

Crystalline condition for A_{inf} -cohomology and ramification bounds

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Abstract. For a prime p and a smooth proper p -adic formal scheme $\mathcal{X}$ over $\mathcal{O}_K$ where K is a p -adic field, we study a series of conditions (Cr_s) , $s \geq 0$ that partially control the G_K -action on the image of the associated Breuil–Kisin prismatic cohomology $\text{R}\Gamma_{\Delta}(\mathcal{X}/\mathfrak{S})$ inside the A_{inf} -prismatic cohomology $\text{R}\Gamma_{\Delta}(\mathcal{X}_{A_{\text{inf}}}/A_{\text{inf}})$. The condition (Cr_0) is a crystallinity criterion for a Breuil–Kisin–Fargues G_K -module of Gee and Liu, and leads to a proof of crystallinity of $H_{\text{ét}}^i(\mathcal{X}_{\bar{\eta}}, \mathbb{Q}_p)$ that avoids the crystalline comparison. Using the higher conditions (Cr_s) , we are able to adapt the strategy of Caruso and Liu to establish ramification bounds for the mod p representations $H_{\text{ét}}^i(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}/p^e\mathbb{Z})$, for arbitrarily large e and i . This extends and/or improves existing bounds in various situations.

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1. Introduction

Let k be a perfect field of characteristic $p > 0$ and $K' = W(k)[1/p]$ the associated absolutely unramified field. Let K/K' be a totally ramified finite extension with ramification index e , and denote by G_K its absolute Galois group. The goal of the present paper is to provide new bounds for ramification of the mod p representations of G_K that arise as the étale cohomology groups $H_{\text{ét}}^i(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}/p\mathbb{Z})$ in terms of p , i and e , where $\mathcal{X}$ is a smooth proper p -adic formal scheme over $\mathcal{O}_K$ (and $\mathcal{X}_{\bar{\eta}}$ is the geometric adic generic fiber). Concretely, let us denote by G_K^{μ} the μ -th ramification group of G_K in the upper numbering (in the standard convention, e.g. [31]) and $G_K^{(\mu)} = G_K^{\mu-1}$. The main result is as follows.

Theorem 1.1 (Theorem 5.15). *Set*

$$\alpha = \left\lfloor \log_p \left(\max \left\{ \frac{ip}{p-1}, \frac{(i-1)e}{p-1} \right\} \right) \right\rfloor + 1, \quad \beta = \frac{1}{p^\alpha} \left(\frac{iep}{p-1} - 1 \right).$$

Then:

- (1) *The group $G_K^{(\mu)}$ acts trivially on $H_{\text{ét}}^i(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}/p\mathbb{Z})$ when $\mu > 1 + e\alpha + \max\{\beta, \frac{e}{p-1}\}$.*
- (2) *Denote by L the field $\bar{K}^H$ where H is the kernel of the G_K -representation ρ given by $H_{\text{ét}}^i(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}/p\mathbb{Z})$. Then*

$$v_K(\mathcal{D}_{L/K}) < 1 + e\alpha + \beta,$$

where $\mathcal{D}_{L/K}$ denotes the different of the extension L/K and v_K denotes the additive valuation on K normalized so that $v_K(K^\times) = \mathbb{Z}$.

In particular, unlike in previous results of this type (discussed below), there are no restrictions on the size of e and i with respect to p .

Remark 1.2. As the constants α, β appearing in Theorem 1.1 are quite complicated, let us draw some non-optimal, but more tractable consequences. The group $G_K^{(\mu)}$ acts trivially on $H_{\text{ét}}^i(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}/p\mathbb{Z})$ when one of the following occurs:

- (1) $e \leq p$ and $\mu > 1 + e(\lfloor \log_p(\frac{ip}{p-1}) \rfloor + 1) + e$,
- (2) $e > p$ and $\mu > 1 + e(\lfloor \log_p(\frac{ie}{p-1}) \rfloor + 1) + p$,¹
- (3) $i = 1$ (e, p are arbitrary) and $\mu > 1 + e(1 + \frac{1}{p-1})$.

Let us briefly summarize the history of related results. Questions of this type originate in Fontaine's paper [15], where he proved that for a finite flat commutative group scheme Γ over $\mathcal{O}_K$ that is annihilated by p^n , $G_K^{(\mu)}$ acts trivially on $\Gamma(\bar{K})$ when $\mu > e(n + 1/(p-1))$; this is a key step in his proof that there are no non-trivial abelian schemes over $\mathbb{Z}$. In the same paper, Fontaine conjectured that general p^n -torsion cohomology would follow the same pattern: given a proper smooth variety X over K with good reduction, $G_K^{(\mu)}$ should act trivially on $H_{\text{ét}}^i(X_{\bar{K}}, \mathbb{Z}/p^n\mathbb{Z})$ when $\mu > e(n + i/(p-1))$.

This conjecture has been partially proved by Fontaine himself [17] in the case when $e = n = 1, i < p-1$ and by Abrashkin ([1]; see also [2]) when $e = 1, i < p-1$ and n is arbitrary. Both results make use of Fontaine–Laffaille modules (introduced in [18]), which parametrize quotients of pairs of G_K -stable lattices in crystalline representations with Hodge–Tate weights in $[0, i]$ (such as $H_{\text{ét}}^i(X_{\bar{K}}, \mathbb{Q}_p)^\vee$). The (duals of the) representations $H_{\text{ét}}^i(X_{\bar{K}}, \mathbb{Z}/p^n\mathbb{Z})$ are included among these thanks to a comparison theorem of Fontaine–Messing [19]. Similarly to the original application, these ramification bounds lead to a scarcity result for existence of smooth proper $\mathbb{Z}$ -schemes.

¹Strictly speaking, to obtain this precise form one has to replace $(i-1)e$ in α from Theorem 1.1 by ie , and modify β appropriately; one can show that such form of Theorem 1.1 is still valid.

Various extensions to the semistable case subsequently followed. Under the assumption $i < p - 1$ (and arbitrary e), Hattori proved in [21] a ramification bound for p^n -torsion quotients of lattices in semistable representations with Hodge–Tate weights in the range $[0, i]$, using (a variant of) Breuil’s filtered (ϕ_r, N) -modules. Thanks to a comparison result between log-crystalline and étale cohomology by Caruso [8], this results in a ramification bound for $H_{\text{ét}}^i(X_{\bar{K}}, \mathbb{Z}/p^n\mathbb{Z})$ when X is proper with semistable reduction, assuming that $ie < p - 1$ when $n = 1$ and $(i + 1)e < p - 1$ when $n \geq 2$.²

These results were further extended by Caruso and Liu in [10] for all p^n -torsion quotients of pairs of semistable lattices with Hodge–Tate weights in $[0, i]$, without any restriction on i or e . The proof uses the theory of $(\varphi, \hat{G})$ -modules [26], which are objects suitable for description of lattices in semistable representations. Roughly speaking, a $(\varphi, \hat{G})$ -module consists of a Breuil–Kisin module M and the datum of an action of the group $\hat{G} = \text{Gal}(K(\mu_{p^\infty}, \pi^{1/p^\infty})/K)$ on $\hat{M} = M \otimes_{\mathcal{O}_K} \hat{\mathcal{R}}$ where $\hat{\mathcal{R}}$ is a suitable subring of Fontaine’s period ring $A_{\text{inf}} = W(\mathcal{O}_{\mathbb{C}_K^b})$ (and $\pi \in K$ is a fixed choice of a uniformizer). However, an obstacle to applying the results of [10] to the torsion étale cohomology $H_{\text{ét}}^i(X_{\bar{K}}, \mathbb{Z}/p\mathbb{Z})$ is that it is not clear when (duals of) such representations come as a quotient of two semistable lattices with Hodge–Tate weights in $[0, i]$. This is indeed the case in the situation when $e = 1$, $i < p - 1$ and X has good reduction by the aforementioned Fontaine–Messing theorem, and it was also shown in the case $i = 1$ (no restriction on e, p) for X with semistable reduction by Emerton and Gee in [13], but in general the question seems open.

Nevertheless, the idea of the proof of Theorem 1.1 is to follow the general strategy of Caruso and Liu. While one does not necessarily have semistable lattices and the associated $(\varphi, \hat{G})$ -modules to work with, a suitable replacement comes from the recently developed cohomology theories of Bhatt–Morrow–Scholze and Bhatt–Scholze [4–6]. Concretely, to a smooth p -adic formal scheme $\mathcal{X}$ one can associate the “ p^n -torsion prismatic cohomologies”

$$\begin{aligned} \text{R}\Gamma_{\Delta, n}(\mathcal{X}/\mathcal{O}) &= \text{R}\Gamma_{\Delta}(\mathcal{X}/\mathcal{O}) \overset{\text{L}}{\otimes} \mathbb{Z}/p^n\mathbb{Z}, \\ \text{R}\Gamma_{\Delta, n}(\mathcal{X}_{A_{\text{inf}}}/A_{\text{inf}}) &= \text{R}\Gamma_{\Delta}(\mathcal{X}_{A_{\text{inf}}}/A_{\text{inf}}) \overset{\text{L}}{\otimes} \mathbb{Z}/p^n\mathbb{Z}, \end{aligned}$$

where $\text{R}\Gamma_{\Delta}(\mathcal{X}_{A_{\text{inf}}}/A_{\text{inf}})$, $\text{R}\Gamma_{\Delta}(\mathcal{X}/\mathcal{O})$ are the prismatic avatars of the A_{inf} - and Breuil–Kisin cohomologies from [4, 5], resp. Taking $M_{\text{BK}} = H_{\Delta, 1}^i(\mathcal{X}/\mathcal{O})$ and $M_{\text{inf}} = H_{\Delta, 1}^i(\mathcal{X}/A_{\text{inf}})$, Li and Liu showed in [24] that M_{BK} is a p -torsion Breuil–Kisin module, M_{inf} is a p -torsion Breuil–Kisin–Fargues G_K -module, and that these modules recover the étale cohomology group $H_{\text{ét}}^i(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}/p\mathbb{Z})$ essentially due to the étale comparison theorem for prismatic cohomology from [6]. The pair $(M_{\text{BK}}, M_{\text{inf}})$ then serves as a suitable replacement of a $(\varphi, \hat{G})$ -module in our context.

²Recently, in [24] Li and Liu extended Caruso’s result to the range $ie < p - 1$ regardless of n , for $\mathcal{X}/\mathcal{O}_K$ proper and smooth (formal) scheme. In view of this, results of [21] should apply in these situations as well.

The most significant deviation from the strategy of [10] then stems from the fact that the pair $(M_{\text{BK}}, M_{\text{inf}})$ obtained this way is “inherently p -torsion”, that is, it does not come equipped with any apparent lift to analogous objects in characteristic 0. This is not the case in [10], where all torsion modules ultimately originate from a free $(\varphi, \hat{G})$ -module $(M, \hat{M})$. A key technical input in *loc. cit.* is to establish a partial control on the Galois action on M inside $\hat{M}$, namely, a condition of the form

$$\forall g \in G_{K(\pi^{1/p^s})}, \forall x \in M: \quad g(x) - x \in (J_{n,s} + p^n A_{\text{inf}})(\hat{M} \otimes_{\hat{\mathcal{R}}} A_{\text{inf}}). \quad (1.1)$$

Here $J_{n,s} \subseteq A_{\text{inf}}$ are certain ideals (that are shrinking with growing s). This is a “rational” fact, in the sense that this claim is a consequence of the description of the Galois action in terms of the monodromy operator on the associated Breuil module $\mathcal{D}(\hat{M})$ (cf. [7], [25, Section 3.2]), a vector space over the characteristic 0 field K' .

As a starting point for replacing (1.1) in our context, we turn to a result by Gee and Liu in [14, Appendix F] (see also [28, Theorem 3.8]). Given a finite free Breuil–Kisin module M_{BK} (of finite height) and a compatible structure of Breuil–Kisin–Fargues G_K -module on $M_{\text{inf}} = M_{\text{BK}} \otimes_{\mathfrak{S}} A_{\text{inf}}$ such that the image of M_{BK} under the natural map lands in $(M_{\text{inf}})^{G_{K(\pi^{1/p^\infty})}}$, the étale realization of M_{inf} is crystalline if and only if

$$\forall g \in G_K, \forall x \in M_{\text{BK}}: \quad g(x) - x \in \varphi^{-1}([\underline{\varepsilon}] - 1)[\underline{\pi}]M_{\text{inf}}. \quad (\text{Cr}_0)$$

Here $[-]$ denotes the Teichmüller lift and $\underline{\varepsilon}, \underline{\pi}$ are the elements of $\mathcal{O}_{\mathbb{C}_K^b}$ given by a collection $(\zeta_{p^n})_n$ of (compatible) p^n -th roots of unity and a collection $(\pi^{1/p^n})_n$ of p^n -th roots of the chosen uniformizer π , resp. We call condition (Cr_0) the *crystalline condition*. As the considered formal scheme $\mathcal{X}$ is assumed to be smooth over $\mathcal{O}_K$, it is reasonable to expect that the same condition applies to the pair $M_{\text{BK}} = H_{\Delta}^i(\mathcal{X}/\mathfrak{S})$ and $M_{\text{inf}} = H_{\Delta}^i(\mathcal{X}_{A_{\text{inf}}}/A_{\text{inf}})$, despite the fact that the Breuil–Kisin and Breuil–Kisin–Fargues modules coming from prismatic cohomology are not necessarily free.

This is indeed the case and, moreover, it can be shown that the crystalline condition even applies to the embedding of the chain complexes $R\Gamma_{\Delta}(\mathcal{X}/\mathfrak{S}) \rightarrow R\Gamma_{\Delta}(\mathcal{X}_{A_{\text{inf}}}/A_{\text{inf}})$: to make sense of this claim, we model the cohomology theories by their associated Čech–Alexander complexes. These were introduced in [6] in the case that $\mathcal{X}$ is affine, but can be extended to (at least) arbitrary separated smooth p -adic formal schemes. We are then able to verify the condition termwise for this pair of complexes. More generally, we introduce a decreasing series of ideals $I_s, s \geq 0$ where $I_0 = \varphi^{-1}([\underline{\varepsilon}] - 1)[\underline{\pi}]A_{\text{inf}}$, and then formulate and prove the analogue of (Cr_0) for I_s and the action of $G_{K(\pi^{1/p^s})}$. As a consequence, we obtain:

Theorem 1.3 (Theorem 4.5, Corollary 4.7, Proposition 4.10). *Let $\mathcal{X}$ be a smooth separated p -adic formal scheme over $\mathcal{O}_K$.*

- (1) *Fix a compatible choice of Čech–Alexander complexes $\check{C}_{\text{BK}}^{\bullet} \subseteq \check{C}_{\text{inf}}^{\bullet}$ that compute $R\Gamma_{\Delta}(\mathcal{X}/\mathfrak{S})$ and $R\Gamma_{\Delta}(\mathcal{X}_{A_{\text{inf}}}/A_{\text{inf}})$, resp. Then for all $s \geq 0$, the pair $(\check{C}_{\text{BK}}^{\bullet}, \check{C}_{\text{inf}}^{\bullet})$ satisfies (termwise) the condition*

$$\forall g \in G_{K(\pi^{1/p^s})}, \forall x \in \check{C}_{\text{BK}}^{\bullet}: \quad g(x) - x \in I_s \check{C}_{\text{inf}}^{\bullet}. \quad (\text{Cr}_s)$$

- (2) *The associated prismatic cohomology groups satisfy the crystalline condition, that is, the condition*

$$\forall g \in G_K, \forall x \in H_{\Delta}^i(\mathcal{X}/\mathfrak{S}): \quad g(x) - x \in \varphi^{-1}([\varepsilon] - 1)[\pi]H_{\Delta}^i(\mathcal{X}_{A_{\text{inf}}}/A_{\text{inf}}).$$

- (3) *For all pairs of integers s, n with $s + 1 \geq n \geq 1$, the p^n -torsion prismatic cohomology groups satisfy the condition*

$$\forall g \in G_{K(\pi^{1/p^s})}, \quad \forall x \in H_{\Delta, n}^i(\mathcal{X}/\mathfrak{S}):$$

$$g(x) - x \in \varphi^{-1}([\varepsilon] - 1)[\pi]^{p^{s+1-n}}H_{\Delta, n}^i(\mathcal{X}_{A_{\text{inf}}}/A_{\text{inf}}).$$

Theorem 1.3(3) specialized to $n = 1$ provides the desired analogue of the property (1.1) of $(\varphi, \hat{G})$ -modules and allows us to carry out the proof of Theorem 1.1.

As a consequence of Theorem 1.3(2), we obtain a proof of crystallinity of the cohomology groups $H_{\text{ét}}^i(\mathcal{X}_{\bar{\eta}}, \mathbb{Q}_p)$ in the proper case partially by means of “internal” p -adic Hodge theory (Corollary 4.8). This fact in this generality is originally due to Bhatt, Morrow and Scholze [4]. Of course, since our setup relies on the machinery of prismatic cohomology and especially the étale comparison, the proof can be considered independent of the one from [4] only in that it avoids the crystalline comparison theorem for (prismatic) A_{inf} -cohomology.

The bounds of Theorem 1.1 compare to the already known bounds as follows. Whenever the bounds of “semistable type” are known to apply to the situation of $H_{\text{ét}}^i(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}/p\mathbb{Z})$ (e.g. [10] when $i = 1$, [21] when $i \leq p - 1$ and $\mathcal{X}$ is a scheme), the bounds from Theorem 1.1 agree with those bounds. The bounds tailored to crystalline representations [1, 17] are slightly better but their applicability is quite limited ($e = 1$ and $i < p - 1$).

The fact that the cohomology groups $H_{\text{ét}}^i(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}/p^n\mathbb{Z})$ have an associated Breuil–Kisin module yields one more source of ramification estimates: in [9], Caruso provides a very general bound for p^n -torsion G_K -modules based on their restriction to $G_{K(\pi^{1/p^\infty})}$ via Fontaine’s theory of étale $\mathcal{O}_{\mathcal{E}}$ -modules. Using the Breuil–Kisin module $H_{\Delta, n}^i(\mathcal{X}/\mathfrak{S})$ attached to $H_{\text{ét}}^i(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}/p^n\mathbb{Z})$, this bound becomes explicit (as discussed in more detail in Remark 5.6). Comparing this result to Theorem 1.1 is more ambiguous due to somewhat different shapes of the estimates, but roughly speaking, the estimate of Theorem 1.1 is approximately the same for $e \leq p$, becomes worse when K is absolutely tamely ramified with large ramification degree, and is expected to outperform Caruso’s bound in case of large wild absolute ramification (see Section 5.2 for a more precise comparison).

In future work, we intend to extend the result of Theorem 1.1 to the case of arbitrary n . This seems plausible thanks to the full statement of Theorem 1.3(3). In a different direction, we plan to extend the results of the present paper to the case of semistable reduction, using the log-prismatic cohomology developed by Koshikawa and Koshikawa–Yao in [22, 23]. An important facts in this regard are that the A_{inf} -log-prismatic cohomology groups are still Breuil–Kisin–Fargues G_K -modules by a result of Česnavičius and Koshikawa [11].

The outline of the paper is as follows. In Section 2 we establish some necessary technical results. Namely, we discuss non-zero divisors and regular sequences on derived complete and completely flat modules with respect to the weak topology of A_{inf} , and establish Čech–Alexander complexes in the case of a separated and smooth formal scheme. Next, Section 3 introduces the conditions (Cr_s) , studies their basic algebraic properties and discusses in particular the crystalline condition (Cr_0) in the case of Breuil–Kisin–Fargues G_K -modules. In Section 4 we prove the conditions (Cr_s) for the Alexander–Čech complexes of a separated smooth p -adic scheme $\mathcal{X}$ over $\mathfrak{S}$ and A_{inf} , and draw some consequences for the individual cohomology groups (especially when $\mathcal{X}$ is proper), proving Theorem 1.3. Finally, in Section 5 we establish the ramification bounds for mod p étale cohomology, proving Theorem 1.1. Subsequently, we discuss in more detail how the bounds from Theorem 1.1 compare to the various bounds from the literature discussed above.

Let us setup some basic notation used throughout the paper. We will use freely the language of prisms and δ -rings from [6], and we adopt much of the related notation and conventions. In particular, a formal scheme $\mathcal{X}$ over a p -adically complete ring A always means a p -adic formal scheme, and it is called smooth if it is locally of the form $\text{Spf } R$ for a (derived³) p -completely smooth A -algebra R – that is, a p -complete A -algebra such that R/p is a smooth A/p -algebra and $\text{Tor}_i^A(R, A/p) = 0$ for all $i > 0$. By the results of Elkik [12] and the discussion in [6, Section 1.2], R is equivalently the p -adic completion of a smooth A -algebra.

We fix a perfect field k of characteristic $p > 0$ and a finite totally ramified extension K/K' of degree e where $K' = W(k)[1/p]$. We fix a uniformizer $\pi \in \mathcal{O}_K$. This choice is arbitrary if $p > 2$, but when $p = 2$, we make a suitable choice of π to be specified later (see Section 3.1). Setting $\mathfrak{S} = W(k)[[u]]$, the choice of π determines a surjective map $\mathfrak{S} \rightarrow \mathcal{O}_K$ given by $u \mapsto \pi$; the kernel of this map is generated by an Eisenstein polynomial $E(u)$ of degree e . $\mathfrak{S}$ is endowed with a Frobenius lift (hence a δ -structure) extending the one on $W(k)$ by $u \mapsto u^p$. The pair $(\mathfrak{S}, E(u)\mathfrak{S})$ then defines a prism (so-called Breuil–Kisin prism).

Denote by A_{inf} the ring $W(\mathcal{O}_{\mathbb{C}_K^b})$ where $\mathbb{C}_K$ is the completion of the algebraic closure of K and

$$\mathcal{O}_{\mathbb{C}_K^b} = \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_K}/p$$

is the tilt of $\mathcal{O}_{\mathbb{C}_K}$. Fix a compatible system $(\pi_n)_n$ of p^n -th roots of π . Such a choice determines an element $\underline{\pi} \in \mathcal{O}_{\mathbb{C}_K^b}$ and an embedding of $\mathfrak{S}$ into A_{inf} via $u \mapsto [\underline{\pi}]$ where $[-]$ denotes the Teichmüller lift. Under this embedding, $E(u)$ is sent to a generator ξ of the kernel of the canonical map $\theta: A_{\text{inf}} \rightarrow \mathcal{O}_{\mathbb{C}_K}$ that lifts the canonical projection

$$\text{pr}_0: \mathcal{O}_{\mathbb{C}_K^b} = \varprojlim_{\varphi} \mathcal{O}_{\mathbb{C}_K}/p \rightarrow \mathcal{O}_{\mathbb{C}_K}/p.$$

³As we will always consider the base A to have bounded p^∞ -torsion, there is no distinction between derived p -completion and p -adic completion in this case.

The pair $(A_{\text{inf}}, \text{Ker } \theta)$ forms a prism, and $(\mathfrak{S}, (E(u))) \rightarrow (A_{\text{inf}}, \text{Ker } \theta)$ is a map of prisms. It is known that under such embedding, A_{inf} is faithfully flat over $\mathfrak{S}$ (see e.g. [14, Proposition 2.2.13]).

Similarly, we fix a choice of a compatible system of primitive p^n -th roots of unity $(\zeta_{p^n})_{n \geq 0}$. This defines an element ε of $\mathcal{O}_{\mathbb{C}_K^\flat}$ in an analogous manner, and the embedding $\mathfrak{S} \hookrightarrow A_{\text{inf}}$ extends to a map (actually still an embedding by [9, Proposition 1.14]) $W(k)[[u, v]] \rightarrow A_{\text{inf}}$ by additionally setting $v \mapsto [\varepsilon] - 1$. Additionally, we denote by ω the element $([\varepsilon] - 1)/([\varepsilon^{1/p}] - 1) = [\varepsilon^{1/p}]^{p-1} + \dots + [\varepsilon^{1/p}] + 1$. It is well known that this is another generator of $\text{Ker } \theta$, therefore ω/ξ is a unit in A_{inf} .

The choices of π, π_n and ζ_{p^n} remain fixed throughout, hence so do the embeddings $\mathfrak{S} \hookrightarrow A_{\text{inf}}$ and $W(k)[[u, v]] \hookrightarrow A_{\text{inf}}$. Consequently, we identify the elements $[\pi], [\varepsilon] - 1, \xi$ with u, v and $E(u)$, respectively.

2. Preparations

2.1. Regularity on $(p, E(u))$ -completely flat modules

The goal of this section is to prove that every $(p, E(u))$ -complete and $(p, E(u))$ -completely flat A_{inf} -module is torsion-free, and that any sequence p, x with $x \in A_{\text{inf}} \setminus (A_{\text{inf}}^\times \cup pA_{\text{inf}})$ is regular on such modules.

Regarding completions and complete flatness, we adopt the terminology of [32, 091N], [6], but since we apply these notions mostly to modules as opposed to objects of derived categories, our treatment is closer in spirit to [29, 30, 34]. Given a ring A and a finitely generated ideal $I = (f_1, f_2, \dots, f_n)$, the derived I -completion⁴ of an A -module M is

$$\hat{M} = M[[X_1, X_2, \dots, X_n]]/(X_1 - f_1, X_2 - f_2, \dots, X_n - f_n)M[[X_1, X_2, \dots, X_n]]. \quad (2.1)$$

M is said to be *derived I -complete* if the natural map $M \rightarrow \hat{M}$ is an isomorphism. This is equivalent to the vanishing of $\text{Ext}_A^i(A_f, M)$ for $i = 0, 1$ and all $f \in I$ (equivalently, for $f = f_j$ for all j), and as a consequence, it can be shown that the category of derived I -complete modules forms a full abelian subcategory of the category of all A -modules with exact inclusion functor (and the derived I -completion is its left adjoint; in particular, derived I -completion is right exact as a functor on A -modules). Another consequence is that derived I -completeness is equivalent to derived J -completeness when I, J are two finitely generated ideals and $\sqrt{I} = \sqrt{J}$. There is always a natural surjection $\hat{M} \rightarrow \hat{M}^{\text{cl}}$ where $(-)^{\text{cl}}$ stands for I -adic completion, which will be referred to as classical I -completion for the rest of the paper. Just like for classically I -complete modules, if M is derived I -complete, then $M/IM = 0$ implies $M = 0$ (this is referred to as *derived Nakayama lemma*).

⁴That is, this is derived I -completion of M as a module. This will be sufficient to consider for our purposes.

