Elemente der Mathematik 80 (2025), 166–169
DOI 10.4171/EM/537

Short note **Generalizations of the Gandhi formula for prime numbers**

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In memory of my mother Marta Bértiz (1945–2022)

1 Introduction

We shall need the Möbius function $\mu(n)$, which is one of the more important arithmetic functions. The Möbius function is defined as follows: $\mu(1) = 1$; if n is the product of r distinct primes, then $\mu(n) = (-1)^r$, and if the square of a prime divides n, then $\mu(n) = 0$. We shall need the following well-known property of the Möbius function:

$$\sum_{d\,|\,n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n > 1. \end{cases}$$

Let $P_{n-1} = p_1 p_2 \cdots p_{n-1}$. In 1971 [1] ([3, pages 182–183]), Gandhi proved the following formula for p_n in terms of the former primes $p_1, p_2, \dots, p_{n-1}$:

$$p_n = \left\lfloor 1 - \log_2\left(-\frac{1}{2} + \sum_{d\,|\,P_{n-1}} \frac{\mu(d)}{2^d - 1}\right)\right\rfloor. \tag{1}$$

Gandhi also proved that p_n is the only integer such that

$$1 < 2^{p_n}\left(-\frac{1}{2} + \sum_{d\,|\,P_{n-1}} \frac{\mu(d)}{2^d - 1}\right) < 2. \tag{2}$$

In 1972 [4] ([3, pages 182–183]), a short and simple proof of Gandhi's formula was given by Vanden Eynden.

In 1974 [2] ([3, pages 184–185]), Golomb gave another short and simple proof of Gandhi's formula. He described it as being the sieve of Eratosthenes performed on the binary expansion of 1, namely $1 = 0,11111\ldots$.

In this note, we generalize Gandhi's formula replacing 2 by any positive integer $k \geq 2$. In our main Theorem 2, we follow Vanden Eynden's proof.

Golomb's proof also works in this generalization if we assign to each positive integer n the weight $W(n) = k^{-n}$. Golomb in his proof used the weight $W(n) = 2^{-n}$, since he worked with $k = 2$.

2 Main results

An almost direct consequence of Gandhi's formula is following theorem.

Theorem 1. *Let $d_n = p_n - p_{n-1}$ and $n > 2$. Then d_n is the even number between the two numbers $\lfloor A \rfloor$ and $\lfloor A \rfloor + 1$, where*

$$A = -\log_2\left(1 - \frac{\sum_{d\mid P_{n-2}} \frac{\mu(d)}{2^{dp_{n-1}}-1}}{-\frac{1}{2} + \sum_{d\mid P_{n-2}} \frac{\mu(d)}{2^d-1}}\right). \tag{3}$$

Proof. Gandhi's formula (1) can be written in the form

$$\begin{aligned} p_n &= \left\lfloor 1 - \log_2\left(-\frac{1}{2} + \sum_{d\mid P_{n-2}} \frac{\mu(d)}{2^d - 1} + \sum_{d\mid P_{n-2}} \frac{\mu(dp_{n-1})}{2^{dp_{n-1}} - 1}\right)\right\rfloor \\ &= \left\lfloor 1 - \log_2\left(-\frac{1}{2} + \sum_{d\mid P_{n-2}} \frac{\mu(d)}{2^d - 1}\right) - \log_2\left(1 - \frac{\sum_{d\mid P_{n-2}} \frac{\mu(d)}{2^{dp_{n-1}}-1}}{-\frac{1}{2} + \sum_{d\mid P_{n-2}} \frac{\mu(d)}{2^d-1}}\right)\right\rfloor. \end{aligned}$$

Therefore, $p_n + \varepsilon_1 = p_{n-1} + \varepsilon_2 + A$, where $0 \le \varepsilon_1 < 1$ and $0 \le \varepsilon_2 < 1$. It is well known that $\lfloor x \rfloor + \lfloor y \rfloor \le \lfloor x + y \rfloor \le \lfloor x \rfloor + \lfloor y \rfloor + 1$. Hence we have

$$p_{n-1} + \lfloor A \rfloor \le p_n \le p_{n-1} + \lfloor A \rfloor + 1,$$

that is,

$$\lfloor A \rfloor \le d_n = p_n - p_{n-1} \le \lfloor A \rfloor + 1.$$

This concludes the proof of the theorem. ∎

The following theorem is our main result. The proof follows very closely the proof of Vanden Eynden.

