vee(fX) &= 1 + (-1)^r\chi_\phi(f)c_1 X + \cdots + (-1)^r\chi_\phi(f)c_{r-1} f^{r-2}X^{r-1} \\ &\quad + (-1)^r\chi_\phi(f) f^{r-1} X^r. \end{aligned} \tag{2.4.7}$$

Moreover,

$$\mu_\phi(f) = (-1)^{r+1}\chi_\phi(f)c_1, \quad \nu_\phi(f) = -c_{r-1}. \tag{2.4.8}$$

We record the induced recursive relations (cf. [14, Lem. 3.5]) on μ_ϕ and ν_ϕ, where taking $m+r \geqslant 1$ and using the convention that $\mu_\phi(b) = \nu_\phi(b) = 0$ if $b \in K \setminus A_+$,

$$\begin{aligned}\mu_\phi(f^{m+r}) = \mu_\phi(f)\mu_\phi(f^{m+r-1}) &- (-1)^r \chi_\phi(f) \sum_{j=2}^{r-1} c_j f^{j-1} \mu_\phi(f^{m+r-j}) \\ &- (-1)^r \chi_\phi(f) f^{r-1} \mu_\phi(f^m), \end{aligned} \tag{2.4.9}$$

$$\begin{aligned}\nu_\phi(f^{m+r}) = \nu_\phi(f)\nu_\phi(f^{m+r-1}) &- \sum_{j=2}^{r-1} c_{r-j} \nu_\phi(f^{m+r-j}) \\ &- (-1)^r \overline{\chi}_\phi(f) f \nu_\phi(f^m). \end{aligned} \tag{2.4.10}$$

2.5. Schur polynomials

We review properties of symmetric polynomials and especially Schur polynomials. For more details on symmetric polynomials see [1, Ch. 8], [37, §2.5], [48, Ch. 7]. Letting $\mathbf{x} = \{x_1, \dots, x_n\}$ be independent variables, the *elementary symmetric polynomials* $\{e_i\}_{i=0}^n = \{e_{n,i}\}_{i=0}^n \subseteq \mathbb{Z}[\mathbf{x}]$ are defined by

$$\sum_{i=0}^{n} e_i(\mathbf{x}) T^i = (1 + x_1 T)(1 + x_2 T) \cdots (1 + x_n T). \tag{2.5.1}$$

We adopt the convention that $e_i = 0$ if $i < 0$ or $i > n$. The *complete homogeneous symmetric polynomials* $\{h_i\}_{i \geqslant 0} = \{h_{n,i}\}_{i \geqslant 0} \subseteq \mathbb{Z}[x_1, \dots, x_n]$ are defined by

$$\sum_{i=0}^{\infty} h_i(\mathbf{x}) T^i = \frac{1}{(1 - x_1 T)(1 - x_2 T) \cdots (1 - x_n T)}, \tag{2.5.2}$$

and similarly if $i < 0$ then we take $h_i = 0$. Then h_i consists of the sum of all monomials in $x_1, \dots, x_n$ of degree i. The *Vandermonde determinant* is

$$V(\mathbf{x}) = \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j).$$

When nonzero we have $\deg e_i = i$ and $\deg h_i = i$, and $\deg V = \binom{n}{2}$.

Definition 2.5.3. For polynomials $P(T) = (T - x_1) \cdots (T - x_k)$ and $Q(T) = (T - y_1) \cdots (T - y_\ell)$, we set

$$(P \otimes Q)(T) := \prod_{\substack{1 \leqslant i \leqslant k \\ 1 \leqslant j \leqslant \ell}} (T - x_i y_j).$$

We further set

$$(\mathrm{Sym}^2 P)(T) = \prod_{1 \leqslant i \leqslant j \leqslant k} (T - x_i x_j),$$

$$\Big(\bigwedge^2 P\Big)(T) = \prod_{1 \leqslant i < j \leqslant k} (T - x_i x_j).$$

Letting B_m be the coefficient of T^m in $(P \otimes Q)(T)$, we find that B_m is symmetric in both $x_1, \dots, x_k$ and $y_1, \dots, y_\ell$, its total degree in $x_1, \dots, x_k$ is $k\ell - m$, and its total degree in $y_1, \dots, y_\ell$ is also $k\ell - m$. As such,

$$B_m \in \mathbb{Z}\big[e_{k,1}(\mathbf{x}), \dots, e_{k,k\ell-m}(\mathbf{x}); e_{\ell,1}(\mathbf{y}), \dots, e_{\ell,k\ell-m}(\mathbf{y})\big].$$

The coefficients of $(P \otimes Q)(T)$ and its inverse $(P \otimes Q)(T)^{-1}$ are expressible in terms of Schur polynomials (see Theorem 2.5.12 and Corollary 2.5.13 for $(P \otimes Q)(T)^{-1}$).

2.5.4. Schur polynomials. Let λ denote an integer partition of length n, i.e.,

$$\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}_{\geqslant 0}^n$$

satisfying $\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0$. We set

$$s_\lambda(\mathbf{x}) = s_{\lambda_1 \cdots \lambda_n}(\mathbf{x}) := V(\mathbf{x})^{-1} \cdot \det \begin{pmatrix} x_1^{\lambda_1+n-1} & \cdots & x_n^{\lambda_n+n-1} \\ \vdots & & \vdots \\ x_1^{\lambda_{n-1}+1} & \cdots & x_n^{\lambda_{n-1}+1} \\ x_1^{\lambda_n} & \cdots & x_n^{\lambda_n} \end{pmatrix} \tag{2.5.5}$$

We have the following properties (see [1, §8.3] and [48, §7.15]).

- $s_\lambda(\mathbf{x})$ is a symmetric polynomial in $\mathbb{Z}[x_1, \dots, x_n]$.
- $\deg s_\lambda(\mathbf{x}) = \lambda_1 + \cdots + \lambda_n$.
- For $0 \leqslant i \leqslant n$ we have $s_{\underbrace{1\cdots1}_{i}\underbrace{0\cdots0}_{n-i}}(\mathbf{x}) = e_i(\mathbf{x})$.
- For $i \geqslant 0$ we have $s_{i\underbrace{0\cdots0}_{n-1}}(\mathbf{x}) = h_i(\mathbf{x})$.

The polynomial s_λ is called the *Schur polynomial for* λ. Following the exposition of Bump and Goldfeld [12, 28], when $n \geqslant 2$ (which we now assume), we consider the subset of Schur polynomials where $\lambda_n = 0$ as follows. For integers $k_1, \dots, k_{n-1} \geqslant 0$, form

$$\lambda = (k_1 + \cdots + k_{n-1}, k_2 + \cdots + k_{n-1}, \dots, k_{n-1}, 0, 0).$$

We set $S_{k_1,\dots,k_{n-1}}(\mathbf{x})$ to be the Schur polynomial s_λ, i.e.,

$$S_{k_1,\dots,k_{n-1}}(\mathbf{x}) := V(\mathbf{x})^{-1} \cdot \det \begin{pmatrix} x_1^{k_1+\cdots+k_{n-1}+n-1} & \cdots & x_n^{k_1+\cdots+k_{n-1}+n-1} \\ x_1^{k_2+\cdots+k_{n-1}+n-2} & \cdots & x_n^{k_2+\cdots+k_{n-1}+n-2} \\ \vdots & & \vdots \\ x_1^{k_{n-1}+1} & \cdots & x_n^{k_{n-1}+1} \\ 1 & \cdots & 1 \end{pmatrix}. \tag{2.5.6}$$

The degree of $S_{k_1,\dots,k_{n-1}}(\mathbf{x})$ is $k_1 + 2k_2 + \cdots + (n-1)k_{n-1}$.

Lemma 2.5.7. *Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be an integer partition. Then*

$$s_\lambda(\mathbf{x}) = (x_1 \cdots x_n)^{\lambda_n} \cdot S_{\lambda_1-\lambda_2, \lambda_2-\lambda_3, \dots, \lambda_{n-1}-\lambda_n}(\mathbf{x}).$$

As a result, we see from the properties of s_λ above that

$$S_{\underbrace{0,\ldots,0,1,0,\ldots,0}_{i\text{-th place}}}(\mathbf{x}) = e_i(\mathbf{x}), \quad 1 \leqslant i \leqslant n-1, \tag{2.5.8}$$

$$S_{i,\underbrace{0,\ldots,0}_{n-2}}(\mathbf{x}) = h_i(\mathbf{x}), \quad i \geqslant 0. \tag{2.5.9}$$

Lemma 2.5.10. *For $k_1,\ldots,k_{n-1} \geqslant 0$, we have*

$$(x_1\cdots x_n)^{k_1+\cdots+k_{n-1}} \cdot S_{k_1,\ldots,k_{n-1}}(x_1^{-1},\ldots,x_n^{-1}) = S_{k_{n-1},\ldots,k_1}(\mathbf{x}).$$

2.5.11. Cauchy's identities.

Theorem 2.5.12 (Cauchy's identity, see [1, Cor. 8.16], [12, §2.2], [48, Thm. 7.12.1]). *For variables $\mathbf{x} = \{x_1,\ldots,x_n\}$ and $\mathbf{y} = \{y_1,\ldots,y_n\}$, let $X = x_1\cdots x_n$ and $Y = y_1\cdots y_n$. Then as power series in $\mathbb{Z}[\mathbf{x},\mathbf{y}][\![T]\!]$,*

$$\prod_{1\leqslant i,j\leqslant n}(1-x_iy_jT)^{-1}$$
$$= (1-XYT^n)^{-1}\sum_{\substack{k_1=0\\ k=(k_1,\ldots,k_{n-1})}}^{\infty}\cdots\sum_{k_{n-1}=0}^{\infty} S_k(\mathbf{x})S_k(\mathbf{y})T^{k_1+2k_2+\cdots+(n-1)k_{n-1}}.$$

If instead we have $\mathbf{x} = \{x_1,\ldots,x_n\}$ and $\mathbf{y} = \{y_1,\ldots,y_\ell\}$ with $n < \ell$, then Cauchy's identity reduces to the following result by setting $x_{n+1} = \cdots = x_\ell = 0$ and simplifying.

Corollary 2.5.13 (Bump [12, §2.2]). *For variables $\mathbf{x} = \{x_1,\ldots,x_n\}$ and $\mathbf{y} = \{y_1,\ldots,y_\ell\}$ with $n < \ell$, let $X = x_1\cdots x_n$. Then as power series in $\mathbb{Z}[\mathbf{x},\mathbf{y}][\![T]\!]$,*

$$\prod_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant \ell}}(1-x_iy_jT)^{-1} = \sum_{\substack{k_1=0\\ k=(k_1,\ldots,k_{n-1})\\ k'=(k_1,\ldots,k_n,0,\ldots,0)}}^{\infty}\cdots\sum_{k_n=0}^{\infty} S_k(\mathbf{x})S_{k'}(\mathbf{y})X^{k_n}T^{k_1+2k_2+\cdots+nk_n}.$$

2.5.14. Littlewood's identities.

Theorem 2.5.15 (Littlewood [40, (11.9;2), (11.9;4)]). *For variables $\mathbf{x} = \{x_1,\ldots,x_n\}$, let $X = x_1\cdots x_n$. Then the following identities hold as power series in $\mathbb{Z}[\mathbf{x}][\![T]\!]$.*

(a) *For $n \geqslant 2$, we have*

$$\prod_{1\leqslant i\leqslant j\leqslant n}(1-x_ix_jT)^{-1}$$
$$= (1-X^2T^n)^{-1}\sum_{\substack{k_1=0\\ k=(k_1,\ldots,K_{n-1})\\ 2|k_i \text{ for all } i}}^{\infty}\cdots\sum_{k_{n-1}=0}^{\infty} S_k(\mathbf{x})T^{k_1+2k_2+\cdots+(n-1)k_{n-1}}.$$

(b) *For $n \geqslant 2$, we have*

$$\prod_{1 \leqslant i < j \leqslant n} (1 - x_i x_j T)^{-1} = (1 - XT^{n/2})^{-\varepsilon} \sum_{\substack{k_1,\dots,k_{n-1} \in \mathbb{Z}_+ \\ k=(k_1,\dots,k_{n-1}) \\ k_i = 0 \text{ if } 2 \nmid i}} S_k(\mathbf{x}) T^{k_2 + 2k_4 + \cdots + \frac{n-1}{2} k_{n-1}},$$

where $\varepsilon = \frac{(-1)^n + 1}{2}$.

2.6. The function $\boldsymbol{\mu}_\phi$

We review the function $\boldsymbol{\mu}_\phi$ and its properties explored in [37, §6.1] by Papanikolas and the author. They in fact defined the function $\boldsymbol{\mu}_\phi$ and its "dual" version $\boldsymbol{\nu}_\phi$. For the purpose of the present paper, we only list the properties for the function $\boldsymbol{\mu}_\phi$.

Let $f \in A_+$ be irreducible, and let $P_{\phi,f}(X)$ and $P^\vee_{\phi,f}(X)$ be defined as in (2.4.6). We let $\alpha_1, \dots, \alpha_r \in \bar{K}$ be the roots of $P^\vee_{\phi,f}(X)$. For $k_1, \dots, k_{r-1} \geqslant 0$, we define

$$\boldsymbol{\mu}_\phi(f^{k_1}, \dots, f^{k_{r-1}}) := S_{k_1,\dots,k_{r-1}}(\alpha_1, \dots, \alpha_r) \cdot f^{k_1 + \cdots + k_{r-1}}, \tag{2.6.1}$$

where $S_{k_1,\dots,k_{r-1}}$ is the Schur polynomial of (2.5.6). We note that by (2.4.7) and (2.5.8),

$$\begin{aligned} Q^\vee_{\phi,f}(fX) = 1 &- \boldsymbol{\mu}_\phi(f, 1, \dots, 1)X + \boldsymbol{\mu}_\phi(1, f, 1, \dots, 1) f X^2 \\ &+ \cdots + (-1)^{r-1} \boldsymbol{\mu}_\phi(1, \dots, 1, f) f^{r-2} X^{r-1} \\ &+ (-1)^r \chi_\phi(f) f^{r-1} X^r. \end{aligned} \tag{2.6.2}$$

We then extend $\boldsymbol{\mu}_\phi$ uniquely to functions on $(A_+)^{r-1}$, by requiring that if $a_1, \dots, a_{r-1}$, $b_1, \dots, b_{r-1} \in A_+$ satisfy

$$\gcd(a_1 \cdots a_{r-1}, b_1 \cdots b_{r-1}) = 1,$$

then

$$\boldsymbol{\mu}_\phi(a_1 b_1, \dots, a_{r-1} b_{r-1}) = \boldsymbol{\mu}_\phi(a_1, \dots, a_{r-1}) \boldsymbol{\mu}_\phi(b_1, \dots, b_{r-1}).$$

Proposition 2.6.3 ([37, Prop. 6.1.5]). *For $a, a_1, \dots, a_{r-1} \in A_+$, the following hold.*

(a) $\boldsymbol{\mu}_\phi(a_1, \dots, a_{r-1}) \in A$.

(b) $\boldsymbol{\mu}_\phi(a, 1, \dots, 1) = \mu_\phi(a)$.

(c) *We have*

$$\deg_\theta \boldsymbol{\mu}_\phi(a_1, \dots, a_{r-1}) \leqslant \frac{1}{r}\big((r-1)\deg_\theta a_1 + (r-2)\deg_\theta a_2 + \cdots + \deg_\theta a_{r-1}\big).$$

We also list some recursive relations of the function $\boldsymbol{\mu}_\phi$ induced by relations on Schur polynomials ([37, (6.1.7), (6.1.9), (6.1.11)], cf. [28, p. 278]). Fix $f \in A_+$ irreducible, and for $k, k_1, \dots, k_{r-1} \geqslant 0$,

$$\begin{aligned}
&\boldsymbol{\mu}_\phi(f^k, 1, \dots, 1)\boldsymbol{\mu}_\phi(f^{k_1}, \dots, f^{k_{r-1}}) \\
&\quad = \sum_{\substack{m_0+\dots+m_{r-1}=k \\ m_1\leqslant k_1,\dots,m_{r-1}\leqslant k_{r-1}}} \boldsymbol{\mu}_\phi(f^{k_1+m_0-m_1}, f^{k_2+m_1-m_2}, \dots, f^{k_{r-1}+m_{r-2}-m_{r-1}}) \\
&\qquad\qquad\qquad\qquad \cdot \chi_\phi(f)^{m_{r-1}} f^{k-m_0}.
\end{aligned}$$

For $0 \leqslant k \leqslant r-1$,

$$\begin{aligned}
&\boldsymbol{\mu}_\phi(\underbrace{1, \dots, 1, f, 1, \dots, 1}_{k\text{-th place}})\boldsymbol{\mu}_\phi(f^{k_1}, \dots, f^{k_{r-1}}) \\
&\quad = \sum_{\substack{m_0+\dots+m_{r-1}=k \\ (m_0,\dots,m_{r-1})\in\mathcal{I}_{k_1,\dots,k_{r-1}}}} \boldsymbol{\mu}_\phi(f^{k_1+m_0-m_1}, f^{k_2+m_1-m_2}, \dots, f^{k_{r-1}+m_{r-2}-m_{r-1}}) \\
&\qquad\qquad\qquad\qquad \cdot \chi_\phi(f)^{m_{r-1}} f^{1-m_0}.
\end{aligned}$$

In particular for $k \geqslant 1$ (cf. [28, p. 278]),

$$\begin{aligned}
&\boldsymbol{\mu}_\phi(f^k, 1, \dots, 1)\boldsymbol{\mu}_\phi(f, 1, \dots, 1) \\
&\quad = \boldsymbol{\mu}_\phi(f^{k+1}, 1, \dots, 1) + \boldsymbol{\mu}_\phi(f^{k-1}, f, 1, \dots, 1) \cdot f, \\
&\boldsymbol{\mu}_\phi(f^k, 1, \dots, 1)\boldsymbol{\mu}_\phi(1, f, 1, \dots, 1) \\
&\quad = \boldsymbol{\mu}_\phi(f^k, f, 1, \dots, 1) + \boldsymbol{\mu}_\phi(f^{k-1}, 1, f, 1, \dots, 1) \cdot f, \\
&\boldsymbol{\mu}_\phi(f^k, 1, \dots, 1)\boldsymbol{\mu}_\phi(1, 1, f, 1, \dots, 1) \\
&\quad = \boldsymbol{\mu}_\phi(f^k, 1, f, 1, \dots, 1) + \boldsymbol{\mu}_\phi(f^{k-1}, 1, 1, f, 1, \dots, 1) \cdot f, \\
&\boldsymbol{\mu}_\phi(f^k, 1, \dots, 1)\boldsymbol{\mu}_\phi(1, \dots, 1, f) \\
&\quad = \boldsymbol{\mu}_\phi(f^k, 1, \dots, 1, f) + \boldsymbol{\mu}_\phi(f^{k-1}, 1, \dots, 1) \cdot \chi_\phi(f) f.
\end{aligned}$$

2.7. Matrix operations

Fixing a subring $R \subseteq \mathbb{L}_t$ with 1, we say $M \in \mathrm{Mat}_r(R)$ *represents an R-module homomorphism $f\colon R^r \to R^r$ with respect to a basis* $\mathbf{v} = (v_1, \dots, v_r)^\intercal \in \mathrm{Mat}_{r\times 1}(R^r)$ if

$$f \cdot \mathbf{v} := \begin{pmatrix} f(v_1) \\ \vdots \\ f(v_r) \end{pmatrix} = M\mathbf{v}.$$

Remark 2.7.1. This is slightly different from the usual sense in linear algebra. For example, if we let $M_f, M_g \in \mathrm{Mat}_r(R)$ represent two R-module homomorphisms $f, g\colon R^r \to R^r$, respectively, then $M_g M_f$ represents $f \circ g$. In fact, M_f is the transpose of the matrix representation of f in the usual sense.

For $i, j = 1, \dots, r$, we define, assuming characteristic of R is not 2,

$$\alpha_{ij} = v_i \otimes v_j, \quad \beta_{ij} = \frac{1}{2}(v_i \otimes v_j + v_j \otimes v_i), \quad \gamma_{ij} = \frac{1}{2}(v_i \otimes v_j - v_j \otimes v_i),$$

and consider the following basis of R-modules in lexicographical order

$$\begin{aligned} \mathfrak{Y}_1 &:= \{\alpha_{ij}\}_{i,j} \subseteq (R^r)^{\otimes 2}, \\ \mathfrak{Y}_2 &:= \{\beta_{ij}\}_{i,j} \subseteq \operatorname{Sym}^2(R^r), \\ \mathfrak{Y}_3 &:= \{\gamma_{ij}\}_{i,j} \subseteq \bigwedge^2(R^r). \end{aligned} \tag{2.7.2}$$

Definition 2.7.3. Let $M \in \operatorname{Mat}_r(R)$, and let $\mathfrak{Y}_1$, $\mathfrak{Y}_2$ and $\mathfrak{Y}_3$ as in (2.7.2). Then M represents some R-module homomorphism $f\colon R^r \to R^r$. We define the following matrix operations

$$M^{\otimes 2} \in \operatorname{Mat}_{r^2}(R), \quad \operatorname{Sym}^2(M) \in \operatorname{Mat}_{\frac{r(r+1)}{2}}(R), \quad \bigwedge^2(M) \in \operatorname{Mat}_{\frac{r(r-1)}{2}}(R)$$

to be matrices representing $f^{\otimes 2}$, $\operatorname{Sym}^2(f)$ and $\bigwedge^2(f)$ with respect to $\mathfrak{Y}_1$, $\mathfrak{Y}_2$ and $\mathfrak{Y}_3$, respectively.

