The Hilbert symbol in the Hodge standard conjecture

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Abstract. We study the Hodge standard conjecture for varieties over finite fields admitting a CM lifting, such as abelian varieties or products of K3 surfaces. For those varieties we show that the signature predicted by the conjecture holds true modulo 4. This amounts to determining the discriminant and the Hilbert symbol of the intersection product. The first is obtained by ℓ-adic arguments whereas the second needs a careful computation in p-adic Hodge theory.

To the memory of Jean-Pierre Wintenberger

1. Introduction

The standard conjecture of Hodge type predicts the signature of the intersection product of algebraic classes on a smooth projective variety. In this paper we study the discriminant and the Hilbert symbol of the intersection product and we show that they coincide with those predicted by the conjecture for varieties over finite fields which admit a CM lifting, in particular for abelian varieties and for products of K3 surfaces. This can be reformulated by saying that the expected signature holds true modulo 4.

This conjecture was formulated by Grothendieck in the sixties [23]. It is modelled on positivity results such as the Hodge index theorem, the Hodge–Riemann bilinear relations and the positivity of the Rosati involution. For a panorama on the history, the original motivation and potential applications of this conjecture, see [2, 4, 26]. This conjecture did not progress until the 21st century. Then, Milne showed that the classical Hodge conjecture for complex abelian varieties would imply the Hodge standard conjecture for abelian varieties in positive characteristic [36] and Ito studied the behaviour of this conjecture under blow-ups [27].

Very recently, some progress has been made. The first author proved the conjecture for motives of rank two using p-adic Hodge theory [2, Theorem 8.1]. This

result implies the conjecture for abelian fourfolds [2, Theorem 1.3] and for some other abelian varieties [1, 33]. On the other hand, Ito–Ito–Koshikawa proved the conjecture for the square of a K3 surface, using the Kuga–Satake construction and ultimately relying on the positivity of the Rosati involution [26].

The purpose of this article is to push the p-adic methods initiated in [2] to CM motives of higher rank and to describe completely the Hilbert symbol of the intersection product. To be more precise, let us recall the formulation of the standard conjecture of Hodge type.

Definition 1.1. Let X be a smooth, projective and geometrically connected variety of dimension d and let L be a hyperplane section of X. We denote by $\mathcal{Z}^n_{\mathrm{num}}(X)_{\mathbb{Q}}$ the finite-dimensional $\mathbb{Q}$-vector space of $\mathbb{Q}$-algebraic cycles on X of codimension n modulo numerical equivalence. For $n \le d/2$, we define the space of primitive cycles as

$$\mathcal{Z}^{n,\mathrm{prim}}_{\mathrm{num}}(X, L)_{\mathbb{Q}} := \left\{ \alpha \in \mathcal{Z}^n_{\mathrm{num}}(X)_{\mathbb{Q}},\ \alpha \cdot L^{d-2n+1} = 0 \text{ in } \mathcal{Z}^{d-n+1}_{\mathrm{num}}(X)_{\mathbb{Q}} \right\}$$

and we define the pairing $q_{X,L,n} : \mathcal{Z}^{n,\mathrm{prim}}_{\mathrm{num}}(X, L)_{\mathbb{Q}} \times \mathcal{Z}^{n,\mathrm{prim}}_{\mathrm{num}}(X, L)_{\mathbb{Q}} \to \mathbb{Q}$ via the intersection product

$$\alpha, \beta \mapsto (-1)^n \alpha \cdot \beta \cdot L^{d-2n}.$$

Conjecture 1.2 (Hodge standard conjecture, [23, Conjecture 2]). *Let X be a smooth, projective and geometrically connected variety of dimension d and let L be a hyperplane section of X. Then for all $n \le d/2$, the quadratic form $q_{X,L,n}$ is positive definite.*

Remark 1.3. The original formulation of [23] is with cycles modulo homological equivalence. As another standard conjecture predicts that homological and numerical equivalence should coincide, the two formulations should be equivalent. See also [2, Proposition 3.11 and Corollary 3.12].

Our main result is the following.

Theorem 1.4. *The following statements hold.*

(1) *Let (X, L) be an abelian variety over a finite field together with a hyperplane section induced by a CM lifting. Let $n \le \dim(X)/2$ be any integer and (s_+, s_-) be the signature of $q_{X,L,n}$. Then s_- is divisible by 4.*

(2) *Let $X = S_1 \times \cdots \times S_m$ be the product of K3 surfaces over a finite field, L_i be a hyperplane section induced by a CM lifting of S_i (cf. [25]), and let $L = \bigoplus_i p_i^* L_i$ be the induced hyperplane section on X. Let $n \le \dim(X)/2 = m$ be any integer and (s_+, s_-) be the signature of $q_{X,L,n}$. Then s_- is divisible by 4.*

The above theorem follows from a general result on motives endowed with quadratic forms (see Theorem 3.2 for the precise definitions).

Theorem 1.5. *Let M be a CM motive in mixed characteristic with coefficients in $\mathbb{Q}$. Suppose that its special fiber M_p is supersingular and that M is endowed with a CM quadratic form q whose Betti realization is a polarisation of the underlying Hodge structure. Let q_Z be the restriction of q to all algebraic classes of M_p. Write (s_+, s_-) for the signature of q_Z, then s_- is divisible by 4.*

Let us give a sketch of the proof of Theorem 1.5. For a quadratic form over $\mathbb{Q}$, the condition $4|s_-$ is equivalent to the fact that the discriminant is positive and that the Hilbert symbol at infinity is $+1$. Both conditions can be checked by studying the quadratic form in the non-archimedean places, thanks to the classical product formula on Hilbert symbols. In particular, Theorem 1.5 boils down to the study of the quadratic form q_Z after its ℓ-adic and crystalline realization. The ℓ-adic computation is obtained from general results on ℓ-adic cohomology, and it is sufficient to prove that the discriminant is positive, as in [2]. Instead, to compute the Hilbert symbol, we need to control the crystalline contribution. Our arguments rely heavily on p-adic Hodge theory. Let us give the mainlines below.

We begin by extending the coefficients of the motive M from $\mathbb{Q}$ to $\mathbb{Q}_p$. Precisely, let F be the CM number field acting on M, the motive $M \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is endowed with an action of $F \otimes_{\mathbb{Q}} \mathbb{Q}_p$. Such an action decomposes the motive into a sum of direct factors and we study them separately. Thus, we reduce to the situation where the motive M has an action of a finite extension of $\mathbb{Q}_p$, which we will denote again by F, endowed with a non-trivial involution $*\colon F \to F$. The crystalline and p-adic étale realisations of M are endowed by hypothesis with a non-degenerate quadratic form. Faltings's crystalline comparison theorem [17] allows to compare these realisations via a matrix of periods laying in Fontaine's ring B_{cris} of p-adic periods. By using the CM action we characterize this matrix by a single invertible period $\lambda \in B_{\mathrm{cris}, F}^{\times}$, where $B_{\mathrm{cris}, F}$ is the smallest subring of B_{dR} containing B_{cris} and F. This period is unique up to a scalar in $F^{\times}$ and it controls completely the arithmetic of our problem. Namely it produces a "re-normalisation factor" $\lambda \lambda^*$ belonging to $F_0^{\times}$, where F_0 denotes the subfield of F of elements fixed by $*$. Thanks to a theorem of Milnor on CM quadratic forms on p-adic fields [37], we show that our problem is equivalent to control precisely whether the re-normalisation factor is a norm of an element of $F^{\times}$. We verify that this problem is multiplicative with respect to the tensor product of motives: hence, we are reduced to treat the case of small Hodge weights. For those motives we need to prove that the re-normalisation factor $\lambda \lambda^*$ is not a norm.

This is the crucial point of the article. Even if the definition of the period λ, based on the crystalline comparison theorem, is not explicit, this period satisfies a strong uniqueness property with respect to the behaviour under the Frobenius and the de Rham filtration. Such a uniqueness follows from the fundamental exact sequences in p-adic Hodge theory. Therefore, we can describe λ through any other periods with

similar behaviour. For this purpose, we use Lubin–Tate periods of Colmez [13] to get some control of λ. It turns out that such a control is precisely what we need to apply a theorem of Dwork [15] in order to compute the image of the re-normalisation factor $\lambda\lambda^*$ through the local reciprocity map $F_0^\times \to \mathrm{Gal}(F/F_0)$. As the kernel of the reciprocity map is the group of norms $N_{F/F_0}(F^\times)$, we obtain that $\lambda\lambda^*$ is not a norm.

Remark 1.6. The main result of [2] is an instance of Theorem 1.5, namely the case where the motive M has rank two. In that particular case the field F is a quadratic extension of $\mathbb{Q}_p$. As there are finitely many such extensions, it was possible in [2, Section 12] to compute the period λ case by case. Moreover, those periods could be described in terms of Lubin–Tate periods and elements algebraic over F.

In higher rank, F is a quadratic extension of F_0, whose degree over $\mathbb{Q}_p$ is equal to the half of the rank of the motive, with a priori no restriction on the ramification of $F_0/\mathbb{Q}_p$ nor of F/F_0. Hence an analysis case by case is impossible. Moreover, the period λ is computed in terms of Lubin–Tate periods and a non-explicit element in $\widehat{F}^{\mathrm{nr}}$, the completion of the maximal unramified extension of F. In general such an element is transcendental over F; see Remark 9.6.

Organisation of the paper

We begin by setting up notation and conventions in Section 2. In Section 3 we give the precise statement of Theorem 1.5 and explain how to deduce Theorem 1.4 from it. In Section 4 we recall classical results on quadratic forms and use them, combined with general theorems on ℓ-adic cohomology, to translate Theorem 1.5 into a p-adic question. Section 5 presents the first reduction step. We extend the coefficients of the motive from $\mathbb{Q}$ to $\mathbb{Q}_p$ and decomposes it into a sum of direct factors. The goal of the section is to show that it is enough to study each factor separately.

In the rest of the paper we fix one of these factors. It is endowed with the action of a finite extension F of $\mathbb{Q}_p$. Section 6 recalls classical results on CM quadratic forms over the p-adic numbers. At the end of the section we define the re-normalisation factor. This is used in Section 7 where we study the behaviour of the problem under tensor product. This allows to reduce the problem to motives whose crystalline realization has small Hodge weights. Section 8 gives a characterization of the p-adic periods of those motives. This allows in Section 9 to describe them explicitly by using Lubin–Tate periods and to apply Dwork's theorem computing the local reciprocity.

In Section 10 we put all the ingredients together and give the proof of the main result. This section can also be read first in order to have a global picture of the strategy.

We finish with an appendix concerning the filtered φ-modules attached to a Lubin–Tate character of a p-adic field. We added it to keep the paper as self-contained as possible, by giving an alternative construction of Lubin–Tate periods of [13].

2. Conventions

Throughout the paper we will work with the following notation and conventions.

2.1. Involutions

We usually denote by $*$ an involution acting on a set. By abuse of notation we will still write $*$ for an endomorphism of a ring B extending an involution of a subfield $F \subset B$. We will denote by F_0 the subfield of F where $*$ acts as the identity.

We will often write z^* for the image $*(z)$ of an element z through $*$.

2.2. p-adic fields

Let p be a prime number. In this text, by p-adic field we mean a finite extension of the field of p-adic numbers $\mathbb{Q}_p$. For such a field L, we denote by $\mathcal{O}_L$ its ring of integers and by k_L the residue field. The degree of k_L over $\mathbb{F}_p$ is called the residual degree of $L/\mathbb{Q}_p$ and it will be denoted by f_L. We denote[1] by L_a the maximal subfield of L unramified over $\mathbb{Q}_p$. It is equal to the field of fractions of the ring of Witt vectors $W(k_L)$. The degree $e_L := [L : L_a]$ is called the ramification index of $L/\mathbb{Q}_p$, or the *absolute* ramification index of L. When there is no risk of confusion, we may drop the subscript L in the notation.

We choose an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and we denote by $\overline{\mathbb{F}}_p$ its residue field. We put $\hat{\mathbb{Q}}_p^{\mathrm{nr}} := \mathrm{Frac}\, W(\overline{\mathbb{F}}_p)$. It is a completion of the maximal unramified extension $\mathbb{Q}_p^{\mathrm{nr}}$ of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$. The fields $\hat{\mathbb{Q}}_p^{\mathrm{nr}}$ and $\mathbb{Q}_p^{\mathrm{nr}}$ are endowed with an automorphism φ, called the absolute Frobenius, which is the unique map lifting the p-power map on the residue field $\overline{\mathbb{F}}_p$. For any subextension L of $\overline{\mathbb{Q}}_p/\mathbb{Q}_p$ we put $G_L := \mathrm{Gal}(\overline{\mathbb{Q}}_p/L)$.

2.3. Motives

We will work with the category $\mathcal{M}ot(S)_{\mathbb{Q}}$ of homological motives over a base S with coefficients in $\mathbb{Q}$. This category is defined as the quotient (in the sense of [5, §2.3]) of the category of Chow motives over S (cf. [39, §5.1]) with respect to the homological equivalence.

In our article the base S will be $\mathbb{C}$, a finite field or the ring of integers of a p-adic field. For generalities on $\mathcal{M}ot(S)_{\mathbb{Q}}$ when S is a field, we refer to [4, §4]. When S is

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[1]In the literature, L_a is often denoted with subscript 0 instead of a, but the former is already used in the context of involutions, as Convention 2.1. The subscript "a" stands for "absolute unramified" as L_a is also called the absolute unramified subextension of L.

the ring of integers of a p-adic field, see also the conventions in [3]. In general these categories depend on the chosen Weil cohomology, but we will use only cohomologies for which the classical comparison theorems will ensure that those are independent of the choice.

We will also use the quotient category of motives with respect to numerical equivalence and denote it by $\mathrm{NUM}(S)_{\mathbb{Q}}$. We will use it only with S being a finite field.

2.4. Realisations

We will make use of the classical realization functors, namely the de Rham realization R_{dR}, the Betti realization R_B, the ℓ-adic realization R_ℓ and the crystalline realization R_{cris}. We consider them with their enriched structures, as in [4, §7.1]. In particular, the functor R_{dR} will land in the category of filtered vector spaces, R_B in the category of Hodge structures, R_ℓ in the category of Galois representations and R_{cris} in the category of modules endowed with an action of an absolute Frobenius.

For a motive M, we will denote by $\dim M$ the dimension of any classical realization of M and we call it the dimension of the motive.

2.5. Unit object

The unit object in any tensor category we will consider will be denoted by $\mathbb{1}$.

2.6. Filtered Modules

Let M be a module over some ring and $(\mathrm{Fil}^i M)_{i \in \mathbb{Z}}$ a decreasing, exhaustive and separated filtration by submodules. We will use the following conventions: for any subset $S \subset M$, we put

$$v_{\mathrm{dR}}(S) := \sup\{i \in \mathbb{Z} \mid S \subset \mathrm{Fil}^i M\}.$$

For m in M, we write $v_{\mathrm{dR}}(m) := v_{\mathrm{dR}}(\{m\})$. (The subscript dR comes from the fact that the filtrations in this article will be the de Rham filtration on some cohomology group or on some ring of periods.)

2.7. p-adic Hodge theory

We denote Fontaine's rings of periods (associated with $\overline{\mathbb{Q}}_p/\mathbb{Q}_p$) by B_{dR} and $\mathrm{B}_{\mathrm{cris}}$. They were introduced in [18, 19]. We gather below some properties of these rings that we will use in this article. For a detailed account on these rings and on p-adic Hodge theory we refer for example to [21, 45].

(1) B_{dR} is a complete discrete valuation field, its ring of integer is denoted by B_{dR}^+ and the residual field identifies to the p-adic completion $\mathbb{C}_p$ of $\overline{\mathbb{Q}}_p$.

(2) B_{dR} is filtered by the (fractional) powers of the maximal ideal of B_{dR}^+, and we denote this filtration by $(\mathrm{Fil}^i B_{dR})_{i \in \mathbb{Z}}$. The map v_{dR} defined in Section 2.6 on B_{dR} is its discrete valuation.

