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Isoperiodic meromorphic forms: two simple poles

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Abstract. In this paper, we prove that isoperiodic moduli spaces of meromorphic differentials with two simple poles on homologically marked smooth curves are non-empty and connected unless they correspond to double covers of $\mathbb{C}/\mathbb{Z}$ on curves of genus at least two. We deduce dynamical consequences for the corresponding isoperiodic foliation.

1. Introduction

1.1. Overview

Abelian differentials, their moduli spaces and associated dynamical systems have attracted much interest in the last years. This work is the second of a series aiming at understanding the dynamics and topology of the isoperiodic foliations on moduli spaces of meromorphic one forms on algebraic curves. The isoperiodic equivalence relation is defined locally by the condition that the integrals on *every* closed cycle in the domain where the form is holomorphic remain constant.

The case of moduli spaces of meromorphic forms with an even number of simple poles appears naturally as boundary components in the Deligne–Mumford compactification of moduli spaces of holomorphic one forms (see [6,7]). The understanding of these foliations will complete the picture of the dynamics of isoperiodic foliations on moduli spaces of stable forms given in [11].

Eluding the general case for the third memoir of the series, we consider here the case of moduli spaces of forms on curves of any genus with two poles, both simple. It will constitute the initial step for an inductive argument to similar statements in any number of simple poles (see work in progress in collaboration with Liza Arzakhova in [5]). Restricting our attention to the case of two poles might seem artificial, but there is a conceptual reason to treat this case separately: up to scaling, any form with two simple poles has peripheral integrals ± 1 (i.e., residues $\pm\frac{1}{2i\pi}$) and therefore the integral along a closed cycle defines an element in the topological group $\mathbb{C}/\mathbb{Z}$ that depends only on its homology class of the *closed* surface. In the case of more poles, the group generated by the residues is not necessarily cyclic, and there are other tools we need to use.

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Recall that in the case of holomorphic one forms on compact curves, the dynamics of the isoperiodic foliation is related to the topology of the period mapping, defined on some Torelli cover of the moduli space (see [11, 34]). In the present paper, we show that, apart from isoperiodic sets consisting of double covers of a bi-infinite cylinder $\mathbb{C}/\mathbb{Z}$ of genus at least two, the isoperiodic moduli spaces of homologically marked meromorphic differentials with two simple poles are non-empty and connected. Related results have been proven recently, see [11, 34, 38]. As in [11], this allows us to show that the fundamental group of the leaves of the isoperiodic foliation surjects onto the stabilizer in $\mathrm{Sp}(2g, \mathbb{Z})$ of their associated period. Another consequence of interest for the dynamics is that we can transfer the dynamical properties of the action of the group $\mathrm{Sp}(2g, \mathbb{Z})$ on $(\mathbb{C}/\mathbb{Z})^{2g}$ to properties satisfied by the isoperiodic foliation.

Ratner's theory can be used to compute the closed invariant sets of this action. We have included a full proof of this computation (see Section 11) as that in [22] was incomplete. From it we deduce the closed subsets of moduli spaces that are saturated/invariant by the isoperiodic foliation. We derive that they are real analytic submanifolds. A qualitative difference with the case of holomorphic differentials is that some transcendental leaves are closed in these quasi-projective manifolds. They correspond to forms whose periods describe a given discrete lattice in $\mathbb{C}$. Such phenomenon occurs for some algebraic foliations, for instance, the one defined by $dy/dx = y$ in $\mathbb{C}^2$, but it seems to be quite rare to have a transcendental leaf of an algebraic foliation which is closed in a Zariski open subset. See for instance [12, 13] or the work of Cousin on the Garnier system [17]. We note that these leaves are examples of affine manifolds (in the sense of [35]) that are not algebraic. Bakker and Mullane have such an example in moduli spaces with marked points. We do not know an example of a non-algebraic affine manifold which is also invariant by the $\mathrm{SL}(2, \mathbb{R})$-action in the moduli space of meromorphic forms (such examples do not exist in moduli spaces of abelian differentials by a result of Filip, see [21]).

Our second application is the ergodicity with respect to Lebesgue measure of the restriction of the isoperiodic foliation to every closed saturated subset. A particularly interesting closed saturated subset that does not arise in the moduli space of abelian differentials is the moduli space of meromorphic differentials with two poles and *real periods*. For every genus g, this set identifies with a copy of moduli space $\mathcal{M}_{g,2}$ and plays a key role in the alternative proof [23] by Grushevsky and Krichever of Diaz's theorem (the non-existence of a complete subvariety of dimension $\geq g-1$ in moduli space $\mathcal{M}_g$ of curves of genus g), and in its generalization by Krichever [32] that provides a solution of Arbarello's conjecture, stating that a compact holomorphic submanifold of $\mathcal{M}_g$ of dimension $g-n$ intersects the Weierstrass locus of curves admitting a meromorphic function with a single pole of order at most n. Recently, Krichever, Lando and Skripchenko proved that the isoperiodic foliation on the moduli space of meromorphic differentials with a single double pole is ergodic, by studying the combinatorics of cut diagrams, see [33]. Ergodicity of isoperiodic foliations on various moduli spaces of holomorphic differentials has been established recently, see [11, 14, 25, 34, 39, 40]. A case of special interest is that of strata of forms with fixed combinatorics of zero and polar divisor. Each closed stratum is transverse to the isoperiodic foliation defined above and the induced foliation is regular, holomorphic, and of dimension $m-1$, where m is the number of zeros. The results in this paper imply trivially that the Transfer Principle can be applied in the open stratum of the moduli space of forms with two simple poles and only simple zeros. We are currently working to generalize this result to the open stratum when there are more simple poles (in [5]). In the case of holomorphic forms, Winsor has recently proved in [38] that the same strategy can be applied to connected strata with at least two different zeros (for generic isoperiodic moduli spaces, at least). A natural question in the present context is whether the same is true for connected components of strata of meromorphic forms with given poles and at least two different zeros. In the open stratum of elliptic differentials with a single double pole, the answer is positive (see [19]). Unfortunately, the methods used in this paper do not seem to shed light on that problem.

1.2. Statement of results

Let $\mathcal{M}_{g,n}$ denote the moduli space of smooth genus g complex curves with n marked ordered points. For $g \geq 1$ consider the bundle $\Omega^{\pm}\mathcal{M}_{g,2} \to \mathcal{M}_{g,2}$ whose fiber over a point (C, x_-, x_+) is the space of meromorphic forms ω on C having simple poles on x_-, x_+ with peripheral integrals of values $-1, +1$ respectively (in other words, having residues $-\frac{1}{2\pi i}, \frac{1}{2\pi i}$ respectively). It is an affine bundle directed by the Hodge (vector) bundle $\Omega\mathcal{M}_{g,2} \to \mathcal{M}_{g,2}$ of holomorphic one forms, hence a complex manifold of dimension $4g-1$.

The choice of (± 1)-peripheral integrals at the two poles of ω allows us to define a period homomorphism $p(C,\omega) \in H^1(C, \mathbb{C}/\mathbb{Z})$ by the arrow

$$H_1(C,\mathbb{Z}) \ni \gamma \mapsto \int_\gamma \omega \in \mathbb{C}/\mathbb{Z}.$$

A family of forms $\{(C_t,\omega_t)\}_t$ in $\Omega^{\pm}\mathcal{M}_{g,2}$ is said to be isoperiodic if $t \mapsto p(C_t,\omega_t)$ is locally constant (the homology groups $H_1(C_t,\mathbb{Z})$ are locally identified by the use of the Gauss–Manin connection). The aim of this paper is the study of the topological and dynamical properties of the isoperiodic foliation, whose leaves are the maximal isoperiodic analytic families.

To be able to compare homology classes on different curves, we need to identify their homology groups. It is well known that the orbifold universal cover $\mathcal{T}_{g,n} \to \mathcal{M}_{g,n}$ of the moduli space is biholomorphic to the Teichmüller space $\mathcal{T}_{g,n}$. The fiber over a point $(C,(q_1,\dots,q_n)) \in \mathcal{M}_{g,n}$ is given by the different possible identifications of the fundamental group of C with the fundamental group of a reference surface of genus g with n marked ordered points $\Sigma_{g,n}$. In this representation, the covering group is the mapping class group $\mathrm{Mod}(\Sigma_{g,n})$ of isotopy classes of orientation preserving diffeomorphisms of Σ_g that fix each marked point.

Let $\Sigma_{g,2} = (\Sigma_g, p_-, p_+)$ be an oriented closed surface of genus g with two ordered distinct marked points. Since we only want to keep the information of the identification at the homology level of the curve without the information on the marked points, we consider the quotient of $\mathcal{T}_{g,2}$ by the subgroup $\mathcal{I}(\Sigma_{g,\cancel{2}}) \subset \mathrm{Mod}(\Sigma_{g,2})$ formed by the elements that act trivially on $H_1(\Sigma_g, \mathbb{Z})$.[1] We denote by

$$\mathcal{S}_{g,\cancel{2}} = \frac{\mathcal{T}_{g,2}}{\mathcal{I}(\Sigma_{g,\cancel{2}})}$$

the quotient space. To each point in it there corresponds a tuple (C, x_-, x_+, m) where (C, x_-, x_+) is a smooth complex curve of genus g with two marked points and $m : H_1(\Sigma_g, \mathbb{Z}) \to H_1(C, \mathbb{Z})$ is an isomorphism, up to the equivalence $(C, x_-, x_+, m) \sim (C', x'_-, x'_+, m')$ if there exists a biholomorphism $h : (C, x_-, x_+) \to (C', x'_-, x'_+)$ such that $m' = h_* \circ m$. The mapping class group of $\Sigma_{g,2}$ acts on $\mathcal{S}_{g,\cancel{2}}$ by precomposition on the marking and the quotient map gives a covering map

$$\mathcal{S}_{g,\cancel{2}} \to \mathcal{M}_{g,2} \tag{1}$$

with covering group being the group $\mathrm{Aut}(H_1(\Sigma_g, \mathbb{Z}))$ of linear automorphisms that preserve the intersection product on $H_1(\Sigma_g, \mathbb{Z})$. The bundle $\Omega^{\pm}\mathcal{M}_{g,2} \to \mathcal{M}_{g,2}$ can be pulled back to $\mathcal{S}_{g,\cancel{2}}$ by using the map (1) producing a bundle $\Omega^{\pm}\mathcal{S}_{g,\cancel{2}} \to \mathcal{S}_{g,\cancel{2}}$ whose fiber over a point (C, x_-, x_+, m) is the vector space $\Omega^{\pm}C$ of meromorphic forms ω on C having two simple poles of peripheral integrals -1 and $+1$ at x_- and x_+, respectively. An element in $\Omega^{\pm}\mathcal{S}_{g,\cancel{2}}$ will sometimes be denoted as a triple (C, m, ω) since the pole information is already in ω.

Definition 1.1. The period of $(C, m, \omega) \in \Omega^{\pm}\mathcal{S}_{g,\cancel{2}}$ is the homomorphism

$$\mathrm{Per}(C, m, \omega) : H_1(\Sigma_g, \mathbb{Z}) \to \mathbb{C}/\mathbb{Z}$$

defined by

$$\gamma \mapsto \left[\int_{m(\gamma)} \omega\right].$$

The period map Per_g on $\Omega^{\pm}\mathcal{S}_{g,\cancel{2}}$ is the map

$$\mathrm{Per}_g : \Omega^{\pm}\mathcal{S}_{g,\cancel{2}} \to H^1(\Sigma_g, \mathbb{C}/\mathbb{Z}) \tag{2}$$

defined by $(C, m, \omega) \mapsto \mathrm{Per}(C, m, \omega) \in \mathrm{Hom}(H_1(\Sigma_g, \mathbb{Z}); \mathbb{C}/\mathbb{Z}) \simeq H^1(\Sigma_g, \mathbb{C}/\mathbb{Z})$.

In Proposition 3.4, we will prove that Per_g is a holomorphic submersion. The underlying foliation is therefore regular and holomorphic.

Definition 1.2. The degree of a homomorphism $p : H_1(\Sigma_g, \mathbb{Z}) \to \mathbb{C}/\mathbb{Z}$ with image Λ is the cardinality of Λ, which we denote as $|\Lambda|$.

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[1] The notation is chosen to distinguish it from other known subgroups of $\mathrm{Mod}(\Sigma_{g,2})$ characterized by the (trivial) action on some other homology groups: $\mathcal{I}(\Sigma_{g,2})$ is the subgroup that fixes every class in the relative homology group $H_1(\Sigma_g, p_-, p_+; \mathbb{Z})$ and $\mathcal{I}(\Sigma_{g,2^*})$ the subgroup that acts trivially on the punctured surface homology group $H_1(\Sigma_g \setminus \{p_-, p_+\}, \mathbb{Z})$.

Remark 1.3. If the period homomorphism of an element $(C, m, \omega) \in \Omega^{\pm}\mathcal{S}_{g,\underline{2}}$ has finite degree $d < \infty$, then the map $z \mapsto e^{\frac{1}{2\pi i}\int_{z_0}^{z}\omega}$ extends to a branched cover $C \to \mathbb{C} \cup \infty$ of the Riemann sphere of topological degree d. In particular, if $d = 1$, the only possibility is $g = 0$.

This is actually the only obstruction.

Theorem 1.4. *Let $g \geq 1$. The fiber of the period map* Per_g *over a non-trivial homomorphism is non-empty.*

Theorem 1.4 can be deduced from the results of [15, 16]. We will provide another proof along this paper.

Aiming at applying a Transfer Principle analogous to the case of holomorphic forms in [11], we consider the problem of the connectedness of the fibers of Per_g.

Theorem 1.5. *Let $g \geq 1$. The fiber of the period map* Per_g *over a homomorphism of degree at least three is connected.*

Theorem 1.6. *The fiber of* Per_g *over a degree two homomorphism is disconnected if $g \geq 2$ and connected if $g = 1$.*

The mapping class group $\mathrm{Mod}(\Sigma_g)$ acts on source and target of the map Per_g equivariatly, that is, for every $(C, m, \omega) \in \Omega^{\pm}\mathcal{S}_{g,\underline{2}}$ and $\varphi \in \mathrm{Mod}(\Sigma_g)$,

$$\mathrm{Per}_g\left(\varphi \cdot (C, m, \omega)\right) = \mathrm{Per}_g\left((C, m, \omega)\right) \circ \varphi_*^{-1}$$

where the star denotes the action of φ in homology, i.e., up to a choice of basis, the $\mathrm{Sp}(2g, \mathbb{Z})$ action on the symplectic group $\mathbb{Z}^{2g}$.

The fibration induced by Per_g thus induces a regular holomorphic foliation $\mathcal{F}_g$ on the quotient (moduli) space $\Omega^{\pm}\mathcal{M}_{g,2}$ called the isoperiodic foliation. Theorem 1.5 immediately implies the following corollary.

Corollary 1.7. *If $L \subset \Omega^{\pm}\mathcal{M}_{g,2}$ is the leaf of $\mathcal{F}_g$ associated with a period $p \in H^1(\Sigma_g, \mathbb{C}/\mathbb{Z})$ of degree at least three, then*

$$\pi_1(L) \to \mathrm{Stab}(p) \subset \mathrm{Aut}\left(H_1(\Sigma_g)\right) \textit{ is surjective.}$$

Theorem 1.5 allows us to relate some saturated sets by $\mathcal{F}_g$ and subsets of $H^1(\Sigma_g, \mathbb{C}/\mathbb{Z})$ invariant by the action of $\mathrm{Mod}(\Sigma_g)$ by means of equivariance of the map (2). This is what we call the Transfer Principle in [11]. Indeed, denote the quotient projection $\pi : \Omega^{\pm}\mathcal{S}_{g,\underline{2}} \to \Omega^{\pm}\mathcal{M}_{g,2}$. Then, every $\mathrm{Sp}(2g, \mathbb{Z})$-invariant subset

$$A \subset \left\{p \in H^1(\Sigma_g, \mathbb{C}/\mathbb{Z}) : \#p\left(H_1(\Sigma_g)\right) \geq 3\right\}$$

corresponds to a $\mathcal{F}_g$-saturated subset

$$B = \pi\left(\mathrm{Per}_g^{-1}(A)\right) \subset \Omega^{\pm}\mathcal{M}_{g,2} \setminus \{\text{degree two coverings over the Riemann sphere}\}, \quad (3)$$

and, moreover, the correspondence $A \mapsto B$ defined in (3) constitutes a bijection between these two types of subsets.

Thanks to the Transfer Principle, the closure of a leaf of $\mathcal{F}_g$ that is not formed by degree two covers of the Riemann sphere corresponds to the $\mathrm{Sp}(2g, \mathbb{Z})$ orbit closure of some element of $H^1(\Sigma_g, \mathbb{C}/\mathbb{Z})$ having at least three points in the image. With the aim of describing those orbit closures, we introduce two definitions.

Definition 1.8. Given a period $p \in H^1(\Sigma_g, \mathbb{C}/\mathbb{Z})$, a discrete factor of p is a continuous surjective morphism $\varphi : \mathbb{C}/\mathbb{Z} \to A$ to a Lie group such that $\varphi \circ p(H_1(\Sigma_g, \mathbb{Z}))$ is discrete. Given two periods $p, q \in H^1(\Sigma_g, \mathbb{C}/\mathbb{Z})$ having both φ as discrete factor, we say that they have the same image in φ if $\varphi \circ p(H_1(\Sigma_g, \mathbb{Z})) = \varphi \circ q(H_1(\Sigma_g, \mathbb{Z}))$.

Up to the isomorphism of the target group A, the candidates to non-trivial discrete factors are $\Im : \mathbb{C}/\mathbb{Z} \to \mathbb{R}$ and for each $\alpha \in \mathbb{R}$, $\Re + \alpha\Im : \mathbb{C}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$.

Definition 1.9. For any $p \in H^1(\Sigma_g, \mathbb{C}/\mathbb{Z})$, $V(p) \in \mathbb{R}/\Im(p)(H_1(\Sigma_g, \mathbb{Z}))$ is defined as

$$V(p) := \Re P \cdot \Im P \mod \Im(p)\big(H_1(\Sigma_g, \mathbb{Z})\big),$$

where P is any lift of p, namely, any $P \in H^1(\Sigma_g, \mathbb{C})$ whose reduction modulo $\mathbb{Z}$ is p. It is relevant for the dynamics of mapping class group only if $\Im : \mathbb{C}/\mathbb{Z} \to \mathbb{R}$ is a discrete factor of p.

Discrete factors are obviously invariant under the action of $\mathrm{Mod}(\Sigma_g)$.

Definition 1.10. Given $(C, \omega) \in \Omega^{\pm}\mathcal{M}_{g,2}$, a discrete factor of (C, ω) is a discrete factor of $\mathrm{Per}(C, m, \omega) \in H^1(\Sigma_g, \mathbb{C}/\mathbb{Z})$ where $(C, m, \omega) \in \Omega^{\pm}\mathcal{S}_{g,\underline{2}}$. If $\Im$ is a discrete factor of (C, ω), define $V(\omega) = V(\mathrm{Per}(C, m, \omega))$.

Theorem 1.11. *Let $g \geq 1$, L a leaf of $\mathcal{F}_g$ and $p \in H^1(\Sigma_g, \mathbb{C}/\mathbb{Z})$ the period of some point in L. A point $(C, \omega) \in \Omega^{\pm}\mathcal{M}_{g,2}$ lies in $\overline{L}$, the closure of L, if and only if the following hold:*

- *any discrete factor of p is a discrete factor of (C, ω) and both have the same image in it;*
- *if $\Im : \mathbb{C}/\mathbb{Z} \to \mathbb{R}$ is a discrete factor of p, then $V((C, \omega)) = V(p)$.*

The number of different discrete factors of an element $p \in H^1(\Sigma_g, \mathbb{C}/\mathbb{Z})$ tells us what the closure Λ of the image of p in $\mathbb{C}/\mathbb{Z}$ is. If p has no discrete factors, then $\Lambda = \mathbb{C}/\mathbb{Z}$, if it has at least two different discrete factors, then Λ is discrete and if it has a unique discrete factor, then Λ is the projection of an $\mathbb{R}$-linear copy of $\mathbb{R} + i\mathbb{Z}$ in $\mathbb{C}$ to $\mathbb{C}/\mathbb{Z}$.

Definition 1.12. Given $(C, \omega) \in \Omega^{\pm}\mathcal{M}_{g,2}$, we define

$$\Lambda_\omega = \left\{ \left[\int_\gamma \omega \right] \in \mathbb{C}/\mathbb{Z} : \gamma \in H_1(C) \right\}$$

and if L is a leaf of $\mathcal{F}_g$, denote $\Lambda_L = \Lambda_\omega$ for any $(C, \omega) \in L$.

From Theorem 1.11 and the previous tricotomy on the number of different factors, we easily deduce the following.

Corollary 1.13. *Let L be a leaf of $\mathcal{F}_g$ and let $\Lambda = \overline{\Lambda}_L \subset \mathbb{C}/\mathbb{Z}$ denote the topological closure of Λ_L.*

(1) *If Λ is discrete, then $\overline{L} = \{(C,\omega) \in \Omega^{\pm}\mathcal{M}_{g,2} : \Lambda_\omega = \Lambda\} = L$.*

(2) *If $\Lambda = \mathbb{C}/\mathbb{Z}$, then $\overline{L} = \Omega^{\pm}\mathcal{M}_{g,2}$.*

(3) *If Λ is neither discrete nor dense, then the following hold:*

  (a) *If $\Im\Lambda = 0$, then $\overline{L} = \{(C,\omega) \in \Omega^{\pm}\mathcal{M}_{g,2} : \Im\Lambda_\omega = 0\}$.*

  (b) *If $\Im\Lambda \neq 0$ is discrete, and we denote V_L as the value of V on an element of L, then $\overline{L} = \{(C,\omega) \in \Omega^{\pm}\mathcal{M}_{g,2} : \Im(\Lambda_\omega) = \Im(\Lambda)$ and $V(\omega) = V_L\}$.*

  (c) *If $\Im\Lambda$ is dense, then for a unique $\alpha \in \mathbb{R}$, $(\Re + \alpha\Im)(\Lambda)$ is finite in $\mathbb{R}/\mathbb{Z}$ and*

$$\overline{L} = \big\{(C,\omega) \in \Omega^{\pm}\mathcal{M}_{g,2} : (\Re + \alpha\Im)(\Lambda_\omega) = (\Re + \alpha\Im)(\Lambda) \bmod \mathbb{Z}\big\}.$$

In either case, $\overline{L}$ is a real analytic subset and the restriction of $\mathcal{F}_g$ to it is ergodic. The leaf L is closed if and only if Λ is discrete and algebraic if and only if Λ is finite (it corresponds to a Hurwitz space of branched covers over the sphere of degree $|\Lambda|$).

2. Strategy of the proof of Theorem 1.5

The proof follows by induction on the genus.

The cases of genera one and two are treated separately by analytic methods and the Torelli map. Each fiber of Per_1 is biholomorphic to a Zariski open set of $\mathcal{T}_{1,1}$ and is therefore connected. On the other hand, each fiber of Per_2 is a branched double cover, over a Zariski open subset of the Siegel space $\mathfrak{S}_2$. The branch points correspond precisely to odd forms with respect to the hyperelliptic involution. In the case of a fiber over a degree two homomorphism, all forms are even and we manage to prove that the cover is disconnected. In the case of degree at least three, we manage to construct an example of branch point (see Section 4.2) to deduce the connectedness of the cover.

Fix some $g \geq 3$ and assume the inductive hypothesis; i.e., Theorem 1.5 is true up to genus $g-1$. Then run the following program for any given homomorphism

$$p \in H^1(\Sigma_g, \mathbb{Z}) \to \mathbb{C}/\mathbb{Z}$$

of degree at least three.

Step 1. Bordify $\mathbf{Per}^{-1}(p)$ (see Section 6). The Teichmüller space $\mathcal{T}_{g,2}$ admits a topological bordification $\overline{\mathcal{T}}_{g,2}$ formed by marked stable curves of genus g with two marked points. It is stratified by the number of nodes of the underlying curves and each stratum is a complex manifold. The action of $\mathrm{Mod}(\Sigma_{g,2})$ preserves the stratification and the quotient is isomorphic to the Deligne–Mumford compactification $\overline{\mathcal{M}}_{g,2}$ of $\mathcal{M}_{g,2}$. The added points, called the boundary, form a normal crossing divisor in $\overline{\mathcal{M}}_{g,2}$ at the orbifold chart level. Quotienting $\overline{\mathcal{T}}_{g,2}$ by $\mathcal{I}(\Sigma_{g,\underline{2}})$ produces a stratified bordification $\overline{\mathcal{S}}_{g,\underline{2}}$ of $\mathcal{S}_{g,\underline{2}}$. The bundle of stable meromorphic forms with two poles over the Deligne–Mumford compactification of $\mathcal{M}_{g,2}$ can be pulled back to a bundle $\Omega^{\pm}\overline{\mathcal{S}}_{g,\underline{2}}$.

Any stable form in the boundary has a non-trivial local isoperiodic deformation space. Two conditions that guarantee that this local isoperiodic deformation space leaves the boundary are that the form has no zero components and the residue of the form at each non-separating node is zero. The bordification of $\mathrm{Per}^{-1}(p)$ that we are interested in is its closure in the space $\Omega_0^{\pm,*}\overline{\mathcal{S}}_{g,\underline{2}}$ of forms in $\Omega^{\pm}\mathcal{S}_{g,\underline{2}}$ having zero residues at all non-separating nodes and no zero components. In fact, the local isoperiodic deformation space projects to a smooth complex manifold in the orbifold charts of the moduli space transverse to each boundary component of $\Omega\overline{\mathcal{M}}_{g,2}$ passing through the point (see Theorem 6.13 and the appendix). The stratification of the boundary (defined by the number of nodes) induces a stratification of the bordification of the isoperiodic set and for each stratum of the ambient space passing through the point there is one isoperiodic component of the stratum that lies in it. Around a point having only separating nodes, the local picture of the stratification is that of a normal crossing divisor. The picture changes by an abelian ramified cover over the divisor when the curve underlying the form has at least one non-separating node (see the local model of this local branched cover in [11, Section 4.4]). The abelian ramified cover does not brake a nice property of the local stratification of a normal crossing divisor: a point in the codimension $k \geq 1$ stratum lies at the intersection of the closure of k codimension one (local) connected components of the divisor. Any other local connected component of a non-open stratum accumulating the point has codimension $1 \leq l \leq k$ and is *precisely* the set of points that belong to the closure of l of the k codimension one components accumulating the point, and not more. Moreover, the open stratum is locally connected at the point.

To any stratified space X that is a locally abelian ramified cover over a normal crossing divisor, we can define its dual boundary graph $\mathcal{C}(X)$. It has a vertex for each (global) connected component of the codimension one stratum, and a simplex between k vertices for each connected component of the codimension k stratum lying in the closure of the corresponding k components. It is well known that the boundary complex associated with $\overline{\mathcal{T}}_{g,2}$ is isomorphic to the *curve complex* $\mathcal{C}_{g,2}$ on the genus g compact surface with two marked points $\Sigma_{g,2}$ (see [20, Chapter 4.1]).

The closure of $\mathrm{Per}^{-1}(p)$ in $\Omega_0^{\pm,*}\overline{\mathcal{S}}_{g,\underline{2}}$ is also shown to be stratified and a locally abelian ramified cover over a normal crossing divisor. In particular, the connectedness of $\mathrm{Per}^{-1}(p)$ is equivalent to that of its bordification.

Moreover, the transversality condition allows us to define a continuous map of complexes

$$\mathcal{C}\big(\mathrm{Per}^{-1}(p)\big) \to \mathcal{C}(\Omega^{\pm}\overline{\mathcal{S}}_{g,\underline{2}}) \tag{4}$$

that associates with each component of an isoperiodic stratum the component of the ambient stratification where it sits. There is a subfamily of components of the codimension one stratum of the ambient space $\Omega^{\pm}\mathcal{S}_{g,\underline{2}}$ where we will be able to prove inductively that there is a single connected isoperiodic component of the isoperiodic stratum of codimension one associated with p. These are the so-called p-simple boundary components and they correspond to forms over stable curves with one node that leaves a pole on each side, and moreover the form restricted to each part has degree at least three. They define a subfamily of vertices of $\mathcal{C}(\mathrm{Per}^{-1}(p))$ that span the subcomplex $\mathcal{C}'(\mathrm{Per}^{-1}(p))$ of p-simple boundary points.