Remark 2.7.4. In Definition 2.7.3, by a direct checking, the matrix $M^{\otimes 2}$ is the Kronecker tensor square of M.

Example 2.7.5. Suppose $r = 2$ and $M = (M_{ij})_{i,j=1,2} \in \operatorname{Mat}_2(R)$. Then

(a) $M^{\otimes 2} = \begin{pmatrix} M_{11}M_{11} & M_{11}M_{12} & M_{12}M_{11} & M_{12}M_{12} \\ M_{11}M_{21} & M_{11}M_{22} & M_{12}M_{21} & M_{12}M_{22} \\ M_{21}M_{11} & M_{21}M_{12} & M_{22}M_{11} & M_{22}M_{12} \\ M_{21}M_{21} & M_{21}M_{22} & M_{22}M_{21} & M_{22}M_{22} \end{pmatrix}$.

(b) $\operatorname{Sym}^2(M) = \begin{pmatrix} M_{11}^2 & 2M_{11}M_{12} & M_{12}^2 \\ M_{11}M_{21} & M_{11}M_{22}+M_{12}M_{21} & M_{12}M_{22} \\ M_{21}^2 & 2M_{21}M_{22} & M_{22}^2 \end{pmatrix}$.

(c) $\bigwedge^2(M) = \det(M)$.

Lemma 2.7.6. *For $M \in \operatorname{Mat}_r(R)$, we let $\mathcal{T}(M)$ denote $M^{\otimes 2}$, $\operatorname{Sym}^2(M)$ or $\bigwedge^2(M)$. Then for $M_1, M_2 \in \operatorname{Mat}_r(R)$, we have the following properties.*

(a) *$\mathcal{T}(M_1M_2) = \mathcal{T}(M_1)\mathcal{T}(M_2)$. Furthermore, $\mathcal{T}(M^{-1}) = \mathcal{T}(M)^{-1}$ if $M \in \operatorname{GL}_r(R)$.*

(b) *$\mathcal{T}(M^{(n)}) = \mathcal{T}(M)^{(n)}$ for $n \geqslant 0$.*

Proof. The first part follows from Remark 2.7.1, and the second part follows from the fact that entries of $\mathcal{T}(M)$ are polynomials in entries of M over $\mathbb{F}_q$. ∎

3. Tensor products of Drinfeld modules

Tensor powers of Drinfeld modules were initiated by Anderson [2] from the aspect of t-motives. Later Hamahata [34] further studied symmetric powers and alternating powers of Drinfeld modules, and provided explicit models as t-modules. In this section, we recover Hamahata's models from the aspect of t-motives.

We fix two Drinfeld modules $\phi, \psi\colon \mathsf{A} \to A[\tau]$ defined over A with everywhere good reduction as in (1.2.1), and their t-motives $\mathcal{M}_\phi = \mathbb{C}_\infty[\tau]$, $\mathcal{M}_\psi = \mathbb{C}_\infty[\tau]$. For convenience, $\kappa_0 = \eta_0 := \theta$.

3.1. Tensor products of Drinfeld modules defined over A

The tensor product of $\mathcal{M}_\phi$ and $\mathcal{M}_\psi$ is

$$\mathcal{M}_\phi \otimes \mathcal{M}_\psi := \mathcal{M}_\phi \otimes_{\mathbb{C}_\infty[t]} \mathcal{M}_\psi$$

equipped with a left $\mathbb{C}_\infty[t,\tau]$-module structure by using the $\mathbb{C}_\infty[t]$-module structure from the tensor product over $\mathbb{C}_\infty[t]$ and setting

$$\tau \cdot (a_1 \otimes a_2) := \tau a_1 \otimes \tau a_2, \quad a_1 \in \mathcal{M}_\phi,\ a_2 \in \mathcal{M}_\psi.$$

In the sense of [2], $\mathcal{M}_\phi \otimes \mathcal{M}_\psi$ is a *pure t-motive*, and its *weight*, denoted by $w(\mathcal{M}_\phi \otimes \mathcal{M}_\psi)$, is defined to be

$$w(\mathcal{M}_\phi \otimes \mathcal{M}_\psi) = \frac{\operatorname{rank}_{\mathbb{C}_\infty[\tau]} \mathcal{M}_\phi \otimes \mathcal{M}_\psi}{\operatorname{rank}_{\mathbb{C}_\infty[t]} \mathcal{M}_\phi \otimes \mathcal{M}_\psi}.$$

Combining this with [2, Prop. 1.11.1], the dimension of $\mathcal{M}_\phi \otimes \mathcal{M}_\psi$ is

$$\begin{aligned}\operatorname{rank}_{\mathbb{C}_\infty[\tau]} \mathcal{M}_\phi \otimes \mathcal{M}_\psi &= \big(w(\phi) + w(\psi)\big) \cdot \operatorname{rank}_{\mathbb{C}_\infty[t]} \mathcal{M}_\phi \otimes \mathcal{M}_\psi \\ &= \left(\frac{1}{r} + \frac{1}{\ell}\right) \cdot r \cdot \ell = r + \ell.\end{aligned}$$

More generally, *the n-th tensor power of $\mathcal{M}_\phi$* is

$$\mathcal{M}_\phi^{\otimes n} := \underbrace{\mathcal{M}_\phi \otimes_{\mathbb{C}_\infty[t]} \cdots \otimes_{\mathbb{C}_\infty[t]} \mathcal{M}_\phi}_{n \text{ times}}$$

equipped with a left $\mathbb{C}_\infty[t,\tau]$-module by using the $\mathbb{C}_\infty[t]$-module structure from the tensor power over $\mathbb{C}_\infty[t]$ and setting

$$\tau \cdot (a_1 \otimes \cdots \otimes a_n) := \tau a_1 \otimes \cdots \otimes \tau a_n, \quad a_i \in \mathcal{M}_\phi.$$

$\mathcal{M}_\phi^{\otimes n}$ is a pure t-motive of weight

$$w(\mathcal{M}_\phi^{\otimes n}) = \frac{n}{r}. \tag{3.1.1}$$

Lemma 3.1.2 (Khaochim [38, Lem. 4.4]). *The set*

$$\{s_i\}_{i=1}^{r+\ell} = \{1 \otimes 1, \ldots, 1 \otimes \tau^{\ell-1}, \tau \otimes 1, \ldots, \tau^r \otimes 1\}$$

is a basis of $\mathcal{M}_\phi \otimes \mathcal{M}_\psi$ as a $\mathbb{C}_\infty[\tau]$-module.

With the $\mathbb{C}_\infty[\tau]$-basis in Lemma 3.1.2, we obtain a model E of the tensor product of ϕ and ψ which is a t-module of dimension $r + \ell$ defined over A by solving the following equation for E_t:

$$t \cdot \mathbf{u}(s_1, \dots, s_{r+\ell})^{\mathsf{T}} = \mathbf{u} E_t (s_1, \dots, s_{r+\ell})^{\mathsf{T}} \tag{3.1.3}$$

for all $\mathbf{u} \in \mathrm{Mat}_{1\times d}(\mathbb{C}_\infty[\tau])$.

Definition 3.1.4. Suppose $r \leqslant \ell$. The *tensor product of ϕ and ψ* is the t-module $\phi \otimes \psi \colon \mathsf{A} \to \mathrm{Mat}_{r+\ell}(A[\tau])$ of dimension $r+\ell$ defined over A satisfying (3.1.3). Explicitly, from [38, Def. 4.5]

$$(\phi \otimes \psi)_t = \left(\begin{array}{c|c} X_1 & X_2 \\ \hline X_3 & X_4 \end{array} \right),$$

where $X_1 \in \mathrm{Mat}_{\ell\times\ell}(A[\tau])$, $X_2 \in \mathrm{Mat}_{\ell\times r}(A[\tau])$, $X_3 \in \mathrm{Mat}_{r\times\ell}(A[\tau])$, $X_4 \in \mathrm{Mat}_{r\times r}(A[\tau])$ are defined by

$$X_1 = \begin{pmatrix} \theta & & & & & & \\ \kappa_1\tau & \theta & & & & & \\ \vdots & \ddots & \ddots & & & & \\ \kappa_{r-1}\tau^{r-1} & \cdots & \kappa_1\tau & \theta & & & \\ \kappa_r\tau^r & \cdots & \cdots & \kappa_1\tau & \theta & & \\ & \ddots & & & & \ddots & \\ & & \kappa_r\tau^r & \cdots & \cdots & \kappa_1\tau & \theta \end{pmatrix},$$

$$X_2 = \begin{pmatrix} \kappa_1 & \cdots & \kappa_{r-1} & \kappa_r \\ \kappa_2\tau & \cdots & \kappa_r\tau & \\ \vdots & \iddots & & \\ \kappa_r\tau^{r-1} & & & \\ 0 & & & \\ \vdots & & & \\ 0 & & & \end{pmatrix},$$

$$X_3 = \begin{pmatrix} \eta_1\tau & \cdots & \cdots & \cdots & \eta_{\ell-1}\tau & \eta_\ell\tau \\ \eta_2\tau^2 & \cdots & \cdots & \cdots & \eta_\ell\tau^2 & \\ \vdots & & & \iddots & & \\ \eta_r\tau^r & \cdots & \eta_\ell\tau^r & & & \end{pmatrix}, \quad X_4 = \begin{pmatrix} \theta & & & \\ \eta_1\tau & \theta & & \\ \vdots & \ddots & \ddots & \\ \eta_{r-1}\tau^{r-1} & \cdots & \eta_1\tau & \theta \end{pmatrix}.$$

Remark 3.1.5. As pointed out in [38, Rem. 4.8], our model for $\phi \otimes \psi$ is in fact Hamahata's $\psi \otimes \phi$, but they are isomorphic as t-modules by [34, Prop. 2.5].

3.2. Symmetric and alternating powers of Drinfeld modules defined over A

Let $\mathfrak{S}_n$ be the n-th symmetric group and $\mathrm{sgn}\colon \mathfrak{S}_n \to \{\pm 1\}$ be the sign function. *The n-th symmetric power of $\mathcal{M}_\phi$* is the $\mathbb{C}_\infty[t,\tau]$-submodule

$$\mathrm{Sym}^n \mathcal{M}_\phi := \mathrm{Span}_{\mathbb{C}_\infty[t]} \left\{ \sum_{\sigma \in \mathfrak{S}_n} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)} \right\} \subseteq \mathcal{M}_\phi^{\otimes n}. \tag{3.2.1}$$

It is also a pure t-motive of weight $w(\operatorname{Sym}^n \mathcal{M}_\phi) = w(\mathcal{M}_\phi^{\otimes n})$ by [2, Prop. 1.10.2]. Together with (3.1.1), the dimension of $\operatorname{Sym}^n \mathcal{M}_\phi$ is

$$\begin{aligned}\operatorname{rank}_{\mathbb{C}_\infty[\tau]} \operatorname{Sym}^n \mathcal{M}_\phi &= \frac{n}{r} \cdot \operatorname{rank}_{\mathbb{C}_\infty[t]} \operatorname{Sym}^n \mathcal{M}_\phi \\ &= \frac{n}{r} \cdot \binom{r+n-1}{n} = \binom{r+n-1}{n-1}.\end{aligned} \tag{3.2.2}$$

The n-th alternating power of $\mathcal{M}_\phi$ is the $\mathbb{C}_\infty[t,\tau]$-submodule

$$\bigwedge^n \mathcal{M}_\phi := \operatorname{Span}_{\mathbb{C}_\infty[t]} \Big\{ \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)(a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}) \Big\} \subseteq \mathcal{M}_\phi^{\otimes n}.$$

Similar to symmetric powers, it is a pure t-motive of weight $w(\bigwedge^n \mathcal{M}_\phi) = w(\mathcal{M}_\phi^{\otimes n})$ and dimension

$$\operatorname{rank}_{\mathbb{C}_\infty[\tau]} \bigwedge^n \mathcal{M}_\phi = n \cdot w(\phi) \cdot \operatorname{rank}_{\mathbb{C}_\infty[t]} \bigwedge^n \mathcal{M}_\phi = \frac{n}{r} \cdot \binom{r}{n} = \binom{r-1}{n-1}. \tag{3.2.3}$$

Lemma 3.2.4. *Suppose $p \neq 2$.*

(a) *The set*

$$\mathfrak{X}_1 := \Big\{1 \otimes 1, \frac{1}{2}(1 \otimes \tau + \tau \otimes 1), \ldots, \frac{1}{2}(1 \otimes \tau^r + \tau^r \otimes 1)\Big\}$$

is a $\mathbb{C}_\infty[\tau]$-basis for $\operatorname{Sym}^2 \mathcal{M}_\phi$.

(b) *The set*

$$\mathfrak{X}_2 := \Big\{\frac{1}{2}(1 \otimes \tau - \tau \otimes 1), \ldots, \frac{1}{2}(1 \otimes \tau^{r-1} - \tau^{r-1} \otimes 1)\Big\}$$

is a $\mathbb{C}_\infty[\tau]$-basis for $\bigwedge^2 \mathcal{M}_\phi$.

Proof. By (3.2.2) and (3.2.3), it suffices to show that the two sets span the corresponding t-motives as $\mathbb{C}_\infty[\tau]$-modules.

For the first part, we observe that the elements in $\operatorname{Sym}^2 \mathcal{M}_\phi$ are of the form, $a_i, b_i \in \mathcal{M}_\phi = \mathbb{C}_\infty[\tau]$, $f_i \in \mathbb{C}_\infty[t]$,

$$\begin{aligned}&\sum_i f_i \cdot (a_i \otimes b_i + b_i \otimes a_i) \\ &= \sum_i \big((f_i \cdot a_i) \otimes b_i + b_i \otimes (f_i \cdot a_i)\big) \\ &= \sum_i \sum_{j_1} \sum_{j_2} h_{i,j_1,j_2} \tau^{\min(j_1,j_2)} (1 \otimes \tau^{|j_1-j_2|} + \tau^{|j_1-j_2|} \otimes 1),\end{aligned}$$

where $h_{i,j_1,j_2} \in \mathbb{C}_\infty$. So it suffices to show, for $m \geqslant 0$,

$$\xi_m := 1 \otimes \tau^m + \tau^m \otimes 1 \in \operatorname{Span}_{\mathbb{C}_\infty[\tau]} \mathfrak{X}_1. \tag{3.2.5}$$

We proceed by induction on m. It is clear that (3.2.5) holds for $m = 0, \ldots, r$. For $m > r$, we consider

$$t \cdot \xi_{m-r} = 1 \otimes (\tau^{m-r}\phi_t) + (\tau^{m-r}\phi_t) \otimes 1 = \sum_{i=0}^{r} \kappa_i^{(m-r)} \xi_{m-r+i}. \tag{3.2.6}$$

On the other hand,

$$\begin{aligned} t \cdot \xi_{m-r} &= \phi_t \otimes \tau^{m-r} + \tau^{m-r} \otimes \phi_t \\ &= \begin{cases} \sum_{i=0}^{r} \kappa_i \tau^i \xi_{m-r-i} & \text{if } r \leqslant m-r, \\ \sum_{i=0}^{m-r} \kappa_i \tau^i \xi_{m-r-i} + \sum_{i=1}^{2r-m} \kappa_{m-r+i} \tau^{m-r} \xi_i & \text{if } r > m-r. \end{cases} \end{aligned} \tag{3.2.7}$$

Combining (3.2.6) and (3.2.7), and using that $\kappa_r \neq 0$, in both cases ξ_m is a $\mathbb{C}_\infty[\tau]$-linear combination of $\{\xi_i\}_{i=0}^{m-1}$, and the result follows by the induction hypothesis.

For the second part, by a similar observation, it suffices to show, for $m \geqslant 1$,

$$\zeta_m := 1 \otimes \tau^m - \tau^m \otimes 1 \in \operatorname{Span}_{\mathbb{C}_\infty[\tau]} \mathfrak{X}_2. \tag{3.2.8}$$

We again proceed by induction on m. It is clear that (3.2.8) holds for $m = 1, \ldots, r-1$. Note that $\zeta_0 = 0$, so we cannot use the same process for the case $m = r$. In fact, it follows by expressing $t \cdot (1 \otimes 1)$ in two ways. Note that

$$t \cdot (1 \otimes 1) = \phi_t \otimes 1 = 1 \otimes \phi_t,$$

which gives

$$\zeta_r = -\kappa_r^{-1} \sum_{i=1}^{r-1} \kappa_i \zeta_i.$$

For $m > r$, we consider

$$t \cdot \zeta_{m-r} = 1 \otimes (\tau^{m-r}\phi_t) - (\tau^{m-r}\phi_t) \otimes 1 = \sum_{i=0}^{r} \kappa_i^{(m-r)} \zeta_{m-r+i}. \tag{3.2.9}$$

On the other hand,

$$\begin{aligned} t \cdot \zeta_{m-r} &= \phi_t \otimes \tau^{m-r} - \tau^{m-r} \otimes \phi_t \\ &= \begin{cases} \sum_{i=0}^{r} \kappa_i \tau^i \zeta_{m-r-i} & \text{if } r \leqslant m-r, \\ \sum_{i=0}^{m-r} \kappa_i \tau^i \zeta_{m-r-i} - \sum_{i=1}^{2r-m} \kappa_{m-r+i} \tau^{m-r} \zeta_i & \text{if } r > m-r. \end{cases} \end{aligned} \tag{3.2.10}$$

Similarly, combining (3.2.9) and (3.2.10), the result follows by the induction hypothesis. ■

In the same fashion as for tensor products, we define symmetric and alternating squares of ϕ as follows.

Definition 3.2.11. We assume $p \neq 2$.

(a) The *symmetric square of* ϕ is the t-module $\operatorname{Sym}^2 \phi\colon \mathsf{A} \to \operatorname{Mat}_{r+1}(A[\tau])$ of dimension $r+1$ defined over A given by

$$(\operatorname{Sym}^2 \phi)_t = \begin{pmatrix} \theta & & & & \\ \kappa_1\tau & \theta & & & \\ \kappa_2\tau^2 & \kappa_1\tau & \theta & & \\ \vdots & & \ddots & \ddots & \\ \kappa_r\tau^r & \cdots & \cdots & \kappa_1\tau & \theta \end{pmatrix} + \begin{pmatrix} 0 & \kappa_1 & \cdots & \kappa_{r-1} & \kappa_r \\ 0 & \kappa_2\tau & \cdots & \kappa_r\tau & \\ \vdots & \vdots & \iddots & & \\ 0 & \kappa_r\tau^{r-1} & & & \\ 0 & & & & \end{pmatrix}.$$

(b) The *alternating square of* ϕ is the t-module $\bigwedge^2 \phi\colon \mathsf{A} \to \operatorname{Mat}_{r-1}(A[\tau])$ of dimension $r-1$ defined over A given by

$$\Big(\bigwedge^2 \phi\Big)_t = \begin{pmatrix} \theta & & & & \\ \kappa_1\tau & \theta & & & \\ \kappa_2\tau^2 & \kappa_1\tau & \theta & & \\ \vdots & & \ddots & \ddots & \\ \kappa_{r-2}\tau^{r-2} & \cdots & \cdots & \kappa_1\tau & \theta \end{pmatrix} - \begin{pmatrix} \kappa_2\tau & \kappa_3\tau & \cdots & \kappa_{r-1}\tau & \kappa_r\tau \\ \kappa_3\tau^2 & \kappa_4\tau^2 & \cdots & \kappa_r\tau^2 & \\ \vdots & \vdots & \iddots & & \\ \kappa_{r-1}\tau^{r-2} & \kappa_r\tau^{r-2} & & & \\ \kappa_r\tau^{r-1} & & & & \end{pmatrix}.$$

In other word, the τ-expansions of $(\operatorname{Sym}^2 \phi)_t$ and $(\bigwedge^2 \phi)_t$ are

$$\begin{aligned} (\operatorname{Sym}^2 \phi)_t &= B_0 + B_1\tau + \cdots + B_r\tau^r, \\ \Big(\bigwedge^2 \phi\Big)_t &= \theta \mathrm{I}_{r-1} + C_1\tau + \cdots + C_{r-1}\tau^{r-1}, \end{aligned}$$

where $B_i = \left(\begin{array}{c|c} 0 & 0 \\ \hline B_i' & 0 \end{array}\right) \in \operatorname{Mat}_{r+1}(A)$ and $C_i = \left(\begin{array}{c|c} 0 & 0 \\ \hline C_i' & 0 \end{array}\right) \in \operatorname{Mat}_{r-1}(A)$ with

$$B_i' = \begin{pmatrix} \kappa_i & \kappa_{i+1} & \cdots & \kappa_r \\ & \kappa_i & \cdots & \kappa_{r-1} \\ & & \ddots & \vdots \\ & & & \kappa_i \end{pmatrix}, \quad C_i' = \begin{pmatrix} -\kappa_{i+1} & \cdots & -\kappa_{r-1} & -\kappa_r \\ \kappa_i & & & \\ & \ddots & & \\ & & \kappa_i & \end{pmatrix},$$

of sizes $(r-i+1) \times (r-i+1)$ and $(r-i) \times (r-i)$, respectively.

