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Weighted Weyl estimates near an elliptic trajectory

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Abstract. Let $\psi_j^\hbar$ and $E_j^\hbar$ denote the eigenfunctions and eigenvalues of a Schrödinger-type operator $H_\hbar$ with discrete spectrum. Let $\psi_{(x,\xi)}$ be a coherent state centered at a point (x,ξ) belonging to an elliptic periodic orbit, γ of action S_γ and Maslov index σ_γ. We consider "weighted Weyl estimates" of the following form: we study the asymptotics, as $\hbar \longrightarrow 0$ along any sequence

$$\hbar = \frac{S_\gamma}{2\pi l - \alpha + \sigma_\gamma},$$

$l \in \mathbb{N}$, $\alpha \in \mathbb{R}$ fixed, of

$$\sum_{|E_j - E| \leq c\hbar} |\, (\psi_{(x,\xi)}, \psi_j^h)\, |^2 .$$

We prove that the asymptotics depend strongly on α-dependent arithmetical properties of c and on the angles θ of the Poincaré mapping of γ. In particular, under irrationality assumptions on the angles, the limit exists for a non-open set of full measure of c's. We also study the regularity of the limit as a function of c.

1. Introduction and results.

Consider a Schrödinger operator $H = -\hbar^2 \Delta + V(x)$ with V smooth, either on $M = \mathbb{R}^m$ (in which case we assume V tends to infinity at infinity and therefore H has discrete spectrum) or on a compact Riemannian manifold, M. In [7] we considered "trace formulae" associated to projectors on coherent states in the following sense. For $(x, \xi) \in \mathbb{R}^{2m}$ and $a \in \mathcal{S}(\mathbb{R}^m)$ define the coherent state $\psi^a_{x\xi}$ as:

$$(1) \quad \psi^a_{(x,\xi)}(y) = \rho(y-x)\,(2\pi\hbar)^{-3m/4}\,2^{-m/4}\,e^{-ix\xi/2\hbar}\,e^{i\xi y/\hbar}\,\hat{a}\Big(\frac{y-x}{\sqrt{\hbar}}\Big)\,.$$

Here ρ is a cut-off function near zero and $\hat{a}$ is the Fourier transform of a, (in the manifold case $(x, \xi) \in T^*M$ and the above definition is in local coordinates near x). Let ψ_j and E_j the eigenfunctions and eigenvalues of H. Then if φ is a Schwartz function whose Fourier transform is compactly supported and $E = |\xi|^2 + V(x)$, we have

$$(2) \qquad \sum_j \varphi\Big(\frac{E_j - E}{\hbar}\Big)\;|\,(\psi_{(x,\xi)}, \psi_j)\,|^2 \sim \sum_{j=0} c^{\varphi}_j(x,\xi)\,\hbar^{-m+1/2+j}\,,$$

for $\hbar \longrightarrow 0$. (If $E \neq |\xi|^2 + V(x)$, the left-hand side tends to 0 rapidly in $\hbar$.) Although the form of the asymptotic expansion does not depend on (x, ξ), the coefficient $c_0(x, \xi)$ is highly sensitive to the point (x, ξ) being periodic or not with respect to the classical flow. In case (x, ξ) is either not periodic or is on a *hyperbolic* trajectory, we proved in [7] (using a Tauberian theorem) that, for every $c \in \mathbb{R}$,

$$(3) \qquad \sum_{|E_j - E| \le c\hbar} |\,(\psi_{(x,\xi)}, \psi_j)\,|^2 = c_0^{\chi_{[-c,c]}}(x,\xi)\,\hbar^{-m+1/2} + o(\hbar^{-m+1/2})\,,$$

as $\hbar \to 0$ possibly along certain sequence. (Here $\chi_{[-c,c]}$ is the characteristic function of the interval $[-c, c]$.) The main goal of this paper is to study the case where (x, ξ) belongs to an elliptic closed trajectory.

Our results are related to the existence of quasi-modes near an elliptic trajectory. Recall that if H is as before and γ is a closed elliptic trajectory of the Hamiltonian $|\xi|^2 + V(x)$ with energy E, period T_γ, action S_γ, Maslov index σ_γ and Poincaré mapping of angles θ_j, $j = 1, \dots, m-1$, then one can construct (see [9], [3], [8], [7]) quasi-modes of H (namely solutions of the Schrödinger equation modulo a remainder), microlocalized near γ, of quasi-energies

$$(4) \qquad E^{k,l}_{QM} = E + \frac{\hbar}{T_\gamma}\Big(\Big(2\pi l - \frac{S_\gamma}{\hbar}\Big) + \sum_{j=1}^{m-1}\Big(k_j + \frac{1}{2}\Big)\theta_j + \sigma_\gamma\Big)\,,$$

for $(k,l)\in\mathbb{Z}^m$, l large. The remainder is $O(\hbar^2)$ uniformly as

$$\Big|2\pi l-\frac{S\gamma}{\hbar}\Big| \qquad \text{and} \qquad |k|:=\sum k_j$$

remain bounded. The existence of these quasi-modes implies that part of the spectral density of H concentrates near the quasi-energies defined by (4), but this doesn't say anything about $E_{QM}^{k,l}$ as $\mid k\mid\longrightarrow\infty$ and does not involve the rest of the spectrum. The results of this paper will indicate that the rescaled localized spectral density

$$\sum_j \delta\Big(\frac{E_j-\lambda}{\hbar}\Big)\mid(\psi_{(x,\xi)},\psi_j)\mid^2 \tag{5}$$

(which is the rescaled spectral density microlocalized at the point in phase space (x,ξ)) has a certain semiclassical limit whose singularities are indeed precisely the quasi-energies (4), and this time with no restriction on $\mid k\mid$.