Step 2. Isoperiodic degeneration toward boundary points (see Section 7). We will first prove that any point in $\mathrm{Per}^{-1}(p)$ can be isoperiodically deformed in $\Omega_0^{\pm *}\overline{\mathcal{S}}_{g,\underline{2}}$ to a point belonging to a p-simple boundary component. In this step, we use Schiffer variations, a way of deforming the singular flat metric underlying a stable meromorphic form with isolated zeros without changing the associated period homomorphism. To achieve this step, we first degenerate to any boundary point and then prove that any boundary point can be joined to a p-simple boundary point.

Step 3. Connectedness of the boundary. This will be achieved by showing that the complex $\mathcal{C}'(\mathrm{Per}^{-1}(p))$ is connected. The inductive hypothesis of Theorem 1.5 allows us to prove that the restriction of the map (4) to the subcomplex $\mathcal{C}'(\mathrm{Per}^{-1}(p))$ of p-simple boundary points is injective at the level of the vertices. We will prove that it has connected image under the map (4). The fact that there are only two poles allows us to rephrase the problem in algebraic terms: the complex $\mathcal{C}(\Omega^{\pm}\overline{\mathcal{S}}_{g,\underline{2}})$ is isomorphic to $\mathcal{C}(\overline{\mathcal{S}}_{g,\underline{2}})$ which in turn is isomorphic to the quotient

$$\frac{\mathcal{C}_{g,2}}{\mathcal{I}(\Sigma_{g,\underline{2}})} \tag{5}$$

of the curve complex $\mathcal{C}_{g,2}$ under the natural action of the group $\mathcal{I}(\Sigma_{g,\underline{2}})$. Each k-simplex in (5) is characterized by a $\mathcal{I}(\Sigma_{g,\underline{2}})$-orbit of a family of k disjoint essential simple closed curves $c = c_1 \sqcup \cdots \sqcup c_k$ in $\Sigma_{g,2}$. Whenever every c_i separates the (two!) ordered marked points (and therefore the surface), we can characterize the simplex by the ordered splitting $H_1(\Sigma_{g,2}) = V_1 \oplus \cdots \oplus V_{k+1}$ into pairwise orthogonal symplectic submodules of rank at least two induced by the parts of $\Sigma_{g,2} \setminus c$. We suppose that the first marked point belongs to V_1 and the other to V_{k+1}. The order of the rest of factors is determined by imposing that V_j has a common boundary component with V_{j-1} and another with V_{j+1}.

The complex $\mathcal{C}'(\mathrm{Per}^{-1}(p))$ has its vertices corresponding to separating essential simple closed curves that separate the marked points, and any higher-dimensional simplex in it corresponds to marked stable forms having only separating nodes that separate the marked points. Therefore, the image of any k-simplex in $\mathcal{C}'(\mathrm{Per}^{-1}(p))$ by (4) is characterized by a splitting of $H_1(\Sigma_{g,2}) = V_1 \oplus \cdots \oplus V_{k+1}$ into $k+1$ pairwise orthogonal symplectic submodules of rank at least two. Moreover, for the vertices, i.e., when $k = 1$, the restriction $p_{|V_i}$ has at least three points in the image. The condition on the size of $p(V_i)$'s can be rephrased as follows.

Remark 2.1. A homomorphism p having image in $\mathbb{C}/\mathbb{Z}$ has at least three points in the image if and only if its composition with $\mathbb{C}/\mathbb{Z} \to \mathbb{C}/\frac{1}{2}\mathbb{Z}$ is non-trivial. Call this composition $[p]$.

Definition 2.2. Given a homomorphism $p : V \to A$ from a symplectic unimodular module to an abelian group A, we define the following:

- A decomposition $V = \oplus_i V_i$ into non-trivial symplectic unimodular submodules is said to be p-admissible if $p_{|V_i} \neq 0$ for all i.
- The graph of p-admissible decompositions has a vertex for every p-admissible decomposition with two factors, and an edge between the vertices corresponding to $V_1 \oplus V_2$ and $V_1' \oplus V_2'$ if there exists a p-admissible decomposition with more factors having one of the V_i's as factor and also one of the V_i''s.

A word of warning about this definition: we do not regard the order of the factors.

Proposition 2.3. *Let $g \geq 3$ and $p \in H^1(\Sigma_g, \mathbb{C}/\mathbb{Z})$. If the graph of $[p]$-admissible decompositions is connected and Theorem 1.5 is true up to genus $g-1$, then the complex $\mathcal{C}'(\mathrm{Per}^{-1}(p))$ is connected.*

The proof of this proposition is contained in Section 8. The following theorem (proven in Section 9) allows us to use Proposition 2.3 to deduce the connectedness of $\mathrm{Per}^{-1}(p)$.

Theorem 2.4. *Given a non-trivial homomorphism $p : V \to A$ from a symplectic unimodular $\mathbb{Z}$-module V of rank at least six to an abelian group A, the graph of p-admissible decompositions is non-empty and connected.*

A posteriori, it would have been enough to bordify only by adding the p-simple boundary points to prove the connectedness of the bordification. Unfortunately, we have not found a proof of the degeneration part that avoids passing through some other boundary components in $\Omega_0^{\pm *}\overline{\mathcal{S}}_{g,\underline{2}}$ as explained at the end of Step 2.

3. Some preliminary tools

3.1. Criterion for p-admissible elements

Definition 3.1. Let $p : V \to A$ be a non-trivial homomorphism from a symplectic unimodular $\mathbb{Z}$-module V to an abelian group A. An element $v \in V \setminus 0$ is p-admissible if it belongs to a factor of a p-admissible decomposition.

Lemma 3.2. *Suppose V has rank at least four and $p : V \to A$ is non-trivial. A non-zero element in V is p-admissible if and only if p does not vanish identically on the orthogonal of v. In this case, it belongs to a factor of rank two of a p-admissible decomposition. In particular, the set $\mathrm{NA}(p) \subset V$ of non-p-admissible elements is a submodule of rank at most one.*

Proof. Let v be a non-zero element of V. Assume v is p-admissible, namely, $v \in W_1$ where $V = W_1 \oplus W_2$ is a p-admissible decomposition. Then, the restriction of p to W_2 does not vanish and $W_2 \subset v^\perp$ so p does not vanish on the orthogonal of v.

Reciprocally, assume that p is not identically zero on $v^\perp$. We will assume that v is primitive, and consider a symplectic basis $a_1, b_1, \ldots,$ such that $v = a_1$.

Case 1 ($p(a_1) \neq 0$). In this case, we are done if p is not identically zero on $(\mathbb{Z}a_1 \oplus \mathbb{Z}b_1)^\perp$ since in this case we can take $W_1 = \mathbb{Z}a_1 \oplus \mathbb{Z}b_1$ and $W_2 = W_1^\perp$. If p is identically zero on

$(\mathbb{Z}a_1 \oplus \mathbb{Z}b_1)^\perp = \mathbb{Z}a_2 \oplus \mathbb{Z}b_2 \oplus \cdots$, we define $W_1 = \mathbb{Z}a_1 \oplus \mathbb{Z}(b_1 - b_2)$ and $W_2 = W_1^\perp$. The symplectic decomposition $V = W_1 \oplus W_2$ is p-admissible in this case since $p(a_1) \neq 0$, $p(a_2 + a_1) \neq 0$ and $a_1 \in W_1$, $a_2 + a_1 \in W_1^\perp$. We are done since $v = a_1 \in W_1$.

Case 2 ($p(a_1) = 0$). In this case, p does not vanish on $\mathbb{Z}a_2 \oplus \mathbb{Z}b_2 + \cdots$ since otherwise it would vanish on $a_1^\perp = v^\perp$, and up to changing the basis $a_2, b_2, \ldots,$ we can assume that $p(a_2) \neq 0$. If $p(b_1) \neq 0$, we are done by setting $W_1 = \mathbb{Z}a_1 + \mathbb{Z}b_1$ and $W_2 = W_1^\perp$. If $p(b_1) = 0$, we consider $W_1 = \mathbb{Z}a_1 \oplus \mathbb{Z}(b_1 + a_2)$ and $W_2 = W_1^\perp$ and we are done since $b_1 + a_2 \in W_1 \setminus \ker(p)$ and $a_2 \in W_2 \setminus \ker(p)$. So we conclude that v belongs to a p-admissible submodule of rank two.

To end the proof of the Lemma, notice that if two non-zero elements v, v' are not p-admissible, then both $v^\perp$ and $(v')^\perp$ are contained in $\ker(p)$. Being corank one primitive submodules, they need to be equal since otherwise they would generate the whole V, and so p would vanish identically. In particular, this proves that v and v' are rationally colinear, and the lemma follows. ∎

Corollary 3.3. *Let V be a symplectic unimodular module of rank at least four. Then, for any non-trivial homomorphism $p : V \to A$, there exists a p-admissible decomposition.*

3.2. Preliminaries on Per_g

We begin by proving the following.

Proposition 3.4. *The period map* Per_g *on* $\Omega^\pm \mathcal{S}_{g,\underline{2}}$ *is a holomorphic submersion.*

Proof. The statement can be checked locally. By the description of $\Omega^\pm \mathcal{S}_{g,\underline{2}}$ as affine bundle over the Hodge bundle, the map Per_g is, up to an adequate choice of local affine coordinate, defined by a germ of the map

$$\Omega\mathcal{S}_g \to \mathrm{Hom}\left(H_1(\Sigma_g, \mathbb{Z}); \mathbb{C}\right) \text{ given by } (C, m, \omega) \mapsto \left\{\gamma \mapsto \int_{m(\gamma)} \omega\right\}$$

where $\Omega\mathcal{S}_g \to \mathcal{S}_g$ denotes the Hodge bundle of *holomorphic* one forms over the Torelli cover $\mathcal{S}_g \to \mathcal{M}_g$. This map is holomorphic everywhere and submersive around any non-zero form (see e.g. [11]). By appropriately choosing the base point of the affine model, we get the desired result. ∎

Proposition 3.5. *Given $p \in H^1(\Sigma_g, \mathbb{C}/\mathbb{Z}) \setminus H^1(\Sigma_g, \mathbb{R}/\mathbb{Z})$, the set of lifts $P \in H^1(\Sigma_g, \mathbb{C})$ that are the periods of a holomorphic differential on a smooth homologically marked curve, and whose reductions modulo $\mathbb{Z}$ is p, is infinite countable. If $p \in H^1(\Sigma_g, \mathbb{R}/\mathbb{Z}) \setminus 0$, the set is empty and if $p = 0$, the only possibility is $P = 0$.*

Proof. Let P be one such lift for any $p \in H^1(\Sigma_g, \mathbb{C}/\mathbb{Z})$ and write $P = u + iv$ with $u, v \in H^1(\Sigma_g, \mathbb{R})$. A necessary and sufficient condition on P to be the period of a non-zero holomorphic one form on a homologically marked curve is that

(1) the symplectic product $u \cdot v$ is positive, and

(2) in the case the image of P in $\mathbb{C}$, considered as a map from $H_1(\Sigma_g, \mathbb{Z})$ to $\mathbb{C}$, is discrete, the symplectic product $u \cdot v$ is strictly greater than the covolume in $\mathbb{C}$ of the image of P.

See Otto Haupt's theorem in [27]. Hence, in the case $p = 0$, the only possibility is that the form is zero. The case where p has values in $\mathbb{R}$ has no lifts that are periods of holomorphic one forms. Suppose $p \in H^1(\Sigma_g, \mathbb{C}/\mathbb{Z}) \setminus H^1(\Sigma_g, \mathbb{R}/\mathbb{Z})$.

Let now $P' = P + w$ another lift, where $w \in H^1(\Sigma_g, \mathbb{Z})$, and $P' = u' + iv'$ where $u' = u + w$ and $v' = v$. Since $u' \cdot v' = u \cdot v + w \cdot v$, and that $v \neq 0$ by assumption (otherwise p would belong to $H^1(\Sigma_g, \mathbb{R}/\mathbb{Z})$), there is a half space in $H^1(\Sigma_g, \mathbb{Z})$ of choices of w so that P' satisfies the volume positivity condition (1) of Haupt's criterion.

Suppose now that for one P', say P, the first condition is satisfied but not the second. In that case, we know that the kernel of P is a symplectic unimodular submodule of $H_1(\Sigma_g, \mathbb{Z})$ of rank $2g-2$, or equivalently that there exist two elements $a, b \in H^1(\Sigma_g, \mathbb{Z})$ such that $a \cdot b = 1$ and two elements $\alpha, \beta \in \mathbb{C}$ such that $\Im(\beta\overline{\alpha}) > 0$ and $P = a\alpha + b\beta$. Now let $P' = P + w$. Assume that P' is also of the form $a'\alpha' + b'\beta'$ with $a' \cdot b' = 1$ and $\Im(\beta'\overline{\alpha'}) > 0$. We then have $a'\alpha' + b'\beta' = a\alpha + b\beta + w$. This means that w is a rational combination of $a\alpha$ and $b\beta$. So, apart from a submodule of rank at most two, all the elements w in a half-space of $H^1(\Sigma_g, \mathbb{Z})$ give rise to a period $P' = P + w$ that satisfies conditions (1) and (2).

The proposition follows. ■

Let $\mathcal{S}_{g,0} \to \mathcal{M}_{g,0}$ denote the Torelli cover of the moduli space of genus g curves. Its covering group is the Torelli group $\mathcal{I}(\Sigma_{g,0}) \subset \mathrm{Mod}(\Sigma_{g,0})$ formed by elements that act trivially on the symplectic group $H_1(\Sigma_g)$. To each element in $\mathcal{S}_{g,0}$ we can associate a pair (C, m) where m is an isomorphism from $H_1(\Sigma_g)$ to $H_1(C)$. Denote by $\Omega\mathcal{S}_{g,0} \to \mathcal{S}_{g,0}$ the pullback of the Hodge (vector) bundle of abelian differentials over $\mathcal{M}_{g,0}$. The period map is well defined on $\Omega\mathcal{S}_{g,0}$ and the restriction of the period map to a fiber of $\Omega\mathcal{S}_{g,0} \to \mathcal{S}_{g,0}$ is linear and injective (recall that on a fixed compact Riemann surface a holomorphic one form is completely determined by its period homomorphism). Therefore, the restriction of the bundle projection to a fiber of Per is a biholomorphism onto its image. The next proposition describes the image of this projection by using the injectivity of the Torelli map

$$J : \mathcal{S}_{g,0} \to \mathfrak{S}_g$$

to Siegel space of symmetric complex $g \times g$ matrices with positive definite imaginary part defined as follows: consider a symplectic basis $a_1, b_1, \dots, a_g, b_g$ of $H_1(\Sigma_g, \mathbb{Z})$, i.e., a basis such that the only non-zero symplectic products of its elements are $a_i \cdot b_i = +1$ and $b_i \cdot a_i = -1$. For each homologically marked compact type genus g curve (C, m), let $\omega_1, \dots, \omega_g$ be a basis of the space of holomorphic one forms on C such that

$$\int_{m(a_i)} \omega_j = \delta_{i,j},$$

$\delta_{i,j}$ being the Kronecker symbol. The Jacobian $g \times g$ matrix is $J(C) = (\int_{m(b_i)} \omega_j)_{i,j}$. The fact that it belongs to the Siegel space $\mathfrak{S}_g$ is a consequence of Riemann's relations.

Proposition 3.6 (McMullen). *Given $p \in H^1(\Sigma_g, \mathbb{C}/\mathbb{Z}) \setminus H^1(\Sigma_g, \mathbb{R}/\mathbb{Z})$ for each lift $P \in H^1(\Sigma_g, \mathbb{C})$ as in Proposition 3.5, the set of elements of $\mathcal{S}_{g,0}$ supporting a holomorphic form having period equal to P is biholomorphic (via J) to a slice of the image of J by a linear Siegel subspace $\mathfrak{S}_P \subset \mathfrak{S}_g$ of genus $g-1$.*

Proof. Let P be a lift of p. Suppose there exists a holomorphic one form ω_P on (C, m) such that $(\int \omega) \circ m = P$. Since $\omega_1, \ldots, \omega_g$ is a basis of $\Omega(C) = H^0(C, K_C)$, we need to have $\omega_P = P(a_1)\omega_1 + \cdots + P(a_g)\omega_g$ so that we get the relations

$$P(b_i) = P(a_1)J(C)_{i,1} + \cdots + P(a_g)J(C)_{i,g} \quad \text{for } i = 1, \ldots, g.$$

Reciprocally, these relations clearly imply the existence of a holomorphic one form with period P on (C, m). The set of Jacobians of homologically marked curves of compact type satisfying these relations is a copy of the $(g-1)^2$-dimensional Siegel space denoted by $\mathfrak{S}_P$. ∎

4. Proof of Theorem 1.5 for $g = 1, 2$

4.1. Genus one

Lemma 4.1. *Given an elliptic curve E, and a morphism $p : H_1(E, \mathbb{Z}) \to \mathbb{C}/\mathbb{Z}$, there exists a unique meromorphic form on E, up to translation, that satisfies the following:*

1. *it is either holomorphic or it has two distinct simple poles,*
2. *its period modulo $\mathbb{Z}$ equals p.*

If p takes real values and is non-vanishing, then we are always in the second case.

Proof. To the period p one can associate the flat unitary line bundle (L, ∇) having monodromy $\exp(2i\pi p) : H_1(E, \mathbb{Z}) \to \mathbb{C}^\star$. Depending on whether this bundle is trivial or not, it has, up to multiplication by a non-zero constant, a unique holomorphic section, or a unique meromorphic section with a simple zero and a simple pole, which can be explicitly described by the quotient of two theta series. Denote this section as $s : E \to L$; the form defined by $\nabla s = \omega s$ is either holomorphic or it has two distinct poles and satisfies $p(\omega) = p$.

In the case p is non-zero but takes real values, the bundle L is never trivial; hence, the form ω cannot be holomorphic in that case. ∎

Corollary 4.2. *A fiber of the period map* Per_1 *is biholomorphic to a Zariski dense subset of the Teichmüller space $\mathcal{T}_{1,1}$ of genus one curves with a marked point. If the fiber is over a non-zero homomorphism with real values, it is biholomorphic to $\mathcal{T}_{1,1}$. In particular, any fiber of* Per_1 *is connected.*

Proof. This is an immediate corollary of Lemma 4.1, as soon as one sees that in a level set of the period map, the one corresponding to case (1) is a Zariski closed set of dimension zero. But this set is easily described: it corresponds exactly to the possible lifts in

$H^1(E, \mathbb{C})$ of p having positive volume, see Section 3.2. In the case where p takes real values, one cannot lift p to a period of positive volume, so the corollary follows. ■

Remark 4.3. A more geometric argument can be developed to connect every pair of isoperiodic forms by a more explicit isoperiodic path. The argument uses a surgery – Schiffer variations – to connect first every form to a form with a single zero in its isoperiodic set. A few further Schiffer variations between forms with a single zero and an analysis of the action of the mapping class group allows us to prove that all isoperiodic forms with a single zero lie in the same connected component of the isoperiodic set. The advantage of this geometric argument with respect to the previous analytic argument is that it can be generalized to the case of genus one with any number of poles. This approach will be dealt with in [5].

4.2. Genus two

Recall that in the genus two case, there is a holomorphic embedding $J : \mathcal{S}_{2,0} \to \mathfrak{S}_2$ of the Torelli space – the cover of $\mathcal{M}_{2,0}$ whose covering group is the Torelli group $\mathcal{I}(\Sigma_{2,0})$ – in the Siegel space consisting of symmetric two by two matrices with complex coefficients and positive definite imaginary part. The map J associates with a homologically marked smooth curve of genus two its Jacobian matrix. The complement $\mathfrak{S}_2 \setminus J(\mathcal{S}_{2,0})$ is the set of Jacobian matrices of a nodal genus two homologically marked curve of compact type: this set is the union, parametrized by $\mathrm{Sp}(4, \mathbb{Z})/(\mathrm{Sp}(2, \mathbb{Z}) \times \mathrm{Sp}(2, \mathbb{Z}))$, of the images of the set of diagonal matrices with coefficients in the upper half-plane by an element of the group $\mathrm{Sp}(4, \mathbb{Z})$.

Theorem 4.4. *Given any $p \in H^1(\Sigma_2, \mathbb{C}/\mathbb{Z})$ of degree at least two, the restriction of the forgetful map*

$$\Omega^{\pm}\mathcal{S}_{2,\cancel{2}} \to \mathcal{S}_{2,0} \text{ defined by } (C, x_-, x_+, m, \omega) \mapsto (C, m)$$

to the isoperiodic set $\mathrm{Per}_2^{-1}(p)$ is a ramified double cover over the open (connected) subset

$$\mathcal{S}_{2,0} \setminus \bigcup_{P \text{ lift of } p} J^{-1}(\mathfrak{S}_P).$$

The critical points correspond to homologically marked meromorphic 1-forms of period p that are odd with respect to the hyperelliptic involution on the corresponding genus two curve.

Before giving the proof, let us recall the following classical lemma.

Lemma 4.5. *Let C be a genus two smooth complex curve and $i : C \to C$ the hyperellptic involution. The map $(q_1, q_2) \in C^2 \mapsto q_1 - q_2 \in J(C)$ is invariant by the involution $I(q_1, q_2) = (i(q_2), i(q_1))$, and the induced map $C^2/I \to J(C)$ is the blow-up of $J(C)$ at the point 0. The exceptional divisor is the quotient by I of the diagonal in C^2.*

Proof. Take two distinct couples of points of C, $(q_1, q_2) \neq (q_1', q_2')$, such that $q_1 - q_2 = q_1' - q_2'$ in $J(C)$. Then the divisor $(q_1 + q_2') - (q_1' + q_2)$ is principal; namely, either

$\{q_1, q_2'\} = \{q_1', q_2\}$ or there exists a non-constant meromorphic function $f : X \to \mathbb{P}^1$ such that $(q_1 + q_2') - (q_1' + q_2) = (f)$. The first case occurs when $q_2 = q_1$ and $q_2' = q_1'$, corresponding to the level set of the trivial bundle in the Jacobian. In the second case, by the uniqueness of the hyperelliptic involution on C, the function f is the quotient of the hyperelliptic involution post composed by a biholomorphism $C/i \to \mathbb{P}^1$, and we then have $\{q_1', q_2\} = i(\{q_1, q_2'\})$, which shows that $(q_1', q_2') = I(q_1, q_2)$. ■

Proof of Theorem 4.4. Let $(C, m) \in \mathcal{S}_{2,0}$ a homologically marked genus two curve. We denote by $i_C : C \to C$ the hyperelliptic involution on C, and by (L_p, ∇_p) the flat line bundle over C whose monodromy is given by the character $\rho_p = \exp(2i\pi p) \circ m^{-1} \in H^1(\Sigma_g, \mathbb{C}^*)$. Forgetting the flat connection, the line bundle L_p has degree zero and defines an element of the Jacobian $J(C)$.

Given a meromorphic form ω on C with two poles, both simple and having peripheral periods ± 1, the function $\exp(2i\pi \int \omega)$ is a meromorphic function on $\widetilde{C}$ which is ρ_p-equivariant. Hence, it induces a meromorphic section $s_\omega : C \to L_p$, whose divisor is $2i\pi$ times the residue divisor of ω. In particular, the bundle L_p is non-trivial by Lemma 4.5.

Reciprocally, suppose that L_p is non-trivial. Then Lemma 4.5 shows that it has two meromorphic sections, both having one simple zero and one simple pole. The logarithmic derivative of any such section with respect to ∇_p gives the two desired meromorphic one forms on C with two poles, both simple and with peripheral periods ± 1, that have a period modulo $\mathbb{Z}$ equal to p.

Notice that if we denote by (q_1, q_2) the two poles of the first section and by (q_1', q_2') the two poles of the second one, ordered so that the residues are $1/2i\pi$ at q_1 and q_1' and $-1/2i\pi$ at q_2 and q_2', then $I(q_1, q_2) = (q_1', q_2')$. Hence, the critical points of the projection of the isoperiodic set $\mathrm{Per}^{-1}(p)$ to the Torelli space $\mathcal{S}_{2,0}$ correspond to the meromorphic differentials whose couple of ordered poles satisfy $I(q_1, q_2) = (q_1, q_2)$. This means that the two poles are exchanged by the hyperelliptic involution. Hence, if we denote by ω such a meromorphic form, the form $-i_C^*\omega$ has period p (this is because the hyperelliptic involution acts by multiplication by -1 on the first cohomology group of C), and the same residue divisor as ω. Hence, the sum $\omega + i_C^*\omega$ is a holomorphic one form with trivial period homomorphism: it needs to be identically zero, and thus ω is odd with respect to the hyperelliptic involution.

To conclude, notice that if L_p is trivial, then it has a section that is nowhere vanishing. The logarithmic derivative of that section is a holomorphic one form on C whose period modulo $\mathbb{Z}$ is equal to p. Hence, the period of that differential is a lift P of p, and $J(C)$ belongs to $\mathcal{S}_P$. Reciprocally, each curve C whose Jacobian lies in $\mathfrak{S}_P$ for some lift P of p has a holomorphic differential ω whose period is P. Then the multi-valued function $\exp(2i\pi \int \omega)$ is ρ_p-equivariant and does not vanish and defines a holomorphic section of L_p that vanishes nowhere. Hence, the bundle L_p is trivial in this case. ■

Corollary 4.6. *If $p \in H^1(\Sigma_2, \mathbb{C}/\mathbb{Z})$ has degree at least three (it might be infinite), then the branched double cover of Theorem 4.4 has a branch point. It is therefore connected. If p has degree two, there are no branch points.*

In the next section, we will determine an invariant that shows that the branched cover is disconnected when the degree is two.

Proof. Recall that a branch point of the map is a homologically marked meromorphic one form of period p which is odd with respect to the hyperelliptic involution on the corresponding genus two curve (see Theorem 4.4).

As is well known (see [9] for details), there is a dictionary between meromorphic forms on a smooth genus g curve and singular translation structures with some particular types of singularities on a compact topological surface of genus g. The period homomorphism of the form corresponds to the holonomy of the translation structure. The "odd" point will be realized by producing a translation structure with the prescribed singularities, holonomy and symmetry (corresponding to the hyperelliptic involution). One of the advantages is that the construction is not analytic, but geometric in spirit. The complex structure is obtained *a posteriori*.

First assume that the image of p is not contained in the real line. Let P be a lift of p for which there exists a homologically marked holomorphic differential (C, m, ω) with period $(\int \omega) \circ m = P$, whose existence has been established in Proposition 3.5. The form ω is odd with respect to the hyperelliptic involution; namely, it satisfies $i_C^* \omega = -\omega$. Among the six points fixed by the hyperelliptic involution on C, there are at least four at which the form ω does not vanish. At one of these points, slit C along a small vertical segment of length l invariant under the hyperelliptic involution, slit the infinite cylinder $\mathbb{C}/\mathbb{Z}$ along a vertical segment of length l and glue these two slit surfaces together in order to get a homologically marked meromorphic one form which is odd with respect to the hyperelliptic involution. So we are done in this case.

Now let us assume that p takes only real values and that its degree is at least three. We begin by the following very elementary claim.

Claim 1. There exists a symplectic basis a_1, b_1, a_2, b_2 of $H_1(\Sigma_2, \mathbb{Z})$ such that none of the periods $p(a_1)$, $p(a_2)$ nor $p(a_1 + a_2)$ vanish.

Proof. Since p is not zero modulo $\frac{1}{2}\mathbb{Z}$, by Corollary 3.3, there exists a symplectic decomposition $H_1(\Sigma_2, \mathbb{Z}) = H_1 \oplus H_2$ so that the restriction of p to each module H_k, $k = 1, 2$, is not zero modulo $\frac{1}{2}\mathbb{Z}$. In each of the symplectic submodules H_k, let a_k, b_k be a symplectic basis so that $p(b_k) = 0$. Then, $p(a_k)$ does not belong to $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$ by construction. If $p(a_1 + a_2) \neq 0$, we are done. Otherwise, $p(a_2) = -p(a_1)$, so define $a_2' = -a_2$ and $b_2' = -b_2$. We then have $p(a_1 + a_2') = 2p(a_1) \neq 0$, and the basis a_1, b_1, a_2', b_2' works. ■

Now let us denote by p_-, p_+ two distinct points in Σ_2, and let

$$\widetilde{a_1}, \widetilde{b_1}, \widetilde{a_2}, \widetilde{b_2} \in H_1\big(\Sigma_2 \setminus \{p_-, p_+\}, \mathbb{Z}\big)$$

lifts of the elements a_1, b_1, a_2, b_2. Denoting by $\pi^{\pm} \in H_1(\Sigma_2 \setminus \{p_-, p_+\}, \mathbb{Z})$ the peripheral homology classes around $p_{\pm}$, the family $\widetilde{a_1}$, $\widetilde{b_1}$, $\widetilde{a_2}$, $\widetilde{b_2}$, π^+ is a basis of $H_1(\Sigma_2 \setminus \{p_-, p_+\}, \mathbb{Z})$. We have the following.