(3) B_{dR} is endowed with an action of $G_{\mathbb{Q}_p}$. We have canonical inclusions $\overline{\mathbb{Q}}_p \subset B_{dR}$ and $\widehat{\mathbb{Q}}_p^{nr} \subset B_{dR}$ compatible with the action of $G_{\mathbb{Q}_p}$.

(4) $B_{cris} \subset B_{dR}$ is a sub-$\widehat{\mathbb{Q}}_p^{nr}$-algebra stable under the $G_{\mathbb{Q}_p}$-action.

(5) We have an endomorphism

$$\varphi_{cris} \colon B_{cris} \to B_{cris},$$

semi-linear with respect to the absolute Frobenius φ of $\widehat{\mathbb{Q}}_p^{nr}$. We call it the Frobenius of B_{cris} and denote it simply by φ.

(6) There exists an element $t \in B_{cris}$ such that $\varphi(t) = pt$ and $t \in \mathrm{Fil}^1 B_{dR}$ (actually, $\mathrm{Fil}^1 B_{dR} = t B_{dR}^+$). For any g in $G_{\mathbb{Q}_p}$, we have $g(t) = \chi(g)t$, where $\chi \colon G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ denotes the cyclotomic character. We set $\mathbb{Q}_p(1) := \mathbb{Q}_p \cdot t \subset B_{cris}$.

Let $L \subset \overline{\mathbb{Q}}_p$ be a finite extension over $\mathbb{Q}_p$. Keep the notation of Section 2.2.

(7) We will denote by $B_{cris,L}$ the smallest subring of B_{dR} containing B_{cris} and L. We recall that the natural map

$$B_{cris} \otimes_{L_a} L \xrightarrow{\;\simeq\;} B_{cris,L} \subset B_{dR}$$

is an isomorphism.

(8) The Frobenius of B_{cris} does not extend to $B_{cris,L}$ in general, nevertheless its power $\varphi_{cris}^{f_L}$ is L_a-linear, hence we can extend it to $B_{cris,L}$ by $\varphi_{cris}^{f_L} \otimes \mathrm{Id}_L$. We will denote it still by $\varphi_{cris}^{f_L}$ or φ^{f_L}. Again, we may drop the subscript L in the notation if there is no risk of confusion.

3. Main statements

In this section we state our main result (Theorem 3.2) and then give some geometric consequences. To put the result into perspective, we formulate a conjecture (Conjecture 3.1) and discuss the role of the different hypothesis (Remark 3.3).

Conjecture 3.1. *Let K be a p-adic field, $\mathcal{O}_K$ its ring of integers and k its residue field. Let (M, q) be a motive in mixed characteristic $M \in \mathcal{M}\mathrm{ot}(\mathcal{O}_K)_{\mathbb{Q}}$ together with a map $q \colon \mathrm{Sym}^2 M \to \mathbb{1}$. Write $\cdot_{/k}$ for the restriction functor to the category $\mathcal{M}\mathrm{ot}(k)_{\mathbb{Q}}$ and R_B for the Betti realization induced by a fixed embedding $\sigma \colon K \hookrightarrow \mathbb{C}$.*

Define two $\mathbb{Q}$-quadratic spaces (V_B, q_B) and (V_Z, q_Z) as follows:

$$(V_B, q_B) := R_B(M, q), \quad V_Z := \mathrm{Hom}_{\mathrm{NUM}(k)_{\mathbb{Q}}}(\mathbb{1}, M_{/k}),$$

and

$$q_Z \colon V_Z \to \mathrm{End}(\mathbb{1}) = \mathbb{Q} \quad \text{defined by } z \mapsto q_{/k} \circ \mathrm{Sym}^2(z).$$

We conjecture that if q_B is a polarization of Hodge structures then q_Z is positive definite.

Theorem 3.2. *Let M, q, q_B and q_Z be as in Conjecture 3.1. Let F be a number field which acts on M and which is endowed with a non-trivial involution $*$. Assume the following:*

(1) *The equality $\dim_{\mathbb{Q}} V_Z = \dim M$ holds.*

(2) *The equality $[F : \mathbb{Q}] = \dim M$ holds.*

(3) *For every z in F the adjoint of z with respect to q is z^*.*

(4) *The pairing q_B on V_B is a polarization of Hodge structures.*

Then the signature (s_+, s_-) of the quadratic form q_Z satisfies $4 | s_-$.

The proof of Theorem 3.2 is given in Section 10.

Remark 3.3. Hypothesis (4) is crucial and absolutely necessary. Indeed, in order to conclude a positivity statement one has to assume some positivity property. The action of F makes hypothesis (1) essentially automatic. Indeed if $\dim V_Z \neq \dim M$, then $V_Z = 0$ and the statement is trivial. See [2, proof of Proposition 6.8] for details.

The statement should be true without hypotheses (2) and (3); see Conjecture 3.1. On the other hand, the standard conjecture of Hodge type can be reduced to the case of varieties over finite fields [2, Proposition 3.16] and for such a variety the Tate conjecture predicts that its motive should be a motive of abelian type [35]. In particular, the hypothesis of Theorem 3.2 should not be restrictive in the study of the standard conjecture of Hodge type.

Finally, homological and numerical equivalence should always coincide hence one would like to replace V_Z with the space of cycles modulo homological equivalence. The following proposition is the crucial reason for which one needs to work with numerical equivalence.

Lemma 3.4. *Keep notation from Theorem 3.2. Then hypothesis (1) is equivalent to the existence of an isomorphism $M_{/k} \cong \mathbb{1}^{\oplus \dim M}$ of homological motives. In particular, V_B and all classical realizations of M have weight zero and if E is the field of coefficients of a given realization R, then*

$$R(M_{/k}) = V_Z \otimes_{\mathbb{Q}} E \quad \text{and} \quad R(q_{/k}) = q_Z \otimes_{\mathbb{Q}} E.$$

Proof. As numerical motives form a semi-simple category (cf. [28]), hypothesis (1) is equivalent to the existence of an isomorphism $M_{/k} \cong \mathbb{1}^{\oplus \dim M}$ of numerical motives. Let us fix such an isomorphism

$$f \colon M_{/k} \to \mathbb{1}^{\oplus \dim M}$$

and let g be its inverse. Consider a map $\widetilde{f} \colon M_{/k} \to \mathbb{1}^{\oplus \dim M}$ at the level of homological motives whose reduction modulo numerical equivalence is f, and similarly consider $\widetilde{g}$ a lifting of g.

The ring of endomorphisms of the unit object $\mathbb{1}$ is $\mathbb{Q}$, both under numerical or homological equivalence. In particular, passing from homological to numerical equivalence in such an endomorphisms ring does not kill any non-zero map. Hence, the composition $\widetilde{f} \circ \widetilde{g}$ is the identity, as so is $f \circ g$. This implies that the map $\widetilde{g} \circ \widetilde{f}$ is a projector inducing a decomposition of the form

$$M_{/k} \cong \mathbb{1}^{\oplus \dim M} \oplus N$$

for some homological motive N. For dimensional reasons, the realization of N is zero, and so is N by the very definition of homological motives. Conversely, the existence of a homological isomorphism $M_{/k} \cong \mathbb{1}^{\oplus \dim M}$ clearly implies hypothesis (1). ∎

Let us now conclude the section with some applications of Theorem 3.2.

Corollary 3.5. *Let A be an abelian variety over a finite field, $\widetilde{A}$ be a CM-lifting, $\widetilde{L}$ be a hyperplane section of $\widetilde{A}$ and L be the restriction of $\widetilde{L}$ to A. For a positive integer $n \leq \dim(A)/2$, let (s_+, s_-) be the signature of the quadratic form $q_{A,L,n}$ from Definition 1.1, then s_- is divisible by 4.*

Proof. We argue as in [2, §8] and use the complex multiplication to decompose the motive of A in an orthogonal sum of smaller motives. Among these factors, the ones we need to study are the so-called exotic. We can conclude as they all fit in the hypotheses of Theorem 3.2. ∎

Remark 3.6. A priori Corollary 3.5 does not apply to a hyperplane sections L which does not lift to characteristic zero. Nevertheless, it is possible sometimes to extend it to all hyperplane sections, as for example in [2, Proposition 3.15] and [33, Lemma 2.2].

Corollary 3.7. *Let $X = S_1 \times \cdots \times S_m$ be the product of K3 surfaces over a finite field, L_i be a hyperplane section induced, as in Corollary 3.5, by a CM lifting of S_i, and let $L = \bigoplus_i p_i^* L_i$ be the induced hyperplane section on X. Let $n \leq \dim(X)/2 = m$ be any integer and (s_+, s_-) be the signature of the quadratic form $q_{X,L,n}$ from Definition 1.1, then s_- is divisible by 4.*

Proof. The argument follows the same lines as in [2, §8], namely we use the complex multiplication to decompose the motive into summands to which we can apply Theorem 3.2.

First, each S_i has a CM lifting $\widetilde{S}_i$ by [25]. By [29], the motive $\mathfrak{h}(\widetilde{S}_i)$ of $\widetilde{S}_i$ admits a decomposition

$$\mathfrak{h}(\widetilde{S}_i) = \mathbb{1} \oplus \mathbb{1}(-2) \oplus \mathbb{1}(-1)^{\oplus \rho_i} \oplus \mathfrak{h}^t(\widetilde{S}_i),$$

where ρ_i is the Picard number of $\widetilde{S}_i$ and $\mathfrak{h}^t(\widetilde{S}_i)$ is the motive whose realization is the transcendental part $H^{2,t}(\widetilde{S}_i)$ of the Hodge structure $H^2(\widetilde{S}_i)$.

Now by [10, Corollary 1.3], there is a CM field L_i acting on $\mathfrak{h}^t(\widetilde{S}_i)$ with the property that the $\mathbb{Q}$-dimensions of L_i and $H^{2,t}(\widetilde{S}_i)$ are the same. Hence, after extending the scalars to $\overline{\mathbb{Q}}$, the motive $\mathfrak{h}^t(\widetilde{S}_i)$ decomposes into a sum of motives of rank one; see [2, Proposition 6.6]. By the Künneth formula, such a decomposition in motives of rank one holds true for the whole motive of X, see [2, Proposition 6.7(1)]. By taking the orbits under the Galois action on the coefficients, we can descend such a decomposition into a decomposition with $\mathbb{Q}$-coefficients, see [2, Proposition 6.7(2)]. Notice that the factors are not anymore of rank one in general but the decomposition is orthogonal, hence it is enough to work with a single factor M.

Assume that $M_{/k}$ contains an algebraic class which is non-zero modulo numerical equivalence (otherwise the factor has no interest for the quadratic form $q_{X,L,n}$). Then M satisfies the hypothesis (1) of Theorem 3.2; see [2, Proposition 6.8]. By construction the hypotheses (2) and (3) are verified for the motives $\mathfrak{h}^t(\widetilde{S}_i)$ and one can check that they are stable under the tensor operations above. Finally, the hypothesis (4) comes from the Hodge–Riemann bilinear relations. We can then apply Theorem 3.2 and deduce the desired conclusion for the factor M. ∎

Remark 3.8. There are probably other varieties to which Theorem 3.2 can be applied. As already mentioned in Remark 3.3, all motives over a finite field should be of abelian type, hence they should have a CM lifting [35]. Some other examples for which this is known are cubic Fermat hypersurfaces [44]. In order to apply Theorem 3.2 to them one has to check that the decomposition induced by the CM action is orthogonal with respect to the quadratic form $q_{X,L,n}$. Similarly, one might try to study Kummer varieties or the Hilbert scheme of points on a K3.

4. From archimedian to p-adic

We recall here some classical facts on quadratic forms. They will allow us to reduce Theorem 3.2 to a p-adic statement. In what follows $\mathbb{Q}_\nu$ denotes the completion of $\mathbb{Q}$ at the place ν. Recall that at every place one defines $\varepsilon_\nu(q) \in \{\pm 1\}$, the Hilbert symbol

(or Hasse symbol) of a non-degenerate $\mathbb{Q}$-quadratic form q at ν; cf. [42, Ch. IV, §§2.1 and 2.4].

Proposition 4.1 ([42, Ch. IV, §2.4]). *Let q be a non-degenerate $\mathbb{Q}$-quadratic form and let (s_+, s_-) be its signature. Then the discriminant is positive if and only if $2|s_-$. In this case, $4|s_-$ if and only if $\varepsilon_{\mathbb{R}}(q) = +1$.*

Proposition 4.2 ([42, Ch. IV, §2.3, Theorem 7]). *Two non-degenerate $\mathbb{Q}_p$-quadratic forms q_1 and q_2 of same rank are isomorphic if and only if the discriminants of q_1 and q_2 coincide and $\varepsilon_p(q_1) = \varepsilon_p(q_2)$.*

Theorem 4.3 ([42, Ch. IV, §3.1]). *Let q be non-degenerate $\mathbb{Q}$-quadratic form. Then for all but a finite number of places ν the equality $\varepsilon_\nu(q) = +1$ holds. Moreover, the following product formula running on all places holds:*

$$\prod_\nu \varepsilon_\nu(q) = +1.$$

Corollary 4.4. *Let q_1 and q_2 be two non-degenerate $\mathbb{Q}$-quadratic forms and let p be a prime number. Suppose that, for all primes ℓ different from p, we have*

$$q_1 \otimes \mathbb{Q}_\ell \cong q_2 \otimes \mathbb{Q}_\ell.$$

Then the two quadratic forms q_1 and q_2 have the same discriminant. Moreover, if this discriminant is positive, then $4|s_-(q_1)$ if and only if the following equality holds:

$$\varepsilon_p(q_1) = \varepsilon_{\mathbb{R}}(q_2)\varepsilon_p(q_2).$$

Proof. The ℓ-adic hypothesis implies in particular that the discriminants of q_1 and q_2 coincide in $\mathbb{Q}_\ell^\times/(\mathbb{Q}_\ell^\times)^2$ for all $\ell \neq p$. This implies that they coincide in $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ by [24, §5.2, Theorem 3]. Suppose from now on that the discriminant is positive. By Proposition 4.1, $4|s_-(q_1)$ if and only if $\varepsilon_{\mathbb{R}}(q_1) = +1$.

Now, Theorem 4.3 implies that

$$\prod_\nu \varepsilon_\nu(q_1) = 1 = \prod_\nu \varepsilon_\nu(q_2).$$

Combining this with the ℓ-adic isomorphisms, we deduce

$$\varepsilon_{\mathbb{R}}(q_1)\varepsilon_p(q_1) = \varepsilon_{\mathbb{R}}(q_2)\varepsilon_p(q_2).$$

This means that $\varepsilon_{\mathbb{R}}(q_1) = +1$ if and only if $\varepsilon_p(q_1) = \varepsilon_{\mathbb{R}}(q_2)\varepsilon_p(q_2)$. ∎

Proposition 4.5. *Let us keep notation from Theorem* 3.2. *Let* p *be the characteristic of* k *and* h_i *be the dimension of the* $(i, -i)$-*part of the Hodge structure* V_B. *Define the positive integer*

$$s_M := \sum_{i \geq 1,\, \text{odd}} h_i.$$

Then the quadratic forms q_B *and* q_Z *have the same discriminant, which is positive. Moreover, Theorem* 3.2 *holds true if and only if the following equality holds true:*

$$\varepsilon_p(q_Z)/\varepsilon_p(q_B) = (-1)^{s_M}.$$

Proof. We want to apply Corollary 4.4 to $q_1 = q_Z$ and $q_2 = q_B$.