We will now state our results, valid for more general quantum Hamiltonians: Let $H_\hbar=\sum_{l=0}^{L}\hbar^l P_l(x,D_x)$ where P_l is a differential operator of order l on $\mathbb{R}^m$ (or M) of principal symbol P_l^0, sub-principal symbol P_l^{-1} (formally P_l is regarded as acting on half-densities) and smooth coefficients. Let $\mathcal{H}(x,\xi)=\sum_{l=0}^{L}P_l^0(x,\xi)$ and $\mathcal{H}_{\mathrm{sub}}(x,\xi)=\sum_{l=0}^{L}P_l^{-1}(x,\xi)$ be the principal and sub-principal symbols of $H_\hbar$. We assume that P_L is elliptic, $\mathcal{H}$ is positive, and in case $M=\mathbb{R}^m$, that $\mathcal{H}$ tends polynomially to infinity at infinity. We will also suppose for simplicity that $\mathcal{H}_{\mathrm{sub}}(x,\xi)=0$.

Let $E_j^\hbar$ and $\psi_j^\hbar$ denote the eigenvalues and eigenvectors of $H_\hbar$. Let us suppose that (x,ξ) belongs to an elliptic trajectory of period T_γ, action S_γ, Maslov index σ_γ and Poincaré mapping of angles $\theta=(\theta_1,\dots,\theta_{m-1})$. We will use throughout the notations

$$k=(k_1,\dots,k_{m-1})\in\mathbb{N}^{m-1}\,,$$

$$k\,\theta:=\sum_{j=1}^{m-1}k_j\theta_j \qquad \text{and} \qquad \Big(k+\frac12\Big)\theta:=\sum_{j=1}^{m-1}\Big(k_j+\frac12\Big)\theta_j\,. \tag{6}$$

Theorem 1.1. *Assume that* $\theta_1/(2\pi), \ldots, \theta_{m-1}/(2\pi)$ *are rational. Then, for every* $\alpha \in [0, 2\pi)$, *as* $\hbar \to 0$ *along the sequence*

$$\hbar = \frac{S_\gamma}{2\pi l - \alpha + \sigma_\gamma}\,, \qquad l \in \mathbb{N}, \tag{7}$$

one has

$$\sum_{|E_j - E| \leq c\hbar} |\, (\psi_{(x,\xi)}, \psi_j)\, |^2 = \hbar^{-m+1/2} \mathcal{L}_\alpha(c) + o(\hbar^{-m+1/2})\,, \tag{8}$$

for all c *such that*

$$c \neq \pm \frac{1}{T_\gamma}\Big(2\pi j + \Big(k + \frac{1}{2}\Big)\theta + \alpha\Big)\,, \qquad \text{for all } j \in \mathbb{Z}\,,\ k \in \mathbb{N}^{m-1}\,. \tag{9}$$

Moreover, as a function of c *the limit* $\mathcal{L}_\alpha(c)$ *is a step function constant on the intervals defined by* (9).

Next we consider the irrational case:

Theorem 1.2. *Assume that* $1, \theta_1/(2\pi), \ldots, \theta_{m-1}/(2\pi)$ *are linearly independent over the rationals. Then there exists a set* $\mathcal{M}^\alpha$ *of values of* c, *of full Lebesgue measure, such that for all* $c \in \mathcal{M}^\alpha$

$$\sum_{|E_j - E| \leq c\hbar} |\, (\psi_{(x,\xi)}, \psi_j)\, |^2 = \hbar^{m-1/2} \mathcal{L}_\alpha(c) + o(\hbar^{m-1/2})\,, \tag{10}$$

for $\hbar$ *as in* (7). *Moreover, as a function of* c, $\mathcal{L}_\alpha(c)$ *is locally Lipschitz on* $\mathcal{M}^\alpha$ *in the sense that for all* $c \in \mathcal{M}^\alpha$ *there exists* $\beta_c > 0$ *such that,*

$$|\, \mathcal{L}_\alpha(c') - \mathcal{L}_\alpha(c)\, | \leq \beta_c\, |\, c' - c\, |\,, \qquad \text{for all } c' \in \mathcal{M}^\alpha\,. \tag{11}$$

Finally there exists a rapidly decreasing family $\{g_k\}_{k \in \mathbb{N}^{m-1}}$ *(related to the microlocalization of the symbol* a *of* $\psi_{(x,\xi)}$*) such that*

$$\{c : \text{ for all } k \in \mathbb{N}^{m-1}\ \ |\, 1 - e^{i(cT_\gamma + (k+1/2)\theta + \alpha)}\, | > \varepsilon\, g_k\} \subset \mathcal{M}^\alpha\,, \tag{12}$$

for all $\varepsilon > 0$. *(For a precise definition of the set* $\mathcal{M}^\alpha$ *see Lemma* 3.3.*).*

REMARK. In the rational case the discontinuities of the function $\mathcal{L}_\alpha$ are located exactly at the values of the $E^{k,l}_{QM}$ defined before by (4), for the values of $\hbar$ given by (7). In the irrational case in order to prove that $\mathcal{L}_\alpha(c)$ exists we need that c be at some distance from the quasi-energies $E^{k,l}_{QM}$ (unless the symbol a of the quasi-mode is chosen very judiciously, in which case we can work with c in the complement of the set of all quasi-energies). In all cases this suggests that the weighted spectral measure, (5), in the semi-classical limit, is particularly singular exactly at the values of the $E^{k,l}_{QM}$ defined before. We hope to provide a rigorous proof of a precise statement of this elsewhere.

The paper is organized as follows: In Section 3 we prove the existence of the functions $\mathcal{L}_\alpha$ which are studied in Section 4. In Section 5 we finish the proof of the main Theorems, using a Tauberian argument that we recall in Section 2. Finally, in the appendix we review and extend slightly a result on Hölder continuity of function such as $\mathcal{L}_\alpha$ using wavelets.

2. A Tauberian lemma.

In this section we refine the Tauberian lemma of [2] and [7].

Consider an expression of the following form

$$\Upsilon^w_{E,\hbar}(\varphi) = \sum_j w_j(\hbar)\, \varphi\Big(\frac{E_j(\hbar) - E}{\hbar}\Big)\,, \tag{13}$$

defined for all $\varphi \in \mathcal{R}$ where $\mathcal{R}$ will henceforth denote the set of all Schwartz functions on the line with compactly supported Fourier transform.

Let $\mathcal{M}^\alpha$ a subset of $\mathbb{R}^+$ of full Lebesgue measure in a bounded interval.