An A -module M is said to be I -completely (faithfully) flat if $\mathrm{Tor}_i^A(M, A/I) = 0$ for all $i > 0$ and M/IM is a (faithfully) flat A/I -module. Just like for derived completeness, I -complete flatness is equivalent to J -complete flatness when J is another finitely generated ideal with $\sqrt{I} = \sqrt{J}$.⁵

Let us start by a brief discussion of regular sequences on derived complete modules in general. For that purpose, given an A -module M and $\underline{f} = f_1, \dots, f_n \in A$, denote by $\mathrm{Kos}(M; \underline{f})$ the usual Koszul complex and let $H_m(M; \underline{f})$ denote the m -th Koszul homology of M with respect to $f_1, f_2, \dots, f_n$.

The first lemma is a straightforward generalization of standard facts about Koszul homology (e.g. [27, Theorem 16.5]) and regularity on finitely generated modules.

Lemma 2.1. *Let A be a ring, $I \subseteq A$ a finitely generated ideal and let M be a non-zero derived I -complete module. Let $\underline{f} = f_1, f_2, \dots, f_n \in I$. Then*

- (1) *$\underline{f}$ forms a regular sequence on M if and only if $H_m(M; \underline{f}) = 0$ for all $m \geq 1$ if and only if $H_1(M; \underline{f}) = 0$.*
- (2) *In this situation, any permutation of $f_1, f_2, \dots, f_n$ is also a regular sequence on M .*

Proof. As Koszul homology is insensitive to the order of the elements $f_1, f_2, \dots, f_n$, part (2) follows immediately from (1).

To prove (1), the forward implications are standard and hold in full generality (see e.g. [27, Theorem 16.5]). It remains to prove that the sequence $f_1, f_2, \dots, f_n$ is regular on M if $H_1(M; f_1, \dots, f_n) = 0$. We proceed by induction on n . The case $n = 1$ is clear ($H_1(M; x) = M[x]$ by definition, and $M/xM \neq 0$ follows by derived Nakayama). Let $n \geq 2$, and denote by $\underline{f}'$ the truncated sequence $f_1, f_2, \dots, f_{n-1}$. Then we have the isomorphism $\mathrm{Kos}(M; \underline{f}) \simeq \mathrm{Kos}(M; \underline{f}') \otimes \mathrm{Kos}(A; f_n)$, which produces a short exact sequence

$$0 \rightarrow \mathrm{Kos}(M; \underline{f}') \rightarrow \mathrm{Kos}(M; \underline{f}) \rightarrow \mathrm{Kos}(M; \underline{f}')[-1] \rightarrow 0$$

of chain complexes. Taking homologies results in a long exact sequence

$$H_1(M; \underline{f}') \xrightarrow{\pm f_n} H_1(M; \underline{f}') \rightarrow H_1(M; \underline{f}) \rightarrow M/(\underline{f}')M \xrightarrow{\pm f_n} M/(\underline{f}')M \rightarrow M/(\underline{f})M \rightarrow 0$$

(as in [27, Theorem 7.4]). By assumption, $H_1(M; \underline{f}) = 0$ and therefore, we have that $f_n H_1(M; \underline{f}') = H_1(M; \underline{f}')$ where $f_n \in I$. Upon observing that $H_1(M; \underline{f}')$ is obtained from finite direct sum of copies of M by repeatedly taking kernels and cokernels, it is derived I -complete. Thus, derived Nakayama implies that $H_1(M; \underline{f}') = 0$ as well, and by induction hypothesis, $\underline{f}'$ is a regular sequence on M . Finally, the above exact sequence also implies that f_n is injective on $M/(\underline{f}')M$, and $M/(\underline{f})M \neq 0$ is satisfied thanks to derived Nakayama again. This finishes the proof. ■

⁵However, note that while (derived) I -completeness more generally implies (derived) I' -completeness when I' is a finitely generated ideal contained in $\sqrt{I}$, the “opposite” works for flatness, i.e., I -complete flatness implies I'' -complete flatness when I'' is a finitely generated ideal with $I \subseteq \sqrt{I''}$.

Corollary 2.2. *Let A be a derived I -complete ring for $I = (\underline{f})$ where $\underline{f} = f_1, f_2, \dots, f_n$ is a regular sequence on A , and let F be a non-zero derived I -complete A -module that is I -completely flat. Then $\underline{f}$ is a regular sequence on F . Consequently, each f_i is a non-zero divisor on F .*

Proof. By Lemma 2.1 (1), $H_m(A; \underline{f}) = 0$ for all $m \geq 1$, hence $\text{Kos}(A; \underline{f})$ is a free resolution of A/I . Thus, the complex $\text{Kos}(F; \underline{f}) = F \otimes_A \text{Kos}(A; \underline{f})$ computes $\text{Tor}_*^A(F, A/I)$, and hence is acyclic in positive degrees by I -complete flatness. We may thus conclude that $H_i(F; \underline{f}) = 0$ for all $i \geq 1$. By Lemma 2.1, $\underline{f}$ is a regular sequence on F , and it remains regular on F after arbitrary permutation. This proves the claim. ■

Now we specialize to the situation $A = A_{\text{inf}}$, which is the case of interest in this paper.

Lemma 2.3. *For every element $x \in A_{\text{inf}} \setminus (A_{\text{inf}}^\times \cup pA_{\text{inf}})$ and for all integers $k, l \geq 0$, we have that $p^k A_{\text{inf}} \cap x^l A_{\text{inf}} = p^k x^l A_{\text{inf}}$, and p, x forms a regular sequence. Furthermore, the ideal $\sqrt{(p, x)}$ is equal to $(p, W(\mathfrak{m}_{\mathbb{C}_K^b}))$, the unique maximal ideal of A_{inf} . In particular, given two choices x, x' as above, we have $\sqrt{(p, x)} = \sqrt{(p, x')}$.*

In particular, the equalities “ $\sqrt{(p, x)} = \sqrt{(p, x')}$ ” imply that all the (p, x) -adic topologies (for x as above) are equivalent to each other; this is the so-called weak topology on A_{inf} (usually defined as (p, u) -adic topology in our notation); it is standard that A_{inf} is complete with respect to this topology.

Proof. By assumption, the image $\bar{x}$ of x in $A_{\text{inf}}/p = \mathcal{O}_{\mathbb{C}_K^b}$ is non-zero and non-unit in A_{inf}/p (non-unit since $x \notin A_{\text{inf}}^\times$ and $p \in \text{rad}(A_{\text{inf}})$). Thus, x^l is a non-zero divisor both on A_{inf} and on A_{inf}/p , hence the claim that $pA_{\text{inf}} \cap x^l A_{\text{inf}} = px^l A_{\text{inf}}$ follows for every l . The element p is itself non-zero divisor on (the domain) A_{inf} and thus, p, x is a regular sequence. It follows that p^k, x^l forms a regular sequence for arbitrary k and l and thus, also the identity $p^k A_{\text{inf}} \cap x^l A_{\text{inf}} = p^k x^l A_{\text{inf}}$.

To prove the second assertion, note that $\sqrt{(\bar{x})} = \mathfrak{m}_{\mathbb{C}_K^b}$ since $A_{\text{inf}}/p = \mathcal{O}_{\mathbb{C}_K^b}$ is a rank 1 valuation ring. It follows that $(p, W(\mathfrak{m}_{\mathbb{C}_K^b}))$ is the unique maximal ideal of A_{inf} above (p) , hence the unique maximal ideal since $p \in \text{rad}(A_{\text{inf}})$, and that $\sqrt{(p, x)}$ is equal to this ideal. ■

We are ready to prove the claim mentioned at the beginning of the section.

Corollary 2.4. *Let F be a derived $(p, E(u))$ -complete and $(p, E(u))$ -completely flat A_{inf} -module, and let $x \in A_{\text{inf}} \setminus (A_{\text{inf}}^\times \cup pA_{\text{inf}})$. Then p, x is a regular sequence on F . In particular, for each $k, l > 0$, we have $p^k F \cap x^l F = p^k x^l F$. Consequently, F is a torsion-free A_{inf} -module.*

Proof. By Lemma 2.3, A_{inf} and F are derived (p, x) -complete, F is (p, x) -completely flat over A_{inf} , and p, x is a regular sequence on A_{inf} . Corollary 2.2 then proves that the sequence p, x , hence also p^k, x^l for any k, l , is regular on F , hence also the claim $p^k F \cap x^l F = p^k x^l F$. To prove the “consequently” part, let y be a non-zero and non-unit element of A_{inf} . Since A_{inf} is classically p -complete, we have $\bigcap_n p^n A_{\text{inf}} = 0$, and so there exist n

such that $y = p^n x$ with $x \notin pA_{\text{inf}}$. If x is a unit, then y is a non-zero divisor on F since so is p^n . Otherwise $x \in A_{\text{inf}} \setminus (A_{\text{inf}}^\times \cup pA_{\text{inf}})$, so p, x is a regular sequence on F , and so is x, p (e.g. by Lemma 2.1). In particular, p, x are both non-zero divisors on F , and hence so is $y = p^n x$. ■

2.2. Čech–Alexander complex

Next, we discuss the construction of Čech–Alexander complexes for computing prismatic cohomology, introduced in [6] in the affine case. Specifically, the goal of this section is to extend this construction to the case of a smooth separated p -adic formal scheme.

Throughout this section, let (A, I) be a fixed bounded base prism, and let $\mathcal{X}$ be a smooth separated p -adic formal scheme over A/I . Recall that $(\mathcal{X}/A)_\Delta$ denotes the site whose underlying category is the opposite of the category of bounded prisms (B, IB) over (A, I) together with a map of formal schemes $\text{Spf}(B/IB) \rightarrow \mathcal{X}$ over A/I . Covers in $(\mathcal{X}/A)_\Delta$ are given by the opposites of faithfully flat maps $(B, IB) \rightarrow (C, IC)$ of prisms, meaning that C is (p, I) -completely faithfully flat over (B, IB) . The prismatic cohomology $\text{R}\Gamma_\Delta(\mathcal{X}, A)$ is then defined as the sheaf cohomology $\text{R}\Gamma((\mathcal{X}/A)_\Delta, \mathcal{O}) (= \text{R}\Gamma((*, \mathcal{O}))$ where $*$ is the terminal sheaf) for the sheaf $\mathcal{O} = \mathcal{O}_\Delta$ on $(\mathcal{X}/A)_\Delta$ defined by $(B, IB) \mapsto B$.

Additionally, let us denote by Δ the site of all bounded prisms, i.e., the opposite of the category of all bounded prisms and their maps, with topology given by faithfully flat maps of prisms.

In order to discuss the Čech–Alexander complex in a non-affine situation, a slight modification of the topology on $(\mathcal{X}/A)_\Delta$ is convenient. The following proposition motivates the change.

Proposition 2.5. *Let (A, I) be a bounded prism.*

- (1) *Given a collection of maps of (bounded) prisms $(A, I) \rightarrow (B_i, IB_i)$, $i = 1, 2, \dots, n$, the canonical map $(A, I) \rightarrow (C, IC) = (\prod_i B_i, I \prod_i B_i)$ is a map of (bounded) prisms.*
- (2) *(C, IC) is flat over (A, I) if and only if each (B_i, IB_i) is flat over (A, I) . In that situation, (C, IC) is faithfully flat prism over (A, I) if and only if the family of maps of formal spectra $\text{Spf}(B_i/IB_i) \rightarrow \text{Spf}(A/I)$ is jointly surjective.*
- (3) *Let $f \in A$ be an element. Then $(\widehat{A_f}, I\widehat{A_f})$, where $\widehat{(-)}$ stands for the derived (equivalently, classical) (p, I) -completion, is a bounded prism,⁶ and the canonical map $(A, I) \rightarrow (\widehat{A_f}, I\widehat{A_f})$ is a flat map of prisms.*
- (4) *Let $f_1, \dots, f_n \in A$ be a collection of elements generating the unit ideal. Then the canonical map $(A, I) \rightarrow (\prod_i \widehat{A_{f_i}}, I \prod_i \widehat{A_{f_i}})$ is a faithfully flat map of (bounded) prisms.*

⁶We do consider the zero ring with its zero ideal a prism, hence allow the possibility of $\widehat{A_f} = 0$, which occurs e.g. when $f \in (p, I)$. Whether the zero ring satisfies Definition 3.2 of [6] depends on whether the inclusion of the empty scheme to itself is considered an effective Cartier divisor; following the usual definitions pedantically, it indeed seems to be the case. Also some related claims, such as [6, Lemma 3.7 (3)] or [3, Lecture 5, Corollary 5.2], suggest that the zero ring is allowed as a prism.

Proof. The proof of (1) is more or less formal. The ring $C = \prod_i B_i$ has a unique A - δ -algebra structure since the forgetful functor from δ -rings to rings preserves limits, and C is as product of (p, I) -complete rings (p, I) -complete. Clearly $IC = \prod_i (IB_i)$ is an invertible ideal since each IB_i is. In particular, $C[I] = 0$, hence C is a prism by [6, Lemma 3.5]. Assuming that all (B_i, IB_i) are bounded, from $C/IC = \prod_i B_i/IB_i$ we have $C/IC[p^\infty] = C/IC[p^k]$ for k big enough so that $B_i/IB_i[p^\infty] = B_i/IB_i[p^k]$ for all i , showing that (C, IC) is bounded.

The $((p, I)$ -complete) flatness part of (2) is clear. For the faithful flatness statement, note that $C/(p, I)C = \prod_i B_i/(p, I)B_i$, hence $A/(p, I) \rightarrow C/(p, I)C$ is faithfully flat if and only if the map of spectra $\coprod_i \text{Spec}(B_i/(p, I)B_i) = \text{Spec}(C/(p, I)C) \rightarrow \text{Spec}(A/(p, I))$ is surjective.

Let us prove (3). Since $\widehat{A}_f$ has $p \in \text{rad}(\widehat{A}_f)$, the equality $\varphi^n(f^k) = f^{kp^n} + p(\dots)$ shows that $\varphi^n(f^k)$ for each $n, k \geq 0$ is a unit in $\widehat{A}_f$. Consequently, as in [6, Remark 2.16], $\widehat{A}_f = \widehat{S^{-1}A}$ for $S = \{\varphi^n(f^k) \mid n, k \geq 0\}$, and the latter has a unique δ -structure extending that of A by [6, Lemmas 2.15 and 2.17]. In particular, $\widehat{A}_f$ is a (p, I) -completely flat A - δ -algebra, hence $(\widehat{A}_f, I\widehat{A}_f)$ is flat prism over (A, I) by [6, Lemma 3.7 (3)].

Part (4) follows formally from parts (1)–(3). $\blacksquare$

Construction 2.6. Denote by $(\mathcal{X}/A)_{\Delta}^{\text{II}}$ the site whose underlying category is $(\mathcal{X}/A)_{\Delta}$. The covers on $(\mathcal{X}/A)_{\Delta}^{\text{II}}$ are given by the opposites of finite families $\{(B, IB) \rightarrow (C_i, IC_i)\}_i$ of flat maps of prisms such that the associated maps $\{\text{Spf}(C_i/IC_i) \rightarrow \text{Spf}(B/IB)\}$ are jointly surjective. Let us call these “faithfully flat families” for short. The covers of the initial object $\emptyset^7$ are the empty cover and the identity. We similarly extend Δ to Δ^{II} , that is, we proclaim the identity cover and the empty cover to be covers of $\emptyset$, and generally proclaim (finite) faithfully flat families to be covers.

Clearly isomorphisms as well as composition of covers are covers in both cases. To check that $(\mathcal{X}/A)_{\Delta}^{\text{II}}$ and Δ^{II} are sites, it thus remains to check the base change axiom. This is trivial for situations involving $\emptyset$, so it remains to check that given a faithfully flat family $\{(B, IB) \rightarrow (C_i, IC_i)\}_i$ and a map of prisms $(B, IB) \rightarrow (D, ID)$, the fiber products⁸ $(C_i, IC_i) \boxtimes_{(B, IB)} (D, ID)$ in Δ^{II} exist and the collection of canonical maps $\{(D, ID) \rightarrow (C_i, IC_i) \boxtimes_{(B, IB)} (D, ID)\}_i$ is a faithfully flat family; the existence and (p, I) -complete flatness follows by the same proof as in [6, Corollary 3.12], only with “ (p, I) -completely faithfully flat” replaced by “ (p, I) -completely flat” throughout, and the fact that the family is faithfully flat follows as well, since $(\prod_i (C_i, IC_i)) \boxtimes_{(B, IB)} (D, ID)$ is canonically identified with $\prod_i ((C_i, IC_i) \boxtimes_{(B, IB)} (D, ID))$ (and using Remark 2.7 (1)).

Remark 2.7. Let us comment further on the topology introduced in Construction 2.6.

- (1) Note that for a finite family of objects (C_i, IC_i) in $(\mathcal{X}/A)_{\Delta}$, the structure map of the product $(A, I) \rightarrow \prod_i (C_i, IC_i)$ together with the map of formal spectra

⁷That is, $\emptyset$ corresponds to the zero ring, which we consider to be a prism as per the previous footnote.

⁸Here we mean fiber products in the variance of the site, i.e., “pushouts of prisms”. We use the symbol $\boxtimes$ to denote this operation.

(induced from the maps for individual i 's)

$$\mathrm{Spf}\left(\prod_i C_i/IC_i\right) = \coprod_i \mathrm{Spf}(C_i/IC_i) \rightarrow \mathcal{X}$$

makes $(\prod_i C_i, I \prod_i C_i)$ into an object of $(\mathcal{X}/A)_\Delta$ that is easily seen to be the coproduct of (C_i, IC_i) 's. In view of Proposition 2.5 (2), one thus arrives at the equivalent formulation

$$\{Y_i \rightarrow Z\}_i \text{ is a } (\mathcal{X}/A)_\Delta^\Pi\text{-cover} \Leftrightarrow \coprod_i Y_i \rightarrow Z \text{ is a } (\mathcal{X}/A)_\Delta\text{-cover.}$$

Similar considerations apply to Δ and Δ^Π .

- (2) The two sites are honestly different in that they define different categories of sheaves. Namely, for every finite coproduct $Y = \coprod_i Y_i$, the collection of canonical maps $\{Y_i \rightarrow \coprod_i Y_i\}_i$ forms a $(\mathcal{X}/A)_\Delta^\Pi$ -cover, and the sheaf axiom forces upon a sheaf $\mathcal{F} \in \mathrm{Shv}((\mathcal{X}/A)_\Delta^\Pi)$ the identity $\mathcal{F}(\coprod_i Y_i) = \prod_i \mathcal{F}(Y_i)$, which is not automatic.⁹ In fact, $\mathrm{Shv}((\mathcal{X}/A)_\Delta^\Pi)$ can be identified with the full category of $\mathrm{Shv}((\mathcal{X}/A)_\Delta)$ consisting of all sheaves compatible with finite disjoint unions in the sense above. In particular, the structure sheaf $\mathcal{O} = \mathcal{O}_\Delta: (B, IB) \mapsto B$ is a sheaf for the $(\mathcal{X}/A)_\Delta^\Pi$ -topology. (Again, the same is true for Δ and Δ^Π , including the fact that the presheaf $\mathcal{O}: (B, IB) \mapsto B$ is a sheaf.)

Despite the above fine distinction, for the purposes of prismatic cohomology, the two topologies are interchangeable. This is a consequence of the following lemma.

Lemma 2.8. *Given an object $(B, IB) \in (\mathcal{X}/A)_\Delta^\Pi$, one has $H^i((B, IB), \mathcal{O}) = 0$ for $i > 0$.*

Proof. The sheaf $\mathcal{O}: (B, I) \mapsto B$ on Δ^Π has vanishing positive Čech cohomology essentially by the proof of [6, Corollary 3.12]: one needs to show acyclicity of the Čech complex for any Δ^Π -cover $\{(B, I) \rightarrow (C_i, IC_i)\}_i$, but the resulting Čech complex is identical to that for the Δ -cover $(B, I) \rightarrow \prod_i (C_i, IC_i)$, for which the acyclicity is proved in [6, Corollary 3.12]. This implies the vanishing of $H^i_{\Delta^\Pi}((B, I), \mathcal{O})$ for all bounded prisms (B, I) and all $i > 0$ (e.g. by [32, 03F9]).

Now we use the fact that cohomology of an object can be computed as the cohomology of the corresponding slice site. Let $(B, IB) \in (\mathcal{X}/A)_\Delta^\Pi$. Forgetting structure, (B, IB) may be viewed as an object of Δ^Π , and then [32, 03F3] implies that for every i , we have the isomorphisms

$$\begin{aligned} H^i_{(\mathcal{X}/A)_\Delta^\Pi}((B, IB), \mathcal{O}) &\simeq H^i((\mathcal{X}/A)_\Delta^\Pi/(B, IB), \mathcal{O}|_{(B, IB)}), \\ H^i_{\Delta^\Pi}((B, IB), \mathcal{O}) &\simeq H^i(\Delta^\Pi/(B, IB), \mathcal{O}|_{(B, IB)}). \end{aligned}$$

⁹For example, every constant presheaf is a sheaf for a topology given by singleton covers only, which is not the case for $(\mathcal{X}/A)_\Delta^\Pi$.

Upon noting that the slice sites $(\mathcal{X}/A)_{\Delta}^{\text{II}}/(B, IB)$, $\Delta^{\text{II}}/(B, IB)$ are equivalent (in a manner that identifies the two versions of the sheaf $\mathcal{O}|_{(B, IB)}$), the claim follows. ■

Corollary 2.9. *One has $\text{R}\Gamma((\mathcal{X}/A)_{\Delta}, \mathcal{O}) = \text{R}\Gamma((\mathcal{X}/A)_{\Delta}^{\text{II}}, \mathcal{O})$.*

Proof. The coverings of $(\mathcal{X}/A)_{\Delta}^{\text{II}}$ contain all the coverings of $(\mathcal{X}/A)_{\Delta}$, so we are in the situation of [32, 0EWW]. Namely, there is a morphism of sites $\varepsilon: (\mathcal{X}/A)_{\Delta}^{\text{II}} \rightarrow (\mathcal{X}/A)_{\Delta}$ given by the identity functor of the underlying categories, where the pushforward functor $\varepsilon_*: \text{Shv}((\mathcal{X}/A)_{\Delta}^{\text{II}}) \rightarrow \text{Shv}((\mathcal{X}/A)_{\Delta})$ is the natural inclusion and the (exact) inverse image functor $\varepsilon^{-1}: \text{Shv}((\mathcal{X}/A)_{\Delta}) \rightarrow \text{Shv}((\mathcal{X}/A)_{\Delta}^{\text{II}})$ is the sheafification with respect to the “II”-topology. One has

$$\Gamma((\mathcal{X}/A)_{\Delta}^{\text{II}}, -) = \Gamma((\mathcal{X}/A)_{\Delta}, -) \circ \varepsilon_*$$

(where ε_* denotes the inclusion of abelian sheaves in this context), hence

$$\text{R}\Gamma((\mathcal{X}/A)_{\Delta}^{\text{II}}, \mathcal{O}) = \text{R}\Gamma((\mathcal{X}/A)_{\Delta}, \text{R}\varepsilon_*\mathcal{O}),$$

and to conclude it is enough to show that $\text{R}^i\varepsilon_*\mathcal{O} = 0 \ \forall i > 0$. But $\text{R}^i\varepsilon_*\mathcal{O}$ is the sheafification of the presheaf given by $(B, IB) \mapsto H^i((B, IB), \mathcal{O})$ [32, 072W], which is 0 by Lemma 2.8. Thus, $\text{R}^i\varepsilon_*\mathcal{O} = 0$, which proves the claim. ■

For an open p -adic formal subscheme $\mathcal{V} \subseteq \mathcal{X}$, denote by $h_{\mathcal{V}}$ the functor sending an object $(B, IB) \in (\mathcal{X}/A)_{\Delta}$ to the set of factorizations of the implicit map $\text{Spf}(B/IB) \rightarrow \mathcal{X}$ through $\mathcal{V} \hookrightarrow \mathcal{X}$; that is,

$$h_{\mathcal{V}}((B, IB)) = \begin{cases} * & \text{if the image of } \text{Spf}(B/IB) \rightarrow \mathcal{X} \text{ is contained in } \mathcal{V}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $(B, IB) \rightarrow (C, IC)$ correspond to a morphism in $(\mathcal{X}/A)_{\Delta}$. If $\text{Spf}(B/IB) \rightarrow \mathcal{X}$ factors through $\mathcal{V}$, then so does $\text{Spf}(C/IC) \rightarrow \text{Spf}(B/IB) \rightarrow \mathcal{X}$. It follows that $h_{\mathcal{V}}$ forms a presheaf on $(\mathcal{X}/A)_{\Delta}$ (with transition maps $h_{\mathcal{V}}((B, IB)) \rightarrow h_{\mathcal{V}}((C, IC))$ given by $* \mapsto *$ when $h_{\mathcal{V}}((B, IB)) \neq \emptyset$, and the empty map otherwise). Note that $h_{\mathcal{X}}$ is the terminal sheaf.

Proposition 2.10. *$h_{\mathcal{V}}$ is a sheaf on $(\mathcal{X}/A)_{\Delta}^{\text{II}}$.*

Proof. Consider a cover in $(\mathcal{X}/A)_{\Delta}^{\text{II}}$ given by a faithfully flat family $\{(B, IB) \rightarrow (C_i, IC_i)\}_i$. One needs to check that the sequence

$$h_{\mathcal{V}}((B, IB)) \rightarrow \prod_i h_{\mathcal{V}}((C_i, IC_i)) \rightrightarrows \prod_{i,j} h_{\mathcal{V}}((C_i, IC_i) \boxtimes_{(B, IB)} (C_j, IC_j))$$

is an equalizer sequence. All the terms have at most one element; consequently, there are just two cases to consider, depending on whether the middle term is empty or not. In both cases, the pair of maps on the right necessarily agree, and so one needs to see that the map on the left is an isomorphism. This is clear in the case when the middle term is empty

(since the only map into an empty set is an isomorphism). It remains to consider the case when the middle term is non-empty, which means that $h_{\mathcal{V}}((C_i, IC_i)) = *$ for all i . In this case we need to show that $h_{\mathcal{V}}((B, IB)) = *$. Since the maps $\mathrm{Spf}(C_i/IC_i) \rightarrow \mathrm{Spf}(B/IB)$ are jointly surjective and each $\mathrm{Spf}(C_i/IC_i) \rightarrow \mathcal{X}$ lands in $\mathcal{V}$, it follows that so does the map $\mathrm{Spf}(B/IB) \rightarrow \mathcal{X}$. Thus, $h_{\mathcal{V}}((B, IB)) = *$, which finishes the proof. $\blacksquare$

When $\mathcal{V}$ is affine, one can cover the sheaf $h_{\mathcal{V}}$ by a representable sheaf. The construction of the representing object is essentially Construction 4.17 of [6] (but note that the resulting object is not weakly initial unless $\mathcal{V} = \mathcal{X}$). We repeat the construction here in order to fix notation and terminology.