Proposition 3.2.12. *The tensor structures* $\phi^{\otimes 2}$, $\operatorname{Sym}^2(\phi)$ *and* $\bigwedge^2(\phi)$ *are uniformizable.*

Proof. By [2, Thm. 4], it suffices to show that $\mathcal{M}_E$ is rigid analytically trivial, in the sense of [37, §2.4.6], for $E = \phi^{\otimes 2}$, $\mathrm{Sym}^2(\phi)$ and $\bigwedge^2(\phi)$. We let $\mathcal{T}(M)$ denote the corresponding matrix operation. Example 2.3.4 provides that $(1, \tau, \dots, \tau^{r-1})^{\mathsf{T}}$ is a basis for $\mathcal{M}_\phi$ with Γ in (2.3.5) representing multiplication by τ on $\mathcal{M}_\phi$. Then by choosing a basis for $\mathcal{M}_E$ in the same way as (2.7.2), we observe that $\mathcal{T}(\Gamma)$ represents multiplication by τ on $\mathcal{M}_E$.

Furthermore, by [44, §4.2] (see also [37, Ex. 2.4.12]), the t-motive $\mathcal{M}_\phi$ is rigid analytically trivial with rigid analytic trivialization denoted by $\Upsilon \in \mathrm{GL}_r(\mathbb{T}_t)$. Then by Lemma 2.7.6,

$$\mathcal{T}(\Upsilon^{\mathsf{T}})^{(1)} = \mathcal{T}\big((\Upsilon^{\mathsf{T}})^{(1)}\big) = \mathcal{T}(\Gamma\Upsilon^{\mathsf{T}}) = \mathcal{T}(\Gamma)\mathcal{T}(\Upsilon^{\mathsf{T}}),$$

which implies $\mathcal{T}(\Upsilon^{\mathsf{T}})^{\mathsf{T}}$ is a rigid analytic trivialization for $\mathcal{M}_E$. ∎

Remark 3.2.13. By a direct computation, the τ-expansion of $(\mathrm{Sym}^2\,\phi)_{t^2}$ is of the form

$$\widetilde{B}_0 + \widetilde{B}_1\tau + \cdots + \widetilde{B}_{2r}\tau^{2r},$$

with $\widetilde{B}_i = 0$ for $r + 1 \leqslant i \leqslant 2r$, and where $\widetilde{B}_r$ is a lower triangular matrix with κ_r^2 on the diagonal. Similarly, the τ-expansion of $(\bigwedge^2\,\phi)_{t^2}$ is of the form

$$\widetilde{C}_0 + \widetilde{C}_1\tau + \cdots + \widetilde{C}_{2r}\tau^{2r-2},$$

with $\widetilde{C}_i = 0$ for $r + 1 \leqslant i \leqslant 2r - 2$, and where $\widetilde{C}_r$ is a lower triangular matrix with κ_r^2 on the diagonal. Therefore, the top coefficients of $(\mathrm{Sym}^2\,\phi)_{t^2}$ and $(\bigwedge^2\,\phi)_{t^2}$ are invertible ($\kappa_r \neq 0$). A similar computation can also be done for $(\phi^{\otimes 2})_{t^2}$, which gives the same conclusion as symmetric and alternating squares. So $\phi^{\otimes 2}$, $\mathrm{Sym}^2\,\phi$ and $\bigwedge^2\,\phi$ are *almost strictly pure* in the sense of [42], which implies pure by [32, Rem. 2.2.3], [33, Rem. 5.5.5], [42, Rem. 4.5.3]. This explicates the purity of their t-motives.

4. Convolution L-series

In a series of articles [29–31], [33, Ch. 8f] Goss defined and investigated function field valued L-series attached to Drinfeld modules and t-modules defined over finite extensions of K. These L-functions possess a rich structure of special values, initiated by Carlitz [13, Thm. 9.3] for the eponymous Carlitz zeta function and continued by Goss [31], [33, Ch. 8]. Anderson and Thakur [4] further revealed the connection between Carlitz zeta values and coordinates of logarithms on tensor powers of the Carlitz module.

Taelman [49–51] discovered a breakthrough on special L-values for Drinfeld modules that related them to the product of an analytic regulator and the A-order of a class module. These results have been extended in several directions, including to t-modules defined over $\overline{K}$ and more refined special value identities [5, 6, 8, 10, 14, 21, 22, 26, 27].

4.1. Goss L-series

Let $\phi : \mathsf{A} \to A[\tau]$ be a Drinfeld module over A with everywhere good reduction as in (1.2.1). Goss [31, §3], [33, §8.6] associated the Dirichlet series

$$L(\phi^\vee, s) = \prod_{f \in A_+,\ \text{irred.}} Q_f^\vee(f^{-s})^{-1}, \quad L(\phi, s) = \prod_{f \in A_+,\ \text{irred.}} Q_f(f^{-s})^{-1},$$

where as in Section 2.4, $Q_f(X) \in A[X]$ is the reciprocal polynomial of the characteristic polynomial $P_f(X)$ of Frobenius acting on the Tate module $T_\lambda(\bar{\phi})$ and $Q_f^\vee(X) \in K[X]$ is the reciprocal polynomial of $P_f^\vee(X)$ arising from $T_\lambda(\bar{\phi})^\vee$. In the future we will write simply "$\prod_f$" to indicate that a product is over all irreducible $f \in A_+$.

Remark 4.1.1. [37, Cor. 3.7.3] makes the calculation of $P_f(X)$ and $P_f^\vee(X)$ reasonable (and hence $Q_f(X)$ and $Q_f^\vee(X)$ also). See [37, Rem. 5.1.2] for the details.

The bounds on the coefficients of $P_f(X)$ from Section 2.4.2 imply that $L(\phi, s)$ converges in K_∞ for $s \in \mathbb{Z}_+$ and that $L(\phi^\vee, s)$ converges for $s \in \mathbb{Z}_{\geqslant 0}$ (e.g., see [14, §3]). Goss extended the definition of these L-series to s in a non-archimedean analytic space, but we will not pursue these extensions here. We will henceforth assume $s \in \mathbb{Z}$.

By (2.4.3), we find that

$$L(\phi^\vee, s) = \sum_{a \in A_+} \frac{\mu_\phi(f)}{a^{s+1}}$$

(see [14, Eqs. (12)–(14)]). In particular, for the Carlitz module $P_{\mathsf{C},f}^\vee(X) = X - 1/f$, so

$$L(\mathsf{C}^\vee, s) = \sum_{a \in A_+} \frac{1}{a^{s+1}} = \zeta_{\mathsf{C}}(s+1)$$

is a shift of the Carlitz zeta function.

Taelman [51, Thm. 1] proved a special value identity for $L(\phi^\vee, 0)$ as follows. First,

$$Q_f^\vee(1)^{-1} = \frac{f}{(-1)^r \,\overline{\chi}(f) \cdot P_f(1)} = \frac{[\mathbb{F}_f]_A}{\left[\bar{\phi}(\mathbb{F}_f)\right]_A}, \tag{4.1.2}$$

where the first equality follows from (2.4.7) and the second from Gekeler [24, Thm. 5.1] (and also from [37, Cor. 3.7.8] combined with the definition of $P_f(X)$). We then have

$$L(\phi^\vee, 0) = \prod_f \frac{[\mathbb{F}_f]_A}{\left[\bar{\phi}(\mathbb{F}_f)\right]_A} = \mathrm{Reg}_\phi \cdot \mathrm{H}(\phi), \tag{4.1.3}$$

where the first equality follows from (4.1.2) and the second is Taelman's identity. The formula on the right contains the regulator $\mathrm{Reg}_\phi \in K_\infty$ and the order of the class module $\mathrm{H}(\phi) \in A$ (see [51] for details). We will use Fang's generalization of Taelman's formula to t-modules. See Theorem 4.2.2.

4.2. Fang's class module formula

In [21], Fang proved an extension of Taelman's class module formula to abelian Anderson t-modules defined over the integral closure of A in a finite extension of K. Later Anglès, Ngo Dac and Tavares Ribeiro [6] extended the class module formula for admissible Anderson modules for more general ring, which comes from global function field over a finite field. For the purpose of the present paper, it suffices to focus on Fang's identity, thus we will mainly provide a summary of it.

Let $E\colon \mathsf{A} \to \mathrm{Mat}_\ell(A[\tau])$ be an abelian Anderson t-module defined over A. The exponential series $\mathrm{Exp}_E \in K[\![\tau]\!]$ of E induces an $\mathbb{F}_q$-linear function

$$\mathrm{Exp}_{E,K_\infty}\colon \mathrm{Lie}(E)(K_\infty) \to E(K_\infty) \iff \mathrm{Exp}_{E,K_\infty}\colon K_\infty^\ell \to K_\infty^\ell.$$

Now $\mathrm{Lie}(E)(K_\infty)$ has a canonical K_∞-vector space structure, but Fang [21, p. 303] pointed out that it has another structure of a vector space over $\mathbb{F}_q(\!(t^{-1})\!)$. Namely we extend $\partial\colon \mathsf{A} \to \mathrm{Mat}_\ell(K_\infty)$ to an $\mathbb{F}_q$-algebra homomorphism,

$$\mathbb{F}_q(\!(t^{-1})\!) \xrightarrow{\partial} \mathrm{Mat}_\ell(K_\infty) : \sum_{j \geqslant j_0} c_j t^{-j} \mapsto \sum_{j \geqslant j_0} c_j \cdot \partial E_t^{-j}.$$

Notably, the series on the right converges by [21, Lem. 1.7]. Furthermore, as Fang subsequently continued, $\mathrm{Lie}(E)(K_\infty)$ obtains an $\mathbb{F}_q(\!(t^{-1})\!)$-vector space structure via ∂. For any $g \in \mathbb{F}_q(\!(t^{-q^\ell})\!)$, we have $\partial g = g \cdot \mathrm{I}_\ell$, and so $\mathrm{Lie}(E)(K_\infty)$ has dimension ℓq^ℓ as over $\mathbb{F}_q(\!(t^{-q^\ell})\!)$, which implies it has dimension ℓ over $\mathbb{F}_q(\!(t^{-1})\!)$. Since $K_\infty = \mathbb{F}_q(\!(\theta^{-1})\!) \cong \mathbb{F}_q(\!(t^{-1})\!)$, we will abuse notation and use the map ∂ to define new K_∞-vector space and A-module structures on $\mathrm{Lie}(E)(K_\infty)$ that are possibly different from scalar multiplication. With respect to this K_∞-structure, Fang showed [21, Thm. 1.10] that

$$\mathrm{Lie}(E)(A) \subseteq \mathrm{Lie}(E)(K_\infty)$$

and

$$\mathrm{Exp}_{E,K_\infty}^{-1}\bigl(E(A)\bigr) \subseteq \mathrm{Lie}(E)(K_\infty)$$

are A-lattices in the sense of [21, Def. 1.9]. In particular, they have rank ℓ as an A-modules via ∂.

Choose A-bases $\{\mathbf{v}_1, \dots, \mathbf{v}_\ell\}$ and $\{\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_\ell\}$ of $\mathrm{Lie}(E)(A)$ and $\mathrm{Exp}_{E,K_\infty}^{-1}(A^\ell)$ via ∂ respectively, and let $V \in \mathrm{GL}_\ell(K_\infty)$ be chosen so that its columns are the coordinates of $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_\ell$ with respect to $\mathbf{v}_1, \dots, \mathbf{v}_\ell$ (via ∂). Following Taelman [50, 51], Fang defined the regulator of E as

$$\mathrm{Reg}_E := \gamma \cdot \det(V) \in K_\infty, \quad \gamma \in \mathbb{F}_q^\times, \tag{4.2.1}$$

where γ is chosen so that Reg_E has sign 1 (leading coefficient as an element of $\mathbb{F}_q(\!(\theta^{-1})\!)$ is 1). This value is independent of the choice of A-bases.

Also following Taelman, Fang [21, Thm. 1.10] defined the class module of E as

$$\mathrm{H}(E) := \frac{E(K_\infty)}{\mathrm{Exp}_{E,K_\infty}\bigl(\mathrm{Lie}(E)(K_\infty)\bigr) + E(A)},$$

and he proved that $\mathrm{H}(E)$ is a finitely generated and torsion A-module. Fang's class module formula is the following.

Theorem 4.2.2 (Fang [21, Thm. 1.10]). *Let $E\colon \mathsf{A} \to \mathrm{Mat}_\ell(A[\tau])$ be an abelian Anderson t-module. Then*

$$\prod_f \frac{\bigl[\mathrm{Lie}(\bar{E})(\mathbb{F}_f)\bigr]_A}{\bigl[\bar{E}(\mathbb{F}_f)\bigr]_A} = \mathrm{Reg}_E \cdot \bigl[\mathrm{H}(E)\bigr]_A,$$

where the left-hand side converges in K_∞.

4.3. L-series of Tensor product of Drinfeld modules over A

Let $\phi, \psi\colon \mathsf{A} \to A[\tau]$ be Drinfeld modules defined over A of ranks r and ℓ respectively as in (1.2.1). Recall the t-module $\phi \otimes \psi\colon \mathsf{A} \to \mathrm{Mat}_{r+\ell}(A[\tau])$ defined over A as in Definition 3.1.4.

4.3.1. Characteristic polynomials of Frobenius and L-series of $\phi \otimes \psi$. Let $f \in A_+$ be irreducible of degree d, and let $\lambda \in \mathsf{A}_+$ be irreducible so that $\lambda(\theta) \neq f$. Let

$$\rho_{\phi,\lambda}\colon \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{Aut}\left(T_\lambda(\phi)\right) \cong \mathrm{GL}_r(\mathsf{A}_\lambda)$$

be the Galois representation associated $T_\lambda(\phi)$, and similarly define $\rho_{\psi,\lambda}\colon \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{Aut}(T_\lambda(\psi))$ for ψ. The λ-adic Tate module

$$T_\lambda(\phi \otimes \psi) = \varprojlim (\phi \otimes \psi)[\lambda^m] \cong \mathsf{A}_\lambda^{r\ell}$$

induces another Galois representation

$$\rho_{\phi\otimes\psi,\lambda}\colon \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \mathrm{Aut}\left(T_\lambda(\phi \otimes \psi)\right) \cong \mathrm{GL}_{r\ell}(\mathsf{A}_\lambda).$$

As outlined in Section 2.4, if $\alpha_f \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ is a Frobenius element for f, then because ϕ and ψ have good reduction at f,

$$\mathrm{Char}\left(\alpha_f, T_\lambda(\phi), X\right)|_{t=\theta} = P_{\phi,f}(X), \quad \mathrm{Char}\left(\alpha_f, T_\lambda(\psi), X\right)|_{t=\theta} = P_{\psi,f}(X),$$

both of which lie in $A[X]$. We define

$$\begin{aligned}\mathbf{P}_{\phi\otimes\psi,f}(X) &:= \mathrm{Char}\left(\alpha_f, T_\lambda(\phi \otimes \psi), X\right)|_{t=\theta},\\ \mathbf{P}^\vee_{\phi\otimes\psi,f}(X) &:= \mathrm{Char}\left(\alpha_f, T^\vee_\lambda(\phi \otimes \psi), X\right)|_{t=\theta}.\end{aligned}$$

Recall the notation $(P \otimes Q)(X)$ from Definition 2.5.3. By [34, Thm. 3.1], we then have

$$\begin{aligned}\mathbf{P}_{\phi\otimes\psi,f}(X) &= (P_{\phi,f} \otimes P_{\psi,f})(X) \in A[X],\\ \mathbf{P}^\vee_{\phi\otimes\psi,f}(X) &= (P^\vee_{\phi,f} \otimes P^\vee_{\psi,f})(X) \in K[X].\end{aligned} \tag{4.3.2}$$

We further set $\mathbf{Q}_{\phi\otimes\psi,f}(X)$ and $\mathbf{Q}^\vee_{\phi\otimes\psi,f}(X)$ to be their reciprocal polynomials. We now consider the L-function

$$L\left((\phi \otimes \psi)^\vee, s\right) := \prod_f \mathbf{Q}^\vee_{\phi\otimes\psi,f}(f^{-s})^{-1}, \quad s \geqslant 0. \tag{4.3.3}$$

4.3.4. Special L-values of tensor product. Fixing $f \in A_+$ irreducible, we let

$$\overline{\phi \otimes \psi}\colon \mathsf{A} \to \mathrm{Mat}_{r+\ell}\left(\mathbb{F}_f[\tau]\right)$$

denote the reduction modulo f. Recall the completely multiplicative functions

$$\chi_\phi, \chi_\psi\colon \mathsf{A}_+ \to \mathbb{F}_q^\times$$

as in (2.4.5). We have the following proposition for determining $[\overline{\phi \otimes \psi}(\mathbb{F}_f)]_A$ in term of the value $\mathbf{P}_{\phi\otimes\psi,f}(1)$.

Proposition 4.3.5. *Let $f \in A_+$ be irreducible. For $\overline{\phi \otimes \psi} \colon \mathsf{A} \to \mathrm{Mat}_{r+\ell}(\mathbb{F}_f[\tau])$ defined above,*

$$\left[\overline{\phi \otimes \psi}(\mathbb{F}_f)\right]_A = \chi_\phi(f)^\ell \chi_\psi(f)^r \cdot \mathbf{P}_{\phi\otimes\psi,f}(1) = \frac{\mathbf{P}_{\phi\otimes\psi,f}(1)}{\mathbf{P}_{\phi\otimes\psi,f}(0)} \cdot f^{r+\ell}.$$

Proof. Let $\lambda \in \mathsf{A}_+$ be irreducible so that $\lambda(\theta) \neq f$. Let K_f^{nr} be the maximal unramified and separable extension of K_f, and let $K_f^{\mathrm{nr}} \supseteq O_f^{\mathrm{nr}} \supseteq M_f^{\mathrm{nr}}$ be its subring of f-integral elements and its maximal ideal. Because ϕ and ψ have everywhere good reduction, we see from [52, Thm. 1] (see also [33, Thm. 4.10.5]) that

$$\phi[\lambda^m], \psi[\lambda^m] \subseteq O_f^{\mathrm{nr}}, \quad \forall\, m \geqslant 1.$$

Moreover, the natural reduction maps

$$\phi(O_f^{\mathrm{nr}})[\lambda^m] = \phi[\lambda^m] \xrightarrow{\sim} \bar{\phi}[\lambda^m], \quad \psi(O_f^{\mathrm{nr}})[\lambda^m] = \psi[\lambda^m] \xrightarrow{\sim} \bar{\psi}[\lambda^m], \quad \forall\, m \geqslant 1,$$

are A-module isomorphisms (e.g., see [52, §2]), which gives isomorphisms on Tate modules commuting with the Galois action

$$T_\lambda(\phi) \xrightarrow{\sim} T_\lambda(\bar{\phi}), \quad T_\lambda(\psi) \xrightarrow{\sim} T_\lambda(\bar{\psi}). \tag{4.3.6}$$

Together with [34, Thm. 3.1], the following diagram of A_λ-modules commutes:

$$\begin{array}{ccc}
T_\lambda(\phi \otimes \psi) & \xrightarrow{\sim} & T_\lambda(\overline{\phi \otimes \psi}) \\
 & & \| \\
\Big\downarrow{\wr} & & T_\lambda(\bar{\phi} \otimes \bar{\psi}) \\
 & & \Big\downarrow{\wr} \\
T_\lambda(\phi) \otimes_{\mathsf{A}_\lambda} T_\lambda(\psi) & \xrightarrow{\sim} & T_\lambda(\bar{\phi}) \otimes_{\mathsf{A}_\lambda} T_\lambda(\bar{\psi}).
\end{array}$$

Furthermore, the maps in the diagram above commute with the Galois action, which implies

$$\mathbf{P}_{\phi\otimes\psi,f}(X) = \mathrm{Char}\left(\tau^d, T_\lambda(\overline{\phi \otimes \psi}), X\right).$$

Now as $\mathbf{P}_f(X) = (P_{\phi,f} \otimes P_{\psi,f})(X)$ by (4.3.2), it follows from (2.4.6) and Definition 2.5.3 that the constant term of $\mathbf{P}_f(X)$ is

$$\mathbf{P}_f(0) = \overline{\chi}_\phi(f)^\ell \overline{\chi}_\psi(f)^r f^{r+\ell}. \tag{4.3.7}$$

By [37, Cor. 3.7.8], it remains to show $\chi_\phi(f)^\ell \chi_\psi(f)^r \cdot \mathbf{P}_{\phi\otimes\psi,f}(1) \in A$ is monic.