Theorem 2. *Let $k \ge 2$ be a positive integer. Then the following formulas hold:*

$$1 < k^{p_n}\left(-\frac{1}{k} + \sum_{d\mid P_{n-1}} \frac{\mu(d)}{k^d - 1}\right) < 2, \tag{4}$$

$$p_n = \left\lfloor \log_k 2 - \log_k\left(-\frac{1}{k} + \sum_{d\mid P_{n-1}} \frac{\mu(d)}{k^d - 1}\right)\right\rfloor, \tag{5}$$

$$\lim_{k\to\infty}\left(k^{p_n}\left(-\frac{1}{k} + \sum_{d\mid P_{n-1}} \frac{\mu(d)}{k^d - 1}\right)\right) = 1, \tag{6}$$

$$p_n = \lim_{k\to\infty}\left(-\log_k\left(-\frac{1}{k} + \sum_{d\mid P_{n-1}} \frac{\mu(d)}{k^d - 1}\right)\right), \tag{7}$$

$$d_n = p_n - p_{n-1} = \lim_{k\to\infty}\left(-\log_k\left(1 - \frac{\sum_{d\mid P_{n-2}} \frac{\mu(d)}{k^{dp_{n-1}}-1}}{-\frac{1}{k} + \sum_{d\mid P_{n-2}} \frac{\mu(d)}{k^d-1}}\right)\right). \tag{8}$$

Proof. We put $Q = P_{n-1} = p_1 p_2 \cdots p_{n-1}$, $p_n = p$ and

$$S = \sum_{d|Q} \frac{\mu(d)}{k^d - 1}.$$

Therefore, we get

$$\begin{aligned}(k^Q - 1)S &= \sum_{d|Q} \mu(d) \frac{k^Q - 1}{k^d - 1} \\ &= \sum_{d|Q} \mu(d)(1 + k^d + k^{2d} + \cdots + k^{Q-d}) = \sum_{t=0}^{Q-1} a_t k^t,\end{aligned}$$

where $a_t = \sum_{d|\gcd(t,Q)} \mu(d)$; in particular, for $t = 0$, this is equal to $\sum_{d|Q} \mu(d)$. Consequently, by well-known properties of the function μ, we have

$$(k^Q - 1)S = \sum_{t=0}^{Q-1} a_t k^t = \sum_{\substack{1 \le t \le Q-p \\ \gcd(t,Q)=1}} k^t + k^{Q-1}$$

and consequently

$$k(k^Q - 1)\Big(-\frac{1}{k} + S\Big) = -(k^Q - 1) + \sum_{\substack{1 \le t \le Q-p \\ \gcd(t,Q)=1}} k^{t+1} + k^Q = 1 + \sum_{\substack{1 \le t \le Q-p \\ \gcd(t,Q)=1}} k^{t+1}.$$

Hence

$$\begin{aligned}1 = k^p \frac{k^{Q-p+1}}{k^{Q+1}} &< k^p\Big(-\frac{1}{k} + S\Big) = \frac{k^p}{k(k^Q - 1)}\Big(1 + \sum_{\substack{1 \le t \le Q-p \\ \gcd(t,Q)=1}} k^{t+1}\Big) \\ &< k^p \frac{1 + k + k^2 + \cdots + k^{Q-p+1}}{k(k^Q - 1)} = \frac{k^p(k^{Q-p+2} - 1)}{(k-1)k(k^Q - 1)} \\ &\le \frac{k}{k-1} \le 2,\end{aligned}$$

that is,

$$1 < k^p\Big(-\frac{1}{k} + S\Big) < \frac{k}{k-1} \le 2. \tag{9}$$

This proves equation (4). Equation (5) is an easy consequence of equation (4). Equation (6) is an easy consequence of equation (9), since $\frac{k}{k-1} \to 1$ as $k \to \infty$. Equation (7) is an easy consequence of equation (6). Equation (8) can be proved as in Theorem 1 and by using equation (7). ∎

Acknowledgments. The author would like to thank the anonymous reviewer for carefully reading the article and for his/her helpful comments. The author is also very grateful to the Universidad Nacional de Luján.

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