Construction 2.11 (Prismatic cover of $\mathcal{V}$). Let us additionally assume that $\mathcal{V} = \mathrm{Spf}(R)$ is affine. Choose a surjection $P_{\mathcal{V}} \rightarrow R$ where $P_{\mathcal{V}} = \widehat{A[X]}$ is a (p, I) -completed free A -algebra. Denote by $J_{\mathcal{V}}$ the kernel of the surjection. Then there is a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & J_{\mathcal{V}} & \longrightarrow & P_{\mathcal{V}} & \longrightarrow & R \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \widehat{J_{\mathcal{V}} P_{\mathcal{V}}^{\delta}} & \longrightarrow & \widehat{P_{\mathcal{V}}^{\delta}} & \longrightarrow & \widehat{R \otimes_{P_{\mathcal{V}}} P_{\mathcal{V}}^{\delta}} \longrightarrow 0, \end{array}$$

where $\widehat{(-)}$ stands for derived (p, I) -completion. Here for an A -algebra S , S^{δ} denotes the “ δ -envelope” of S , that is, the S -algebra initial among S -algebras endowed with an A - δ -algebra structure. Note that $\widehat{P_{\mathcal{V}}^{\delta}} = (\widehat{P_{\mathcal{V}}^0})^{\delta}$, where $P_{\mathcal{V}}^0 = A[X]$ is the polynomial algebra before completion; in particular, since $(P_{\mathcal{V}}^0)^{\delta}$ is a flat $P_{\mathcal{V}}^0$ -algebra (essentially by [6, Lemma 2.11]), it follows that $\widehat{P_{\mathcal{V}}^{\delta}}$ is (p, I) -completely flat $P_{\mathcal{V}}$ -algebra. Consequently, the completions in the lower row of the diagram can be equivalently taken as classical (p, I) -completions (cf. [6, Lemma 3.7]).

Denote by $J_{\mathcal{V}}^{\delta, \wedge} \subseteq \widehat{P_{\mathcal{V}}^{\delta}}$ the image of the map $\widehat{J_{\mathcal{V}} P_{\mathcal{V}}^{\delta}} \rightarrow \widehat{P_{\mathcal{V}}^{\delta}}$, i.e., the (p, I) -complete ideal of $\widehat{P_{\mathcal{V}}^{\delta}}$ topologically generated by $J_{\mathcal{V}}$. Then we have a short exact sequence

$$0 \rightarrow J_{\mathcal{V}}^{\delta, \wedge} \rightarrow \widehat{P_{\mathcal{V}}^{\delta}} \rightarrow \widehat{R \otimes_{P_{\mathcal{V}}} P_{\mathcal{V}}^{\delta}} \rightarrow 0.$$

Let $(\check{C}_{\mathcal{V}}, I\check{C}_{\mathcal{V}})$ be the prismatic envelope of $(\widehat{P_{\mathcal{V}}^{\delta}}, J_{\mathcal{V}}^{\delta, \wedge})$. It follows from [6, Proposition 3.13, Example 3.14] that $(\check{C}_{\mathcal{V}}, I\check{C}_{\mathcal{V}})$ exists and is given by a flat prism over (A, I) . The map

$$R \rightarrow \widehat{R \otimes_{P_{\mathcal{V}}} P_{\mathcal{V}}^{\delta}} = \widehat{P_{\mathcal{V}}^{\delta}} / J_{\mathcal{V}}^{\delta, \wedge} \rightarrow \check{C}_{\mathcal{V}} / I\check{C}_{\mathcal{V}}$$

of p -complete rings corresponds to the map of formal schemes $\mathrm{Spf}(\check{C}_{\mathcal{V}}/I\check{C}_{\mathcal{V}}) \rightarrow \mathcal{V} \hookrightarrow \mathcal{X}$. This defines an object of $(\mathcal{X}/A)_{\Delta}^{\mathrm{II}}$, which we call a *prismatic cover* of $\mathcal{V}$.

Remarks 2.12. Let us record several technical observations regarding Construction 2.11.

- (1) Note that $(\check{C}_{\mathcal{V}}, I\check{C}_{\mathcal{V}})$ is equivalently the prismatic envelope of $(\widehat{P_{\mathcal{V}}^{\delta}}, J_{\mathcal{V}} \widehat{P_{\mathcal{V}}^{\delta}})$. Moreover, when the ideal $J_{\mathcal{V}}$ is finitely generated, one has the equality $J_{\mathcal{V}}^{\delta, \wedge} = J_{\mathcal{V}} \widehat{P_{\mathcal{V}}^{\delta}}$.

- (2) Since the ring R in Construction 2.11 is a p -completely smooth A/I -algebra, it is in particular a p -completion of a finitely presented A/I -algebra. It follows that the map $P_{\mathcal{V}} \rightarrow R$ may be chosen so that $P_{\mathcal{V}}$ is the (derived) (p, I) -completion of a polynomial A -algebra of finite type, with the kernel $J_{\mathcal{V}}$ finitely generated. While such a choice may be preferable, we formulate the construction without imposing it, as it may be convenient to allow non-finite-type free algebras in the construction e.g. for the reasons of functoriality (see the remark at the end of [6, Construction 4.17]).
- (3) As is shown below, the sheaf represented by $(\check{C}_{\mathcal{V}}, I\check{C}_{\mathcal{V}})$ covers the sheaf $h_{\mathcal{V}}$, justifying the term “prismatic cover”. While this term should arguably describe *any* object of $(\mathcal{X}/A)_{\Delta}$ whose represented sheaf covers $\mathcal{V}$, we reserve its usage for the objects obtained as a result of Construction 2.11.

Proposition 2.13. *Denote by $h_{\check{C}_{\mathcal{V}}}$ the sheaf represented by the object $(\check{C}_{\mathcal{V}}, I\check{C}_{\mathcal{V}}) \in (\mathcal{X}/A)_{\Delta}^{\text{II}}$. There exists a unique map of sheaves $h_{\check{C}_{\mathcal{V}}} \rightarrow h_{\mathcal{V}}$, and it is an epimorphism.*

Proof. If $(B, IB) \in (\mathcal{X}/A)_{\Delta}$ with $h_{\check{C}_{\mathcal{V}}}((B, IB)) \neq \emptyset$, this means that $\text{Spf}(B/IB) \rightarrow \mathcal{X}$ factors through $\mathcal{V}$ since it factors through $\text{Spf}(\check{C}_{\mathcal{V}}/I\check{C}_{\mathcal{V}})$. Thus, we also have

$$h_{\mathcal{V}}((B, IB)) = *,$$

and so the (necessarily unique) map $h_{\check{C}_{\mathcal{V}}}((B, IB)) \rightarrow h_{\mathcal{V}}((B, IB))$ is defined. When $h_{\check{C}_{\mathcal{V}}}((B, IB))$ is empty, the map $h_{\check{C}_{\mathcal{V}}}((B, IB)) \rightarrow h_{\mathcal{V}}((B, IB))$ is still defined and unique, namely given by the empty map. Thus, the claimed morphism of sheaves exists and is unique.

By the same argument as in [6, Construction 4.17], we show that this map is an epimorphism. Let $(B, IB) \in (\mathcal{X}/A)_{\Delta}$ such that $h_{\mathcal{V}}((B, IB)) = *$, i.e., $\text{Spf}(B/IB) \rightarrow \mathcal{X}$ factors through $\mathcal{V}$, and consider the map $R \rightarrow B/IB$ associated to the map $\text{Spf}(B/IB) \rightarrow \mathcal{X}$. Since $P_{\mathcal{V}}$ is a p -completed free A -algebra surjecting onto R and B is (p, I) -complete, the map $R \rightarrow B/IB$ admits a lift $P_{\mathcal{V}} \rightarrow B$. This induces an A - δ -algebra map $\widehat{P_{\mathcal{V}}^{\delta}} \rightarrow B$ which gives a morphism of δ -pairs

$$(\widehat{P_{\mathcal{V}}^{\delta}}, J_{\mathcal{V}}\widehat{P_{\mathcal{V}}^{\delta}}) \rightarrow (B, IB),$$

and further the map of prisms $(\check{C}_{\mathcal{V}}, I\check{C}_{\mathcal{V}}) \rightarrow (B, IB)$ using the universal properties of objects involved. It is easy to see that this is indeed (the opposite of) a morphism in $(\mathcal{X}/A)_{\Delta}$. This shows that $h_{\check{C}_{\mathcal{V}}}((B, IB))$ is non-empty whenever $h_{\mathcal{V}}((B, IB))$ is. Thus, the map is an epimorphism. ■

Remark 2.14 (Binary products in $(\mathcal{X}/A)_{\Delta}$). For $(B, IB), (C, IC) \in (\mathcal{X}/A)_{\Delta}$, let us denote their binary product (in the variance of the site) by $(B, IB) \boxtimes (C, IC)$. Let us describe it explicitly at least under the additional assumptions that

- (1) $(B, IB), (C, IC)$ are flat prisms over (A, I) ,
- (2) there are affine opens $\mathcal{U}, \mathcal{V} \subseteq \mathcal{X}$ such that $h_{\mathcal{U}}((B, IB)) = * = h_{\mathcal{V}}((C, IC))$.

Set $\mathcal{W} = \mathcal{U} \cap \mathcal{V}$ and denote the rings corresponding to the affine open sets $\mathcal{U}, \mathcal{V}$ and $\mathcal{W}$ by R, S and T , resp. Then any object $(D, ID) \in (\mathcal{X}/A)_{\Delta}$ with maps both to (B, IB) and (C, IC) lives over $\mathcal{W}$, i.e., satisfies $h_{\mathcal{W}}((D, ID)) = *$. This justifies the following construction. Consider the following commutative diagram, where $\lrcorner$ denotes the pushout of p -complete commutative rings, i.e., taking the classically p -completed tensor product $\hat{\otimes}$ (and $B \hat{\otimes}_A C$ is the derived, but equivalently classical, (p, I) -completion of $B \otimes_A C$):

$$\begin{array}{ccccc}
 & & B \hat{\otimes}_A C & & \\
 & \nearrow & \downarrow & \nwarrow & \\
 B & & (B/IB \hat{\otimes}_R T) \hat{\otimes}_T (C/IC \hat{\otimes}_S T) & & C \\
 \downarrow & & \downarrow & & \downarrow \\
 B/IB & \rightarrow & B/IB \hat{\otimes}_R T & \xrightarrow{\quad} & C/IC \hat{\otimes}_S T \leftarrow C/IC \\
 \uparrow & \lrcorner & \uparrow & & \uparrow \\
 R & \longrightarrow & T & \xlongequal{\quad} & T \longleftarrow S
 \end{array}$$

Let $J \subseteq B \hat{\otimes}_A C$ be the kernel of the map

$$B \hat{\otimes}_A C \rightarrow (B/IB \hat{\otimes}_R T) \hat{\otimes}_T (C/IC \hat{\otimes}_S T).$$

Then the product $(B, IB) \boxtimes (C, IC)$ is given by the prismatic envelope of the δ -pair $(B \hat{\otimes}_A C, J)$.

Let $\mathfrak{V} = \{\mathcal{V}_j\}_{j \in J}$ be an affine open cover of $\mathcal{X}$. For an integer $n \geq 1$ and a multi-index $(j_1, j_2, \dots, j_n) \in J^n$, denote by $\mathcal{V}_{j_1, \dots, j_n}$ the intersection $\mathcal{V}_{j_1} \cap \dots \cap \mathcal{V}_{j_n}$. As $\mathcal{X}$ is assumed to be separated, each $\mathcal{V}_{j_1, \dots, j_n}$ is affine and we write $\mathcal{V}_{j_1, \dots, j_n} = \text{Spf}(R_{j_1, \dots, j_n})$.

Proposition 2.15. *The prismatic covers for the open formal subschemes $\mathcal{V}_{j_1, \dots, j_n}$ as above can be chosen so that for every multi-index $(j_1, \dots, j_n)$, we have*

$$(\check{C}_{\mathcal{V}_{j_1, \dots, j_n}}, I\check{C}_{\mathcal{V}_{j_1, \dots, j_n}}) = (\check{C}_{\mathcal{V}_{j_1}}, I\check{C}_{\mathcal{V}_{j_1}}) \boxtimes (\check{C}_{\mathcal{V}_{j_2}}, I\check{C}_{\mathcal{V}_{j_2}}) \boxtimes \dots \boxtimes (\check{C}_{\mathcal{V}_{j_n}}, I\check{C}_{\mathcal{V}_{j_n}}).$$

Proof. Clearly it is enough to show the statement for binary products. More precisely, given two affine opens $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathcal{X}$ and an arbitrary initial choice of $(\check{C}_{\mathcal{V}_1}, I\check{C}_{\mathcal{V}_1})$ and $(\check{C}_{\mathcal{V}_2}, I\check{C}_{\mathcal{V}_2})$, we show that $P_{\mathcal{V}_{12}} \rightarrow R_{12}$ can be chosen so that the resulting prismatic cover $(\check{C}_{\mathcal{V}_{12}}, I\check{C}_{\mathcal{V}_{12}})$ of $\mathcal{V}_{12}$ is equal to $(\check{C}_{\mathcal{V}_1}, I\check{C}_{\mathcal{V}_1}) \boxtimes (\check{C}_{\mathcal{V}_2}, I\check{C}_{\mathcal{V}_2})$. For the purposes of this proof, let us refer to a prismatic envelope of a δ -pair (S, J) also as “the prismatic envelope of the arrow $S \rightarrow S/J$ ”.

Consider $\alpha_i: P_{\mathcal{V}_i} \twoheadrightarrow R_i, i = 1, 2$ as in Construction 2.11, and set $P_{\mathcal{V}_{12}} = P_{\mathcal{V}_1} \hat{\otimes}_A P_{\mathcal{V}_2}$. Then one has the induced surjection $\alpha_1 \otimes \alpha_2: P_{\mathcal{V}_{12}} \rightarrow R_1 \hat{\otimes}_{A/I} R_2$, which can be followed by the induced map $R_1 \hat{\otimes}_{A/I} R_2 \rightarrow R_{12}$. This latter map is surjective as well since $\mathcal{X}$ is separated, and therefore the composition of these two maps $\alpha_{12}: P_{\mathcal{V}_{12}} \rightarrow R_{12}$ is surjective, with the kernel $J_{\mathcal{V}_{12}}$ that contains $(J_{\mathcal{V}_1}, J_{\mathcal{V}_2})P_{\mathcal{V}_{12}}$. We may construct a diagram analogous

to the one from Remark 2.14, which becomes the diagram

$$\begin{array}{ccccc}
 & & \widehat{P_{\mathcal{V}_{12}}^\delta} & & \\
 & \nearrow & \downarrow & \nwarrow & \\
 \widehat{P_{\mathcal{V}_1}^\delta} & & & & \widehat{P_{\mathcal{V}_2}^\delta} \\
 \downarrow & & & & \downarrow \\
 R_1 \hat{\otimes}_{P_{\mathcal{V}_1}} \widehat{P_{\mathcal{V}_1}^\delta} \rightarrow R_{12} \hat{\otimes}_{P_{\mathcal{V}_1}} \widehat{P_{\mathcal{V}_1}^\delta} & & R_{12} \hat{\otimes}_{P_{\mathcal{V}_{12}}} (\widehat{P_{\mathcal{V}_{12}}^\delta}) & & R_{12} \hat{\otimes}_{P_{\mathcal{V}_2}} \widehat{P_{\mathcal{V}_2}^\delta} \leftarrow R_2 \hat{\otimes}_{P_{\mathcal{V}_2}} \widehat{P_{\mathcal{V}_2}^\delta} \\
 \uparrow & \quad \quad \quad \uparrow & \quad \quad \quad \downarrow & \quad \quad \quad \uparrow & \\
 R_1 & \longrightarrow & R_{12} & \longleftarrow & R_2,
 \end{array}$$

where the expected arrow in the central column is replaced by an isomorphic one, namely the map obtained from the surjection $P_{\mathcal{V}_{12}} \rightarrow R_{12}$ by the procedure as in Construction 2.11. Now $(\check{C}_{\mathcal{V}_{12}}, I\check{C}_{\mathcal{V}_{12}})$ is obtained as the prismatic envelope of this composed central arrow, while $(\check{C}_{\mathcal{V}_1}, I\check{C}_{\mathcal{V}_1}) \boxtimes (\check{C}_{\mathcal{V}_2}, I\check{C}_{\mathcal{V}_2})$ is obtained the same way, but only after replacing the downward arrows on the left and right by their prismatic envelopes. Comparing universal properties, one easily sees that the resulting central prismatic envelope remains unchanged, proving the claim. $\blacksquare$

Remark 2.16. Fix a choice of free bases (i.e., polynomial variables) $\underline{X}_j$ for an initial choice of maps $P_{\mathcal{V}_j} \rightarrow R_j$. For later purposes, let us note that

- (1) The prismatic cover $(\check{C}_{\mathcal{V}_j}, I\check{C}_{\mathcal{V}_j}) \in (\mathcal{X}/A)_{\Delta}^{\text{II}}$ functorially depends on the underlying map of sets $\underline{X}_j \rightarrow R_j$,
- (2) The collection of prismatic covers $(\check{C}_{\mathcal{V}_{j_1, \dots, j_n}}, I\check{C}_{\mathcal{V}_{j_1, \dots, j_n}}) \in (\mathcal{X}/A)_{\Delta}^{\text{II}}$ functorially depends on the initial choice of maps $\underline{X}_j \rightarrow R_j$ for all j (having the formal scheme $\mathcal{X}/A$ fixed). For example, the map of rings $P_{\mathcal{V}_{j_1, j_2}} \rightarrow R_{j_1, j_2}$ and, hence, the formation of $(\check{C}_{\mathcal{V}_{j_1, j_2}}, I\check{C}_{\mathcal{V}_{j_1, j_2}})$, is determined by the map

$$\underline{X}_{j_1} \amalg \underline{X}_{j_2} \subseteq P_{\mathcal{V}_{j_1}} \hat{\otimes}_A P_{\mathcal{V}_{j_2}} \rightarrow R_{j_1} \otimes R_{j_2} \rightarrow R_{j_1, j_2}$$

as in the preceding proof.

Remark 2.17. Suppose that for each j , the initial choice of the map $P_{\mathcal{V}_j} \rightarrow R_j$ has been made as in Remark 2.12(2), that is, $P_{\mathcal{V}_j}$ is the (p, I) -completion of a finite-type free A -algebra and the ideal $J_{\mathcal{V}_j}$ is finitely generated. If now $P_{\mathcal{V}_{j_1, j_2, \dots, j_n}}$ is the (p, I) -completed free A -algebra for $\mathcal{V}_{j_1, j_2, \dots, j_n}$ obtained by iterating the procedure in the proof of Proposition 2.15, it is easy to see that in this case, the algebra $P_{\mathcal{V}_{j_1, j_2, \dots, j_n}}$ is still the (p, I) -completion of a finite-type free A -algebra, and it can be shown that the corresponding ideal $J_{\mathcal{V}_{j_1, j_2, \dots, j_n}}$ is finitely generated.

In more detail, given a ring B and a finitely generated ideal $J \subseteq B$, Let us call a B -algebra C *J-completely finitely presented* if C is derived J -complete and there exists a map $\alpha: B[\underline{X}] \rightarrow C$ from the polynomial ring in finitely many variables $\underline{X} = \{X_1, \dots, X_n\}$ such that the derived J -completed map $\hat{\alpha}: \widehat{B[\underline{X}]} \rightarrow C$ is surjective and with a finitely generated kernel. Then the algebra $R_{j_1, j_2, \dots, j_n}$ corresponding to $\mathcal{V}_{j_1, j_2, \dots, j_n}$ is (p, I) -completely finitely presented by Remark 2.12 (2), and since $P_{\mathcal{V}_{j_1, j_2, \dots, j_n}}$ is the (p, I) -completion of a finite-type polynomial A -algebra, the following lemma shows that $J_{\mathcal{V}_{j_1, j_2, \dots, j_n}}$ is finitely generated.

Lemma 2.18. *Let C be a J -completely finitely presented B -algebra, and consider a map $\beta: B[\underline{Y}] \rightarrow C$ from a polynomial algebra in finitely many variables $\underline{Y} = \{Y_1, \dots, Y_m\}$ such that $\hat{\beta}$ is surjective. Then the kernel of $\hat{\beta}$ is finitely generated.*

Proof. The proof is an adaptation of the proof of [32, 00R2], which is a similar assertion about finitely presented algebras. Consider α as in Remark 2.17, and additionally let us fix a generating set $(f_1, f_2, \dots, f_k) \subseteq \widehat{B[\underline{X}]}$ of $\text{Ker } \hat{\alpha}$.

For $i = 1, \dots, m$, let us choose $g_i \in \widehat{B[\underline{X}]}$ such that $\hat{\alpha}(g_i) = \beta(Y_i)$. Then one can define a surjective map

$$\theta_0: \widehat{B[\underline{X}][\underline{Y}]} \rightarrow C, \quad \theta_0|_{\widehat{B[\underline{X}]}} = \hat{\alpha}, \quad \theta_0(Y_i) = \beta(Y_i),$$

and it is easy to see that $\text{Ker } \theta_0 = (f_1, \dots, f_k, Y_1 - g_1, \dots, Y_m - g_m)$. That is, we have an exact sequence

$$(\widehat{B[\underline{X}][\underline{Y}]})^{\oplus k+m} \rightarrow \widehat{B[\underline{X}][\underline{Y}]} \xrightarrow{\theta_0} C \rightarrow 0,$$

where the map on the left is a module map determined by the finite set of generators of $\text{Ker } \theta_0$. After taking the derived J -completion, the sequence becomes the exact sequence

$$\widehat{B[\underline{X}, \underline{Y}]}^{\oplus k+m} \rightarrow \widehat{B[\underline{X}, \underline{Y}]} \xrightarrow{\theta} C \rightarrow 0.$$

That is, we have a surjective map $\theta: \widehat{B[\underline{X}, \underline{Y}]} \rightarrow C$ determined on topological generators by $\theta(X_j) = \alpha(X_j)$, $\theta(Y_i) = \beta(Y_i)$, and the kernel of θ is $(f_1, \dots, f_k, Y_1 - g_1, \dots, Y_m - g_m)$.

Next, we choose elements $h_j \in \widehat{B[\underline{Y}]}$ such that $\hat{\beta}(h_j) = \alpha(X_j)$ for each j . Then we have a surjective map $\psi: \widehat{B[\underline{X}, \underline{Y}]} \rightarrow \widehat{B[\underline{Y}]}$ given by $X_j \mapsto h_j$ and $Y_i \mapsto Y_i$, which has the property that $\hat{\beta} \circ \psi = \theta$. That is,

$$\text{Ker } \theta = \text{Ker}(\hat{\beta} \circ \psi) = \psi^{-1}(\text{Ker}(\hat{\beta})),$$

and therefore $\psi(\text{Ker } \theta) = \text{Ker } \hat{\beta}$ since ψ is surjective. But $\text{Ker } \theta$ is finitely generated by the previous, and hence so is $\text{Ker } \hat{\beta}$. ■

Proposition 2.19. *The map $\coprod_j h_{\mathcal{V}_j} \rightarrow h_{\mathcal{X}} = *$ (where $\coprod$ denotes the coproduct in the category $\text{Shv}((\mathcal{X}/A)_{\mathbb{A}}^{\text{II}})$) to the final sheaf is an epimorphism, hence so is the map*

$$\coprod_j h_{\check{\mathcal{C}}_{\mathcal{V}_j}} \rightarrow *.$$

Proof. It is enough to show that for a given object $(B, IB) \in (\mathcal{X}/A)_{\Delta}^{\Pi}$, there is a faithfully flat family $(B, IB) \rightarrow (C_i, IC_i)$ in $(\mathcal{X}/A)_{\Delta}^{\Pi, \text{op}}$ such that $\coprod_j^{\text{pre}} h_{\mathcal{V}_j}((C_i, IC_i)) \neq \emptyset$ for all i where $\coprod^{\text{pre}}$ denotes the coproduct of presheaves.

With that aim, let us first consider the preimages $\mathcal{W}_j \subseteq \text{Spf}(B/IB)$ of each $\mathcal{V}_j$ under the map $\text{Spf}(B/IB) \rightarrow \mathcal{X}$. This is an open cover of $\text{Spf}(B/IB)$ that corresponds to an open cover of $\text{Spec } B/(p, I)B$. One can then choose $f_1, f_2, \dots, f_m$ such that $\{\text{Spec}(B/(p, I)B)_{f_i}\}_i$ refines this cover, i.e., every $\text{Spec}(B/(p, I)B)_{f_i}$ corresponds to an open subset of $\mathcal{W}_{j(i)}$ for some index $j(i)$.

The elements $f_1, \dots, f_m$ generate the unit ideal of B since they do so modulo (p, I) which is contained in $\text{rad}(B)$. Thus, the family

$$(B, IB) \rightarrow (C_i, IC_i) := (\widehat{B}_{f_i}, I\widehat{B}_{f_i}), \quad i = 1, 2, \dots, m$$

is easily seen to give the desired faithfully flat family, with each $\coprod_j^{\text{pre}} h_{\mathcal{V}_j}((C_i, IC_i))$ non-empty, since each $\text{Spf}(C_i/IC_i) \rightarrow \mathcal{X}$ factors through $\mathcal{V}_{j(i)}$ by construction. ■

Remark 2.20. The proof of Proposition 2.19 is the one step where we used the refinement of the topology, namely the fact that the faithfully flat cover $(B, IB) \rightarrow \prod_i (C_i, IC_i)$ can be replaced by the family $\{(B, IB) \rightarrow (C_i, IC_i)\}_i$.

Finally, we obtain the Čech–Alexander complexes in the global case.