Indeed, writing $\mathbf{P}_f(X) = \sum_{i=0}^{r\ell} b_i X^i$, $b_i \in A[X]$ and letting $c_0, \dots, c_{r-1} \in A$ be given as in (2.4.1), Definition 2.5.3 implies that, for $0 \leqslant m \leqslant r\ell - 1$, each b_m is a polynomial in $c_0, \dots, c_{r-1}$ with coefficients in $\mathbb{F}_q$. Assigning the weight $r - i$ to each c_i, then as formal expressions, each monomial in $c_0, \dots, c_{r-1}$ in b_m has the same total weight $r\ell - m$.

That is, if $c_0^{n_0}\cdots c_{r-1}^{n_{r-1}}$ is a monomial in b_m, then $\sum_{i=0}^{r-1}(r-i)n_i = r\ell - m$, and so by Section 2.4.2,

$$\deg_\theta(c_0^{n_0}\cdots c_{r-1}^{n_{r-1}}) \leqslant \sum_{i=0}^{r-1}\frac{d}{r}\cdot n_i(r-i) = \frac{d}{r}(r\ell-m) = d\ell - \frac{dm}{r}.$$

From (4.3.7), this is an equality if $m=0$. On the other hand, this inequality implies,

$$0 < m \leqslant r\ell - 1 \implies \deg_\theta b_m < d\ell.$$

Therefore from (4.3.7), $\chi_\phi(f)^\ell\chi_\psi(f)^r\cdot\mathbf{P}_{\phi\otimes\psi,f}(1)\in A$ is monic. ∎

For each irreducible $f\in\mathsf{A}_+$, (4.3.2) implies that $\mathbf{Q}^\vee_{\phi\otimes\psi,f}(1)=\mathbf{P}_{\phi\otimes\psi,f}(1)/\mathbf{P}_{\phi\otimes\psi,f}(0)$. By combining Proposition 4.3.5 and also the relation (4.3.3), we are able to obtain the following identity for $L(\phi\otimes\psi^\vee,0)$, which shows that Fang's class module formula (Theorem 4.2.2) applies to the special values we are considering.

Proposition 4.3.8. *We have*

$$L\big((\phi\otimes\psi)^\vee,0\big) = \prod_f \frac{[\mathbb{F}_f^{r+\ell}]_A}{\big[\,\overline{\phi\otimes\psi}(\mathbb{F}_f)\big]_A}.$$

4.3.9. Convolution L-series of $\phi\otimes\psi$. We investigate

$$L\big((\phi\otimes\psi)^\vee,s\big) = \prod_f \mathbf{Q}^\vee_{\phi\otimes\psi,f}(f^{-s})^{-1} \tag{4.3.10}$$

from (4.3.3). We at first fix $f\in A_+$ irreducible, and we let $\alpha_1,\dots,\alpha_r\in\bar{K}$ be the roots of $P^\vee_{\phi,f}(X)$, and we let $\beta_1,\dots,\beta_\ell\in\bar{K}$ be the roots of $P^\vee_{\psi,f}(X)$. We split it into two cases.

For $r=\ell$. As $\mathbf{Q}^\vee_{\phi\otimes\psi,f}(X)$ is the reciprocal polynomial of $\mathbf{P}^\vee_{\phi\otimes\psi,f}(X) = P^\vee_{\phi,f}(X)\otimes P^\vee_{\psi,f}(X)$, we can expand $\mathbf{Q}^\vee_{\phi\otimes\psi,f}(f^{-s})^{-1}$ using Cauchy's identity (Theorem 2.5.12). We note from (2.4.6) that $\alpha_1\cdots\alpha_r = \chi_\phi(f)f^{-1}$ and $\beta_1\cdots\beta_r = \chi_\psi(f)f^{-1}$. By the definitions of $\boldsymbol{\mu}_\phi$ and $\boldsymbol{\mu}_\psi$ from (2.6.1), Theorem 2.5.12 implies

$$\begin{aligned}
&\mathbf{Q}^\vee_{\phi\otimes\psi,f}(f^{-s})^{-1}\\
&=\left(1-\frac{\chi_\phi(f)\chi_\psi(f)}{f^{rs+2}}\right)^{-1}\sum_{\substack{k_1=0\\ k=(k_1,\dots,k_{r-1})}}^{\infty}\cdots\sum_{k_{r-1}=0}^{\infty} S_k(\boldsymbol{\alpha})S_k(\boldsymbol{\beta})f^{-s(k_1+2k_2+\cdots+(r-1)k_{r-1})}\\
&=\left(1-\frac{\chi_\phi(f)\chi_\psi(f)}{f^{rs+2}}\right)^{-1}\sum_{k_1,\dots,k_{r-1}\geqslant 0}\frac{\boldsymbol{\mu}_\phi(f^{k_1},\dots,f^{k_{r-1}})\boldsymbol{\mu}_\psi(f^{k_1},\dots,f^{k_{r-1}})}{f^{2(k_1+k_2+\cdots+k_{r-1})+s(k_1+2k_2+\cdots+(r-1)k_{r-1})}},
\end{aligned} \tag{4.3.11}$$

where $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_r)$ and $\boldsymbol{\beta}=(\beta_1,\dots,\beta_r)$. We define the twisted Carlitz zeta function

$$L(A,\chi_\phi\chi_\psi,s) := \sum_{a\in A_+}\frac{\chi_\phi(a)\chi_\psi(a)}{a^s}, \tag{4.3.12}$$

and finally we define the L-series,

$$L(\boldsymbol{\mu}_\phi \times \boldsymbol{\mu}_\psi, s) := \sum_{a_1 \in A_+} \cdots \sum_{a_{r-1} \in A_+} \frac{\boldsymbol{\mu}_\phi(a_1, \dots, a_{r-1}) \boldsymbol{\mu}_\psi(a_1, \dots, a_{r-1})}{(a_1 \cdots a_{r-1})^2 (a_1 a_2^2 \cdots a_{r-1}^{r-1})^s}. \tag{4.3.13}$$

The convergence of this series in K_∞ can be deduced from Proposition 2.6.3 (d) for $s \geqslant 0$. By examining the Euler products of the L-series, we arrive at the following result.

Theorem 4.3.14. *Let ϕ, $\psi : \mathsf{A} \to A[\tau]$ be Drinfeld modules both of rank $r \geqslant 2$ with everywhere good reduction, as defined in (1.2.1). Then*

$$L\big((\phi \otimes \psi)^\vee, s\big) = L(A, \chi_\phi \chi_\psi, rs + 2) \cdot L(\boldsymbol{\mu}_\phi \times \boldsymbol{\mu}_\psi, s).$$

We can substitute $s = 0$ into Theorem 4.3.14 and obtain the following special value identity.

Corollary 4.3.15. *Let ϕ, $\psi : \mathsf{A} \to A[\tau]$ be Drinfeld modules both of rank $r \geqslant 2$ with everywhere good reduction, as defined in (1.2.1).*

$$\begin{aligned} L(\boldsymbol{\mu}_\phi \times \boldsymbol{\mu}_\psi, 0) &= \sum_{a_1 \in A_+} \cdots \sum_{a_{r-1} \in A_+} \frac{\boldsymbol{\mu}_\phi(a_1, \dots, a_{r-1}) \boldsymbol{\mu}_\psi(a_1, \dots, a_{r-1})}{(a_1 \cdots a_{r-1})^2} \\ &= \frac{\mathrm{Reg}_{\phi \otimes \psi} \cdot \big[\mathrm{H}(\phi \otimes \psi)\big]_A}{L(A, \chi_\phi \chi_\psi, 2)}. \end{aligned}$$

For $r < \ell$. Using that $\alpha_1 \cdots \alpha_r = \chi_\phi(f) f^{-1}$, we apply Bump's specialization of Cauchy's identity (Corollary 2.5.13), and similar to calculations in Section 4.3.9, we find

$$\begin{aligned} &\mathbf{Q}^\vee_{\phi\otimes\psi, f}(f^{-s})^{-1} \\ &\quad = \sum_{\substack{k_1 = 0 \\ k = (k_1, \dots, k_{r-1}) \\ k' = (k_1, \dots, k_r, 0, \dots, 0)}}^{\infty} \cdots \sum_{k_r = 0}^{\infty} S_k(\boldsymbol{\alpha}) S_{k'}(\boldsymbol{\beta}) (\chi_\phi(f) f^{-1})^{k_r} f^{-s(k_1 + 2k_2 + \cdots + rk_r)} \\ &\quad = \sum_{k_1, \dots, k_r \geqslant 0} \frac{\boldsymbol{\mu}_\phi(f^{k_1}, \dots, f^{k_{r-1}}) \boldsymbol{\mu}_\psi(f^{k_1}, \dots, f^{k_r}, 1, \dots, 1) \cdot \chi_\phi(f^{k_r})}{f^{2(k_1 + k_2 + \cdots + k_r) + s(k_1 + 2k_2 + \cdots + rk_r)}}. \end{aligned}$$

The expression $\boldsymbol{\mu}_\psi(f^{k_1}, \dots, f^{k_r}, 1, \dots, 1)$ generically has 1's in exactly the last $\ell - 1 - r$ places. We thus define the L-series when $r < \ell$,

$$\begin{aligned} &L(\boldsymbol{\mu}_\phi \times \boldsymbol{\mu}_\psi, s) \\ &\quad := \sum_{a_1, \dots, a_r \in A_+} \frac{\chi_\phi(a_r) \boldsymbol{\mu}_\phi(a_1, \dots, a_{r-1}) \boldsymbol{\mu}_\psi(a_1, \dots, a_r, 1, \dots, 1)}{(a_1 \cdots a_r)^2 (a_1 a_2^2 \cdots a_r^r)^s}. \end{aligned} \tag{4.3.16}$$

Similarly, the Euler products of these L-series yield the following theorem.

Theorem 4.3.17. *Let ϕ, $\psi\colon \mathsf{A} \to A[\tau]$ be Drinfeld modules of ranks r and ℓ respectively with everywhere good reduction, as defined in (1.2.1). Assume that $r, \ell \geqslant 2$ and that $r < \ell$. Then*

$$L\big((\phi \otimes \psi)^{\vee}, s\big) = L(\boldsymbol{\mu}_{\phi} \times \boldsymbol{\mu}_{\psi}, s).$$

We can substitute $s = 0$ into Theorem 4.3.17 and obtain the following special value identity.

Corollary 4.3.18. *Let ϕ, $\psi\colon \mathsf{A} \to A[\tau]$ be Drinfeld modules both of rank $r \geqslant 2$ with everywhere good reduction, as defined in (1.2.1). Assume that $r, \ell \geqslant 2$ and that $r < \ell$. Then*

$$\begin{aligned} L(\boldsymbol{\mu}_{\phi} \times \boldsymbol{\mu}_{\psi}, 0) &= \sum_{a_1,\dots,a_r \in A_+} \frac{\chi_{\phi}(a_r)\boldsymbol{\mu}_{\phi}(a_1,\dots,a_{r-1})\boldsymbol{\mu}_{\psi}(a_1,\dots,a_r,1,\dots,1)}{(a_1 \cdots a_r)^2} \\ &= \mathrm{Reg}_{\phi\otimes\psi} \cdot \big[\mathrm{H}(\phi \otimes \psi)\big]_A. \end{aligned}$$

Remark 4.3.19. Note that the case $r > \ell$ is included in the case $r < \ell$ since $\phi \otimes \psi$ is isomorphic to $\phi \otimes \phi$ as t-modules by [34, Prop. 2.5].

4.4. L-series of symmetric and alternating squares of Drinfeld module over A

We assume $p \neq 2$. Let $\phi\colon \mathsf{A} \to A[\tau]$ be Drinfeld modules defined over A of rank r as in (1.2.1). Recall the t-module $\mathrm{Sym}^2 \phi\colon \mathsf{A} \to \mathrm{Mat}_{r+1}(A[\tau])$ and $\bigwedge^2 \phi\colon \mathsf{A} \to \mathrm{Mat}_{r-1}(A[\tau])$ defined over A as in Definition 3.2.11. Following the same process as in Section 4.3.1, We define

$$\begin{aligned} \mathbf{P}_{\mathrm{Sym}^2 \phi, f}(X) &:= \mathrm{Char}\big(\alpha_f, T_\lambda(\mathrm{Sym}^2 \phi), X\big)\big|_{t=\theta}, \\ \mathbf{P}^{\vee}_{\mathrm{Sym}^2 \phi, f}(X) &:= \mathrm{Char}\big(\alpha_f, T^{\vee}_\lambda(\mathrm{Sym}^2 \phi), X\big)\big|_{t=\theta}, \end{aligned}$$

and

$$\begin{aligned} \mathbf{P}_{\bigwedge^2 \phi, f}(X) &:= \mathrm{Char}\Big(\alpha_f, T_\lambda\Big(\bigwedge^2 \phi\Big), X\Big)\Big|_{t=\theta}, \\ \mathbf{P}^{\vee}_{\bigwedge^2 \phi, f}(X) &:= \mathrm{Char}\Big(\alpha_f, T^{\vee}_\lambda\Big(\bigwedge^2 \phi\Big), X\Big)\Big|_{t=\theta} \end{aligned}$$

Recall the notations $(\mathrm{Sym}^2 P)(X)$ and $(\bigwedge^2 P)(X)$ that were introduced earlier in Definition 2.5.3. By [34, Thm. 5.9], we then have

$$\begin{aligned} \mathbf{P}_{\mathrm{Sym}^2 \phi, f}(X) &= (\mathrm{Sym}^2 P_{\phi,f})(X) \in A[X], \\ \mathbf{P}^{\vee}_{\mathrm{Sym}^2 \phi, f}(X) &= (\mathrm{Sym}^2 P^{\vee}_{\phi,f})(X) \in K[X], \end{aligned} \tag{4.4.1}$$

and

$$\begin{aligned} \mathbf{P}_{\bigwedge^2 \phi, f}(X) &= \Big(\bigwedge^2 P_{\phi,f}\Big)(X) \in A[X], \\ \mathbf{P}^{\vee}_{\bigwedge^2 \phi, f}(X) &= \Big(\bigwedge^2 P^{\vee}_{\phi,f}\Big)(X) \in K[X]. \end{aligned} \tag{4.4.2}$$

We further set $\mathbf{Q}_{E,f}(X)$ and $\mathbf{Q}^{\vee}_{E,f}(X)$ to be their reciprocal polynomials for $E = \operatorname{Sym}^2 \phi$ or $\bigwedge^2 \phi$. We now consider the L-functions

$$L\big((\operatorname{Sym}^2 \phi)^{\vee}, s\big) := \prod_f \mathbf{Q}^{\vee}_{\operatorname{Sym}^2 \phi, f}(f^{-s})^{-1}, \quad s \geqslant 0, \tag{4.4.3}$$

$$L\Big(\Big(\bigwedge^2 \phi\Big)^{\vee}, s\Big) := \prod_f \mathbf{Q}^{\vee}_{\bigwedge^2 \phi, f}(f^{-s})^{-1}, \quad s \geqslant 0. \tag{4.4.4}$$

4.4.5. Special L-values of symmetric and alternating squares. Fixing $f \in A_+$ irreducible, similar to the tensor products, we denote

$$\overline{\operatorname{Sym}^2 \phi} \colon \mathsf{A} \to \operatorname{Mat}_{r+1}\left(\mathbb{F}_f[\tau]\right)$$

and

$$\overline{\bigwedge^2 \phi} \colon \mathsf{A} \to \operatorname{Mat}_{r-1}\left(\mathbb{F}_f[\tau]\right)$$

the reductions modulo f. We also have the following useful proposition for determining the desired quantity $\big[\overline{\operatorname{Sym}^2 \phi}(\mathbb{F}_f)\big]_A$ and $\big[\overline{\bigwedge^2 \phi}(\mathbb{F}_f)\big]_A$ in term of the values $\mathbf{P}_{\operatorname{Sym}^2 \phi, f}(1)$ and $\mathbf{P}_{\bigwedge^2 \phi, f}(1)$, respectively.

Proposition 4.4.6. *Let $f \in A_+$ be irreducible. For $\overline{\operatorname{Sym}^2 \phi} \colon \mathsf{A} \to \operatorname{Mat}_{r+1}(\mathbb{F}_f[\tau])$ and $\overline{\bigwedge^2 \phi} \colon \mathsf{A} \to \operatorname{Mat}_{r-1}(\mathbb{F}_f[\tau])$ defined above,*

$$\big[\overline{\operatorname{Sym}^2 \phi}(\mathbb{F}_f)\big]_A = (-1)^{\frac{r(r+1)}{2}} \chi_\phi(f)^{r+1} \cdot \mathbf{P}_{\operatorname{Sym}^2 \phi, f}(1) = \frac{\mathbf{P}_{\operatorname{Sym}^2 \phi, f}(1)}{\mathbf{P}_{\operatorname{Sym}^2 \phi, f}(0)} \cdot f^{r+1},$$

and

$$\Big[\overline{\bigwedge^2 \phi}(\mathbb{F}_f)\Big]_A = (-1)^{\frac{r(r-1)}{2}} \chi_\phi(f)^{r-1} \cdot \mathbf{P}_{\bigwedge^2 \phi, f}(1) = \frac{\mathbf{P}_{\bigwedge^2 \phi, f}(1)}{\mathbf{P}_{\bigwedge^2 \phi, f}(0)} \cdot f^{r-1}.$$

Proof. In a similar fashion to the proof of Proposition 4.3.5, following from [34, Thm. 5.9] the diagrams of A_λ-modules below commutes:

$$\begin{array}{ccc}
T_\lambda(\operatorname{Sym}^2 \phi) & \xrightarrow{\sim} & T_\lambda(\overline{\operatorname{Sym}^2 \phi}) \\
\Big\downarrow\wr & & \Big\| \\
 & & T_\lambda\big(\operatorname{Sym}^2(\bar{\phi})\big) \\
 & & \Big\downarrow\wr \\
\operatorname{Sym}^2\left(T_\lambda(\phi)\right) & \xrightarrow{\sim} & \operatorname{Sym}^2\left(T_\lambda(\bar{\phi})\right)
\end{array}
\qquad
\begin{array}{ccc}
T_\lambda\big(\bigwedge^2 \phi\big) & \xrightarrow{\sim} & T_\lambda\big(\overline{\bigwedge^2 \phi}\big) \\
\Big\downarrow\wr & & \Big\| \\
 & & T_\lambda\big(\bigwedge^2(\bar{\phi})\big) \\
 & & \Big\downarrow\wr \\
\bigwedge^2\left(T_\lambda(\phi)\right) & \xrightarrow{\sim} & \bigwedge^2\left(T_\lambda(\bar{\phi})\right)
\end{array}$$

Furthermore, the maps in the diagram above commute with the Galois action, which implies

$$\mathbf{P}_{\operatorname{Sym}^2 \phi, f}(X) = \operatorname{Char}\big(\tau^d, T_\lambda(\overline{\operatorname{Sym}^2 \phi}), X\big),$$

and

$$\mathbf{P}_{\bigwedge^2 \phi, f}(X) = \mathrm{Char}\Big(\tau^d, T_\lambda\Big(\overline{\bigwedge^2 \phi}\Big), X\Big).$$

The remaining part follows by the same process of the proof of Proposition 4.3.5. ■

For each irreducible $f \in \mathsf{A}_+$, (4.4.1) and (4.4.2) imply that

$$\mathbf{Q}^\vee_{\mathrm{Sym}^2 \phi, f}(1) = \frac{\mathbf{P}_{\mathrm{Sym}^2 \phi, f}(1)}{\mathbf{P}_{\mathrm{Sym}^2 \phi, f}(0)} \quad \text{and} \quad \mathbf{Q}^\vee_{\bigwedge^2 \phi, f}(1) = \frac{\mathbf{P}_{\bigwedge^2 \phi, f}(1)}{\mathbf{P}_{\bigwedge^2 \phi, f}(0)}.$$

By combining Proposition 4.4.6, (4.4.3) and (4.4.4), we obtain the following identities for $L(\mathrm{Sym}^2 \phi^\vee, 0)$ and $L(\bigwedge^2 \phi^\vee, 0)$, which show that Fang's class module formula (Theorem 4.2.2) applies to the special values we are considering.