First of all, by Lemma 3.4, we have that $q_Z \otimes \mathbb{Q}_\ell = R_\ell(q_{/k})$. By Artin comparison theorem, we have $q_B \otimes \mathbb{Q}_\ell = R_\ell(q_{/\mathbb{C}})$. Combining these equalities with smooth proper base change in ℓ-adic cohomology, we deduce that

$$q_B \otimes \mathbb{Q}_\ell = q_Z \otimes \mathbb{Q}_\ell.$$

Now, by hypothesis q_B is a polarization for the Hodge structure V_B and recall that V_B has weight zero (Lemma 3.4). In particular, the Hodge–Riemann relations compute the signature $(s_{B,+}, s_{B,-})$ of q_B as

$$s_{B,+} = \sum_{i \text{ even}} h_i \quad \text{and} \quad s_{B,-} = \sum_{i \text{ odd}} h_i.$$

Note that $s_{B,-}$ is even because of the Hodge symmetry. This implies through Proposition 4.1 that the discriminant is positive and that the Hilbert symbol of $q_B \otimes \mathbb{R}$ is $(-1)^{\sum_{i \geq 1, \text{odd}} h_i}$. We can now conclude by applying Corollary 4.4 to $q_1 = q_Z$ and $q_2 = q_B$. ∎

Remark 4.6. We have

$$\sum_{i \geq 1,\, \text{odd}} h_i \equiv \sum_{i \geq 0} i h_i \quad (\text{mod } 2),$$

in particular the integer s_M has the same parity as the minimum of the Hodge polygon.

The role of the above proposition is to translate a positivity problem (Theorem 3.2) into a p-adic problem. The advantage is that the latter has a cohomological interpretation, as the following proposition shows.

Proposition 4.7. *Let us keep the notation from Theorem* 3.2 *and Section* 2.4. *Then the following hold:*

(1) *The quadratic space* $(V_B, q_B) \otimes_\mathbb{Q} \mathbb{Q}_p$ *is isomorphic to* $(R_p(M), R_p(q))$.

(2) *The Galois representation $R_p(M)$ is crystalline and the admissible filtered-φ-module which corresponds to it is $R_{\mathrm{cris}}(M)$.*

(3) *The $\mathbb{Q}_p$-subspace of Frobenius invariant vectors in $R_{\mathrm{cris}}(M)$ identifies with $V_Z \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and generates $R_{\mathrm{cris}}(M)$.*

(4) *The quadratic form $q_Z \otimes_{\mathbb{Q}} \mathbb{Q}_p$ identifies with the restriction of $R_{\mathrm{cris}}(q)$ to $V_Z \otimes_{\mathbb{Q}} \mathbb{Q}_p$.*

Proof. Point (1) comes from Artin's comparison theorem. Point (2) comes from Falting's comparison theorem in p-adic Hodge theory [17]; see also [2, Theorem 10.2]. For point (3), notice that in general algebraic classes are included in the Frobenius invariant ones, but for dimensional reasons (hypothesis (1) in Theorem 3.2) this inclusion is an equality. Point (4) follows from Lemma 3.4. ∎

5. Reduction to CM-simple objects

In this section we study quadratic forms in the context of p-adic Hodge theory. The setting we choose is inspired by the previous section and in particular by Propositions 4.5 and 4.7.

Definition 5.1. A supersingular pair $(V_{B,p}, V_{Z,p})$ is the collection of the following objects:

(1) A $\mathbb{Q}_p$-vector space $V_{B,p}$ endowed with the action of the absolute Galois group of a p-adic field K which makes the representation crystalline.

(2) A $\mathbb{Q}_p$-vector space $V_{Z,p}$ which generates the (admissible) filtered φ-module corresponding to $V_{B,p}$ and on which the Frobenius acts trivially.

An orthogonal supersingular datum $(V_{B,p}, V_{Z,p}, q_{B,p}, q_{Z,p})$, or simply an orthogonal supersingular pair $(q_{B,p}, q_{Z,p})$ is the collection of the following objects:

(1) A supersingular pair $(V_{B,p}, V_{Z,p})$.

(2) A non-degenerate $\mathbb{Q}_p$-quadratic form $q_{B,p}$ on $V_{B,p}$ for which the Galois action is isometric.

(3) The non-degenerate $\mathbb{Q}_p$-quadratic form $q_{Z,p}$ on $V_{Z,p}$ corresponding to $q_{B,p}$.

Lemma 5.2. *Let $(q_{B,p}, q_{Z,p})$ be an orthogonal supersingular pair. Then the discriminants of $q_{B,p}$ and $q_{Z,p}$ coincide.*

Proof. The quadratic form on $V_{Z,p}$ induces a quadratic form on the one-dimensional vector space $\det V_{Z,p}$. This form is characterized by an element of $\mathbb{Q}_p^{\times}/(\mathbb{Q}_p^{\times})^2$, which is the discriminant of q_Z. (The same holds for $V_{B,p}$.) As the Frobenius acts trivially on $\det V_{Z,p}$, admissibility implies that the filtration is trivial as well. This means that

the comparison isomorphism sends the $\mathbb{Q}_p$-structure $\det V_{Z,p}$ on the $\mathbb{Q}_p$-structure $\det V_{B,p}$. As the comparison isomorphism respects the underlined quadratic forms the discriminants coincide. ∎

Lemma 5.3. *Let $(V_{B,p}, V_{Z,p}, q_{B,p}, q_{Z,p})$ be an orthogonal supersingular datum, let F be a number field and let $*$ be a non-trivial field involution on F. Then the following hold true:*

(1) *The ring $F \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a product of p-adic fields*

$$F \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{i=1}^{n} F_i.$$

(2) *The involution $*$ acts on this product as a composition of disjoint transpositions, i.e. there is an integer s such that, after changing the numbering, one has*

$$\begin{aligned} *(F_i) &= F_{i+s} \quad \text{for } i \leq s, \\ *(F_i) &= F_{i-s} \quad \text{for } s < i \leq 2s, \\ *(F_i) &= F_i \quad\;\; \text{for } i > 2s. \end{aligned}$$

Moreover, the induced involution on F_i for $i > 2s$ is non-trivial.

(3) *Suppose that F acts on $V_{B,p}$ and that this action commutes with the Galois action. Then F acts on $V_{Z,p}$ as well and the two actions are compatible with respect to the comparison theorem. In particular, the decomposition of point (1) induces two decompositions*

$$V_{B,p} = \bigoplus_{i=1}^{n} V_{B,p}^i \quad \text{and} \quad V_{Z,p} = \bigoplus_{i=1}^{n} V_{Z,p}^i,$$

and for all i, the pair $(V_{B,p}^i, V_{Z,p}^i)$ is a supersingular pair.

(4) *Suppose moreover that for all $z \in F$ the adjoint of z with respect to $q_{B,p}$ is z^*. Then the same holds true for $q_{Z,p}$. Moreover, the following decompositions in $(n-s)$ factors are orthogonal:*

$$V_{B,p} = \overset{\perp}{\bigoplus_{s < i \leq 2s}} \left(V_{B,p}^i \oplus V_{B,p}^{i-s}\right) \oplus^{\perp} \overset{\perp}{\bigoplus_{i > 2s}} \left(V_{B,p}^i\right),$$

$$V_{Z,p} = \overset{\perp}{\bigoplus_{s < i \leq 2s}} \left(V_{Z,p}^i \oplus V_{Z,p}^{i-s}\right) \oplus^{\perp} \overset{\perp}{\bigoplus_{i > 2s}} \left(V_{Z,p}^i\right).$$

(5) *Keep the notation and hypothesis from point (4). For all $i > s$, denote by $q_{B,p}^i$ the restriction of $q_{B,p}$ to each orthogonal factor of the above decomposition, and similarly for $q_{Z,p}^i$. Then the pair $(q_{B,p}^i, q_{Z,p}^i)$ is an orthogonal supersingular pair.*

(6) *For $s < i \leq 2s$, consider the orthogonal supersingular pair $(q_{B,p}^i, q_{Z,p}^i)$ defined in point (5). Then the two $\mathbb{Q}_p$-quadratic forms $q_{B,p}^i$ and $q_{Z,p}^i$ are isomorphic. More precisely, the two subspaces $V_{B,p}^i$ and $V_{B,p}^{i-s}$ of $V_{B,p}^i \oplus V_{B,p}^{i-s}$ are maximal isotropic for the quadratic form $q_{B,p}^i$ (and similarly for $q_{Z,p}^i$). Moreover, $q_{B,p}^i$ realizes the Galois representations $V_{B,p}^i$ and $V_{B,p}^{i-s}$ as one the dual of the other.*

Proof. Point (1) comes from the fact that $F/\mathbb{Q}$ is a separable extension.

For point (2), consider $F \otimes_{\mathbb{Q}} \mathbb{Q}_p = F_1 \times \cdots \times F_n$. Its spectrum is a disjoint union of n closed points and $*$ acts on it as a permutation of order two. Consider now $i > 2s$ as in the statement and the inclusion $F \hookrightarrow F_i$ induced from point (1). By construction the involution on F_i extends the one on F, hence it is non-trivial.

For point (3), by the comparison theorem, as F acts on $V_{B,p}$ seen as crystalline Galois representation of G_K, then F must act also on its corresponding filtered φ-module. This means that the action of F must commute with the Frobenius hence F stabilizes the Frobenius invariant part, i.e. $V_{Z,p}$.

Let us now study point (4), let q be one of the quadratic forms we want to study and b be its corresponding bilinear form. Let p_i be the i-th projector in the decomposition $F \otimes_{\mathbb{Q}} \mathbb{Q}_p = F_1 \times \cdots \times F_n$. By hypothesis, we have

$$b\big(p_i(-), p_j(-)\big) = b\big((-), p_i^* p_j(-)\big).$$

But $p_i^* p_j = 0$ except if $p_j = p_i^*$.

Point (5) is automatic from the construction. For point (6) we have to study the quadratic forms on the space $V_{B,p}^i \oplus V_{B,p}^{i-s}$ for $s < i \leq 2s$ (and similarly for $V_{Z,p}^i \oplus V_{Z,p}^{i-s}$). Notice that the proof of point (4) we gave, shows in particular that the two subspaces $V_{B,p}^i$ and $V_{B,p}^{i-s}$ are maximal isotropic. Thus the quadratic forms $q_{B,p}$ and $q_{Z,p}$ are both isomorphic to a sum of hyperbolic planes. The duality statement comes from the fact that $q_{B,p}$ is non-degenerate by hypothesis. ∎

Definition 5.4. Let $M = (q_{B,p}, q_{Z,p})$ be an orthogonal supersingular pair, let $\varepsilon_p(q_{B,p})$ and $\varepsilon_p(q_{Z,p})$ be the Hilbert symbols of the quadratic forms and define s_M to be the minimum of the Hodge polygon of the filtered φ-module underlying M. We say that M is good if the following equality holds

$$\varepsilon_p(q_{Z,p})/\varepsilon_p(q_{B,p}) = (-1)^{s_M}. \tag{5.4.1}$$

Lemma 5.5. *A supersingular pair $M = (q_{B,p}, q_{Z,p})$ is good if and only if one of the following situation happens:*

(1) *The quadratic forms $q_{B,p}$ and $q_{Z,p}$ are isomorphic and s_M is even.*

(2) *The quadratic forms $q_{B,p}$ and $q_{Z,p}$ are not isomorphic and s_M is odd.*

Proof. The two quadratic spaces have the same rank by definition of supersingular pair and the same discriminant by Lemma 5.2. Hence they are isomorphic if and only if they have the same Hilbert symbol (Proposition 4.2). This translates the definition of good to the conditions (1) or (2) of the statement. ∎

Proposition 5.6. *Keep the notation as in Lemma 5.3. If for each $i > 2s$ the pair $(q_{B,p}^i, q_{Z,p}^i)$ is good then the pair $(q_{B,p}, q_{Z,p})$ is good as well.*

Proof. First of all, let us show that the pair $(q_{B,p}^i, q_{Z,p}^i)$ is good also for $s < i \leq 2s$. In this case the two quadratic forms are isomorphic by Lemma 5.3 (6), hence

$$\varepsilon_p(q_{Z,p}^i)/\varepsilon_p(q_{B,p}^i) = 1.$$

We need to show that the height of the corresponding Hodge polygon is even.

Let $D_{Z,p}^i$ be the admissible filtered φ-module associated with the factor $V_{Z,p}^i$. The height of the Newton polygon of each one of those modules is zero by construction as they have a Frobenius invariant basis. In particular, since they are admissible, the height of the Hodge polygon is also zero. Moreover, by Lemma 5.3 (6), $D_{Z,p}^i$ and $D_{Z,p}^{i-s}$ are dual to each other. This duality together with height zero implies that the two Hodge polygons are symmetric to each other. In particular, they have the same minimum, hence the minimum of the Hodge polygon of $D_{Z,p}^i \oplus D_{Z,p}^{i-s}$ is even.

Now, as the right hand side of the equation (5.4.1) is additive on direct sums, let us show that the left hand side is additive as well, this will imply the statement. Consider two orthogonal supersingular pairs M_1 and M_2 which are both partial sums of the factors in the decomposition of Lemma 5.3 (4). Let $\delta_{B,1}, \varepsilon_{B,1}, \delta_{Z,1}, \varepsilon_{Z,1}$ be the discriminants and the Hilbert symbols of the two quadratic forms associated with M_1 and similarly let $\delta_{B,2}, \varepsilon_{B,2}, \delta_{Z,2}, \varepsilon_{Z,2}$ be the invariants for M_2. By the definition of the Hilbert symbol of a quadratic form [42, Ch. IV, §2.1], it follows that the Hilbert symbols of the sum $M_1 \oplus M_2$ are respectively

$$\varepsilon_B(M_1 \oplus M_2) = \varepsilon_{B,1} \cdot \varepsilon_{B,2} \cdot (\delta_{B,1}, \delta_{B,2}),$$
$$\varepsilon_Z(M_1 \oplus M_2) = \varepsilon_{Z,1} \cdot \varepsilon_{Z,2} \cdot (\delta_{Z,1}, \delta_{Z,2}),$$

where (δ_1, δ_2) denotes the Hilbert symbol of a couple of elements $\delta_1, \delta_2 \in \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^2$; cf. [42, Ch. III, §1.1, Theorem 2].

If we write the quotient of the two equalities above the contribution of the discriminants simplifies as $\delta_{B,1} = \delta_{Z,1}$ and $\delta_{B,2} = \delta_{Z,2}$ by Lemma 5.2. ∎

6. CM-quadratic forms over p-adic fields

From now on (except for Section 10), let F be a finite extension of $\mathbb{Q}_p$. We assume that F is endowed with a non-trivial involution $*: F \to F$, $\alpha \mapsto \alpha^*$. We collect here

some basic results on some quadratic forms endowed with an action of F which we call CM (Definition 6.1).

Throughout the section we fix a Galois closure $\widetilde{F} \supset F$ of F, and denote by Γ the set of $\mathbb{Q}_p$-embeddings of F in $\widetilde{F}$. We recall that F_0 denote the subfield of F, where $*$ acts as the identity; cf. Section 2.1. We denote still by $*$ the action of $*$ on Γ by pre-composition. Denote respectively by $\operatorname{tr}_{F_0/\mathbb{Q}_p} \colon F_0 \to \mathbb{Q}_p$ and $N_{F/F_0} \colon F \to F_0$ the trace of the extension $F_0/\mathbb{Q}_p$ and the norm of F/F_0; the extension F/F_0 has degree two and we have $N_{F/F_0}(x) = x^* x$ for every $x \in F$.

Definition 6.1. (1) A CM-space (with respect to F) is the data of a $\mathbb{Q}_p$-vector space V and an action of F on V such that V as an F-vector space has dimension one.

(2) A CM-quadratic space (with respect to F) is the data of a CM-space V and a non-degenerate quadratic form $q \colon V \to \mathbb{Q}_p$ on the $\mathbb{Q}_p$-vector space V, such that for all α in F, the adjoint with respect to q of the multiplication by α is the multiplication by α^*.

For simplicity, we may say that q (or V) is a *CM*-quadratic form without mentioning explicitly the other structures.

Proposition 6.2. *Let V be a CM-quadratic space. Fix a non-zero vector v in V and consider the induced identification $V \cong F$ as F-vector spaces. Under this identification, there exists a unique $a \in F_0^\times$ such that for every $x \in F$, we have*

$$q(x) = \operatorname{tr}_{F_0/\mathbb{Q}_p}\big(a N_{F/F_0}(x)\big).$$

If one changes the choice of the vector v, the element $a \in F_0^\times$ is multiplied by some norm in $N_{F/F_0}(F^\times)$. In particular, the class of a in $F_0^\times / N_{F/F_0}(F^\times)$ is well defined and depends only on V.