Proposition 2.21. *With the notation for $\mathcal{V}_{j_1, \dots, j_n}$ as above and the choice of prismatic covers $\check{\mathcal{C}}_{\mathcal{V}_{j_1, j_2, \dots, j_n}}$ as in Proposition 2.15, $\text{R}\Gamma((\mathcal{X}/A)_{\Delta}, \mathcal{O})$ is modelled by the Čech–Alexander complex*

$$0 \rightarrow \prod_j \check{\mathcal{C}}_{\mathcal{V}_j} \rightarrow \prod_{j_1, j_2} \check{\mathcal{C}}_{\mathcal{V}_{j_1, j_2}} \rightarrow \prod_{j_1, j_2, j_3} \check{\mathcal{C}}_{\mathcal{V}_{j_1, j_2, j_3}} \rightarrow \dots \quad (\check{\mathcal{C}}_{\mathfrak{g}}^{\bullet})$$

(that is, the complex associated to the cosimplicial ring $(\prod_{j_1, \dots, j_n} \check{\mathcal{C}}_{\mathcal{V}_{j_1, \dots, j_n}})_n$).

Proof. By [32, 079Z], the epimorphism of sheaves $\coprod_j h_{\check{\mathcal{C}}_{\mathcal{V}_j}} \rightarrow *$ from Proposition 2.19 implies that there is a spectral sequence with E_1 -page

$$\begin{aligned} E_1^{p, q} &= H^q\left(\left(\coprod_j h_{\check{\mathcal{C}}_{\mathcal{V}_j}}\right)^{\times p}, \mathcal{O}\right) = H^q\left(\coprod_{j_1, j_2, \dots, j_p} h_{\check{\mathcal{C}}_{\mathcal{V}_{j_1, \dots, j_p}}}, \mathcal{O}\right) \\ &= \prod_{j_1, \dots, j_p} H^q((\check{\mathcal{C}}_{\mathcal{V}_{j_1, \dots, j_p}}, I\check{\mathcal{C}}_{\mathcal{V}_{j_1, \dots, j_p}}), \mathcal{O}) \end{aligned}$$

converging to

$$H^{p+q}(*, \mathcal{O}) = H^{p+q}((\mathcal{X}/A)_{\Delta}^{\Pi}, \mathcal{O}) = H^{p+q}((\mathcal{X}/A)_{\Delta}, \mathcal{O}),$$

where we implicitly used Corollary 2.9 and the fact that

$$h_{\check{\mathcal{C}}_{\mathcal{V}_{j_1}}} \times h_{\check{\mathcal{C}}_{\mathcal{V}_{j_2}}} = h_{\check{\mathcal{C}}_{\mathcal{V}_{j_1}} \boxtimes \check{\mathcal{C}}_{\mathcal{V}_{j_2}}} = h_{\check{\mathcal{C}}_{\mathcal{V}_{j_1, j_2}}}$$

as in Proposition 2.15, and similarly for higher multi-indices.

By Lemma 2.8, $H^q((\check{C}_{\mathcal{V}_{j_1, \dots, j_n}}, I\check{C}_{\mathcal{V}_{j_1, \dots, j_n}}), \mathcal{O}) = 0$ for every $q > 0$ and every multi-index $j_1, \dots, j_n$. The first page is therefore concentrated in a single row of the form $\check{C}_{\mathfrak{Y}}^\bullet$ and thus, the spectral sequence collapses on the second page. This proves that the cohomologies of $R\Gamma((\mathcal{X}/A)_{\Delta}, \mathcal{O})$ are computed as cohomologies of $\check{C}_{\mathfrak{Y}}^\bullet$, but in fact, this yields a quasi-isomorphism of the complexes themselves. (For example, analyzing the proof of [32, 079Z] via [32, 03OW], the double complex $E_0^{\bullet\bullet}$ of the above spectral sequence comes equipped with a natural map $\alpha: \check{C}_{\mathfrak{Y}}^\bullet \rightarrow \text{Tot}(E_0^{\bullet\bullet})$, and a natural quasi-isomorphism $\beta: R\Gamma((\mathcal{X}/A)_{\Delta}, \mathcal{O}) \rightarrow \text{Tot}(E_0^{\bullet\bullet})$; when the spectral sequence collapses as above, α is also a quasi-isomorphism.) ■

Notation 2.22. For future use, let us establish the following notation. Given the choice of an affine open cover $\mathfrak{Y} = \{\mathcal{V}_j\}_j$ of $\mathcal{X}$ and prismatic covers $(\check{C}_{\mathcal{V}_{j_1, \dots, j_n}}, I\check{C}_{\mathcal{V}_{j_1, \dots, j_n}})$ chosen as in Proposition 2.15, denote by $N_{\mathfrak{Y}}$ the Čech nerve of $\coprod_j h_{\check{C}_{\mathcal{V}_j}} \rightarrow *$, i.e., the simplicial object in $(\mathcal{X}/A)_{\Delta}^{\text{II}}$ underlying the construction of the Čech–Alexander complex $\check{C}_{\mathfrak{Y}}^\bullet$ of Proposition 2.21 (this is a slight abuse of notation since it depends on the choice of prismatic covers and not just on $\mathfrak{Y}$; however, hopefully such a choice will always be clear from context).

Then for an abelian sheaf $\mathcal{F}$ on $(\mathcal{X}/A)_{\Delta}^{\text{II}}$, we use the notation $\check{C}^\bullet(N_{\mathfrak{Y}}, \mathcal{F})$ for the complex associated with the cosimplicial abelian group obtained from $N_{\mathfrak{Y}}$ by termwise applying $\mathcal{F}$. Thus, for example, the Čech–Alexander complex $\check{C}_{\mathfrak{Y}}^\bullet$ of Proposition 2.21 is also denoted by $\check{C}^\bullet(N_{\mathfrak{Y}}, \mathcal{O})$.

Remark 2.23. Just as in the affine case, the formation of Čech–Alexander complexes is compatible with “termwise flat base-change” on the base prism essentially by [6, Proposition 3.13]. That is, if $(\check{C}^m, \partial)_m$ is a Čech–Alexander complex modelling $R\Gamma_{\Delta}(\mathcal{X}/A)$ and $(A, I) \rightarrow (B, IB)$ is a flat map of prisms, then the complex $(\check{C}^m \hat{\otimes}_A B, \partial \otimes 1)_m$ is a Čech–Alexander complex that computes $R\Gamma_{\Delta}(\mathcal{X}_B/B)$.

In order to understand Čech–Alexander complexes even more explicitly, we employ the following approximation of the prismatic envelope, similar in spirit to the proof of [6, Proposition 3.13].

Definition 2.24. Let B be a δ -ring, $J \subseteq B$ an ideal with a fixed generating set $\underline{x} = \{x_i\}_{i \in \Lambda}$, and let $b \in J$ be an element. Denote by $\mathfrak{b}_0$ be the kernel of the B -algebra map

$$B[\underline{T}] = B[\{T_i\}_{i \in \Lambda}] \rightarrow B\left[\frac{1}{b}\right], \quad T_i \mapsto \frac{x_i}{b}.$$

Denote by $B\{\underline{T}\}$ the free B - δ -algebras on the generators $\underline{T}$, and let $\mathfrak{b}$ be the δ -ideal in $B\{\underline{T}\}$ generated by $\mathfrak{b}_0$. Then we denote by $B\{\frac{\underline{x}}{b}\}$ the δ -ring $B\{\underline{T}\}/\mathfrak{b}$, and call it *weak δ -blowup algebra of $\underline{x}$ and b* .

That is, the above construction adjoins (in δ -sense) the fractions x_i/b to B together with all relations among them that exist in $B[1/b]$, making it possible to naturally compute with fractions.

Note that if $B \rightarrow C$ is a map of B - δ -algebras such that $JC = bC$ and this ideal is invertible, the fact that the localization map $C \rightarrow C[\frac{1}{b}]$ is injective shows that there is a unique map of B - δ -algebras $B\{\frac{x}{b}\} \rightarrow C$.

The purpose of the construction is the following.

Proposition 2.25. *Let (A, I) be a bounded prism with $I = dB$ for a distinguished element $d \in B$. Let $(A, I) \rightarrow (B, J)$ be a map of δ -pairs and assume that (C, IC) is a prismatic envelope for (B, J) that is classically (p, I) -complete. Let $\underline{x} = \{x_i\}_{i \in \Lambda}$ be a system of generators of J . Then there is a surjective map of δ -rings $\widehat{B\{\frac{x}{d}\}}^{\text{cl}} \rightarrow C$, where $(-)^{\text{cl}}$ denotes the classical (p, I) -completion.*

Proposition 2.25 applies in particular when (C, IC) a prismatic cover from Construction 2.11. Since the prismatic cover is (p, I) -completely flat over the base prism, it is classically (p, I) -complete by [6, Proposition 3.7].

Proof. Since $JC = dC$ and d is a non-zero divisor on C , there is an induced map $B\{\frac{x}{d}\} \rightarrow C$ and hence a map of δ -rings $\widehat{B\{\frac{x}{d}\}}^{\text{cl}} \rightarrow C$ (using [6, Lemma 2.17]).

To see that this map is surjective, let C' denote its image in C , and denote by ι the inclusion of C' into C . Then C' is (derived, and, consequently, classically) (p, I) -complete A - δ -algebra with $C'[d] = 0$. It follows that $(C', IC') = (C', (d))$ is a prism by [6, Lemma 3.5] and thus, by the universal property of C , there is a map of B - δ -algebras $r: C \rightarrow C'$ which is easily seen to be right inverse to ι . Hence, ι is surjective, proving the claim. ■

3. The conditions (Cr_s)

3.1. Definition and basic properties

In order to describe the conditions (Cr_s) , we need to fix more notation. Recall that π is a fixed choice of a uniformizer of K . For a natural number s , denote by K_s the field $K(\pi_s)$ (where $(\pi_n)_n$ is the compatible chain of p^n -th roots of π chosen before, i.e., so that $u = [(\pi_n)_n]$ in A_{inf}), and set $K_\infty = \bigcup_s K_s$. Further set $K_{p^\infty} = \bigcup_m K(\zeta_{p^m})$ and for $s \in \mathbb{N} \cup \{\infty\}$, set $K_{p^\infty, s} = K_{p^\infty} K_s$. Note that the field $K_{p^\infty, \infty}$ is the Galois closure of K_∞ . Denote by $\hat{G}$ the Galois group $\text{Gal}(K_{p^\infty, \infty}/K)$ and by G_s the group $\text{Gal}(\bar{K}/K_s)$, for $s \in \mathbb{N} \cup \{\infty\}$.

When $p > 2$, the group $\hat{G}$ is generated by its two subgroups $\text{Gal}(K_{p^\infty, \infty}/K_{p^\infty})$ and $\text{Gal}(K_{p^\infty, \infty}/K_\infty)$ by [25, Lemma 5.1.2]. When $p = 2$, the same can be achieved by a specific choice of π by the following lemma, and we choose π in this manner (this is the specific choice of π mentioned in Section 1).

Lemma 3.1. *Let $p = 2$. Then $K_\infty \cap K_{2^\infty} = K$ for a suitable choice of the uniformizer $\pi \in \mathcal{O}_K$.*

Proof. This is [33, Lemma 2.1]. We just note that while [33] assumes that the residue field k is finite, the proof easily adapts to the more general case of a perfect residue field k . ■

The subgroup $\text{Gal}(K_{p^\infty, \infty}/K_{p^\infty})$ of $\hat{G}$ is normal. An element $g \in \text{Gal}(K_{p^\infty, \infty}/K_{p^\infty})$ is determined by its action on the elements $(\pi_s)_s$, which takes the form $g(\pi_s) = \zeta_{p^s}^{a_s} \pi_s$, with the integers a_s unique modulo p^s and compatible with each other as s increases. It follows that $\text{Gal}(K_{p^\infty, \infty}/K_{p^\infty}) \simeq \mathbb{Z}_p$, with a topological generator τ given by $\tau(\pi_n) = \zeta_{p^n} \pi_n$ (where, again, ζ_{p^n} 's are chosen as before, so that $v = [(\zeta_{p^n})_n] - 1$).

Similarly, the image of G_s in $\hat{G}$ is the subgroup $\hat{G}_s = \text{Gal}(K_{p^\infty, \infty}/K_s)$. Clearly $\hat{G}_s$ contains $\hat{G}_\infty$ and the intersection of $\hat{G}_s$ with $\text{Gal}(K_{p^\infty, \infty}/K_{p^\infty})$ is $\text{Gal}(K_{p^\infty, \infty}/K_{p^\infty, s})$. Just as in the $s = 0$ case, $\hat{G}_s$ is generated by these two subgroups, with the subgroup $\text{Gal}(K_{p^\infty, \infty}/K_{p^\infty, s})$ normal and topologically generated by the element τ^{p^s} .

There is a natural G_K -action on $A_{\text{inf}} = W(\mathcal{O}_{\mathbb{C}_K}^b)$, extended functorially from the natural action on $\mathcal{O}_{\mathbb{C}_K}^b$. This action makes the map $\theta: A_{\text{inf}} \rightarrow \mathcal{O}_{\mathbb{C}_K}$ G_K -equivariant, in particular, the kernel $E(u)A_{\text{inf}}$ is G_K -stable. The G_K -action on the G_K -closure of $\mathfrak{S}$ in A_{inf} factors through $\hat{G}$. Note that the subgroup $\text{Gal}(K_{p^\infty, \infty}/K_\infty)$ of $\hat{G}$ acts trivially on elements of $\mathfrak{S}$, and the action of the subgroup $\text{Gal}(K_{p^\infty, \infty}/K_{p^\infty})$ is determined by the equality

$$\tau(u) = (v + 1)u.$$

For an integer $s \geq 0$ and i between 0 and s , denote by $\xi_{s,i}$ the element

$$\xi_{s,i} = \frac{\varphi^s(v)}{\omega\varphi(\omega) \cdots \varphi^i(\omega)} = \varphi^{-1}(v)\varphi^{i+1}(\omega)\varphi^{i+2}(\omega) \cdots \varphi^s(\omega)$$

(recall that $\omega = v/\varphi^{-1}(v)$), and set

$$I_s = (\xi_{s,0}u, \xi_{s,1}u^p, \dots, \xi_{s,s}u^{p^s}).$$

For convenience of notation, we further set $I_\infty = 0$ and $\varphi^\infty(v)u = 0$.

We are concerned with the following conditions.

Definition 3.2. Let M_{inf} be an A_{inf} -module endowed with a G_K - A_{inf} -semilinear action, let M_{BK} be an $\mathfrak{S}$ -module and let $f: M_{\text{BK}} \rightarrow M_{\text{inf}}$ be an $\mathfrak{S}$ -linear map. Let $s \geq 0$ be an integer or ∞ .

- (1) An element $x \in M_{\text{inf}}$ is called a (Cr_s) -element if for every $g \in G_s$,

$$g(x) - x \in I_s M_{\text{inf}}.$$

- (2) We say that the pair $M_{\text{BK}} \rightarrow M_{A_{\text{inf}}}$ satisfies the condition (Cr_s) if for every element $x \in M_{\text{BK}}$, the image of x in M_{inf} is (Cr_s) .

- (3) An element $x \in M_{\text{inf}}$ is called a (Cr'_s) -element if for every $g \in G_s$,

$$g(x) - x \in \varphi^s(v)u M_{\text{inf}}.$$

- (4) We say that the pair $M_{\text{BK}} \rightarrow M_{A_{\text{inf}}}$ satisfies the condition (Cr'_s) if for every element $x \in M_{\text{BK}}$, the image of x in M_{inf} is (Cr'_s) .

- (5) Additionally, we call (Cr_0) -elements *crystalline elements* and we call the condition (Cr_0) the *crystalline condition*.

Remarks 3.3. Several remarks are in order.

- (1) Since $I_0 = \varphi^{-1}(v)uA_{\text{inf}}$, the crystalline condition equivalently states that for all $g \in G_K$ and all x in the image of M_{BK} ,

$$g(x) - x \in \varphi^{-1}(v)uM_{\text{inf}}.$$

The reason for the extra terminology in the case $s = 0$ is that the condition is connected with a criterion for certain representations to be crystalline, as discussed in Section 3.2. The conditions (Cr_∞) and (Cr'_∞) are clearly both equivalent to the condition $f(M_{\text{BK}}) \subseteq M_{\text{inf}}^{G_\infty}$.

- (2) Strictly speaking, one should talk about the crystalline condition (or (Cr_s)) for the map f , but we choose to talk about the crystalline condition (or (Cr_s)) for the pair $(M_{\text{BK}}, M_{\text{inf}})$ instead, leaving the map f implicit. This is because typically we consider the situation that M_{BK} is an $\mathfrak{S}$ -submodule of $M_{\text{inf}}^{G_\infty}$ and $M_{\text{BK}} \otimes_{\mathfrak{S}} A_{\text{inf}} \simeq M_{A_{\text{inf}}}$ via the natural map (or the derived $(p, E(u))$ -completed variant, $M_{\text{BK}} \hat{\otimes}_{\mathfrak{S}} A_{\text{inf}} \simeq M_{A_{\text{inf}}}$). Also note that $f: M_{\text{BK}} \rightarrow M_{\text{inf}}$ satisfies the condition (Cr_s) if and only if $f(M_{\text{BK}}) \subseteq M_{\text{inf}}$ does.
- (3) For every fixed s , the ideal I_s contains the principal ideal $\varphi^s(v)uA_{\text{inf}}$. Consequently, for every s , (Cr'_s) implies (Cr_s) . To our knowledge, there is no logical dependency between any of the conditions (Cr_s) (and between (Cr'_s) , respectively) for different values of s .

Lemma 3.4. *For any integer s , the ideals $\varphi^s(v)uA_{\text{inf}}$ and I_s are G_K -stable.*

Proof. It is enough to prove that the ideals uA_{inf} and vA_{inf} are G_K -stable. Then G_K -stability of $\varphi^s(v)A_{\text{inf}}$ for any $s \in \mathbb{Z}$ follows since φ is a G_K -equivariant automorphism of A_{inf} . Then $g\varphi^s(v)$ equals to $\varphi^s(v)$ times a unit for every g and s and the same is then true of $\varphi^i(\omega) = \varphi^i(v)/\varphi^{i-1}(v)$, hence also of all the elements $\xi_{i,s}$ and it follows that I_s is G_K -stable.

Given $g \in G_K$, $g(\pi_n) = \zeta_{p^n}^{a_n} \pi_n$ for an integer a_n unique modulo p^n and such that $a_{n+1} \equiv a_n \pmod{p^n}$. It follows that $g(u) = [\varepsilon]^a u$ for a p -adic integer $a (= \lim_n a_n)$. Thus, uA_{inf} is G_K -stable.

Similarly, we have $g(\zeta_{p^n}) = \zeta_{p^n}^{b_n}$, for integers b_n coprime to p , unique modulo p^n and compatible with each other as n grows. It follows that $g([\varepsilon]) = [\varepsilon]^b$ for $b = \lim_n b_n$, and so $g(v) = (v+1)^b - 1 = \lim_n ((v+1)^{b_n} - 1)$. The resulting expression is still divisible by v . To see that, fix the integers b_n to have all positive representatives. Then the claim follows from the formula

$$(v+1)^{b_n} - 1 = v((v+1)^{b_n-1} + (v+1)^{b_n-2} + \cdots + 1),$$

upon noting that the sequence of elements

$$((v+1)^{b_n-1} + (v+1)^{b_n-2} + \cdots + 1) = ((v+1)^{b_n} - 1)/v$$

is still (p, v) -adically convergent, e.g. by Lemma 2.3. ■

Let $f: M_{\text{BK}} \rightarrow M_{\text{inf}}$ be as in Definition 3.2. Lemma 3.4 shows that the modules $M_{\text{inf}}/I_s M_{\text{inf}}$ and $M_{\text{inf}}/\varphi^s(v)u M_{\text{inf}}$ have a well-defined G_K -action. Consequently, we get the following restatement of the conditions (Cr_s) and (Cr'_s) .

Lemma 3.5. *Given $f: M_{\text{BK}} \rightarrow M_{\text{inf}}$ as in Definition 3.2, the pair $(M_{\text{BK}}, M_{\text{inf}})$ satisfies the condition (Cr_s) ((Cr'_s) , resp.) if and only if the image of M_{BK} in $\overline{M_{\text{inf}}} := M_{\text{inf}}/I_s M_{\text{inf}}$ ($\overline{M_{\text{inf}}} := M_{\text{inf}}/\varphi^s(v)u M_{\text{inf}}$, resp.) lands in $\overline{M_{\text{inf}}}^{G_s}$.*

In the case of the above-mentioned condition $f(M_{\text{BK}}) \subseteq M_{\text{inf}}^{G_\infty}$, the G_K -closure of $f(M_{\text{BK}})$ in M_{inf} is contained in the G_K -submodule $M_{\text{inf}}^{G_{K_{p^\infty, \infty}}}$, and thus, the G_K -action on it factors through $\widehat{G}$. Under mild assumptions on M_{inf} , the G_s -action on the elements of $f(M_{\text{BK}})$ is ultimately determined by τ^{p^s} , the topological generator of $\text{Gal}(K_{p^\infty, \infty}/K_{p^\infty, s})$. Consequently, the conditions (Cr'_s) are also determined by the action of this single element.

Lemma 3.6. *Let $f: M_{\text{BK}} \rightarrow M_{\text{inf}}$ be as in Definition 3.2. Additionally, assume that M_{inf} is classically $(p, E(u))$ -complete and $(p, E(u))$ -completely flat, that the action of G_K on M_{inf} is continuous for the $(p, E(u))$ -adic topology, and that the pair $(M_{\text{BK}}, M_{\text{inf}})$ satisfies (Cr_∞) . Then the action of $\widehat{G}$ on elements of $f(M_{\text{BK}})$ makes sense, and $(M_{\text{BK}}, M_{\text{inf}})$ satisfies (Cr'_s) if and only if*

$$\forall x \in f(M_{\text{BK}}): \tau^{p^s}(x) - x \in \varphi^s(v)u M_{\text{inf}}.$$

Proof. Clearly the stated condition is necessary. To prove sufficiency, assume the above condition for τ^{p^s} . By the fixed-point interpretation of the condition (Cr'_s) as in Lemma 3.5, it is clear that the analogous condition holds for every element $g \in \langle \tau^{p^s} \rangle$.

Next, let us assume that g is an element of $\text{Gal}(K_{p^\infty, \infty}/K_{p^\infty, s})$, the p -adic closure of $\langle \tau^{p^s} \rangle$. Then $g = \lim_n \tau^{p^s a_n}$, with the sequence of integers (a_n) p -adically convergent. For $x \in f(M_{\text{BK}})$, by continuity we have $g(x) - x = \lim_n (\tau^{p^s a_n}(x) - x)$, which is equal to $\lim_n \varphi^s(v)u y_n$ with $y_n \in M_{\text{inf}}$. Since the sequence (y_n) is still convergent (using the fact that the $(p, E(u))$ -adic topology is the $(p, \varphi^s(v)u)$ -adic topology, and that $p, \varphi^s(v)u$ is a regular sequence on M_{inf}), we have that $g(x) - x = \varphi^s(v)u y$ where $y = \lim_n y_n$.

To conclude, note that a general element g of $\widehat{G}_s$ can be written in the form $g = g_1 g_2$ where $g_1 \in \text{Gal}(K_{p^\infty, \infty}/K_{p^\infty, s})$ and $g_2 \in \text{Gal}(K_{p^\infty, \infty}/K_\infty)$. Then for $x \in f(M_{\text{BK}})$, by the assumption $f(M_{\text{BK}}) \subseteq M_{\text{inf}}^{G_\infty}$ we have $g_1 g_2(x) - x = g_1(x) - x$, and so the condition (Cr'_s) is proved by the previous part. ■

Let us now discuss some basic algebraic properties of the conditions (Cr_s) and (Cr'_s) . The basic situation when they are satisfied is the inclusion $\mathfrak{S} \hookrightarrow A_{\text{inf}}$ itself.

Lemma 3.7. *The pair $\mathfrak{S} \hookrightarrow A_{\text{inf}}$ satisfies the conditions (Cr'_s) (hence also (Cr_s)) for all $s \geq 0$.*

Proof. Note that $\mathfrak{S} \hookrightarrow A_{\text{inf}}$ satisfies the assumptions of Lemma 3.6, so it is enough to consider the action of the element $\tau^{p^s} \in \widehat{G}_s$. For an element $f = \sum_i a_i u^i \in \mathfrak{S}$ we have

$$\tau^{p^s}(f) - f = \sum_{i \geq 0} a_i ((v+1)^{p^s} u)^i - \sum_{i \geq 0} a_i u^i = \sum_{i \geq 1} a_i ((v+1)^{p^s} - 1) u^i,$$

and thus,

$$\frac{\tau^{p^s}(f) - f}{\varphi^s(v)u} = \sum_{i \geq 1} a_i \frac{(v+1)^{p^s i} - 1}{\varphi^s(v)} u^{i-1} = \sum_{i \geq 1} a_i \frac{(v+1)^{p^s i} - 1}{(v+1)^{p^s} - 1} u^{i-1}.$$

Since $\varphi^s(v) = (v+1)^{p^s} - 1$ divides $(v+1)^{p^s i} - 1$ for each i , the obtained series has coefficients in A_{inf} , showing that $\tau^{p^s}(f) - f \in \varphi^s(v)uA_{\text{inf}}$ as desired. $\blacksquare$

The following lemma shows that in various contexts, it is often sufficient to verify the conditions (Cr_s) and (Cr'_s) on generators.

Lemma 3.8. *Fix an integer $s \geq 0$. Let (C) be either the condition (Cr_s) or (Cr'_s) .*

- (1) *Let M_{inf} be an A_{inf} -module with a G_K - A_{inf} -semilinear action. The set of all (C)-elements forms an $\mathfrak{S}$ -submodule of M_{inf} .*
- (2) *Let C_{inf} be an A_{inf} -algebra endowed with a G_K -semilinear action. The set of (C)-elements of C_{inf} forms an $\mathfrak{S}$ -subalgebra of C_{inf} .*
- (3) *If the algebra C_{inf} from (2) is additionally A_{inf} - δ -algebra such that G_K acts by δ -maps (i.e., $\delta g = g\delta$ for all $g \in G_K$) then the set of all (C)-elements forms a $\mathfrak{S}$ - δ -subalgebra of C_{inf} .*
- (4) *If the algebra C_{inf} as in (2) is additionally classically $(p, E(u))$ -complete and $C_{\text{BK}} \rightarrow C_{\text{inf}}$ is a map of $\mathfrak{S}$ -algebras that satisfies the condition (C), then so does $\widehat{C_{\text{BK}}} \rightarrow C_{\text{inf}}$, where $\widehat{C_{\text{BK}}}$ is the classical $(p, E(u))$ -completion of C_{BK} . In particular, the set of all (C)-elements in C_{inf} forms a classically $(p, E(u))$ -complete $\mathfrak{S}$ -subalgebra of C_{inf} .*

Proof. Let J be the ideal I_s if $(\text{C}) = (\text{Cr}_s)$ and the ideal $\varphi^s(v)uA_{\text{inf}}$ if $(\text{C}) = (\text{Cr}'_s)$. In view of Lemma 3.5, the sets described in (1) and (2) are obtained as the preimages of $(M_{\text{inf}}/JM_{\text{inf}})^{G_s}$ ($(C_{\text{inf}}/JC_{\text{inf}})^{G_s}$, resp.) under the canonical map $M_{\text{inf}} \rightarrow M_{\text{inf}}/JM_{\text{inf}}$ ($C_{\text{inf}} \rightarrow C_{\text{inf}}/JC_{\text{inf}}$, resp.). As these G_s -fixed points form an $\mathfrak{S}$ -module ($\mathfrak{S}$ -algebra, resp.) by Lemma 3.7, this proves (1) and (2).