Claim 2. One can choose the symplectic basis a_1, b_1, a_2, b_2 and its lift and $P \in H^1(\Sigma_2 \setminus \{p_-, p_+\}, \mathbb{R})$ whose reduction modulo $\mathbb{Z}$ equals p, and which is such that

$$P(\widetilde{a_1}), P(\widetilde{a_2}) > 0 \quad \text{and} \quad P(\widetilde{a_1}) + P(\widetilde{a_2}) < 1.$$

Proof. Consider the basis a_1, b_1, a_2, b_2 constructed in the Claim 1. Let P be any lift of p satisfying that $P(\widetilde{a_k})$ is the determination of $p(a_k)$ modulo $\mathbb{Z}$ that belongs to $(0, 1)$. Observe that $P(\widetilde{a_1}) + P(\widetilde{a_2})$ belongs to $(0, 2) \setminus \{1\}$ since $p(a_1) + p(a_2) \neq 0$. If $P(\widetilde{a_1}) + P(\widetilde{a_2}) < 1$, we are done. Otherwise, we have $P(\widetilde{a_1}) + P(\widetilde{a_2}) > 1$. In this case, introduce the new symplectic basis $a'_k = -a_k, b'_k = -b_k$, and its lift $\widetilde{a'_k} = -\widetilde{a_k}, \widetilde{b'_k} = -\widetilde{b_k}$. Define P' so that $P'(\widetilde{a'_k})$ is the determination of $p(a'_k)$ modulo $\mathbb{Z}$ lying in the interval $(0, 1)$. We have $P'(\widetilde{a'_k}) = 1 - P(\widetilde{a_k})$. Hence,

$$P'(\widetilde{a'_1}) + P'(\widetilde{a'_2}) = 2 - \big(P(\widetilde{a_1}) + P(\widetilde{a_2})\big) < 1,$$

and we are done. ■

Let now $\widetilde{a_3} \in H_1(\Sigma_2 \setminus \{p_-, p_+\}, \mathbb{Z})$ be defined by the relation $\widetilde{a_1} + \widetilde{a_2} + \widetilde{a_3} = \pi^+$. Since $P(\pi^+) = 1$, we have

$$P(\widetilde{a_k}) > 0 \text{ for } k = 1, 2, 3 \quad \text{and} \quad P(\widetilde{a_1}) + P(\widetilde{a_2}) + P(\widetilde{a_3}) = 1.$$

Consider a translation surface X^+ made of four vertical cylinders C_0, C_1, C_2, C_3. The first one, C_0, is a semi-infinite cylinder of circumference 1 isomorphic to $\overline{\mathbb{H}}/\mathbb{Z}$, and the other three, C_1, C_2, C_3, are compact cylinders of height 1 and of circumference $P(\widetilde{a_1})$, $P(\widetilde{a_2})$, and $P(\widetilde{a_3})$. We glue these three cylinders below C_0 to form a translation surface homeomorphic to a sphere minus three open discs and one puncture (see Figure 1).

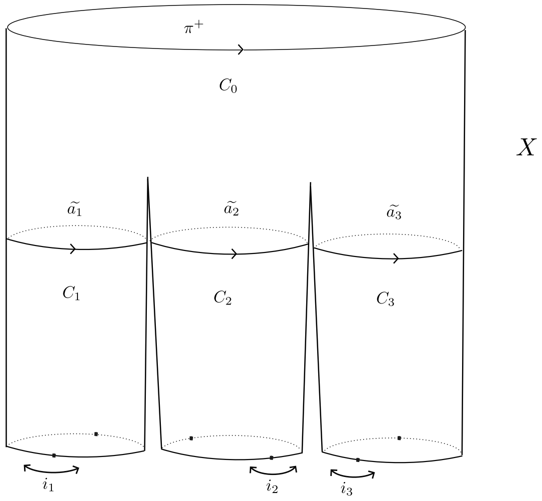

Figure 1. The surface with boundary X^+.

This translation surface is non-compact and has three boundary components $\partial_k X$ that are horizontal geodesics of length $P(\widetilde{a_k})$. Choosing reversing orientation isometries i_k of the boundary components $\partial_k X^+$, one constructs a translation surface X in the following way: let X^- be the translation surface obtained from X^+ by modifying each chart of the translation structure by its composition with $-\mathrm{id}$. The reversing orientation isometries i_k can be viewed as isometries from $\partial_k X^+ \subset \partial X^+$ to $\partial_k X^- \subset \partial X^-$. Use these isometries to glue X^+ to X^- and set X to be the resulting translation surface. The identifying map from X^+ to X^- and from X^- to X^+ defines an involution $i_X : X \to X$ which in the charts of the translation structure of X takes the form $z \mapsto -z + \mathrm{cst}$. Let

$$m : H_1\big(\Sigma_g \setminus \{q^+, q^-\}, \mathbb{Z}\big) \to H_1(X, \mathbb{Z})$$

be any marking of X sending $\pi^{\pm}$ to the peripheral of $C_0^{\pm}$, and $\widetilde{a_k}$ to the boundary $\partial_k X^+$. If we denote by ω the differential on X defined by $\omega = dz$ in any translation chart z, we then have $\int_{m(\widetilde{a_k})} \omega = P(\widetilde{a_k})$.

This construction depends on some continuous parameters that are the isometries i_k. Conjugating i_k by a rotation of angle θ_k results in twisting with angle θ_k along the geodesic $\partial_k X^+$. The periods of the new homologically marked holomorphic form (X', ω', m') (the marking m' is obtained from the marking m using the Gauss–Manin connection) can be computed by the following simple formula:

$$\int_{m(\tilde{\gamma})} \omega' = \int_{m(\tilde{\gamma})} \omega + \sum_l \theta_l \, \widetilde{a_l} \cdot \tilde{\gamma},$$

where as usual the symplectic product is denoted by a dot. We can then choose the angles θ_k, for $k = 1, 2$, and $\theta_3 = 0$, in such a way that

$$\int_{m(\widetilde{b_k})} \omega' = P(\widetilde{b_k}).$$

The form $(\overline{X'}, \omega', \overline{m'}) \in \Omega^{\pm}\mathcal{S}_{g,\underline{2}}$ obtained from (X, ω, m) by adding two points at its two ends is the desired element of $\mathrm{Per}^{-1}(p)$ which is odd with respect to hyperelliptic involution. Hence, $\mathrm{Per}^{-1}(p)$ is connected in this case.

To conclude, we need to show that if the degree of p is two, there are no branch points of the branched double cover. It suffices to establish that there is no meromorphic differential with two simple poles on a genus two curve which is at the same time of degree two and odd with respect to the hyperelliptic involution. Now, given a meromorphic form ω of degree two, the function $\exp(2i\pi \int \omega)$ is the unique hyperelliptic function up to post-composition; hence, ω is even with respect to the hyperelliptic involution in this case, not odd. ■

5. Proof of Theorem 1.6

Given a finite subset $E \subset \mathbb{P}^1$, the homology group $H_1(\mathbb{P}^1 \setminus E, \mathbb{Z}/2\mathbb{Z})$ is naturally isomorphic to $M_E = \frac{\bigoplus_{e \in E} \mathbb{Z}/2\mathbb{Z}e}{\mathbb{Z}/2\mathbb{Z} \sum_{e \in E} e}$. In the sequel, we will consider subsets of the Riemann sphere that contain the points 0 and ∞ of cardinality $2g + 2$. We will fix once for all such a subset, which we denote as E_0. Observe that any bijection σ from E to E_0, which maps the points 0 and ∞ to themselves, induces a linear map $\varphi_\sigma : M_E \to M_{E_0}$. Two such linear maps differ by post-composition by a map of the form φ_σ, where σ is a permutation of E_0 that fixes 0 and ∞. Given a $\mathbb{Z}/2\mathbb{Z}$-module M, we say that two morphisms $f : M \to M_{E_0}$ are equivalent if they differ by post-composition by a morphism of the form φ_σ where σ is a permutation of E_0 that fixes the points 0 and ∞. The set of equivalence classes is denoted as $S_{2g} \setminus \mathrm{Hom}(M, M_{E_0})$.

Definition 5.1. Given a marked meromorphic form (C, m, ω) of degree two on a smooth curve, we define its Arnold invariant

$$\mathrm{Ar} = \mathrm{Ar}(C, m, \omega)$$

as the class $[A]$ in $S_{2g}\setminus\mathrm{Hom}(H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z}), M_{E_0})$ of the homomorphism $A = \varphi_\sigma \circ f_* \circ m$, where

$$f_* : H_1(C, \mathbb{Z}/2\mathbb{Z}) \to H_1\big(\mathbb{P}^1 \setminus \mathrm{VC}(f), \mathbb{Z}/2\mathbb{Z}\big)$$

is the well-defined homomorphism induced in the $\mathbb{Z}/2\mathbb{Z}$-homology of C by the restriction of the branched double covering $f = \exp(2\pi i \int \omega) : C \to \mathbb{P}^1$ to the complement of the ($2g + 2$ simple) branch points (see [4]). In this formula, $\mathrm{VC}(f)$ is the set of critical values of the covering f, and σ is a bijection from $\mathrm{VC}(f)$ to E_0 mapping 0 to 0 and ∞ to ∞.

For each degree two homomorphism $p \in H^1(\Sigma_g, \mathbb{C}/\mathbb{Z})$, the Arnold invariant is *locally* constant on $\mathrm{Per}^{-1}(p)$. On the other hand, the fact that $\ker(p)$ has rank $2g - 1$ implies that there is a non-trivial stabilizer $\mathrm{Stab}(p)$ of p in $\mathrm{Aut}(H_1(\Sigma_g, \mathbb{Z}))$. This stabilizer acts on $\mathrm{Per}^{-1}(p)$ by pre-composition on the marking. The idea of the proof of Theorem 1.6 is to show that in the orbit of a point in $\mathrm{Per}^{-1}(p)$, there are at least two values of the Arnold invariant, and therefore $\mathrm{Per}^{-1}(p)$ has at least as many components as the number of values that the Arnold invariant takes.

Note that we can recover the period $p \in H^1(\Sigma_g, \frac{1}{2}\mathbb{Z}/\mathbb{Z}) \simeq H^1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})$ from Arnold's invariant of a marked form (C, m, ω) of degree two. Indeed, the group $M_{\{0,\infty\}}$ is naturally isomorphic to $\mathbb{Z}/2\mathbb{Z}$ (such a module has no non-trivial automorphisms); the period p is just the composition of a representative A of $\mathrm{Ar}(C, m, \omega)$ with the projection $M_{E_0} \to M_{\{0,\infty\}} \simeq M_{E_0}/\bigoplus_{e \in E_0\setminus\{0,\infty\}} \mathbb{Z}/2\mathbb{Z}e$. In particular, the stabilizer of Arnold's invariant of an element (C, m, ω) of degree two in the symplectic group $\mathrm{Aut}(H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z}))$ is contained in the stabilizer of the period $p = \mathrm{Per}(C, m, \omega)$.

Lemma 5.2. *For any $g \geq 2$, given any $(C, m, \omega) \in \Omega^{\pm}\mathcal{S}_{g,2}$ of degree two, the stabilizer in the group* $\mathrm{Aut}(H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z}))$ *of the reduction modulo two of the period homomorphism* $p = \mathrm{Per}(C, m, \omega)$ *is larger than the stabilizer of the Arnold class* $\mathrm{Ar}(C, m, \omega)$.

Proof. Since $p \neq 0 \bmod 2$, the stabilizer of the reduction mod 2 of p in $\mathrm{Aut}(H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z}))$ is in bijection of the set of symplectic basis of $H^1(\Sigma_g, \mathbb{Z}/2\mathbb{Z})$ that contain the element p as the first member. The number of such basis is the product

$$2^{2g-1}(2^{2g-2} - 1)2^{2g-3} \times \cdots \times (2^2 - 1) \times 2.$$

Let us now bound from below the number of elements in the stabilizer of Arnold's class $\mathrm{Ar} = \mathrm{Ar}(C, m, \omega)$, representated by a map $A : H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z}) \to M_{E_0}$.

Observe that the map A is injective, and that its image is the subset $M_{E_0}^{\mathrm{even}} \subset M_{E_0}$ consisting of formal sums $\sum_{e \in F} e$, where F is a subset of E_0 with an even number of elements.

Given any element $M \in \mathrm{Aut}(2g, \mathbb{Z}/2\mathbb{Z})$ which fixes Ar, there exists a permutation σ of the set E such that

$$A \circ M^{-1} = \varphi_\sigma \circ A. \tag{6}$$

Since M_{E_0} has more than two elements, the action of φ_σ on $M_{E_0}^{\mathrm{even}}$ completely determines σ, the permutation σ satisfying (6) is well defined and the induced map

$$M \in \mathrm{Stab}_{\mathrm{Aut}(H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z}))}(\mathrm{Ar}) \mapsto \sigma \in S_{2g} \tag{7}$$

is a homomorphism. Now, because the map A is injective, the morphism (7) is injective. We thus get that the stabilizer of Ar in $\mathrm{Aut}(H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z}))$ has at most $(2g)!$ elements. The lemma follows. ■

Corollary 5.3. *If $g \geq 2$ and $(C, m, \omega) \in \Omega^{\pm}\mathcal{S}_{g,\underline{2}}$ has degree two period homomorphism $p = \mathrm{Per}(C, m, \omega)$, there exist at least two values of the Arnold invariant in the orbit*

$$\mathrm{Stab}(p) \cdot (C, m, \omega) \subset \mathrm{Per}^{-1}(p).$$

Proof. The reduction modulo two map $\mathrm{Aut}(H_1(\Sigma_g, \mathbb{Z})) \to \mathrm{Aut}(H_1(\Sigma_g, \mathbb{Z}/2\mathbb{Z}))$ is surjective. From Lemma 5.2, we deduce that there exist two elements in the orbit that are not equivalent by post-composition by an element in Sym_{2g}. ■

Theorem 1.6 follows from Corollary 5.3. The latter also implies that the branched double cover of Theorem 4.4 is not only unbranched, as stated in Corollary 4.6, but also disconnected.

6. Extension of the period map and bordification

6.1. Augmented Teichmüller space

As in [11], we need to extend the domain of definition of the period map in to appropriate bordifications of the space $\Omega^{\pm}\mathcal{S}_{g,\underline{2}}$. For the sake of completeness, we have included a general theory of the period map for moduli spaces of forms with simple poles and their extension to the augmented Torelli spaces in the appendix.

In this section, we give a brief outline and focus on some particular features of the case of two poles.

The moduli space of genus g smooth curves with n marked points $\mathcal{M}_{g,n}$ admits a Deligne–Mumford–Knudsen compactification $\overline{\mathcal{M}}_{g,n}$ as a complex orbifold (see [18, 29–31]). Each point in the boundary $\partial\mathcal{M}_{g,n} = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ corresponds to a class under conformal isomorphism of the following geometric object.

Definition 6.1. A connected complex curve C with n marked distinct points $q_1, \dots, q_n \in C$ is said to be stable if its singularities are nodes that do not coincide with any of the marked points, and the closure C_i of each component of $C^* := C \setminus \operatorname{Sing}(C)$, called a part of C, has a *finite* group of automorphisms that fix the marked points and the boundary points. The normalization of C is the (possibly disconnected) smooth curve $\widehat{C} = \bigsqcup C_i$ marked by the points $q_1, \dots, q_n$ and the pairs of points corresponding to the nodes of C. A stable curve C is said to be of compact type if every node separates C in two components. Otherwise, C is said to be of non-compact type.

The arithmetic genus of a stable curve is $g = h^1(C, \mathcal{O})$. When C has δ nodes and its normalization has ν components of genera $g_1, \dots, g_\nu$, the arithmetic genus satisfies (see [26, p. 48])

$$g = \sum_{i=1}^{\nu} (g_i - 1) + \delta + 1.$$

The boundary $\partial \mathcal{M}_{g,n}$ is a normal crossing divisor and, as such, is stratified. Its regular points correspond to the subset of curves with a single node.

There are natural attaching maps

$$\overline{\mathcal{M}}_{g_1, n_1 \cup *} \times \overline{\mathcal{M}}_{g_2, n_2 \cup \star} \to \overline{\mathcal{M}}_{n_1 + n_2} \tag{8}$$

and

$$\mathcal{M}_{g, n \cup \{*, \star\}} \to \overline{\mathcal{M}}_{g+1, n} \tag{9}$$

where the special points are identified into a new node. Their image is therefore in the boundary.

There are also forgetful maps (forget the last marked point with stabilization of the resulting curve, i.e., collapse non-stable components)

$$\overline{\mathcal{M}}_{g, n \cup *} \to \overline{\mathcal{M}}_{g,n}. \tag{10}$$

All of the said maps are holomorphic. By convention, we omit the zero subindex when $n = 0$ and write $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ instead.

Recall from the introduction that the universal cover of $\mathcal{M}_{g,n}$ can be identified with the Teichmüller space $\mathcal{T}_{g,n}$ and its covering group with the mapping class group $\operatorname{Mod}(\Sigma_{g,n})$ for a reference surface $\Sigma_{g,n} = (\Sigma_g, p_1, \dots, p_n)$ of genus g with n ordered marked points that we will sometimes denote as $P = (p_1, \dots, p_n)$. When $n = 0$, we omit the subindex and write $\Sigma_g = \Sigma_{g,0}$.

The Teichmüller space can be bordified to the so-called augmented Teichmüller space $\overline{\mathcal{T}}_{g,n}$, a stratified topological space that is no longer a manifold but where the action of $\operatorname{Mod}(\Sigma_{g,n})$ extends naturally with quotient $\overline{\mathcal{T}}_{g,n} / \operatorname{Mod}(\Sigma_{g,n})$ homeomorphic to $\overline{\mathcal{M}}_{g,n}$ (see [3]). The quotient map

$$\overline{\mathcal{T}}_{g,n} \to \overline{\mathcal{T}}_{g,n} / \operatorname{Mod}(\Sigma_{g,n})$$

is a branched cover, branching over the boundary points. Let us recall its definition and some properties.

Definition 6.2. A homotopical marking (or sometimes a collapse) of a connected genus g stable curve C with n ordered pairwise distinct ordered marked points $Q = (q_1, \dots, q_n) \in (C^*)^n$ is a continuous surjection $f : \Sigma_{g,n} \to (C, q_1, \dots, q_n)$ such that $f(p_i) = q_i$, the preimage of each node is a simple closed curve on $\Sigma_g \setminus P$ and on each component of $\Sigma_g \setminus f^{-1}(N)$ where N is the set of nodes, the map f is a homeomorphism onto a part of C that preserves the orientation.

Definition 6.3. A homotopically marked stable curve with n marked points is a marked stable curve $(C, q_1, \dots, q_n)$ together with a homotopical marking $f : (\Sigma_g, P) \to (C, Q)$. Two homotopically marked stable curves

$$f_i : (\Sigma_g, P) \to (C_i, Q_i) \quad \text{for } i = 1, 2$$

are said to be equivalent if there exists a conformal isomorphism $g : C_1 \to C_2$ such that $g \circ f_1$ is homotopic to f_2 relative to P. The class of a $\Sigma_{g,n}$ marked stable curve will be denoted by $\{f : \Sigma_{g,n} \to (C, q_1, \dots, q_n)\}$ or by $(C, Q, \{f\})$.

When C is non-singular, the homotopy class of a collapse corresponds to a unique isotopy class of homeomorphisms.

Remark 6.4. If $\Delta : \Sigma_{g,n} \to \Sigma_{g,n}$ is a Dehn twist around a simple closed curve in Σ_g that is collapsed by the marking $f : (\Sigma_g, P) \to (C, Q)$ to a point, then (C, Q) marked by $f \circ \Delta$ is equivalent to the same curve marked by f_1.

Definition 6.5. The augmented Teichmüller space $\overline{\mathcal{T}}_{g,n}$ is the set of all homotopically marked stable genus g curves with n marked points up to equivalence. It contains the Teichmüller space $\mathcal{T}_{g,n}$ of isotopy classes of marked *smooth* curves, and the boundary $\partial\mathcal{T}_{g,n} = \overline{\mathcal{T}}_{g,n} \setminus \mathcal{T}_{g,n}$.

The topology of Teichmüller space extends to a topology on augmented Teichmüller space in such a way that around a boundary point it is not locally compact (see [3] and references therein). There is a natural stratification of the boundary by complex manifolds. Each stratum is defined by the number of nodes of the curves and that number gives precisely the codimension of the manifold with respect to the dimension of the open stratum $\mathcal{T}_{g,n}$.

The local structure of the stratification is described in [3]. It is a local abelian ramified cover of a normal crossing divisor (we borrow the terminology from [11, Section 4]). This property allows us to mimic the description of the dual complex associated with a normal crossing divisor as follows: a vertex is associated with each connected component of the codimension one stratum. Each connected component of the stratum of codimension k lies in the closure of precisely k connected components of the codimension one stratum. We attach a k-simplex to k given vertices for each connected component of the intersection of the closures of the k corresponding components of strata. The resulting complex is denoted as $\mathcal{C}(\overline{\mathcal{T}}_{g,n})$. It is isomorphic to the curve complex $\mathcal{C}_{g,n}$ defined on a genus g closed surface with n marked points (see [11]).

6.2. Homology quotients

In this paper, we will need to work on some intermediate quotients of $\overline{\mathcal{T}}_{g,2}$ associated with particular subgroups of $\mathrm{Mod}(\Sigma_{g,2})$ characterized by their (trivial) action on certain homology groups with integral coefficients.

The pointed Torelli group $\mathcal{I}(\Sigma_{g,n^*})$ defined by the exact sequence

$$1 \to \mathcal{I}(\Sigma_{g,n^*}) \to \mathrm{Mod}(\Sigma_{g,n}) \to \mathrm{Aut}\big(H_1\big(\Sigma_g \setminus \{p_1, \dots, p_n\}\big)\big) \tag{11}$$

and the classical Torelli group represented in $\mathrm{Mod}(\Sigma_{g,n})$ by the exact sequence

$$1 \to \mathcal{I}(\Sigma_{g,\cancel{n}}) \to \mathrm{Mod}(\Sigma_{g,n}) \to \mathrm{Aut}\big(H_1(\Sigma_g)\big). \tag{12}$$

are related. Denote by Π_n the rank $n-1$ submodule generated by the n peripheral curves around the punctures in $H_1(\Sigma_g \setminus \{p_1, \dots, p_n\})$, and we have an exact sequence

$$0 \to \Pi_n \to H_1\big(\Sigma_g \setminus \{p_1, \dots, p_n\}\big) \to \frac{H_1\big(\Sigma_g \setminus \{q_1, \dots, q_n\}\big)}{\Pi_n} \cong H_1(\Sigma_g) \to 0. \tag{13}$$

The group $\mathrm{Mod}(\Sigma_{g,n})$ acts on (13) preserving it since it fixes every class in Π_n. In particular, $\mathcal{I}(\Sigma_{g,n^*}) \subset \mathcal{I}(\Sigma_{g,\cancel{n}})$ is a subgroup. We claim that there is an isomorphism

$$\frac{\mathcal{I}(\Sigma_{g,\cancel{n}})}{\mathcal{I}(\Sigma_{g,n^*})} \cong \mathrm{Hom}(H_1(\Sigma_g), \Pi_n).$$

Indeed, take $\phi \in \mathcal{I}(\Sigma_{g,\cancel{n}})$. The endomorphism defined by $\phi_* - \mathrm{Id}$ on $H_1(\Sigma_g \setminus \{p_1, \dots, p_n\})$ has an image in Π_n and Π_n in its kernel. Thanks to the isomorphism in (13), it defines an element in $\mathrm{Hom}(H_1(\Sigma_g), \Pi_n)$. The kernel of the group homomorphism $\phi \mapsto \phi_* - \mathrm{Id}$ is precisely $\mathcal{I}(\Sigma_{g,n^*})$.

The previous analysis implies that the quotient map

$$\frac{\mathcal{T}_{g,n}}{\mathcal{I}(\Sigma_{g,n^*})} \to \frac{\mathcal{T}_{g,n}}{\mathcal{I}(\Sigma_{g,\cancel{n}})} \tag{14}$$

has covering group $\mathrm{Hom}(H_1(\Sigma_g), \Pi_n)$.

Define

$$\overline{\mathcal{S}}_{g,n^*} = \overline{\mathcal{S}}(\Sigma_{g,n^*}) = \frac{\overline{\mathcal{T}}_{g,n}}{\mathcal{I}(\Sigma_{g,n^*})} \quad \text{and} \quad \overline{\mathcal{S}}_{g,\cancel{n}} = \frac{\overline{\mathcal{T}}_{g,n}}{\mathcal{I}(\Sigma_{g,\cancel{n}})}.$$

Then the quotient map (14) extends to a map

$$\overline{\mathcal{S}}_{g,n^*} \to \overline{\mathcal{S}}_{g,\cancel{n}} \tag{15}$$

that has branch points possibly at points with non-separating nodes. The stable curve depicted in Figure 2 is an example of branch point with $(g, n) = (4, 2)$.

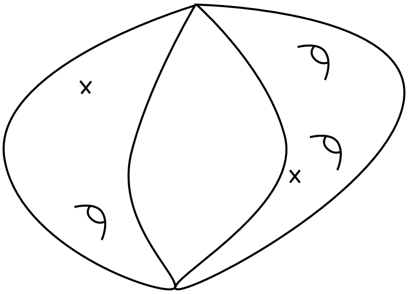

Figure 2. An example of branch point of the extension (15) with $(g, n) = (4, 2)$. The Dehn twist around the two curves of the curve system acts trivially on $\overline{\mathcal{S}}_{4,\not{2}}$ and the depicted boundary point of $\mathcal{S}_{4,2^*}$ but non-trivially in the neighboring points outside the boundary.

6.3. Homological characterization of strata of curves of compact type

The stratification by the number of nodes of $\overline{\mathcal{T}}_{g,n}$ is invariant under the action of the group $\mathcal{I}(\Sigma_{g,n^*})$ (resp. $\mathcal{I}(\Sigma_{g,\not{n}})$) and induces a stratification of $\overline{\mathcal{S}}_{g,n^*}$ (resp. $\overline{\mathcal{S}}_{g,\not{n}}$). Both stratifications are still locally abelian ramified covers of a normal crossing divisor. Therefore, we can define the dual boundary complex $\mathcal{C}(\overline{\mathcal{S}}_{g,n^*})$ (resp. $\mathcal{C}(\overline{\mathcal{S}}_{g,\not{n}})$) which is isomorphic to the quotient of the curve complex $\mathcal{C}_{g,n}/\mathcal{I}(\Sigma_{g,n^*})$ (resp. $\mathcal{C}_{g,n}/\mathcal{I}(\Sigma_{g,\not{n}})$).

To every class of point $(C, Q, \{f\}) \in \overline{\mathcal{S}}_{g,\not{n}}$ there corresponds a surjective homomorphism

$$m = [f]_* : H_1(\Sigma_g, \mathbb{Z}) \to H_1(C, \mathbb{Z}).$$

Lemma 6.6. *If $(C, Q, \{f\})$ and $(C, Q, \{f'\})$ satisfy*

- $[f]_* = [f']_*$
- $\operatorname{rank}(\ker[f]_*) \leq 1$,

then both points define the same class in $\overline{\mathcal{S}}_{g,\not{n}}$.

Proof. There exists $\phi \in \operatorname{Mod}(\Sigma_{g,n})$ such that $f' = f \circ \phi$. By hypothesis, $[f \circ \phi]_* = [f']_* = [f]_*$. The endomorphism of $H_1(\Sigma_g)$ defined by $\phi_* - \operatorname{Id}$ has its image in $\ker[f]_*$. If $\ker[f]_* = 0$, we are done.