Proof. Let $b \colon F \times F \to \mathbb{Q}_p$ the symmetric form associated with q. Since q is a CM-quadratic form, for every $x \in F$, we have

$$q(x) = b(x, x) = b(1, x^* x) = b\big(1, N_{F/F_0}(x)\big).$$

The restriction of $b(1, -)$ to F_0 gives a $\mathbb{Q}_p$-linear form on F_0. As $(u, v) \mapsto \operatorname{tr}_{F_0/\mathbb{Q}_p}(uv)$ is a non-degenerate bilinear form on F_0, we can conclude that there exists such an element $a \in F_0$. It must be non-zero otherwise q would be zero.

Let w be a new non-zero vector in V and $q'(x)$ be the induced quadratic form on F under this new identification. There is a unique non-zero element $z \in F$ such that $zv = w$. This gives the relation $q'(x) = q(zx)$. Hence, one deduces

$$q'(x) = \operatorname{tr}_{F_0/\mathbb{Q}_p}\big(a N_{F/F_0}(zx)\big) = \operatorname{tr}_{F_0/\mathbb{Q}_p}\big((az^* z) \cdot N_{F/F_0}(x)\big). \qquad \blacksquare$$

Definition 6.3. Let q be a CM-quadratic form. We define the gauge of q as the element of $F^\times / N_{F/F_0}(F^\times)$ associated with q through the above proposition.

Theorem 6.4 (Milnor). *The following statements hold.*

(1) *The group $F^\times / N_{F/F_0}(F^\times)$ has cardinality two.*

(2) *All CM-quadratic forms (with respect to the same F) have the same discriminant.*

(3) *Two CM-quadratic forms are isomorphic if and only if they have the same gauge in $F^\times / N_{F/F_0}(F^\times)$.*

(4) *Up to isomorphism there are exactly two CM-quadratic spaces.*

Proof. The extension F/F_0 has degree 2. The reciprocity isomorphism

$$F^\times / N_{F/F_0}(F^\times) \cong \mathrm{Gal}(F/F_0)$$

of local class field theory proves (1).

Consider F as an F_0-vector space of dimension 2 and consider on it the non-degenerate F_0-quadratic forms $\widetilde{q}_a(x) = a N_{F/F_0}(x)$ for $a \in F_0^\times$. By definition a CM-quadratic form q of gauge a is isometric to $\mathrm{tr}_{F_0/\mathbb{Q}_p} \circ \widetilde{q}_a$. Now, notice that all quadratic forms $\widetilde{q}_a$ have the same discriminant in $F_0^\times / (F_0^\times)^2$, independent of a, as $\widetilde{q}_a(x) = a\widetilde{q}_1(x)$ and the quadratic space has dimension 2. (Actually this discriminant is the class of the opposite of the discriminant of the extension F/F_0.) By [37, Lemma 2.2] this implies (2).

For (3) we argue as above and consider the quadratic forms $\widetilde{q}_a$. We claim that $\widetilde{q}_a \cong \widetilde{q}_b$ if and only if a and b are in the same class of $F^\times / N_{F/F_0}(F^\times)$. This will give (3) thanks to [37, Theorem 2.3]. The claim is elementary: if a and b are in the same class, we can write $a/b = N_{F/F_0}(z)$ and the change of variable $x \mapsto z \cdot x$ gives the isometry. Conversely, if the quadratic forms are isomorphic then the images of $F^\times$ by them must be the same, but those images are precisely $a \cdot N_{F/F_0}(F^\times)$ and $b \cdot N_{F/F_0}(F^\times)$. Point (4) is the combination of (1) and (3). ∎

Proposition 6.5. *Let V be a CM-space and consider $V \otimes_{\mathbb{Q}_p} F$. It is endowed with two different actions of F, one on the left, induced by the action of F on V, and one on the right. To distinguish them we will write, for all $w \in V \otimes_{\mathbb{Q}_p} F$ and $z \in F$, $z(w)$ for the left multiplication and $w \cdot z$ for the right multiplication. The inclusion $F \subset \widetilde{F}$ gives a fixed inclusion $V \otimes_{\mathbb{Q}_p} F \subset V \otimes_{\mathbb{Q}_p} \widetilde{F}$ and, for all $\sigma \in \Gamma$, let us denote again by σ the induced embedding $V \otimes_{\mathbb{Q}_p} F \subset V \otimes_{\mathbb{Q}_p} \widetilde{F}$. Then the following hold.*

(1) *There exists a non-zero (eigen)vector $v \in V \otimes_{\mathbb{Q}_p} F$ such that $z(v) = v \cdot z$. Such a v is unique up to (left or right) multiplication by an element of $F^\times$.*

(2) *The (eigen)vectors $\{\sigma(v)\}_{\sigma \in \Gamma}$ form a basis of $V \otimes_{\mathbb{Q}_p} \widetilde{F}$ over $\widetilde{F}$. We set $L_\sigma := \sigma(v) \cdot \widetilde{F}$, which is an eigenline with respect to the embedding σ, i.e. $z(\sigma(v)) = \sigma(v) \cdot \sigma(z)$.*

Proof. Follows immediately from the classical description $F \otimes_{\mathbb{Q}_p} \widetilde{F} = \widetilde{F}^\Gamma$ and the fact that the dimension of V over F is one. ∎

Proposition 6.6. *Let (V, q) be a CM-quadratic space and let $v \in V \otimes_{\mathbb{Q}_p} F$ be the eigenvector constructed in Proposition 6.5 above. Let b be the bilinear pairing induced by q. Then the following hold.*

(1) *One has $b(\sigma(v), \tau(v)) = 0$ except if $\sigma = \tau^*$.*

(2) *The number $2b(v, v^*) \in F_0^\times$ is the gauge of q.*

Proof. For all $z \in F$, we have

$$b\big(\sigma(z(v)), \tau(v)\big) = b\big(\sigma(v), \tau(z^*(v))\big)$$

by definition of CM-quadratic forms. On the other hand, we have the relations

$$b\big(\sigma(z(v)), \tau(v)\big) = b\big(\sigma(v), \tau(v)\big) \cdot \sigma(z),$$
$$b\big(\sigma(v), \tau(z^*(v))\big) = b\big(\sigma(v), \tau(v)\big) \cdot \tau^*(z),$$

by construction of v. These relations together give (1).

For the second point, choose $\{\sigma(v)\}_{\sigma \in \Gamma}$ as basis. A vector $w \in V \otimes_{\mathbb{Q}_p} \widetilde{F}$ is actually in V if and only if there exists a scalar $z \in F$ such that the coordinates of w with respect to this basis are $\{\sigma(z)\}_{\sigma \in \Gamma}$. Let us now compute the quadratic form on such a vector $w \in V$:

$$q(w) = \sum_{\sigma, \tau \in \Gamma} b\big(\sigma(zv), \tau(zv)\big)$$
$$= \sum_{\sigma \in \Gamma} b\big(\sigma(zv), \sigma^*(zv)\big) = \sum_{\sigma \in \Gamma_0} 2\sigma(zz^*)\sigma\big(b(v, v^*)\big),$$

where the second equality comes from part (1) and Γ_0 denotes the set of $\mathbb{Q}_p$-embeddings of F_0 in $\widetilde{F}$. In conclusion, we have

$$q(w) = \operatorname{tr}_{F_0/\mathbb{Q}_p}\big(2b(v, v^*) \cdot N_{F/F_0}(z)\big),$$

which computes the gauge. ∎

Proposition 6.7. *Let V_1 and V_2 be two CM-spaces and let B be an F-algebra. Suppose that it is given an identification of B-modules*

$$V_1 \otimes_{\mathbb{Q}_p} B = V_2 \otimes_{\mathbb{Q}_p} B,$$

which is compatible with the F-actions. Let v_1 and v_2 be the vectors constructed in Proposition 6.5 associated respectively with V_1 and V_2. Then, there exists a unique invertible element $\lambda \in B^\times$ such that $v_1 = \lambda v_2$.

Let now $\widetilde{B}$ be an $\widetilde{F}$-algebra and suppose that each $\sigma \in \Gamma$ extends to a homomorphism $\widetilde{\sigma}: B \to \widetilde{B}$. Then, for all σ, we have

$$\sigma(v_1) = \widetilde{\sigma}(\lambda)\sigma(v_2).$$

Proof. Consider the B-module $V_1 \otimes_{\mathbb{Q}_p} B = V_2 \otimes_{\mathbb{Q}_p} B$ and consider on it the left action of F. By construction, v_1 and v_2 generate the same eigenline with respect to this action, this gives the existence of λ.

For the second part, one applies $\widetilde{\sigma}$ to the equality $v_1 = \lambda v_2$. As the v_i live in $V_i \otimes_{\mathbb{Q}_p} F \subset V_i \otimes_{\mathbb{Q}_p} B$, the action on them coincides with σ. ∎

Proposition 6.8. *Let (V_1, q_1) and (V_2, q_2) be two CM-quadratic forms and let B be an F-algebra. Suppose that it is given an identification of B-modules*

$$V_1 \otimes_{\mathbb{Q}_p} B = V_2 \otimes_{\mathbb{Q}_p} B,$$

which is compatible with the F-actions and with the quadratic forms.

Let $\lambda \in B^\times$ be the scalar constructed in Proposition 6.7 and suppose that the involution $*$ of F extends to an endomorphism $*$ of B as $\mathbb{Q}_p$-algebra. Then $\lambda \cdot *(\lambda)$ belongs to $F_0^\times$ and the two quadratic forms q_1 and q_2 are isomorphic if and only if

$$\lambda \cdot *(\lambda) \in N_{F/F_0}(F^\times). \tag{6.8.1}$$

Remark 6.9. Notice that λ is well determined up to a constant in $F^\times$ since so are v_1 and v_2; cf. Proposition 6.5. Nevertheless, the condition (6.8.1) is independent of such a choice.

Proof. Let v_i be the vectors from Proposition 6.7. By Theorem 6.4, the two quadratic forms are isomorphic if and only if they have the same gauge in $F_0^\times/N_{F/F_0}(F^\times)$. By Proposition 6.6 (2), this means that they are isomorphic if and only if the ratio $2b(v_1, v_1^*)/2b(v_2, v_2^*)$, which belongs to $F_0^\times$, is a norm of an element in F. On the other hand, by Proposition 6.7, this ratio is equal to $\lambda \cdot *(\lambda)$. ∎

7. Reduction to tensor generators

We keep the same notation as in the previous section, in particular, F is a finite extension of $\mathbb{Q}_p$ endowed with a non-trivial involution $*: F \to F$ and an embedding ι in a fixed Galois closure $\widetilde{F} \supset F$. Moreover, $F_0 \subset F$ is the subfield fixed by $*$ and Γ is the set of $\mathbb{Q}_p$-embeddings of F in $\widetilde{F}$. The maximal subfield of $\widetilde{F}$ which is unramified over $\mathbb{Q}_p$ will be denoted by $\widetilde{F}_a$; see Section 2.2.

Definition 7.1. A filtered-CM-space (with respect to F) is the datum of a CM-space V (Definition 6.1) together with an integer n_σ associated with each eigenline $L_\sigma \subset V \otimes_{\mathbb{Q}_p} \widetilde{F}$ of Proposition 6.5 (2). Such a module is called symmetric if $n_\sigma = -n_{\sigma^*}$.

Lemma 7.2. *The datum of a filtered-CM-space V is equivalent to the datum of a filtered φ-module D over $\widetilde{F}$ such that:*

    (1) $\dim_{\widetilde{F}_a}(D) = [F : \mathbb{Q}_p]$;

    (2) *D is endowed with an action of F;*

    (3) *D has a basis of vectors fixed by the Frobenius φ.*

The equivalence goes as follows. To V one associates the module $D := V \otimes_{\mathbb{Q}_p} \widetilde{F}_a$ endowed with the Frobenius which is the identity on V and extends semi-linearly. The filtration Fil^i on $D_{\widetilde{F}} := V \otimes_{\mathbb{Q}_p} \widetilde{F}$ is the sum of the eigenlines L_σ such that $n_\sigma \geq i$. Conversely to D, one associates $V = D^{\varphi=\mathrm{Id}}$ and $n_\sigma = v_{\mathrm{dR}}(L_\sigma)$; see Section 2.6 for the notation v_{dR}.

Proof. By definition of filtered-φ-module each Fil^i is stable under the action of F, hence it is the sum of eigenlines. The equivalence follows. ∎

Lemma 7.3. *We define the tensor product of two filtered-CM-spaces (V, n_σ) and (V', n'_σ) as*

$$(V \otimes_F V', n_\sigma + n'_\sigma).$$

Under the equivalence of Lemma 7.2 this corresponds to taking the two corresponding filtered-φ-modules D and D' and to considering inside the filtered-φ-module $D \otimes D'$ the sub-filtered-φ-module where the two F-actions coincide.

Proof. Let $\{L_\sigma\}_{\sigma \in \Gamma}$ and $\{L'_\sigma\}_{\sigma \in \Gamma}$ be the eigenlines associated with the actions of F on D and D'. Then the action of $F \otimes_{\mathbb{Q}_p} F$ on $D \otimes D'$ has $\{L_\sigma \otimes L'_\tau\}_{\sigma, \tau \in \Gamma}$ as eigenlines. By definition of the filtration on a tensor product, one has

$$v_{\mathrm{dR}}(L_\sigma \otimes L'_\tau) = n_\sigma + n'_\tau.$$

On the other hand, the submodule of $D \otimes D'$ where the two F-actions coincide has $L_\sigma \otimes L'_\sigma$ as eigenline corresponding to $\sigma \in \Gamma$, hence we do have that $n_\sigma + n'_\sigma$ is its de Rham valuation. ∎

Definition 7.4. The fundamental filtered-CM-space (with respect to (F, ι)) is the (symmetric) filtered-CM-space $F(\iota)$ whose underling CM-space is F and whose associated integers are

$$n_\iota = 1, \quad n_{\iota^*} = -1, \quad n_\sigma = 0 \ \ \forall \sigma \neq \iota, \iota^*.$$

Remark 7.5. By construction, the set of fundamental filtered-CM-spaces $\{F(\tau)\}_{\tau \in \Gamma}$ is a family of tensor generators of the category of symmetric filtered-CM-spaces.

Proposition 7.6. *If a filtered-CM-space is symmetric (Definition* 7.1*), then its corresponding filtered-φ-module is admissible.*

Proof. Admissibility is stable under tensor product and direct factor, and hence it is enough to study the set $\{F(\tau)\}_{\tau \in \Gamma}$ of tensor generators of the symmetric filtered-φ-modules; cf. Remark 7.5. Hence, it is enough to show admissibility for the fundamental filtered-CM-space $F(\iota)$ (for all pairs (F, ι)).

Let D be the filtered-φ-module corresponding to $F(\iota)$ by Lemma 7.2. By [14] it is enough to check the condition of weakly-admissibility for D. The Newton polygon is constantly zero by construction. Hence, the only sub-filtered-φ-modules which might contradict the condition of weakly-admissibility are those containing the eigenline L of de Rham valuation one.