Similarly, to prove (3) we need to prove only that the ideal JC_{inf} is a δ -ideal and therefore the canonical projection $C_{\text{inf}} \rightarrow C_{\text{inf}}/JC_{\text{inf}}$ is a map of δ -rings.

Let us argue first in the case (Cr'_s) . As $\delta(u) = 0$, we have

$$\delta(\varphi^s(v)u) = \delta(\varphi^s(v))u^p = \frac{\varphi(\varphi^s(v)) - (\varphi^s(v))^p}{p} u^p = \frac{\varphi^{s+1}(v) - (\varphi^s(v))^p}{p} u^p.$$

Recall that $\varphi^s(v) = [\varepsilon]^{p^s} - 1$ divides $\varphi^{s+1}(v) = ([\varepsilon]^{p^s})^p - 1$. The numerator of the last fraction is thus divisible by $\varphi^s(v)$ and since $\varphi^s(v)A_{\text{inf}} \cap pA_{\text{inf}} = \varphi^s(v)pA_{\text{inf}}$ by Lemma 2.3, $\varphi^s(v)$ divides the whole fraction $(\varphi^{s+1}(v) - (\varphi^s(v))^p)/p$ in A_{inf} . (We note that this is true for every integer s , in particular $s = -1$, as well.)

Let us now prove that the ideal $J = I_s$ (hence also $I_s C_{\text{inf}}$) is a δ -ideal. For any i between 0 and $s - 1$, we have

$$\begin{aligned} \delta(\xi_{s,i}) &= \delta(\varphi^{-1}(v)\varphi^{i+1}(\omega) \cdots \varphi^s(\omega)) \\ &= \frac{\varphi^{-1}(v)\omega\varphi^{i+2}(\omega) \cdots \varphi^{s+1}(\omega) - \varphi^{-1}(v)^p\varphi^{i+1}(\omega)^p \cdots \varphi^s(\omega)^p}{p}. \end{aligned}$$

The numerator is divisible by $\xi_{s,i+1}$, and so is the whole fraction thanks to Lemma 2.3. Thus, we have that $\delta(\xi_{s,i}u^{p^i}) = \delta(\xi_{s,i})u^{p^{i+1}}$ is a multiple of $\xi_{s,i+1}u^{p^{i+1}}$. Finally, when $i = s$, we have $\xi_{s,s} = \varphi^{-1}(v)$, and $\delta(\xi_{s,s})$ is thus a multiple of $\xi_{s,s}$ by the previous. Consequently, $\delta(\xi_{s,s}u^{p^s}) = \delta(\xi_{s,s})u^{p^{s+1}}$ is divisible by $\xi_{s,s}u^{p^s}$. This shows that I_s is a δ -ideal.

Finally, let us prove (4). Note that $E(u) \equiv u^e \pmod{p\mathfrak{S}}$, hence

$$\sqrt{(p, E(u))} = \sqrt{(p, u^e)} = \sqrt{(p, u)}$$

even as ideals of $\mathfrak{S}$; consequently, $(p, E(u))$ -completions agree with (p, u) -completions both for $\mathfrak{S}$ - and A_{inf} -modules. We may therefore replace $(p, E(u))$ -completions with (p, u) -completions throughout.

Since C_{inf} is (p, u) -complete, any power series of the form

$$f = \sum_{i,j} c_{i,j} p^i u^j$$

with $c_{i,j} \in C_{\text{inf}}$ defines a unique element $f \in C_{\text{inf}}$, and f comes from $\widehat{C_{\text{BK}}}$ if and only if the coefficients $c_{i,j}$ may be chosen in the image of the map $C_{\text{BK}} \rightarrow C_{\text{inf}}$. Assuming this, for $g \in G_s$ we have

$$\begin{aligned} g(f) - f &= \sum_{i,j} g(c_{i,j}) p^i (\gamma u)^j - \sum_{i,j} c_{i,j} p^i u^j \\ &= \sum_{i,j} (g(c_{i,j})\gamma^j - g(c_{i,j}) + g(c_{i,j}) - c_{i,j}) p^i u^j, \end{aligned}$$

where γ is the A_{inf} -unit such that $g(u) = \gamma u$. Thus, it is clearly enough to show, upon assuming the condition (C) for $(C_{\text{BK}}, C_{\text{inf}})$, that the terms $(g(c_{i,j})\gamma^j - g(c_{i,j}))p^i u^j$ and $(g(c_{i,j}) - c_{i,j})p^i u^j$ are in $J C_{\text{inf}}$ when $g \in G_s$. (Note that an element $d = \sum_{i,j} d_{i,j} p^i u^j$ with $d_{i,j} \in J C_{\text{inf}}$ is itself in $J C_{\text{inf}}$, since J is finitely generated.)

Since $g(c_{i,j}) - c_{i,j} \in J C_{\text{inf}}$ by assumption, it remains to treat the term $g(c_{i,j})(\gamma^j - 1)$. Since $(\gamma^j - 1)$ is divisible by $\gamma - 1$, it is also divisible by $\varphi^s(v)$ by Lemma 3.7. Thus, the terms $g(c_{i,j})(\gamma^j - 1)p^i u^j$ are divisible by $\varphi^s(v)u$ when $j \geq 1$, and are 0 when $j = 0$; in either case, they are members of $J C_{\text{inf}}$.

To prove the second assertion of (4), let now $C_{\text{BK}} \subseteq C_{\text{inf}}$ be the $\mathfrak{S}$ -subalgebra of all (C)-elements. By the previous, the map $\widehat{C_{\text{BK}}} \rightarrow C_{\text{inf}}$ satisfies (C), and hence the image C_{BK}^+ of this map consists of (C)-elements. Thus, we have $C_{\text{BK}} \subseteq C_{\text{BK}}^+ \subseteq C_{\text{BK}}$, and hence, C_{BK} is derived $(p, E(u))$ -complete since so is C_{BK}^+ . However, as a subring of C_{inf} it is also separated for the $(p, E(u))$ -adic topology, hence it is classically $(p, E(u))$ -complete. ■

For future use in applications to p^n -torsion modules, we consider the following approximation of the ideals I_s appearing in the conditions (Cr_s) .

Lemma 3.9. *Consider a pair of integers n, s with $s \geq 0, n \geq 1$. Set $t = \max\{0, s + 1 - n\}$. Then $I_s \subseteq \varphi^{-1}(v)u^{p^t}A_{\text{inf}} + p^n A_{\text{inf}}$.*

Proof. When $t = 0$ there is nothing to prove, therefore we may assume that

$$t = s + 1 - n > 0.$$

In the definition of I_s , we may replace the elements

$$\xi_{s,i} = \varphi^{-1}(v)\varphi^{i+1}(\omega)\varphi^{i+2}(\omega) \cdots \varphi^s(\omega)$$

by the elements

$$\xi'_{s,i} = \varphi^{-1}(v)\varphi^{i+1}(E(u))\varphi^{i+2}(E(u)) \cdots \varphi^s(E(u)),$$

since the quotients $\xi_{s,i}/\xi'_{s,i}$ are A_{inf} -units.

It is thus enough to show that for every i with $0 \leq i \leq s$, the element

$$\vartheta_{s,i} = \frac{\xi'_{s,i}u^{p^i}}{\varphi^{-1}(v)} = \varphi^{i+1}(E(u))\varphi^{i+2}(E(u)) \cdots \varphi^s(E(u))u^{p^i}$$

taken modulo p^n is divisible by $u^{p^{s+1-n}}$.

This is clear when $i \geq s + 1 - n$, and so it remains to discuss the cases when $i \leq s - n$. Write $\varphi^j(E(u)) = (u^e)^{p^j} + px_j$ (with $x_j \in \mathfrak{S}$). Then it is enough to show that

$$\frac{\vartheta_{s,i}}{u^{p^i}} = ((u^e)^{p^{i+1}} + px_{i+1})((u^e)^{p^{i+2}} + px_{i+2}) \cdots ((u^e)^{p^s} + px_s) \quad (*)$$

taken modulo p^n is divisible by

$$u^{p^{s+1-n}-p^i} = u^{p^i(p-1)(1+p+\cdots+p^{s-n-i})}.$$

Since we are interested in the product $(*)$ only modulo p^n , in expanding the brackets we may ignore the terms that use the expressions of the form px_j at least n times. Each of the remaining terms contains the product of at least $s - i - n + 1$ distinct terms from the following list:

$$(u^e)^{p^{i+1}}, (u^e)^{p^{i+2}}, \dots, (u^e)^{p^s}.$$

Thus, each of the remaining terms is divisible by (at least)

$$(u^e)^{p^{i+1}+p^{i+2}+\cdots+p^{s-n+1}} = (u^e)^{p^i \cdot (p) \cdot (1+p+\cdots+p^{s-n-i})},$$

which is more than needed. This finishes the proof. ■

3.2. Crystalline condition for Breuil–Kisin–Fargues G_K -modules

The situation of central interest for the crystalline condition is the inclusion $M_{\text{BK}} \subseteq M_{\text{inf}}^{G_\infty}$ such that $A_{\text{inf}} \otimes_{\mathfrak{S}} M_{\text{BK}} \rightarrow M_{\text{inf}}$ is an isomorphism, where M_{BK} is a Breuil–Kisin module and M_{inf} is a Breuil–Kisin–Fargues G_K -module. The version of these notions used in this paper is tailored to the context of prismatic cohomology.

Definition 3.10. Breuil–Kisin(–Fargues, resp.) modules are defined as follows.

- (1) A *Breuil–Kisin module* consists of a finitely generated $\mathfrak{S}$ -module M together with a $\mathfrak{S}[1/E(u)]$ -linear isomorphism

$$\varphi = \varphi_{M[1/E]}: (\varphi_{\mathfrak{S}}^* M)[1/E(u)] \xrightarrow{\sim} M[1/E(u)].$$

For a positive integer i , the Breuil–Kisin module M is said to be *of height $\leq i$* if $\varphi_{M[1/E]}$ is induced (by linearization and localization) by a $\varphi_{\mathfrak{S}}$ -semilinear map $\varphi_M: M \rightarrow M$ such that, denoting $\varphi_{\text{lin}}: \varphi^* M \rightarrow M$ its linearization, there exists an $\mathfrak{S}$ -linear map $\psi: M \rightarrow \varphi^* M$ such that both the compositions $\psi \circ \varphi_{\text{lin}}$ and $\varphi_{\text{lin}} \circ \psi$ are multiplication by $E(u)^i$. A Breuil–Kisin module is *of finite height* if it is of height $\leq i$ for some i .

- (2) A *Breuil–Kisin–Fargues module* is a finitely presented A_{inf} -module M such that $M[1/p]$ is a free $A_{\text{inf}}[1/p]$ -module, together with an $A_{\text{inf}}[1/E(u)]$ -linear isomorphism

$$\varphi = \varphi_{M[1/E]}: (\varphi_{A_{\text{inf}}}^* M)[1/E(u)] \xrightarrow{\sim} M[1/E(u)].$$

Similarly, the Breuil–Kisin–Fargues module is *of height $\leq i$* if $\varphi_{M[1/E]}$ comes from a $\varphi_{A_{\text{inf}}}$ -semilinear map $\varphi_M: M \rightarrow M$ admitting an A_{inf} -linear map $\psi: M \rightarrow \varphi^* M$ such that $\psi \circ \varphi_{\text{lin}}$ and $\varphi_{\text{lin}} \circ \psi$ are multiplication maps by $E(u)^i$, where φ_{lin} is the linearization of φ_M . A Breuil–Kisin–Fargues module is *of finite height* if it is of height $\leq i$ for some i .

- (3) A *Breuil–Kisin–Fargues G_K -module* (of height $\leq i$, of finite height, resp.) is a Breuil–Kisin–Fargues module (of height $\leq i$, of finite height, resp.) that is additionally endowed with a continuous, A_{inf} -semilinear G_K -action that makes $\varphi_{M[1/E]}$ G_K -equivariant (that makes also φ_M G_K -equivariant in the finite height cases).

That is, the definition of a Breuil–Kisin module agrees with the one in [4], and M_{inf} is a Breuil–Kisin–Fargues module in the sense of the above definition if and only if $\varphi_{A_{\text{inf}}}^* M_{\text{inf}}$ is a Breuil–Kisin–Fargues module in the sense of [4].¹⁰ The notion of Breuil–Kisin module of height $\leq i$ agrees with what is called “(generalized) Kisin modules of height i ” in [24].

¹⁰This is to account for the fact that while Breuil–Kisin–Fargues modules in the sense of [4] appear as A_{inf} -cohomology groups of smooth proper formal schemes, Breuil–Kisin–Fargues modules in the above sense appear as *prismatic* A_{inf} -cohomology groups of smooth proper formal schemes.

The above notion of finite height Breuil–Kisin–Fargues modules agrees with the one from [14, Appendix F] except that the modules are not assumed to be free. Also note that under these definitions, for a Breuil–Kisin module M_{BK} (of height $\leq i$, resp.), the A_{inf} -module $M_{\text{inf}} = A_{\text{inf}} \otimes_{\mathfrak{S}} M_{\text{BK}}$ is a Breuil–Kisin–Fargues module (of height $\leq i$, resp.), without the need to twist the embedding $\mathfrak{S} \rightarrow A_{\text{inf}}$ by φ .

The connection between Breuil–Kisin-, Breuil–Kisin–Fargues G_K -modules and the crystalline condition (justifying its name) is the following theorem.

Theorem 3.11 ([14, Appendix F], [20]). *Let M_{inf} be a free Breuil–Kisin–Fargues G_K -module which admits as an $\mathfrak{S}$ -submodule a free Breuil–Kisin module $M_{\text{BK}} \subseteq M_{\text{inf}}^{G_{\infty}}$ of finite height, such that $A_{\text{inf}} \otimes_{\mathfrak{S}} M_{\text{BK}} \xrightarrow{\sim} M_{\text{inf}}$ (as Breuil–Kisin–Fargues modules) via the natural map, and such that the pair $(M_{\text{BK}}, M_{\text{inf}})$ satisfies the crystalline condition. Then the étale realization of M_{inf} ,*

$$V(M_{\text{inf}}) = (W(\mathbb{C}_K^b) \otimes_{A_{\text{inf}}} M_{\text{inf}})^{\varphi=1} \left[\frac{1}{p} \right],$$

is a crystalline representation.

Remarks 3.12. In more detail, Theorem 3.11 relates to the results of [14, 20] as follows.

- (1) Theorem 3.11 is actually an equivalence: If $V(M_{\text{inf}})$ is crystalline, it can be shown that the pair $(M_{\text{BK}}, M_{\text{inf}})$ satisfies the crystalline condition. We state the theorem in the one direction since this is the one that we use. However, the converse direction motivates why it is reasonable to expect the crystalline condition for prismatic cohomology groups that is discussed in Section 4.
- (2) Strictly speaking, in [14, Appendix F] one assumes extra conditions on the pair M_{inf} (“satisfying all descents”); however, these extra assumptions are used only for a semistable version of the statement. Theorem 3.11 in its equivalence form is therefore only implicit in the proof of [14, Theorem F.11]. (See also [28, Theorem 3.8] for a closely related result.)
- (3) On the other hand, Theorem 3.11 in the one-sided form as above is a consequence of [20, Proposition 7.11] that states that $V(M_{\text{inf}})$ is crystalline if and only if the weaker condition

$$\forall g \in G_K: \quad (g-1)M_{\text{BK}} \subseteq \varphi^{-1}(v)W(\mathfrak{m}_{\mathbb{C}_K^b})M_{\text{inf}}$$

is satisfied.

It will be convenient later to have version of Theorem 3.11 that applies to not necessarily free Breuil–Kisin and Breuil–Kisin–Fargues modules. Recall that, by [4, Proposition 4.3], any Breuil–Kisin module M_{BK} is related to a free Breuil–Kisin module $M_{\text{BK}, \text{free}}$ by a functorial exact sequence

$$0 \rightarrow M_{\text{BK}, \text{tor}} \rightarrow M_{\text{BK}} \rightarrow M_{\text{BK}, \text{free}} \rightarrow \overline{M_{\text{BK}}} \rightarrow 0,$$

where $M_{\text{BK},\text{tor}}$ is a p^n -torsion module for some n and $\overline{M_{\text{BK}}}$ is supported at the maximal ideal (p, u) . Taking the base-change to A_{inf} , one obtains an analogous exact sequence

$$0 \rightarrow M_{\text{inf},\text{tor}} \rightarrow M_{\text{inf}} \rightarrow M_{\text{inf},\text{free}} \rightarrow \overline{M_{\text{inf}}} \rightarrow 0$$

(also described by [4, Proposition 4.13]) where $M_{\text{inf},\text{free}}$ is a free Breuil–Kisin–Fargues module. Clearly the maps $M_{\text{BK}} \rightarrow M_{\text{BK},\text{free}}$ and $M_{\text{inf}} \rightarrow M_{\text{inf},\text{free}}$ become isomorphisms after inverting p .

Assume that M_{inf} is endowed with a G_K -action that makes it a Breuil–Kisin–Fargues G_K -module.

By the functoriality of the latter exact sequence, one obtains an induced G_K -action on $M_{\text{inf},\text{free}}$, making it a free Breuil–Kisin–Fargues G_K -module and the map $M_{\text{inf}} \rightarrow M_{\text{inf},\text{free}}$ G_K -equivariant. Note that this map becomes an isomorphism of φ - G_K -modules after inverting p .

Proposition 3.13. *Assume that the pair $M_{\text{BK}} \hookrightarrow M_{\text{inf}}$ satisfies the crystalline condition. Then so does the pair $M_{\text{BK},\text{free}} \hookrightarrow M_{\text{inf},\text{free}}$.*

Proof. Notice that the crystalline condition is satisfied for $M_{\text{BK}}[1/p] \rightarrow M_{\text{inf}}[1/p]$ and by [4, Propositions 4.3, 4.13], this map can be identified with $M_{\text{BK},\text{free}}[1/p] \hookrightarrow M_{\text{inf},\text{free}}[1/p]$. Thus, the following lemma finishes the proof. ■

Lemma 3.14. *Let F_{inf} be a free A_{inf} -module endowed with A_{inf} -semilinear G_K -action and let $F_{\text{BK}} \subseteq F_{\text{inf}}$ be a free $\mathfrak{S}$ -submodule such that $F_{\text{BK}}[1/p] \hookrightarrow F_{\text{inf}}[1/p]$ satisfies the crystalline condition. Then the pair $F_{\text{BK}} \hookrightarrow F_{\text{inf}}$ satisfies the crystalline condition.*

Proof. Fix an element $a \in F_{\text{BK}}$ and $g \in G_K$. The crystalline condition holds after inverting p , and so we have the equality

$$b := (g - 1)a = \varphi^{-1}(v)u \frac{c}{p^k}$$

with $c \in F_{\text{inf}}$. In other words (using that p^k is a non-zero divisor on F_{inf}), we have

$$p^k b = \varphi^{-1}(v)uc \in p^k F_{\text{inf}} \cap \varphi^{-1}(v)u F_{\text{inf}} = p^k \varphi^{-1}(v)u F_{\text{inf}},$$

where the last equality follows by Lemma 2.3 since F_{inf} is a free module. In particular, we have

$$p^k b = p^k \varphi^{-1}(v)ud$$

for yet another element $d \in F_{\text{inf}}$. As p^k is a non-zero divisor on A_{inf} , hence on F_{inf} , we may cancel out to conclude

$$(g - 1)a = b = \varphi^{-1}(v)ud \in \varphi^{-1}(v)u F_{\text{inf}},$$

as desired. ■

Proposition 3.13 leads to the following strengthening of Theorem 3.11.

Theorem 3.15. *The “free” assumption in Theorem 3.11 is superfluous. That is, given a Breuil–Kisin–Fargues G_K -module M_{inf} and a Breuil–Kisin $\mathfrak{S}$ -submodule $M_{\text{BK}} \subseteq M_{\text{inf}}^{G_{\text{inf}}^\infty}$ of finite height such that $A_{\text{inf}} \otimes_{\mathfrak{S}} M_{\text{BK}} \xrightarrow{\sim} M_{\text{inf}}$ and such that the pair $(M_{\text{BK}}, M_{\text{inf}})$ satisfies the crystalline condition, the representation*

$$V(M_{\text{inf}}) = (W(\mathbb{C}_K^b) \otimes_{A_{\text{inf}}} M_{\text{inf}})^{\varphi=1} \left[\frac{1}{p} \right]$$

is crystalline.

Proof. With the notation as above, upon realizing that $V(M_{\text{inf}})$ and $V(M_{\text{inf}, \text{free}})$ agree, the result is a direct consequence of Proposition 3.13. ■

4. Conditions (Cr_s) for cohomology

4.1. Conditions (Cr_s) for Čech–Alexander complexes

Let $\mathcal{X}$ be a smooth separated p -adic formal scheme over $\mathcal{O}_K$. $\text{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_{c_K}}/A_{\text{inf}})$ comes with a natural G_K -action by functoriality of cohomology, which we now discuss in detail.

Definition 4.1. For $g \in G_K$, the semilinear map $g: \text{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_{c_K}}/A_{\text{inf}}) \rightarrow \text{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_{c_K}}/A_{\text{inf}})$ is given as follows.

- (0) As $\mathcal{X}_{\mathcal{O}_{c_K}}$ comes from $\mathcal{X}/\mathcal{O}_K$, there is a canonical identification of $g^*\mathcal{X}_{\mathcal{O}_{c_K}}$ with $\mathcal{X}_{\mathcal{O}_{c_K}}$ as a formal $\mathcal{O}_{c_K}$ -scheme. In more detail, there is a Cartesian square of the form

$$\begin{array}{ccc} \mathcal{X}_{\mathcal{O}_{c_K}} & \xrightarrow{\tilde{g}} & \mathcal{X}_{\mathcal{O}_{c_K}} \\ \downarrow s' & & \downarrow s' \\ \text{Spf}\mathcal{O}_{c_K} & \xrightarrow{g} & \text{Spf}\mathcal{O}_{c_K}. \end{array}$$

(where $s': \mathcal{X}_{\mathcal{O}_{c_K}} \rightarrow \text{Spf}\mathcal{O}_{c_K}$ is the structure map). From now on, let us denote the map $\tilde{g}$ again simply by g . Note that for an affine open $\mathcal{V} = \text{Spf}(R) \subseteq \mathcal{X}$ and $\mathcal{V}' = \mathcal{V}_{\mathcal{O}_{c_K}} \subseteq \mathcal{X}_{\mathcal{O}_{c_K}}$, one again has $g^*\mathcal{V}'$ canonically identified with $\mathcal{V}'$ (even as formal subschemes of $\mathcal{X}_{\mathcal{O}_{c_K}}$), and the resulting map $g: \mathcal{V}' \rightarrow \mathcal{V}'$ is described in terms of coordinate rings as $R \hat{\otimes}_{\mathcal{O}_K} \mathcal{O}_{c_K} \rightarrow R \hat{\otimes}_{\mathcal{O}_K} \mathcal{O}_{c_K}$, $x \otimes a \mapsto x \otimes g(a)$ (equivalently, the canonical isomorphism $\mathcal{V}' \xrightarrow{\sim} g^*\mathcal{V}'$ is given by the map of coordinate rings $R \hat{\otimes}_{\mathcal{O}_K} \mathcal{O}_{c_K} \otimes_{\mathcal{O}_{c_K}, g} \mathcal{O}_{c_K} \rightarrow R \hat{\otimes}_{\mathcal{O}_K} \mathcal{O}_{c_K}$, $x \otimes a \otimes b \mapsto x \otimes g(a)b$).

- (1) The map $g: \mathcal{X}_{\mathcal{O}_{c_K}} \rightarrow \mathcal{X}_{\mathcal{O}_{c_K}}$ from the previous step induces a (continuous and co-continuous) functor of sites $g: (\mathcal{X}_{\mathcal{O}_{c_K}}/A_{\text{inf}})_{\Delta}^{\Pi} \rightarrow (\mathcal{X}_{\mathcal{O}_{c_K}}/A_{\text{inf}})_{\Delta}^{\Pi}$ given as follows. An object described by $(A_{\text{inf}}, (\xi)) \rightarrow (B, \xi B)$ and $\text{Spf}(B/\xi B) \rightarrow \mathcal{X}_{\mathcal{O}_{c_K}}$ is sent to the object obtained by the compositions $(A_{\text{inf}}, (\xi)) \xrightarrow{g} (A_{\text{inf}}, (\xi)) \rightarrow (B, \xi B)$ and $\text{Spf}(B/\xi B) \rightarrow \mathcal{X}_{\mathcal{O}_{c_K}} \xrightarrow{g} \mathcal{X}_{\mathcal{O}_{c_K}}$. Let us denote this object by $g_*(B, \xi B)$ for brevity.

(2) The pullback functor

$$g^{-1}: \mathrm{Shv}(\mathcal{X}_{\mathcal{O}_{c_K}}/A_{\mathrm{inf}})_{\Delta}^{\Pi} \rightarrow \mathrm{Shv}(\mathcal{X}_{\mathcal{O}_{c_K}}/A_{\mathrm{inf}})_{\Delta}^{\Pi}$$

(as in [32, 00XN]) sends a sheaf $\mathcal{F} \in \mathrm{Shv}(\mathcal{X}_{\mathcal{O}_{c_K}}/A_{\mathrm{inf}})_{\Delta}^{\Pi}$ to the sheaf $g^{-1}\mathcal{F}$, which takes any object $(B, \xi B)$ of $(\mathcal{X}_{\mathcal{O}_{c_K}}/A_{\mathrm{inf}})_{\Delta}^{\Pi}$ to $\mathcal{F}(g_*(B, \xi B))$. In particular, the sheaf $\mathcal{O} = \mathcal{O}_{\Delta}$ is mapped to the sheaf $g^{-1}\mathcal{O}$, which takes an object $(B, \xi B) \in (\mathcal{X}_{\mathcal{O}_{c_K}}/A_{\mathrm{inf}})_{\Delta}^{\Pi}$ to g_*B (i.e., B viewed as an A_{inf} -algebra via $A_{\mathrm{inf}} \xrightarrow{g} A_{\mathrm{inf}} \rightarrow B$).