If $\ker[f]_*$ has rank one, it is a primitive module, generated by the class $[c]$ of a non-separating simple closed curve c and

$$\phi - \operatorname{Id} = [c]\varphi \quad \text{where } \varphi \in H^1(\Sigma_g, \mathbb{Z}).$$

By duality, φ is the symplectic dual of some element of $\ker[f]_* \subset H_1(\Sigma_g)$; hence, there exist $k \in \mathbb{Z}$ such that

$$\phi(a) = a + k(a \cdot c)[c] \quad \forall a \in H_1(\Sigma_g).$$

This homomorphism is precisely the action in homology induced by the composition $\Delta_c^{\circ k}$ of k times the Dehn twist around the curve c. By Remark 6.4, $f'' = (f \circ \phi) \circ (\Delta_c^{-1})^{\circ k}$ is an equivalent topological marking to f' that falls in the initial case. ■

In particular, in $\overline{\mathcal{S}}_{g,\cancel{2}}$, every point over a curve of compact type C is characterized by a tuple (C, x_-, x_+, m) where $m : H_1(\Sigma_g, \mathbb{Z}) \to H_1(C, \mathbb{Z})$ is a surjective homomorphism. There is a natural direct sum decomposition into symplectic submodules

$$H_1(C, \mathbb{Z}) = H_1(C_1, \mathbb{Z}) \oplus \cdots \oplus H_1(C_k, \mathbb{Z})$$

where $C_1, \ldots, C_k$ are the parts of C, each of genus $g_i = \mathrm{genus}(C_i)$. This decomposition induces, via the marking m, a decomposition

$$H_1(\Sigma_g, \mathbb{Z}) = W_1 \oplus \cdots \oplus W_k \tag{16}$$

into pairwise orthogonal symplectic submodules of ranks $\mathrm{rank}(W_i) = 2g_i$.

As proven in [11], using the homological characterization of Lemma 6.6, the restriction of attaching maps as in equation (8) can be defined at the level of the coverings

$$\mathcal{S}_{g_1,\cancel{n_1 \cup *}} \times \mathcal{S}_{g_2,\cancel{n_2 \cup \star}} \to \overline{\mathcal{S}}_{g_1+g_2,\cancel{n_1+n_2}} \tag{17}$$

by the relation

$$(C_1, Q_1 \cup *, m_1) \times (C_2, Q_2 \cup \star, m_2) \mapsto (C_1 \vee_{*=\star} C_2, Q_1 \cup Q_2, m_1 \oplus m_2). \tag{18}$$

All points in the image have the same associated symplectic decomposition. Moreover, if $n_1 + n_2 = 2$, then the image is contained in $\overline{\mathcal{S}}_{g,\cancel{2}}$ and the range is a product of spaces $\mathcal{S}_{g,\cancel{n}}$ where $n = 0, 1$ or 2.

Remark 6.7. Using Lemma 6.6, we can extend the domain of the map (17) to pairs of curves with marked points such that at most one has one non-separating node.

6.4. Stable meromorphic forms with two simple poles

Definition 6.8. A stable meromorphic form with two simple poles and peripheral integrals ± 1 on a stable curve with two marked points $(C, x_-, x_+) \in \overline{\mathcal{M}}_{g,2}$ is a meromorphic form on each component of C^* having at worst simple poles at the points corresponding to the nodes, such that the sum of residues at the branches of C at any point $x \neq x_\pm$ is zero. Moreover, the peripheral integral is -1 at x_- and $+1$ at x_+. The space of all such forms on a given stable curve (C, x_-, x_+) is denoted by $\Omega^{\pm}(C, x_-, x_+)$. It is an affine space directed by the g-dimensional vector space $\Omega(C)$ of stable (holomorphic) forms on C (with zero residue sum at all points).

On any component of C^* the residue theorem holds for the restricted form, telling us that the sum of residues in the component is zero. To be able to integrate ω along a path in C, it needs to avoid all poles of the restrictions to the components of C^*.

In contrast with meromorphic forms on smooth curves, a stable meromorphic form can have a zero component, but not be zero globally. It can even have zero components and poles. On the complement of the nodes, zeros and poles, integration defines a translation structure. The underlying flat metric has the structure of a conical point angle $2\pi(1 + \mathrm{ord}_z(\omega))$ at a point z of positive order or of an infinite half cylinder $\mathbb{H}^{\pm}/\mathbb{Z}$ at a simple pole of peripheral integral ± 1. At a node, the metric is either a union of two conical points of possibly a different angle (if the residue of the node is zero) or the disjoint union of two infinite half cylinders.

Definition 6.9. $\Omega_0^{\pm}(C, x_-, x_+)$ is the subset of forms in $\Omega^{\pm}(C, x_-, x_+)$ with zero residues at non-separating nodes. $\Omega_0^{\pm *}(C, x_-, x_+)$ is the subset of forms in $\Omega_0^{\pm}(C, x_-, x_+)$ with isolated zeros.

If $\omega \in \Omega_0^{\pm}(C, x_-, x_+)$ has some separating node, then its peripheral integral is, up to sign, zero or one depending on whether x_-, x_+ lie on the same side of the node or not, i.e., depending on whether the node also separates the poles or not.

Remark that the form ω can be integrated along any path avoiding the poles of the restrictions to components of C^*, hence, any path avoiding x_-, x_+ and the nodes that separate the poles x_-, x_+. Remark also that, thanks to the Mayer–Vietoris sequence, any homology class in $H_1(C \setminus \{x_-, x_+\}, \mathbb{Z})$ is represented by some path avoiding points with non-zero residue of ω (see the appendix for details).

The rank g affine bundle

$$\Omega^{\pm}\overline{\mathcal{M}}_{g,2} \to \overline{\mathcal{M}}_{g,2} \tag{19}$$

is the bundle that has $\Omega^{\pm}(C, x_-, x_+)$ as fiber over the point (C, x_-, x_+). It is a natural holomorphic extension of the affine bundle $\Omega^{\pm}\mathcal{M}_{g,2} \to \mathcal{M}_{g,2}$, directed by the Hodge vector bundle $\Omega\overline{\mathcal{M}}_{g,2} \to \overline{\mathcal{M}}_{g,2}$ of stable (holomorphic) forms. An element in $\Omega^{\pm}\overline{\mathcal{M}}_{g,2}$ will be denoted as a tuple (C, x_-, x_+, ω) or, since the information of the poles is already in ω, by (C, ω). In [6, 7], the reader can find more details on the structure of bundles of meromorphic one forms over the Deligne–Mumford compactification and on augmented Teichmüller space. We are going to describe the relevant properties for our interest in the covers where the period map is naturally defined.

The fiber bundle (19) can be pulled back via branched covers to topological bundles

$$\Omega^{\pm}\overline{\mathcal{S}}_{g,2^*} \to \overline{\mathcal{S}}_{g,2^*} \quad \text{and} \quad \Omega^{\pm}\overline{\mathcal{S}}_{g,\cancel{2}} \to \overline{\mathcal{S}}_{g,\cancel{2}}.$$

Elements in those covers will be denoted by $(C, [f], \omega)$ where $[f]$ denotes an equivalence class of topological marking under the covering group that defines the space.

Given a subset $\mathcal{K}$ of $\overline{\mathcal{M}}_{g,2}$ or any of its covers, we denote by $\Omega^{\pm}\mathcal{K}$ the restriction of the $\Omega^{\pm}$-bundle to the given subset. Its intersection with the set of forms having zero residues at all non-separating nodes will be denoted by $\Omega_0^{\pm}\mathcal{K}$, and the intersection of the latter with the subset of forms isolated zeros by $\Omega_0^{\pm,*}\mathcal{K}$.

On $\Omega_0^{\pm}\overline{\mathcal{S}}_{g,2^*}$ there is a well-defined period map (see the appendix for details)

$$\mathcal{P}\mathrm{er}_{g,2^*} : \Omega_0^{\pm}\overline{\mathcal{S}}_{g,2^*} \to \mathrm{Hom}\left(H_1(\Sigma_g \setminus \{p_-, p_+\}, \mathbb{Z}); \mathbb{C}\right)$$

that is equivariant with respect to the action of $\mathrm{Mod}(\Sigma_{g,2})$ on source and target. It is defined by associating with any class in the homology group, and a fixed marked form in $\Omega_0^{\pm}\overline{\mathcal{S}}_{g,2^*}$, the integral of the form along the corresponding class in the stable curve (up to an appropriate choice of representative to allow for integration). The correspondence of classes is induced by the marking in homology. By construction, the peripheral module Π_2 has rank one and its generator is sent to $\pm 1 \in \mathbb{C}$.

Similarly, we can define the period of an element in $\Omega_0^{\pm}\overline{\mathcal{S}}_{g,\underset{\sim}{2}}$ as follows.

Definition 6.10. Given $(C, m, \omega) \in \Omega_0^{\pm}\overline{\mathcal{S}}_{g,\underset{\sim}{2}}$ having zero residues at non-separating nodes, there is a well-defined homomorphism

$$\mathrm{Per}(C, m, \omega) : H_1(\Sigma_g, \mathbb{Z}) \to \mathbb{C}/\mathbb{Z} \quad \text{by } \gamma \mapsto \int_{m(\gamma)} \omega \bmod 1.$$

Definition 6.11. The period map on $\Omega_0^{\pm}\overline{\mathcal{S}}_{g,\underset{\sim}{2}}$ is the map

$$\mathcal{P}\mathrm{er}_g : \Omega_0^{\pm}\overline{\mathcal{S}}_{g,\underset{\sim}{2}} \to H^1(\Sigma_g, \mathbb{C}/\mathbb{Z})$$

sending each point to its period homomorphism.

It extends the map Per_g defined by Proposition 3.4 to a part of the boundary of $\Omega^{\pm}\overline{\mathcal{S}}_{g,\underset{\sim}{2}}$ where it makes sense to talk about "having the same period homomorphism" as a form in $\Omega^{\pm}\mathcal{S}_{g,\underset{\sim}{2}}$.

Proposition 6.12. *The branched cover $\Omega^{\pm}\overline{\mathcal{S}}_{g,2^*} \to \Omega^{\pm}\overline{\mathcal{S}}_{g,\underset{\sim}{2}}$ sends period fibers to period fibers. Its restriction to a fiber of $\mathcal{P}\mathrm{er}_{g,2^*}$ is a homeomorphism onto a fiber of $\mathcal{P}\mathrm{er}_g$.*

In particular, the branch points of $\Omega^{\pm}\overline{\mathcal{S}}_{g,2^*} \to \Omega^{\pm}\overline{\mathcal{S}}_{g,\underset{\sim}{2}}$ do not occur at points with zero residues at the non-separating nodes (compare with Figure 2).

Proof. The first part of the statement is a consequence of the isomorphism in (13) and the fact that the condition on the residues implies that the peripheral module

$$\Pi_2 \subset H_1\big(\Sigma_g \setminus \{p_-, p_+\}\big)$$

has periods in $\mathbb{Z}$. Remark also that, by (14), the monodromy of the cover on the unbranched part is $\mathrm{Hom}(H_1(\Sigma_g), \Pi_2)$ and the branch points correspond to some fixed point of this action. However, the action of the non-trivial elements of the covering group does not leave any fiber of the period map invariant. Indeed, the action of $h \in \mathrm{Hom}(H_1(\Sigma_g), \Pi_2)$ on a homomorphism $p : H_1(\Sigma_g, \setminus \{p_-, p_+\}) \to \mathbb{C}$ is the homomorphism $p \circ (\mathrm{Id} + h \circ \pi)$ where

$$\pi : H_1\big(\Sigma_g \setminus \{p_-, p_+\}\big) \to \frac{H_1\big(\Sigma_g \setminus \{p_-, p_+\}\big)}{\Pi_2} \cong H_1(\Sigma_g).$$

Suppose p is the period of some element in $\Omega^{\pm}\overline{\mathcal{S}}_{g,2^*}$. In particular, $p_{|\Pi_2} : \Pi_2 \to \mathbb{Z}$ is injective. Suppose p is fixed by the action of h. Then $p + p \circ h \circ \pi = p$. Since the image of h lies in Π_2, we deduce $h \circ \pi = 0$. But since π is surjective, this implies $h = 0$.

Let F be a non-empty fiber of $\mathcal{P}\mathrm{er}_g$ and $\widehat{F}$ its preimage under the branched map, a union of fibers. The restriction of the branched cover to $\widehat{F} \to F$ has no branch points and trivial monodromy. This concludes the proof. ■

6.5. The stratification of augmented Teichmüller space and period fibers

The stratification of the ambient space $\Omega^{\pm}\overline{\mathcal{S}}_{g,2^*}$ induces a partition of the fibers of the period map. The next theorem describes this partition.

Theorem 6.13. *The local fiber L of the period map in $\Omega^{\pm}\overline{\mathcal{S}}_{g,2^*}$ (resp. $\Omega^{\pm}\overline{\mathcal{S}}_{g,\underline{2}}$) at a point with isolated zeros projects to the orbifold chart of $\Omega\overline{\mathcal{M}}_{g,2}$ as a complex manifold transverse to all boundary divisors through the point. Therefore, L is an abelian ramified cover of a normal crossing divisor in $(\mathbb{C}^{2g-2}, 0)$ having precisely one component of codimension one in each component of codimension one of the ambient space through the point.*

The statement is not true in general for forms with zero components.

Proof. The proof in $\Omega^{\pm}\overline{\mathcal{S}}_{g,2^*}$ (and more generally for $\Omega_0\overline{\mathcal{S}}_{g,n^*}$ with $n \geq 2$, i.e., for forms with at least two simple poles whose non-separating nodes have zero residue) follows exactly the same lines of the proof of the equivalent theorem for abelian differentials in [11, Section 4.18]. For completeness and future reference, we include a proof in the appendix.

As for the case of $\Omega^{\pm}\overline{\mathcal{S}}_{g,\underline{2}}$, the restriction of the homeomorphism of Proposition 6.12 to the neighborhood of a point provides a homeomorphism that preserves boundary points. Hence, the result. ∎

Corollary 6.14. *Let $F \subset \Omega^{\pm}\mathcal{S}_{g,\underline{2}}$ be a fiber of* Per_g *and $\overline{F}$ its closure in $\Omega_0^{\pm *}\overline{\mathcal{S}}_{g,\underline{2}}$. Then F is connected if and only if $\overline{F}$ is connected.*

Proof. The homeomorphism of Proposition 6.12 sends boundary points to boundary points, so it suffices to prove the equivalent statement for a fiber of the period map on the cover $\Omega^{\pm}\mathcal{S}_{g,2^*}$ and its closure in the space $\Omega_0^{\pm *}\overline{\mathcal{S}}_{g,2^*}$ of homologically marked meromorphic stable forms with zero residues at non-separating nodes and isolated zeros. This equivalent statement is proven in Corollary 12.12 of the appendix. ∎

Definition 6.15. The closure of the fiber of Per_g in $\Omega_0^{\pm *}\overline{\mathcal{S}}_{g,\underline{2}}$ is what we consider as bordification of the fiber. It coincides with the fiber of $\mathcal{P}\mathrm{er}_g$ on $\Omega_0^{\pm *}\overline{\mathcal{S}}_{g,\underline{2}}$.

6.6. Smoothings, simple boundary points and the proof of Theorem 1.4

Definition 6.16. A path in $\Omega_0^{\pm}\overline{\mathcal{S}}_{g,\underline{2}}$ is said to be isoperiodic if the value of $\mathcal{P}\mathrm{er}_g$ is constant along it (i.e., it lies in a fiber of $\mathcal{P}\mathrm{er}_g$). To simplify the notations, two marked forms ω_1 and ω_2 in $\Omega_0^{\pm *}\overline{\mathcal{S}}_{g,\underline{2}}$ are equivalent and denoted as $\omega_1 \sim \omega_2$ if they can be joined by an isoperiodic path in the total space.

Recall that every node of a stable curve C determines a boundary component of $\overline{\mathcal{S}}_{g,\underline{2}}$ passing through C.

Definition 6.17. A smoothing of a node of a stable form is an equivalent form that does not belong to the boundary component corresponding to the node but that belongs to the other boundary components passing through the point.

The metric description of the neighborhood of the node when smoothing it is a flat cylinder of finite area. If the residue at the node is non-zero, when we get closer to the form ω, the circumference of the annulus stays constant but its volume tends to grow, so the cylinder becomes long.

Definition 6.18. A smoothing of a boundary point ω with isolated zeros is an equivalent form on a non-singular curve. They do always exist thanks to Theorem 6.13.

Remark that the dual graph of a curve of compact type supporting a meromorphic form with two simple poles, each of whose nodes separate the two poles, is a segment. We denote such a form as an *ordered* sequence $\omega_1 \vee \cdots \vee \omega_k$ where ω_i is a meromorphic form with two poles on a smooth curve C_i, ω_1 (resp. ω_k) has a pole of peripheral integral -1 (resp. $+1$) and the pole of peripheral integral $+1$ of ω_j is glued to the pole of peripheral integral -1 of ω_{j+1} to form a node. Each ω_i is called a part of the form and has isolated zeros.

The maps (17) extend at the level of forms

$$\Omega^{\pm}\mathcal{S}_{g_1,\cancel{n_1 \cup *}} \times \Omega^{\pm}\mathcal{S}_{g_2,\cancel{n_2 \cup \star}} \to \Omega_0^{\pm *}\overline{\mathcal{S}}_{g_1+g_2,\cancel{n_1+n_2}} \tag{20}$$

by the relation

$$\begin{aligned}(C_1, Q_1 \cup *, m_1, \omega_1) &\times (C_2, Q_2 \cup \star, m_2, \omega_2) \\ &\mapsto (C_1 \vee_{*=\star} C_2, Q_1 \cup Q_2, m_1 \oplus m_2, \omega_1 \vee \omega_2)\end{aligned} \tag{21}$$

by imposing that the two marked points where we glue are of opposite residue. In particular, if one of the poles is chosen as a marked point, the other is completely determined.

Remark 6.19. The map (20) is continuous and sends products of period fibers to period fibers. In particular, it sends products of isoperiodic paths to an isoperiodic path. We use the following shorthand notation:

$$\omega_1 \sim \omega_1',\ \omega_2 \sim \omega_2' \implies \omega_1 \vee \omega_2 \sim \omega_1' \vee \omega_2'. \tag{22}$$

A generalization of the following remark motivates the inductive proof of Theorems 1.4 and 1.5.

Remark 6.20. Given a holomorphic form ω on an elliptic curve E, the form $\omega \vee \frac{dz}{2i\pi z}$ on the stable curve $E \vee \mathbb{P}^1$ defines a boundary point in $\Omega_0^{\pm *}\overline{\mathcal{S}}_{1,\cancel{2}}$, and the same periods in $\mathbb{C}/\mathbb{Z}$ as ω. The smoothing of the latter defines an isoperiodic form with two poles on a smooth elliptic curve.

Definition 6.21. A boundary point of $\Omega_0^{\pm *}\overline{\mathcal{S}}_{g,\cancel{2}}$ is said to be simple if the underlying nodal curve is of compact type, each node separates the poles and the degree of each part is at least three.

Simple boundary points do not exist for fibers corresponding to periods of degree two. Let us analyze the existence of simple boundary points for other fibers. Recall the following definition.

Definition 6.22. Let $g \geq 2$ and $p : H_1(\Sigma_g, \mathbb{Z}) \to A$ be a non-trivial homomorphism of abelian groups. A symplectic decomposition $H_1(\Sigma_g, \mathbb{Z}) = V_1 \oplus \cdots \oplus V_k$ is said to be p-admissible if none of the restrictions $p_{|V_i}$ is trivial.

A simple boundary point $(C_1 \vee \cdots \vee C_k, m, \omega_1 \vee \cdots \vee \omega_k)$ induces a $[p]$-admissible *ordered* decomposition of the homology group $H_1(\Sigma_g, \mathbb{Z})$

$$V_1 \oplus \cdots \oplus V_k \quad \text{where } V_i = m^{-1}\big(H_1(C_i, \mathbb{Z})\big)$$

where $[p]$ is the class modulo $\frac{1}{2}\mathbb{Z}$ of the period homomorphism

$$p = \mathcal{P}\mathrm{er}_g(C_1 \vee \cdots \vee C_k, m, \omega_1 \vee \cdots \vee \omega_k) : H_1(\Sigma_g, \mathbb{Z}) \to \mathbb{C}/\mathbb{Z}.$$

The order of the factors is determined by the order of the forms, bearing in mind that the first one (resp. last one) has a pole of -1-peripheral integral (resp. $+1$-peripheral integral).

Lemma 6.23. *For $g \geq 2$ and $p : H_1(\Sigma_g, \mathbb{Z}) \to \mathbb{C}/\mathbb{Z}$ non-trivial, there exists a p-admissible decomposition. If* $\deg p \geq 3$*, there exists a p-admissible decomposition whose factors have also degree at least three.*

Any such decomposition is induced by some stable meromorphic form with two poles on a curve of compact type whose nodes separate the two poles.

Proof. The algebraic part is proven in Corollary 3.3. By inductively applying the algebraic result, we can find a sub-decomposition $H_1(\Sigma_g, \mathbb{Z}) = V_1 \oplus \cdots \oplus V_k$ of any given decomposition into symplectic submodules of rank two such that $p_{|V_i}$ is non-trivial. Lemma 4.1 implies that there is either a form with two simple poles or a holomorphic form ω_i realizing $p_{|V_i}$. By Remark 6.20, we can suppose that ω_i has two simple poles. By considering forms on smooth genus one curves E_i with two poles ω_i and periods p_i, we can construct the boundary form of compact type $\omega_1 \vee \cdots \vee \omega_k$ of period p by connecting the negative pole of one with the positive of the next to a stable meromorphic form. Smoothing the relevant nodes, we can realize any decomposition obtained by joining factors of the one induced by $V_i's$. ■

Proof of Theorem 1.4. For $g = 1$, we use Remark 6.20. For $g \geq 2$, we realize p as the period of a form on a marked stable curve of compact type by using Lemma 6.23. By smoothening its nodes, we obtain the desired form on a smooth curve. ■

7. Degenerating to a simple boundary point

A Schiffer variation (see [11] and references therein for details) allows us to define isoperiodic paths in $\Omega^{\pm}\bar{\mathcal{M}}_{g,n}$ (and its cover $\Omega_0^{\pm *}\mathcal{S}_{g,\underline{2}}$) by deforming any marked stable meromorphic form having an isolated zero continuously in its fiber of the period map. It is defined by a continuous surgery for any given family of twin paths, i.e., paths $\gamma_1, \gamma_2, \ldots, \gamma_k$ with parameter space $[0, 1)$ starting at an isolated zero (nodes with zero residue are allowed) of a meromorphic form (C, ω) such that the k paths in $\mathbb{C}$ defined by integration

$$t \mapsto \int_{\gamma_{i|[0,t]}} \omega \quad \text{coincide for } i = 1, \dots, k.$$

It actually deforms the flat structure underlying the form. For more details and examples of deformations that include nodal curves, we refer to [11]. If the form is marked, the surgery allows us to follow the marking along the surgery. In this section, we use Schiffer variations to connect any meromorphic form to some form on a nodal curve.

Proposition 7.1. *Any meromorphic form on a smooth genus $g \geq 2$ Riemann surface with two simple poles of peripheral integrals ± 1 can be deformed via a sequence of Schiffer variations to a stable meromorphic form on a singular stable curve with one node and isolated zeros. If the periods are real, we can further suppose that the node is separating and separates the poles. Otherwise, we can suppose that the node has zero residue.*

7.1. Proof of Proposition 7.1 in the case of real periods

The integral of the imaginary part of a meromorphic form ω with real periods and two poles provides a harmonic surjective first integral $h_\omega : X \to \mathbb{R}$ for the horizontal foliation. We claim that up to a Schiffer variation we can suppose that h_ω has a regular connected fiber (that is necessarily a closed leaf) that separates X in two surfaces of positive genera. The cylinder it generates has a height. By moving all zeros in each side of the leaf, in the imaginary direction, toward the pole in the corresponding side, at the same speed, we have that the height of the cylinder of closed leaves generated by the chosen leaf is growing (as depicted in Figure 3). In the limit we get a stable form on a singular nodal curve with a separating node none of whose components is the zero form.

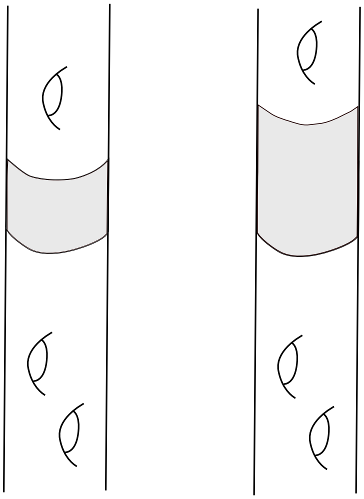

Figure 3. Isoperiodic stretching of a cylinder by appropriate Schiffer variations.

To get the desired closed leaf, first apply some Schiffer variations in the imaginary direction to guarantee that all zeros of ω are simple and have different values $-\infty < x_1 < x_2 < \cdots < x_{2g} < \infty$ of the first integral h_ω. The fiber $h_\omega^{-1}(x)$ over a point $x \neq x_i$ is a disjoint union of circles. The number of circles in each fiber is constant in each interval of $\bigsqcup_{i=0}^{2g} I_i = \mathbb{R} \setminus \{x_1, \dots, x_{2g}\}$ and generates cylinders of closed leaves. In each unbounded interval it corresponds to one of the cylinders defined by a pole. Since all zeros are simple, the number of cylinders changes at each x_i by a unit: on one side, there is one cylinder approaching, on the other, two. Hence, over I_1 and I_{2g-1}, there are two connected components of the fiber.

The graph of horizontal cylinders of ω is the trivalent connected graph having $2g$ vertices (one for each zero) and we put an edge between two zeros if there is a cylinder of closed leaves that has both zeros in the boundary (see Figure 4 for an example).

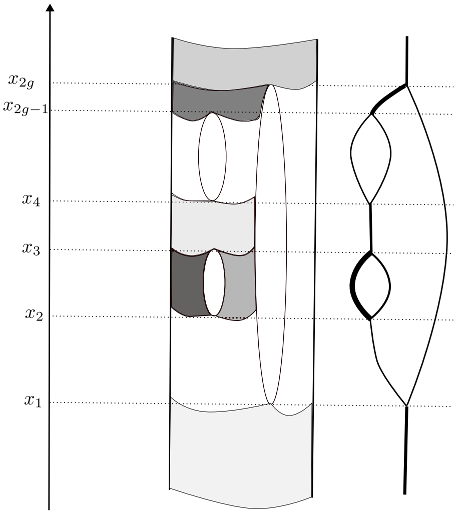

Figure 4. An example of the graph of cylinders.

If there is an intermediate segment $I_2, \dots, I_{2g-2}$ for which the fiber $h_\omega^{-1}(I_j)$ is one cylinder, we will be done. We will analyze how the graph of horizontal cylinders changes along a movement of zeros, via Schiffer variations, and verify that up to this further surgery we fall in the desired situation.

Let us analyze the change of graph we can obtain by applying a particular Schiffer variation at a zero z_0 that has two components going upward. We denote by v_0 the vertex of the cylinder graph corresponding to z_0. Among the two edges e_0, e_0' that go from v_0 upward, one has the smallest length, say e_0. Its other extremity is denoted as v_1. Let e_1 be an edge going upward at v_1. There exists a pair of twins at z_0 whose projections to the cylinder graph are

(1) the concatenation of e_0 with a small interval in e_1,

(2) a portion of e_0.

Indeed, first choose a geodesic starting at z_0 whose projection to the cylinder graph satisfies (1), and then take its twin. The Schiffer variation along this pair of twin produces a new meromorphic differential whose cylinder graph is depicted in Figure 5.

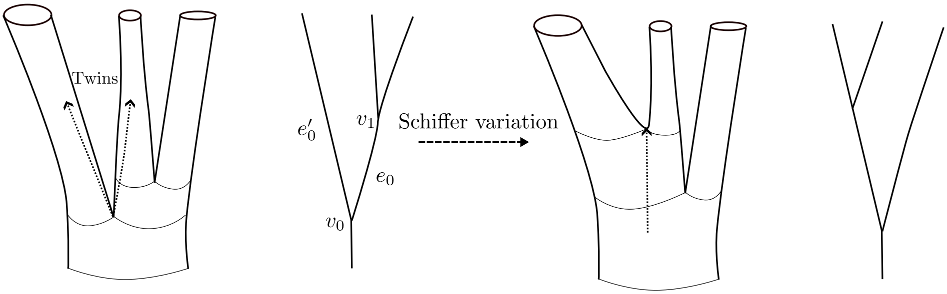

Figure 5. Graph changes when a zero moves upward.