Let N be such a module and let us show that $N = D$, which will give the admissibility. It is not restrictive to assume that N is stable through the action of F, by [9, Proposition 3.1.1.5] (or by a short direct argument). Then the inclusion

$$N^{\varphi = \mathrm{Id}} \subset D^{\varphi = \mathrm{Id}}$$

is F-equivariant, hence $N^{\varphi = \mathrm{Id}} = 0$ or $N^{\varphi = \mathrm{Id}} = D^{\varphi = \mathrm{Id}}$. On the other hand, as D is generated by its φ-invariant part, the Newton slopes of N are also zero; cf. [31, §(1.3) and Lemma 1.3.4]. Now, it is well known that for any φ-module Q, we have

$$\dim_{\mathbb{Q}_p}(Q^{\varphi = \mathrm{Id}}) \leq \dim_{\widetilde{F}_a}(Q).$$

By construction, $\dim_{\mathbb{Q}_p}(D^{\varphi = \mathrm{Id}}) = \dim_{\widetilde{F}_a}(D)$ and N is not zero (it contains L); thus by arguing on the dimension over $\mathbb{Q}_p$ in the exact sequence

$$0 \to N^{\varphi = \mathrm{Id}} \to D^{\varphi = \mathrm{Id}} \to (D/N)^{\varphi = \mathrm{Id}},$$

we get $N^{\varphi = \mathrm{Id}} = D^{\varphi = \mathrm{Id}}$. Therefore, $N = D$. ■

Definition 7.7. Let V be a symmetric filtered-CM-space, D be the corresponding admissible filtered φ-module (cf. Lemma 7.2 and Proposition 7.6). Let W be the crystalline representation of $G_{\widetilde{F}}$ corresponding to D via Fontaine's p-adic comparison theorem. Consider the identification

$$W \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{cris}} = D \otimes_{\widetilde{F}_a} \mathrm{B}_{\mathrm{cris}} = V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{cris}},$$

and the induced one by tensoring it with F over F_a,

$$V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{cris}, F} = W \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{cris}, F},$$

where $B_{\mathrm{cris},F}$ is defined in Section 2.7 (7). Then we define the period associated with V as the invertible element $\lambda_V \in B_{\mathrm{cris},F}^{\times}$ deduced from Proposition 6.7 applied to $V_1 = V$, $V_2 = W$ and $B = B_{\mathrm{cris},F}$. We recall that λ_V is well determined up to a constant in $F^{\times}$ since it is defined as the period such that

$$v_1 = \lambda_V v_2$$

for a given choice of a basis v_1 (resp. v_2) of the F-line in $V \otimes_{\mathbb{Q}_p} F$ (resp. $W \otimes_{\mathbb{Q}_p} F$), where the two F-actions coincide; see Proposition 6.5.

We define *the fundamental period* (with respect to (F, ι)) as the period

$$\lambda = \lambda_{F(\iota)}, \tag{7.7.1}$$

associated with the fundamental filtered-CM-space $F(\iota)$ of Definition 7.4.

The goal of the next two sections is to establish the following theorem, whose proof is finally given in Section 9.8.

Theorem 7.8. *Let $\lambda \in B_{\mathrm{cris},F}$ be the fundamental period defined in Definition 7.7 above. For every endomorphism $*: B_{\mathrm{cris},F} \to B_{\mathrm{cris},F}$ extending the involution $*$ of F, we have*

$$\lambda \cdot *(\lambda) \notin N_{F/F_0}(F^{\times}). \tag{7.8.1}$$

Remark 7.9. (1) Notice that we have $\lambda \cdot *(\lambda) \in F_0^{\times}$ by applying Proposition 6.8, with B equal to $B_{\mathrm{cris},F}$.

(2) It is enough to prove (7.8.1) for one extension $*: B_{\mathrm{cris},F} \to B_{\mathrm{cris},F}$, since by Proposition 6.8, the condition (7.8.1) is equivalent to a statement not involving the choice of the extension.

(3) In Section 8.2 we will construct such an extension $* = *_{\mathrm{cris}}$ and we will work with it through the next two sections.

Corollary 7.10. *Let V be a symmetric filtered-CM-space, λ_V be the corresponding period (Definition 7.7) and $*$ as in Theorem 7.8. Then $\lambda_V \cdot *(\lambda_V) \in F_0^{\times}$ belongs to $N_{F/F_0}(F^{\times})$ if and only if the non-negative integer*

$$\sum_{\sigma, n_\sigma \geq 0} n_\sigma$$

is even, where the sum is taken over all $\sigma \in \Gamma$ such that $n_\sigma \geq 0$.

Proof. Recall that the group $F_0^{\times}/N_{F/F_0}(F^{\times})$ has cardinality two (Theorem 6.4). By construction the map $V \mapsto \lambda_V$ from symmetric filtered-CM-spaces to $B_{\mathrm{cris},F}^{\times}$ is multiplicative on tensor products. In particular, the statement is stable under tensor product, hence it is enough to check it on tensor generators. By Remark 7.5 it is enough to

show the statement for the fundamental filtered-CM-space $F(\iota)$, which is Theorem 7.8 above. ∎

Remark 7.11. By construction, the integer $-\sum_{\sigma, n_\sigma \geq 0} n_\sigma$ is the minimum of the Hodge polygon of the filtered φ-module associated with the symmetric filtered-CM-space V of Corollary 7.10 above.

8. Characterization of p-adic periods

In (7.7.1) we defined a period λ in $B_{\mathrm{cris}, F}$, which we called the fundamental period. The goal of this section is to give some properties of λ as element of $B_{\mathrm{cris}, F}$ which are enough to characterize it up to a constant in $F^\times$, see Propositions 8.7 and 8.8.

We keep the same notation as in Sections 6 and 7, in particular F is a finite extension of $\mathbb{Q}_p$ with ramification index e and residual degree f. Recall that F is endowed with an involution $*\colon F \to F$, whose subfield of fixed points is denoted by F_0, which has not to be confused with the maximal unramified subfield of F, denoted by F_a. The degree of F over F_0 is 2, so $ef = 2[F_0 : \mathbb{Q}_p]$.

We start this section with some preliminary constructions and lemmas, certainly known to experts, that we recall here for the convenience of the reader. For the notation on the rings of p-adic periods we refer to Section 2.7.

8.1. Embeddings of p-adic fields

In Sections 6 and 7, we have fixed a Galois closure $\widetilde{F}$ of F and an embedding $\iota\colon F \hookrightarrow \widetilde{F}$; we have then denoted by Γ the set of $\mathbb{Q}_p$-embeddings of F in $\widetilde{F}$ and by $*$ the embedding $\iota \circ * \in \Gamma$. From now on, let us consider $\widetilde{F}$ as a subfield of B_{dR} by choosing an embedding of $\widetilde{F}$ in B_{dR}. We can then identify Γ to the set of $\mathbb{Q}_p$-embeddings of F in B_{dR}. We will also identify F to its image $\iota(F) \subset \widetilde{F} \subset B_{\mathrm{dR}}$. Since $F_a/\mathbb{Q}_p$ is a cyclic unramified extension, every σ in Γ stabilizes F_a and its restriction to F_a is a power of the absolute Frobenius φ of F_a: we will set $\sigma_{|F_a} = \varphi^{\nu(\sigma)}$ for a unique integer $0 \leq \nu(\sigma) \leq f - 1$. By construction, we have $\nu(\iota) = 0$. We put $\nu := \nu(*)$, it is either 0 or $f/2$ whether F/F_0 is ramified or unramified.

8.2. Some endomorphisms of period rings

We keep the notation of Section 8.1. We denote by $B_{\mathrm{cris}, F}$ the smallest subring of B_{dR} containing B_{cris} and F; cf. Section 2.7 (7); the ring $B_{\mathrm{cris}, F}$ is identified to the image of the natural map

$$B_{\mathrm{cris}} \otimes_{F_a} F \to B_{\mathrm{dR}}.$$

Similarly, we will use $B_{\mathrm{cris},\widetilde{F}}$, which is isomorphic to $B_{\mathrm{cris}} \otimes_{\widetilde{F}_a} \widetilde{F}$, where $\widetilde{F}_a$ is the absolute unramified subfield of $\widetilde{F}$.

We recall that the ring $B_{\mathrm{cris},F}$ is endowed with an F-linear endomorphism

$$\varphi^f : B_{\mathrm{cris},F} \to B_{\mathrm{cris},F},$$

defined as $\varphi_{\mathrm{cris}}^f \otimes_{\mathrm{Id}_{F_a}} \mathrm{Id}_F$, where φ_{cris} is the Frobenius of B_{cris}; cf. Section 2.7 (8). Furthermore, every σ in Γ extends to a homomorphism $\sigma_{\mathrm{cris}} : B_{\mathrm{cris},F} \to B_{\mathrm{cris},\widetilde{F}}$, defined as

$$\sigma_{\mathrm{cris}} : B_{\mathrm{cris},F} = B_{\mathrm{cris}} \otimes_{F_a} F \xrightarrow{\varphi_{\mathrm{cris}}^{\nu(\sigma)} \otimes \sigma} B_{\mathrm{cris}} \otimes_{\widetilde{F}_a} \widetilde{F} = B_{\mathrm{cris},\widetilde{F}}, \qquad (8.2.1)$$

where the product $\varphi_{\mathrm{cris}}^{\nu(\sigma)} \otimes \sigma$ is taken over the inclusion $F_a \hookrightarrow \widetilde{F}_a$; cf. [7, A.II, §3, n°3, p. 53]. Note that ι_{cris} coincides with the inclusion $B_{\mathrm{cris},F} \subset B_{\mathrm{cris},\widetilde{F}}$ as subrings of B_{dR}. Moreover, although we will not need it in following, it is not difficult to show that σ_{cris} is injective by using the fact that the natural maps $B_{\mathrm{cris}} \otimes_{F_a} F \to B_{\mathrm{dR}}$, $B_{\mathrm{cris}} \otimes_{\widetilde{F}_a} \widetilde{F} \to B_{\mathrm{dR}}$ and the Frobenius φ_{cris} are injective.

Finally, the involution $*$ of F also extends to an endomorphism

$$*_{\mathrm{cris}} : B_{\mathrm{cris},F} \to B_{\mathrm{cris},F}$$

by putting $*_{\mathrm{cris}} := \varphi_{\mathrm{cris}}^{\nu} \otimes_{\varphi^{\nu}} *$. Beware that in general it is not an involution of $B_{\mathrm{cris},F}$: precisely, as follows from Lemma 8.3 below, if F/F_0 is unramified, then $*_{\mathrm{cris}}^2 = \varphi_{\mathrm{cris}}^f$; if F/F_0 is ramified, then $*_{\mathrm{cris}}^2 = \mathrm{Id}_{B_{\mathrm{cris},F}}$.

Lemma 8.3. *We keep the notation of Sections* 8.1 *and* 8.2.

(1) *For all σ in Γ, we have $\sigma_{\mathrm{cris}} \circ *_{\mathrm{cris}} = (\sigma \circ *)_{\mathrm{cris}}$ except if F/F_0 is unramified and $\nu(\sigma) \geq f/2$, as in that case $\sigma_{\mathrm{cris}} \circ *_{\mathrm{cris}} = (\sigma \circ *)_{\mathrm{cris}} \circ \varphi_{\mathrm{cris}}^f$.*

(2) $*_{\mathrm{cris}} \circ \varphi_{\mathrm{cris}}^f = \varphi_{\mathrm{cris}}^f \circ *_{\mathrm{cris}}$.

Proof. Property (2) follows directly by definition:

$$*_{\mathrm{cris}} \circ \varphi_{\mathrm{cris}}^f = (\varphi_{\mathrm{cris}}^{\nu} \otimes *) \circ (\varphi_{\mathrm{cris}}^f \otimes \mathrm{Id}_F) = \varphi_{\mathrm{cris}}^{\nu+f} \otimes *$$
$$= (\varphi_{\mathrm{cris}}^f \otimes \mathrm{Id}_F) \circ (\varphi_{\mathrm{cris}}^{\nu} \otimes *) = \varphi_{\mathrm{cris}}^f \circ *_{\mathrm{cris}}.$$

Let us prove (1). We have

$$\sigma_{\mathrm{cris}} \circ *_{\mathrm{cris}} = \left(\varphi_{\mathrm{cris}}^{\nu(\sigma)} \otimes_{\varphi^{\nu(\sigma)}} \sigma\right) \circ (\varphi_{\mathrm{cris}}^{\nu} \otimes_{\varphi^{\nu}} *)$$
$$= \varphi_{\mathrm{cris}}^{\nu(\sigma)+\nu} \otimes_{\varphi^{\nu(\sigma)+\nu}} (\sigma \circ *).$$

If F/F_0 is ramified, then $\nu = 0$ and $\nu(\sigma \circ *) = \nu(\sigma)$ (cf. Section 8.1) and the statement is clear. If F/F_0 is unramified, then $\nu = f/2$: if $\nu(\sigma) < f/2$, the statement is also clear. Finally, if F/F_0 is unramified and $\nu(\sigma) \geq f/2$, we have

$$\nu(\sigma) + f/2 = f + \nu(\sigma \circ *),$$

and so

$$\varphi_{\text{cris}}^{\nu(\sigma)+f/2} \otimes_{\varphi^{\nu(\sigma)+f/2}} (\sigma \circ *) = \varphi_{\text{cris}}^{\nu(\sigma\circ*)+f} \otimes_{\varphi^{\nu(\sigma\circ*)+f}} (\sigma \circ *)$$

$$= (\sigma \circ *)_{\text{cris}} \circ \varphi_{\text{cris}}^{f}. \qquad \blacksquare$$

Convention 8.4. When there is no risk of confusion we will write abusively σ (resp. $*$, resp. φ) instead of σ_{cris} (resp. $*_{\text{cris}}$, resp. φ_{cris}); also, for any $\lambda \in B_{\text{cris},F}$, we will sometimes write $\lambda*$ (resp. $\lambda*_{\text{cris}}$) instead of $*(\lambda)$ (resp. $*_{\text{cris}}(\lambda)$).

Lemma 8.5 (Colmez). *We keep the notation of Sections* 8.1 *and* 8.2, *and Convention* 8.4. *Let* $\mu \in B_{\text{cris},F}$ *be an element such that:*

(1) *for all* $\sigma \in \Gamma$, $v_{\text{dR}}(\sigma(\mu)) = 0$;

(2) $\varphi^{f}(\mu) = \mu$.

Then μ *belongs to* $F^{\times} \subset B_{\text{cris},F}$.

Proof. This follows from the fundamental exact sequence [13, Lemma 9.25 (SEF3E)]. We briefly recall the argument. By construction B_{cris} is the localization $A_{\text{cris}}[t^{-1}]$ of Fontaine's ring A_{cris}, where t is the period of $\mathbb{Q}_p(1)$; see Section 2.7 (6). Thus there exists some integer $n \geq 0$, such that

$$\mu t^{n} \in B_{\text{cris},F}^{+} := A_{\text{cris}} \otimes_{W(k_F)} F.$$

By evaluating the sequence of [13, Lemma 9.16] in $\mathbb{C}_p$ and by multiplying with t^{n-1}, we get a short exact sequence of $\mathbb{Q}_p$-vector spaces

$$0 \to F \cdot t^{n} \to (B_F)^{\varphi^{f}=p^{nf}} \xrightarrow{\Theta_F} \bigoplus_{\Gamma} \mathbb{C}_p \to 0,$$

where $B_F \subset B_{\text{dR}}^{+}$ is some ring of periods containing $B_{\text{cris},F}^{+}$ (cf. [13, §8.5][2]); and the map Θ_F is given by

$$\Theta_F : x \mapsto \big(\theta(\sigma(xt^{-n+1}))\big)_{\sigma \in \Gamma},$$

where $\theta : B_{\text{dR}}^{+} \to \mathbb{C}_p$ denotes the reduction map and $\sigma : B_F \to B_{\text{dR}}^{+}$ extends the homomorphism σ_{cris} of (8.2.1). By hypothesis (2) of the statement, the element $x = \mu t^{n}$ belongs to $(B_{\text{cris},F}^{+})^{\varphi^{f}=p^{nf}}$; we get

$$\Theta_F(x) = \big(\theta(\sigma(\mu t))\big)_{\sigma \in \Gamma}.$$

By hypothesis (1) of the statement, we have $v_{\text{dR}}(\sigma(\mu)) = 0$ and since

$$\sigma_{\text{cris}}(t) = \varphi_{\text{cris}}^{\nu(\sigma)}(t) = p^{\nu(\sigma)}t,$$

---

[2]The ring B_F is the ring $B_{\text{max},F}^{+}$ in the notation of loc. cit.

we get $\Theta_F(x) = 0$. Hence, $\mu \in F$; moreover, μ is non-zero otherwise hypothesis (1) above would fail. ∎

Definition 8.6. An element β in $B_{\mathrm{cris},F}$ is called a *fundamental period* if it satisfies the following properties:

(a) $v_{\mathrm{dR}}(\beta) = 1$;

(b) $v_{\mathrm{dR}}(\beta^*) = -1$;

(c) for all $\sigma \in \Gamma \backslash \{\iota, *\}$, $v_{\mathrm{dR}}(\sigma(\beta)) = 0$;

(d) $\varphi^f(\beta) = \beta$,

where $\sigma(\beta)$, φ^f, and $\beta^* \in B_{\mathrm{cris},\widetilde{F}} \subset B_{\mathrm{dR}}$ are defined in Section 8.2 and Convention 8.4.