We introduce the following notations. Fix a positive function $f \in \mathcal{R}$ satisfying $f(0) = 1$ and $\hat{f}(0) = 1$. For every $a > 0$, define

$$f_a(r) := a^{-1} f\Big(\frac{r}{a}\Big) \tag{14}$$

and for every $a > 0$ and $c > 0$

$$\varphi_{a,c} := f_a * \chi_{[-c,c]}\,, \tag{15}$$

where $\chi_{[-c,c]}$ is the characteristic function of the interval $[-c,c]$.

The Tauberian lemma in question is:

Theorem 2.1 (See [2] and [7]). *Let $\mathcal{M}^\alpha$ a subset of $\mathbb{R}^+$ of full Lebesgue measure in a bounded interval. Suppose $w_j(\hbar)$, $E_j(\hbar)$, E and $\Upsilon^w_\hbar$ itself satisfy all of the following:*

1) *There exists a positive function $\omega(\hbar)$, defined on an interval $(0,\hbar_0)$, and a functional $\mathcal{F}_0$ on $\mathcal{R}$, such that for all $\varphi\in\mathcal{R}$*

$$\Upsilon^w_{E,\hbar}(\varphi)=\mathcal{F}_0(\varphi)\,\omega(\hbar)+o(\omega(\hbar))\,,\qquad \hbar\longrightarrow 0\,. \tag{16}$$

2) *for all $c\in\mathcal{M}^\alpha$ the limit*

$$\mathcal{L}_\alpha(c)=\lim_{a\to 0}\mathcal{F}_0(\varphi_{a,c})$$

exists.

3) *$\mathcal{L}_\alpha$ is a continuous function on $\mathcal{M}^\alpha$.*

4) *There exists a $k\in\mathbb{Z}$ such that $\hbar^k=\mathcal{O}(\omega(\hbar))$, $\hbar\to 0$.*

5) *There exists an $\varepsilon>0$ such that for every φ there is a constant C_φ such that for all $E'\in[E-\varepsilon,E+\varepsilon]$*

$$|\Upsilon^w_{E',\hbar}(\varphi)|\le C_\varphi\,\omega(\hbar) \tag{17}$$

(rough uniformity in E).

6) *The $w_j(\hbar)$ are non-negative and bounded: there exists a constant $C\ge 0$ such that for all j and all $\hbar, 0<\hbar<\hbar_0$*

$$0\le w_j(\hbar)\le C\,. \tag{18}$$

7) *The eigenvalues $E_j(\hbar)$ satisfy the following rough estimate: for each C_1 there exist constants C_2, N_0 such that for all k*

$$\#\{j: E_j(\hbar)\le C_1+k\hbar\,\}\le C_2(\hbar^{-1}k)^{N_0}\,. \tag{19}$$

Define the weighted counting function by

$$N^w_{E,c}(\hbar)=\sum_{j\,;\,|x_j(\hbar)|\le c} w_j(\hbar)\,, \tag{20}$$

where

$$x_j(\hbar) := \frac{E_j(\hbar) - E}{\hbar}\,. \tag{21}$$

Then the conclusion is: for all $c \in \mathcal{M}^\alpha$,

$$N^w_{E,c}(\hbar) = \mathcal{L}_\alpha(c)\,\omega(\hbar) + o(\omega(\hbar))\,, \qquad \hbar \longrightarrow 0\,. \tag{22}$$

PROOF. Except for the fact that the set $\mathcal{M}^\alpha$ of allowed c's is not $\mathbb{R}^+$, this theorem is precisely [2, Theorem 6.3]. Proceeding exactly as in the proof of the [2, inequalities (188)], one shows that for all $R > 0$, for all $N \in \mathbb{N}$ exists $C > 0$, $C_N > 0$ such that for all $a \in (0, R)$ and for all η, $0 < \eta < c$,

$$\begin{aligned} \frac{1}{\omega(\hbar)}\Big(1 - C\,\frac{a}{\eta}\Big) N_{E,c-\eta}(\hbar) &\le \frac{1}{\omega(\hbar)}\,\Upsilon_{E,\hbar}(\varphi_{a,c}) \\ &\le \frac{1}{\omega(\hbar)}\,N_{E,c+\eta}(\hbar) + C_N\Big(\frac{a}{\eta}\Big)^N\,. \end{aligned} \tag{23}$$

Let $c \in \mathcal{M}^\alpha$ be given. We begin by observing that by the first of the inequalities (23)

$$\frac{1}{\omega(\hbar)}\,N_{E,c}(\hbar) \le \frac{1}{\omega(\hbar)}\,\Upsilon_{E,\hbar}(\varphi_{a,c+\eta}) + C_1\,\frac{a}{\eta}\,, \tag{24}$$

where we have also used the fact that $N_{E,c}(\hbar)/\omega(\hbar)$ is bounded (a trivial consequence of (16)). For every η such that $0 < \eta < c$ one can take the limit in (24) as $\hbar \longrightarrow 0$ to obtain that

$$\limsup_{\hbar \to 0} \frac{1}{\omega(\hbar)} N_{E,c}(\hbar) \le \mathcal{F}_0(\varphi_{a,c+\eta}) + C_1\,\frac{a}{\eta}\,. \tag{25}$$

If we now assume that $\eta + c \in \mathcal{M}^\alpha$ we can take the limit as $a \longrightarrow 0$ to obtain

$$\limsup_{\hbar \to 0} \frac{1}{\omega(\hbar)}\,N_{E,c}(\hbar) \le \mathcal{L}_\alpha(c + \eta)\,. \tag{26}$$

By the assumption that $\mathcal{M}^\alpha$ has full measure, we can find a sequence $\{\eta_j\}$ such that for all j, $c + \eta_j \in \mathcal{M}^\alpha$ and $\eta_j \longrightarrow 0$. Taking the limit in

(26) of $\mathcal{L}_\alpha(c+\eta_j)$ as $j \longrightarrow \infty$ and using the fact that $\mathcal{L}_\alpha$ is continuous at c we obtain

$$\limsup_{\hbar\to 0} \frac{1}{\omega(\hbar)} N_{E,c}(\hbar) \le \mathcal{L}_\alpha(c)\,. \tag{27}$$

A similar argument starting with the second inequality (23) shows that

$$\liminf_{\hbar\to 0} \frac{1}{\omega(\hbar)} N_{E,c}(\hbar) \ge \mathcal{L}_\alpha(c)\,, \tag{28}$$

which finishes the proof.