Choose two paths in the graph, each leaving the top vertex of the cylinder graph on different edges, and choose any way of going down the graph with each of them. At the first moment they meet (they always do since the graph ends at a unique vertex), we find a point y_0 that has two edges upward. Let v be the number of vertices that we find in the union of the paths. If $v = 2$, we are done since there is a cylinder just under y_0. Use the choice of paths to move y_0 up the graph. As explained in Figure 5, we can move the point y_0 upward along the given choice of paths. At each step, we transform the cylinder graph and reduce the number v by one. Proceeding inductively, we reach the situation where this number is two.

7.2. Degeneration in the case of non-real periods

The proof works in a more general case. Suppose (C, m, ω) is a meromorphic form with simple poles on a smooth connected curve, whose residues at the poles lie in a real line. Suppose further that there is some period not lying in the same real line. In particular, this is the case for the case of two simple poles of peripheral integrals ± 1 and non-real periods.

We claim that by Schiffer variations we can either find a pair of distinct non-closed twin paths with common end-point (at a zero) or join all zeros to a point. By changing the point locally in the period fiber using period coordinates in the stratum (see [9, Remark 3.8]), we can suppose that the coordinates corresponding to saddle connections between distinct zeros of ω have different values in $\mathbb{C}$, and, moreover, none coincides with the period of a cycle, i.e., lies in the (countable) image of the period map $\mathrm{Per}(C, m, \omega)$ in $\mathbb{C}$. Consider one of the shortest saddle connections, and the set of all its twins at its starting point. By the choices made so far, the only possibility for endpoints of any of those paths is a regular point, or the endpoint of the original saddle connection. If one of the twin paths has the same endpoint, we have found a pair of twins whose endpoints coincide. Otherwise, the Schiffer variation along the set of all twins at the point produces a meromorphic differential with one zero less (of higher order). Continuing the process with the obtained form inductively, we end up in one of the desired situations. If we have found the couple of twins between distinct zeros, the Schiffer variation will degenerate the surface to a node with zero residue and no zero components and we are done.

Next we claim that if the form has a single zero, we can also find, up to some Schiffer variations, a pair of twins having the same endpoints (but this time the starting point and endpoints might coincide as well). Indeed, consider the horizontal foliation induced by ω on C. Around each pole, it has an open cylinder of infinite volume formed by closed regular geodesics. The complement $\mathcal{U} \subset C$ of the cylinders in C has finite volume. It cannot be empty since the form ω has some non-real period. The boundary is formed by a collection of saddle connections at the unique zero of ω. According to [37, Proposition 5.5], in each connected component of $\mathcal{U} \setminus \{\text{saddle connections}\}$, the real foliation is either a finite volume cylinder formed by closed regular geodesics, or minimal. We claim that, up to changing the direction of the foliation, we can find a cylinder by closed regular geodesics. Indeed, if there are none for the real foliation, we can apply [37, Lemma 5.12] to one of the minimal components and find a finite volume cylinder in $\mathcal{U}$ formed by closed regular geodesics (necessarily in a different direction).

The boundary curves b^+ and b^- of a finite volume cylinder by closed geodesics have the same length. Each defines a finite sequence of saddle connections based at the zero of ω. In this situation, up to some Schiffer variations, we can always find a pair of twin curves having the same endpoints. It suffices to carry exactly the same proof by cases of [11, Section 5.3]. As in that case, the Schiffer variation along this pair of twins degenerates the surface to an isoperiodic stable form with no zero components and a node of zero residue.

7.3. From boundary points to simple boundary points

Proposition 7.2. *Let $g \geq 2$. Then any form in $\Omega_0^{\pm *}\overline{\mathcal{S}}_{g,2}$ with period homomorphism of degree at least three is equivalent to a simple boundary point.*

Proof. By induction on the genus. Let $p : H_1(\Sigma_2, \mathbb{Z}) \to \mathbb{C}/\mathbb{Z}$ with $\deg(p) \geq 3$. Since $\mathrm{Per}_2^{-1}(p)$ is connected by Corollary 4.6, it is enough to prove that there exists a simple boundary point in the bordification.

By Lemma 6.23, there exists a decomposition $H_1(\Sigma_2, \mathbb{Z}) = W_1 \oplus W_2$ into rank two symplectic submodules such that each $p(W_i)$ has at least three elements. Up to identifying W_i with $H_1(\Sigma_1, \mathbb{Z})$, $p_{|W_i}$ can be thought of as an element in $H^1(\Sigma_1, \mathbb{C}/\mathbb{Z})$ of degree at least three. By Theorem 1.4, we can consider a form ω_i on a smooth curve with two poles and periods $p_{|W_i}$. Hence, $\omega_1 \vee \omega_2$ is a simple boundary point of the fiber $\mathrm{Per}_2^{-1}(p)$.

Fix some $g \geq 3$ and suppose that the statement of the proposition is true up to genus $g - 1$. Consider $p : H_1(\Sigma_g, \mathbb{Z}) \to \mathbb{C}/\mathbb{Z}$ with $\deg(p) \geq 3$.

Along this proof, and by abuse of notation, we will denote the stable form obtained by gluing two meromorphic forms ω_1,ω_2 on smooth curves C_1, C_2 glued at points that are poles or not by $\omega_1 \vee \omega_2$ and omit the information of the points where they are glued. A smoothing of such a form will be denoted by $\omega_1 \tilde{\vee} \omega_2$.

Case 1. Suppose $\mathrm{Im}(p) \subset \mathbb{R}$.

Up to applying Proposition 7.1, we can suppose that we start at a form $\omega = \omega_1 \vee \omega_2$ with precisely one node that separates the surface and has one pole on each side. The decomposition of the form provides a decomposition $p = p_1 \oplus p_2$. Suppose that one of p_i has degree two. Since $\deg p \geq 3$, we cannot have both of degree two. If the component which is not of degree two, say ω_1, is of genus at least two, we can use the hypothesis to join ω_1 to a nodal stable form $\omega_3 \vee \omega_4$ with non-zero residue at the node, and each part of which has periods of degree at least three. The form $\omega_2 \vee \omega_3$ has periods of degree at least three, and so does ω_4. Then $\omega \sim (\omega_2 \mathbin{\tilde{\vee}} \omega_3) \vee \omega_4$.

If the form ω_1 (of degree at least three) has genus one, then ω_2 has genus $g - 1$ and degree two. We can still find an equivalent $\omega_3 \vee \omega_4$ each part of which will have degree two. However, the sum $\omega_1 \vee \omega_3$ will have degree at least three and genus at least two. The previous case can be applied to the form $(\omega_1 \mathbin{\tilde{\vee}} \omega_3) \vee \omega_4$, which is equivalent to ω.

Case 2. Suppose p has a non-real number in the image.

Case 2.1. From a non-simple boundary point to either a simple boundary point or a form on a curve with a non-separating node with zero residue.

Up to applying Proposition 7.1, we can suppose that we start at a form ω with precisely one node with zero residue. If it is non-separating, we are done. Suppose the node is separating and write $\omega = \omega_1 \vee \omega_2$. Let g_i be the genus of ω_i. One of the parts, say ω_2, has no poles and the other has two poles. Therefore, ω_2 has infinite degree.

If $g_1 \geq 2$ and ω_1 has degree at least three, then we can connect ω_1 to a simple boundary point $\omega_3 \vee \omega_4$ in genus g_1. The form ω is therefore connected to $\omega_3 \vee (\omega_4 \mathbin{\tilde{\vee}} \omega_2)$ which has a separating node with non-zero residue and degree at least three on each part. It corresponds to a simple boundary point.

If $g_1 = 1$, then $g_2 \geq 2$ and we can connect ω_2 to a stable holomorphic form on a nodal curve by an isoperiodic path (see section degeneration in [10]). If the node is non-separating, we are in the second possibility of Case 2.1. Otherwise, we can write this nodal curve as $\omega_3 \vee \omega_4$ both holomorphic with non-zero components. The form ω is connected to $(\omega_1 \mathbin{\tilde{\vee}} \omega_3) \vee \omega_4$ which falls in the previous case.

Next we suppose that ω_1 has a period map of degree two. Up to using Lemma 7.1 repeatedly to ω_1, we can further suppose $g_1 = 1$ and therefore $g_2 \geq 2$. Since the periods p are not all real, ω_2 has non-real periods and has infinite degree. If it has no poles, we fall in a previous case. If it has poles, we can use the hypothesis to connect ω_2 to $\omega_3 \vee \omega_4$ with both parts of degree at least three and two poles. The form $(\omega_1 \mathbin{\tilde{\vee}} \omega_3) \vee \omega_4$ is a simple boundary point that can be connected to ω by an isoperiodic path.

Case 2.2. From a form with zero residue on a non-separating node to a simple boundary point.

Given a form without zero components ω with some non-separating nodes of zero residue, we denote by $\tilde{\omega}$ some form obtained by smoothing the non-separating nodes of ω.

Let (C, m, ω) be a form on a curve with a single non-separating node of zero residue. We claim first that ω is equivalent to a stable form $\omega_1 \vee \omega_2$ where ω_1 is a meromorphic stable form with two poles on a singular curve of genus one (containing the non-separating node of zero residue) and ω_2 is a meromorphic form on a smooth curve with two poles. This can be achieved if we find a smooth closed geodesic for ω that separates the surface C in two parts with the said properties. We claim that up to deforming ω by an isoperiodic path in its stratum, we can find such a geodesic. Indeed, consider a simple closed curve γ on C with base-point at the node that defines a non-trivial homology class in $H_1(C, \mathbb{Z})$. Let γ_0 denote the path with distinct endpoints obtained in the normalization (C_0, m_0, ω_0) of (C, m, ω). Any continuous deformation γ_t of γ_0 in the space of paths in C_0 with distinct endpoints such that $t \mapsto \int_{\gamma_t} \omega_0$ is constant allows us to define an isoperiodic path starting at (C, m, ω). It suffices, for each t, to glue the endpoints of the path γ_t to obtain a curve with a single non-separating node of zero residue (C_t, ω_t) and follow the same rule that marked C_0 from C to mark the homology C_t from C_0. By construction, geodesics of ω_0 are sent to geodesics of ω_t for each t. Therefore, if we manage to find such a deformation γ_t for which both endpoints lie in the cylinder around one of the poles of ω_0, we will be able to find the closed geodesic as one of the geodesics of the cylinder leaving both endpoints at the same side of the pole. Consider the oriented directional foliation in C_0 defined by ω_0 on a generic direction in $\mathbb{C}$ that is not tangent to the real line and such that the endpoints q_1, q_2 of γ_0 do not belong to saddle connections (see [36, Section 11.4]). All geodesics in that direction that are not saddle connections converge to the same pole under the geodesic flow G_t of the chosen direction. Let t_1 be a time such that $G_{t_1}(q_i)$ belongs to the interior of the infinite cylinder around that pole for $i = 1, 2$. Then for each $t \in [0, t_1]$, the path γ_t obtained by joining $G_t(q_1)$ with q_1 along the geodesic, γ_0 and then q_2 with $G_t(q_2)$ along the geodesic, satisfies $\int_{\gamma_t} \omega_0 = \int_{\gamma_0} \omega_0$ for all $t \in [0, t_1]$ and γ_t has distinct endpoints.

If ω_2 has a period map of degree at least three, we can write $\omega_2 \sim \omega_3 \vee \omega_4$ with each part of degree at least three and with non-zero residues. The form ω is equivalent to the form $(\widetilde{\omega_1} \tilde{\vee} \omega_3) \vee \omega_4$ which determines a simple boundary point.

If ω_2 has degree two, then ω_1 cannot have degree two (otherwise, ω would). Write $\omega_2 \sim \omega_3 \vee \omega_4$ where now ω_3 and ω_4 have degree two. The form $\omega_1 \vee \omega_3$ has periods of degree at least three over a curve of genus at least two. By inductive hypothesis, it is equivalent to a form $\omega_5 \vee \omega_6$ where each part has degree at least three. The initial form equivalent to $\omega_5 \vee (\omega_6 \tilde{\vee} \omega_4)$ has parts of degree at least three and non-zero residue. Hence, it determines a simple boundary point. ∎

8. Sufficient conditions for the connectedness of the set of simple boundary points of a period fiber

8.1. Equivalence of simple boundary points inducing the same factors

Proposition 8.1. *Let $g \geq 3$ and $p \in H^1(\Sigma_g, \mathbb{C}/\mathbb{Z})$ of degree at least three. Suppose Theorem 1.5 is true up to genus $g - 1$. Then any pair of p-simple boundary points inducing the same factors $V_1, \ldots, V_k$ in the decomposition of $H_1(\Sigma_g, \mathbb{Z})$ (maybe in different order)* *lie in the same connected component of the set of p-simple boundary points of the period fiber.*

Proof. The homomorphism $p_i := p_{|V_i}$ has degree at least three for each i. Identify V_i with the homology group $H_1(\Sigma_{g_i}, \mathbb{Z})$ for the appropriate $g_i < g$. By Theorem 1.5 applied to p_i, the fiber $\mathrm{Per}_{g_i}^{-1}(p_i)$ is connected.

Suppose first that the decompositions associated with the two simple boundary points have the factors in the same order. Then, both lie in the image of the same (restricted) attaching map

$$\mathrm{Per}_{g_1}^{-1}(p_1) \times \cdots \times \mathrm{Per}_{g_k}^{-1}(p_k) \to \Omega^{\pm}\overline{\mathcal{S}}_{g,\cancel{2}}$$

that attaches following the same rule of attaching of the given forms: the $(+1)$-pole of a part is glued to the (-1)-pole of the next. Any form in the image has some node and period homomorphism $p_1 \oplus p_2 \oplus \cdots \oplus p_k = p$. Since the source is connected, both points belong to the same connected component of $\mathcal{P}\mathrm{er}^{-1}(p)$ and are therefore equivalent.

Next we claim that changing the order of the parts of a given form $\omega_1 \vee \cdots \vee \omega_k$ of genus at least three with parts of degree at least three defines forms that lie in the same connected component of the boundary of $\mathrm{Per}^{-1}(p)$. Up to applying Proposition 7.2 to each of the parts and using (22), we can suppose that all the parts of the form have genus one. It suffices to check that when $g \geq 3$, we can change the order of two consecutive factors of genus one without leaving the boundary to deduce the result.

Now, for any pair of forms ω_1, ω_2 of genus one with two poles of peripheral integrals ± 1, we know by Corollary 4.6 that there is an isoperiodic path $\{\eta_t\}_{t\in[1,2]}$ joining $\omega_1 \vee \omega_2$ with $\omega_2 \vee \omega_1$. This path will, in general, leave the boundary. However, for any other pair of forms ω, η (not both zero) with two poles of residues ± 1, the path $\eta \vee \eta_t \vee \omega$ joins $\eta \vee \omega_1 \vee \omega_2 \vee \omega$ with $\eta \vee \omega_2 \vee \omega_1 \vee \omega$ along p-simple boundary points. ■

8.2. Proof of Proposition 2.3

It remains to find equivalences between simple boundary points whose associated decompositions have distinct factors.

The realization of decompositions given by Lemma 6.23 allows us to algebraize the problem completely. Recall the definition of admissible decompositions given in Definition 2.2.

By smoothing all except for one node, we reduce the problem to finding equivalences between simple boundary points with one node. For each such form, we can associate a vertex of the graph of $[p]$-admissible submodules, namely, the vertex corresponding to the pair $\{V, V^{\perp}\}$ where $V \oplus V^{\perp} = H_1(\Sigma_g, \mathbb{Z})$ is the decomposition induced by the simple boundary point. By Proposition 8.1, two simple boundary points having the same associated vertex are equivalent. It suffices to find equivalences between distinct vertices.

If two simple boundary points are associated with distinct vertices $\{V_1, V_2\}$ and $\{V_1', V_2'\}$ that are joined by an edge in the graph of $[p]$-admissible decompositions, it means that there exists a decomposition $W_1 \oplus \cdots \oplus W_k$ with at least three factors one of which is in the first pair and another in the other pair. Using Lemma 6.23, let $\omega_1 \vee \cdots \vee \omega_k$ be a simple boundary point of period homomorphism p whose associated decomposition is $W_1 \oplus \cdots \oplus W_k$. By smoothing the nodes appropriately, we can find an equivalence between this form and a form having the factors $\{W_j, W_j^{\perp}\}$ for any $j = 1, \ldots, k$ and period homomorphism p. Hence, the two simple boundary points are equivalent.

The connectedness of the graph of $[p]$-admissible decompositions implies that all simple boundary points are equivalent.

9. Connectedness of graphs of p-admissible decompositions

In this section, we prove Theorem 2.4.

We consider V a symplectic unimodular module over $\mathbb{Z}$, of rank at least four, and $p : V \to A$ a non-zero morphism to an abelian group. (Ultimately, this morphism will be the reduction modulo $\mathbb{Z}$ (resp. modulo $\frac{1}{2}\mathbb{Z}$) of the period of a meromorphic differential with two poles around which the periods are 1 and -1, so $A = \mathbb{C}/\mathbb{Z}$ or $\frac{\mathbb{C}}{\frac{1}{2}\mathbb{Z}}$.)

Recall the definition of the graph of p-admissible decompositions and elements in Definitions 2.2 and 3.1. To lighten the notation, we introduce an equivalence relation among p-admissible submodules.

Definition 9.1. Two p-admissible submodules V_1 and V_2 of V are said to be equivalent and denoted by $V_1 \sim V_2$ if the vertices corresponding to $V_1 \oplus V_1^{\perp}$ and $V_2 \oplus V_2^{\perp}$ lie in the same connected component of the graph of p-admissible submodules.

We start with a corollary of Lemma 3.2.

Corollary 9.2. *Any p-admissible symplectic decomposition is equivalent to one with a rank two factor.*

The rest of the section will be devoted to finding equivalences between rank two p-admissible decompositions.

9.1. Forcing intersection

Lemma 9.3. *Assume* $\operatorname{rank}(V) \geq 6$. *Then, given any pair W, W' of p-admissible submodules of rank two, there exists another pair W_1, W_1' which is such that*

- *W_1 and W_1' are p-admissible submodules of rank two,*
- *$W_1 \cap W_1' \neq \{0\}$,*
- *$W_1 \sim W$ and $W_1' \sim W'$.*

Proof. The spaces $W^{\perp}$ and $(W')^{\perp}$ intersect on a linear subspace of dimension at least two. In particular, the set $(W^{\perp} \cap (W')^{\perp}) \setminus (\mathrm{NA}(p_{|W^{\perp}}) \cup \mathrm{NA}(p_{|(W')^{\perp}}))$ is non-empty. By Lemma 3.2, an element in this set belongs to the spaces W_1 and W_1' of p-admissible symplectic decompositions $W^{\perp} = W_1 \oplus W_2$ and $(W')^{\perp} = W_1' \oplus W_2'$. The three items are satisfied for the pair W_1, W_1' and we are done. ∎

9.2. Turning around a line

Lemma 9.4. *Assume that* $\dim(V) \geq 6$. *Let* W_1, W_1' *be a pair of* p*-admissible submodules of rank two such that* $W_1 \cap W_1' \neq \{0\}$. *Then* W_1' *is equivalent to* W_1.

Proof. The statement is obvious if $W_1 = W_1'$. Next suppose $a_1 \in V$ is a primitive element such that $W_1 \cap W_1' = \mathbb{Z}a_1$, and $b_1 \in W_1$ such that $W_1 = \mathbb{Z}a_1 \oplus \mathbb{Z}b_1$. Let $a_2 \in W_1^{\perp}$ be a primitive element such that $W_1' = \mathbb{Z}a_1 \oplus \mathbb{Z}(b_1 + \alpha_2 a_2)$ for some $\alpha_2 \in \mathbb{Z} \setminus 0$.

Complete $a_, b_1, a_2$ into a symplectic basis $a_1, b_1, \ldots, a_g, b_g$. We have $(W_1 \oplus W_1')^{\perp} = \mathbb{Z}a_2 \oplus \sum_{i\geq 3}(\mathbb{Z}a_i \oplus \mathbb{Z}b_i)$. Notice that we have the following indeterminacy on the choice of basis: given elements $m_i, n_i \in \mathbb{Z}$ for $i \geq 3$, define

$$a_1' = a_1, \quad b_1' = b_1, \quad a_2' = a_2, \quad b_2' = b_2 + \sum_{i\geq 3} n_i a_i - m_i b_i$$

and

$$a_i' = a_i + m_i a_2, \quad b_i' = b_i + n_i a_2 \quad \text{for } i \geq 3.$$

We then have the symplectic decompositions $V = W_1 \oplus W_2 \oplus W_3$ and $V = W_1' \oplus W_2' \oplus W_3$ with $W_2 = \mathbb{Z}a_2 \oplus \mathbb{Z}b_2'$, $W_2' = \mathbb{Z}a_2 \oplus \mathbb{Z}(\alpha_2 a_1 + b_2')$ and $W_3 = \sum_{i\geq 3} \mathbb{Z}a_i' \oplus \mathbb{Z}b_i'$.

Case 0. If $p(a_2) \neq 0$, it suffices to choose m_3 such that $p(a_3') \neq 0$. Then none of the restrictions of p to any of the modules W_1, W_2, W_3 (resp. W_1', W_2', W_3) is trivial; hence, $W_1 \sim W_3 \sim W_1'$.

In the sequel, we suppose $p(a_2) = 0$. For any choice of the $m_j's$ and n_j's and any $i \geq 3$, we have $p(a_i') = p(a_i)$, $p(b_i') = p(b_i)$.

Case 1. If $p(W_3) \neq 0$.

Case 1.1. If $p(a_1) = 0$. Then, since W_1 is p-admissible, $p(b_1) \neq 0$ and we can choose $m_i, n_i \in \mathbb{Z}$ appropriately to guarantee that $p(b_2') \neq 0$. Since $p(\alpha_2 a_1 + b_2') = p(b_2') \neq 0$, all submodules $W_1, W_1', W_2, W_2', W_3$ are p-admissible, so $W_1 \sim W_3 \sim W_1'$.

Case 1.2. If $p(a_1) \neq 0$.

Case 1.2.1. If $p(W_3)$ contains at least three elements. We can still choose the $m_i, n_i \in \mathbb{Z}$ appropriately to guarantee that $p(\alpha_2 a_1 + b_2') \neq 0$, $p(b_2') \neq 0$ to conclude as in the last case.

Case 1.2.2. If $p(W_3)$ has at most two elements, i.e., it is either trivial or $\mathbb{Z}/2\mathbb{Z}$. The intersection of $\mathbb{Z}a_2 \oplus W_3 = (W_1 + W_1')^{\perp}$ with the kernel of p contains a primitive rank two subgroup K. We then follow a route similar to that of Lemma 9.3. There exists an element in K which does not belong to *the union* of the spaces $\mathrm{NA}(p_{|(W_1)^{\perp}})$ and $\mathrm{NA}(p_{|(W_1')^{\perp}})$ which have each rank bounded by one by Lemma 3.2. Such an element belongs to a symplectic $p_{|(W_1)^{\perp}}$-admissible rank two submodule $W_2 \subset (W_1)^{\perp}$, and to a symplectic $p_{|(W_1')^{\perp}}$-admissible rank two submodule $W_2' \subset (W_1')^{\perp}$. We then have $W_2 \sim W_1$ and $W_2' \sim W_1'$. By construction, the intersection $W_2 \cap W_2'$ contains a non-zero element in the kernel of p and $v = a_1$ belongs to $(W_2 + W_2')^{\perp}$ and satisfies $p(v) \neq 0$. By Case 0 or Case 1.1, we conclude that $W_2' \sim W_2$ and therefore $W_1' \sim W_1$.

Case 2. If $p(W_3) = 0$. We then have $p(a_2) = p(a_i) = p(b_i) = 0$ for $i = 3, \dots, g$.

Case 2.1. If $p(a_1) \neq 0$. We apply the same arguments of Case 1.2.2.

Case 2.2. If $p(a_1) = 0$. Since W_1 is p-admissible, $p(b_1) \neq 0$. In particular, $p(a_2) = p(a_i) = p(b_i) = 0$ for $i \geq 3$ and $p(b_1) \neq 0$. Define $W_1'' = \mathbb{Z}a_1 \oplus \mathbb{Z}(b_1 + \alpha_2 a_2 + \sum_{i \geq 3} m_i a_i + n_i b_i)$, for some choices of m_i, n_i's. The orthogonal of W_1'' contains the elements

$$a_2, \alpha_2 a_1 + b_2 \text{ and for } i \geq 3 \text{ by } \alpha_2 a_i + n_i b_2, \alpha_2 b_i - m_i b_2.$$

So since $p(b_1 + \alpha_2 a_2 + \sum_{i \geq 3} m_i a_i + n_i b_i) = p(b_1) \neq 0$ and $p(\alpha_2 a_1 + b_2) = p(b_2) \neq 0$, W_1'' is p-admissible.

The module $(W_1 + W_1'')^{\perp}$ contains $\alpha_2 a_3 + n_3 b_2$ for instance, so if $n_3 = 1$, the restriction of p to $(W_1 + W_1'')^{\perp}$ does not vanish identically. In particular, by Case 0 or Case 1, we conclude that $W_1'' \sim W_1$. On the other hand, $(W_1')^{\perp}$ is the module generated by $a_2, \alpha_2 a_1 + b_2$ and the a_i, b_i's for $i \geq 3$, so $(W_1' + W_1'')^{\perp}$ contains $\alpha_2 a_1 + b_2$ which does not belong to the kernel of p. Again by Case 0 or Case 1, we show that $W_1'' \sim W_1'$.

This finishes the proof. ■

Proof of Theorem 2.4. By Lemma 9.3, any pair of p-admissible symplectic modules is equivalent to a pair of p-admissible modules that intersect on a non-trivial subspace. We then deduce from Lemma 9.4 that such a pair are equivalent, showing that the original two modules were equivalent as well. Hence, the graph of p-admissible symplectic submodules of rank two is connected. The fact that the whole graph is connected comes from Corollary 9.2. ■

10. Proof of Theorem 1.5

Proof of Theorem 1.5. By induction on the genus. The cases $g = 1$ and $g = 2$ are treated in Corollaries 4.2 and 4.6. Suppose $g \geq 3$ and Theorem 1.5 is true up to genus $g - 1$. Let $p \in H^1(\Sigma_g, \mathbb{C}/\mathbb{Z})$ with $\deg(p) \geq 3$.

By Proposition 7.2, any point of $\mathrm{Per}_g^{-1}(p)$ is equivalent to a simple boundary point with precisely one node. The homomorphism $[p] : H_1(\Sigma_g, \mathbb{Z}) \to \frac{\mathbb{Z}}{\frac{1}{2}\mathbb{Z}}$ is non-trivial by hypothesis. By Theorem 2.4, the graph of $[p]$-admissible decompositions is connected. Therefore, we can apply Proposition 2.3 to deduce that all simple boundary points are equivalent. In other words, the bordification of $\mathrm{Per}^{-1}(p)$ is connected. By Corollary 6.14, we deduce that $\mathrm{Per}^{-1}(p)$ is connected. ■

11. Dynamics of $\mathrm{Aut}(H_1(\Sigma_g, \mathbb{Z}))$ on $H^1(\Sigma_g, \mathbb{C}/\mathbb{Z})$ and Transfer Principle

Thanks to Theorem 1.5, we can apply the Transfer Principle (see equation (3) and immediate consequences) in the following context: each leaf closure of $\mathcal{F}_g$ corresponds to an orbit closure of the group $\mathrm{Aut}(H_1(\Sigma_g, \mathbb{Z}))$ on $H^1(\Sigma_g, \mathbb{C}/\mathbb{Z})$. The proof of Theorem 1.11 is therefore equivalent to the proof of the following statement.