We denote by $\mathcal{P} \subset B_{\mathrm{cris},F}$ the set of fundamental periods.

Proposition 8.7. *The fundamental period* $\lambda \in B_{\mathrm{cris},F}^{\times}$ *defined in* (7.7.1) *is a fundamental period according to Definition* 8.6.

Proof. In this proof let us write V for the fundamental filtered-CM-space $F(\iota)$; cf. Definition 7.7. Let $D := V \otimes_{\mathbb{Q}_p} \widetilde{F}_a$ be the admissible filtered-φ-module corresponding to $F(\iota)$ by Lemma 7.2 and Proposition 7.6. Let W be the Galois representation associated with D, which comes with an identification of $B_{\mathrm{cris},\widetilde{F}}$-modules

$$V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris},\widetilde{F}} = W \otimes_{\mathbb{Q}_p} B_{\mathrm{cris},\widetilde{F}}.$$

In this context, Proposition 6.7 gives vectors $v_1 \in V_F := V \otimes_{\mathbb{Q}_p} F$ and $v_2 \in W_F := W \otimes_{\mathbb{Q}_p} F$ and the relation $v_1 = \lambda v_2$. By applying $\sigma \in \Gamma$, we get $\sigma(v_1) = \sigma(\lambda)\sigma(v_2)$. Now, by construction we know that

- for all $\sigma \in \Gamma$, $v_{\mathrm{dR}}(\sigma(v_2)) = 0$;

- $v_{\mathrm{dR}}(v_1) = 1$ and $v_{\mathrm{dR}}(v_1^*) = -1$;

- for all $\sigma \in \Gamma \backslash \{\iota, *\}$, $v_{\mathrm{dR}}(\sigma(v_1)) = 0$.

Therefore, we get

- for $\sigma = \iota$, $v_1 = \lambda v_2$, so $v_{\mathrm{dR}}(\lambda) = 1$;

- for $\sigma = *$, $v_1^* = \lambda^* v_2^*$, so $v_{\mathrm{dR}}(\lambda^*) = -1$;

- for $\sigma \in \Gamma \backslash \{\iota, *\}$, $\sigma(v_1) = \sigma(\lambda)\sigma(v_2)$, so $v_{\mathrm{dR}}(\sigma(\lambda)) = 0$.

Finally, by applying φ^f to $v_1 = \lambda v_2$, since by construction both v_1 and v_2 are fixed by φ^f, we get

$$v_1 = \varphi_D^f(v_1) = \varphi^f(\lambda)\varphi^f(v_2) = \varphi^f(\lambda)v_2,$$

hence $\varphi^f(\lambda) = \lambda$. ∎

Proposition 8.8 (Uniqueness of fundamental periods). *Let $\lambda \in B_{\mathrm{cris},F}^{\times}$ be as in Proposition 8.7 and $\mathcal{P}$ the set of fundamental periods of Definition 8.6, then*

$$\mathcal{P} = F^{\times} \cdot \lambda.$$

In particular, all fundamental periods are invertible.

Proof. Since λ is invertible in $B_{\mathrm{cris},F}$, we can consider the subset $\lambda^{-1}\mathcal{P}$ of $B_{\mathrm{cris},F}$. We have to prove $\lambda^{-1}\mathcal{P} = F^{\times}$, which is exactly the statement of Lemma 8.5. ∎

Proposition 8.9. *Let $\lambda \in B_{\mathrm{cris},F}$ be a fundamental period. Then $\lambda\lambda^{*} \in F_0^{\times}$.*

Remark 8.10. For the fundamental period λ defined in (7.7.1), the statement of Proposition 8.9 follows directly from Proposition 6.8. Thanks to Proposition 8.8 the same is true for any fundamental period. We give below an alternative proof based on the p-adic properties of the period which does not use quadratic forms.

Proof. In this proof in order to avoid any possible confusion we will write $\lambda^{*\mathrm{cris}}$ instead of λ^{*}. First, let us show that the period $\lambda\lambda^{*\mathrm{cris}} \in B_{\mathrm{cris},F}$ satisfies the hypothesis of Lemma 8.5. For any σ in Γ, we have

$$\sigma_{\mathrm{cris}}(\lambda\lambda^{*\mathrm{cris}}) = \sigma_{\mathrm{cris}}(\lambda) \cdot (\sigma_{\mathrm{cris}} \circ *_{\mathrm{cris}})(\lambda),$$

and by Lemma 8.3 (1), $(\sigma_{\mathrm{cris}} \circ *_{\mathrm{cris}})(\lambda)$ is equal to $(\sigma \circ *)_{\mathrm{cris}}(\lambda)$ or $(\sigma \circ *)_{\mathrm{cris}}(\varphi_{\mathrm{cris}}^{f}(\lambda))$. Since $\varphi_{\mathrm{cris}}^{f}(\lambda) = \lambda$ by property (d) of Definition 8.6, we get

$$\sigma_{\mathrm{cris}}(\lambda\lambda^{*\mathrm{cris}}) = \sigma_{\mathrm{cris}}(\lambda) \cdot (\sigma \circ *)_{\mathrm{cris}}(\lambda).$$

Applying ν_{dR} and considering the properties (a), (b) and (c) of Definition 8.6 for λ, we get that $\lambda\lambda^{*\mathrm{cris}}$ satisfies the property (1) of Lemma 8.5. Let us check property (2) of Lemma 8.5:

$$\begin{aligned} \varphi_{\mathrm{cris}}^{f}(\lambda\lambda^{*\mathrm{cris}}) &= \varphi_{\mathrm{cris}}^{f}(\lambda)\varphi_{\mathrm{cris}}^{f}(\lambda^{*\mathrm{cris}}) \\ &= \lambda\lambda^{*\mathrm{cris}}, \end{aligned}$$

because $\varphi_{\mathrm{cris}}^{f}$ and $*_{\mathrm{cris}}$ commute, cf. Lemma 8.3 (2), and $\varphi_{\mathrm{cris}}^{f}(\lambda) = \lambda$. Therefore, by Lemma 8.5, the period $\lambda\lambda^{*\mathrm{cris}} \in B_{\mathrm{cris},F}$ belongs to $F^{\times} \subset B_{\mathrm{cris},F}$. Applying $*$, we get

$$\begin{aligned} (\lambda\lambda^{*\mathrm{cris}})^{*} &= \lambda^{*\mathrm{cris}}\lambda^{*^2_{\mathrm{cris}}} \\ &= \lambda^{*\mathrm{cris}}\varphi_{\mathrm{cris}}^{2\nu}(\lambda) = \lambda^{*\mathrm{cris}}\lambda, \end{aligned}$$

where $\nu = 0$ or $f/2$ (cf. Sections 8.1 and 8.2); and we again used $\varphi_{\mathrm{cris}}^{f}(\lambda) = \lambda$. Being stable under $*$, the element $\lambda\lambda^{*\mathrm{cris}} \in F^{\times}$ actually belongs to $F_0^{\times}$. ∎

9. Lubin–Tate periods and local reciprocity

The goal of this section is the proof of Theorem 7.8, which is given in 9.8. Before that, we need to relate the fundamental period λ of (7.7.1) to the Lubin–Tate periods; cf. Proposition 9.5 and Corollary 9.7.

We keep the notation of Section 8 and specifically we refer to Sections 8.1 and 8.2, and Convention 8.4. In particular, recall that F is a p-adic field endowed with non-trivial involution $*$ and we denoted by F_0 the subfield of F of points fixed by $*$, whereas we denoted by F_a the absolute unramified subfield of F.

Definition 9.1. Let π be a uniformizer of F. We say that α in $B_{\mathrm{cris},F}$ is a Lubin–Tate period (relative to $\pi \in F$) if

(1) $v_{\mathrm{dR}}(\alpha) = 1$;

(2) for all $\sigma \in \Gamma \setminus \{\iota\}$, $v_{\mathrm{dR}}(\sigma(\alpha)) = 0$;

(3) $\varphi^f(\alpha) = \pi\alpha$;

(4) α is invertible in $B_{\mathrm{cris},F}$.

Theorem 9.2 (Colmez). *There exist Lubin–Tate periods relative to any choice of a uniformizer π of F.*

Remark 9.3. (1) Lubin–Tate periods are constructed in [13, §§9.3–9.5][3], via a direct computation in periods rings using Lubin–Tate formal groups [34]; see also [12, §2] and [22, §3.6]. Therefore the action of G_F on $\alpha \in B_{\mathrm{cris},F}$ is given precisely by multiplication with the Lubin–Tate character. We will not need this property, so we did not mention it in Definition 9.1 above.

(2) To keep this article more self-contained, we recall in Appendix A another construction of Lubin–Tate periods, relying on the theorem "weakly admissible $\Rightarrow$ admissible" (cf. [14]) applied to a well-chosen filtered φ-module over F, see Definition A.5.

9.4. Arithmetic Frobenius

Let us denote by $\widehat{F}^{\mathrm{nr}}$ (resp. $\widehat{F}_a^{\mathrm{nr}}$) the completion of maximal unramified extension of F (resp. F_a) and by $\mathrm{Gal}(\widehat{F}^{\mathrm{nr}}/F)$ the group of continuous F-automorphisms of $\widehat{F}^{\mathrm{nr}}$. We have

$$\widehat{F}^{\mathrm{nr}} = \widehat{F}_a^{\mathrm{nr}} \otimes_{F_a} F,$$

---

[3]Precisely, cf. [13, Proposition 9.10 and Lemma 9.18]. Note that these periods are constructed in $B_{\mathrm{max},F}$, but actually they belong to $B_{\mathrm{cris},F}$, since, by Definition 9.1 (3) above, they belong to $\varphi^f(B_{\mathrm{max},F}) \subset B_{\mathrm{cris},F}$.

which we embed canonically in $B_{\text{cris},F}$. By definition, the endomorphism φ^f_{cris} of $B_{\text{cris},F}$, restricted to $\widehat{F}^{\text{nr}}$, is the arithmetic Frobenius φ^f in $\text{Gal}(\widehat{F}^{\text{nr}}/F)$, i.e. the unique isomorphism of $\widehat{F}^{\text{nr}}$ lifting the map $x \mapsto x^q$, $q = p^f$, on the residue field.

Proposition 9.5 (Explicit construction of the fundamental period). *We assume that $\alpha \in B_{\text{cris},F}$ is a Lubin–Tate period, for some uniformizer $\pi \in F$. Then there exists $c \in (\widehat{F}^{\text{nr}})^\times$ such that $c(\alpha/\alpha^*)$ is a fundamental period; see Definition* 8.6. *Moreover, c can be chosen as follows:*

- *if F/F_0 is unramified, and π belongs to F_0, then $c = 1$;*
- *if F/F_0 is (totally) ramified, then c satisfies*

$$\varphi^f(c) = \frac{\pi^*}{\pi} c, \tag{9.5.1}$$

where $\varphi^f \in \text{Gal}(\widehat{F}^{\text{nr}}/F)$ is the arithmetic Frobenius; see Section 9.4 *above.*

Proof. The Lubin–Tate period α is invertible in $B_{\text{cris},F}$, so is $*_{\text{cris}}(\alpha)$. Consider the period

$$\frac{\alpha}{\alpha^*} := \alpha \cdot *_{\text{cris}}(\alpha)^{-1} \in B_{\text{cris},F}.$$

First let us show that α/α^* satisfies properties (a), (b) and (c) of Definition 8.6; then, we will multiply it by a well-chosen constant c in $\widehat{F}^{\text{nr}}$ in order to get property (d) of Definition 8.6. Since, for every σ in Γ,

$$\sigma_{\text{cris}}(\widehat{F}^{\text{nr}}) = \widehat{F}^{\text{nr}} \subset B_{\text{cris},\widetilde{F}} \cap B^+_{\text{dR}},$$

normalizing the period by a non-zero constant in $\widehat{F}^{\text{nr}}$ will not change the first three properties.

As α is a Lubin–Tate period, by using Definition 9.1, we get the following:

(a) $v_{\text{dR}}(\alpha/\alpha^*) = v_{\text{dR}}(\alpha) - v_{\text{dR}}(*_{\text{cris}}(\alpha)) = 1 - 0 = 1$.

(b) $v_{\text{dR}}(*_{\text{cris}}(\alpha/\alpha^*)) = v_{\text{dR}}(*_{\text{cris}}(\alpha)) - v_{\text{dR}}(*^2_{\text{cris}}(\alpha)) = -v_{\text{dR}}(*^2_{\text{cris}}(\alpha))$. Since $*^2_{\text{cris}}$ is either $\text{Id}_{B_{\text{cris},F}}$ or φ^f_{cris} (cf. Section 8.2) and $\varphi^f_{\text{cris}}(\alpha) = \pi\alpha$, we get

$$-v_{\text{dR}}(*^2_{\text{cris}}(\alpha)) = -1.$$

(c) For every σ in $\Gamma \setminus \{\iota, *\}$, we have

$$v_{\text{dR}}\left(\sigma_{\text{cris}}\left(\frac{\alpha}{\alpha^*}\right)\right) = v_{\text{dR}}(\sigma_{\text{cris}}(\alpha)) - v_{\text{dR}}\left(\sigma_{\text{cris}}(*_{\text{cris}}(\alpha))\right)$$
$$= -v_{\text{dR}}\left(\sigma_{\text{cris}}(*_{\text{cris}}(\alpha))\right),$$

as $\sigma \neq \iota$. By using Lemma 8.3 (1), the period $\sigma_{\text{cris}}(*_{\text{cris}}(\alpha))$ is either

$$(\sigma \circ *)_{\text{cris}}(\alpha) \quad \text{or} \quad (\sigma \circ *)_{\text{cris}}(\varphi^f_{\text{cris}}(\alpha)) = (\sigma \circ *)_{\text{cris}}(\pi\alpha) = \sigma(\pi^*)(\sigma \circ *)_{\text{cris}}(\alpha).$$

As $\sigma(\pi^*) \in \widetilde{F} \subset B_{\mathrm{cris}, \widetilde{F}} \cap B_{\mathrm{dR}}^+$ and $\sigma \circ * \neq \iota$, we have, in both cases,

$$v_{\mathrm{dR}}\big(\sigma_{\mathrm{cris}}(*_{\mathrm{cris}}(\alpha))\big) = 0$$

by Definition 9.1 (3).

Finally, we need to check property (d) of Definition 8.6. As π^* and π have the same valuation (one), we have $\pi^* \pi^{-1} \in \mathcal{O}_F^\times$. In particular, the equation

$$\frac{\varphi^f(c)}{c} = \frac{\pi^*}{\pi}$$

has a solution $c \in (\widehat{F}^{\mathrm{nr}})^\times$; see for example [38, Ch. V, §2, Lemma (2.1)]. Note that if F/F_0 is unramified and π is in F_0, we can take $c = 1$. We finish by computing

$$\varphi_{\mathrm{cris}}^f\left(c\,\frac{\alpha}{\alpha^{*_{\mathrm{cris}}}}\right) = \varphi^f(c)\,\frac{\varphi_{\mathrm{cris}}^f(\alpha)}{\varphi_{\mathrm{cris}}^f(*_{\mathrm{cris}}(\alpha))} = c\,\frac{\pi^*}{\pi}\,\frac{\varphi_{\mathrm{cris}}^f(\alpha)}{*_{\mathrm{cris}}(\varphi_{\mathrm{cris}}^f(\alpha))}$$

$$= c\,\frac{\pi^*}{\pi}\,\frac{\pi\alpha}{(\pi\alpha)^{*_{\mathrm{cris}}}} = c\,\frac{\alpha}{\alpha^{*_{\mathrm{cris}}}},$$

where we used $\varphi_{\mathrm{cris}}^f(\alpha) = \pi\alpha$ and Lemma 8.3 (2). ∎

Remark 9.6. If F/F_0 is tamely ramified, we can choose a uniformizer $\pi \in F$, such that $\pi^* = -\pi$ and the constant c can be chosen in a quadratic unramified extension of F, such that $\varphi^f(c) = -c$. In general, the constant c is algebraic over F if and only if its orbit under $\mathrm{Gal}(\widehat{F}^{\mathrm{nr}}/F)$ is finite. Since φ^f is a topological generator, the relation (9.5.1) implies that c is algebraic if and only if $\pi^* \pi^{-1}$ is a root of unity. When the break of the ramification filtration of $\mathrm{Gal}(F/F_0)$ is large, it may happen that for any choice of a uniformizer π in F, $\pi^* \pi^{-1}$ is never a root of unity (e.g. $F = \mathbb{Q}_2(\zeta_8)$ and $F_0 = \mathbb{Q}_2(\zeta_4)$, where ζ_8 is a primitive 8th-root of unity and $\zeta_8^2 = \zeta_4$).