- (3) For any sheaf $\mathcal{F} \in \mathrm{Shv}(\mathcal{X}_{\mathcal{O}_{c_K}}/A_{\mathrm{inf}})_{\Delta}^{\Pi}$, there is an induced map of global sections $\Gamma(\mathcal{X}_{\mathcal{O}_{c_K}}, \mathcal{F}) \rightarrow \Gamma(\mathcal{X}_{\mathcal{O}_{c_K}}, g^{-1}\mathcal{F})$ (sending a section $*$ to $\mathcal{F}$ to $* = g^{-1}* \rightarrow g^{-1}\mathcal{F}$) and hence, a map $\mathrm{R}\Gamma(\mathcal{X}_{\mathcal{O}_{c_K}}, \mathcal{F}) \rightarrow \mathrm{R}\Gamma(\mathcal{X}_{\mathcal{O}_{c_K}}, g^{-1}\mathcal{F})$. Taking $\mathcal{F} = \mathcal{O}$, we get a map $\mathrm{R}\Gamma(\mathcal{X}_{\mathcal{O}_{c_K}}, \mathcal{O}) \rightarrow \mathrm{R}\Gamma(\mathcal{X}_{\mathcal{O}_{c_K}}, g^{-1}\mathcal{O})$ (tautologically A_{inf} -linear when the A_{inf} -action on $g^{-1}\mathcal{O}$ is induced by the A_{inf} -action on $\mathcal{O}$).
- (4) Finally, there is a morphism of sheaves $g^{-1}\mathcal{O} \rightarrow \mathcal{O}$ (making the morphism of sites g from step (1) into a morphism of ringed sites) given by the “identity map”

$$(g^{-1}\mathcal{O})((B, \xi B)) = g_*B \xrightarrow{\mathrm{id}} B = \mathcal{O}((B, \xi B))$$

on every $(B, \xi B) \in (\mathcal{X}_{\mathcal{O}_{c_K}}/A_{\mathrm{inf}})_{\Delta}^{\Pi}$ (the map can, thus, be thought of as an A_{inf} - g -semilinear identification of $g^{-1}\mathcal{O}$ with $\mathcal{O}$). Composing the map from step (3) with the map on global sections induced by $g^{-1}\mathcal{O} \rightarrow \mathcal{O}$, the result is an A_{inf} - g -semilinear map $g: \mathrm{R}\Gamma(\mathcal{X}_{\mathcal{O}_{c_K}}, \mathcal{O}) \rightarrow \mathrm{R}\Gamma(\mathcal{X}_{\mathcal{O}_{c_K}}, \mathcal{O})$, which is the map defining the action of g on $\mathrm{R}\Gamma(\mathcal{X}_{\mathcal{O}_{c_K}}, \mathcal{O}) = \mathrm{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_{c_K}}/A_{\mathrm{inf}})$.

The resulting action can be described in terms of a (suitably chosen) Čech–Alexander complex modelling $\mathrm{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_{c_K}}/A_{\mathrm{inf}})$. Roughly speaking, this is due to the pullback functoriality of Čech complexes together with an initial choice of a cover that is stable under the natural G_K -action.

Construction 4.2. Denote by $\check{C}_{\mathrm{BK}}^{\bullet}$ a Čech–Alexander complex that models $\mathrm{R}\Gamma_{\Delta}(\mathcal{X}/\mathfrak{S})$. Denote the affine open cover of $\mathcal{X}$ used in the construction of $\check{C}_{\mathrm{BK}}^{\bullet}$ by $\mathfrak{V} = \{\mathcal{V}_j\}_j$, and the chosen compatible prismatic covers by $(\check{C}_{\mathcal{V}_{j_1, \dots, j_n}}, E(u)\check{C}_{\mathcal{V}_{j_1, \dots, j_n}})$. Set $\check{C}_{\mathrm{inf}}^{\bullet} = \check{C}_{\mathrm{BK}}^{\bullet} \hat{\otimes}_{\mathfrak{S}} A_{\mathrm{inf}}$, computed termwise – by Remark 2.23, this is a Čech–Alexander complex modelling $\mathrm{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_{c_K}}/A_{\mathrm{inf}})$, associated similarly with the affine open cover $\mathfrak{V}' = \{\mathcal{V}'_j\}_j$, obtained by base-change of $\{\mathcal{V}_j\}_j$ along $\mathcal{O}_K \rightarrow \mathcal{O}_{c_K}$. Likewise, denote the associated prismatic covers $(\check{C}_{\mathcal{V}'_{j_1, \dots, j_n}} \hat{\otimes}_{A_{\mathrm{inf}}} \xi \check{C}_{\mathcal{V}_{j_1, \dots, j_n}} \hat{\otimes}_{\mathfrak{S}} A_{\mathrm{inf}})$ simply by $(\check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}, \xi \check{C}_{\mathcal{V}'_{j_1, \dots, j_n}})$.

Further, denote the coordinate ring of $\mathcal{V}_{j_1, \dots, j_n}$ by $R_{j_1, \dots, j_n}$ and of $\mathcal{V}'_{j_1, \dots, j_n}$ by $R'_{j_1, \dots, j_n}$, respectively. Let us denote by $P_{j_1, \dots, j_n} \rightarrow R_{j_1, \dots, j_n}$ the initial choices of surjective maps from $(p, E(u))$ -completely free $\mathfrak{S}$ -algebras made in the construction of prismatic covers (Construction 2.11; but note that these are chosen compatibly in the sense of Proposition 2.15), and by $P'_{j_1, \dots, j_n} \rightarrow R'_{j_1, \dots, j_n}$ the $(p, E(u))$ -completed base change to A_{inf} . By Remark 2.17, we may assume that the set of variables for each of the underlying poly-

nomial algebras is finite. Although not strictly necessary at this stage, we do make this assumption from now on.

For $g \in G_K$, denote by $g^*\mathfrak{Y}'$ the affine open cover $\{g^*\mathcal{V}'_j\}_j$ of $\mathcal{X}_{\mathcal{O}_{c_K}}$, and denote by $g^*(\check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}, \xi \check{C}_{\mathcal{V}'_{j_1, \dots, j_n}})$ the pullback of

$$(\check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}, \xi \check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}) \in (\mathcal{X}_{\mathcal{O}_{c_K}}/A_{\text{inf}})_{\Delta}^{\Pi}$$

(that is, the object given by the prism $g^*\check{C}_{\mathcal{V}'_{j_1, \dots, j_n}} = \check{C}_{\mathcal{V}'_{j_1, \dots, j_n}} \hat{\otimes}_{A_{\text{inf}}, g} A_{\text{inf}}$ and where the map $\text{Spf}(\check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}/\xi \check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}) \rightarrow \mathcal{X}_{\mathcal{O}_{c_K}}$ is base-changed along $\mathcal{X}_{\mathcal{O}_{c_K}} \xrightarrow{g} \mathcal{X}_{\mathcal{O}_{c_K}}$). These assemble into a simplicial object $g^*N_{\mathfrak{Y}'}$ obtained by applying g^* termwise to $N_{\mathfrak{Y}'}$ (equivalently, this is $N_{g^*\mathfrak{Y}'}$, the Čech nerve of the cover $h_{g^*\check{C}_{\mathcal{V}'_j}} \rightarrow *$). Consider now the following.

- (1) For a sheaf $\mathcal{F}$ on $(\mathcal{X}_{\mathcal{O}_{c_K}}/A_{\text{inf}})_{\Delta}^{\Pi}$, a section $s \in \mathcal{F}((\check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}, \xi \check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}))$ corresponds by Yoneda lemma to a map of sheaves $f_s: h_{(\check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}, \xi \check{C}_{\mathcal{V}'_{j_1, \dots, j_n}})} \rightarrow \mathcal{F}$.

Taking pullback by g yields a map

$$g^{-1}f_s: g^{-1}h_{(\check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}, \xi \check{C}_{\mathcal{V}'_{j_1, \dots, j_n}})} \rightarrow g^{-1}\mathcal{F},$$

where $g^{-1}h_{(\check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}, \xi \check{C}_{\mathcal{V}'_{j_1, \dots, j_n}})}$ is naturally isomorphic to $h_{g^*(\check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}, \xi \check{C}_{\mathcal{V}'_{j_1, \dots, j_n}})}$. This yields a section

$$g^{-1}s \in g^{-1}\mathcal{F}(g^*(\check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}, \xi \check{C}_{\mathcal{V}'_{j_1, \dots, j_n}})).$$

The resulting assignment $s \mapsto g^{-1}s$ induces a map of chain complexes

$$\check{C}^{\bullet}(N_{\mathfrak{Y}'}, \mathcal{F}) \rightarrow \check{C}^{\bullet}(g^*N_{\mathfrak{Y}'}, g^{-1}\mathcal{F}).$$

Taking $\mathcal{F} = \mathcal{O}$, one obtains a map $\check{C}_{\text{inf}}^{\bullet} = \check{C}^{\bullet}(N_{\mathfrak{Y}'}, \mathcal{O}) \rightarrow \check{C}^{\bullet}(g^*N_{\mathfrak{Y}'}, g^{-1}\mathcal{O})$. Unwrapping all the definitions, this morphism is the canonical map $\check{C}_{\text{inf}}^{\bullet} \rightarrow g_*g^*\check{C}_{\text{inf}}^{\bullet}$ given by $x \mapsto x \otimes 1$ (but note that it is A_{inf} -linear thanks to g_* on the right).

- (2) From the map of sheaves of rings $g^{-1}\mathcal{O} \rightarrow \mathcal{O}$ from Definition 4.1 (4), one obtains a map $\check{C}^{\bullet}(g^*N_{\mathfrak{Y}'}, g^{-1}\mathcal{O}) \rightarrow \check{C}^{\bullet}(g^*N_{\mathfrak{Y}'}, \mathcal{O})$. This map pre-composed with the map from step (1) yields the (A_{inf}, g) -semilinear canonical map $\check{C}_{\text{inf}} \rightarrow g^*\check{C}_{\text{inf}}$ given by $x \mapsto x \otimes 1$.
- (3) The cover $g^*\mathfrak{Y}'$ is, in fact, canonically isomorphic to the cover $\mathfrak{Y}'$ as outlined in Definition 4.1 (0). Moreover, the prismatic covers $g^*(\check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}, \xi \check{C}_{\mathcal{V}'_{j_1, \dots, j_n}})$ are also canonically isomorphic to the prismatic covers $(\check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}, \xi \check{C}_{\mathcal{V}'_{j_1, \dots, j_n}})$ of $\mathcal{V}'_{j_1, \dots, j_n}$. To see this, note that the completed polynomial algebra $P'_{j_1, \dots, j_n}$ comes from a polynomial $W(k)$ -algebra $P_{j_1, \dots, j_n}^0 = W(k)[\{X_i\}_i]$ by the (p, ξ) -completed base-change $-\hat{\otimes}_{W(k)} A_{\text{inf}}$. The algebra $g^*(P_{j_1, \dots, j_n}^0 \otimes_{W(k)} A_{\text{inf}})$ is then canonically

identified, as an A_{inf} -algebra, with $P_{j_1, \dots, j_k}^0 \otimes_{W(k)} A_{\text{inf}}$, for the same reasons as in Definition 4.1 (0). In particular, the canonical isomorphism

$$P_{j_1, \dots, j_k}^0 \otimes_{W(k)} A_{\text{inf}} \otimes_{A_{\text{inf}}, g} A_{\text{inf}} = g^*(P_{j_1, \dots, j_k}^0 \otimes_{W(k)} A_{\text{inf}}) \rightarrow (P_{j_1, \dots, j_k}^0 \otimes_{W(k)} A_{\text{inf}})$$

is given by $f \otimes a \otimes b \mapsto f \otimes g(a)b$; if we treat $g^* A_{\text{inf}}[\{X_i\}_i]$ on the left as the polynomial algebra $A_{\text{inf}}[\{X_i\}_i]$ with twisted A_{inf} -action, the map takes the form $\sum_i a_i X^i \mapsto \sum_i g(a_i) X^i$.

This, in particular, agrees with the canonical isomorphism $g^* R'_{j_1, \dots, j_n} \rightarrow R'_{j_1, \dots, j_n}$, i.e., there is a commutative square

$$\begin{array}{ccc} g^* P_{j_1, \dots, j_n}^0 \otimes_{W(k)} A_{\text{inf}} & \longrightarrow & g^* R'_{j_1, \dots, j_n} \\ \downarrow & & \downarrow \\ P_{j_1, \dots, j_n}^0 \otimes_{W(k)} A_{\text{inf}} & \longrightarrow & R'_{j_1, \dots, j_n}, \end{array}$$

where the horizontal maps come from $P'_{j_1, \dots, j_n} \rightarrow R'_{j_1, \dots, j_n}$ and its g -twist, and the vertical maps are the canonical isomorphisms just described. Taking (p, ξ) -adic completion, one thus obtains the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & g^*(JP'_{j_1, \dots, j_n}) & \longrightarrow & g^*P'_{j_1, \dots, j_n} & \longrightarrow & g^*R'_{j_1, \dots, j_n} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & JP'_{j_1, \dots, j_n} & \longrightarrow & P'_{j_1, \dots, j_n} & \longrightarrow & R'_{j_1, \dots, j_n} \longrightarrow 0. \end{array}$$

Taking δ -envelopes and prismatic envelopes as in Construction 2.11, one obtains a canonical isomorphism of prisms $g^*(\check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}, \xi \check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}) \xrightarrow{\sim} (\check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}, \xi \check{C}_{\mathcal{V}'_{j_1, \dots, j_n}})$ over A_{inf} . Moreover, the maps from $R'_{j_1, \dots, j_n}$ to $\check{C}_{\mathcal{V}'_{j_1, \dots, j_n}} / \xi \check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}$ as in Construction 2.11 are also clearly compatible, i.e., there is a commutative square

$$\begin{array}{ccc} g^* R'_{j_1, \dots, j_n} & \longrightarrow & g^* \check{C}_{\mathcal{V}'_{j_1, \dots, j_n}} / \xi (g^* \check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}) \\ \downarrow & & \downarrow \\ R'_{j_1, \dots, j_n} & \longrightarrow & \check{C}_{\mathcal{V}'_{j_1, \dots, j_n}} / \xi \check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}, \end{array}$$

where the vertical maps are the canonical maps. This shows that the canonical isomorphism $g^*(\check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}, \xi \check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}) \rightarrow (\check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}, \xi \check{C}_{\mathcal{V}'_{j_1, \dots, j_n}})$ is in fact an isomorphism of objects of $(\mathcal{X}_{\mathcal{O}_K} / A_{\text{inf}})_{\Delta}^{\text{II}}$ (in the opposite direction in the variance of the site). Moreover, since this isomorphism is functorial in the maps $P' \rightarrow R'$ (or, more precisely, in the map of sets $\underline{X} \rightarrow R'$ where $\underline{X}$ is the given chosen set of free variables of P' , cf. Remark 2.16), these isomorphisms assemble to an isomorphism of simplicial objects $N_{\mathfrak{Y}'} \rightarrow N_{g^* \mathfrak{Y}'}$.

- (4) Finally, given $\mathcal{F} \in \text{Shv}(\mathcal{X}_{\mathcal{O}_{c_K}}/A_{\text{inf}})_{\Delta}^{\text{II}}$, the canonical isomorphism $N_{\mathfrak{Y}'} \rightarrow N_{g^*\mathfrak{Y}'}$ yields an isomorphism $\check{C}^\bullet(N_{g^*\mathfrak{Y}'}, \mathcal{F}) \rightarrow \check{C}^\bullet(N_{\mathfrak{Y}'}, \mathcal{F})$; in particular, for $\mathcal{F} = \mathcal{O}$ one obtains an isomorphism $g^*\check{C}_{\text{inf}}^\bullet \rightarrow \check{C}_{\text{inf}}^\bullet$. Composing the maps from steps (1) and (2) with this map yields the A_{inf} - g -semilinear action map $g: \check{C}_{\text{inf}}^\bullet \rightarrow \check{C}_{\text{inf}}^\bullet$.

Proposition 4.3. *Definition 4.1 and Construction 4.2 are compatible, that is, the action-by- g map $g: \text{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_{c_K}}/A_{\text{inf}}) \rightarrow \text{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_{c_K}}/A_{\text{inf}})$ from Definition 4.1 is modelled by the action map $g: \check{C}_{\text{inf}}^\bullet \rightarrow \check{C}_{\text{inf}}^\bullet$ from Construction 4.2.*

Proof. Just like in the proof of Proposition 2.21, we invoke proofs of [32, 079Z, 03OW]. That is, upon choosing a resolution $\mathcal{O} \rightarrow \mathcal{I}^\bullet$ by injective abelian sheaves, there is a diagram

$$\text{R}\Gamma_{\Delta}(\mathcal{X}_{\mathcal{O}_{c_K}}/A_{\text{inf}}) = \Gamma(*, \mathcal{I}^\bullet) \xrightarrow{a} \check{C}^\bullet(N_{\mathfrak{Y}'}, \mathcal{I}^\bullet) \xleftarrow{\alpha} \check{C}^\bullet(N_{\mathfrak{Y}'}, \mathcal{O}) = \check{C}_{\text{inf}}^\bullet,$$

where

- α is the natural map to the left (zeroth) column (i.e., $\mathcal{I}^\bullet(\coprod_j (\check{C}_{V_j}))$) of the double complex $\check{C}^\bullet(N_{\mathfrak{Y}'}, \mathcal{I}^\bullet)$, and becomes a quasi-isomorphism after totalization of this double complex,
- a is the map to the bottom (zeroth) row (i.e., $\check{C}^\bullet(N_{\mathfrak{Y}'}, \mathcal{I}^0)$) of the double complex $\check{C}^\bullet(N_{\mathfrak{Y}'}, \mathcal{I}^\bullet)$ induced by the map of sheaves $\mathcal{O} \rightarrow \mathcal{I}^0$, and becomes a quasi-isomorphism after totalization of the double complex.

This diagram can be extended to the commutative diagram

$$\begin{array}{ccccc} \Gamma(*, \mathcal{I}^\bullet) & \xrightarrow{\alpha} & \check{C}^\bullet(N_{\mathfrak{Y}'}, \mathcal{I}^\bullet) & \xleftarrow{a} & \check{C}^\bullet(N_{\mathfrak{Y}'}, \mathcal{O}) \\ \downarrow \gamma_1 & & \downarrow d_1 & & \downarrow c_1 \\ \Gamma(*, g^{-1}\mathcal{I}^\bullet) & \xrightarrow{\alpha_1} & \check{C}^\bullet(N_{g^*\mathfrak{Y}'}, g^{-1}\mathcal{I}^\bullet) & \xleftarrow{a_1} & \check{C}^\bullet(N_{g^*\mathfrak{Y}'}, g^{-1}\mathcal{O}) \\ \downarrow \gamma_2 & & \downarrow d_2 & & \downarrow c_2 \\ \Gamma(*, \mathcal{I}^\bullet) & \xrightarrow{\alpha_2} & \check{C}^\bullet(N_{g^*\mathfrak{Y}'}, \mathcal{I}^\bullet) & \xleftarrow{a_2} & \check{C}^\bullet(N_{g^*\mathfrak{Y}'}, \mathcal{O}) \\ & \searrow \alpha & \downarrow d_3 & & \downarrow c_3 \\ & & \check{C}^\bullet(N_{\mathfrak{Y}'}, \mathcal{I}^\bullet) & \xleftarrow{a} & \check{C}^\bullet(N_{\mathfrak{Y}'}, \mathcal{O}), \end{array}$$

where

- (1) The map γ_1 is the pullback map from Definition 4.1 (3) (applied to the complex of sheaves $\mathcal{I}^\bullet$) and, in parallel, the maps d_1, c_1 are the pullback of sections operation on Čech complexes described in Construction 4.2 (1) (applied to the complex of sheaves $\mathcal{I}^\bullet$ and to $\mathcal{O}$, respectively).
- (2) The complex of sheaves $g^{-1}\mathcal{I}^\bullet$ is an injective resolution of $g^{-1}\mathcal{O}$ (e.g. because the sheaf pullback functor g^{-1} is an auto-equivalence on the category of abelian sheaves on $(\mathcal{X}_{\mathcal{O}_{c_K}}/A_{\text{inf}})_{\Delta}^{\text{II}}$) and thus, one can choose a lift $l: g^{-1}\mathcal{I}^\bullet \rightarrow \mathcal{I}^\bullet$ of

the map of sheaves $g^{-1}\mathcal{O} \rightarrow \mathcal{O}$ from Definition 4.1 (4). Then the maps γ_2, d_2 are obtained from l (by functoriality of global sections and of the Čech complex formation, resp.) and likewise, c_2 is obtained by applying the Čech construction to the original map $g^{-1}\mathcal{O} \rightarrow \mathcal{O}$.

- (3) The maps d_3, c_3 are induced by the isomorphism of Čech nerves $N_{\mathfrak{Y}'} \rightarrow N_{g^*\mathfrak{Y}'}$ described in Construction 4.2 (3).
- (4) The maps α_i, a_i are the natural maps into zeroth column and row, resp., defined analogously to α and a but for different complex of sheaves ($g^{-1}\mathcal{O} \rightarrow g^{-1}\mathcal{I}^\bullet$) and/or different choice of the Čech nerve ($N_{g^*\mathfrak{Y}'}$).

The composition $\gamma_2\gamma_1$ is the map g on $\mathrm{R}\Gamma_\Delta(\mathcal{X}_{\mathcal{O}_{c_K}}/A_{\mathrm{inf}})$ described in Definition 4.1 while the composition $c_3c_2c_1$ is the map g on $\check{C}_{\mathrm{inf}}^\bullet$. After totalization of the double complex $\check{C}^\bullet(N_{\mathfrak{Y}'}, \mathcal{I}^\bullet)$, the maps a, α become quasi-isomorphisms (altogether, the canonical quasi-isomorphism between cohomology and the Čech–Alexander complex), showing that $c_3c_2c_1$ models the map g on $\mathrm{R}\Gamma_\Delta(\mathcal{X}_{\mathcal{O}_{c_K}}/A_{\mathrm{inf}})$. ■

As a consequence of Construction 4.2, the Galois action on the complex $\check{C}_{\mathrm{inf}}^\bullet$ is fairly explicit. In particular, we have the following properties.

Corollary 4.4. *Let us keep the notation from Definition 4.1 and Construction 4.2.*

- (1) *The Galois action on each term $\check{C}_{\mathrm{inf}}^{n-1}$ makes each direct factor $\check{C}_{\mathcal{V}'_{j_1, \dots, j_n}} G_K$ -stable and, in fact, for every $g \in G_K$, the map $g: \check{C}_{\mathrm{inf}}^{n-1} \rightarrow \check{C}_{\mathrm{inf}}^{n-1}$ is the direct product of the maps $g: \check{C}_{\mathcal{V}'_{j_1, \dots, j_n}} \rightarrow \check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}$ on these direct factors.*
- (2) *On each such $\check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}$, the action is given by (A_{inf} -semilinear) maps of prisms. Moreover, if $\{X_i\}_i$ is the free basis of the polynomial algebra used in construction of $\check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}$ (as in Remark 2.16), then $g(X_i) = X_i$ for all $g \in G_K$ and all i .*

Proof. In terms of the notation from proof of Proposition 4.3, the map $c_2c_1: \check{C}_{\mathrm{inf}}^\bullet \rightarrow g^*\check{C}_{\mathrm{inf}}^\bullet$ has the explicit description $x \mapsto x \otimes 1$ (cf. Construction 4.2 (2)) and thus, it is a direct product of maps $\check{C}_{\mathcal{V}'_{j_1, \dots, j_n}} \rightarrow g^*\check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}, x \mapsto x \otimes 1$, which are g -semilinear maps of prisms. Moreover, it is easy to see that this map is obtained from the commutative square

$$\begin{array}{ccc} P'_{j_1, \dots, j_n} & \longrightarrow & R'_{j_1, \dots, j_n} \\ \downarrow x \mapsto x \otimes 1 & & \downarrow x \mapsto x \otimes 1 \\ g^*P'_{j_1, \dots, j_n} & \longrightarrow & g^*R'_{j_1, \dots, j_n} \end{array}$$

by the functorial construction of prismatic covers (Construction 2.11). The map c_3 has also the product form from its definition, and is obtained by the functoriality of prismatic covers from the square

$$\begin{array}{ccc} g^*P'_{j_1, \dots, j_n} & \longrightarrow & g^*R'_{j_1, \dots, j_n} \\ \downarrow & & \downarrow \\ P'_{j_1, \dots, j_n} & \longrightarrow & R'_{j_1, \dots, j_n}, \end{array}$$

where the vertical maps are the canonical isomorphisms described in Construction 4.2 (3).

Altogether, the map $g = c_3 c_2 c_1: \check{C}_{\text{inf}}^{n-1} \rightarrow \check{C}_{\text{inf}}^{n-1}$ therefore decomposes as a product of maps $g: \check{C}_{\mathcal{V}'_{j_1, \dots, j_n}} \rightarrow \check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}$, proving part (1). Furthermore, these are maps of prisms since they are functorially obtained from the composite square

$$\begin{array}{ccc} P'_{j_1, \dots, j_n} & \longrightarrow & R'_{j_1, \dots, j_n} \\ \downarrow x \mapsto x \otimes 1 & & \downarrow x \mapsto x \otimes 1 \\ g^* P'_{j_1, \dots, j_n} & \longrightarrow & g^* R'_{j_1, \dots, j_n} \\ \downarrow & & \downarrow \\ P'_{j_1, \dots, j_n} & \longrightarrow & R'_{j_1, \dots, j_n} \end{array}$$

by functoriality of Construction 2.11. Moreover, for the free basis $\{X_i\}_i$ used in the construction, it is easy to see that the left vertical map $P'_{j_1, \dots, j_n} \rightarrow P'_{j_1, \dots, j_n}$ sends each X_i to X_i (in fact, the vertical map is obtained from the endomorphism $\sum_{\alpha} a_{\alpha} \underline{X}^{\alpha} \mapsto \sum_{\alpha} g(a_{\alpha}) \underline{X}^{\alpha}$ of the polynomial algebra $A_{\text{inf}}[\{X_i\}_i]$ by (p, ξ) -completion). This proves part (2). ■

With the Galois action on $\check{C}_{\text{inf}}^{\bullet}$ established, our next aim is to prove the following.