Theorem 11.1. *Let* $p \in H^1(\Sigma_g, \mathbb{C}/\mathbb{Z})$. *The closure* $\overline{\mathrm{Aut}(H_1(\Sigma_g, \mathbb{Z}))p}$ *is the set of periods* $q \in H^1(\Sigma_g, \mathbb{C}/\mathbb{Z})$ *such that*

- *any discrete factor of* p *is a discrete factor of* q *and both have the same image in it (see Definition 1.8);*
- *if* $\Im : \mathbb{C}/\mathbb{Z} \to \mathbb{R}$ *is a discrete factor of* p, *then* $V(q) = V(p)$ *(see Definition 1.9).*

The goal of this section is to provide a proof of this result. It reproduces the argument presented in [22] and aims at clarifying some issues.

For technical reasons, it will be better to use the natural identification between the homology group $H_1(\Sigma_g, \mathbb{Z})$ and the cohomology group $H^1(\Sigma_g, \mathbb{Z})$ together with their natural unimodular symplectic structures. Under this identification, the action of $\mathrm{Aut}(H_1(\Sigma_g, \mathbb{Z}))$ on $H^1(\Sigma_g, \mathbb{C}/\mathbb{Z})$ translates to the action of $\mathrm{Aut}(H^1(\Sigma_g, \mathbb{Z}))$ on $H^1(\Sigma_g, \mathbb{C}/\mathbb{Z})$ given by the action of $\mathrm{Aut}(H^1(\Sigma_g, \mathbb{Z}))$ on real and imaginary part.

Let G be the group consisting of affine transformations of $H^1(\Sigma_g, \mathbb{R})$ that preserve the symplectic structure. We denote such an affine map as

$$x \in H^1(\Sigma_g, \mathbb{R}) \mapsto Lx + t \in H^1(\Sigma_g, \mathbb{R}).$$

The group G is isomorphic to the semi-direct product $\mathrm{Aut}(H^1(\Sigma_g, \mathbb{R})) \ltimes H^1(\Sigma_g, \mathbb{R})$ with the twisted group law structure

$$(L, t) \cdot (L', t') = (LL', Lt' + t). \tag{23}$$

Let us consider the action of G on $H^1(\Sigma_g, \mathbb{R}) \times H^1(\Sigma_g, \mathbb{R}) \cong H^1(\Sigma_g, \mathbb{C})$ given in real and imaginary coordinates $u + iv$ by

$$(L, t) \cdot (u, v) = (Lu + t, Lv).$$

Understanding the orbit closures of the action of $\mathrm{Aut}(H^1(\Sigma_g, \mathbb{Z}))$ on $H^1(\Sigma_g, \mathbb{C}/\mathbb{Z})$ is equivalent to understanding the orbit closures of the action of the lattice

$$G^{\mathbb{Z}} := \mathrm{Aut}\left(H^1(\Sigma_g, \mathbb{Z})\right) \ltimes H^1(\Sigma_g, \mathbb{Z}) \subset G \quad \text{on } H^1(\Sigma_g, \mathbb{C}).$$

Indeed, note that the natural map $\pi : H^1(\Sigma_g, \mathbb{C}) \to H^1(\Sigma_g, \mathbb{C}/\mathbb{Z})$ is a covering, with abelian covering group $H^1(\Sigma_g, \mathbb{Z})$, and that there is a bijective correspondence between $\mathrm{Aut}(H^1(\Sigma_g, \mathbb{Z}))$-invariant subsets $A \subset H^1(\Sigma_g, \mathbb{C}/\mathbb{Z})$ and $G^{\mathbb{Z}}$-invariant subsets $B \subset H^1(\Sigma_g, \mathbb{C})$ given by $A \mapsto B = \pi^{-1}(A)$ with inverse $B \mapsto A = \pi(B)$. Moreover, this correspondence induces a bijective correspondence between *closed* $\mathrm{Aut}(H^1(\Sigma_g, \mathbb{Z}))$-invariant subsets of $H^1(\Sigma_g, \mathbb{C}/\mathbb{Z})$ and *closed* $G^{\mathbb{Z}}$-invariant subsets $B \subset H^1(\Sigma_g, \mathbb{C})$.

The action of G on $H^1(\Sigma_g, \mathbb{C})$ has an invariant Zariski open subset $\mathcal{P} \subset H^1(\Sigma_g, \mathbb{C})$ consisting of classes with non-zero imaginary part. The complement, $\mathcal{Q} \simeq H^1(\Sigma_g, \mathbb{R})$ consisting of real periods, is also invariant. In the sequel, we will separately study the orbit closures of $G^{\mathbb{Z}}$ in restriction to $\mathcal{Q}$ and $\mathcal{P}$, using Ratner's theory, and then collect the consequences on the orbit closures of $G^{\mathbb{Z}}$ on $H^1(\Sigma_g, \mathbb{C})$.

The proof of Theorem 11.1 will proceed by a case by case analysis of the different possibilities for the orbit closures. At the end of each case, we will recap the information to show that the orbit closure is as stated in Theorem 11.1.

11.1. Orbit closures on $\mathcal{Q}$

To understand orbit closures on $\mathcal{Q}$, we will focus on understanding orbit closures of $\mathrm{Aut}(H^1(\Sigma_g,\mathbb{Z}))$ on the torus $H^1(\Sigma_g,\mathbb{R}/\mathbb{Z})$, and it is well known that for this, latter orbits of irrational points are dense and orbits of rational points are finite.

Let us review the argument as a preparation for the case of $\mathcal{P}$. The stabilizer of a point $P=(u,0)\in\mathcal{Q}$ is the connected group

$$S_P := \big\{(L,t) \mid t = u - Lu\big\}$$

which is generated by unipotents. So the classification of orbit closures is reminiscent to Ratner's theory: there exists a Lie subgroup $G_P\subset G$ such that

- G_P contains S_P,
- $G_P^{\mathbb{Z}} = G_P \cap G^{\mathbb{Z}}$ is a lattice in G_P,
- $\overline{G^{\mathbb{Z}}P} = G^{\mathbb{Z}} G_P P$.

See [28, Section 3].

Denoting $f : G \to \mathrm{Aut}(H^1(\Sigma_g,\mathbb{R}))$ as the natural projection, we have that

$$f(G^{\mathbb{Z}}) = \mathrm{Aut}\big(H^1(\Sigma_g,\mathbb{Z})\big) \quad \text{and} \quad f(G_P) = \mathrm{Aut}\big(H^1(\Sigma_g,\mathbb{R})\big).$$

So the image $f(G_P^{\mathbb{Z}})$ is a finite index subgroup Γ of $\mathrm{Aut}(H^1(\Sigma_g,\mathbb{Z}))$. Note also that it implies that the group $B_P = G_P\cap \mathrm{Ker}(f)$ is invariant by the action of $\mathrm{Aut}(H^1(\Sigma_g,\mathbb{Z}))$ on $\mathrm{Ker}(f)\simeq H^1(\Sigma_g,\mathbb{R})$. There are two possibilities for this group: either $B_P=\{0\}$ or $B_P=\mathrm{Ker}(f)$.

Case $B_P=\{0\}$. We have $G_P=S_P$, and so the lattice $G_P^{\mathbb{Z}}$ is the graph over Γ of the function $L\mapsto u-Lu$. In particular, we have that $u-Lu\in H^1(\Sigma_g,\mathbb{Z})$ for each $L\in\Gamma$. Using that there exists in Γ an element having no eigenvalue equal to 1, we see that this implies that $u\in H^1(\Sigma_g,\mathbb{Q})$. In this case, the orbit of P by $G^{\mathbb{Z}}$ is discrete, and the orbit of $p = P \bmod \mathbb{Z}$ by $\mathrm{Aut}(H^1(\Sigma_g,\mathbb{Z}))$ is finite.

To prove Theorem 11.1, in this case, we remark that $\Re$ and $\Im$ are two distinct discrete factors of P. We need to show that all $Q\in\mathcal{Q}$ with the same (finite) image mod $\mathbb{Z}$ as P lie in the $G^{\mathbb{Z}}$ orbit of P. Let $\frac{1}{k}\mathbb{Z}/\mathbb{Z}$ with $k\in\mathbb{N}^*$ be the image group. Up to changing P and Q by some integral translates, we can suppose that their image in $\mathbb{R}$ is precisely $\frac{1}{k}\mathbb{Z}$. The forms kP and kQ can be thought of as primitive classes in $H^1(\Sigma_g,\mathbb{Z})$ and are therefore equivalent by some symplectic automorphism.

Case $B_P=\mathrm{Ker}(f)$. We have that $G_P=G$, and so $\overline{G^{\mathbb{Z}}P}=\mathcal{Q}$.

The proof of Theorem 11.1 reduces to checking the condition to the single discrete factor of P, namely, $\Im$ with trivial image. The elements of $\mathcal{Q}$ satisfy this condition.

11.2. Orbit closures on $\mathscr{P}$

Since the stabilizer of a point $P = (u, v) \in \mathscr{P}$,

$$S_P = \{(L, t) \mid Lv = v \text{ and } t = u - Lu\}, \tag{24}$$

which is also generated by unipotents: so again there exists a Lie subgroup $G_P \subset G$ such that

- G_P contains S_P,
- $G_P^{\mathbb{Z}} = G_P \cap G^{\mathbb{Z}}$ is a lattice in G_P,
- $\overline{G^{\mathbb{Z}} \cdot P} = G^{\mathbb{Z}} G_P P$.

See [28, Section 3]. As in [22], we denote by B_P the intersection of G_P and of the kernel of the projection $G \to \mathrm{Aut}(H^1(\Sigma_g, \mathbb{R}))$ that sends (L, t) to L; this is a closed normal subgroup of G_P. We also denote by A_P the quotient of G_P by B_P as an abstract Lie group (there is a natural immersion of Lie group $i : A_P \to \mathrm{Aut}(H^1(\Sigma_g, \mathbb{R}))$ but it is unclear that the image is a Lie subgroup *a priori*).

Lemma 11.2. *The intersection* $B_P^{\mathbb{Z}} := G_P^{\mathbb{Z}} \cap B_P$ *is a lattice in* B_P*, and the image* $A_P^{\mathbb{Z}}$ *of* $G_P^{\mathbb{Z}}$ *in* A_P *is a lattice in* A_P.

Proof. The group $B_P^{\mathbb{Z}}$ is discrete in B_P since discreteness is preserved by intersecting with a Lie subgroup. We also have that $A_P^{\mathbb{Z}}$ is discrete in A_P since $A_P^{\mathbb{Z}} \subset i^{-1} \mathrm{Aut}(H^1(\Sigma_g, \mathbb{Z}))$ and i is an immersion. One can construct a Haar measure on G_P as an integral over A_P with respect to a Haar measure on A_P of B_P-Haar measures on the fibers of $B_P \to G_P \to A_P$, so

$$\mathrm{Vol}(G_P^{\mathbb{Z}} \setminus G_P) = \mathrm{Vol}(A_P^{\mathbb{Z}} \setminus A_P) \cdot \mathrm{Vol}(B_P^{\mathbb{Z}} \setminus B_P).$$

Since $\mathrm{Vol}(G_P^{\mathbb{Z}} \setminus G_P)$ is finite, so are $\mathrm{Vol}(A_P^{\mathbb{Z}} \setminus A_P)$ and $\mathrm{Vol}(B_P^{\mathbb{Z}} \setminus B_P)$. ∎

We use the following notation: given a Lie group K, K^0 is the connected component of K containing the neutral element.

Lemma 11.3. $i(A_P^0)$ *is either* $\mathrm{Aut}(H^1(\Sigma_g, \mathbb{R}))$ *or the stabilizer* S_v *in* $\mathrm{Aut}(H^1(\Sigma_g, \mathbb{R}))$ *of* v.

Proof. Since the image of S_P by i is the group S_v, and that $S_P \subset G_P^0$, we have that $S_v \subset A_P^0$. Assume first that $i(A_P^0)$ is contained in $S_{\mathbb{R}v}$. Since S_v is of codimension 1 in $S_{\mathbb{R}v}$, and that both are connected, the image of A_P^0 by the immersion i is either S_v or $S_{\mathbb{R}v}$. The second possibility cannot occur since $S_{\mathbb{R}v}$ is not unimodular, but A_P^0 is since it contains a lattice. Hence, we conclude in that case that $i(A_P^0) = S_v$.

Assume now that $i(A_P^0)$ is not contained in the stabilizer $S_{\mathbb{R}v}$ in $\mathrm{Aut}(H^1(\Sigma_g, \mathbb{R}))$ of the line $\mathbb{R}v$. Note that $S_v \subset i(A_P^0)$. So if we denote $O = i(A_P^0)v$, then for any $w \in O$, we have $S_w \subset i(A_P^0)$.

We claim that O contains an element v' that is not orthogonal to v. This is clear if $g = 1$ since in this case one has $v^{\perp} = \mathbb{R}v$. If $g \geq 2$, take $w \in O \setminus \mathbb{R}v$. If $w \notin v^{\perp}$, we are done. Otherwise, $w \in v^\perp \setminus \mathbb{R}v$, so $S_v w = v^\perp \setminus \mathbb{R}v$. In particular, $v^\perp \setminus \mathbb{R}v$ is contained in O. Since $v^\perp \setminus \mathbb{R}v$ is not isotropic, we deduce that there exist $w', w'' \in O$ such that $w' \cdot w'' \neq 0$. By transitivity of the action of $i(A_P^0)$ on O, $v = Lw'$ for some $L \in i(A_P^0)$. The element $v' = Lw''$ then belongs to $O \setminus v^\perp$ since L preserves the product, and the claim follows.

The orbit $S_v v'$ is equal to the set $v' + v^\perp$. In particular, the set $O \setminus v^\perp$ is invariant by the fibration by affine hyperplanes directed by $v^\perp$. Similarly, $O \setminus (v')^\perp$ is invariant by the fibration by affine hyperplanes directed by $(v')^\perp$. We deduce that O contains $H^1(\Sigma_g, \mathbb{R}) \setminus (v^\perp \cap (v')^\perp)$. This being valid for any $v, v' \in O$ that are not collinear, we infer that $O = H^1(\Sigma_g, \mathbb{R}) \setminus \{0\}$, and in particular that $i(A_P^0) = \mathrm{Aut}(H^1(\Sigma_g, \mathbb{R}))$. ■

11.2.1. Case where $i(A_P^0) = \mathrm{Aut}(H^1(\Sigma_g, \mathbb{R}))$.

Lemma 11.4. *In this case, B_P is either 0 or the whole $H^1(\Sigma, \mathbb{R})$.*

Proof. B_P is a closed subgroup of $H^1(\Sigma_g, \mathbb{R})$ invariant under $i(A_P^0)$, so it is also invariant under $\mathrm{Aut}(H^1(\Sigma_g, \mathbb{R}))$. The only such subgroups are 0 or $H^1(\Sigma_g, \mathbb{R})$. ■

Case $B_P = H^1(\Sigma_g, \mathbb{R})$. In this case, $G_P = G$, and the closure of $G^{\mathbb{Z}} P$ in $\mathcal{P}$ is equal to $G_P P = GP = \mathcal{P}$. Notice that for this situation to happen, the period P has no discrete factor since otherwise, all periods of the closure of $G^{\mathbb{Z}} P$ would have the same discrete factor. The absence of a discrete factor is equivalent to the image of P being dense.

Since $\mathcal{P}$ is dense in $H^1(\Sigma_g, \mathbb{C})$, we conclude that Theorem 11.1 is true in this case (there are no discrete factors to test).

Case $B_P = 0$. In this case, the group G_P is the image of $\mathrm{Aut}(H^1(\Sigma_g, \mathbb{R}))$ by a section of the morphism $G \to \mathrm{Aut}(H^1(\Sigma_g, \mathbb{R}))$. If $\sigma : \mathrm{Aut}(H^1(\Sigma_g, \mathbb{R})) \to G$ denotes this section, one can write $\sigma(L) = (L, t(L))$ in the coordinates of G given by (23), where $t : \mathrm{Aut}(H^1(\Sigma_g, \mathbb{R})) \to H^1(\Sigma_g, \mathbb{R})$ is a certain function. We then have

$$\big(L, t(L)\big) \cdot \big(L', t(L')\big) = \big(LL', Lt(L') + t(L)\big) = \big(LL', t(LL')\big)$$

so t is a 1-cocycle, namely,

$$t(LL') = t(L) + Lt(L') \text{ for every couple of elements } L, L' \in \mathrm{Aut}\big(H^1(\Sigma_g, \mathbb{R})\big). \quad (25)$$

Since S_P is contained in G_P, from (24), we deduce that

$$t(L) = u - L(u) \quad \text{for every } L \in S_v. \quad (26)$$

Introduce the function $T(L) = t(L) + L(u) - u$, which satisfies also the same cocycle equation (25) and is identically vanishing on S_v. We then have $T(LL') = T(L)$ for every $L \in \mathrm{Aut}(H^1(\Sigma_g, \mathbb{R}))$ and $L' \in S_v$, so since $\mathrm{Aut}(H^1(\Sigma_g, \mathbb{R}))/S_v \simeq H^1(\Sigma_g, \mathbb{R}) \setminus \{0\}$, there exists a function $\tau : H^1(\Sigma_g, \mathbb{R}) \setminus \{0\} \to H^1(\Sigma_g, \mathbb{R})$ such that $T(L) = \tau(Lv)$. We have $\tau(v) = 0$. Given $L \in \mathrm{Aut}(H^1(\Sigma_g, \mathbb{R}))$, and $z \in H^1(\Sigma, \mathbb{R}) \setminus \{0\}$, writing $z = L'v$ for some $L' \in \mathrm{Aut}(H^1(\Sigma_g, \mathbb{R}))$, we have

$$\tau(Lz) = \tau(LL'y) = T(LL') = T(L) + LT(L') = \tau(Lv) + L\tau(z). \quad (27)$$

In particular, τ is equivariant with respect to the group S_v. The functions equivariant with respect to S_v take the following form: for z not colinear to v, we have

$$\tau(z) = \alpha(z \cdot v)v + \beta(z \cdot v)z$$

where $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ are some functions (to prove this, use (1) equivariance, (2) that any point fixed by the whole group $S_v \cap S_z$ belongs to the space $\mathbb{R}v + \mathbb{R}z$ and (3) that the group S_v acts transitively on the sets $\{z \mid z \cdot v = c\}$ if $c \neq 0$ and on the set $v^{\perp} \setminus \mathbb{R}v$). And $\mathbb{R}v \setminus \{0\}$ is mapped to $\mathbb{R}v$ without any additional restriction *a priori*.

Now equation (27) writes, assuming z, Lz and Lv do not belong to $\mathbb{R}v$:

$$\alpha(Lz \cdot v)v + \beta(Lz \cdot v)Lz = \alpha(Lv \cdot v)v + \beta(Lv \cdot v)Lv + \alpha(z \cdot v)Lv + \beta(z \cdot v)Lz.$$

Assuming moreover that v, Lv and Lz are not colinear, we get the equations

$$\alpha(Lz \cdot v) = \alpha(Lv \cdot v), \quad \beta(Lz \cdot v) = \beta(z \cdot v), \quad \beta(Lv \cdot v) = -\alpha(z \cdot v).$$

From there we see that indeed the functions α, β are constant with $\alpha + \beta = 0$. In particular, τ is a function which preserves $\mathbb{R}v$ and which is equal to $\alpha v - \alpha\mathrm{id}$ on the complementary of $\mathbb{R}v$.

Let us rewrite equation (27) with this information: taking any $L \in \mathrm{Aut}(H^1(\Sigma_g, \mathbb{R}))$, we have for $z \in H^1(\Sigma_g, \mathbb{R}) \setminus (\mathbb{R}v \cup L^{-1}(\mathbb{R}v))$

$$\tau(Lv) = \tau(Lz) - L\tau(z) = \alpha(v - Lz - Lv + Lz) = \alpha(v - Lv)$$

and so the formula

$$\tau = \alpha(v - \mathrm{id}) \tag{28}$$

is valid on $H^1(\Sigma_g, \mathbb{R}) \setminus \{0\}$. This gives the following expression for the cocycle t:

$$t(L) = \tau(Lv) + u - Lu = \alpha(v - Lv) + u - Lu = (I - L)(u + \alpha v).$$

Since the intersection of G_P with $G^{\mathbb{Z}}$ is a lattice, there exists a finite index subgroup of $\mathrm{Aut}(H^1(\Sigma_g, \mathbb{Z}))$ in restriction to which the function $L \mapsto (I - L)(u + \alpha v)$ takes integral values. In particular, $z = u + \alpha v \in H^1(\Sigma_g, \mathbb{Q})$, which is equivalent to the fact that $\Re + \alpha\Im$ mod $\mathbb{Z}$ is a discrete factor of P mod $\mathbb{Z}$. Then the G_P-orbit of P is made of the periods $Q \in \mathcal{P}$ such that $\Re Q + \alpha\Im Q = z$, and consequently the closure $\overline{G^{\mathbb{Z}}P}^{\mathcal{P}}$ of $G^{\mathbb{Z}}P$ in $\mathcal{P}$ consists of the periods $Q \in H^1(\Sigma_g, \mathbb{C})$ with non-zero imaginary part and such that $\Re Q + \alpha\Im Q$ is any form which differs from an element of $\mathrm{Aut}(H^1(\Sigma_g, \mathbb{Z}))u$ by an element of $H^1(\Sigma_g, \mathbb{Z})$. Equivalently, those are periods Q with non-zero imaginary part such that $\Re Q + \alpha\Im Q$ has a reduction modulo $\mathbb{Z}$ with the same image as the reduction modulo $\mathbb{Z}$ of $u + \alpha v$ (which is finite since $u + \alpha v$ is rational). Consequently, the closure of $\overline{G^{\mathbb{Z}}P}^{\mathcal{P}}$ in $H^1(\Sigma_g, \mathbb{C})$ also contains all points Q with $\Im Q = 0$ satisfying the condition that $\Re Q + \alpha\Im Q$ has the same image mod $\mathbb{Z}$ as $\Re P + \alpha\Im P$.

Notice that for the situation considered in this paragraph, i.e., $i(A_P^0) = \mathrm{Aut}(H^1(\Sigma_g, \mathbb{R}))$ and $B_P = 0$, we have a discrete factor for the reduction modulo $\mathbb{Z}$ of the form $\Re +$

$\alpha\Im$ mod $\mathbb{Z}$, but $\Im$ is not a discrete factor of this reduction (or equivalently the space $\mathbb{R}v$ is not rational) since otherwise, the whole period P would have discrete image, and its orbit $G^{\mathbb{Z}}P$ would be discrete in $\mathcal{P}$. The previously described closure of $G^{\mathbb{Z}}P$ in $H^1(\Sigma_g,\mathbb{C})$ is precisely the subset having the same image under the discrete factor $\Re + \alpha\Im$ mod $\mathbb{Z}$, thus proving Theorem 11.1 in this case.

11.2.2. Case where $i(A_P^0) = S_v$. In this case, the intersection $S_v \cap \operatorname{Aut}(H^1(\Sigma_g,\mathbb{Z}))$ is a lattice in S_v. Borel density theorem shows that the line $\mathbb{R}v$ is a rational subspace of $H^1(\Sigma_g,\mathbb{R})$ (or equivalently, $\Im$ is a discrete factor of P modulo $\mathbb{Z}$). In particular, there exists $\nu > 0$ so that the image of the period v is the infinite cyclic group $\mathbb{Z}\nu$.

Consider the closed $G^{\mathbb{Z}}$-invariant subset $\mathcal{P}^{\nu} \subset \mathcal{P}$ formed by those periods Q so that the image of $\Im Q$ is equal to $\mathbb{Z}\nu$. On the set $\mathcal{P}^{\nu}$, the function

$$V : \mathcal{P}^{\nu} \to \mathbb{R}/\nu\mathbb{Z} \text{ defined by } V(q) = \Re q \cdot \Im q \text{ mod } \nu\mathbb{Z}$$

is continuous and invariant under the group $G^{\mathbb{Z}}$.

Lemma 11.5. *B_P^0 is either 0, $\mathbb{R}v$, $v^{\perp}$ or the whole $H^1(\Sigma_g,\mathbb{R})$.*

Proof. We have seen in the proof of Lemma 11.3 that $i(A_p^0)$ contains S_y, so $B_p \subset H^1(\Sigma,\mathbb{R})$ is a S_y-invariant closed subgroup of $H^1(\Sigma,\mathbb{R})$. The result follows from the fact that B_p^0 is a linear subspace of $H^1(\Sigma,\mathbb{R})$ and that the only linear S_y-invariant subspaces of $H^1(\Sigma,\mathbb{R})$ are those indicated in the statement. ■

The case $B_p^0 = H^1(\Sigma,\mathbb{R})$ cannot occur. Indeed, assume by contradiction the contrary. The closure of the orbit $G^{\mathbb{Z}}P$ is contained in a level subset of V. However, the function V takes any value in $\mathbb{R}/\nu\mathbb{Z}$ on the B_P^0-orbit of P, hence, on its G_P-orbit, which contradicts $\overline{G^{\mathbb{Z}}P} = G^{\mathbb{Z}}G_P P$.

Case $B_P^0 = v^{\perp}$. In this case, we claim that the closure of $G^{\mathbb{Z}}P$ is the level of V containing P (namely, the set $V^{-1}(V(P))$). By $G^{\mathbb{Z}}$-invariance of the function V, we already know that $\overline{G^{\mathbb{Z}}P} \subset V^{-1}(\{V(P)\})$. For the opposite inclusion, take any $Q \in V^{-1}(\{V(P)\})$. Since $\frac{1}{\nu}\Im P$ and $\frac{1}{\nu}\Im Q$ are primitive integer classes in $H^1(\Sigma_g,\mathbb{Z})$, and that the group $\operatorname{Aut}(H^1(\Sigma_g,\mathbb{Z}))$ acts transitively on those classes, there exists $L \in \operatorname{Aut}(H^1(\Sigma_g,\mathbb{Z}))$ such that $L\Im Q = \Im P$. We know that $\Re LQ \cdot L\Im Q = \Re Q \cdot \Im Q = \Re P \cdot \Im P + n\nu$ for some integer $n \in \mathbb{Z}$ since $V(P) = V(Q)$. Again using that $\frac{1}{\nu}\Im LQ$ is a primitive integer cohomology class, there exists a class $r \in H^1(\Sigma_g,\mathbb{Z})$ so that $r \cdot (\frac{1}{\nu}\Im LQ) = 1$. In particular, considering $Q' = LQ - nr \in G^{\mathbb{Z}}Q$, we have

$$\Re Q' \cdot \Im Q' = (\Re Q - nr) \cdot \Im Q = \Re P \cdot \Im P \quad \text{and} \quad \Im Q' = \Im P.$$

In particular, $\Re Q' - \Re P \in v^{\perp}$, $Q' \in B_P^0 P$, and consequently $Q \in G^{\mathbb{Z}}B_P P = \overline{G^{\mathbb{Z}}P}$.

We notice that this case only occurs when $\Im$ is a discrete factor of P, but there is no other discrete factor (since otherwise P would have discrete image and its $G^{\mathbb{Z}}$-orbit would be discrete too). The argument in the last paragraph proves that Theorem 11.1 is true in this case for the points of $\mathscr{P}$. No point in $\mathscr{Q}$ can be in the closure of the latter set since the image of the discrete factor $\mathfrak{I}$ for elements of $\mathscr{Q}$ is 0.

Before proceeding to the next case for B_P^0, remark that when $g = 1$, $v^\perp = \mathbb{R}v$, so we have covered the case $B_P^0 = \mathbb{R}v$ in genus $g = 1$.

The case $B_P^0 = \mathbb{R}v$ does not occur if $g \geq 2$. Assume by contradiction that $g \geq 2$ and $B_P^0 = \mathbb{R}v$. In this case, the connected component of G_P is the group $S_v \ltimes \mathbb{R}v$. Since the intersection $G_P^0 \cap G^{\mathbb{Z}}$ is a lattice in G_P^0, its image in S_v by the natural projection is a finite index subgroup Γ of the lattice $S_v \cap \mathrm{Aut}(H^1(\Sigma_g, \mathbb{Z}))$ (this is due to the fact that $G^{\mathbb{Z}}$ projects to $\mathrm{Aut}(H^1(\Sigma_g, \mathbb{Z}))$). As a consequence, for every $L \in \Gamma$, there exists $\beta_L \in \mathbb{R}$ such that, I being the identity operator,

$$(I - L)u + \beta_L v = z_L \in H^1(\Sigma_g, \mathbb{Z}). \tag{29}$$

Indeed, the last expression is the translation part of a lift of L in $G_P^0 \cap G^{\mathbb{Z}}$.