Corollary 9.7. *Let λ be a fundamental period in $B_{\mathrm{cris}, F}$, then*

$$\lambda \in F^\times \cdot c\,\frac{\alpha}{\alpha^*},$$

with c and α as in Proposition 9.5.

Proof. This is an immediate consequence of Propositions 8.8 and 9.5. ∎

9.8. Proof of Theorem 7.8

Let $\lambda \in B_{\mathrm{cris}, F}^\times$ be the fundamental period defined in (7.7.1). By Proposition 8.7, it is a fundamental period in the sense of Definition 8.6. By Proposition 8.9 (or by Proposition 6.8), the period $\lambda\lambda^*$ belongs to $F_0^\times$. By Remark 7.9 (2), it is enough to

prove that $\lambda\lambda^*$ does not belong to $N_{F/F_0}(F^\times)$, for $* = *_{\text{cris}}$, the endomorphism of $B_{\text{cris},F}$ defined in Section 8.2.

By Corollary 9.7, we have $\lambda = bc(\alpha/\alpha^*)$, for some $b \in F^\times$, a Lubin–Tate period $\alpha \in B^\times_{\text{cris},F}$ and $c \in \widehat{F}^{\text{nr}}$ as in Proposition 9.5. Considering that $bb^* = N_{F/F_0}(b) \in F_0^\times$ is a norm we can suppose $b = 1$. Following Proposition 9.5, we need to treat separately two cases, depending on whether the extension F/F_0 is unramified or ramified.

(1) Assume that the extension F/F_0 is unramified. We have $*^2_{\text{cris}} = \varphi^f$; cf. Section 8.2. Choose π in F_0 so that we can have $c = 1$; cf. Proposition 9.5. We get

$$\lambda\lambda^* = \frac{\alpha}{(\alpha^*)}\frac{\alpha^*}{(\alpha^*)^*} = \frac{\alpha}{(\alpha^*)^*} = \frac{\alpha}{\varphi^f(\alpha)} = \frac{\alpha}{\pi\alpha} = \frac{1}{\pi}.$$

Since F/F_0 is unramified every norm has even valuation in F_0, hence $\lambda\lambda^*$ cannot be a norm.

(2) Assume that the extension F/F_0 is (totally) ramified. In particular, the endomorphism $*_{\text{cris}}$ of $B_{\text{cris},F}$ is an involution; cf. Section 8.2. We get

$$\lambda\lambda^* = c\frac{\alpha}{(\alpha^*)}c^*\frac{\alpha^*}{(\alpha^*)^*} = cc^* = N_{\widehat{F}^{\text{nr}}/\widehat{F}_0^{\text{nr}}}(c),$$

where the last equality follows given that the restriction of $*_{\text{cris}}$ to

$$\widehat{F}^{\text{nr}} = \widehat{F}_a^{\text{nr}} \otimes_{F_a} F \subset B_{\text{cris},F}$$

is the unique non-trivial element in $\text{Gal}(\widehat{F}^{\text{nr}}/\widehat{F}_0^{\text{nr}})$. Let

$$(-, F/F_0)\colon F_0^\times \to \text{Gal}(F/F_0)$$

be the local reciprocity map. The equation $\varphi^f(c)c^{-1} = \pi^*\pi^{-1}$ from (9.5.1) fits exactly the hypothesis of a theorem of Dwork (cf. [41, Ch. XIII, §5, Corollary of Theorem 2]); whence we get

$$\left(N_{\widehat{F}^{\text{nr}}/\widehat{F}_0^{\text{nr}}}(c), F/F_0\right) = *^{-1} = *.$$

Since $* \neq \text{Id}_F$, that exactly means that $N_{\widehat{F}^{\text{nr}}/\widehat{F}_0^{\text{nr}}}(c) \in F_0^\times$ is not a norm, i.e. it does not belong to $N_{F/F_0}(F^\times)$, and that finishes the proof. ∎

Remark 9.9. For $* = *_{\text{cris}}$, the proof above, as all the statements of this section, does not use that λ is coming from a CM-quadratic space (cf. Definition 6.1), but only the properties of Definition 8.6; see also Remark 8.10.

10. Proof of the main theorem

We now put all the ingredients together to prove Theorem 3.2.

Proof of Theorem 3.2

By Proposition 4.5, the result is reduced to a p-adic statement. In particular, following Proposition 4.7, it is enough to study the p-adic étale realization and the Frobenius invariant part of the crystalline realization of the motive M. By this same proposition, this pair of quadratic spaces is an orthogonal supersingular pair, in the language of Definition 5.1. We can reformulate Proposition 4.5 using Definition 5.4: we have to show that our orthogonal supersingular pair is good (see also Remark 4.6).

These two quadratic spaces are endowed with the action of $F \otimes_{\mathbb{Q}} \mathbb{Q}_p$, where F is the number field given by hypothesis. By Proposition 5.6 we can use the action of $F \otimes_{\mathbb{Q}} \mathbb{Q}_p$ to decompose the two quadratic spaces and study them separately. More precisely it is enough to study the case where the two quadratic spaces are endowed with an action of a p-adic field (which is a factor of $F \otimes_{\mathbb{Q}} \mathbb{Q}_p$) and is stable by the involution $*$ induced by F. By abuse of notation we will denote such a p-adic field again by F.

We have now an orthogonal supersingular pair endowed with an action of a p-adic field. Both quadratic space are now CM-quadratic spaces with respect to F in the sense of Definition 6.1. We can now apply Proposition 6.7 to our two quadratic spaces and to the ring $B = \mathrm{B}_{\mathrm{cris},F}$. This constructs a period $\lambda \in B^\times$, well defined up to multiplication by an element of $F^\times$.

By Proposition 6.8 applied to the endomorphism $* = *_{\mathrm{cris}}$ defined in Section 8.2, the two orthogonal spaces are isomorphic if and only if $\lambda \cdot *(\lambda)$ lies in the group of norms $N_{F/F_0}(F^\times)$. By Corollary 7.10, this norm condition is equivalent to the fact that the minimum of the Hodge polygon of the underlying filtered φ-module is an even number (see also Remark 7.11). Altogether we have that the two orthogonal spaces are isomorphic if and only if the minimum of the Hodge polygon is even. By Lemma 5.5 this means precisely that the pair is good. ∎

Remark 10.1. Note that Corollary 7.10 is a consequence of Theorem 7.8 whose proof is the object of Sections 8 and 9.

A. Lubin–Tate filtered φ-modules

Lubin–Tate periods were constructed by Colmez in [13], via a direct construction based on Lubin–Tate's formal group law [34]. The goal of this appendix is to present to the reader an alternative construction of these periods as self-contained as possible. Beyond [13], there is a vast literature for the p-adic representations associated with Lubin–Tate groups and their periods; cf., for example, [6, 12, 22, 32].

Let π be a uniformizer of a p-adic field F. The plan is the following:

(1) Describe concretely a filtered φ-module $D = (D, \varphi_\pi, \mathrm{Fil}^\bullet(D_F))$ over F using only semi-linear algebra data, see Definitions A.1 and A.5.

(2) Show that D is weakly-admissible, see Proposition A.6; and hence admissible by [14].

(3) Show that a Lubin–Tate period as in Definition 9.1 appears as period $\alpha = \alpha_\pi$ of D; see (A.9.1) and Proposition A.12.

(4) Relate α to Colmez' original construction, hence D to the Galois representation given by the Lubin–Tate character; see Proposition A.14.

In this appendix we do not assume anymore that F is endowed with a non-trivial involution. Keep the notation of Sections 2.2 and 2.7. In particular, recall that we denoted by F_a the absolute unramified subfield of F, which has degree f over $\mathbb{Q}_p$. Let $E_\pi(x) \in F_a[x]$ be the minimal (monic) polynomial of π over F_a, which is an Eisenstein polynomial of degree e.

Definition A.1 (The φ-module). Let us define a filtered φ-module: we set

$$D := (F_a)^{ef},$$

as F_a-vector space and we endow it with the semi-linear Frobenius $\varphi_\pi \colon D \to D$ given, in the canonical basis $\mathcal{C} := (e_1, \ldots, e_{ef})$, by the block matrix

$$A := \left(\begin{array}{c|c} 0 & C \\ \hline I_{e(f-1)} & 0 \end{array} \right) \in \mathrm{M}_{ef}(F_a), \tag{A.1.1}$$

where $I_{e(f-1)} \in M_{e(f-1)}(F_a)$ is the identity matrix, 0 stands for the zero (rectangular) blocks, and $C \in M_e(F_a)$ is the companion matrix of the minimal polynomial $E_\pi(x)$ of π.

Lemma A.2. *The following statements hold.*

(1) *The φ-module (D, φ_π) is endowed with an F-action, i.e. a linear action of F on the F_a-vector space D which commutes with φ_π.*

(2) *The action of $\pi \in F$ is equal to the action of φ_π^f on D.*

Proof. Let us define an F-action on the φ-module D. For any square matrix $M \in M_e(F_a)$, we denote by $\varphi_{F_a}(M)$ the matrix on which the Frobenius φ_{F_a} is applied to all entries. We then define the diagonal block matrix

$$\widetilde{M} := \mathrm{Diag}\big(M, \varphi_{F_a}(M), \ldots, \varphi_{F_a}^{f-1}(M)\big) \in \mathrm{M}_{ef}(F_a), \tag{A.2.1}$$

where the diagonal blocks are the matrices $\varphi_{F_a}^i(M)$, for i going from 0 to $f-1$. The F_a-linear map $D \to D$ given by $\widetilde{M}$ in the canonical basis $\mathcal{C}$ of D commutes with the

(semi-linear) Frobenius φ_π if and only if

$$\tilde{M} A = A \varphi_{F_a}(\tilde{M}),$$

by Definition A.1. It is elementary to check that this holds exactly when M commutes with the matrix C from (A.1.1). Since C is cyclic, its commutator in $M_e(F_a)$ is precisely $F_a[C] \cong F$. This gives an F_a-linear F-action on the φ-module D and proves the statement (1) above.

We can now prove (2). First, note that by construction the action $\pi \in F$ is given by the matrix $\tilde{C}$ from (A.2.1), since C is the companion matrix of the minimal polynomial of π.

Now, since φ_π is semi-linear, its f power φ_π^f is F_a-linear and it is given in the canonical basis by the (twisted) product

$$A \cdot \varphi_{F_a}(A) \cdots \varphi_{F_a}^{f-1}(A). \tag{A.2.2}$$

An elementary computation shows that the product (A.2.2) is equal to the diagonal block matrix

$$\mathrm{Diag}(C, \varphi_{F_a}(C), \ldots, \varphi_{F_a}^{f-1}(C)) \in M_{ef}(F_a), \tag{A.2.3}$$

which is the matrix $\tilde{C}$ from (A.2.1). As remarked above this shows exactly that φ_π^f acts as π. ∎

Remark A.3. By construction we have two field actions on D: the (left) F-action defined in Lemma A.2 (1), which does commute with the Frobenius; and another one of F_a via its original structure of F_a-vector space, which does not commute in general with the Frobenius. To distinguish them, we denote the latter as a right action as we did in Proposition 6.5. Those actions clearly coincide on $\mathbb{Q}_p \subset F$, hence D is an $F \otimes_{\mathbb{Q}_p} F_a$-module.

Lemma A.4. *The following statements hold.*

(1) *The $F \otimes_{\mathbb{Q}_p} F_a$-module D is free of rank one.*

(2) *Set $D_F := D \otimes_{F_a} F$. It is a free $F \otimes_{\mathbb{Q}_p} F$-module of rank one. In particular, there is a unique F-line $W \subset D_F$, where the right F-structure of D_F coincide with the left one.*

Proof. Statement (2) follows directly from (1). (In particular, the uniqueness of the line W comes from the fact that D_F has rank one).

Let us show that D is free of rank one. Indeed the ring $R := F \otimes_{\mathbb{Q}_p} F_a$ is a product of fields, so the module D splits as a product of vector spaces on those fields. We claim that those vector spaces have all the same dimension, thus D is free as R-module by [8, AC.II, §5.3, Proposition 5, p. 113]; furthermore, since $[F : \mathbb{Q}_p] = \dim_{F_a}(D)$,

the rank of D must be one. Let us prove the claim. It follows from the fact that D is endowed with a Frobenius: if we twist the action of F_a on D by the absolute Frobenius of F_a, then the Frobenius φ_π of D induces an R-linear isomorphism

$$\Phi_\pi : \varphi^* D := D \otimes_\varphi F_a \to D,$$

mapping $m \otimes a$ to $\varphi_\pi(m) \cdot a$. Since $F \supset F_a$ and $F_a/\mathbb{Q}_p$ is a Galois extension with Galois group generated by φ, we see that the ring R splits exactly as a product of f fields (all abstractly isomorphic to F); thus the Frobenius $\mathrm{Id}_F \otimes \varphi$ acts transitively on $\mathrm{Spec}(R)$ as an f-cycle. Therefore, the claim follows from the existence of the isomorphism Φ_π. ∎

Definition A.5 (The filtered φ-module). Let us define a filtration on $D_F = D \otimes_{F_a} F$. We set

$$\mathrm{Fil}^i(D_F) := \begin{cases} D_F & \text{if } i \leq 0, \\ W & \text{if } i = 1, \\ 0 & \text{if } i \geq 2, \end{cases}$$

where the F-line W is constructed in Lemma A.4 (2). The datum

$$D_\pi := (D, \varphi_\pi, \mathrm{Fil}^\bullet(D_F))$$

altogether with the F-action, defined in Lemmas A.2–A.4, form a filtered φ-module D_π over F with F-coefficients (cf. [9][4]) that we call *the Lubin–Tate filtered φ-module associated with $\pi \in F$*. We may denote it D for brevity.

Proposition A.6. *The Lubin–Tate filtered φ-module D is admissible. Its Newton and Hodge polygons are given as in Figure* 1.
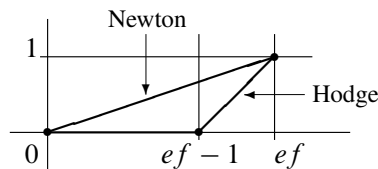

Figure 1. Newton and Hodge polygons of the Lubin–Tate filtered φ-module.

Proof. By the definition of the filtration, the Hodge–Tate slopes are: the slope zero with multiplicity $ef - 1$ and the slope one with multiplicity one. Let us compute the Newton slopes. The slope of the Frobenius φ_π are equal to the slopes of φ_π^f divided

---

[4]In their definition, set $N := 0$ as monodromy operator.

by f; cf. [30, §2.1.3]. Since φ_π^f is an F_a-linear endomorphism of D, its slopes are given by the p-adic valuation of the eigenvalues of φ_π^f, where the p-adic valuation v is normalized by $v(F_a^\times) = \mathbb{Z}$; cf. loc. cit. The eigenvalues of the matrix (A.2.3) are the eigenvalues of the diagonal block matrices $\varphi_{F_a}^i(C)$. The matrix $\varphi_{F_a}^i(C)$ is the companion matrix of the polynomial $\varphi_{F_a}^i(E_\pi(x)) \in F_a[x]$ and its eigenvalues are the roots of it. For any i, this is an Eisenstein polynomial of degree e; whence, by using the theory of the Newton polygon (for polynomials with coefficients on local fields, cf. [16, Ch. I, §6, Theorem 6.1]), we get that all the roots of $\varphi_{F_a}^i(E_\pi(x))$ have the same valuation $1/e$. Therefore, the Frobenius φ_π has only one slope, equal to $1/ef$, with multiplicity ef, and the φ-module D must be absolutely irreducible: indeed its Newton polygon does not meet points with integral coordinates, except for the vertexes $(0,0)$ and $(ef, 1)$, see the picture above. Finally, the Newton polygon of D has the same height as the Hodge polygon and lies above it; by irreducibility there are no non-trivial submodules; thus D satisfies the condition of weakly admissibility. Therefore, it is admissible by [14]. ∎

Remark A.7. The φ-module D is actually an F_a-form of the irreducible φ-module of slope $1/ef$ over $\widehat{F}^{\mathrm{nr}}$, in the Dieudonné–Manin classification; cf., for example, [30, Theorem 2]. Here $\widehat{F}^{\mathrm{nr}}$ denotes the completion of maximal unramified extension of F. Its endomorphisms as a φ-module form the central division algebra over $\mathbb{Q}_p$ with invariant $1/ef$ (cf. [40, Chapitre VI, §3, Lemma 3.3.2.2]); whereas its endomorphisms as a *filtered* φ-module gives the maximal abelian subfield F of this algebra.