3. The existence of $\mathcal{L}_\alpha(c)$.

In this section we prove the existence of the coefficients $\mathcal{L}_\alpha(c)$ in the limits (8) and (10) (see (36) below).

Lemma 3.1. *There exists a rapidly decreasing family of non-negative numbers, $\{c_k\}_{k\in\mathbb{N}^{m-1}}$, such that for all $\varphi \in \mathcal{R}$ the first coefficient $c_k^\varphi(x,\xi)$ in* (2) *can be written as*

$$c_0^\varphi(x,\xi) = \sum_{n=-\infty}^{+\infty} \sum_{k\in\mathbb{N}^{m-1}} \hat\varphi(n\,T_\gamma)\, c_k\, e^{in((k+1/2)\theta+\alpha)}\,. \tag{29}$$

PROOF. In [7] we proved that the first coefficient $c_0^\varphi(x,\xi)$ in (2) can be written as

$$2^{2n}\,\pi^{(3n+1)/2}\,c_0^\varphi(x,\xi) \\ = \sum_{n=-\infty}^{+\infty} \hat\varphi(nT_\gamma)\, e^{inS_\gamma/\hbar+\sigma_\gamma} \int_{-\infty}^{+\infty} (a, Z((s\,\dot x, s\,\dot\xi))\, U^n a)\, ds\,, \tag{30}$$

where $(\dot x,\dot\xi)$ is the tangent vector to the classical flow at (x,ξ), Z is the Weyl/Heisenberg operator defined by

$$Z(e,f)(a)(\eta) = e^{-ief/2}\, e^{ie\eta}\, a(\eta - f) \tag{31}$$

and U is the metaplectic representation of the linearized flow at time T_γ. (We should point out that in the manifold case a defines intrinsically a smooth vector in the metaplectic representation of $T_{(x,\xi)}(T^*M)$, and U and Z are operators in that representation space.) Denoting by S the linearized flow at time T_γ, we also showed that one can find a symplectic mapping R such that $R^{-1}SR$ is block-diagonal of the form

$$R^{-1}SR = \begin{pmatrix} 1 & \mu & 0 \\ 0 & 1 & 0 \\ 0 & 0 & A_\theta \end{pmatrix}, \tag{32}$$

where $\mu \in \mathbb{R}$ and A_θ is the direct sum of rotations of angles $\theta_1, \dots, \theta_{m-1}$. Furthermore, the transformation R maps the vector $(s\,\dot{x}, s\,\dot{\xi})$ to the vector $(s, 0)$.

Let us denote $a' := \mathrm{Mp}(R)^{-1}a$ and $V := \mathrm{Mp}(R^{-1}SR)$, where $\mathrm{Mp}(R)$ denotes the metaplectic representation of the mapping R. Then, letting $Z(s) := Z((s\,\dot{x}, s\,\dot{\xi}))$ and

$$W(s) := \mathrm{Mp}(R)^{-1}\, Z(s)\, \mathrm{Mp}(R) = Z(s, 0, 0, 0)\,,$$

one has

$$\Big(a, Z(s)\, U^n a\Big) = \Big(a', W(s)\, V^n a'\Big)\,.$$

Denote the variables of a' by (η_1, η_2) where $\eta_1 \in \mathbb{R}$ and $\eta_2 \in \mathbb{R}^{m-1}$, and let $e^{i\theta(D^2_{\eta_2} + {\eta_2}^2)/2}$ denote the direct sum of the propagators of one-dimensional Harmonic oscillators at times $\theta_1, \dots \theta_{m-1}$, acting on a' by acting on the η_2 variables. If $e^{in\mu\partial^2_{\eta_1}/2}$ denotes the metaplectic quantization of

$$\begin{pmatrix} 1 & n\mu \\ 0 & 1 \end{pmatrix}, \tag{33}$$

we get that (30) becomes

$$\begin{aligned} &2^{2n}\, \pi^{(3n+1)/2}\, c_0^{\varphi}(x,\xi) \\ &\quad = \sum_{n=-\infty}^{+\infty} \hat{\varphi}(n\,T_\gamma)\, e^{in\alpha} \\ &\qquad\qquad \cdot \int \overline{a'(\eta)}\, \big(e^{in\mu\partial^2_{\eta_1}/2}\, e^{in\theta(D^2_{\eta_2} + {\eta_2}^2)/2}\, (a')\big)\, (\eta_1 - s, \eta_2)\, d\eta\, ds\,. \end{aligned}$$

The integral over ds is a convolution and the integral over $d\eta_1$ is the integral of that convolution. Therefore, using the Fourier inversion formula plus the fact that on the Fourier transform side the operator $e^{in\mu\partial^2_{\eta_1}/2}$ is multiplication by $e^{-in\mu\zeta^2/2}$ (ζ being the dual variable), one gets

$$(34)\quad \begin{aligned} &2^{2n}\,\pi^{(3n+1)/2}\,c_0^{\varphi}(x,\xi)\\ &\quad= \sum_{n=-\infty}^{+\infty} \hat\varphi(n\,T_\gamma)\,e^{in\alpha}\int \overline{a'^{\wedge}(0,\eta_2)}\,e^{in\theta(D^2_{\eta_2}+{\eta_2}^2)/2}a'^{\wedge}(0,\eta_2)\,d\eta_2\;,\end{aligned}$$

where $a'^{\wedge}$ is the Fourier transform of a' with respect to η_1. Let $b(x) := a'^{\wedge}(0,x)$ and let us decompose b on the Hermite basis, h_k, of eigenfunctions of the harmonic oscillator

$$(35)\qquad b = \sum_{k\in\mathbb{N}^{m-1}} b_k\,h_k\;.$$

Then, letting $c_k := |b_k|^2$ we get (29) and the family $\{c_k\}$ is non-negative. It is also rapidly decreasing since the function b is Schwartz.