We claim that the space $\mathbb{R}u + \mathbb{R}v$ is rational. This is clear if u and v are $\mathbb{R}$-collinear since $\mathbb{R}v$ is rational. So in the sequel we will assume that $\mathbb{R}u + \mathbb{R}v$ is two-dimensional. Notice that $\mathbb{R}v$ and $v^\perp$ are rational subspaces of $H^1(\Sigma_g, \mathbb{R})$. So if we could invert $I - L$, that would prove the claim since $\mathbb{R}u$ would be a rational subspace too due to equation (29). Unfortunately, $I - L$ is not invertible since $L(v) = v$. So to make this idea work, we will need to invert $I - L$ in another space: the space $v^\perp/\mathbb{R}v$ indeed, on which the map $I - L$ acts naturally and is invertible for sufficiently generic $L \in S_v \cap \mathrm{Aut}(H^1(\Sigma_g, \mathbb{Z}))$.

Let us proceed to the formal proof of the claim. We will use the following elements $L \in S_v$: let $a \in \mathbb{R}v \cap H^1(\Sigma_g, \mathbb{Z})$ be a primitive element, and let $b \in H^1(\Sigma_g, \mathbb{Z})$ such that $a \cdot b = 1$. Consider a map $L_b \in \Gamma$ such that $L_b(a) = a$, $L_b(b) = b + na$ for a certain non-zero integer n and the restriction of L_b to $(\mathbb{R}a + \mathbb{R}b)^\perp$ has no (complex) eigenvalue equal to 1. Such an element exists since Γ has finite index in $S_v^{\mathbb{Z}}$. For such a L_b, the induced action of $I - L_b$ on $y^\perp/\mathbb{R}y$ is invertible (since it is conjugate to the action of L_b on the space $(\mathbb{R}a + \mathbb{R}b)^\perp$).

Notice now that the three terms appearing in equation (29), namely, $(I - L_b)u$, v and z_{L_b}, all belong to $v^\perp$. In the quotient $v^\perp/\mathbb{R}v$, we have the equation

$$(I - L_b)(u) \bmod \mathbb{R}v = z_{L_b} \bmod \mathbb{R}v$$

and in particular $(I - L_b)(u) \bmod \mathbb{R}v$ is rational there. Since the action of $I - L_b$ on $v^\perp/\mathbb{R}v$ is invertible, there exists a rational element $w_b \in v^\perp$ such that

$$u - L(u) = w_b - L_b(w_b) \bmod \mathbb{R}v,$$

or equivalently

$$u - w_b \in (I - L_b)^{-1}(\mathbb{R}v).$$

With our special choice of L_b, notice that $(I - L_b)^{-1}(\mathbb{R}v) = \mathbb{R}a + \mathbb{R}b$. If for some b as before the corresponding L_b satisfies $w_b \in \mathbb{R}a + \mathbb{R}b$, then either $u \in \mathbb{R}v$ or $\mathbb{R}u + \mathbb{R}v = \mathbb{R}a + \mathbb{R}b$, so we are done. Otherwise, we see that all the spaces $\mathbb{R}u + \mathbb{R}v + \mathbb{R}b$ are rational since they are freely generated by a, b, w_b. Recall that this is valid for every

$b \in H^1(\Sigma, \mathbb{Z})$ such that $a \cdot b = 1$. But we have

$$\mathbb{R}u + \mathbb{R}v = \bigcap_{b \in H^1(\Sigma,\mathbb{Z}),\, a\cdot b=1} \mathbb{R}u + \mathbb{R}v + \mathbb{R}b$$

and the proof of the claim follows since an intersection of rational subspaces is a rational space.

Notice that the claim of the rationality of the space $\mathbb{R}u + \mathbb{R}v$ with $u = \Re P$ and $v = \Im P$ holds true on the whole orbit $G^{\mathbb{Z}} P$, in particular on $P + H^1(\Sigma_g, \mathbb{Z})$. This proves that the spaces $\mathbb{R}(u + w) + \mathbb{R}v$ are all rational for $w \in H^1(\Sigma, \mathbb{Z})$. Using this fact, we deduce that the affine line $u + \mathbb{R}v$ contains a rational point. Indeed, the claim is obvious if u and v are $\mathbb{R}$-collinear since $\mathbb{R}v$ is rational. If u and v are linearly independent over $\mathbb{R}$, let us take a rational basis $a, b, \dots$ of $\mathbb{R}u + \mathbb{R}v$ such that $v \in \mathbb{R}a$ and $u = \mu a + \nu b$ with $\nu \neq 0$. Since $g \geq 2$, extend a, b to a basis $a, b, c, d, \dots$ of $H^1(\Sigma_g, \mathbb{Q})$. Write $u = \mu a + \nu b$, for some $\mu, \nu \in \mathbb{R}$. For $N \in \mathbb{N}^*$ large enough, Nc belongs to $H^1(\Sigma_g, \mathbb{Z})$, so the space $\mathbb{R}(\mu a + \nu b + Nc) + \mathbb{R}a$ is rational. This also imposes ν rational, and in either case, $\nu b \in (u + \mathbb{R}v) \cap H^1(\Sigma_g, \mathbb{Q})$. In particular, there exists $\alpha \in \mathbb{R}$ such that $u + \alpha v$ is rational, and as in the end of the previously treated case $i(A^0_P) = \mathrm{Aut}(H^1(\Sigma_g, \mathbb{R}))$ and $B_P = 0$, we are in the case where $\Re + \alpha \Im$ is a discrete factor of P mod $\mathbb{Z}$. Since $\Im$ is also a discrete factor of P mod $\mathbb{Z}$, this proves that P mod $\mathbb{Z}$ has a discrete image in $\mathbb{C}/\mathbb{Z}$, and thus the $G^{\mathbb{Z}}$-orbit of P is discrete. But this is contradictory with $B^0_P = \mathbb{R}v$ since this latter implies that the $G^{\mathbb{Z}}$-orbit closure of P is one-dimensional.

The case $B^0_p = 0$. In this case, we can follow the arguments of the previous case $B^0_P = \mathbb{R}v$ (with $\beta_l = 0$ in equation (29)). We deduce that the period P mod $\mathbb{Z}$ has a discrete image in $\mathbb{C}/\mathbb{Z}$, and infinite. We claim that the orbit $G^{\mathbb{Z}} P$ consists in the periods $Q \in \mathcal{P}$ so that

(1) Q mod $\mathbb{Z}$ has the same image as the one of P mod $\mathbb{Z}$, and

(2) $V(Q) = V(P)$.

This claim implies that Theorem 11.1 is true also in this case.

Let us prove the claim. Each point $Q \in G^{\mathbb{Z}} P$ satisfies (1) and (2). For the converse, we first normalize the image of P. Up to isomorphism of $\mathbb{C}/\mathbb{Z}$, we can suppose that the image of P mod $\mathbb{Z}$ is $(\frac{1}{k}\mathbb{Z} + i\mathbb{Z})/\mathbb{Z}$ for an appropriate $k \in \mathbb{N}^*$. Next assume that $Q \in \mathcal{P}$ satisfies (1) and (2). We will show that up to the action of $G^{\mathbb{Z}}$, we can suppose that both P and Q have the same discrete image, and both are either periods of some holomorphic differential as in Proposition 3.5 or have symplectic kernel. In either case, we can use the results in [11] to conclude.

Step 1. There exist $u, v \in H^1(\Sigma_g, \mathbb{Z})$ such that $P' = P + u$ and $Q' = Q + v$ satisfy

$$\Re P' \cdot \Im P' = \Re Q' \cdot \Im Q' > 0.$$

Proof of Step 1. We carry it for P, and the same argument applies to Q. Fix any $V > 0$ such that $V(P) = V$ in $\mathbb{R}/\Im(P)(H_1(\Sigma_g, \mathbb{Z}))$. The equation for u can be read as

$$u \cdot \Im(P) = V - \Re(P) \cdot \Im(P) \in \Im(P)\big(H_1(\Sigma_g, \mathbb{Z})\big) = \mathbb{Z}.$$

As $\Im(P)$ is primitive, the duality of the pairing $H^1(\Sigma_g,\mathbb{Z}) \times H^1(\Sigma_g,\mathbb{Z}) \to \mathbb{Z}$ guarantees the existence of u. Remark that the image of P' mod $\mathbb{Z}$ is still $(\frac{1}{k}\mathbb{Z} + i\mathbb{Z})/\mathbb{Z}$.

Step 2. There exist $u' \in \Im(P)^\perp$ and $v' \in \Im(Q)^\perp$ such that the image of both $P'' = P' + u'$ and $Q'' = Q' + v'$ in $\mathbb{C}$ is precisely $\frac{1}{k}\mathbb{Z} + i\mathbb{Z}$.

Proof of Step 2. We carry it for P', and the same argument applies to Q'. First remark that $\Im(P') = \Im(P)$. We can find $u' \in \Im(P)^\perp$ such that there exists $x \in \ker P'$ with $u'(x) = 1$. Indeed, since $\ker P'$ is primitive, there exists a supplementary $S \subset H_1(\Sigma_g,\mathbb{Z})$ such that

$$H_1(\Sigma_g,\mathbb{Z}) = S \oplus \ker P'.$$

Since $S^\perp$ is a module of rank at least two, $S^\perp \cap \Im(P)^\perp$ has rank at least one. Take $u' \in S^\perp \cap \Im(P)^\perp$ a primitive element. By abuse of notation, we denote its dual as u' as well. Since $u'_{|S} \equiv 0$, we have that $u'_{|\ker P'}$ is also primitive, and there exists $x \in \ker P'$ such that $u'(x) = 1$ as desired. With this choice of u', the form P'' of the statement of Step 2 satisfies $P''(x) = 1$ and its reduction mod $\mathbb{Z}$ has image group $(\frac{1}{k}\mathbb{Z} + i\mathbb{Z})/\mathbb{Z}$. Therefore, P'' has $\frac{1}{k}\mathbb{Z} + i\mathbb{Z}$ as image group.

Step 3. There exists a symplectic $M \in \operatorname{Aut}(H_1(\Sigma_g,\mathbb{Z}))$ such that $Q'' = P'' \circ M$.

Proof of Step 3. The elements P'' and Q'' have the same discrete image $\frac{1}{k}\mathbb{Z} + i\mathbb{Z}$. The covolume of this image group is

$$\operatorname{vol}\left(\frac{\mathbb{C}}{\frac{1}{k}\mathbb{Z} + i\mathbb{Z}}\right) = \frac{1}{k}.$$

On the other hand, the products $\Re(P'') \cdot \Im(P'') = \Re(Q'') \cdot \Im(Q'') > 0$ define an element in $\frac{1}{k}\mathbb{Z}$. We therefore obtain

$$D = \frac{\Re(P'') \cdot \Im(P'')}{\operatorname{vol}\left(\mathbb{C}/(\frac{1}{k}\mathbb{Z} + i\mathbb{Z})\right)} = \frac{\Re(Q'') \cdot \Im(Q'')}{\operatorname{vol}\left(\mathbb{C}/(\frac{1}{k}\mathbb{Z} + i\mathbb{Z})\right)} \in \mathbb{N}^*.$$

If $D > 1$, the two conditions in the proof of Proposition 3.5 are satisfied for P'' and Q'', so both are periods of some holomorphic differentials. We can apply [11, Lemma 9.1] to P'' and Q'' to conclude that there exists a symplectic $M \in \operatorname{Aut}(H_1(\Sigma_g,\mathbb{Z}))$ such that $Q'' = P'' \circ M$. Otherwise, $D = 1$ and, as shown in [11], $\ker P''$ (resp. $\ker Q''$) is a rank $2g - 2$ symplectic submodule. We can find a symplectic isomorphism M with $M(\ker Q'') = \ker P''$ that also satisfies $Q'' = P'' \circ M$.

Therefore, $Q \in G^{\mathbb{Z}} P$, as was claimed.

11.3. Proof of Corollary 1.13

The three cases correspond to the existence of two different discrete factors (case (1)), no discrete factors (case (2)) and a unique discrete factor (case (3)). The reader can check along the proof of Theorem 11.1 that each case is covered.

Moore's theorem allows us to deduce the ergodicity property. The closedness of the leaves with discrete Λ is easy. If Λ is infinite, then the leaf cannot be algebraic, with respect to the Deligne–Mumford algebraic structure on moduli space. More precisely, let $\Omega\overline{\mathcal{M}_{g,2}}$ be the Hodge bundle over the Deligne–Mumford compactification. In the boundary component $\partial \subset \Omega\overline{\mathcal{M}_{g,2}}$ consisting of forms having a node separating the curve in a curve of genus g and a rational curve containing the two marked points, let us consider the Zariski open subset ∂^* defined by the curve of genus g being smooth. Given any lift P of p satisfying Proposition 3.5, let us introduce the subset $\mathcal{H}_P \subset \partial^*$ consisting of nodal forms constructed by attaching the form $(\mathbb{P}^1, \frac{dz}{2i\pi z})$ with an abelian differential having a period equal to P after conveniently marking it. Since the image of P is a lattice, the set $\mathcal{H}_P$ is a copy of the Hurwitz space of ramified coverings of degree $\deg(P) := \frac{\Re P \cdot \Im P}{\mathrm{vol}(\mathbb{C}/\operatorname{Im}(P))}$ over the elliptic curve $\mathbb{C}/\operatorname{Im}(P)$. We claim that

$$\bar{L} \cap \partial^* = \bigcup_{P \text{ lift of } p} \mathcal{H}_P,$$

which explains the reason why L is not algebraic since a union of a countable number of disjoint algebraic subvarieties is not algebraic. The inclusion $\subset$ is merely the fact already mentioned that the limit of a sequence of elements of L is a nodal meromorphic differential whose period is equal to p in a certain marking. The inclusion $\supset$ comes from the fact that one can deform isoperiodically a form in any element of ∂^* in the smooth part, together with the fact that, by Theorem 1.5, the leaf L is exactly the set of elements in $\Omega\mathcal{M}_{g,2}$ having period p in a certain marking. ■

12. Appendix: The isoperiodic foliation on moduli spaces of stable meromorphic forms with simple poles

In this appendix, we review, adapt and extend [11, Section 3] to the case of moduli spaces of stable meromorphic forms with simple poles.

12.1. Augmented Teichmüller space and its stratification

Recall the definition of the augmented Teichmüller space from Section 6.1.

Definition 12.1. The augmented Teichmüller space $\overline{\mathcal{T}}_{g,n}$ is the set of all homotopically marked stable genus g curves with n marked points up to equivalence.

The Teichmüller space $\mathcal{T}_{g,n}$ is the subset of $\overline{\mathcal{T}}_{g,n}$ formed by curves without nodes. Its complement, denoted by $\partial\mathcal{T}_{g,n} = \overline{\mathcal{T}}_{g,n} \setminus \mathcal{T}_{g,n}$, is called the boundary. Given a subset $\mathcal{K}$ of $\overline{\mathcal{T}}_{g,n}$, its boundary is defined as $\partial\mathcal{K} := \mathcal{C} \cap \partial\mathcal{T}_{g,n}$.

Definition 12.2. A curve system $c = \sqcup c_i$ in $\Sigma_{g,n} = (\Sigma_g, p_1, \dots, p_n)$ is a disjoint collection of simple closed curves c_i on $\Sigma_g \setminus \{p_1, \dots, p_n\}$, none of which is isotopic to any other, to a point or to a cylinder in $\Sigma_g \setminus \{p_1, \dots, p_n\}$. To a curve system c we can associate the subset $B_c \subset \partial\mathcal{T}_g$ of the boundary consisting of homotopically marked stable curves topologically equivalent to a collapse $\Sigma_g \to \Sigma_g/c$ obtained by identifying each curve in c to a distinct point (note that we do not mean all curves in c to a point).

Given a curve system c, for each component Σ^i of $\Sigma_g \setminus c$, we define (Σ_{g_i}, P_i) to be the closed surface of genus g_i with a set of marked points P_i obtained by collapsing each boundary component of Σ^i to a (marked) point and keeping the marked points of Σ_g lying on Σ^i in Q_i. By using attaching maps, there is a natural identification

$$B_c \cong \Pi_i \mathcal{T}_{g_i,n_i}. \tag{30}$$

The boundary $\partial\mathcal{T}_{g,n}$ is the disjoint union of all boundary strata $\bigsqcup_c B_c$ where c varies in the set of non-empty curve systems. $\mathcal{T}_{g,n}$ corresponds to the empty curve system. Each stratum B_c has a topology and complex structure given by the bijection (30). In particular, it is connected. All the curves that appear in a stratum have the same number of separating and non-separating nodes. When all the nodes are non-separating, we say that the stratum is of compact type.

Given a simple closed curve $c' \subset \Sigma_{g,n}$, we denote by

$$D_{c'} = \bigsqcup_{c' \subset \mathbf{c}} B_{\mathbf{c}}$$

the union of all strata that collapse c' to a node.

12.2. Topology on $\overline{\mathcal{T}}_{g,n}$

For a detailed description of the topology, we refer to [3, pp. 485–493] and references therein.

The restriction of the given topology to $\mathcal{T}_{g,n}$ produces the so-called conformal topology (see [2]). Abikoff showed (in [1, Theorem 1]) that this topology is equivalent to the Teichmüller topology.

The restriction of the topology to the boundary set B_c corresponding to a curve system c is equivalent to the product topology obtained from (30).

The topology is not locally compact around any boundary point. Indeed, if U is a neighborhood of a point in B_c, the action of the Dehn twist $\Delta_a : \Sigma_g \to \Sigma_g$ around a simple closed curve $a \in c$ fixes all the points in U for which the marking collapses a to a point but has infinite orbits at any other point in U, see Remark 6.4. Therefore, there is no manifold structure in $\overline{\mathcal{T}}_{g,n}$ compatible with the given topology.

Definition 12.3. Given a curve system c, the distinguished neighborhood of the stratum B_c is the set

$$U_c = \bigsqcup_{c' \subset c} B_{c'}.$$

We think of U_c as the union of $\mathcal{T}_{g,n}$ with some of the boundary strata. These open sets will be useful to define and work with the complex structure on quotients of the augmented Teichmüller space.

12.3. Complex structure and Deligne–Mumford–Knudsen compactification

The mapping class group of $\Sigma_{g,n}$, i.e., the group $\mathrm{Mod}(\Sigma_{g,n})$ of isotopy classes of orientation preserving diffeomorphisms that fix each marked point, acts on $\overline{\mathcal{T}}_{g,n}$ by homeomorphisms that preserve the stratification and are holomorphic in restriction to any stratum. The action is defined by pre-composition on the marking. The quotient

$$\overline{\mathcal{M}}_{g,n} = \overline{\mathcal{T}}_{g,n} / \mathrm{Mod}(\Sigma_{g,n})$$

is a compact topological space. It can be endowed with a complex orbifold structure that extends the usual orbifold structure induced on $\mathcal{M}_{g,n}$ by the complex structure of $\mathcal{T}_{g,n}$ described in [2].

Consider a curve system c and define Γ_c as the abelian group generated by Dehn twists around the curves in c. Following [8], the quotient U_c/Γ_c is equivalent to a bounded domain in $\mathbb{C}^{3g-3}$. Under this equivalence, each stratum $B_{c'}$ associated with a simple closed curve $c' \subset c$ has an image contained in a regular divisor $D_{c'}$. These divisors intersect normally and their intersections define the other different strata: the stratum associated with $c' \subset c$ is the intersection of all the divisors associated with the simple curves in c'. The complement of this divisor is the stratum $B_{\emptyset} = \mathcal{T}_{g,n}$ formed by smooth marked curves.

The union of all natural maps $U_c/\Gamma_c \to \overline{\mathcal{M}}_{g,n}$ induce a system of (orbifold) charts with holomorphic transition maps on $\overline{\mathcal{M}}_{g,n}$.

The boundary $\partial\mathcal{M}_{g,n} = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ is a normal crossing divisor each of whose components correspond to the image of one of the D_c's of $\overline{\mathcal{T}}_{g,n}$ in the quotient $\overline{\mathcal{M}}_{g,n}$.

12.4. The subgroup of $\mathrm{Mod}(\Sigma_{g,n})$ acting trivially on punctured homology

Let $g, n \in \mathbb{N}$. Choose an oriented closed genus g reference surface Σ_g with a set of n ordered distinct marked points $P = (p_1, \dots, p_n)$. Denote by Σ_{g,n^*} the (possibly non-closed) surface $\Sigma_g \setminus P$. We will use the following notation for certain relative homology groups

$$H_1(\Sigma_{g,n^*}) = H_1(\Sigma_g \setminus P, \mathbb{Z})$$

that we call punctured homology groups. The mapping class group $\mathrm{Mod}(\Sigma_{g,n})$ induces an action on $H_1(\Sigma_{g,n^*})$ that fixes every element in the peripheral module, i.e., the module Π_n generated by cycles turning positively once around each puncture p_i for $i = 1, \dots, n$.

The subgroup of $\mathrm{Mod}(\Sigma_{g,n})$ that acts trivially on $H_1(\Sigma_{g,n^*})$ will be denoted by $\mathcal{I}(\Sigma_{g,n^*})$ and called the (punctured) Torelli group of Σ_{g,n^*}. For future reference, we explicit the associated exact sequence of groups

$$0 \to \mathcal{I}(\Sigma_{g,n^*}) \to \mathrm{Mod}(\Sigma_{g,n}) \to \mathrm{Aut}\big(H_1(\Sigma_{g,n^*}), \cdot\big) \tag{31}$$

where the product preserved in homology is the intersection product. The action of the Dehn twist Δ_γ around a simple closed curve γ in Σ_{g,n^*} in $H_1(\Sigma_{g,n^*})$ is the module morphism defined by

$$a \mapsto a + \big(a \cdot [\gamma]\big)[\gamma]. \tag{32}$$

The quotient denoted by

$$\overline{\mathcal{S}}_{g,n^*} := \overline{\mathcal{T}}_{g,n}/\mathcal{I}(\Sigma_{g,n^*})$$

is what we call the augmented Torelli space of Σ_{g,n^*}. A point in this space will be denoted by a triple $(C, Q, [f])$ where $[f]$ denotes the equivalence class of the homotopical collapse map f under the action of the corresponding Torelli group. To each such point, there corresponds an exact sequence

$$0 \to \ker[f]_* \to H_1(\Sigma_{g,n^*}) \xrightarrow{[f]_*} H_1(C \setminus Q, \mathbb{Z}) \to 0 \tag{33}$$

where $\ker[f]_*$ is the isotropic subgroup generated by the homology classes induced by the curves in the curve system c collapsed by f to the nodes of C.

The action of the Torelli group on augmented Teichmüller space preserves the stratification. The class of each stratum B_c in the quotient is characterized by the equivalence class $\mathbf{c}$ of curve system c under the action of the Torelli group and denoted by $B_{\mathbf{c}}$.

The open sets U_c defined at the level of augmented Teichmüller space induce open sets that we denote by $U_{\mathbf{c}}$ that are characterized by the Torelli class $\mathbf{c}$ of curve system c. The action in $U_{\mathbf{c}}$ of the group Γ_c generated by Dehn-twists around the curves in the curve system coincides with that of the group generated only by non-separating curves of the curve system. Indeed, the Dehn twists around separating curves of c define elements in $\mathcal{I}(\Sigma_{g,n^*})$ (even when the separating curve induces a non-trivial element in $H_1(\Sigma_{g,n^*})$). The quotient $U_{\mathbf{c}}/\Gamma_{\mathbf{c}}$ is a complex manifold. In particular, $U_{\mathbf{c}}$ is a complex manifold whenever all curves in $\mathbf{c}$ are separating.

12.5. The stratification of $\overline{\mathcal{S}}_{g,n^*}$ and its dual boundary complex

Lemma 12.4. *The stratification induced on $\overline{\mathcal{S}}_{g,n^*}$ by that of $\overline{\mathcal{T}}_{g,n}$ is a local abelian ramified covering of a normal crossing divisor and its dual boundary complex $\mathcal{C}(\overline{\mathcal{S}}_{g,n^*})$ is isomorphic to the quotient $\mathcal{C}_{g,n}/\mathcal{I}(\Sigma_{g,n^*})$ of the curve complex $\mathcal{C}_{g,n}$ of the surface Σ_g deprived of the n marked points.*

Proof. The stratification of the open set U_c is invariant under the action of the subgroup $\Gamma_c \cap \mathcal{I}(\Sigma_{g,n^*}) \subset \Gamma_c$ and a local abelian ramified cover of a normal crossing divisor. Therefore, the quotient on $U_c/(\Gamma_c \cap \mathcal{I}(\Sigma_{g,n^*}))$ has the same local property. On the other hand, as a consequence of the fact that any automorphism of a stable curve that acts trivially on homology is equivalent to some Dehn twist around the pinched curves,

$$U_c/\big(\Gamma_c \cap \mathcal{I}(\Sigma_{g,n^*})\big) \to U_c/\mathcal{I}(\Sigma_{g,n^*}) \subset \overline{\mathcal{S}}_{g,n^*}$$

is a local homeomorphism at every point of the stratum B_c. A connected component of stratum of $\overline{\mathcal{S}}_{g,n^*}$ corresponds to one orbit of connected components of a stratum of $\overline{\mathcal{T}}_{g,n}$ under the action of $\mathcal{I}(\Sigma_{g,n^*})$. The isomorphism of complexes then follows from the isomorphism $\mathcal{C}(\overline{\mathcal{T}}_{g,n}) \simeq \mathcal{C}_{g,n}$. ■

12.6. Moduli of stable forms with simple poles and their substratification

Let us denote by $\Omega\bar{\mathcal{M}}_{g,n} \to \bar{\mathcal{M}}_{g,n}$ the bundle whose fiber over a point (C, Q) is the vector space of meromorphic stable forms on the marked curve (C, Q) having at worst simple poles on the n distinct ordered points of Q, in other words, the space of meromorphic sections of the twisted line bundle $K_C(-(q_1 + \cdots + q_n))$. A point in $\Omega\bar{\mathcal{M}}_{g,n}$ will be denoted by (C, Q, ω). When $(\omega)_\infty = q_1 + \cdots + q_{n_1}$, we will omit Q and write (C, ω). In the next lemma, we analyze the domain and analytic properties of functions defined by integrating stable forms along certain paths.

Lemma 12.5. *Let c be a curve system in Σ_{g,n^*} and $U_c/\Gamma_c \to U \subset \bar{\mathcal{M}}_{g,n}$ a distinguished (orbifold) chart. Given γ a path in $\Sigma_g \setminus \{c, P\}$ (avoiding possible nodes and poles), the map $\Omega U_c \to \mathbb{C}$ defined by*

$$\big(C, Q, \{f\}, \omega\big) \mapsto \int_{f_*(\gamma)} \omega \in \mathbb{C},$$

where f is a representative in its isotopy class that collapses some curves of c, is well defined and invariant by the action of Γ_c. It induces a holomorphic map $\Omega(U_c/\Gamma_c) \to \mathbb{C}$.

Proof. The map is well defined on U_c/Γ_c because γ does not intersect any of the simple closed curves in the curve system c that is collapsed to the nodes of the curve. It is holomorphic around any point with finite value outside the boundary and bounded in the neighborhood of every boundary point with finite value. On the other hand, we know that the boundary forms a divisor. Hence, by Riemann's extension theorem, we deduce that there is a unique holomorphic extension to the boundary. ■

The boundary stratification of the bundle $\Omega\bar{\mathcal{M}}_{g,n}$ by the number of nodes is substratified by the orders of the form at zeros and nodes. The order of ω at a regular point $q \in C$ is defined to be the $\mathrm{ord}_q(f)$ where $\omega(z) = f(z)dz$ in a holomorphic coordinate $z : (C_1, q) \to \mathbb{C}$ around q. The order of ω at a node $q \in C$ is

$$\mathrm{ord}_q(\omega) = 2 + \mathrm{ord}_q(\omega_{|C_1}) + \mathrm{ord}_q(\omega_{|C_2})$$

where C_1 and C_2 are the branches of C at q. The order of the form at any point is clearly invariant by biholomorphism.

Definition 12.6. Two stable forms belong to the same substratum if there is a homeomorphism between the underlying curves with marked points preserving the order of the forms at each point (nodes to nodes, zero components to zero components, isolated zeros (resp. poles) are preserved with the same order).