Proposition 4.4.7. *We have*

$$L\big((\mathrm{Sym}^2 \phi)^\vee, 0\big) = \prod_f \frac{[\mathbb{F}_f^{r+1}]_A}{\big[\overline{\mathrm{Sym}^2 \phi}(\mathbb{F}_f)\big]_A}, \quad L\Big(\Big(\bigwedge^2 \phi\Big)^\vee, 0\Big) = \prod_f \frac{[\mathbb{F}_f^{r-1}]_A}{\big[\overline{\bigwedge^2 \phi}(\mathbb{F}_f)\big]_A}.$$

4.4.8. Convolution L-series of symmetric and alternating squares. We investigate

$$L\big((\mathrm{Sym}^2 \phi)^\vee, s\big) = \prod_f \mathbf{Q}^\vee_{\mathrm{Sym}^2 \phi, f}(f^{-s})^{-1}, \quad L\Big(\Big(\bigwedge^2 \phi\Big)^\vee, s\Big) = \prod_f \mathbf{Q}^\vee_{\bigwedge^2 \phi, f}(f^{-s})^{-1}$$

from (4.4.3) and (4.4.4). Same as we did in tensor product cases, we at first fix $f \in A_+$ irreducible, and we let $\alpha_1, \ldots, \alpha_r \in \bar{K}$ be the roots of $P^\vee_{\phi, f}(X)$, and we let $\beta_1, \ldots, \beta_\ell \in \bar{K}$ be the roots of $P^\vee_{\psi, f}(X)$. Instead of applying Cauchy's identity, we should apply Littlewood's identities to analyze $\mathbf{Q}^\vee_{\mathrm{Sym}^2 \phi, f}(X)^{-1}$ and $\mathbf{Q}^\vee_{\bigwedge^2 \phi, f}(X)^{-1}$.

Symmetric square. As $\mathbf{Q}^\vee_{\mathrm{Sym}^2 \phi, f}(X)$ is the reciprocal polynomial of

$$\mathbf{P}^\vee_{\mathrm{Sym}^2 \phi, f}(X) = (\mathrm{Sym}^2 P^\vee_{\phi, f})(X),$$

we can expand $\mathbf{Q}^\vee_{\mathrm{Sym}^2 \phi, f}(f^{-s})^{-1}$ using Littlewood's identity (Theorem 2.5.15 (a)). We note from (2.4.6) that $\alpha_1 \cdots \alpha_r = \chi_\phi(f) f^{-1}$. By the definitions of $\boldsymbol{\mu}_\phi$ from (2.6.1), Theorem 2.5.15 (a) implies

$$\begin{aligned} &\mathbf{Q}^\vee_{\mathrm{Sym}^2 \phi, f}(f^{-s})^{-1} \\ &\quad = \left(1 - \frac{\chi_\phi(f)^2}{f^{rs+2}}\right)^{-1} \sum_{\substack{k_1=0 \\ k=(k_1,\ldots,k_{r-1}) \\ 2|k_i \text{ for all } i}}^{\infty} \cdots \sum_{k_{r-1}=0}^{\infty} S_k(\boldsymbol{\alpha}) f^{-s(\frac{k_1}{2} + \frac{2k_2}{2} + \cdots + \frac{(r-1)k_{r-1}}{2})} \\ &\quad = \left(1 - \frac{\chi_\phi(f)^2}{f^{rs+2}}\right)^{-1} \sum_{k_1,\ldots,k_{r-1} \geqslant 0} \frac{\boldsymbol{\mu}_\phi(f^{2k_1}, \ldots, f^{2k_{r-1}})}{f^{2(k_1+k_2+\cdots+k_{r-1}) + s(k_1+2k_2+\cdots+(r-1)k_{r-1})}}, \end{aligned}$$

where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r)$. Recalling the twisted Carlitz zeta function

$$L(A, \chi_\phi^2, s) := \sum_{a \in A_+} \frac{\chi_\phi(a)^2}{a^s},$$

and finally we define the L-series,

$$L(\widetilde{\boldsymbol{\mu}}_\phi, s) := \sum_{a_1 \in A_+} \cdots \sum_{a_{r-1} \in A_+} \frac{\boldsymbol{\mu}_\phi(a_1^2, \dots, a_{r-1}^2)}{(a_1 \cdots a_{r-1})^2 (a_1 a_2^2 \cdots a_{r-1}^{r-1})^s}. \tag{4.4.9}$$

The convergence of this series in K_∞ can be deduced from Proposition 2.6.3 (d) for $s \geqslant 0$. After some straightforward simplification we arrive at the following result.

Theorem 4.4.10. *Let $\phi\colon \mathsf{A} \to A[\tau]$ be Drinfeld module of rank $r \geqslant 2$ with everywhere good reduction, as defined in* (1.2.1)*. Then*

$$L\big((\operatorname{Sym}^2 \phi)^\vee, s\big) = L(A, \chi_\phi^2, rs + 2) \cdot L(\widetilde{\boldsymbol{\mu}}_\phi, s).$$

We can substitute $s = 0$ into Theorem 4.4.10 and obtain the following special value identities.

Corollary 4.4.11. *Let $\phi\colon \mathsf{A} \to A[\tau]$ be Drinfeld module of rank $r \geqslant 2$ with everywhere good reduction, as defined in* (1.2.1)*.*

$$L(\widetilde{\boldsymbol{\mu}}_\phi, 0) = \sum_{a_1 \in A_+} \cdots \sum_{a_{r-1} \in A_+} \frac{\boldsymbol{\mu}_\phi(a_1^2, \dots, a_{r-1}^2)}{(a_1 \cdots a_{r-1})^2} = \frac{\operatorname{Reg}_{\operatorname{Sym}^2 \phi} \cdot \big[\mathrm{H}(\operatorname{Sym}^2 \phi)\big]_A}{L(A, \chi_\phi^2, 2)}.$$

Alternating square. We splits it into three cases. As $\mathbf{Q}^\vee_{\wedge^2 \phi, f}(X)$ is the reciprocal polynomial of

$$\mathbf{P}^\vee_{\wedge^2 \phi, f}(X) = \Big(\bigwedge^2 P^\vee_{\phi, f}\Big)(X),$$

we can expand $\mathbf{Q}^\vee_{\wedge^2 \phi, f}(f^{-s})^{-1}$ using Littlewood's identity (Theorem 2.5.15 (b)) for the first two cases. We recall from (2.4.6) that $\alpha_1 \cdots \alpha_r = \chi_\phi(f) f^{-1}$, and $\boldsymbol{\mu}_\phi$ from (2.6.1). Letting $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r)$ and $w = \lfloor \frac{n}{2} \rfloor$, we have the following identities.

Case 1. For $r \geqslant 3$ and $2 \nmid r - 1$,

$$\begin{aligned} &\mathbf{Q}^\vee_{\wedge^2 \phi, f}(f^{-s})^{-1} \\ &\quad = \left(1 - \frac{\chi_\phi(f)}{f^{\frac{rs}{2}+1}}\right)^{-1} \sum_{\substack{k_1 = 0 \\ k = (0, k_1, 0, k_2, \dots, k_w, 0)}}^{\infty} \cdots \sum_{k_w = 0}^{\infty} S_k(\boldsymbol{\alpha}) f^{-s(k_1 + 2k_2 + \cdots + wk_w)} \\ &\quad = \left(1 - \frac{\chi_\phi(f)}{f^{\frac{rs}{2}+1}}\right)^{-1} \sum_{k_1, \dots, k_w \geqslant 0} \frac{\boldsymbol{\mu}_\phi(1, f^{k_1}, 1, f^{k_2}, \dots, f^{k_w}, 1)}{f^{k_1 + k_2 + \cdots + k_w + s(k_1 + 2k_2 + \cdots + wk_w)}}, \end{aligned}$$

Case 2. For $r \geqslant 3$ and $2 \mid r-1$,

$$\mathbf{Q}^{\vee}_{\bigwedge^2 \phi, f}(f^{-s})^{-1} = \sum_{\substack{k_1=0 \\ k=(0,k_1,0,k_2,\dots,k_w)}}^{\infty} \cdots \sum_{k_w=0}^{\infty} S_k(\boldsymbol{\alpha}) f^{-s(k_1+2k_2+\cdots+wk_w)}$$
$$= \sum_{k_1,\dots,k_w \geqslant 0} \frac{\boldsymbol{\mu}_\phi(1, f^{k_1}, 1, f^{k_2}, \dots, f^{k_w})}{f^{k_1+k_2+\cdots+k_w+s(k_1+2k_2+\cdots+wk_w)}},$$

Case 3. If $r = 2$, then

$$\mathbf{Q}^{\vee}_{\bigwedge^2 \phi, f}(f^{-s})^{-1} = \left(1 - \frac{\chi_\phi(f)}{f^{s+1}}\right)^{-1}.$$

Considering the twisted Carlitz zeta function

$$L(A, \chi_\phi, s) := \sum_{a \in A_+} \frac{\chi_\phi(a)}{a^s},$$

and finally we define the L-series,

$$L(\widehat{\boldsymbol{\mu}}_\phi, s) := \sum_{\substack{a_1,\dots,a_{r-1} \in A_+ \\ a_i = 1 \text{ if } 2 \nmid i}} \frac{\boldsymbol{\mu}_\phi(a_1, \dots, a_{r-1})}{a_1 \cdots a_{r-1}(a_1 a_2^2 \cdots a_{r-1}^{r-1})^s}. \tag{4.4.12}$$

The convergence of this series in K_∞ can be deduced from Proposition 2.6.3 (d) for $s \geqslant 0$. After some straightforward simplification we arrive at the following result.

Theorem 4.4.13. *Let $\phi\colon \mathsf{A} \to A[\tau]$ be Drinfeld module of rank $r \geqslant 2$ with everywhere good reduction, as defined in* (1.2.1). *Then*

$$L\Big(\Big(\bigwedge^2 \phi\Big)^{\vee}, s\Big) = L\Big(A, \chi_\phi, \frac{rs}{2} + 1\Big)^{\frac{(-1)^r+1}{2}} \cdot L(\widehat{\boldsymbol{\mu}}_\phi, s).$$

We can substitute $s = 0$ into Theorem 4.4.13 and obtain the following special value identities.

Corollary 4.4.14. *Let $\phi\colon \mathsf{A} \to A[\tau]$ be Drinfeld module of rank $r \geqslant 2$ with everywhere good reduction, as defined in* (1.2.1).

$$L(\widehat{\boldsymbol{\mu}}_\phi, 0) = \sum_{\substack{a_1,\dots,a_{r-1} \in A_+ \\ a_i = 1 \text{ if } 2 \nmid i}} \frac{\boldsymbol{\mu}_\phi(a_1, \dots, a_{r-1})}{a_1 \cdots a_{r-1}} = \frac{\mathrm{Reg}_{\bigwedge^2 \phi} \cdot \big[\mathrm{H}\big(\bigwedge^2 \phi\big)\big]_A}{L(A, \chi_\phi, 1)^{\frac{(-1)^r+1}{2}}}.$$

Remark 4.4.15. In the case $r = 2$, from (4.4.12), the L-series $L(\widehat{\boldsymbol{\mu}}_\phi, s) = 1$. Theorem 4.4.13 and Corollary 4.4.14 imply

$$L\Big(\Big(\bigwedge^2 \phi\Big)^{\vee}, s\Big) = L(A, \chi_\phi, s+1),$$

and

$$L(A, \chi_\phi, 1) = \mathrm{Reg}_{\bigwedge^2 \phi} \cdot \Big[\mathrm{H}\Big(\bigwedge^2 \phi\Big)\Big]_A.$$

In fact, the alternating square $\bigwedge^2 \phi$ is a Drinfeld module of rank 1 defined by $(\bigwedge^2 \phi)_t = \theta - \kappa_2\tau$. In this case, the class module $\mathrm{H}(\bigwedge^2 \phi)$ is trivial, and $\mathrm{Reg}_{\bigwedge^2 \phi} = \mathrm{Log}_{\bigwedge^2 \phi}(1)$. We refer readers to the next section for more details on regulators of tensor products, symmetric and alternating squares. In conclusion,

$$L(A, \chi_\phi, 1) = \mathrm{Log}_{\bigwedge^2 \phi}(1).$$

5. Regulators of tensor products, symmetric and alternating squares

In this section, we provide explicit expressions for the regulators of the tensor, symmetric, and alternating squares of Drinfeld modules of rank 2, which require explicit formulas for logarithms derived in Section 5.2. To compute logarithms, we review Anderson's exponentiation theorem in Section 5.1.

5.1. Anderson's exponentiation theorem

We let $E\colon \mathsf{A} \to \mathrm{Mat}_\ell(\mathbb{C}_\infty[\tau])$ be a uniformizable almost strictly pure t-module of rank r in the sense of [42], and let $\mathbf{m} = (m_1, \dots, m_r)^\mathsf{T} \in \mathrm{Mat}_{r\times 1}(\mathcal{M}_E)$ be a basis of its t-motive $\mathcal{M}_E := \mathrm{Mat}_{1\times\ell}(\mathbb{C}_\infty[\tau])$ with $\widetilde{\Phi}_E$ denoting multiplication by τ on $\mathcal{M}_E$. Picking $\mathbf{n} = (n_1, \dots, n_r)^\mathsf{T} \in \mathrm{Mat}_{r\times 1}(\mathcal{N}_E)$ to be a basis of its dual t-motive $\mathcal{N}_E := \mathrm{Mat}_{1\times\ell}(\mathbb{C}_\infty[\sigma])$, a *$t$-frame* for E is a pair (ι_E, Φ_E), where Φ_E represents multiplication by σ with respect to $\mathbf{n}$, and $\iota_E\colon \mathrm{Mat}_{1\times r}(\mathbb{C}_\infty[t]) \to \mathcal{N}_E$ is a map given by for $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathrm{Mat}_{1\times r}(\mathbb{C}_\infty[t])$,

$$\iota_E(\boldsymbol{\alpha}) = \boldsymbol{\alpha} \cdot \mathbf{n} = \alpha_1 n_1 + \cdots + \alpha_r n_r.$$

For $n = \sum_{i=0}^{l} a_i \sigma^i \in \mathcal{N}_E$ with $a_i \in \mathrm{Mat}_{1\times\ell(\mathbb{C}_\infty)}$, we then define two maps $\varepsilon_0, \varepsilon_1\colon \mathcal{N}_E \to \mathbb{C}_\infty^\ell$ by setting

$$\varepsilon_0(n) := a_0^\mathsf{T}, \quad \varepsilon_1(n) := \left(\sum_{i=0}^{l} a_i^{(i)}\right)^\mathsf{T}.$$

We have the following two results by Anderson.

Lemma 5.1.1 (Anderson [35, Prop. 2.5.8] and [42, Lem. 3.4.1]). *There exists a unique bounded $\mathbb{C}_\infty$-linear map*

$$\mathcal{E}_0 = \mathcal{E}_{0,E}\colon \big(\mathrm{Mat}_{1\times\ell}(\mathbb{T}_\theta), \|\cdot\|_\theta\big) \to (\mathbb{C}_\infty^\ell, |\cdot|_\infty)$$

of normed vector spaces such that $\mathcal{E}_0|_{\mathrm{Mat}_{1\times r}(\mathbb{C}_\infty[t])} = \varepsilon_0 \circ \iota_E$.

We further let $\mathcal{E}_1 = \mathcal{E}_{1,E} := \varepsilon_1 \circ \iota_E\colon \mathrm{Mat}_{1\times r}(\mathbb{C}_\infty[t]) \to \mathbb{C}_\infty^\ell$. Then we state the Anderson's exponentiation theorem below.

Theorem 5.1.2 (Anderson [35, Thm. 2.5.21], [42, Thm. 3.4.2]). *Let $E\colon \mathsf{A} \to \mathrm{Mat}_\ell(\mathbb{C}_\infty[\tau])$ be an A-finite t-module of rank r with t-frame (ι_E, Φ_E). Fix $\mathbf{h} \in \mathrm{Mat}_{1\times r}(\mathbb{C}_\infty[t])$, and suppose there exists $\mathbf{g} \in \mathrm{Mat}_{1\times r}(\mathbb{T}_\theta)$ such that*

$$\mathbf{g}^{(-1)}\Phi_E - \mathbf{g} = \mathbf{h}.$$

Then

$$\mathrm{Exp}_E\left(\mathcal{E}_0(\mathbf{g}+\mathbf{h})\right)=\mathcal{E}_1(\mathbf{h}).$$

Let $\xi \in \mathbb{C}_\infty^\ell$, and define $\mathbf{h}_\xi \in \mathrm{Mat}_{1\times r}(\mathbb{C}_\infty[t])$ as in [42, (4.4.21)]. Considering

$$\mathbf{g}:=\sum_{m\geqslant 0}\mathbf{h}_\xi^{(m+1)}(\Phi_E^{-1})^{(m+1)}\cdots(\Phi_E^{-1})^{(1)}\in \mathrm{Mat}_{1\times r}(\mathbb{T}_\theta),$$

by verifying directly, we have

$$\mathbf{g}^{(-1)}\Phi_E-\mathbf{g}=\mathbf{h}_\xi.$$

Theorem 5.1.2 and [42, Prop. 4.5.22] imply

$$\mathrm{Exp}_E\left(\mathcal{E}_{0,E}\left(\mathbf{h}_\xi+\sum_{m\geqslant 1}\mathbf{h}_\xi^{(m)}(\Phi_E^{-1})^{(m)}\cdots(\Phi_E^{-1})^{(1)}\right)\right)=\xi. \tag{5.1.3}$$

Remark 5.1.4. The construction above generalizes Chen's construction for Drinfeld modules [16, Rem. 3.1.8] to uniformizable almost strictly pure t-modules.

5.2. Logarithms of tensor structures

From now on, we fix a Drinfeld module $\phi\colon \mathsf{A}\to A[\tau]$ of rank 2, given by $\phi_t=\theta+\kappa_1\tau+\kappa_2\tau^2$, where $\kappa_2\in\mathbb{F}_q^\times$. In this subsection, we follow the processes in [42] to compute $\mathcal{E}_0$ and $\mathbf{h}_\xi$ in (5.1.3), which allow us to provide expressions of logarithms of tensor structures. It requires that the t-modules $E:=\phi^{\otimes 2}$, $\mathrm{Sym}^2(\phi)$, $\bigwedge^2(\phi)$ are uniformizable and almost strictly pure, which are followed by Proposition 3.2.12 and Remark 3.2.13.