REMARK. For a given quantum Hamiltonian H, the coefficients $\{c_k\}$ depend only on the symbol a of the coherent state. Observe that the proof shows that given any rapidly decreasing family $\{c_k\}$ one can find an a giving rise to it.

We next prove the existence of the limit

$$(36)\qquad \mathcal{L}_\alpha(c) := \lim_{a\to 0} c_0^{(f_a*\chi_{[-c,c]})}(x,\xi)\,,$$

for f as in the Tauberian lemma and $c_0^{\varphi}(x,\xi)$ as in (29). Let $\phi_a(c) := c_0^{(f_a*\chi_{[-c,c]})}(x,\xi)$, that is

$$(37)\qquad \phi_a(c) :=\; c\;+\sum_{n\neq 0,k}\hat f(a\,n)\,\frac{\sin\,(n\,c\,T_\gamma)}{n\,T_\gamma}\,c_k\,e^{in((k+1/2)\theta+\alpha)}\;.$$

We must then prove that the limit $\mathcal{L}_\alpha(c) = \lim_{a\to 0}\phi_a(c)$ exists.

To lighten up the notation a bit, let us define

$$(38)\qquad d_k := \Big(k+\frac{1}{2}\Big)\theta+\alpha\,,\qquad k\in\mathbb{N}^{m-1}\,,$$

keeping in mind the notation (6). Let $0 < a < 1$, then

$$\begin{aligned}\phi_1(c) - \phi_a(c) &= \frac{1}{T_\gamma} \sum_{(n,k)\in\mathbb{Z}\times\mathbb{N}^{m-1}} \sin(c\,n\,T_\gamma)\, c_k\, e^{ind_k} \int_a^1 \hat{f}'(t\,n)\,dt \\ &= \frac{1}{T_\gamma} \sum_{(n,\,k)} \Big(\frac{e^{incT_\gamma} - e^{-incT_\gamma}}{2\,i}\Big) c_k\, e^{ind_k} \int_a^1 \hat{f}'(t\,n)\,dt\,. \qquad (39)\end{aligned}$$

Applying the Poisson summation formula to the series over n, we get (after a calculation)

$$\begin{aligned}\phi_1(c) - \phi_a(c) = \frac{-\pi}{T_\gamma} \sum_{(j,\,k)\in\mathbb{Z}\times\mathbb{N}^{m-1}} c_k \int_a^1 \Big(g\Big(\frac{1}{t}\,(2\pi j + c\,T_\gamma + d_k)\Big) \\ - g\Big(\frac{1}{t}\,(2\pi j - c\,T_\gamma + d_k)\Big)\Big)\frac{dt}{t}\,, \qquad (40)\end{aligned}$$

where $g(x) := x\,f(x)$.

Lemma 3.2. *Define*

$$\mathcal{M}_0^\alpha = \Big\{c \in \mathbb{R} :\ \textit{for all } (j,k) \in \mathbb{Z}\times\mathbb{N}^{m-1}\,,\ c \neq \pm\frac{1}{T_\gamma}\,(2\pi j + d_k)\Big\}\,.$$

If $\theta_1/(2\pi), \dots, \theta_{m-1}/(2\pi)$ are rational and $c \in \mathcal{M}_0^\alpha$, then each of the limits

$$\lim_{a\to 0} \sum_{(j,k)\in\mathbb{Z}\times\mathbb{N}^{m-1}} c_k \int_a^1 g\Big(\frac{1}{t}\,(2\pi j \pm c\,T_\gamma + d_k)\Big)\frac{dt}{t}\,, \qquad (41)$$

exists (and is finite). Moreover, the convergence is locally uniform in c.

PROOF. By the rationality assumption the complement of $\mathcal{M}_0^\alpha$ is discrete. Therefore, if $c \in \mathcal{M}_0^\alpha$ there exists ε such that

$$0 < \varepsilon \le |2\pi j \pm c\,T_\gamma + d_k|\,, \qquad \text{for all } (j,k) \in \mathbb{Z}\times\mathbb{N}^{m-1}\,.$$

The function g is rapidly decreasing: for all $N \in \mathbb{N}$ $\exists C_N > 0$ such that for all $x \in \mathbb{R}$, $|g(x)| \le C_N\,(1 + |x|)^{-N}$. Therefore

$$\begin{aligned}\Big|g\Big(\frac{2\pi j \pm c\,T + d_k}{t}\Big)\Big| &\le C_N\,\frac{t^N}{t^N + (2\pi j \pm c\,T + d_k)^N} \\ &\le C_N\,\frac{t^N}{(2\pi j \pm c\,T + d_k)^N}\,, \qquad (42)\end{aligned}$$

and so for all $(j,k) \in \mathbb{Z} \times \mathbb{N}^{m-1}$ and for all $a \in (0,1)$

$$\int_a^1 \left| g\Big(\frac{1}{t}\,(2\pi j \pm cT_\gamma + d_k)\Big)\right| \frac{dt}{t} \le \frac{C_N}{N}\,\frac{1-a^{N+1}}{|2\pi j \pm cT_\gamma + d_k|^N}\,. \tag{43}$$

This shows that each of the integrals in the series (41) extends to a continuous function of $a \in [0,1)$. Moreover, since the family

$$M_{k,j} := \frac{c_k}{|2\pi j \pm cT_\gamma + d_k|^N}\,, \qquad (j,k) \in \mathbb{Z} \times \mathbb{N}^{m-1}$$

is absolutely convergent (for N sufficiently large) and it dominates the absolute values of the terms of (41), we are done.

We now turn to the irrational case.

Lemma 3.3. *Assume that $1, \theta_1/(2\pi), \ldots, \theta_{m-1}/(2\pi)$ are linearly independent over the rationals. Let*

$$\mathcal{M}^\alpha_\pm := \Big\{ c \in \mathcal{M}^\alpha_0 : \sum_{k\in\mathbb{N}^{m-1}} c_k \Big(\pm \Big(d_k + \frac{cT}{2\pi}\Big)\Big)^{-2} < \infty \Big\}, \tag{44}$$

where $\{x\}$ denotes the fractional part of x, and let

$$\mathcal{M}^\alpha := \mathcal{M}^\alpha_+ \cap \mathcal{M}^\alpha_-\,. \tag{45}$$

Then, if $c \in \mathcal{M}^\alpha$, each of the limits

$$\lim_{a\to 0} \sum_{j,k} c_k \int_a^b g\Big(\frac{1}{t}\,(2\pi j \pm cT_\gamma + d_k)\Big)\frac{dt}{t}$$

exists and is finite. Moreover, the convergence is locally uniform in c.