Theorem 4.5. *For every $m \geq 0$ and $s \in \mathbb{N} \cup \{\infty\}$, the pair $\check{C}_{\text{BK}}^m \rightarrow \check{C}_{\text{inf}}^m$ satisfies the condition (Cr_s) .*

By Corollary 4.4, Theorem 4.5 immediately reduces to the case of a single prismatic cover $\check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}$. Let us fix a choice of this cover (i.e., fix n , multi-index $j_1, \dots, j_n$, and consider $\check{C}_{\mathcal{V}'_{j_1, \dots, j_n}}$, appearing in $\check{C}_{\text{inf}}^{\bullet}$ as a direct factor). To simplify notation from the one previously used, let us omit the multi-index everywhere, i.e.,

$$\text{Spf}(R) = \mathcal{V} = \mathcal{V}_{j_1, \dots, j_n}, \quad \text{Spf}(R') = \mathcal{V}' = \mathcal{V}'_{j_1, \dots, j_n}, \quad P = P_{j_1, \dots, j_n}, \quad P' = P'_{j_1, \dots, j_n}, \dots$$

and so on. Let us denote the basis used for constructing the map $P \rightarrow R$, by $\{X_i\} = \{X_1, X_2, \dots, X_m\}$. That is, the map $P \rightarrow R$ is obtained from the map $A_{\text{inf}}[\{X_i\}_i] \rightarrow R$ by $(p, E(u))$ -completion on the left. Further, let us denote

$$(\check{C}_{\text{BK}}, E(u)\check{C}_{\text{BK}}) = (\check{C}_{j_1, \dots, j_n}, E(u)\check{C}_{j_1, \dots, j_n}), \quad (\check{C}_{\text{inf}}, E(u)\check{C}_{\text{inf}}) = (\check{C}'_{j_1, \dots, j_n}, E(u)\check{C}'_{j_1, \dots, j_n}).$$

It is now enough to verify the following:

Proposition 4.6. *The pair $\check{C}_{\text{BK}} \rightarrow \check{C}_{\text{inf}}$ satisfies the conditions (Cr_s) for every $s \in \mathbb{N} \cup \{\infty\}$.*

Proof. Fix a generating set $y_1, y_2, \dots, y_l$ of $J = \text{Ker}(P \rightarrow R)$, and set $P_1 = \widehat{P}^{\delta}$, $P'_1 = (\widehat{P}')^{\delta}$. Using the weak δ -blowup construction from Section 2.2, we obtain a commutative diagram

$$\begin{array}{ccc} \widehat{P_1 \left\{ \frac{y}{E(u)} \right\}^{\text{cl}}} & \longrightarrow & \widehat{P'_1 \left\{ \frac{y}{E(u)} \right\}^{\text{cl}}} \\ \downarrow \sigma & & \downarrow \sigma' \\ \check{C}_{\text{BK}} & \longrightarrow & \check{C}_{\text{inf}}, \end{array} \tag{4.1}$$

where the vertical maps are the surjective maps from Proposition 2.25, and the horizontal maps come from the $(p, E(u))$ -completed base change $-\hat{\otimes}_{\mathfrak{S}} A_{\text{inf}}$. In particular, $\check{C}_{\text{BK}}$ and $\check{C}_{\text{inf}}$ are $(p, E(u))$ -adically topologically generated as an $\mathfrak{S}$ - δ -algebra and A_{inf} - δ -algebra, resp., by

$$X_1, X_2, \dots, X_m, \quad t_1 = \sigma(y_1/E(u)), \quad t_2 = \sigma(y_2/E(u)), \dots, \quad t_l = \sigma(y_l/E(u)).$$

By Lemma 3.8 (3), (4), it is enough to check the conditions for these generators. By Corollary 4.4, we have $g(X_i) = X_i$ for all i . Regarding the action on t_j 's, note that $t_j \in \check{C}_{\text{inf}}$ is the unique element with

$$t_j E(u) = y_j.$$

Thus, for $g \in G_K$ we obtain

$$g(t_j) \gamma E(u) = g(y_j),$$

where γ is the A_{inf} -unit such that $g(E(u)) = \gamma E(u)$. Note that, since $y_j \in JP'$ and $JP' = \text{Ker}(P' \rightarrow R')$ is G_K -stable (under the action considered e.g. in the proof of Corollary 4.4, compatible with the G_K -action on $\check{C}_{\text{inf}}$), $g(y_j) \in JP'$. Consequently, there exists a (necessarily unique) $r_j \in \check{C}_{\text{inf}}$ with $r_j E(u) = g(y_j)$, given by $r_j = \sigma'(g(y_j)/E(u))$. We may therefore write

$$g(t_j) = \gamma^{-1} r_j.$$

Let us now fix an integer $s \in \mathbb{N} \cup \{\infty\}$ and verify (Cr_s) for the above fixed generators. Since the elements X_i satisfy $g(X_i) - X_i = 0$ for every $g \in G_s$, these are in fact (Cr'_s) -elements; by Lemma 3.8, so are the elements $y_1, y_2, \dots, y_n$ since they come from the $\mathfrak{S}$ -algebra topologically generated by $X_1, \dots, X_m$, i.e., from the image of P in $\check{C}_{\text{BK}}$. From now on, let us denote this image simply by P , and similarly, refer to the image of P' in $\check{C}_{\text{inf}}$ simply as P' .

Let us now fix an index $1 \leq j \leq l$ and verify (Cr_s) for t_j . For an element $g \in G_s$, we may write $g(y_j) - y_j = \varphi^s(v)uz_j = \xi_{s,0}uE(u)\tilde{z}_j$ for some $z_j, \tilde{z}_j \in P'$ (that are equal up to a multiplication by an A_{inf} -unit). Similarly, we have

$$g^{-1}(E(u)) - E(u) = (\gamma^{-1} - 1)E(u) = \varphi^s(v)ua = \xi_{s,0}uE(u)\tilde{a}$$

with $a, \tilde{a} \in A_{\text{inf}}$ (again equal up to a unit).

Thus, we have that

$$\begin{aligned} g(t_j) - t_j &= \gamma^{-1} r_j - t_j = \sigma' \left(\frac{\gamma^{-1} g(y_j) - y_j}{E(u)} \right) \\ &= \sigma' \left(\frac{\gamma^{-1} g(y_j) - \gamma^{-1} y_j + \gamma^{-1} y_j - y_j}{E(u)} \right) \\ &= \gamma^{-1} \sigma' \left(\frac{g(y_j) - y_j}{E(u)} \right) + (\gamma^{-1} - 1) \sigma' \left(\frac{y_j}{E(u)} \right) \\ &= \gamma^{-1} \xi_{s,0} u \tilde{z}_j + \xi_{s,0} u \tilde{a} y_j \in I_s \check{C}_{\text{inf}}. \end{aligned}$$

This shows that each of the remaining generators t_j is a (Cr_s) -element, which finishes the proof. ■

4.2. Consequences for cohomology groups

Let us now use Theorem 4.5 to draw some conclusions for individual cohomology groups. The first is the crystalline condition for the prismatic cohomology groups and its consequence for p -adic étale cohomology. As before, let $\mathcal{X}$ be a separated smooth p -adic formal scheme over $\mathcal{O}_K$. Denote by $\mathcal{X}_{A_{\text{inf}}}$ the base change $\mathcal{X} \times_{\mathcal{O}_K} \mathcal{O}_{\mathbb{C}_K} = \mathcal{X} \times_{\mathfrak{S}} A_{\text{inf}}$, and by $\mathcal{X}_{\bar{\eta}}$ the geometric adic generic fiber.

Corollary 4.7. *For every $i \geq 0$, the pair $H_{\Delta}^i(\mathcal{X}/\mathfrak{S}) \rightarrow H_{\Delta}^i(\mathcal{X}_{A_{\text{inf}}}/A_{\text{inf}})$ satisfies the conditions (Cr₀) and (Cr_∞).*

Proof. By the results of Section 2.2, we may and do model the cohomology theories by the Čech–Alexander complexes

$$\check{C}_{\text{BK}}^{\bullet} \rightarrow \check{C}_{\text{inf}}^{\bullet} = \check{C}_{\text{BK}}^{\bullet} \hat{\otimes}_{\mathfrak{S}} A_{\text{inf}},$$

and by Theorem 4.5 the conditions (Cr₀) and (Cr_∞) termwise hold for this pair. The condition (Cr_∞) for $H_{\Delta}^i(\mathcal{X}/\mathfrak{S}) \subseteq H_{\Delta}^i(\mathcal{X}/A_{\text{inf}})$ thus follows immediately, and it remains to verify the condition (Cr₀).

Each of the terms $\check{C}_{\text{inf}}^i$ is $(p, E(u))$ -completely flat over A_{inf} , which means in particular that the terms $\check{C}_{\text{inf}}^i$ are torsion-free by Corollary 2.4. Denote the differentials on $\check{C}_{\text{BK}}^{\bullet}, \check{C}_{\text{inf}}^{\bullet}$ by ∂ and ∂' , resp.

To prove the crystalline condition for cohomology groups, it is clearly enough to verify the condition at the level of cocycles. Given $x \in Z^i(\check{C}_{\text{BK}}^{\bullet})$, denote by x' its image in $Z^i(\check{C}_{\text{inf}}^{\bullet})$. For $g \in G_K$ we have $g(x') - x' = \varphi^{-1}(v)uy'$ for some $y' \in \check{C}_{\text{inf}}^i$. As $g(x') - x' \in Z^i(\check{C}_{\text{inf}}^{\bullet})$, we have

$$\varphi^{-1}(v)u\partial'(y') = \partial'(\varphi^{-1}(v)uy') = 0,$$

and the torsion-freeness of $\check{C}_{\text{inf}}^{i+1}$ implies that $\partial'(y') = 0$. Thus, $y' \in Z^i(\check{C}_{\text{inf}}^{\bullet})$ as well, showing that $g(x') - x' \in \varphi^{-1}(v)uZ^i(\check{C}_{\text{inf}}^{\bullet})$, as desired. ■

When $\mathcal{X}$ is proper over $\mathcal{O}_K$, we use Corollary 4.7 to reprove the result from [4] that the étale cohomology groups $H_{\text{ét}}^i(\mathcal{X}_{\bar{\eta}}, \mathbb{Q}_p)$ are in this case crystalline representations.

Corollary 4.8. *Assume that $\mathcal{X}$ is additionally proper over $\mathcal{O}_K$. Then for any $i \geq 0$, the p -adic étale cohomology $H_{\text{ét}}^i(\mathcal{X}_{\bar{\eta}}, \mathbb{Q}_p)$ is a crystalline representation.*

Proof. It follows from [6, Theorem 1.8] (and faithful flatness of $A_{\text{inf}}/\mathfrak{S}$) that $M_{\text{BK}} = H_{\Delta}^i(\mathcal{X}/\mathfrak{S})$ and $M_{\text{inf}} = H_{\Delta}^i(\mathcal{X}_{A_{\text{inf}}}/A_{\text{inf}})$ are Breuil–Kisin and Breuil–Kisin–Fargues modules, resp., such that $M_{\text{inf}} = M_{\text{BK}} \otimes_{\mathfrak{S}} A_{\text{inf}}$. Moreover, M_{inf} has the structure of a Breuil–Kisin–Fargues G_K -module with

$$V(M_{\text{inf}}) := (W(\mathbb{C}_K^{\flat}) \otimes_{A_{\text{inf}}} M_{\text{inf}})^{\varphi=1} \left[\frac{1}{p} \right] \simeq H_{\text{ét}}^i(\mathcal{X}_{\bar{\eta}}, \mathbb{Q}_p)$$

as G_K -representations. By Corollary 4.7, the pair $(M_{\text{BK}}, M_{\text{inf}})$ satisfies all the assumptions of Theorem 3.15. The claim thus follows. ■

For the purposes of obtaining a bound on ramification of p -torsion étale cohomology in Section 5, let us recall the notion of torsion prismatic cohomology as defined in [24], and discuss the consequences of the conditions (Cr_s) in this context.

Definition 4.9. Given a bounded prism (A, I) and a smooth p -adic formal scheme $\mathcal{X}$ over A/I , the p^n -torsion prismatic cohomology of $\mathcal{X}$ is defined as

$$\mathrm{R}\Gamma_{\Delta, n}(\mathcal{X}/A) = \mathrm{R}\Gamma_{\Delta}(\mathcal{X}/A) \overset{\mathrm{L}}{\otimes}_{\mathbb{Z}} \mathbb{Z}/p^n \mathbb{Z}.$$

We denote the cohomology groups of $\mathrm{R}\Gamma_{\Delta, n}(\mathcal{X}/A)$ by $H_{\Delta, n}^i(\mathcal{X}/A)$ (and refer to them as p^n -torsion prismatic cohomology groups).

Proposition 4.10. *Let $s \geq 0$, $n \geq 1$ be a pair of integers. Set $t = \max\{0, s + 1 - n\}$. Then the torsion prismatic cohomology groups $H_{\Delta, n}^i(\mathcal{X}/\mathfrak{S}) \rightarrow H_{\Delta, n}^i(\mathcal{X}_{A_{\mathrm{inf}}}/A_{\mathrm{inf}})$ satisfy the following:*

$$\forall g \in G_s: \quad (g-1)H_{\Delta, n}^i(\mathcal{X}/\mathfrak{S}) \subseteq \varphi^{-1}(v)u^{p^t}H_{\Delta, n}^i(\mathcal{X}_{A_{\mathrm{inf}}}/A_{\mathrm{inf}}).$$

Proof. The proof is a slightly refined variant of the proof of Corollary 4.7. Consider again the associated Čech–Alexander complexes over $\mathfrak{S}$ and A_{inf} ,

$$\check{C}_{\mathrm{BK}}^{\bullet} \rightarrow \check{C}_{\mathrm{inf}}^{\bullet} = \check{C}_{\mathrm{BK}}^{\bullet} \hat{\otimes}_{\mathfrak{S}} A_{\mathrm{inf}}.$$

Both of these complexes are given by torsion-free, hence $\mathbb{Z}$ -flat, modules by Corollary 2.4. Consequently, $\mathrm{R}\Gamma_{\Delta, n}(\mathcal{X}/\mathfrak{S})$ is modelled by $\check{C}_{\mathrm{BK}, n}^{\bullet} := \check{C}_{\mathrm{BK}}^{\bullet}/p^n \check{C}_{\mathrm{BK}}^{\bullet}$, and similarly for $\mathrm{R}\Gamma_{\Delta, n}(\mathcal{X}_{A_{\mathrm{inf}}}/A_{\mathrm{inf}})$ and $\check{C}_{\mathrm{inf}, n}^{\bullet} = \check{C}_{\mathrm{inf}}^{\bullet}/p^n \check{C}_{\mathrm{inf}}^{\bullet}$. That is, the considered maps between cohomology groups are obtained as the maps on cohomology for the base-change map of chain complexes

$$\check{C}_{\mathrm{BK}, n}^{\bullet} \rightarrow \check{C}_{\mathrm{inf}, n}^{\bullet} = \check{C}_{\mathrm{BK}, n}^{\bullet} \hat{\otimes}_{\mathfrak{S}} A_{\mathrm{inf}},$$

and as in the proof of Corollary 4.7, it is enough to establish the desired condition for the respective groups of cocycles.

Set $\alpha = \varphi^{-1}(v)u^{p^t}$. Note that by Lemma 3.9, the condition (Cr_s) for the pair of complexes $\check{C}_{\mathrm{BK}, n}^{\bullet} \rightarrow \check{C}_{\mathrm{inf}, n}^{\bullet}$ implies the condition

$$\forall g \in G_s: \quad (g-1)\check{C}_{\mathrm{BK}, n}^{\bullet} \subseteq \alpha \check{C}_{\mathrm{inf}, n}^{\bullet}$$

(meant termwise as usual), and since the terms of the complex $\check{C}_{\mathrm{inf}}^{\bullet}$ are $(p, E(u))$ -complete and $(p, E(u))$ -completely flat, α is a non-zero divisor on the terms of $\check{C}_{\mathrm{inf}, n}^{\bullet}$ by Corollary 2.4.

So pick any element $x \in Z^i(\check{C}_{\mathrm{BK}, n}^{\bullet})$. The image x' of x in $\check{C}_{\mathrm{inf}, n}^i$ lies in $Z^i(\check{C}_{\mathrm{inf}, n}^{\bullet})$ and for any chosen $g \in G_s$ we have $g(x') - x' = \alpha y'$ for some $y' \in \check{C}_{\mathrm{inf}, n}^i$. Now $g(x') - x'$ lies in $Z^i(\check{C}_{\mathrm{inf}, n}^{\bullet})$, so $\alpha y' = g(x') - x'$ satisfies

$$0 = \partial'(\alpha y') = \alpha \partial'(y').$$

Since α is a non-zero divisor on $\check{C}_{\mathrm{inf}, n}^{i+1}$, it follows that $\partial'(y') = 0$, that is, y' lies in $Z^i(\check{C}_{\mathrm{inf}, n}^{\bullet})$. We thus infer that $g(x') - x' = \alpha y' \in \alpha Z^i(\check{C}_{\mathrm{inf}, n}^{\bullet})$, as desired. $\blacksquare$

5. Ramification bounds for mod p étale cohomology

5.1. Ramification bounds

We are ready to discuss the implications to the question of ramification bounds for mod p étale cohomology groups $H_{\text{ét}}^i(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}/p\mathbb{Z})$ when $\mathcal{X}$ is smooth and proper p -adic formal scheme over $\mathcal{O}_K$.

We define an additive valuation v^b on $\mathcal{O}_{\mathbb{C}_K}^b$ by $v^b(x) = v(x^\#)$ where v is the valuation on $\mathcal{O}_{\mathbb{C}_K}$ normalized so that $v(\pi) = 1$, and $(-)^{\#}: \mathcal{O}_{\mathbb{C}_K}^b \rightarrow \mathcal{O}_{\mathbb{C}_K}$ is the multiplicative lift of $\text{pr}_0: \mathcal{O}_{\mathbb{C}_K}^b \rightarrow \mathcal{O}_{\mathbb{C}_K}/p$. This way, we have $v^b(\pi) = 1$ and $v^b(\varepsilon - 1) = pe/(p-1)$. For a real number $c \geq 0$, denote by $\alpha^{>c}$ ($\alpha^{\geq c}$, resp.) the ideal of $\mathcal{O}_{\mathbb{C}_K}^b$ formed by all elements x with $v^b(x) > c$ ($v^b(x) \geq c$, resp.).

Similarly, we fix an additive valuation v_K of K normalized by $v_K(\pi) = 1$. Then for an algebraic extension L/K and a real number $c \geq 0$, we denote by $\alpha_L^{>c}$ the ideal consisting of all elements $x \in \mathcal{O}_L$ with $v_K(x) > c$ (and similarly, for “ $\geq$ ” as well).

For a finite extensions $M/F/K$ and a real number $m \geq 0$, let us recall (a version of¹¹) Fontaine’s property ($P_m^{M/F}$):

($P_m^{M/F}$) For any algebraic extension E/F , the existence of an $\mathcal{O}_F$ -algebra map $\mathcal{O}_M \rightarrow \mathcal{O}_E/\alpha_E^{>m}$ implies the existence of an F -injection of fields $M \hookrightarrow E$.

We also recall the upper ramification numbering in the convention used in [10]. For $G = \text{Gal}(M/F)$ and a non-negative real number λ , set

$$G_{(\lambda)} = \{g \in G \mid v_M(g(x) - x) \geq \lambda \ \forall x \in \mathcal{O}_M\},$$

where v_M is again the additive valuation of M normalized by $v_M(M^\times) = \mathbb{Z}$.

For $t \geq 0$, set

$$\phi_{M/F}(t) = \int_0^t \frac{ds}{[G_{(1)} : G_{(s)}]}$$

(which makes sense as $G_{(s)} \subseteq G_{(1)}$ for all $s > 0$). Then $\phi_{M/F}$ is a piecewise-linear increasing continuous concave function. Denote by $\psi_{M/F}$ its inverse, and set $G^{(\mu)} = G_{(\psi_{M/F}(\mu))}$.

Denote by $\lambda_{M/F}$ the infimum of all $\lambda \geq 0$ such that $G_{(\lambda)} = \{\text{id}\}$, and by $\mu_{M/F}$ the infimum of all $\mu \geq 0$ such that $G^{(\mu)} = \{\text{id}\}$. Clearly one has $\mu_{M/F} = \phi_{M/F}(\lambda_{M/F})$.

Remark 5.1. If $G^{S(\mu)}$, $G^{F(\mu)}$ are the upper-index ramification groups in [31] and [15], resp., and similarly $G_{S(\lambda)}$ and $G_{F(\lambda)}$ are the corresponding lower-index ramification groups, then we have

$$G^{(\mu)} = G^{S(\mu-1)} = G^{F(\mu)}, \quad G_{(\lambda)} = G_{S(\lambda-1)} = G_{F(\lambda/\bar{e})},$$

where $\bar{e} = e_{M/F}$ is the ramification index of M/F .

¹¹Fontaine’s original condition uses the ideals $\alpha_E^{\geq m}$ instead. Up to changing some inequalities from “ $<$ ” to “ $\leq$ ” and vice versa, the conditions are fairly equivalent.

In particular, the claims that lower indexing is compatible with restrictions to subgroups and upper indexing is compatible with passing to quotients remain valid. Thus, it make sense to set

$$G_F^{(\mu)} = \varprojlim_{M'/F} \text{Gal}(M'/F)^{(\mu)},$$

where M'/F varies over finite Galois extensions M'/F contained in a fixed algebraic closure $\bar{K}$ of K (and $G_F = \varprojlim_{M'/F} \text{Gal}(M'/F)$ is the absolute Galois group).

Regarding μ , the following transitivity formula is useful.

Lemma 5.2 ([10, Lemma 4.3.1]). *Let $N/M/F$ be a pair of finite extensions with both N/F and M/F Galois. Then we have $\mu_{N/F} = \max(\mu_{M/F}, \phi_{M/F}(\mu_{N/M}))$.*

The property $(P_m^{M/F})$ is connected with the ramification of the field extension M/F as follows.

Proposition 5.3. *Let $M/F/K$ be finite extensions of fields with M/F Galois and let $m > 0$ be a real number. If the property $(P_m^{M/F})$ holds, then:*

- (1) ([35, Proposition 3.3]) $\mu_{M/F} \leq e_{F/K} m$. In fact, $\mu_{M/F}/e_{F/K}$ is the infimum of all $m > 0$ such that $(P_m^{M/F})$ is valid.
- (2) ([10, Corollary 4.2.2]) $v_K(\mathcal{D}_{M/F}) < m$, where $\mathcal{D}_{M/F}$ is the different of the extension M/F .

Corollary 5.4. *Both the assumptions and the conclusions of Proposition 5.3 are insensitive to replacing F by any unramified extension of F contained in M .*

Proof. Let F'/F be an unramified extension such that $F' \subseteq M$. The fact that $(P_m^{M/F})$ is equivalent to $(P_m^{M/F'})$ is proved in [35, Proposition 2.2]. To show that also the conclusions are the same for F and F' , it is enough to observe that $e_{F'/K} = e_{F/K}$, $e_{M/F'} = e_{M/F}$, $v_K(\mathcal{D}_{M/F'}) = v_K(\mathcal{D}_{M/F})$ and $\mu_{M/F'} = \mu_{M/F}$. The first two equalities are clear since F'/F is unramified. The third equality follows from $\mathcal{D}_{M/F} = \mathcal{D}_{M/F'} \mathcal{D}_{F'/F}$, since $\mathcal{D}_{F'/F}$ is the unit ideal. Finally, by Lemma 5.2, we have

$$\mu_{M/F} = \max(\mu_{F'/F}, \phi_{F'/F}(\mu_{M/F'})).$$

As F'/F is unramified, we have $\mu_{F'/F} = 0$ and $\phi_{F'/F}(t) = t$ for all $t \geq 0$. The fourth equality thus follows as well. ■

Let $\mathcal{X}$ be a proper and smooth p -adic formal scheme over $\mathcal{O}_K$. Fix the integer i , and denote by T' the Galois module $H_{\text{ét}}^i(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}/p\mathbb{Z})$. Let L be the splitting field of T' , i.e., $L = \bar{K}^{\text{Ker } \rho}$ where $\rho: G_K \rightarrow \text{Aut}_{\mathbb{F}_p}(T')$ is the associated representation. The goal is to provide an upper bound on $v_K(\mathcal{D}_{L/K})$, and a constant $\mu_0 = \mu_0(e, i, p)$ such that $G_K^{(\mu)}$ acts trivially on T' for all $\mu > \mu_0$.

We follow rather closely the strategy of [10], but replacing the datum of $(\varphi, \hat{G})$ -module in [10] by a pair consisting of a p -torsion Breuil–Kisin module and a Breuil–Kisin–Fargues G_K -module that arise from the p -torsion prismatic $\mathfrak{S}$ - and A_{inf} -cohomology, whose relation to T' is described below.

Proposition 5.5 ([24, Proposition 7.2, Corollary 7.4, Remark 7.5]). *Let $\mathcal{X}$ be a smooth and proper p -adic formal scheme over $\mathcal{O}_K$. Then*

- (1) $M_{\text{BK}} = H_{\Delta, n}^i(\mathcal{X}/\mathcal{S})$ is a p^n -torsion Breuil–Kisin module of height $\leq i$, and we have

$$H_{\text{ét}}^i(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}/p^n\mathbb{Z}) \simeq T_n(M_{\text{BK}}) := (M_{\text{BK}} \otimes_{W_n(k)[[u]]} W_n(\mathbb{C}_K^b))^{\varphi=1}$$

as $\mathbb{Z}/p^n\mathbb{Z}[G_{\infty}]$ -modules.

- (2) $M_{\text{inf}} = H_{\Delta, n}^i(\mathcal{X}_{A_{\text{inf}}}/A_{\text{inf}})$ is a p^n -torsion Breuil–Kisin–Fargues G_K -module of height $\leq i$, and we have

$$H_{\text{ét}}^i(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}/p^n\mathbb{Z}) \simeq T_n(M_{\text{inf}}) := (M_{\text{inf}} \otimes_{W_n(\mathcal{O}_{\mathbb{C}_K^b})} W_n(\mathbb{C}_K^b))^{\varphi=1}$$

as $\mathbb{Z}/p^n\mathbb{Z}[G_K]$ -modules.