5.2.1. Calculation of $\mathcal{E}_0$.

The case tensor square. We let $E=\phi^{\otimes 2}$, and let $\mathcal{N}_\phi=\mathbb{C}_\infty[\sigma]$ be the dual t-motive of ϕ. First of all, we define a t-module $\widetilde{E}\colon \mathsf{A}\to \mathrm{Mat}_4(\mathbb{C}_\infty[\tau])$ from the dual t-motive

$$\mathcal{N}_{\widetilde{E}}:=\mathcal{N}_\phi^{\otimes 2}:=\mathbb{C}_\infty[\sigma]\otimes_{\mathbb{C}_\infty[t]}\mathbb{C}_\infty[\sigma]$$

in a similar way as in Section 3.1. To be precise, one can verify that

$$\{s_i\}_{i=1}^4=\{1\otimes 1, 1\otimes\sigma,\sigma\otimes 1,\sigma^2\otimes 1\} \tag{5.2.2}$$

is a $\mathbb{C}_\infty[\sigma]$-basis of $\mathcal{N}_{\widetilde{E}}$ (cf. Lemma 3.1.2). Then we obtain $\widetilde{E}$ by solving the following equation for $\widetilde{E}_t$:

$$t\cdot\mathbf{u}(s_1,\ldots,s_4)^\intercal=\mathbf{u}\widetilde{E}_t^*(s_1,\ldots,s_4)^\intercal$$

for all $\mathbf{u}\in\mathrm{Mat}_{1\times d}(\mathbb{C}_\infty[\sigma])$. Explicitly, we have

$$\widetilde{E}_t=\begin{pmatrix}\theta & \kappa_1\tau & \kappa_1\tau & \kappa_2\tau^2\\ & \theta & \kappa_2\tau & \\ \kappa_1^{(-1)} & \kappa_2\tau & \theta & \kappa_1\tau\\ \kappa_2 & & & \theta\end{pmatrix}.$$

By calculating directly, we have an isomorphism $U_E^* = U_E^{\mathsf{T}}$ from $\widetilde{E}$ to E given by

$$U_E = \begin{pmatrix} & & & 1 \\ & & 1 & -\frac{\kappa_1}{\kappa_2} \\ & 1 & & \\ 1 & -\frac{\kappa_1^{(-1)}}{\kappa_2} & & \end{pmatrix} \in \mathrm{GL}_4(\mathbb{C}_\infty) \subseteq \mathrm{GL}_4\left(\mathbb{C}_\infty[\tau]\right).$$

By the $\mathbb{C}_\infty[\sigma]$-basis in (5.2.2), we identify $\mathcal{N}_{\widetilde{E}}$ with $\mathrm{Mat}_{1\times 4}(\mathbb{C}_\infty[\sigma])$ by setting $s_i \mapsto \mathbf{s}_i$, where $\mathbf{s}_1, \ldots, \mathbf{s}_4$ is the standard basis vectors in $\mathrm{Mat}_{1\times 4}(\mathbb{C}_\infty[\sigma])$. In this way, the $\mathbb{C}_\infty[t]$-basis

$$\{1 \otimes 1, 1 \otimes \sigma, \sigma \otimes 1, \sigma \otimes \sigma\}$$

is identified with

$$\{n_i\}_{i=1}^4 = \{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \sigma\mathbf{s}_1\}.$$

Furthermore, by Example 2.3.4 we note that

$$\Phi_E = \Phi_{\widetilde{E}} = \begin{pmatrix} 0 & 1 \\ \frac{t-\theta}{\kappa_2} & -\frac{\kappa_1^{(-1)}}{\kappa_2} \end{pmatrix}^{\otimes 2} = \begin{pmatrix} & & & 1 \\ & & \frac{t-\theta}{\kappa_2} & -\frac{\kappa_1^{(-1)}}{\kappa_2} \\ & \frac{t-\theta}{\kappa_2} & & -\frac{\kappa_1^{(-1)}}{\kappa_2} \\ \frac{(t-\theta)^2}{\kappa_2^2} & -\frac{\kappa_1^{(-1)}(t-\theta)}{\kappa_2^2} & -\frac{\kappa_1^{(-1)}(t-\theta)}{\kappa_2^2} & \frac{(\kappa_1^{(-1)})^2}{\kappa_2^2} \end{pmatrix}$$

represents multiplication by σ on $\mathcal{N}_E$ and $\mathcal{N}_{\widetilde{E}}$.

We observe that there exists $C \in \mathrm{GL}_4(\mathbb{C}_\infty[t])$ so that

$$C\Phi_E = \begin{pmatrix} (t-\theta)^2 & & & \\ & (t-\theta) & & \\ & & (t-\theta) & \\ & & & 1 \end{pmatrix}.$$

To calculate $\mathcal{E}_0$ by [42, Prop. 3.5.7], we follow [42, Rem. 3.5.11] to define

$$V_E := \begin{pmatrix} \partial n_1(\partial \widetilde{E}_t^* - \theta \mathrm{I}_4) \\ \partial n_1 \\ \partial n_2 \\ \partial n_3 \end{pmatrix} = \begin{pmatrix} & & \kappa_1^{(-1)} & \kappa_2 \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{pmatrix} \in \mathrm{GL}_4(\mathbb{C}_\infty) \subseteq \mathrm{GL}_4\left(\mathbb{C}_\infty[\tau]\right).$$

Note that $V_E^{\mathsf{T}}\colon \rho \to \widetilde{E}$ is an isomorphism of t-modules with $\rho\colon \mathsf{A} \to \mathrm{Mat}_4(\mathbb{C}_\infty[\tau])$ defined by

$$\rho_t = (V_E^{\mathsf{T}})^{-1} \widetilde{E}_t V_E^{\mathsf{T}}.$$

Now we have isomorphisms of t-modules:

$$\rho \xrightarrow{V_E^T} \widetilde{E} \xrightarrow{U_E^T} E,$$

which gives isomorphisms of dual t-motives

$$\mathcal{N}_\rho \xrightarrow{(\cdot)V_E} \mathcal{N}_{\widetilde{E}} \xrightarrow{(\cdot)U_E} \mathcal{N}_E.$$

Then the $\mathbb{C}_\infty[t]$-basis $\mathbf{n}_{\widetilde{E}} = (n_1, \ldots, n_4)^\mathsf{T}$ of $\mathcal{N}_{\widetilde{E}}$ gives $\mathbb{C}_\infty[t]$-bases

$$\mathbf{n}_E = \mathbf{n}_{\widetilde{E}} U_E \quad \text{and} \quad \mathbf{n}_\rho = \mathbf{n}_{\widetilde{E}} V_E^{-1} \tag{5.2.3}$$

of $\mathcal{N}_E$ and $\mathcal{N}_\rho$, respectively. The t-frame $\iota_E \colon \mathrm{Mat}_{1\times 4}(\mathbb{C}_\infty[t]) \to \mathcal{N}_E$ can be computed by

$$\iota_E(\mathbf{f}) = \iota_{\widetilde{E}}(\mathbf{f})U_E = \iota_\rho(\mathbf{f})V_E U_E, \quad \text{for } \mathbf{f} \in \mathrm{Mat}_{1\times 4}\big(\mathbb{C}_\infty[t]\big).$$

We let $\partial_t \colon \mathbb{F}_q(t) \to \mathbb{F}_q(t)$ be the first hyperderivative with respect to t. By applying [42, Prop. 3.5.7] to ρ and using $U_E, V_E \in \mathrm{Mat}_4(\mathbb{C}_\infty)$,

$$\begin{aligned}
\mathcal{E}_0(\mathbf{f}) &= \varepsilon_0\big(\iota_E(\mathbf{f})\big) = (V_E U_E)^\mathsf{T} \mathcal{E}_{0,\rho}(\mathbf{f}) \\
&= (V_E U_E)^\mathsf{T} \begin{pmatrix} \partial_t(f_1) \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}\Bigg|_{t=\theta} = \begin{pmatrix} \kappa_2 \partial_t(f_1)|_{t=\theta} \\ f_3(\theta) \\ f_2(\theta) \\ f_1(\theta) - \frac{\kappa_1}{\kappa_2} f_2(\theta) \end{pmatrix},
\end{aligned} \tag{5.2.4}$$

for $\mathbf{f} = (f_1, \ldots, f_4) \in \mathrm{Mat}_{1\times 4}(\mathbb{C}_\infty(t))$.

The case symmetric square. We let $E = \mathrm{Sym}^2(\phi)$. Similar to the tensor square case, one check directly that

$$\{s_i\}_{i=1}^3 = \left\{1 \otimes 1, \frac{1}{2}(1 \otimes \sigma + \sigma \otimes 1), \frac{1}{2}(1 \otimes \sigma^2 + \sigma^2 \otimes 1)\right\} \tag{5.2.5}$$

is a $\mathbb{C}_\infty[\sigma]$-basis of $\mathcal{N}_{\widetilde{E}} := \mathrm{Sym}^2(\mathcal{N}_\phi) \subseteq \mathcal{N}_\phi^{\otimes 2}$ (cf. (3.2.1), Lemma 3.2.4). In the same fashion for the tensor square, we define the t-module $\widetilde{E} \colon \mathsf{A} \to \mathrm{Mat}_3(\mathbb{C}_\infty[\tau])$, given by

$$\widetilde{E}_t = \begin{pmatrix} \theta & \kappa_1 \tau & \kappa_2 \tau^2 \\ \kappa_1^{(-1)} & \theta + \kappa_2 \tau & \kappa_1 \tau \\ \kappa_2 & & \theta \end{pmatrix},$$

which is isomorphic to E by $U_E^* \colon \widetilde{E} \to E$, where

$$U_E = \begin{pmatrix} & & 1 \\ & \frac{1}{2} & -\frac{\kappa_1}{2\kappa_2} \\ 1 & -\frac{\kappa_1^{(-1)}}{2\kappa_2} & \frac{\kappa_1 \kappa_1^{(-1)}}{2\kappa_2^2} \end{pmatrix} \in \mathrm{GL}_3(\mathbb{C}_\infty) \subseteq \mathrm{GL}_3\big(\mathbb{C}_\infty[\tau]\big).$$

We identify $\mathcal{N}_{\widetilde{E}}$ with $\mathrm{Mat}_{1\times 3}(\mathbb{C}_\infty[\sigma])$ by setting the $\mathbb{C}_\infty[\sigma]$-basis in (5.2.5) $s_i \mapsto \mathbf{s}_i$, where $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$ is the standard basis vectors in $\mathrm{Mat}_{1\times 3}(\mathbb{C}_\infty[\sigma])$. In this way, the $\mathbb{C}_\infty[t]$-basis

$$\left\{1 \otimes 1, \frac{1}{2}(1 \otimes \sigma + \sigma \otimes 1), \sigma \otimes \sigma\right\}$$

is identified with

$$\{n_i\}_{i=1}^3 = \{\mathbf{s}_1, \mathbf{s}_2, \sigma\mathbf{s}_1\}.$$

Furthermore, by Example 2.3.4 we note that

$$\Phi_E = \Phi_{\widetilde{E}} = \mathrm{Sym}^2 \begin{pmatrix} 0 & 1 \\ \frac{t-\theta}{\kappa_2} & -\frac{\kappa_1^{(-1)}}{\kappa_2} \end{pmatrix} = \begin{pmatrix} & & 1 \\ & \frac{t-\theta}{\kappa_2} & -\frac{\kappa_1^{(-1)}}{\kappa_2} \\ \frac{(t-\theta)^2}{\kappa_2^2} & -2\frac{\kappa_1^{(-1)}(t-\theta)}{\kappa_2^2} & \frac{(\kappa_1^{(-1)})^2}{\kappa_2^2} \end{pmatrix}$$

represents multiplication by σ on $\mathcal{N}_E$ and $\mathcal{N}_{\widetilde{E}}$.

We observe that there exists $C \in \mathrm{GL}_3(\mathbb{C}_\infty[t])$ so that

$$C\Phi_E = \begin{pmatrix} (t-\theta)^2 & & \\ & (t-\theta) & \\ & & 1 \end{pmatrix}.$$

Again, we follow [42, Rem. 3.5.11] to define

$$V_E := \begin{pmatrix} \partial n_1(\partial \widetilde{E}_t^* - \theta \mathrm{I}_3) \\ \partial n_1 \\ \partial n_2 \end{pmatrix} = \begin{pmatrix} & \kappa_1^{(-1)} & \kappa_2 \\ 1 & & \\ & 1 & \end{pmatrix} \in \mathrm{GL}_3(\mathbb{C}_\infty) \subseteq \mathrm{GL}_3\left(\mathbb{C}_\infty[\tau]\right).$$

Note that $V_E^{\mathsf{T}}\colon \rho \to \widetilde{E}$ is an isomorphism of t-modules with $\rho\colon \mathsf{A} \to \mathrm{Mat}_3(\mathbb{C}_\infty[\tau])$ defined by

$$\rho_t = (V_E^{\mathsf{T}})^{-1}\widetilde{E}_t V_E^{\mathsf{T}}.$$

Now we have isomorphisms of t-modules:

$$\rho \xrightarrow{V_E^T} \widetilde{E} \xrightarrow{U_E^T} E,$$

which gives isomorphisms of dual t-motives

$$\mathcal{N}_\rho \xrightarrow{(\cdot)V_E} \mathcal{N}_{\widetilde{E}} \xrightarrow{(\cdot)U_E} \mathcal{N}_E.$$

Then the $\mathbb{C}_\infty[t]$-basis $\mathbf{n}_{\widetilde{E}} = (n_1, n_2, n_3)^{\mathsf{T}}$ of $\mathcal{N}_{\widetilde{E}}$ gives $\mathbb{C}_\infty[t]$-bases

$$\mathbf{n}_E = \mathbf{n}_{\widetilde{E}} U_E \quad \text{and} \quad \mathbf{n}_\rho = \mathbf{n}_{\widetilde{E}} V_E^{-1} \tag{5.2.6}$$

of $\mathcal{N}_E$ and $\mathcal{N}_\rho$ respectively. So the t-frame $\iota_E\colon \mathrm{Mat}_{1\times 3}(\mathbb{C}_\infty[t]) \to \mathcal{N}_E$ can be computed by

$$\iota_E(\mathbf{f}) = \iota_{\widetilde{E}}(\mathbf{f})U_E = \iota_\rho(\mathbf{f})V_E U_E, \quad \text{for } \mathbf{f} \in \mathrm{Mat}_{1\times 3}\left(\mathbb{C}_\infty[t]\right).$$

By applying [42, Prop. 3.5.7] to ρ and using $U_E, V_E \in \mathrm{Mat}_3(\mathbb{C}_\infty)$,

$$\mathcal{E}_0(\mathbf{f}) = (V_E U_E)^{\mathsf{T}} \mathcal{E}_{0,\rho}(\mathbf{f}) = (V_E U_E)^{\mathsf{T}} \begin{pmatrix} \partial_t(f_1) \\ f_1 \\ f_2 \end{pmatrix}\Bigg|_{t=\theta} = \begin{pmatrix} \kappa_2 \partial_t(f_1)|_{t=\theta} \\ \frac{1}{2} f_2(\theta) \\ f_1(\theta) - \frac{\kappa_1}{2\kappa_2} f_2(\theta) \end{pmatrix},$$

for $\mathbf{f} = (f_1, f_2, f_3) \in \mathrm{Mat}_{1\times 3}(\mathbb{C}_\infty(t))$.

The case alternating square. By Definition 3.2.11, the alternating square of ϕ is given by $\bigwedge^2 \phi_t = \theta - \kappa_2\tau$, which is a Drinfeld module of rank 1. It follows by [42, Ex. 3.5.14] that, for $\mathbf{f} \in \mathbb{C}_\infty(t)$,

$$\mathcal{E}_0(\mathbf{f}) = \mathbf{f}(\theta).$$

5.2.7. Calculation of $\mathbf{h}_{\mathfrak{E}}$. The key step involves applying [42, (4.4.21)], which utilizes the matrix V representing Hartl–Juschka's isomorphism, as described in [35, Thm. 2.5.13] (see also [42, Thm. 4.4.9]). To compute the matrix V, we apply [42, Cor. 4.5.20 (a)], reducing the problem to computing the matrices X, Y, and B, which are defined in [42, §4.5]. We provide detailed computations for each case of tensor structures.

The case tensor square. We let $E = \phi^{\otimes 2}$. In the same fashion as the $\mathcal{N}_{\widetilde{E}}$ in Section 5.2.1, we identify $\mathcal{M}_E := \mathcal{M}_\phi^{\otimes 2}$ with $\mathrm{Mat}_{1\times 4}(\mathbb{C}_\infty[\tau])$ by setting $s_i \mapsto \mathbf{s}_i$, where $\{s_i\}_{i=1}^4$ is the $\mathbb{C}_\infty[\tau]$-basis in Lemma 3.1.2. Then the $\mathbb{C}_\infty[t]$-basis

$$\{1 \otimes 1, 1 \otimes \tau, \tau \otimes 1, \tau \otimes \tau\}$$

is identified with

$$\{m_i\}_{i=1}^4 = \{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \tau\mathbf{s}_1\}. \tag{5.2.8}$$

On the other hand, by a direct computation, the $\mathbb{C}_\infty[t]$-basis of $\mathcal{N}_E$ in (5.2.3) is

$$\left\{\mathbf{s}_4, \mathbf{s}_3 - \frac{\kappa_1}{\kappa_2}\mathbf{s}_4, \mathbf{s}_2, \sigma\mathbf{s}_4\right\}.$$

By Remark 3.2.13, the top coefficient of E_{t^2} is invertible, so the matrices $X, Y \in \mathrm{Mat}_{8\times 4}(\mathbb{C}_\infty[t])$ in [42, Cor. 4.5.20 (a)] come from solving the following equations.

$$\begin{pmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \\ \mathbf{s}_3 \\ \mathbf{s}_4 \\ \tau\mathbf{s}_1 \\ \tau\mathbf{s}_2 \\ \tau\mathbf{s}_3 \\ \tau\mathbf{s}_4 \end{pmatrix} = X \begin{pmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \\ \mathbf{s}_3 \\ \tau\mathbf{s}_1 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \\ \mathbf{s}_3 \\ \mathbf{s}_4 \\ \sigma\mathbf{s}_1 \\ \sigma\mathbf{s}_2 \\ \sigma\mathbf{s}_3 \\ \sigma\mathbf{s}_4 \end{pmatrix} = Y \begin{pmatrix} \mathbf{s}_4 \\ \mathbf{s}_3 - \frac{\kappa_1}{\kappa_2}\mathbf{s}_4 \\ \mathbf{s}_2 \\ \sigma\mathbf{s}_4 \end{pmatrix},$$

which give

$$X = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ * & * & * & * \\ & & & 1 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}, \quad Y = \begin{pmatrix} * & * & * & * \\ & & 1 & \\ \frac{\kappa_1}{\kappa_2} & 1 & & \\ 1 & & & \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ & & & 1 \end{pmatrix}.$$

Remark 5.2.9. The entries $*$'s in X and Y can be found explicitly, but we just do not need them for the later computations.

By Definition 3.1.4, the matrix $B \in \mathrm{Mat}_8(\mathbb{C}_\infty)$ in [42, Cor. 4.5.20] is given by

$$B = \left(\begin{array}{c|c} B_1 & B_2 \\ \hline B_2 & 0 \end{array}\right),$$

where if we write $E_t = B_0 + B_1\tau + B_2\tau^2$, then

$$B_0 = \begin{pmatrix} \theta & & \kappa_1 & \kappa_2 \\ & \theta & & \\ & & \theta & \\ & & & \theta \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \kappa_1 & & \kappa_2 & \\ \kappa_1 & \kappa_2 & & \\ & & \kappa_1 & \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \kappa_2 & 0 & 0 & 0 \end{pmatrix}. \tag{5.2.10}$$

Then by [42, Cor. 4.5.20],

$$V = (X^{(1)})^\intercal B^\intercal Y = \begin{pmatrix} \frac{\kappa_1^2}{\kappa_2} & \kappa_1 & \kappa_1 & \kappa_2 \\ \kappa_1 & \kappa_2 & & \\ \kappa_1 & & \kappa_2 & \\ \kappa_2 & & & \end{pmatrix}. \tag{5.2.11}$$

By the proof of Proposition 3.2.12,

$$\widetilde{\Phi}_E = \widetilde{\Phi}_\phi^{\otimes 2} = \begin{pmatrix} 0 & 1 \\ \frac{t-\theta}{\kappa_2} & -\frac{\kappa_1}{\kappa_2} \end{pmatrix}^{\otimes 2} = \begin{pmatrix} & & & 1 \\ & & \frac{t-\theta}{\kappa_2} & -\frac{\kappa_1}{\kappa_2} \\ & \frac{t-\theta}{\kappa_2} & & -\frac{\kappa_1}{\kappa_2} \\ \frac{(t-\theta)^2}{\kappa_2^2} & -\frac{\kappa_1(t-\theta)}{\kappa_2^2} & -\frac{\kappa_1(t-\theta)}{\kappa_2^2} & \frac{\kappa_1^2}{\kappa_2^2} \end{pmatrix}. \tag{5.2.12}$$

We let $\mathbf{m} = (m_1, \ldots, m_4)^\intercal$ be the $\mathbb{C}_\infty[t]$-basis for $\mathcal{M}_E$ in (5.2.8), and write $\widetilde{\Phi}_E = \sum_{i=0}^2 \widetilde{U}_i t^i$ with $\widetilde{U}_i \in \mathrm{Mat}_4(\mathbb{C}_\infty)$. By [42, (4.4.21)], together with (5.2.10)–(5.2.12), and recall that $\xi = (\xi_1, \ldots, \xi_4)^\intercal \in \mathbb{C}_\infty^4$, we have

$$\begin{aligned} \mathbf{h}_\xi &= \left(\widetilde{U}_1\mathbf{m}\xi + t\widetilde{U}_2\mathbf{m}\xi + \widetilde{U}_2\mathbf{m}(B_0\xi + B_1\xi^{(1)} + B_2\xi^{(2)})\right)^\intercal \cdot V \\ &= \left(0, \frac{\xi_3}{\kappa_2}, \frac{\xi_2}{\kappa_2}, -\frac{1}{\kappa_2^2}\left((t-\theta)\xi_1 - \kappa_1\xi_2 + \kappa_2\xi_4\right)\right) \cdot V \\ &= \left(\frac{(t-\theta)\xi_1 + \kappa_1\xi_3 + \kappa_2\xi_4}{\kappa_2}, \xi_3, \xi_2, 0\right) \in \mathrm{Mat}_{1\times 4}\left(\mathbb{C}_\infty[t]\right). \end{aligned} \tag{5.2.13}$$