PROOF. It is enough to consider one of the series above, say the one with the plus sign. Let $c \in \mathcal{M}^\alpha$ and define

$$O^+ := \{(j,k) \in \mathbb{Z} \times \mathbb{N}^{m-1} :\ 2\pi j + cT_\gamma + d_k > 0\}\,,$$

and

$$O^- := \{(j,k) \in \mathbb{Z} \times \mathbb{N}^{m-1} :\ 2\pi j + cT_\gamma + d_k < 0\}\,.$$

Since $c \in \mathcal{M}_0^\alpha$, $\mathbb{Z} \times \mathbb{N}^{m-1} = O^+ \cup O^-$. Recalling that $g(x) = x\,f(x)$ and that f as well as the c_k are non-negative, we see that the terms with $(j,k) \in O^\pm$ have the sign $\pm$ and therefore each of

$$\sum_{(j,k)\in O^\pm} c_k \int_a^1 g\Big(\frac{1}{t}\,(2\pi j + c\,T_\gamma + d_k)\Big)\frac{dt}{t}$$

is a decreasing function of a. It therefore suffices to show that

$$\lim_{a\to 0} \sum_{(j,k)\in O^+} c_k \int_a^1 g\Big(\frac{1}{t}\,(2\pi j + c\,T_\gamma + d_k)\Big)\frac{dt}{t} < \infty$$

and similarly for the series over O^-.

Specializing (43) to $N = 2$, we see that exists $C > 0$ such that for all $a \in (0,1)$ and for all $(j,k) \in O^+$

$$\int_a^1 g\Big(\frac{2\pi j + c\,T + d_k}{t}\Big)\frac{dt}{t} \le \frac{C}{(2\pi j + c\,T + d_k)^2}\,. \tag{46}$$

(The last denominator is not zero if $(j,k) \in O^+$.) Therefore, the Lemma will be proved provided we show the convergence of the double series of scalars

$$\sum_{(j,k)\in O^+} M_{k,j}\,, \tag{47}$$

where

$$M_{k,j} = c_k\,\Big(j + \frac{c\,T + d_k}{2\pi}\Big)^{-2},$$

that is

$$M_{k,j} = \frac{c_k}{(j + k\,\xi + \beta)^2}\,, \tag{48}$$

where

$$\xi = \Big(\frac{\theta_1}{2\pi}, \dots, \frac{\theta_{m-1}}{2\pi}\Big) \qquad \text{and} \qquad \beta = \frac{1}{2\pi}\Big(c\,T + \alpha + \sum_{j=1}^{m-1}\frac{\theta_j}{2}\Big)\,. \tag{49}$$

Since the terms in (47) are positive, we can prove its convergence by first summing over j with k fixed, and then summing over $k \in \mathbb{N}^{m-1}$. Observe that

$$(j,k) \in O^+ \qquad \text{if and only if} \qquad j \ge [-k\,\xi - \beta] + 1\,, \tag{50}$$

where $[x]$ denotes the greatest integer less than or equal to x. For every k consider the series

$$\sum_{j=-[k\xi+\beta]}^{+\infty} M_{k,j} \ . \tag{51}$$

(If $x \notin \mathbb{Z}$, then $[-x] = -[x] - 1$, and since $c \in \mathcal{M}_0^\alpha$, for all $k \in \mathbb{N}^{m-1}$, $k\,\xi + \beta \notin \mathbb{Z}$.) Comparing this series with the integral

$$\int_{-[k\xi+\beta]}^{+\infty} \frac{dx}{(x + k\,\xi + \beta)^2} \ , \tag{52}$$

we find that

$$\sum_{j=-[k\xi+\beta]}^{+\infty} M_{k,j} \leq M_{k,[-k\xi+\beta]} + \frac{c_k}{-[k\xi+\beta] + k\,\xi + \beta} \ , \tag{53}$$

or with the notation $\{x\}$ = fractional part of $x = x - [x]$,

$$\sum_{j=-[k\,\xi+\beta]+1}^{+\infty} M_{k,j} \leq \frac{c_k}{\{k\,\xi+\beta\}^2} + \frac{c_k}{\{k\,\xi+\beta\}} \ . \tag{54}$$

Therefore convergence of (47) follows from the convergence of

$$\sum_{k\in\mathbb{N}^{m-1}} \frac{c_k}{\{k\,\xi+\beta\}^2} \ .$$

But since by assumption $c \in \mathcal{M}_+^\alpha$, this series converges.

In conclusion we have shown that $\mathcal{L}_\alpha(c)$ exists for c as defined by the Lemmas.

REMARKS. In the irrational case:

1) To find examples of numbers c in $\mathcal{M}_\pm^\alpha$, it suffices to find a family $\{g_k\}_{k\in\mathbb{N}^{m-1}}$ of positive numbers such that $\sum g_k^{-2}\, c_k < \infty$. Then if

$$\left|2\pi j \pm c\,T_\gamma + \left(k + \frac{1}{2}\right)\theta + \alpha\right| > g_k \ , \qquad \text{for all } (j,k) \in \mathbb{Z}\times\mathbb{N}^{m-1} \ , \tag{55}$$

then $c \in \mathcal{M}^\alpha$. Defining $\hat{g}_k = \varepsilon g_k$, $\varepsilon > 0$, one see that still $\sum \hat{g}_k^{-2} c_k < \infty$ and therefore associated to $\hat{g}_k$ (by (55)) is a subset of $\mathcal{M}_\alpha$ whose intersection with any interval I has a co-measure in I arbitrary small as $\varepsilon \longrightarrow 0$; therefore $\mathcal{M}^\alpha$ has full measure.