Some strata admit a distinguished atlas produced by integration and a use of the Gauss–Manin connection to identify homology groups.

Lemma 12.7. *Let $(C, Q, \omega) \in \Omega\bar{\mathcal{M}}_{g,n}$ be a stable meromorphic form ω on a marked stable curve (C, Q) having simple poles on $P(\omega) = Q$, discrete zero set $Z(\omega)$ and zero* *residues at the nodes $N(C)$. Then, integration on cycles produces a well-defined map from a neighborhood R in its stratum of one forms*

$$R \to \operatorname{Hom}\big(H_1\big(C \setminus P(\omega), N(C) \cup Z(\omega); \mathbb{Z}\big); \mathbb{C}\big) \tag{34}$$

that is a homeomorphism onto its image.

Proof. The result is well known if C is smooth [23, 24]. There is a slight generalization in this case that will be useful for our purposes: the case where $C \setminus Q$ has also a set A of m marked points (that can coincide or not with the zeros of ω). Integration allows us to define a map on the neighborhood R of $(C, Q \cup A, \omega)$ in its stratum of $\Omega\mathcal{M}_{n+m}$, namely,

$$R \to \operatorname{Hom}\big(H_1(C \setminus Q, Z(\omega) \cup A, \mathbb{Z}); \mathbb{C}\big). \tag{35}$$

It is also a homeomorphism onto its image.

Let us assume C has some node. Consider the normalization $\widehat{C}$ marked by the pairs $N_j = \{n_j^+, n_j^-\}$ of points that produce the jth node of C. Denote that $N = \bigcup N_j$ and $C_1, \dots, C_k$ are the (smooth) components of the normalization $\widehat{C}$, $\omega_1, \dots, \omega_k$ are the restriction of ω to C_j, $Q_j = C_j \cap Q$ and $A_j = C_j \cap N$. Let R denote the neighborhood of (C, Q, ω) in its stratum and let R_j denote the neighborhood of the point $(C_j, Q_j \cup A_j, \omega_j)$ in its stratum. Up to reducing R, we can construct a homeomorphism

$$R_1 \times \dots \times R_k \to R$$

by applying the rule of attaching a map given by that of C. Since each $(C_j, Q_j \cup A_j, \omega_j)$ is a non-zero form on a smooth compact curve with marked points, we can apply the lemma in the case of a smooth curve with marked points to deduce that, up to reducing the size of the R_j's, we have a homeomorphism

$$R_1 \times \dots \times R_k \simeq \bigoplus_{j=1}^{k} \operatorname{Hom}\big(H_1(C_j \setminus P(\omega_j), Z(\omega_j) \cup A_j, \mathbb{Z}); \mathbb{C}\big).$$

The latter is isomorphic to

$$\operatorname{Hom}\left(\left(\bigoplus_{j=1}^{k} H_1\big(C_j \setminus P(\omega_j), Z(\omega_j) \cup A_j, \mathbb{Z}\big)\right); \mathbb{C}\right).$$

To finish the proof, we claim that there exists a natural isomorphism

$$\bigoplus_{j=1}^{k} H_1\big(C_j \setminus P(\omega_j), Z(\omega_j) \cup A_j, \mathbb{Z}\big) \simeq H_1\big(C \setminus P(\omega), N(C) \cup Z(\omega), \mathbb{Z}\big). \tag{36}$$

We sketch the last equivalence in the case of one node and leave the recurrence argument for the reader. Suppose that C has one node N_1 and its normalization one or two components C_j's. Let A_1 be the marked points in the normalization corresponding to N_1. The relative homology of the pair $((\sqcup C_j \setminus P(\omega_j))/A_1, \{A_1\} \cup Z(\omega))$ is isomorphic to the sum of relative homologies of the left-hand side of (36). On the other hand, the pair is homeomorphic to the $(C \setminus P(\omega), N(C) \cup Z(\omega))$. ■

12.7. Homology coverings of bundles of stable forms and period map

The pullback of the bundle $\Omega\overline{\mathcal{M}}_{g,n}$ by the (branched) cover $\overline{\mathcal{S}}_{g,n^*} \to \overline{\mathcal{M}}_{g,n}$ will be denoted as

$$\Omega\overline{\mathcal{S}}_{g,n^*} \to \overline{\mathcal{S}}_{g,n^*}.$$

A point in $\Omega\overline{\mathcal{S}}_{g,n^*}$ will be denoted by $(C, Q, [f], \omega)$. We want to describe the set of forms in this space for which all integrals on classes of $H_1(\Sigma_{g,n^*})$ are well-defined complex numbers that coincide with those of some form on a *smooth* curve. A problem that arises is that stable forms can have non-zero residues at nodes, and this does not allow us to integrate any path passing through the node. The Mayer–Vietoris exact sequence can be used to overcome this difficulty in the case of separating nodes. Indeed, consider a family of separating curves $c_1, \dots, c_k$ of some curve system c and write $\Sigma_g \setminus \{c_1 \cup \dots \cup c_k\} = \Sigma^{(1)} \sqcup \dots \sqcup \Sigma^{(\nu)}$, where each $\Sigma^{(i)}$ is connected and non-empty. Each $\Sigma^{(i)}$ can be identified with some Σ_{g_i,n_i^*}, a genus g_i compact surface with n_i punctures (we think of a boundary component as a point). For each c_i, we consider two peripheral curves γ_i^-, γ_i^+ on each of its sides. Then

$$H_1(\Sigma_{g,n^*}) \cong \frac{H_1(\Sigma_{g_1,n_1^*}) \oplus \dots \oplus H_1(\Sigma_{g_\nu,n_\nu^*})}{\langle [\gamma_i^+] + [\gamma_{i+1}^-] : i = 1, \dots, k-1 \rangle} \tag{37}$$

where the class $[\gamma_i^-]$ (resp. $[\gamma_i^+]$) is in the homology group of the (punctured) component to which it belongs.

We deduce that every homology class in $H_1(\Sigma_{g,n^*})$ can be written as a sum of classes disjoint from the separating nodes. However, this is not true for non-separating nodes. To be able to integrate a stable form on any class in $H_1(\Sigma_{g,n^*})$, we need to suppose that the residues are zero at non-separating nodes. The union of strata with this property will be useful for our purposes:

$$\Omega_0\overline{\mathcal{S}}_{g,n^*} = \left\{ (C, Q, [f], \omega) \in \Omega\overline{\mathcal{S}}_{g,n^*} : \mathrm{Res}_q(\omega) = 0 \ \forall \text{ non-separating node } q \text{ of } C \right\}.$$

Unfortunately, this set is neither open (its interior is formed by stable forms on curves of compact type) nor closed (it is dense) in $\Omega\overline{\mathcal{S}}_{g,n^*}$. However, it is the set where forms can be integrated in *all* classes of $H_1(\Sigma_{g,n^*})$

Definition 12.8. Given $(C, Q, [f], \omega) \in \Omega_0\overline{\mathcal{S}}_{g,n^*}$, we define an element

$$p = \mathrm{Per}\left(C, Q, [f], \omega\right) \in \mathrm{Hom}\left(H_1(\Sigma_{g,n^*}); \mathbb{C}\right)$$

called the period homomorphism of $(C, Q, [f], \omega)$ as follows:

- Consider the decomposition as in (37) using all separating nodes of C.
- Define the homomorphism

$$\bigoplus_{i=1}^{\nu} H_1(\Sigma_{g_i,n_i^*}) \ni a \mapsto \int_{[f]_*(a)} \omega \in \mathbb{C}.$$

The opposite residue condition of stable forms at nodes implies that the homomorphism defines a homomorphism p on the quotient on the right-hand side of (37) that can be thought of as an element in $\mathrm{Hom}(H_1(\Sigma_{g,n^*}); \mathbb{C})$.

Definition 12.9. The period map on $\Omega_0\overline{\mathcal{S}}_{g,n^*}$ is the map

$$\mathcal{P}\mathrm{er}_{g,n^*} : \Omega_0\overline{\mathcal{S}}_{g,n^*} \to \mathrm{Hom}\left(H_1(\Sigma_{g,n^*}); \mathbb{C}\right)$$

sending each point to its period homomorphism.

By the decomposition in (37), and Lemma 12.5, the period map is holomorphic in the neighborhood of a marked stable form on a curve of *compact type*. Moreover, if the residues at the nodes are all zero and the zeros of the form are isolated, the restriction of the period map to the stratum, written in the coordinates of Lemma 12.7, is just a linear projection. This implies that the period map is submersive in the neighborhood of such a point (even in restriction to the stratum or to any smooth manifold containing it). In particular, the local fiber at such a point is a regular holomorphic manifold transverse to any boundary component passing thorough it.

This description of the local fibers of the period map is also true in more generality (possibly up to a branched cover) for forms with isolated zeros.

Theorem 12.10. *The local fiber of the period map in $\Omega_0\overline{\mathcal{S}}_{g,n^*}$ at a point $(C, Q, [f], \omega) \in \Omega_0\overline{\mathcal{S}}_{g,n^*}$ with isolated zeros projects to the orbifold chart of $\Omega\overline{\mathcal{M}}_{g,n}$ as a complex manifold transverse to all boundary divisors through the point. Therefore, it is an abelian ramified cover of a normal crossing divisor in $(\mathbb{C}^{2g+n-3}, 0)$ having precisely one component of codimension one in each component of codimension one of the ambient space through the point.*

Proof. The local period fiber on $\Omega_0\overline{\mathcal{S}}_{g,n^*}$ at $(C, Q, [f], \omega) \in \Omega_0\overline{\mathcal{S}}_{g,n^*}$ can be lifted to a local period fiber $L \subset \Omega_0\overline{\overline{\mathcal{T}}}_{g,n}$ of the map

$$\Omega_0\overline{\overline{\mathcal{T}}}_{g,n} \to H^1(\Sigma_g, q_1, \ldots, q_n, \mathbb{C}) \tag{38}$$

that associates with any stable form with zero residues at non-separating nodes its period homomorphism. If the lift of $(C, Q, [f], \omega)$ belongs to the stratum ΩB_c, the covering group is the subgroup $\Gamma_c \cap \mathcal{I}_g \subset \Gamma_c$. Hence, to prove the result, we just need to prove that L is locally an abelian ramified cover over a normal crossing divisor.

Let U_c denote the distinguished neighborhood of B_c. It suffices in fact to show that at the level of the quotient $\Omega_0(U_c/\Gamma_c)$, the period fiber is a holomorphic manifold transverse to every boundary component passing through the point. Thanks to the normal crossing condition of the ambient space, this is guaranteed if the period map is submersive in restriction to the (regular) stratum of the normal crossing divisor where the point belongs to. In fact, we will not use the boundary stratum but a smaller regular submanifold, consisting of the substratum through the point $(C, Q, \{f\}, \omega) \in L \subset \Omega_0\overline{\overline{\mathcal{T}}}_{g,n}$.

Next we prove the analyticity of $L/\Gamma_{\mathbf{c}}$ in two steps corresponding to the dual exact sequence of (33), namely,

$$0 \to \operatorname{Hom}\big(H_1(C \setminus Q), \mathbb{C}\big) \to \operatorname{Hom}\big(H_1(\Sigma_{g,n^*}), \mathbb{C}\big) \to \operatorname{Hom}\big(\ker[f]_*, \mathbb{C}\big) \to 0.$$

Remark that by definition, the intersection of any pair of elements of $\ker[f]_*$ is zero.

(1) Integrating on the (peripheral) curves of $\mathbf{c}$ and using Riemann–Roch's theorem as in [11, Section 3.16], we show that there is a well-defined map

$$\operatorname{NRes} : \Omega(U_{\mathbf{c}}/\Gamma_{\mathbf{c}}) \to \operatorname{Hom}\big(\ker[f]_*; \mathbb{C}\big)$$

that is holomorphic thanks to Lemma 12.5. It extends the residue map at the nodes of C for forms in $\Omega(C)$. The same application of Riemann–Roch's theorem allows us to show that the (linear) restriction of NRes to $\Omega(C)$ is surjective. Denote by $p = \mathcal{P}\mathrm{er}(C, Q, [f], \omega)$ the period homomorphism. By construction, $\operatorname{NRes}(C, Q, [f], \omega) = p_{|\ker[f]_*}$. The base point belongs to the smooth complex manifold $\operatorname{NRes}^{-1}(p_{|\ker[f]_*}) \subset \Omega_0(U_{\mathbf{c}}/\Gamma_{\mathbf{c}})$. Denote by $\widetilde{\operatorname{NRes}^{-1}}(p_{|\ker[f]_*})$ the lift of $\operatorname{NRes}^{-1}(p_{|\ker[f]_*})$ to ΩU_c. It is contained in $\Omega_0 U_c$, and by construction, it is a union of fibers of the period map.

(2) The restriction of the period map to $\widetilde{\operatorname{NRes}^{-1}}(p_{|\ker[f]_*})$ is well defined. It is $\Gamma_{\mathbf{c}}$-invariant because for every curve c_i of the curve system c, we have that either its intersection with any other class is zero (if c_i is separating) or the period along c_i of a form in $\widetilde{\operatorname{NRes}^{-1}}(p_{|\ker[f]_*})$ is zero. Therefore, it induces a holomorphic map

$$h : \operatorname{NRes}^{-1}(p_{|\ker[f]_*}) \to \operatorname{Hom}_{p_{|\ker[f]_*}}\big(H_1(\Sigma_{g,n^*}), \mathbb{C}\big) \tag{39}$$

with image in the set $\operatorname{Hom}_{p_{|\ker[f]_*}}(H_1(\Sigma_{g,n^*}), \mathbb{C})$ of homomorphisms that extend $p_{|\ker[f]_*}$ to a homomorphism on $H_1(\Sigma_{g,n^*})$. The fibers of h are analytic sets. Locally at the base point, the fiber is $L/\Gamma_{\mathbf{c}}$.

Next we prove that under the hypothesis of isolated zeros of ω, the set $L/\Gamma_{\mathbf{c}}$ is smooth and transverse to each boundary component. This is a consequence of the only stronger lemma.

Lemma 12.11. *The map h defined by (39) restricted to the intersection with the substratum at a point with isolated zeros is submersive.*

Hence, the local fiber H of the map (39) at the base point is a smooth complex manifold transverse to every smooth manifold containing the substratum of the base point. In particular, H is transverse to each irreducible boundary component passing through the base point. We deduce that ∂H is a normal crossing divisor in H, and moreover ∂H, H and $H \setminus \partial H$ are connected.

Proof of Lemma 12.11. Let R denote a neighborhood in its stratum of the form $(C, Q, \omega) \in \Omega\bar{\mathcal{M}}_{g,n}$ that has zero residues at the non-separating nodes. Recall that if a composition of maps is submersive, then the last map of the composition is submersive as well. We want to show that

$$h_{|R} : R \cap \mathrm{NRes}^{-1}(p_{|\ker[f]_*}) \to \mathrm{Hom}_{p_{|\ker[f]_*}}\big(H_1(\Sigma_{g,n^*});\mathbb{C}\big)$$

is submersive whenever ω has isolated zeros.

If all nodes of C have zero residue for ω, we can apply Lemma 12.7 and find coordinates in R where the map $h_{|R}$ is written as the linear projection

$$\mathrm{Hom}\big(H_1\big(C \setminus Q, N(C) \cup Z(\omega);\mathbb{Z}\big);\mathbb{C}\big) \to \mathrm{Hom}\big(H_1(C \setminus Q, ;\mathbb{Z});\mathbb{C}\big)$$

induced by inclusion $H_1(C \setminus Q, \mathbb{Z}) \to H_1(C \setminus Q, N(C) \cup Z(\omega), \mathbb{Z})$. Therefore, it is submersive.

Suppose there is some node with non-zero residue. Let $\mathrm{PN}(C) = \{N_1, \dots, N_{k-1}\}$ denote the nodes of C with non-zero residue for ω (i.e., polar nodes). All of them are separating by hypothesis. Write $C \setminus \mathrm{PN}(C)$ as a disjoint union $\bigsqcup_{j=1}^{\nu}(C_j \setminus \mathrm{PN}(C_j))$ where C_j is a stable curve where the restricted form $\omega_j = \omega_{|C_j}$ has only zero residues at the nodes and simple poles on the set $Q_j = (Q \cup \mathrm{PN}(C)) \cap C_j$. Let R_j be the stratum of the form (C_j, Q_j, ω_j) in $\Omega\overline{\mathcal{M}}_{g_j,n_j}$, where g_j is the genus of C_j and n_j the cardinality of Q_j. Consider the subset $R'_j \subset R_j$ of forms having the same residues at the poles $\mathrm{PN}(C_j)$ as ω_j has.

The attaching rule to obtain C from the C_j's can be used to define a holomorphic map

$$\varphi : R'_1 \times \cdots \times R'_k \to R \cap \mathrm{NRes}^{-1}(p_{|\ker[f]_*}).$$

In its image, we find only forms in the stratum having the same residues at the nodes corresponding to $\mathrm{PN}(C)$ as ω has. We claim that

$$h \circ \varphi : R'_1 \times \cdots \times R'_k \to \mathrm{Hom}_{p_{|\ker[f]_*}}\big(H_1(\Sigma_{g,n^*});\mathbb{C}\big)$$

is submersive. Recall from Lemma 12.7 that in each R_j, we have holomorphic coordinates with values in $\mathrm{Hom}(H_1(C_j \setminus P(\omega_j), N(C_j) \cup Z(\omega_j), \mathbb{Z});\mathbb{C})$. In the given coordinates, the set R'_j corresponds to a codimension one or two linear subspace, corresponding to fixing the residues at the poles $\mathrm{PN}(C_j)$. As before, the projection

$$\pi_j : \mathrm{Hom}\big(H_1\big(C_j \setminus P(\omega_j), N(C_j) \cup Z(\omega_j), \mathbb{Z}\big);\mathbb{C}\big) \to \mathrm{Hom}\big(H_1\big(C_j \setminus P(\omega_j), \mathbb{Z}\big);\mathbb{C}\big) \quad (40)$$

is linear, surjective. The restriction of π_j to R'_j is submersive onto the subspace

$$\mathrm{Hom}_{\mathrm{PN}(\omega_j)}\big(H_1\big(C_j \setminus P(\omega_j), \mathbb{Z}\big);\mathbb{C}\big)$$

of homomorphisms whose value around the peripherals of $\mathrm{PN}(C_j)$ is the same as that of ω_j. The projections allow us to define a map on the cartesian product

$$R'_1 \times \cdots \times R'_k \to \bigoplus_{j=1}^{k} \mathrm{Hom}_{\mathrm{PN}(\omega_j)}\big(H_1\big(C_j \setminus P(\omega_j), \mathbb{Z}\big);\mathbb{C}\big) \quad (41)$$

that is submersive. By using a similar argument as in equation (37), but only with the separating nodes of non-zero residue, we deduce that the target space of the map (41) is homomorphic to $\mathrm{Hom}_{p_{|\ker[f]_*}}(H_1(\Sigma_{g,n^*});\mathbb{C})$. In this expression, the map $h \circ \varphi$ is the map (41), so we have proven that $h \circ \varphi$ is submersive. Therefore, so are $h_{|R}$ and h. ■

This finishes the proof of Theorem 12.10. ■

Corollary 12.12. *Let $F \subset \Omega\mathcal{S}_{g,n^*}$ be a fiber of* Per_{g,n^*} *and $\overline{F}$ its closure in $\Omega_0^*\overline{\mathcal{S}}_{g,n^*}$. Then F is connected if and only if $\overline{F}$ is connected.*

Proof. All the points in $\overline{F}$ have the same value for $\mathcal{P}\mathrm{er}_{g,n^*}$. Suppose F is connected. Thanks to Theorem 12.10, the connected component of $\overline{F}$ at any boundary point contains points of F. Hence, $\overline{F}$ is connected. For the converse, suppose $\overline{F}$ is connected. It is also path connected. Let $t \mapsto \gamma(t)$ be a path in $\overline{F}$ joining a pair of points of F. By Theorem12.10, there are points of F in any neighborhood of any of the points $\gamma(t)$. All of them lie in the same component of F. Hence, the starting point and endpoint lie in the same connected component of F. This shows that F is path connected. ■

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References

[1] W. Abikoff, Degenerating families of Riemann surfaces. *Ann. of Math. (2)* **105** (1977), no. 1, 29–44 Zbl 0347.32010 MR 0442293

[2] L. V. Ahlfors, The complex analytic structure of the space of closed Riemann surfaces. In *Analytic functions*, pp. 45–66, Princeton Math. Ser 24, Princeton University Press, Princeton, NJ, 1960 Zbl 0100.28903 MR 0124486

[3] E. Arbarello, M. Cornalba, and P. A. Griffiths, *Geometry of algebraic curves. Volume II*. Grundlehren Math. Wiss. 268, Springer, Heidelberg, 2011 Zbl 1235.14002 MR 2807457

[4] V. I. Arnold, A remark on the branching of hyperelliptic integrals as functions of the parameters. *Funkcional. Anal. i Priložen.* **2** (1968), no. 3, 1–3 Zbl 0192.58201 MR 0244266

[5] L. Arzhakova, G. Calsamiglia, and B. Deroin, Isoperiodic meromorphic forms with at least three simple poles. 2025, arXiv:2508.00099v1

[6] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Möller, Compactification of strata of Abelian differentials. *Duke Math. J.* **167** (2018), no. 12, 2347–2416 Zbl 1403.14058 MR 3848392

[7] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky, and M. Möller, The moduli space of multi-scale differentials. [v1] 2019, [v2] 2022, arXiv:1910.13492v2

[8] L. Bers, Finite-dimensional Teichmüller spaces and generalizations. *Bull. Amer. Math. Soc. (N.S.)* **5** (1981), no. 2, 131–172 Zbl 0485.30002 MR 0621883

[9] C. Boissy, Connected components of the strata of the moduli space of meromorphic differentials. *Comment. Math. Helv.* **90** (2015), no. 2, 255–286 Zbl 1323.30060 MR 3351745

[10] G. Calsamiglia, B. Deroin, and S. Francaviglia, Branched projective structures with Fuchsian holonomy. *Geom. Topol.* **18** (2014), no. 1, 379–446 Zbl 1286.30031 MR 3159165

[11] G. Calsamiglia, B. Deroin, and S. Francaviglia, A transfer principle: from periods to isoperiodic foliations. *Geom. Funct. Anal.* **33** (2023), no. 1, 57–169 Zbl 1511.14047 MR 4561148

[12] C. Camacho and B. Azevedo Scárdua, Complex foliations with algebraic limit sets. *Astérisque* **261** (2000), 57–88 Zbl 0947.32022 MR 1755437

[13] C. Camacho, A. Lins Neto, and P. Sad, Foliations with algebraic limit sets. *Ann. of Math. (2)* **136** (1992), no. 2, 429–446 Zbl 0769.57017 MR 1185124

[14] J. Chaika and B. Weiss, On the ergodic theory of the real Rel foliation. *Forum Math. Pi* **12** (2024), article no. e7 Zbl 07834161 MR 4726499

[15] D. Chen and G. Faraco, Period realization of meromorphic differentials with prescribed invariants. *Forum Math. Sigma* **12** (2024), article no. e90 Zbl 1553.30040 MR 4819727

[16] S. Chenakkod, G. Faraco, and S. Gupta, Translation surfaces and periods of meromorphic differentials. *Proc. Lond. Math. Soc. (3)* **124** (2022), no. 4, 478–557 Zbl 1528.30014 MR 4413502

[17] G. Cousin, Algebraic isomonodromic deformations of logarithmic connections on the Riemann sphere and finite braid group orbits on character varieties. *Math. Ann.* **367** (2017), no. 3-4, 965–1005 Zbl 1360.14028 MR 3623216

[18] P. Deligne and D. Mumford, The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.* (1969), no. 36, 75–109 Zbl 0181.48803 MR 0262240

[19] G. Faraco, G. Tahar, and Y. Zhang, Isoperiodic foliation of the stratum $\mathcal{H}(1,1,-2)$. [v1] 2023, [v2] 2024, arXiv:2305.06761v2

[20] B. Farb and D. Margalit, *A primer on mapping class groups*. Princeton Math. Ser. 49, Princeton University Press, Princeton, NJ, 2012 Zbl 1245.57002 MR 2850125

[21] S. Filip, Splitting mixed Hodge structures over affine invariant manifolds. *Ann. of Math. (2)* **183** (2016), no. 2, 681–713 Zbl 1342.14015 MR 3450485

[22] S. Ghazouani, Mapping class group dynamics and the holonomy of branched affine structures. *Math. Z.* **289** (2018), no. 1–2, 1–23 Zbl 1430.37032 MR 3803779

[23] S. Grushevsky and I. Krichever, The universal Whitham hierarchy and the geometry of the moduli space of pointed Riemann surfaces. In *Surveys in differential geometry. Vol. XIV. Geometry of Riemann surfaces and their moduli spaces*, pp. 111–129, Surv. Differ. Geom. 14, International Press, Somerville, MA, 2009 Zbl 1213.14055 MR 2655324

[24] S. Grushevsky and I. Krichever, Real-normalized differentials and the elliptic Calogero-Moser system. [v1] 2011, [v2] 2018, arXiv:1108.4211v2

[25] U. Hamenstädt, Ergodicity of the absolute period foliation. *Israel J. Math.* **225** (2018), no. 2, 661–680 Zbl 1398.32016 MR 3805661

[26] J. Harris and I. Morrison, *Moduli of curves*. Grad. Texts in Math. 187, Springer, New York, 1998 Zbl 0913.14005 MR 1631825

[27] O. Haupt, Ein Satz über die Abelschen Integrale 1. Gattung. *Math. Z.* **6** (1920), no. 3-4, 219–237 Zbl 47.0351.02 MR 1544406

[28] M. Kapovich, Periods of abelian differentials and dynamics. In *Dynamics: topology and numbers*, pp. 297–315, Contemp. Math. 744, American Mathematical Society, Providence, RI, 2020 Zbl 1436.37047 MR 4062570

[29] F. F. Knudsen, The projectivity of the moduli space of stable curves. II. The stacks $M_{g,n}$. *Math. Scand.* **52** (1983), no. 2, 161–199 Zbl 0544.14020 MR 0702953

[30] F. F. Knudsen, The projectivity of the moduli space of stable curves. III. The line bundles on $M_{g,n}$, and a proof of the projectivity of $\overline{M}_{g,n}$ in characteristic 0. *Math. Scand.* **52** (1983), no. 2, 200–212 Zbl 0544.14021 MR 0702954

[31] F. F. Knudsen and D. Mumford, The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div". *Math. Scand.* **39** (1976), no. 1, 19–55 Zbl 0343.14008 MR 0437541

[32] I. M. Krichever, Real normalized differentials and Arbarello's conjecture. *Funct. Anal. Appl.* **46** (2012), no. 2, 110–120 Zbl 1278.30041 MR 2978059

[33] I. M. Krichever, S. Lando, and A. Skripchenko, Real-normalized differentials with a single order 2 pole. *Lett. Math. Phys.* **111** (2021), no. 2, article no. 36 Zbl 1462.14032 MR 4232857

[34] C. T. McMullen, Moduli spaces of isoperiodic forms on Riemann surfaces. *Duke Math. J.* **163** (2014), no. 12, 2271–2323 Zbl 1371.30037 MR 3263035

[35] M. Möller, Linear manifolds in the moduli space of one-forms. *Duke Math. J.* **144** (2008), no. 3, 447–487 Zbl 1148.32007 MR 2444303

[36] K. Strebel, *Quadratic differentials*. Ergeb. Math. Grenzgeb. (3) 5, Springer, Berlin, 1984 Zbl 0547.30001 MR 0743423

[37] G. Tahar, Counting saddle connections in flat surfaces with poles of higher order. *Geom. Dedicata* **196** (2018), 145–186 Zbl 1403.32003 MR 3853632

[38] K. Winsor, Dynamics of the absolute period foliation of a stratum of holomorphic 1-forms. 2021, arXiv:2109.12669v1

[39] K. Winsor, Saturated orbit closures in the Hodge bundle. *Int. Math. Res. Not. IMRN* **2024** (2024), no. 6, 4710–4737 MR 4721654

[40] F. Ygouf, Isoperiodic dynamics in rank 1 affine invariant orbifolds. *Int. Math. Res. Not. IMRN* **2022** (2022), no. 24, 19854–19861 Zbl 1510.37058 MR 4523261

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