The case symmetric square. We let $E = \mathrm{Sym}^2 \phi$. In this case, we identify $\mathcal{M}_E := \mathrm{Sym}^2(\mathcal{M}_\phi)$ with $\mathrm{Mat}_{1\times 3}(\mathbb{C}_\infty[\tau])$ by assigning the $\mathbb{C}_\infty[\tau]$-basis in Lemma 3.2.4 (a) to the standard basis vectors $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$. Then the $\mathbb{C}_\infty[t]$-basis

$$\left\{1 \otimes 1, \frac{1}{2}(1 \otimes \tau + \tau \otimes 1), \tau \otimes \tau\right\}$$

is identified with

$$\{m_i\}_{i=1}^3 = \{\mathbf{s}_1, \mathbf{s}_2, \tau\mathbf{s}_1\}. \tag{5.2.14}$$

On the other hand, by a direct computation, the $\mathbb{C}_\infty[t]$-basis of $\mathcal{N}_E$ in (5.2.6) is

$$\left\{\mathbf{s}_3, \frac{1}{2}\mathbf{s}_2 - \frac{\kappa_1}{2\kappa_2}\mathbf{s}_3, \sigma\mathbf{s}_3\right\}.$$

By Remark 3.2.13, the top coefficient of E_{t^2} is invertible, so the matrices $X, Y \in \mathrm{Mat}_{6\times 3}(\mathbb{C}_\infty[t])$ in [42, Cor. 4.5.20 (a)] come from solving the following equations.

$$\begin{pmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \\ \mathbf{s}_3 \\ \tau\mathbf{s}_1 \\ \tau\mathbf{s}_2 \\ \tau\mathbf{s}_3 \end{pmatrix} = X \begin{pmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \\ \tau\mathbf{s}_1 \end{pmatrix}, \quad \begin{pmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \\ \mathbf{s}_3 \\ \sigma\mathbf{s}_1 \\ \sigma\mathbf{s}_2 \\ \sigma\mathbf{s}_3 \end{pmatrix} = Y \begin{pmatrix} \mathbf{s}_3 \\ \frac{1}{2}\mathbf{s}_2 - \frac{\kappa_1}{2\kappa_2}\mathbf{s}_3 \\ \sigma\mathbf{s}_3 \end{pmatrix},$$

which give

$$X = \begin{pmatrix} 1 & & \\ & 1 & \\ * & * & * \\ & & 1 \\ * & * & * \\ * & * & * \end{pmatrix}, \quad Y = \begin{pmatrix} * & * & * \\ \frac{\kappa_1}{\kappa_2} & 2 & \\ 1 & & \\ * & * & * \\ * & * & * \\ & & 1 \end{pmatrix}.$$

By Definition 3.2.11 (a), the matrix $B \in \mathrm{Mat}_6(\mathbb{C}_\infty)$ in [42, Cor. 4.5.20] is given by

$$B = \left(\begin{array}{c|c} B_1 & B_2 \\ \hline B_2 & 0 \end{array}\right),$$

where if we write $E_t = B_0 + B_1\tau + B_2\tau^2$, then

$$B_0 = \begin{pmatrix} \theta & \kappa_1 & \kappa_2 \\ & \theta & \\ & & \theta \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & 0 \\ \kappa_1 & \kappa_2 & 0 \\ 0 & \kappa_1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \kappa_2 & 0 & 0 \end{pmatrix}. \tag{5.2.15}$$

Then by [42, Cor. 4.5.20],

$$V = (X^{(1)})^\mathsf{T} B^\mathsf{T} Y = \begin{pmatrix} \frac{\kappa_1^2}{\kappa_2} & 2\kappa_1 & \kappa_2 \\ 2\kappa_1 & 2\kappa_2 & \\ \kappa_2 & & \end{pmatrix}. \tag{5.2.16}$$

By the proof of Proposition 3.2.12,

$$\widetilde{\Phi}_E = \mathrm{Sym}^2(\widetilde{\Phi}_\phi) = \mathrm{Sym}^2 \begin{pmatrix} 0 & 1 \\ \frac{t-\theta}{\kappa_2} & -\frac{\kappa_1}{\kappa_2} \end{pmatrix} = \begin{pmatrix} & & 1 \\ & \frac{t-\theta}{\kappa_2} & -\frac{\kappa_1}{\kappa_2} \\ \frac{(t-\theta)^2}{\kappa_2^2} & -2\frac{\kappa_1(t-\theta)}{\kappa_2^2} & \frac{\kappa_1^2}{\kappa_2^2} \end{pmatrix}. \tag{5.2.17}$$

We let $\mathbf{m} = (m_1, m_2, m_3)^{\mathsf{T}}$ be the $\mathbb{C}_\infty[t]$-basis for $\mathcal{M}_E$ in (5.2.14), and write $\widetilde{\Phi}_E = \sum_{i=0}^{2} \widetilde{U}_i t^i$ with $\widetilde{U}_i \in \mathrm{Mat}_3(\mathbb{C}_\infty)$. By [42, (4.4.21)], together with (5.2.15)–(5.2.17), and recall that $\xi = (\xi_1, \xi_2, \xi_3)^{\mathsf{T}} \in \mathbb{C}_\infty^3$, we have

$$\begin{aligned} \mathbf{h}_\xi &= \left(\widetilde{U}_1 \mathbf{m}\xi + t\widetilde{U}_2 \mathbf{m}\xi + \widetilde{U}_2 \mathbf{m}(B_0\xi + B_1\xi^{(1)} + B_2\xi^{(2)})\right)^{\mathsf{T}} \cdot V \\ &= \left(0, \frac{\xi_2}{\kappa_2}, -\frac{1}{\kappa_2^2}\left((t-\theta)\xi_1 - \kappa_1\xi_2 + \kappa_2\xi_3\right)\right) \cdot V \\ &= \left(\frac{(t-\theta)\xi_1 + \kappa_1\xi_2 + \kappa_2\xi_3}{\kappa_2}, 2\xi_2, 0\right) \in \mathrm{Mat}_{1\times 3}\left(\mathbb{C}_\infty[t]\right). \end{aligned} \tag{5.2.18}$$

The case alternating square. By [42, (4.6.12)], for $\xi \in \mathbb{C}_\infty$,

$$\mathbf{h}_\xi = \xi$$

since $\bigwedge^2 \phi$ is a Drinfeld module of rank 1.

5.2.19. Conclusion. We let $E = \phi^{\otimes 2}$, $\mathrm{Sym}^2 \phi$ or $\bigwedge^2 \phi$ of dimension ℓ with the Drinfeld module $\phi\colon \mathsf{A} \to A[\tau]$ given by $\phi_t = \theta + \kappa_1\tau + \kappa_2\tau^2$ with $\kappa_2 \in \mathbb{F}_q^\times$, and let $\mathcal{T}_E$ be the corresponding matrix operator defined in Section 2.7, and write $\mathrm{Log}_\phi(z) = \sum_{m=0}^{\infty} \beta_m z^{q^m} \in K[\![z]\!]$.

El-Guindy–Papanikolas [20, (6.4), (6.5)] constructed rational functions $\mathcal{B}_m(t) \in K(t)$ such that $\mathcal{B}_m(\theta) = \beta_m$ for $m \geqslant 0$. By their construction, for $m < 0$, $\mathcal{B}_m(t) = 0$ and $\mathcal{B}_0(t) = 1$. We state a recursive formula for $\mathcal{B}_m(t)$ with rank 2 assumption below.

Lemma 5.2.20 (El-Guindy–Papanikolas [20, Lem. 6.12 (b)]). *For $m \geqslant 1$, the sequence $\mathcal{B}_m(t)$ satisfies the following recurrence:*

$$\mathcal{B}_m(t) = \frac{\kappa_1^{(m-1)}}{t-\theta^{(m)}}\mathcal{B}_{m-1}(t) + \frac{\kappa_2}{t-\theta^{(m)}}\mathcal{B}_{m-2}(t).$$

The proposition below provides an expression of

$$\mathcal{R}_{E,m} := (\Phi_E^{-1})^{(m)} \cdots (\Phi_E^{-1})^{(1)}$$

in terms of these rational functions.

Proposition 5.2.21. *For $m \geqslant 1$,*

$$\mathcal{R}_{E,m} = \mathcal{T}_E \begin{pmatrix} \mathcal{B}_m(t) & \frac{\kappa_2}{t-\theta^{(1)}}\mathcal{B}_{m-1}^{(1)}(t) \\ \mathcal{B}_{m-1}(t) & \frac{\kappa_2}{t-\theta^{(1)}}\mathcal{B}_{m-2}^{(1)}(t) \end{pmatrix}.$$

Proof. By Lemma 2.7.6, it suffice to show that

$$\mathcal{R}_{\phi,m} := (\Phi_\phi^{-1})^{(m)} \cdots (\Phi_\phi^{-1})^{(1)} = \begin{pmatrix} \mathcal{B}_m(t) & \frac{\kappa_2}{t-\theta^{(1)}}\mathcal{B}_{m-1}^{(1)}(t) \\ \mathcal{B}_{m-1}(t) & \frac{\kappa_2}{t-\theta^{(1)}}\mathcal{B}_{m-2}^{(1)}(t) \end{pmatrix}.$$

Indeed, a direct calculation shows that

$$\Phi_\phi^{-1} = \begin{pmatrix} \frac{\kappa_1^{(-1)}}{t-\theta} & \frac{\kappa_2}{t-\theta} \\ 1 & 0 \end{pmatrix},$$

which gives, by applying Lemma 5.2.20 with $m = 1$,

$$\mathcal{R}_{\phi,1} = (\Phi_\phi^{-1})^{(1)} = \begin{pmatrix} \mathcal{B}_1(t) & \frac{\kappa_2}{t-\theta^{(1)}}\mathcal{B}_0^{(1)}(t) \\ \mathcal{B}_0(t) & 0 \end{pmatrix}.$$

For $m \geqslant 2$, the first column of $\mathcal{R}_{\phi,m}$ follows by [16, Lem. 3.1.4], and the second column of $\mathcal{R}_{\phi,m}$ follows from the results of the first column and the following formula:

$$[\mathcal{R}_{\phi,m}]_{i2} = \left[\mathcal{R}_{\phi,m-1}^{(1)}(\Phi_\phi^{-1})^{(1)}\right]_{i2} = [\mathcal{R}_{\phi,m-1}^{(1)}]_{i1}\left[(\Phi_\phi^{-1})^{(1)}\right]_{12} = \mathcal{B}_{m-i}^{(1)}\frac{\kappa_2}{t-\theta^{(1)}}. \qquad \blacksquare$$

Remark 5.2.22. Instead of looking at dual t-motives, Khaochim and Papanikolas [39, Lem. 4.2] provided a similar expression for the first column of, for $m \geqslant 0$,

$$\frac{1}{t-\theta^{(m)}}\widetilde{\Phi}^{-1}(\widetilde{\Phi}^{-1})^{(1)}\cdots(\widetilde{\Phi}^{-1})^{(m-1)},$$

for Drinfeld modules.

Theorem 5.2.23. *Suppose that $E = \phi^{\otimes 2}$, $\operatorname{Sym}^2\phi$ or $\bigwedge^2\phi$ of dimension ℓ with the Drinfeld module $\phi\colon \mathsf{A} \to A[\tau]$ given by $\phi_t = \theta + \kappa_1\tau + \kappa_2\tau^2$ with $\kappa_2 \in \mathbb{F}_q^\times$. We let $\mathbf{z} = (z_1,\dots,z_\ell)^\mathsf{T}$. Then*

$$\operatorname{Log}_E(\mathbf{z}) = \mathbf{z} + \sum_{m\geqslant 1}\mathcal{E}_{0,E}\left(\mathbf{h}_{\mathbf{z}}^{(m)}\mathcal{T}_E\begin{pmatrix}\mathcal{B}_m(t) & \frac{\kappa_2}{t-\theta^{(1)}}\mathcal{B}_{m-1}^{(1)}(t) \\ \mathcal{B}_{m-1}(t) & \frac{\kappa_2}{t-\theta^{(1)}}\mathcal{B}_{m-2}^{(1)}(t)\end{pmatrix}\right) \in K[\![\mathbf{z}]\!]^\ell.$$

Proof. Note that $\mathcal{E}_{0,E}$ is additive and by a direct computation that $\mathcal{E}_{0,E}(\mathbf{h}_\mathbf{z}) = \mathbf{z}$ in each case. Since $\beta_m \in K$, the result follows by (5.1.3), Proposition 5.2.21. $\blacksquare$

We define $\widetilde{\beta}_m := \mathcal{B}_m^{(1)}(t)|_{t=\theta} \in K$, and $\beta'_m := \partial_t(\mathcal{B}_m(t))|_{t=\theta} \in K$, and the following $\mathbb{F}_q$-linear series in $K[\![z]\!]$:

$$L_1(z) := \sum_{m\geqslant 1} L_{1,m}z^{q^m} := \sum_{m\geqslant 1}\beta_m^2 z^{q^m}, \tag{5.2.24}$$

$$L_2(z) := \sum_{m\geqslant 1} L_{2,m}z^{q^m} := \sum_{m\geqslant 1}\beta_m\beta_{m-1} z^{q^m}, \tag{5.2.25}$$

$$L'_1(z) := \sum_{m\geqslant 1} L'_{1,m}z^{q^m} := \sum_{m\geqslant 1}2\beta_m\beta'_m z^{q^m}, \tag{5.2.26}$$

$$L'_2(z) := \sum_{m\geqslant 1} L'_{2,m}z^{q^m} := \sum_{m\geqslant 1}(\beta'_m\beta_{m-1} + \beta_m\beta'_{m-1}) z^{q^m}, \tag{5.2.27}$$

$$\widetilde{L}_0(z) := \sum_{m\geqslant 1} \widetilde{L}_{0,m} z^{q^m} := \frac{\kappa_2}{\theta - \theta^{(1)}} \sum_{m\geqslant 1} (\beta_m \widetilde{\beta}_{m-2} - \beta_{m-1}\widetilde{\beta}_{m-1}) z^{q^m}, \tag{5.2.28}$$

$$\widetilde{L}_1(z) := \sum_{m\geqslant 1} \widetilde{L}_{1,m} z^{q^m} := \frac{\kappa_2}{\theta - \theta^{(1)}} \sum_{m\geqslant 1} \beta_m \widetilde{\beta}_{m-1} z^{q^m}, \tag{5.2.29}$$

$$\widetilde{L}_2(z) := \sum_{m\geqslant 1} \widetilde{L}_{2,m} z^{q^m} := \frac{\kappa_2}{\theta - \theta^{(1)}} \sum_{m\geqslant 1} \beta_{m-1} \widetilde{\beta}_{m-1} z^{q^m}. \tag{5.2.30}$$

Combining Theorem 5.2.23 and the formulas for $\mathcal{T}$, $\mathcal{E}_0$, $\mathbf{h}_{\mathbf{z}}$ in previous subsubsections, we obtain explicit expressions for logarithms of tensor structures.

Corollary 5.2.31. *Let $\phi\colon \mathsf{A} \to A[\tau]$ be a Drinfeld module $\phi\colon \mathsf{A} \to A[\tau]$ given by $\phi_t = \theta + \kappa_1\tau + \kappa_2\tau^2$ with $\kappa_2 \in \mathbb{F}_q^{\times}$, and let*

$$\mathbf{L}_m := \begin{pmatrix} L_{1,m} + (\theta-\theta^{(m)})L'_{1,m} & \kappa_2 L'_{2,m} & \kappa_1^{(m)} L'_{1,m} + \kappa_2 L'_{2,m} & \kappa_2 L'_{1,m} \\ \frac{(\theta-\theta^{(m)})}{\kappa_2}\widetilde{L}_{1,m} & \widetilde{L}_{0,m} + \widetilde{L}_{2,m} & \frac{\kappa_1^{(m)}}{\kappa_2}\widetilde{L}_{1,m} + \widetilde{L}_{2,m} & \widetilde{L}_{1,m} \\ \frac{(\theta-\theta^{(m)})}{\kappa_2}\widetilde{L}_{1,m} & \widetilde{L}_{2,m} & \widetilde{L}_{0,m} + \frac{\kappa_1^{(m)}}{\kappa_2}\widetilde{L}_{1,m} + \widetilde{L}_{2,m} & \widetilde{L}_{1,m} \\ \frac{(\theta-\theta^{(m)})}{\kappa_2} L_{1,m} & L_{2,m} & \frac{\kappa_1^{(m)}}{\kappa_2} L_{1,m} + L_{2,m} & L_{1,m} \end{pmatrix} \in \mathrm{Mat}_4(K)$$

for $m \geqslant 1$, then

(a) $\mathrm{Log}_{\phi^{\otimes 2}} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} + \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -\frac{\kappa_1}{\kappa_2} & 1 \end{pmatrix} \sum_{m\geqslant 1} \mathbf{L}_m \begin{pmatrix} z_1^{q^m} \\ z_2^{q^m} \\ z_3^{q^m} \\ z_4^{q^m} \end{pmatrix}$,

(b) $\mathrm{Log}_{\mathrm{Sym}^2\phi} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} 1 & & & \\ & \frac{1}{2} & \frac{1}{2} & \\ & -\frac{\kappa_1}{2\kappa_2} & -\frac{\kappa_1}{2\kappa_2} & 1 \end{pmatrix} \sum_{m\geqslant 1} \mathbf{L}_m \begin{pmatrix} z_1^{q^m} \\ z_2^{q^m} \\ z_2^{q^m} \\ z_3^{q^m} \end{pmatrix}$,

(c) $\mathrm{Log}_{\wedge^2\phi}(z) = z + \widetilde{L}_0(z)$.

5.3. Expressions for regulators

We maintain the same notation as in Section 5.2.19, let $\mathbf{e}_1, \dots, \mathbf{e}_\ell$ be standard basis vectors of $\mathbb{C}_\infty^\ell$, and denote $\partial_\theta\colon K \to K$ to be the first hyperderivative with respect to θ.

Lemma 5.3.1 (cf. [19, Prop. 2.5]). *The set $\{\mathbf{e}_i\}_{i=1}^{\ell}$ is an A-basis of* $\mathrm{Lie}(E)(A)$ *via* ∂.

Proof. It suffices to show that, for $\boldsymbol{\alpha} \in \mathrm{Lie}(E)(A) = A^\ell$, there exist $b_1, \dots, b_\ell \in A$ such that

$$\boldsymbol{\alpha} = \sum_{i=1}^{\ell} \partial E_{b_i(t)} \mathbf{e}_i. \tag{5.3.2}$$

We only show the case $E = \mathrm{Sym}^2\phi$. The remaining two cases follow by similar methods.

Observe that

$$\partial E_{t^i} = (\partial E_t)^i = \begin{pmatrix} \theta & \kappa_1 & \kappa_2 \\ & \theta & \\ & & \theta \end{pmatrix}^i = \begin{pmatrix} \theta^i & i\theta^{i-1}\kappa_1 & i\theta^{i-1}\kappa_2 \\ & \theta^i & \\ & & \theta^i \end{pmatrix},$$

which implies, for $b \in A$,

$$\partial E_{b(t)} = \begin{pmatrix} b & \kappa_1 \partial_\theta(b) & \kappa_2 \partial_\theta(b) \\ & b & \\ & & b \end{pmatrix}.$$

Therefore, the existence of b_i follows by rewriting (5.3.2) to

$$\boldsymbol{\alpha} = \begin{pmatrix} b_1 + \kappa_1 \partial_\theta(b_2) + \kappa_2 \partial_\theta(b_3) \\ b_2 \\ b_3 \end{pmatrix}. \qquad \blacksquare$$

Lemma 5.3.3. *Suppose that* $\{\mathbf{v}_1, \dots, \mathbf{v}_\ell\} \subseteq K_\infty^\ell$ *is an* A*-basis of* $\operatorname{Exp}_{E,K_\infty}^{-1}(A^\ell)$ *via* ∂. *If we write*

$$(\mathbf{r}_1, \dots, \mathbf{r}_\ell)^\intercal := (\mathbf{v}_1, \dots, \mathbf{v}_\ell) \in \operatorname{Mat}_\ell(K_\infty),$$

then we have

(a) $\operatorname{Reg}_{\phi^{\otimes 2}} = \gamma \cdot \det \begin{pmatrix} \mathbf{r}_1 - \kappa_1 \partial_\theta(\mathbf{r}_3) - \kappa_2 \partial_\theta(\mathbf{r}_4) \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \mathbf{r}_4 \end{pmatrix}$.

(b) $\operatorname{Reg}_{\operatorname{Sym}^2 \phi} = \gamma \cdot \det \begin{pmatrix} \mathbf{r}_1 - \kappa_1 \partial_\theta(\mathbf{r}_2) - \kappa_2 \partial_\theta(\mathbf{r}_3) \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{pmatrix}$.

(c) $\operatorname{Reg}_{\wedge^2 \phi} = \gamma \cdot \det(\mathbf{r}_1)$.