2) The set $\mathcal{M}^\alpha$ is related to the rate of decay of the c_k (that is to the properties of the symbol, a, of the coherent states), as well as to irrationality properties of $\theta/(2\pi)$. At one extreme, we can choose a such that only finitely-many of the coefficients c_k are non-zero (see the remark following Lemma 3.1). In that case $\mathcal{M}^\alpha = \mathcal{M}_0^\alpha$ is just the complement of the set of quasi-energies of the quasi-modes associated with the trajectory.

4. Properties of the function $\mathcal{L}_\alpha$.

Having established the existence of the function $\mathcal{L}_\alpha(c)$, we now derive some of its properties.

Rational case. Let us go back to the identity $\mathcal{L}_\alpha(c) = \lim_{a\to 0} \phi_a(c)$ where ϕ_a is defined in (37). Applying in (37) the Poisson summation formula to the series over n with k fixed one obtains

$$\mathcal{L}_\alpha(c) = \lim_{a\to 0} \frac{1}{a}\Big(F_c * f\Big(\frac{\cdot}{a}\Big)\Big)(0)\,, \tag{56}$$

where

$$F_c(y) = \int_{-c}^{c} \sum_{j,k} c_k\, \delta(T_\gamma(x-y) - 2\pi j - d_k)\, dx\,. \tag{57}$$

For each $c > 0$ the function F_c is a step function; indeed

$$F_c = \sum_{j,k} c_k\, \chi_{[-c-(2\pi j + d_k)/T\,,\, c-(2\pi j+d_k)/T]}\,. \tag{58}$$

Since $f(\cdot/a)/a \longrightarrow \delta$, we obtain

$$\mathcal{L}_\alpha(c) = \sum_{\{j,k\,:\, -cT < 2\pi j + d_k < cT\}} c_k\,, \qquad \text{for all } c \in \mathcal{M}_0^\alpha\,, \tag{59}$$

which is clearly a step function (*i.e.* a locally constant function) of $c \in \mathcal{M}_0^\alpha$.

Irrational case. To study the function $\mathcal{L}_\alpha(c)$ on $\mathcal{M}^\alpha$ as defined by (36), we will use a wavelet decomposition.

Let $g \in L^2$ be a function satisfying $\int g(x)\,dx = 0$ and $\int x\,g(x)\,dx = 0$. If it exists, the wavelet coefficient of $\mathcal{L}_\alpha(c) - c$ is

$$T(a,b) = \frac{1}{a}\int g\Big(\frac{x-b}{a}\Big)(\mathcal{L}_\alpha(x) - x)\,dx\,. \tag{60}$$

Plugging in (60) the expression

$$\mathcal{L}_\alpha(x) - x = \sum_{\substack{n\neq 0\\ k}} \frac{\sin(n\,x\,T_\gamma)}{n\,T_\gamma}\,e^{ind_k}\,c_k\,, \tag{61}$$

one finds, supposing $\hat{g}$ even

$$T(a,b) = \frac{1}{2\,i}\sum_{\substack{n\neq 0\\ k}} \frac{1}{n\,T_\gamma}\,\hat{g}(a\,n)\sin(n\,b\,T_\gamma)\,c_k\,e^{in(d_k)}\,. \tag{62}$$

The following result shows that such a decomposition is indeed valid.

Proposition 4.1. *Let g as before, $\hat{g}$ being compactly supported and even, and let us suppose that φ is a compactly supported function satisfying*

$$\int \overline{\hat{\varphi}}(a)\,\hat{g}(a)\,\frac{da}{a} = \int \overline{\hat{g}}(-a)\,\hat{\varphi}(-a)\,\frac{da}{a} = 1\,. \tag{63}$$

Then, for all $c \in \mathcal{M}^\alpha$,

$$\mathcal{L}_\alpha(c) - c = \lim_{\varepsilon\to 0}\int_\varepsilon^{+\infty}\frac{da}{a}\int_{-\infty}^{+\infty}\varphi\Big(\frac{c-b}{a}\Big)T(a,b)\,db\,, \tag{64}$$

where

$$T(a,b) = \frac{1}{2\,i}\sum_{\substack{n\neq 0\\ k}} \frac{1}{n\,T_\gamma}\,\hat{g}(a\,n)\sin(n\,b\,T_\gamma)\,c_k\,e^{ind_k}\,. \tag{65}$$

PROOF.

$$\int_\varepsilon^{+\infty} \frac{da}{a} \int_{-\infty}^{+\infty} db\, \varphi\Big(\frac{c-b}{a}\Big) T(a,b)$$

$$= \int_\varepsilon^{+\infty} \frac{da}{a} \sum_{\substack{n\neq 0\\ k}} \frac{1}{n\,T_\gamma} \left(\hat\varphi(a\,n)\, e^{incT_\gamma} - \hat\varphi(-a\,n)\, e^{-incT_\gamma}\right)$$

$$\cdot\, e^{ind_k}\, \hat g(a\,n)\, c_k$$

$$= \int_\varepsilon^{+\infty} \frac{da}{a} \sum_{\substack{n\neq 0\\ k}} \hat\varphi(a\,n)\, \hat g(a\,n) \sin(n\,c\,T_\gamma)\, e^{in((k+1/2)\theta+\alpha)}\, c_k \tag{66}$$

$$= \sum_{\substack{n\neq 0\\ k}} \psi(\varepsilon\, n) \sin(n\,c\,T_\gamma)\, e^{ind_k}\, c_k\ ,$$

where

$$\psi(\varepsilon) := \int_\varepsilon^{+\infty} \frac{da}{a}\, \hat\varphi(a)\, \hat g(a)\,.$$

Noting that $\psi'(a) = \hat\varphi(a)\,\hat g(a)/a$ is compactly supported and $\psi(0) = 1$ by hypothesis one get the result, thanks to Lemma 3.3.

The next result, thanks to the result of the Appendix will enable us to prove the Lipschitz continuity on $\mathcal{M}^\alpha$.

Proposition 4.2.

(67) $T(a,b) = O(a)\,,$ *near* 0 *almost everywhere and uniformly in* b.