A.8. Contravariant Fontaine formalism

Before attaching a period to D, let us recall the contravariant Fontaine formalism; see [20, §5.3.7]. Set
$$V_{\mathrm{cris}}^\vee(D) := \mathrm{Hom}_{\mathrm{fil}\text{-}\varphi\text{-mod}}(D, \mathrm{B}_{\mathrm{cris}}),$$
where the morphisms are taken in the category of filtered φ-modules over F; cf. [20, §4.3.3] The $\mathbb{Q}_p$-vector space $V_{\mathrm{cris}}^\vee(D)$ is a crystalline representation of G_F which is of dimension ef over $\mathbb{Q}_p$, since D is admissible. Moreover, it is endowed with an action of F, thus it is an F-vector space necessarily of dimension one.

A.9. Definition of the period

Keep previous notation; moreover, we use the notation of Sections 8.1 and 8.2, except that we are not assuming anymore that F is endowed with a non-trivial involution.

Let us attach a period α_π to $D = D_\pi$. Any morphism $x \in V_{\mathrm{cris}}^\vee(D)$ can be seen as the element
$$x_F := x \otimes \iota \in D_{\mathrm{dR}}^\vee := \mathrm{Hom}_F(D_F, \mathrm{B}_{\mathrm{dR}}),$$

where $\iota\colon F \hookrightarrow B_{\mathrm{dR}}$ is the embedding fixed in Section 8.1. It is clear by construction that for every $x \in V^\vee_{\mathrm{cris}}(D)$, we have $x_F(D_F) \subset B_{\mathrm{cris},F} \cap B^+_{\mathrm{dR}}$. Let us choose a basis w of the line $W = \mathrm{Fil}^1(D_F)$ over F and a basis $x \in V^\vee_{\mathrm{cris}}(D)$ over F. We define a period

$$\alpha_\pi := x_F(w) \in B_{\mathrm{cris},F} \cap \mathrm{Fil}^1 B_{\mathrm{dR}}. \tag{A.9.1}$$

It does not depend on the choices of w and x up to a constant in $F^\times$ and we call it the period associated with the filtered φ-module D_π. We will see in Proposition A.12 that it is a Lubin–Tate period.

A.10. A basis of eigenvectors

Recall from Section 8.1 that we have denoted by $\widetilde{F} \subset B_{\mathrm{dR}}$ the Galois closure of F and by Γ the set of $\mathbb{Q}_p$-embeddings of F in $\widetilde{F}$. For any $\sigma \in \Gamma$, consider the map

$$\varphi_\pi^{\nu(\sigma)} \otimes \sigma \colon D \otimes_{F_a} F \to D \otimes_{F_a} \widetilde{F},$$

and set $w_\sigma := (\varphi_\pi^{\nu(\sigma)} \otimes \sigma)(w) \in D_{\widetilde{F}} := D_F \otimes_F \widetilde{F}$, where w is the basis of $\mathrm{Fil}^1(D_F)$ chosen in (A.9.1). We have $w_\sigma \neq 0$ since $w \neq 0$ and $\varphi_\pi^{\nu(\sigma)} \otimes \sigma$ is injective. By definition of ι, we get $w_\iota = w \otimes 1$.

Lemma A.11. *For any $z \in F$ and $d \in D_{\widetilde{F}} = D \otimes_{F_a} \widetilde{F}$, denote by $z(d)$ the action induced by the action of F on D and by $d \cdot z$ the action by multiplication by the scalar $z \in \widetilde{F}$ on the right. Let $w_\sigma \in D_{\widetilde{F}}$ be the element defined in Section A.10. The following are true.*

(1) *For every $\sigma \in \Gamma$, we have $\varphi_\pi^f(w_\sigma) = \sigma(\pi)w_\sigma$.*

(2) *For every $\sigma \in \Gamma$ and for every $z \in F$, we have $z(w_\sigma) = w_\sigma \cdot \sigma(z)$.*

(3) *The family $(w_\sigma)_{\sigma \in \Gamma}$ forms a basis of $D_{\widetilde{F}}$ as vector space over $\widetilde{F}$.*

Proof. The proof of (1) is a formal argument:

$$\varphi_\pi^f(w_\sigma) = (\varphi_\pi^{f+\nu(\sigma)} \otimes \sigma)(w) = (\varphi_\pi^{\nu(\sigma)} \otimes \sigma)(\varphi_\pi^f(w))$$
$$= (\varphi_\pi^{\nu(\sigma)} \otimes \sigma)(\pi w) = \sigma(\pi)w_\sigma.$$

Let us prove (2):

$$z(w_\sigma) = z((\varphi_\pi^{\nu(\sigma)} \otimes \sigma)(w)) = (z\varphi_\pi^{\nu(\sigma)} \otimes \sigma)(w)$$
$$= (\varphi_\pi^{\nu(\sigma)}z \otimes \sigma)(w) = (\varphi_\pi^{\nu(\sigma)} \otimes \sigma)(z(w)),$$

since the F-action on D commutes with φ_π by construction. Now by definition of w, we have $z(w) = w \cdot z$, hence we finish by

$$(\varphi_\pi^{\nu(\sigma)} \otimes \sigma)(z(w)) = (\varphi_\pi^{\nu(\sigma)} \otimes \sigma)(w \cdot z) = w_\sigma \cdot \sigma(z).$$

Point (3) follows from (2): the w_σ are eigenvectors of multiplicity one diagonalizing the left action of F on $D_{\widetilde{F}}$. ∎

Proposition A.12. *Let* $\pi \in F$ *be a uniformizer and* D *the associated Lubin–Tate filtered* φ-*module from Definition* A.5. *Then the element* $\alpha := \alpha_\pi \in \mathrm{B}_{\mathrm{cris},F} \cap \mathrm{Fil}^1\mathrm{B}_{\mathrm{dR}}$ *associated with it in* (A.9.1) *is a Lubin–Tate period relative to* $\pi \in F$ *in the sense of Definition* 9.1.

Proof. We compute

$$
\begin{aligned}
\varphi_{\mathrm{cris}}^f(\alpha) &= \big((\varphi_{\mathrm{cris}}^f \otimes \mathrm{Id}_F) \circ (x \otimes \iota)\big)(w) \\
&= \big((x \otimes \iota) \circ (\varphi_\pi^f \otimes \mathrm{Id}_F)\big)(w) \\
&= (x \otimes \iota)(w \cdot \pi) = \pi\alpha,
\end{aligned}
$$

so condition (3) of Definition 9.1 holds.

Let us check the other properties of Definition 9.1. By construction $v_{\mathrm{dR}}(\alpha) \geq 1$ and, for all $\sigma \in \Gamma\backslash\{\iota\}$, we compute

$$
\begin{aligned}
\sigma_{\mathrm{cris}}(\alpha) &= \big((\varphi_{\mathrm{cris}}^{\nu(\sigma)} \otimes \sigma) \circ (x \otimes \iota)\big)(w) \\
&= \big((x \otimes \mathrm{Id}_{\widetilde{F}}) \circ (\varphi_\pi^{\nu(\sigma)} \otimes \sigma)\big)(w) = (x \otimes \mathrm{Id}_{\widetilde{F}})(w_\sigma), \quad\quad\text{(A.12.1)}
\end{aligned}
$$

and in particular $v_{\mathrm{dR}}(\sigma_{\mathrm{cris}}(\alpha)) \geq 0$, since $(x \otimes \mathrm{Id}_{\widetilde{F}})(D_{\widetilde{F}}) \subset \mathrm{B}_{\mathrm{cris},\widetilde{F}} \cap \mathrm{B}_{\mathrm{dR}}^+$.

Let us define

$$
P := \prod_{\sigma \in \Gamma} \sigma_{\mathrm{cris}}(\alpha) \in \mathrm{B}_{\mathrm{cris},\widetilde{F}}. \quad\quad\text{(A.12.2)}
$$

We claim the following:

(1) P belongs to $\mathrm{B}_{\mathrm{cris},F}$;

(2) P is invertible in $\mathrm{B}_{\mathrm{cris},F}$;

(3) $v_{\mathrm{dR}}(P) = 1$.

These claims conclude the proof. Indeed the equality

$$
\sum_{\sigma \in \Gamma} v_{\mathrm{dR}}(\sigma_{\mathrm{cris}}(\alpha)) = v_{\mathrm{dR}}(P) = 1
$$

implies that the inequalities we have shown above on the $v_{\mathrm{dR}}(\sigma_{\mathrm{cris}}(\alpha))$ must be equalities. Moreover, α is invertible in $\mathrm{B}_{\mathrm{cris},F}$, since it divides P which is invertible.

Let us first prove claim (1). Consider the Galois group $G := \mathrm{Gal}(\widetilde{F}/F')$, where $F' := \widetilde{F}_a F$. Let G act on $\mathrm{B}_{\mathrm{cris},\widetilde{F}} = \mathrm{B}_{\mathrm{cris}} \otimes_{\widetilde{F}_a} \widetilde{F}$ as the identity on $\mathrm{B}_{\mathrm{cris}}$ and via its natural action on the right. By its very construction P is stable under G, hence it belongs to

$$
(\mathrm{B}_{\mathrm{cris},\widetilde{F}})^G = \mathrm{B}_{\mathrm{cris},F'} = \mathrm{B}_{\mathrm{cris}} \otimes_{\widetilde{F}_a} \widetilde{F}_a F = \mathrm{B}_{\mathrm{cris}} \otimes_{F_a} F = \mathrm{B}_{\mathrm{cris},F}.
$$

In order to show claims (2) and (3), consider the determinant Δ of the filtered φ-module D:

$$\Delta := \det{}_{F_a}(D) \overset{ef}{=} \bigwedge D.$$

It is admissible of slope one (both Newton and Hodge), hence $V_{\mathrm{cris}}^{\vee}(\Delta)$ is a rank one crystalline representation of Hodge–Tate weight one. To conclude, it is enough to show that, up to a constant in $F^{\times}$, P is the period $\delta \in B_{\mathrm{cris}}^{\times}$ of Δ.

The left F-action on $V_{\widetilde{F}} := V_{\mathrm{cris}}^{\vee}(D) \otimes_{\mathbb{Q}_p} \widetilde{F}$ decomposes the element $x \otimes \mathrm{Id}_{\widetilde{F}}$ as a sum of eigenvectors

$$x \otimes \mathrm{Id}_{\widetilde{F}} = \sum_{\sigma \in \Gamma} x_{\sigma}.$$

By Lemma A.11 (3), $(w_{\sigma})_{\sigma \in \Gamma}$ is a basis of $D_{\widetilde{F}}$, and similarly $(x_{\sigma})_{\sigma \in \Gamma}$ is a basis of $V_{\widetilde{F}}$. We have $x_{\tau}(w_{\sigma}) = 0$ whenever $\tau \neq \sigma$, because x_{τ} and w_{σ} live in eigenlines of different eigenvalues. Hence, up to a constant u in $\widetilde{F}^{\times}$, we can compute the period δ by using these bases, as

$$u\delta = \prod_{\sigma \in \Gamma}(x_{\sigma} \otimes \mathrm{Id}_{\widetilde{F}})(w_{\sigma}) = \prod_{\sigma \in \Gamma}(x \otimes \mathrm{Id}_{\widetilde{F}})(w_{\sigma}),$$

which is equal to P by (A.12.1). Finally, by claim (1) the constant u must belong to $F^{\times}$. ∎

Remark A.13. The period δ of Δ can be computed directly by taking the determinant of the matrix A of (A.1.1). This leads to a finer expression of P as

$$P = uc_{\eta}t \in B_{\mathrm{cris},F}^{\times},$$

where $u \in F^{\times}$, t is the period of $\mathbb{Q}_p(1)$ (cf. Section 2.7 (6)), and $c_{\eta} \in W(\overline{\mathbb{F}}_p)^{\times}$ is the period of an unramified character $\eta \colon G_F \to \mathbb{Z}_p^{\times}$. Notice that such a period c_{η} lies in $W(\overline{\mathbb{F}}_p)^{\times} \subset B_{\mathrm{cris}}^{\times}$ since the cohomology group $H^1(G_{k_F}, W(\overline{\mathbb{F}}_p)^{\times})$ is trivial by [43, III-33 Lemma].

More precisely, the period c_{η} can be chosen in $W(\overline{\mathbb{F}}_p)^{\times}$, up to an invertible in $\mathbb{Z}_p$, as the solution of the equation

$$\frac{\varphi(c_{\eta})}{c_{\eta}} = \frac{\det A}{p},$$

which exists by [38, Ch. V, §2, Lemma (2.1)] since $\det A$ is a uniformizer of $W(k_F)$.

Furthermore, notice that the action of the Frobenius φ^f of $B_{\mathrm{cris},F}$ on P is easy to compute as

$$\varphi^f(P) = N_{F/\mathbb{Q}_p}(\pi)P.$$

Proposition A.14. *Let $\pi \in F$ be a uniformizer and D the associated Lubin–Tate filtered φ-module, from Definition* A.5. *Then the crystalline representation $V_{\mathrm{cris}}^{\vee}(D)$ is given by the Lubin–Tate character associated with $\pi \in F$.*

Proof. Let $\mathfrak{F}$ be a Lubin–Tate group over $\mathcal{O}_F$ associated with π. Let T be its Tate module, given by the Lubin–Tate character $\psi \colon G_F \to \mathcal{O}_F^{\times}$; set $V := T[1/p]$. We want to show that $V_{\mathrm{cris}}^{\vee}(D) = V$. Colmez constructs an element $t_{\mathfrak{F}} \in \mathrm{B}_{\mathrm{cris}, F}$, satisfying Definition 9.1 and such that for every $g \in G$, we have

$$g(t_{\mathfrak{F}}) = \psi(g)t_{\mathfrak{F}}\,;$$

see, for example, [22, §3.6].

Let $\alpha = \alpha_{\pi} \in \mathrm{B}_{\mathrm{cris}, F}^{\times}$ be the Lubin–Tate period in Proposition A.12. By Lemma 8.5, the period $t_{\mathfrak{F}}\alpha^{-1}$ belongs to $F^{\times}$, whence for every $g \in G_F$, we also have

$$g(\alpha) = \psi(g)\alpha.$$

The map

$$V_{\mathrm{cris}}^{\vee}(D) \to \mathrm{B}_{\mathrm{cris}, F}\,,$$

defined by $x \mapsto (x \otimes \iota)(w)$, is G_F-equivariant by definition of the action of G_F on $V_{\mathrm{cris}}^{\vee}(D)$ and it is F-linear by definition of w; cf. (A.9.1). Therefore, this map identifies $V_{\mathrm{cris}}^{\vee}(D)$ to $F \cdot \alpha \subset \mathrm{B}_{\mathrm{cris}, F}$ and that completes the proof. ∎

Remark A.15. The representation $V_{\mathrm{cris}}^{\vee}(D)$ is an example of F-crystalline representation; cf. [32, Introduction]. For a more general construction relating F-crystalline representations to their filtered φ-modules, see [32, §(3.3)] and [11, §4.1].

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