Proof. By the definition of Reg_E in (4.2.1), the lemma follows by expressing $\mathbf{v}_i$ in terms of standard basis vectors via ∂ using the same method as in the proof of Lemma 5.3.1. ■

Similar to (5.2.26) and (5.2.27), we further define $\widehat{\beta}_m := \partial_\theta(\mathscr{B}_m(t))|_{t=\theta} \in K$, and the following $\mathbb{F}_q$-linear series in $K[\![z]\!]$:

$$\operatorname{Log}_{\phi,2}(z) := z + L_1(z) = \sum_{m \geqslant 0} \beta_m^2 z^{q^m}, \tag{5.3.4}$$

$$\widehat{L}_1(z) := \sum_{m \geqslant 1} \widehat{L}_{1,m} z^{q^m} := \sum_{m \geqslant 1} 2\beta_m \widehat{\beta}_m z^{q^m}, \tag{5.3.5}$$

$$\widehat{L}_2(z) := \sum_{m \geqslant 1} \widehat{L}_{2,m} z^{q^m} := \sum_{m \geqslant 1} (\widehat{\beta}_m \beta_{m-1} + \beta_m \widehat{\beta}_{m-1}) z^{q^m}. \tag{5.3.6}$$

Theorem 5.3.7. *We have the following formulas for regulators.*

(a) *Assume that* $\deg(\kappa_1) \leqslant (q+1)/2$. *Then*

$$\operatorname{Reg}_{\wedge^2 \phi} = \operatorname{Log}_{\wedge^2 \phi}(1) = 1 + \frac{\kappa_2}{\theta - \theta^{(1)}} \sum_{m \geq 1} (\beta_m \widetilde{\beta}_{m-2} - \beta_{m-1} \widetilde{\beta}_{m-1}).$$

(b) *Assume that* $\deg(\kappa_1) \leqslant 1$. *Then*

(i) $\mathrm{Reg}_{\mathrm{Sym}^2\phi} = \gamma \cdot \det(M)$, *where* $\gamma \in \mathbb{F}_q^\times$ *is chosen so that it has sign* 1, *and*

$$M = \begin{pmatrix} 1 + \widehat{L}_1(\theta) & -\widehat{L}_1(\kappa_1) - 2\kappa_2\widehat{L}_2(1) - \partial_\theta(\kappa_1) & -\kappa_2\widehat{L}_1(1) \\ -\kappa_2^{-1}\widetilde{L}_1(\theta) & \mathrm{Log}_{\wedge^2\phi}(1) + \kappa_2^{-1}\widetilde{L}_1(\kappa_1) + 2\widetilde{L}_2(1) & \widetilde{L}_1(1) \\ -\kappa_2^{-1}\mathrm{Log}_{\phi,2}(\theta) & \kappa_2^{-1}\mathrm{Log}_{\phi,2}(\kappa_1) + 2L_2(1) & \mathrm{Log}_{\phi,2}(1) \end{pmatrix}.$$

(ii) $\mathrm{Reg}_{\phi^{\otimes 2}} = \mathrm{Reg}_{\mathrm{Sym}^2\phi} \cdot \mathrm{Reg}_{\wedge^2\phi}$.

Proof. By [20, Cor. 6.9], the assumptions on $\deg(\kappa_1)$ imply that the logarithm series $\mathrm{Log}_{\phi^{\otimes 2}}(\mathbf{z})$, $\mathrm{Log}_{\mathrm{Sym}^2\phi}(\mathbf{z})$ and $\mathrm{Log}_{\wedge^2\phi}(\mathbf{z})$ converge at standard basis vectors. We may choose

$$\left\{\mathrm{Log}_E(\mathbf{e}_1), \ldots, \mathrm{Log}_E(\mathbf{e}_\ell)\right\} \tag{5.3.8}$$

as an A-basis of $\mathrm{Exp}_{E,K_\infty}^{-1}(A^\ell)$. We start with calculating $\mathrm{Reg}_{\phi^{\otimes 2}}$. By Corollary 5.2.31 (a), the matrix

$$\left(\mathrm{Log}_{\phi^{\otimes 2}}(\mathbf{e}_1), \ldots, \mathrm{Log}_{\phi^{\otimes 2}}(\mathbf{e}_4)\right) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & -\frac{\kappa_1}{\kappa_2} & 1 \end{pmatrix} \times$$
$$\begin{pmatrix} \mathrm{Log}_{\phi,2}(1) + \theta L_1'(1) - L_1'(\theta) & \kappa_2 L_2'(1) & L_1'(\kappa_1) + \kappa_2 L_2'(1) & \kappa_2 L_1'(1) \\ \frac{1}{\kappa_2}\left(\theta\widetilde{L}_1(1) - \widetilde{L}_1(\theta)\right) & \mathrm{Log}_{\wedge^2\phi}(1) + \widetilde{L}_2(1) & \frac{1}{\kappa_2}\widetilde{L}_1(\kappa_1) + \widetilde{L}_2(1) & \widetilde{L}_1(1) \\ \frac{1}{\kappa_2}\left(\theta\widetilde{L}_1(1) - \widetilde{L}_1(\theta)\right) & \widetilde{L}_2(1) & \mathrm{Log}_{\wedge^2\phi}(1) + \frac{1}{\kappa_2}\widetilde{L}_1(\kappa_1) + \widetilde{L}_2(1) & \widetilde{L}_1(1) \\ \frac{1}{\kappa_2}\left(\theta L_1(1) - L_1(\theta)\right) & L_2(1) & \frac{\kappa_1}{\kappa_2} + \frac{1}{\kappa_2}L_1(\kappa_1) + L_2(1) & \mathrm{Log}_{\phi,2}(1) \end{pmatrix}. \tag{5.3.9}$$

For $f \in K(t)$, by [42, Lem. 2.4.6], we have the chain rule

$$\partial_t(f)|_{t=\theta} - \partial_\theta\big(f(\theta)\big) = -\partial_\theta(f)|_{t=\theta}. \tag{5.3.10}$$

Then by applying Lemma 5.3.3 (a) with the A-basis of $\mathrm{Exp}_{E,K_\infty}^{-1}(A^4)$ given in (5.3.8), (5.3.9) and (5.3.10) as well as (5.3.5) and (5.3.6) give the following expression for $\mathrm{Reg}_{\phi^{\otimes 2}}$, for some $\gamma_{\phi^{\otimes 2}} \in \mathbb{F}_q^\times$,

$$\mathrm{Reg}_{\phi^{\otimes 2}} = \gamma_{\phi^{\otimes 2}} \times$$
$$\begin{vmatrix} 1 - \theta\widehat{L}_1(1) + \widehat{L}_1(\theta) & -\kappa_2\widehat{L}_2(1) & -\widehat{L}_1(\kappa_1) - \kappa_2\widehat{L}_2(1) - \partial_\theta(\kappa_1) & -\kappa_2\widehat{L}_1(1) \\ \frac{1}{\kappa_2}\left(\theta\widetilde{L}_1(1) - \widetilde{L}_1(\theta)\right) & \mathrm{Log}_{\wedge^2\phi}(1) + \widetilde{L}_2(1) & \frac{1}{\kappa_2}\widetilde{L}_1(\kappa_1) + \widetilde{L}_2(1) & \widetilde{L}_1(1) \\ \frac{1}{\kappa_2}\left(\theta\widetilde{L}_1(1) - \widetilde{L}_1(\theta)\right) & \widetilde{L}_2(1) & \mathrm{Log}_{\wedge^2\phi}(1) + \frac{1}{\kappa_2}\widetilde{L}_1(\kappa_1) + \widetilde{L}_2(1) & \widetilde{L}_1(1) \\ \frac{1}{\kappa_2}\left(\theta L_1(1) - L_1(\theta)\right) & L_2(1) & \frac{\kappa_1}{\kappa_2} + \frac{1}{\kappa_2}L_1(\kappa_1) + L_2(1) & \mathrm{Log}_{\phi,2}(1) \end{vmatrix}. \tag{5.3.11}$$

We then proceed the following row and column operations

(a) $\mathrm{R}_2 \mapsto \mathrm{R}_2 - \mathrm{R}_3$,

(b) $C_3 \mapsto C_2 + C_3$,

(c) $C_1 \mapsto C_1 - \frac{\theta}{\kappa_2} C_4$.

Then the determinant becomes

$$\mathrm{Log}_{\bigwedge^2 \phi}(1) \cdot \det(M).$$

One can check, by similar calculations for $\mathrm{Sym}^2 \phi$ and $\bigwedge^2 \phi$, that the first factor gives $\mathrm{Reg}_{\bigwedge^2 \phi}$, and the second factor gives $\mathrm{Reg}_{\mathrm{Sym}^2 \phi}$, which complete the proof. ■

Remark 5.3.12. We conclude with the following two remarks about special values.

(a) Theorem 5.3.7 shows that special values of the dilogarithm function $\mathrm{Log}_{\phi,2}(z)$ of ϕ appear in the regulators of $\phi^{\otimes 2}$ and $\bigwedge^2 \phi$.

(b) The formulas in Theorem 5.3.7 give explicit expressions for special values of convolution L-series appearing in Corollaries 4.3.15, 4.3.18, 4.4.11 and 4.4.14.

Acknowledgments. The author gratefully acknowledges M. Papanikolas for his valuable suggestions and encouragement throughout this project. Special thanks are also due to D. Thakur for numerous insightful discussions and comments. The author further thanks the referees for their constructive suggestions, which significantly improved the exposition of this paper.

References

[1] M. Aigner, *A course in enumeration*. Grad. Texts in Math. 238, Springer, Berlin, 2007 Zbl 1123.05001 MR 2339282

[2] G. W. Anderson, t-motives. *Duke Math. J.* **53** (1986), no. 2, 457–502 Zbl 0679.14001 MR 0850546

[3] G. W. Anderson, W. D. Brownawell, and M. A. Papanikolas, Determination of the algebraic relations among special Γ-values in positive characteristic. *Ann. of Math. (2)* **160** (2004), no. 1, 237–313 Zbl 1064.11055 MR 2119721

[4] G. W. Anderson and D. S. Thakur, Tensor powers of the Carlitz module and zeta values. *Ann. of Math. (2)* **132** (1990), no. 1, 159–191 Zbl 0713.11082 MR 1059938

[5] B. Anglès, T. Ngo Dac, and F. Tavares Ribeiro, On special L-values of t-modules. *Adv. Math.* **372** (2020), article no. 107313 Zbl 1458.11095 MR 4128574

[6] B. Anglès, T. Ngo Dac, and F. Tavares Ribeiro, A class formula for admissible Anderson modules. *Invent. Math.* **229** (2022), no. 2, 563–606 Zbl 1501.11063 MR 4448991

[7] B. Anglès, F. Pellarin, and F. Tavares Ribeiro, Arithmetic of positive characteristic L-series values in Tate algebras. *Compos. Math.* **152** (2016), no. 1, 1–61 Zbl 1336.11042 MR 3453387

[8] B. Anglès and L. Taelman, Arithmetic of characteristic p special L-values. *Proc. Lond. Math. Soc. (3)* **110** (2015), no. 4, 1000–1032 Zbl 1328.11065 MR 3335293

[9] B. Anglès and F. Tavares Ribeiro, Arithmetic of function field units. *Math. Ann.* **367** (2017), no. 1-2, 501–579 Zbl 1382.11043 MR 3606448

[10] T. Beaumont, On equivariant class formulas for Anderson modules. *Res. Number Theory* **9** (2023), no. 4, article no. 68 Zbl 1536.11135 MR 4646438

[11] W. D. Brownawell and M. A. Papanikolas, A rapid introduction to Drinfeld modules, t-modules, and t-motives. In *t-motives: Hodge structures, transcendence and other motivic aspects*, pp. 3–30, EMS Ser. Congr. Rep., European Mathematical Society, Berlin, 2020 Zbl 1440.14020 MR 4321964

[12] D. Bump, The Rankin–Selberg method: A survey. In *Number theory, trace formulas and discrete groups (Oslo, 1987)*, pp. 49–109, Academic Press, Boston, MA, 1989 Zbl 0668.10034 MR 0993311

[13] L. Carlitz, On certain functions connected with polynomials in a Galois field. *Duke Math. J.* **1** (1935), no. 2, 137–168 Zbl 61.0127.01 MR 1545872

[14] C.-Y. Chang, A. El-Guindy, and M. A. Papanikolas, Log-algebraic identities on Drinfeld modules and special L-values. *J. Lond. Math. Soc. (2)* **97** (2018), no. 2, 125–144 Zbl 1450.11057 MR 3789840

[15] C.-Y. Chang and M. A. Papanikolas, Algebraic independence of periods and logarithms of Drinfeld modules. *J. Amer. Math. Soc.* **25** (2012), no. 1, 123–150 Zbl 1271.11079 MR 2833480

[16] Y.-T. Chen, On Furusho's analytic continuation of Drinfeld logarithms. *Math. Z.* **307** (2024), no. 3, article no. 48 Zbl 1557.11060 MR 4754368

[17] J. Coates and C.-G. Schmidt, Iwasawa theory for the symmetric square of an elliptic curve. *J. Reine Angew. Math.* **375/376** (1987), 104–156 Zbl 0609.14013 MR 0882294

[18] F. Demeslay, *Formules de classes en caractéristique positive*. Ph.D. thesis, Université de Caen Basse-Normandie, 2015

[19] F. Demeslay, A class formula for L-series in positive characteristic. *Ann. Inst. Fourier (Grenoble)* **72** (2022), no. 3, 1149–1183 Zbl 1507.11050 MR 4485822

[20] A. El-Guindy and M. A. Papanikolas, Identities for Anderson generating functions for Drinfeld modules. *Monatsh. Math.* **173** (2014), no. 4, 471–493 Zbl 1322.11062 MR 3177942

[21] J. Fang, Special L-values of abelian t-modules. *J. Number Theory* **147** (2015), 300–325 Zbl 1360.11074 MR 3276327

[22] J. Ferrara, N. Green, Z. Higgins, and C. D. Popescu, An equivariant Tamagawa number formula for Drinfeld modules and applications. *Algebra Number Theory* **16** (2022), no. 9, 2215–2264 Zbl 1515.11056 MR 4523328

[23] M. Flach, A finiteness theorem for the symmetric square of an elliptic curve. *Invent. Math.* **109** (1992), no. 2, 307–327 Zbl 0781.14022 MR 1172693

[24] E.-U. Gekeler, On finite Drinfel'd modules. *J. Algebra* **141** (1991), no. 1, 187–203 Zbl 0731.11034 MR 1118323

[25] O. Gezmiş, Taelman L-values for Drinfeld modules over Tate algebras. *Res. Math. Sci.* **6** (2019), no. 1, article no. 18 Zbl 1465.11184 MR 3910186

[26] O. Gezmiş, Special values of Goss L-series attached to Drinfeld modules of rank 2. *J. Théor. Nombres Bordeaux* **33** (2021), no. 2, 511–552 Zbl 1496.11086 MR 4371529

[27] O. Gezmiş and C. Namoijam, On the transcendence of special values of goss l-functions attached to drinfeld modules. [v1] 2021, [v3] 2024, arXiv:2110.02569v3

[28] D. Goldfeld, *Automorphic forms and L-functions for the group* $\mathrm{GL}(n, \mathbf{R})$. Cambridge Stud. Adv. Math. 99, Cambridge University Press, Cambridge, 2006 Zbl 1108.11039 MR 2254662

[29] D. Goss, v-adic zeta functions, L-series and measures for function fields. *Invent. Math.* **55** (1979), no. 2, 107–119 Zbl 0402.12006 MR 0553704

[30] D. Goss, On a new type of L-function for algebraic curves over finite fields. *Pacific J. Math.* **105** (1983), no. 1, 143–181 Zbl 0571.14010 MR 0688411

[31] D. Goss, L-series of t-motives and Drinfel'd modules. In *The arithmetic of function fields (Columbus, OH, 1991)*, pp. 313–402, Ohio State Univ. Math. Res. Inst. Publ. 2, de Gruyter, Berlin, 1992 Zbl 0806.11028 MR 1196527

[32] D. Goss, Drinfel'd modules: Cohomology and special functions. In *Motives (Seattle, WA, 1991)*, pp. 309–362, Proc. Sympos. Pure Math. 55, American Mathematical Society, Providence, RI, 1994 Zbl 0827.11035 MR 1265558

[33] D. Goss, *Basic structures of function field arithmetic*. Ergeb. Math. Grenzgeb. (3) 35, Springer, Berlin, 1996 Zbl 0874.11004 MR 1423131

[34] Y. Hamahata, Tensor products of Drinfel'd modules and v-adic representations. *Manuscripta Math.* **79** (1993), no. 3-4, 307–327 Zbl 0797.11055 MR 1223025

[35] U. Hartl and A.-K. Juschka, Pink's theory of Hodge structures and the Hodge conjecture over function fields. In *t-motives: Hodge structures, transcendence and other motivic aspects*, pp. 31–182, EMS Ser. Congr. Rep., European Mathematical Society, Berlin, 2020 Zbl 1451.14013 MR 4321965

[36] L.-C. Hsia and J. Yu, On characteristic polynomials of geometric Frobenius associated to Drinfeld modules. *Compos. Math.* **122** (2000), no. 3, 261–280 Zbl 0965.11026 MR 1781330

[37] W.-C. Huang and M. A. Papanikolas, Convolutions of Goss and Pellarin L-series. [v1] 2022, [v2] 2025, arXiv:2206.14931v2

[38] C. Khaochim, *Rigid analytic trivializations and periods of Drinfeld modules and their tensor products*. Ph.D. thesis, Texas A&M University, 2021

[39] C. Khaochim and M. A. Papanikolas, Effective rigid analytic trivializations for Drinfeld modules. *Canad. J. Math.* **75** (2023), no. 3, 713–742 Zbl 1527.11046 MR 4586830

[40] D. E. Littlewood, *The theory of group characters and matrix representations of groups*. AMS Chelsea, Providence, RI, 2006 Zbl 1090.20001 MR 2213154

[41] A. Maurischat, Abelian equals A-finite for Anderson A-modules. [v1] 2021, [v3] 2024, arXiv:2110.11114v3

[42] C. Namoijam and M. A. Papanikolas, Hyperderivatives of periods and quasi-periods for Anderson t-modules. *Mem. Amer. Math. Soc.* **302** (2024), no. 1517, v+121 Zbl 07946547 MR 4813037

[43] O. Ore, On a special class of polynomials. *Trans. Amer. Math. Soc.* **35** (1933), no. 3, 559–584 Zbl 59.0163.02 MR 1501703

[44] F. Pellarin, Aspects de l'indépendance algébrique en caractéristique non nulle (d'après Anderson, Brownawell, Denis, Papanikolas, Thakur, Yu, et al.). *Astérisque* **317** (2008), 205–242 Zbl 1185.11048 MR 2487735

[45] F. Pellarin, Values of certain L-series in positive characteristic. *Ann. of Math. (2)* **176** (2012), no. 3, 2055–2093 Zbl 1336.11064 MR 2979866

[46] B. Poonen, Fractional power series and pairings on Drinfeld modules. *J. Amer. Math. Soc.* **9** (1996), no. 3, 783–812 Zbl 0861.13010 MR 1333295

[47] J. H. Silverman, *Advanced topics in the arithmetic of elliptic curves*. Grad. Texts in Math. 151, Springer, New York, 1994 Zbl 0911.14015 MR 1312368

[48] R. P. Stanley, *Enumerative combinatorics. Vol. 2*. Cambridge Stud. Adv. Math. 62, Cambridge University Press, Cambridge, 1999 Zbl 0928.05001 MR 1676282

[49] L. Taelman, Special L-values of t-motives: A conjecture. *Int. Math. Res. Not. IMRN* **2009** (2009), no. 16, 2957–2977 Zbl 1236.11082 MR 2533793

[50] L. Taelman, A Dirichlet unit theorem for Drinfeld modules. *Math. Ann.* **348** (2010), no. 4, 899–907 Zbl 1217.11062 MR 2721645

[51] L. Taelman, Special L-values of Drinfeld modules. *Ann. of Math. (2)* **175** (2012), no. 1, 369–391 Zbl 1323.11039 MR 2874646

[52] T. Takahashi, Good reduction of elliptic modules. *J. Math. Soc. Japan* **34** (1982), no. 3, 475–487 Zbl 0476.14010 MR 0659616

[53] D. S. Thakur, *Function field arithmetic*. World Scientific, River Edge, NJ, 2004 Zbl 1061.11001 MR 2091265

Communicated by Takeshi Saito

Received 1 August 2024; revised 27 March 2025.

Wei-Cheng Huang
Department of Mathematics, University of Rochester, 500 Joseph C. Wilson Blvd., Rochester, NY 14627, USA; w.huang@rochester.edu