PROOF. Since $\int x\, g(x)\, dx = \int g(x)\, dx = 0$, $g'(0) = 0$. So one can find a C^∞ function f such that $\hat g(\xi) = \xi\, f(\xi)$ and $f(0) = 0$. Then

$$T(a,b) = a \sum_{\substack{n\neq 0\\ k}} f(a\,n) \sin(b\,n\,T_\gamma)\, e^{ind_k}\, c_k\ , \tag{68}$$

and it is easy to check, by the same argument as in Lemma 3.3, that if $b\in\mathcal{M}^\alpha$,

$$\sum_{\substack{n\neq 0\\ k}} f(a\,n) \sin(b\,n\,T_\gamma)\, e^{in((k+1/2)\theta+\alpha)}\, c_k$$

is bounded.

5. End of proofs.

The convergence statements in both theorems are immediate consequences of the Tauberian lemma of Section 2, applied to the following objects

$$\Upsilon_\hbar(a,c) = \sum_j w_j(\hbar)\,\varphi\Big(\frac{E_j(\hbar)-E}{\hbar}\Big)\,, \tag{69}$$

where

$$w_j(\hbar) = |(\psi_{(x,\xi)},\psi_j)|^2\,. \tag{70}$$

The weighted counting function is therefore

$$\sum_{\substack{j\\ |E_j(\hbar)-E|\leq c\hbar}} |(\psi_{(x,\xi)},\psi_j)|^2\,. \tag{71}$$

The functional of the Tauberian lemma is

$$\mathcal{F}_0(\varphi) := c_0^\varphi(x,\xi) \tag{72}$$

as defined by (29). We must check that the above objects satisfy the assumptions of the Tauberian lemma.

a) Theorem 1.1. It is easy to see that the functional $\mathcal{F}_0$ defined where $c_0^\varphi(x,\xi)$ is defined by (29) satisfies the hypothesis 2 of the Tauberian Lemma of Section 2 if we take for $\mathcal{M}^\alpha$ the set defined by (9). Moreover the other hypotheses are satisfied as in [7]. Then just apply the Tauberian Lemma.

b) Theorem 1.2. The Lipschitz continuity of $\mathcal{F}_0$ is an immediate consequence of Proposition 4.2 together with Theorem A.1 below. The fact that $\mathcal{M}^\alpha$ is of full Lebesgue measure, is a classical result of Diophantine analysis (recall that the sequence $\{g_k\}$ in the remark 1, Section 3 is rapidly decreasing).

Appendix. Wavelets and Hölder continuity.

Int his appendix we will prove an easy extension of results of [6], [5] and [4].

Let $\mathcal{M}^\alpha$ a bounded subset of $\mathbb{R}$ of full Lebesgue measure.

Theorem A.1. *Let g be a be a continuously differentiable compactly supported function. Let f defined and bounded on $\mathcal{M}^\alpha$. Let us suppose that f admits a "scale-space coefficient T(a,b)" decomposition with respect to g, namely*

$$f(x) = \int_0^\infty \int_{-\infty}^{+\infty} g\Big(\frac{x-b}{a}\Big) T(a,b)\, \frac{da}{a}\, db\,, \qquad \text{for all } x \in \mathcal{M}^\alpha\,. \tag{73}$$

Let us suppose moreover that

$$T(a,b) = o(a^\alpha)\,, \tag{74}$$

near 0 *almost everywhere and uniformly in b. Then F is α-Hölder continuous on $\mathcal{M}^\alpha$; by this we mean*

$$|f(x_1) - f(x_2)| = O_{x_1}(|x_2 - x_1|^\alpha)\,, \qquad \text{for all } x_1, x_2 \in \mathcal{M}^\alpha\,. \tag{75}$$

Proof. The proof is absolutely equivalent to the one in [4], so we will only sketch it. Let us write first:

$$\begin{aligned} f(x) &= \Big(\int_0^1 \frac{da}{a} + \int_1^\infty \frac{da}{a}\Big) \int db\, g\Big(\frac{x-b}{a}\Big) T(a,b) \\ &= f_s(x) + f_l(x)\,, \end{aligned} \tag{76}$$

f_l is obviously C^∞. We concentrate on f_s.

Let $x_1, x_2 \in \mathcal{M}^\alpha$, $x_1 < x_2$, we cut f_s in three pieces.

$$\begin{aligned} f_s(x_1) - f_s(x_2) = &\int_0^{x_2 - x_1} \frac{da}{a} \int db\, g\Big(\frac{x_2 - b}{a}\Big) T(a,b) \\ &- \int_0^{x_1 - x_1} \frac{da}{a} \int db\, g\Big(\frac{x_2 - b}{a}\Big) T(a,b) \\ &+ \int_{x_2 - x_1}^1 da \int db\, \Big(\frac{1}{a}\, g\Big(\frac{x_2 - b}{a}\Big) - \frac{1}{a}\, g\Big(\frac{x_1 - b}{a}\Big)\Big) \\ &\qquad \cdot T(a,b) \end{aligned} \tag{77}$$

$$=: T_1 - T_2 + T_3\,. \tag{78}$$

We now analyze each term:

- T_1 and T_2. Since $T(a,b) = O(a^\alpha)$ almost everywhere, we have

$$|T_i| = \int_0^{x_2-x_1} \frac{da}{a} \int db \left|\frac{1}{a}\, g\Big(\frac{x_i-b}{a}\Big)\right| C\, a^\alpha \\ = O(|x_2-x_1|^\alpha)\, \|g\|_{L_1}\, \frac{C}{\alpha}\,. \tag{79}$$

- T_3. If g is continuously differentiable let us write

$$g\Big(\frac{x_2-b}{a}\Big) - g\Big(\frac{x_1-b}{a}\Big) = \frac{x_2-x_1}{a}\, g'\Big(\frac{x'-b}{a}\Big) \tag{80}$$

with $x_1 \le x' \le \overline{x}_2$. So

$$|T_3| \le \int_{x_2-x_1}^{1} \frac{da}{a} \int db \left|\frac{1}{a^2}\, g'\Big(\frac{x'-b}{a}\Big)\right| |T(a,b)|\, |x_2-x_1| \\ = O(|x_2-x_1|)\, \|g'\|_{L_1} \int_{x_2-x_1}^{1} \frac{da}{a}\, a^{\alpha-1} \\ = O(|x_2-x_1|^\alpha)\,. \tag{81}$$

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