- (3) We have $M_{\text{BK}} \otimes_{\mathcal{S}} A_{\text{inf}} = M_{\text{BK}} \otimes_{W_n(k)[[u]]} W_n(\mathcal{O}_{\mathbb{C}_K^b}) \simeq M_{\text{inf}}$, and the natural map $M_{\text{BK}} \hookrightarrow M_{\text{inf}}$ has the image contained in $M_{\text{inf}}^{G_{\infty}}$.

Given the parallel with the strategy of [10], we refer to the proofs in *loc. cit.* whenever possible, and describe the needed modifications where necessary. We note that while [10, 21] are concerned with the case $p > 2$, the arguments and results we will use apply for $p = 2$ as well, with identical proofs. (To facilitate this approach further, the notation used will usually closely reflect the notation of [10].)

Setting $M_{\text{BK}}^0 = H_{\Delta, 1}^i(\mathcal{X}/\mathcal{S})$, $M_{\text{inf}}^0 = H_{\Delta, 1}^i(\mathcal{X}_{A_{\text{inf}}}/A_{\text{inf}})$, we have

$$T' = H_{\text{ét}}^i(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}/p\mathbb{Z}) = T_1^{\text{inf}}(M_{\text{inf}}^0) = T_1(M_{\text{BK}}^0).$$

Moreover, since u is a unit in $W_1(\mathbb{C}_K^b) = \mathbb{C}_K^b$, we have

$$T_1(M_{\text{BK}}^0) = T_1(M_{\text{BK}}) \quad \text{and} \quad T_1^{\text{inf}}(M_{\text{inf}}^0) = T_1^{\text{inf}}(M_{\text{inf}}),$$

where $M_{\text{BK}} = M_{\text{BK}}^0/M_{\text{BK}}^0[u^{\infty}]$ and $M_{\text{inf}} = M_{\text{inf}}^0/M_{\text{inf}}^0[u^{\infty}]$ are again a Breuil–Kisin module and a Breuil–Kisin–Fargues G_K -module, resp., of height $\leq i$. The isomorphism $M_{\text{inf}} \simeq M_{\text{BK}} \otimes_{\mathcal{S}} A_{\text{inf}}$ still remains true, and the pair $(M_{\text{BK}}, M_{\text{inf}})$ satisfies the conditions

$$\forall g \in G_s \quad \forall x \in M_{\text{BK}}: \quad g(x) - x \in \varphi^{-1}(v)u^{p^s} M_{\text{inf}} \quad (5.1)$$

for all $s \geq 0$, since the pair $(M_{\text{BK}}^0, M_{\text{inf}}^0)$ satisfies the analogous conditions by Proposition 4.10. Finally, the module M_{BK} is finitely generated and u -torsion-free $k[[u]]$ -module, hence a finite free $k[[u]]$ -module (and, consequently, M_{inf} is a finite free $\mathcal{O}_{\mathbb{C}_K^b}$ -module).

Instead of using $T' = T_1(M_{\text{inf}}) = H_{\text{ét}}^i(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}/p\mathbb{Z})$ directly, we work instead with the dual module

$$T := T_1^{*, \text{inf}}(M_{\text{inf}}) = \text{Hom}_{A_{\text{inf}}, \varphi}(M_{\text{inf}}, \mathcal{O}_{\mathbb{C}_K^b}) \simeq H_{\text{ét}}^i(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}/p\mathbb{Z})^{\vee}.$$

This is equivalent, as the splitting field of T is still L . Note that

$$T \simeq T_1^*(M_{\text{BK}}) = \text{Hom}_{\mathfrak{S}, \varphi}(M_{\text{BK}}, \mathcal{O}_{\mathbb{C}_K^b})$$

as a $\mathbb{Z}/p\mathbb{Z}[G_\infty]$ -module.

Remark 5.6 (Ramification bounds of [9]). Similarly to the discussion above we may take, for any $n \geq 1$, $M_{\text{BK}}^0 = H_{\Delta, n}^i(\mathcal{X}/\mathfrak{S})$, and $M_{\text{BK}} = M_{\text{BK}}^0/M_{\text{BK}}^0[u^\infty]$. Then the G_∞ -module

$$T := T_n^*(M_{\text{BK}}) = \text{Hom}_{\mathfrak{S}, \varphi}(M_{\text{BK}}, W_n(\mathcal{O}_{\mathbb{C}_K^b}))$$

is the restriction of $H_{\text{ét}}^i(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}/p^n\mathbb{Z})^\vee$ to G_∞ . Denoting by $\mathcal{O}_{\mathfrak{S}}$ the p -adic completion of $\mathfrak{S}[1/u]$, $M_{\mathfrak{S}} := M_{\text{BK}} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathfrak{S}}$ then becomes an étale φ -module over $\mathcal{O}_{\mathfrak{S}}$ in the sense of [16, Section A], with the natural map $M_{\text{BK}} \rightarrow M_{\mathfrak{S}}$ injective; thus, in terminology of [9], M_{BK} serves as a φ -lattice of height dividing $E(u)^i$. Upon observing that T is the G_∞ -representation associated with $M_{\mathfrak{S}}$ (see e.g. [9, Section 2.1.3]), Theorem 2 of [9] implies the ramification bound

$$\mu_{L/K} \leq 1 + c_0(K) + e(s_0(K) + \log_p(ip)) + \frac{e}{p-1}.$$

Here $c_0(K)$, $s_0(K)$ are constants that depend on the field K and that generally grow with increasing e . (Their precise meaning is described in Section 5.2.)

We employ the following approximations of the functors T_1^* and $T_1^{*, \text{inf}}$.

Notation 5.7. For a real number $c \geq 0$, we define

$$\begin{aligned} J_c(M_{\text{BK}}) &= \text{Hom}_{\mathfrak{S}, \varphi}(M_{\text{BK}}, \mathcal{O}_{\mathbb{C}_K^b}/\alpha^{>c}), \\ J_c^{\text{inf}}(M_{\text{inf}}) &= \text{Hom}_{A_{\text{inf}}, \varphi}(M_{\text{inf}}, \mathcal{O}_{\mathbb{C}_K^b}/\alpha^{>c}). \end{aligned}$$

We further set $J_\infty(M_{\text{BK}}) = T_1^*(M_{\text{BK}})$ and $J_\infty^{\text{inf}}(M_{\text{inf}}) = T_1^{*, \text{inf}}(M_{\text{inf}})$. Given $c, d \in \mathbb{R}^{\geq 0} \cup \{\infty\}$ with $c \geq d$, we denote by $\rho_{c,d}: J_c(M_{\text{BK}}) \rightarrow J_d(M_{\text{BK}})$ ($\rho_{c,d}^{\text{inf}}: J_c^{\text{inf}}(M_{\text{inf}}) \rightarrow J_d^{\text{inf}}(M_{\text{inf}})$, resp.) the map induced by the quotient map $\mathcal{O}_{\mathbb{C}_K^b}/\alpha^{>c} \rightarrow \mathcal{O}_{\mathbb{C}_K^b}/\alpha^{>d}$.

Since $M_{\text{inf}} \simeq M_{\text{BK}} \otimes_{\mathfrak{S}} A_{\text{inf}}$ as φ -modules, it is easy to see that for every $c \in \mathbb{R}^{\geq 0} \cup \{\infty\}$, we have a natural isomorphism $\theta_c: J_c(M_{\text{BK}}) \xrightarrow{\sim} J_c^{\text{inf}}(M_{\text{inf}})$ of abelian groups; the biggest point of distinction between the two is that $J_c^{\text{inf}}(M_{\text{inf}})$ naturally attains G_K -action from the one on M_{inf} by the rule

$$g(f)(x) := g(f(g^{-1}(x))), \quad g \in G_K, \quad f \in J_c^{\text{inf}}(M_{\text{inf}}), \quad x \in M_{\text{inf}}.$$

On $J_c(M_{\text{BK}})$, there is a natural action given similarly by the formula $g(f)(x) := g(f(x))$ where $f \in J_c(M_{\text{BK}})$ and $x \in M_{\text{BK}}$. However, in order for this action to make sense, one needs that each $g(f)$ defined this way is still an $\mathfrak{S}$ -linear map, which comes down to requiring that $g(u) = u$ (that is, $g(\pi) = \pi$) in the ring $\mathcal{O}_{\mathbb{C}_K^b}/\alpha^{>c}$. This condition holds for $g \in G_s$ for s depending on c as follows.

Proposition 5.8 ([10, Proposition 2.5.3]). *Let $s \geq 0$ be an integer with $s > \log_p(\frac{c(p-1)}{ep})$. Then the natural action of G_s on $\mathcal{O}_{\mathbb{C}_K^b}/\alpha^{>c}$ induces an action of G_s on $J_c(M_{\text{BK}})$. Furthermore, when $d \leq c$, the map $\rho_{c,d}: J_c(M_{\text{BK}}) \rightarrow J_d(M_{\text{BK}})$ is G_s -equivariant, and when $s' \geq s$, the $G_{s'}$ -action on $J_c(M_{\text{BK}})$ defined in this manner is the restriction of the G_s -action to $G_{s'}$.*

The crucial connection between the actions on $J_c(M_{\text{BK}})$ and $J_c^{\text{inf}}(M_{\text{inf}})$ is established using (the consequences of) the conditions (Cr_s).

Proposition 5.9. *For*

$$s > \max \left\{ \log_p \left(\frac{c(p-1)}{ep} \right), \log_p \left(c - \frac{e}{p-1} \right) \right\},$$

the natural isomorphism $\theta_c: J_c(M_{\text{BK}}) \xrightarrow{\sim} J_c^{\text{inf}}(M_{\text{inf}})$ is G_s -equivariant.

Proof. Identifying M_{inf} with $M_{\text{BK}} \otimes_{\mathbb{C}} A_{\text{inf}}$, θ_c takes the form $f \mapsto \tilde{f}$ where

$$\tilde{f}(x \otimes a) := af(x), \quad x \in M_{\text{BK}}, \quad a \in A_{\text{inf}}.$$

Note that we have $\varphi^{-1}(v)u^{p^s}\mathcal{O}_{\mathbb{C}_K^b} = \alpha^{\geq p^s+e/(p-1)}$. The condition (5.1) then states that for all $x \in M_{\text{BK}}$ and all $g \in G_s$, $g(x \otimes 1) - x \otimes 1$ lies in $\alpha^{\geq p^s+e/(p-1)}M_{\text{inf}}$ and therefore in $\alpha^{>c}M_{\text{inf}}$ thanks to the assumption on s . It then follows that for every $\tilde{f} \in J_c^{\text{inf}}(M_{\text{inf}})$, $\tilde{f}(g(x \otimes 1)) = \tilde{f}(x \otimes 1)$, and hence

$$\begin{aligned} g(\tilde{f})(x \otimes a) &= g(\tilde{f}(g^{-1}(x \otimes a))) = g(g^{-1}(a)\tilde{f}(g^{-1}(x \otimes 1))) \\ &= ag(\tilde{f}(x \otimes 1)) = ag(f(x)) \end{aligned}$$

for every $g \in G_s$, $x \in M_{\text{BK}}$ and $a \in A_{\text{inf}}$. Thus, we have that $g(\tilde{f}) = \widetilde{g(f)}$ for every $g \in G_s$ and $f \in J_c(M_{\text{BK}})$, proving the equivariance of θ_c . ■

From now on, set $b := ie/(p-1)$ and $a := iep/(p-1)$. Then T is determined by $J_a(M)$, $J_b(M)$ in the following sense.

Proposition 5.10. *Keeping all the previous notation, we have:*

- (1) *The map $\rho_{\infty,b}: T_1^*(M_{\text{BK}}) \rightarrow J_b(M_{\text{BK}})$ is injective, with the image being precisely the subgroup $\rho_{a,b}(J_a(M_{\text{BK}}))$ of $J_b(M_{\text{BK}})$.*
- (2) *Similarly, the map $\rho_{\infty,b}^{\text{inf}}: T_1^{*,\text{inf}}(M_{\text{inf}}) \rightarrow J_b^{\text{inf}}(M_{\text{inf}})$ is injective with $\rho_{\infty,b}^{\text{inf}}(M_{\text{inf}}) = \rho_{a,b}^{\text{inf}}(J_a^{\text{inf}}(M_{\text{inf}}))$.*
- (3) *For $s > \log_p(i)$, $T_1^*(M_{\text{BK}})$ has a natural action of G_s that extends the usual G_{∞} -action.*
- (4) *For $s > \max(\log_p(i), \log_p((i-1)e/(p-1)))$, the action from (3) agrees with $T|_{G_s}$.*

Proof. Part (1) is proved in [10, Proposition 2.3.3]. Then $T_1^*(M_{\text{BK}})$ attains the action of G_s with $s > \log_p(i)$ by identification with $\rho_{a,b}(J_a(M_{\text{BK}}))$ and using Proposition 5.8 (see also [10, Theorem 2.5.5]), which proves (3). Finally, the proof of (2), (4) is analogous

to [10, Corollary 3.3.3] and [10, Theorem 3.3.4]. Explicitly, consider the commutative diagram

$$\begin{array}{ccccc}
 T_1^*(M_{\text{BK}}) & \xrightarrow{\rho_{\infty,a}} & J_a(M_{\text{BK}}) & \xrightarrow{\rho_{a,b}} & J_b(M_{\text{BK}}) \\
 \sim \downarrow \theta_\infty & & \sim \downarrow \theta_a & & \sim \downarrow \theta_b \\
 T_1^{*,\text{inf}}(M_{\text{inf}}) & \xrightarrow{\rho_{\infty,a}^{\text{inf}}} & J_a^{\text{inf}}(M_{\text{inf}}) & \xrightarrow{\rho_{a,b}^{\text{inf}}} & J_b^{\text{inf}}(M_{\text{inf}}),
 \end{array}$$

where the composition of the rows gives $\rho_{\infty,b}$ and $\rho_{\infty,b}^{\text{inf}}$, resp. This immediately proves (2) using (1). Finally, the map $\rho_{\infty,b}^{\text{inf}}$ is G_K -equivariant, the map $\rho_{\infty,b}$ is (tautologically) G_s -equivariant for $s > \log_p(i)$ by the proof of (3), and both maps are injective. Since θ_b is G_s -equivariant when $s > \log_p((i-1)e/(p-1))$ by Proposition 5.9, it follows that so is θ_∞ , which proves (4). ■

We employ further approximations of $J_c(M_{\text{BK}})$ defined as follows.

Notation 5.11. Let $s \geq 0$ be an integer. Consider a real number $c \in [0, ep^s)$ and an algebraic extension E/K_s . We consider the ring

$$(\varphi_k^s)^* \mathcal{O}_E / \alpha_E^{>c/p^s} = k \otimes_{\varphi_k^s, k} \mathcal{O}_E / \alpha_E^{>c/p^s}$$

(note that the condition on c implies that $p \in \alpha_E^{>c/p^s}$, making $\mathcal{O}_E / \alpha_E^{>c/p^s}$ a k -algebra). We endow this ring with an $\mathfrak{S}$ -algebra structure via

$$\mathfrak{S} \xrightarrow{\text{mod } p} k[[u]] \xrightarrow{\alpha} (\varphi_k^s)^* \mathcal{O}_E / \alpha_E^{>c/p^s}$$

where α extends the k -algebra structure map by the rule $u \mapsto 1 \otimes \pi_s$. Then we set

$$J_c^{(s),E}(M_{\text{BK}}) = \text{Hom}_{\mathfrak{S}, \varphi} (M_{\text{BK}}, (\varphi_k^s)^* \mathcal{O}_E / \alpha_E^{>c/p^s}).$$

In the case when E/K_s is Galois, the module $J_c^{(s),E}(M_{\text{BK}})$ attains a G_s -action induced by the G_s -action on $\mathcal{O}_E / \alpha_E^{>c/p^s}$.

When c, d are two real numbers satisfying $ep^s > c \geq d \geq 0$, there is an obvious transition map $\rho_{c,d}^{(s),E}(M_{\text{BK}}): J_c^{(s),E}(M_{\text{BK}}) \rightarrow J_d^{(s),E}(M_{\text{BK}})$, which is G_s -equivariant in the Galois case.

The relation to $J_c(M_{\text{BK}})$ is the following.

Proposition 5.12. *Let s, c be as above. Then*

- (1) *Given an algebraic extension E/K_s , $J_c^{(s),E}(M_{\text{BK}})$ naturally embeds into $J_c(M_{\text{BK}})$ as a submodule (G_s -submodule when E/K_s is Galois).*
- (2) *Given a tower of algebraic extensions $F/E/K_s$, $J_c^{(s),E}(M_{\text{BK}})$ naturally embeds into $J_c^{(s),F}(M_{\text{BK}})$ as a submodule (G_s -submodule if both E/K_s and F/K_s are Galois).*
- (3) *$J_c^{(s),\bar{K}}(M_{\text{BK}})$ is naturally isomorphic to $J_c(M_{\text{BK}})$ as a G_s -module.*

Proof. Part (2) follows from the observation that the map $\mathcal{O}_E/\alpha_E^{>c/p^s} \rightarrow \mathcal{O}_F/\alpha_F^{>c/p^s}$ induced by the inclusion $\mathcal{O}_E \hookrightarrow \mathcal{O}_F$ remains injective (and is clearly G_s -equivariant in the Galois case). Similarly, part (3) follows from the fact that the map

$$\text{pr}_s: \mathcal{O}_{\mathbb{C}_K^b} = \varprojlim_{s, \varphi} \mathcal{O}_{\bar{K}}/p \rightarrow \mathcal{O}_{\bar{K}}/p$$

induces a (G_s -equivariant) isomorphism $\mathcal{O}_{\mathbb{C}_K^b}/\alpha^{>c} \rightarrow (\varphi_K^s)^* \mathcal{O}_{\bar{K}}/\alpha_{\bar{K}}^{>c/p^s}$ when $s > \log_p(c/e)$ (so a fortiori when $s > \log_p(c)$), which is proved in [10, Lemma 2.5.1]. Part (1) is then obtained as a direct combination of (2) and (3). ■

For a non-negative integer s , denote by L_s the composite of the fields K_s and L . The following adaptation of Theorem 4.1.1 of [10] plays a key role in establishing the ramification bound.

Theorem 5.13. *Let E/K_s be an algebraic extension where s is an integer satisfying*

$$s > M_0 := \max \left\{ \log_p \left(\frac{a}{e} \right), \log_p \left(b - \frac{e}{p-1} \right) \right\} = \max \left\{ \log_p \left(\frac{ip}{p-1} \right), \log_p \left(\frac{(i-1)e}{(p-1)} \right) \right\}.$$

Then the inclusion $\rho_{a,b}^{(s),E}(J_a^{(s),E}(M_{\text{BK}})) \hookrightarrow \rho_{a,b}(J_a(M_{\text{BK}}))$, facilitated by the inclusions $J_a^{(s),E}(M_{\text{BK}}) \hookrightarrow J_a(M_{\text{BK}})$ and $J_b^{(s),E}(M_{\text{BK}}) \hookrightarrow J_b(M_{\text{BK}})$ from Proposition 5.12, is an isomorphism if and only if $L_s \subseteq E$.

Proof. The proof of [10, Theorem 4.1.1] applies in our context as well, as we now explain. In [10, Section 4.1], for every F/K_s algebraic, an auxiliary set $\tilde{J}_1^{(s),F}(M_{\text{BK}})$ is constructed, together with maps of sets

$$\tilde{\rho}_c^{(s),F}: \tilde{J}_1^{(s),F}(M_{\text{BK}}) \rightarrow J_c^{(s),F}(M_{\text{BK}}) \quad \text{for every } c \in (0, ep^s).$$

Notably, the construction relies only on the fact that M_{BK} is a Breuil–Kisin module that is free over $k[[u]]$ and the assumption $s > \log_p(a/e)$. When F is Galois over K , this set is naturally a G_s -set and the maps are G_s -equivariant. Moreover, the sets have the property that $(\tilde{J}_1^{(s),F}(M_{\text{BK}}))^{G_{F'}} = \tilde{J}_1^{(s),F'}(M_{\text{BK}})$ when $F/F'/K_s$ is an intermediate extension.

Subsequently, it is shown in [10, Lemma 4.1.4] that

$$\tilde{\rho}_b^{(s),F} \text{ is injective and its image is } \rho_{a,b}^{(s),F}(J_a^{(s),F}(M_{\text{BK}})), \quad (*)$$

where the only restriction on s is again $s > \log_p(a/e)$.

Finally, one obtains a series of G_s -equivariant bijections:

$$\begin{aligned} \tilde{J}_1^{(s),\bar{K}}(M_{\text{BK}}) &\simeq \rho_{a,b}^{(s),\bar{K}}(J_a^{(s),\bar{K}}(M_{\text{BK}})) \quad (\text{by } (*)) \\ &\simeq \rho_{a,b}(J_a(M_{\text{BK}})) \quad (\text{Proposition 5.12 (3)}) \\ &\simeq \rho_{a,b}^{\text{inf}}(J_a^{\text{inf}}(M_{\text{inf}})) \quad (\text{Proposition 5.9}) \\ &\simeq T \quad (\text{Proposition 5.10 (2)}) \end{aligned}$$

(where the next-to-last step relies on the assumption $s > \log_p(b - e/(p - 1))$). Applying $(-)^{G_E}$ to both sides and using $(*)$ again then yields

$$\rho_{a,b}^{(s),E}(J_a^{(s),E}(M_{\text{BK}})) \simeq T^{G_E}.$$

Therefore, we may replace the inclusion from the statement of the theorem by the inclusion $T^{G_E} \subseteq T$, and the claim now easily follows. $\blacksquare$

Finally, we are ready to establish the desired ramification bound. Let $N_s = K_s(\zeta_{p^s})$ be the Galois closure of K_s over K , and set $M_s = L_s N_s$. Then we have

Proposition 5.14. *Let s be as in Theorem 5.13, and set $m = a/p^s$. Then the properties $(P_m^{L_s/K_s})$ and $(P_m^{M_s/N_s})$ hold.*

Proof. The proof of $(P_m^{L_s/K_s})$ is the same as in [10], which refers to an older version of [21] for parts of the proof. Let us therefore reproduce the argument for convenience. By Corollary 5.4, it is enough to prove $(P_m^{L_s/K_s^{\text{un}}})$ where K_s^{un} denotes the maximal unramified extension of K_s in L_s .

Let E/K_s^{un} be an algebraic extension and $f: \mathcal{O}_{L_s} \rightarrow \mathcal{O}_E/\alpha_K^{>m}$ be an $\mathcal{O}_{K_s^{\text{un}}}$ -algebra map. Setting $c = a$ or $c = b$, one can consider an induced map

$$f_c: \mathcal{O}_{L_s}/\alpha_{L_s}^{>c/p^s} \rightarrow \mathcal{O}_E/\alpha_K^{>c/p^s},$$

and we claim is that this map is well defined and injective.

Indeed, let ϖ be a uniformizer of L_s , satisfying the relation

$$\varpi^{e'} = c_1 \varpi^{e'-1} + c_2 \varpi^{e'-2} + \cdots + c_{e'-1} \varpi + c_{e'},$$

where $P(T) = T^{e'} - \sum_i c_i T^{e'-i}$ is an Eisenstein polynomial over K_s^{un} . Applying f one thus obtains $t^{e'} = \sum_i c_i t^{e'-i}$ in $\mathcal{O}_E/\alpha_K^{>m}$ where $t = f(\varpi)$, and thus, lifting t to $\tilde{t} \in \mathcal{O}_E$, we obtain the equality

$$\tilde{t}^{e'} = c_1 \tilde{t}^{e'-1} + c_2 \tilde{t}^{e'-2} + \cdots + c_{e'-1} \tilde{t} + c_{e'} + r$$

with $v_K(r) > m > 1/p^s$. It follows that $v_K(\tilde{t}) = v_K(\varpi) = 1/p^s e'$, and so $\varpi^n \in \alpha_{L_s}^{>c/p^s}$ if and only if $\tilde{t}^n \in \alpha_E^{>c/p^s}$, proving that f_c is both well defined as well as injective.

The map f_c then induces a k -algebra injection $(\varphi_k^s)^* \mathcal{O}_{L_s}/\alpha_{L_s}^{>c/p^s} \hookrightarrow (\varphi_k^s)^* \mathcal{O}_E/\alpha_E^{>c/p^s}$ which in turn gives an injection $J_c^{(s),L_s}(M_{\text{BK}}) \rightarrow J_c^{(s),E}(M_{\text{BK}})$, where $c = a$ or $c = b$; consequently, we obtain an injection

$$\rho_{a,b}^{(s),L_s}(J_a^{(s),L_s}(M_{\text{BK}})) \hookrightarrow \rho_{a,b}^{(s),E}(J_a^{(s),E}(M_{\text{BK}})).$$

Combining this with Propositions 5.10 and 5.12, we have the series of injections

$$\begin{aligned} \rho_{a,b}^{(s),L_s}(J_b^{(s),L_s}(M_{\text{BK}})) &\hookrightarrow \rho_{a,b}^{(s),E}(J_b^{(s),E}(M_{\text{BK}})) \hookrightarrow \rho_{a,b}^{(s),\bar{K}}(J_b^{(s),\bar{K}}(M_{\text{BK}})) \\ &\hookrightarrow \rho_{a,b}(J_b(M_{\text{BK}})) \simeq T. \end{aligned}$$

Since $\rho_{a,b}^{(s),L_s}(J_b^{(s),L_s}(M_{\text{BK}})) \simeq T$ by Theorem 5.13, this is actually an injection $T \hookrightarrow T$, hence an isomorphism since T is finite. In particular, the natural morphism

$$\rho_{a,b}^{(s),E}(J_b^{(s),E}(M_{\text{BK}})) \hookrightarrow \rho_{a,b}(J_b(M_{\text{BK}}))$$

is an isomorphism, and Theorem 5.13 thus implies that $L_s \subseteq E$. This finishes the proof of (1).

Similarly as in [10], the property $(P_m^{M_s/N_s})$ is deduced from $(P_m^{L_s/K_s})$ as follows. Given an algebraic extension E/N_s and an $\mathcal{O}_{N_s}$ -algebra morphism $\mathcal{O}_{M_s} \rightarrow \mathcal{O}_E/\alpha_E^{>m}$, by restriction we obtain an $\mathcal{O}_{K_s}$ -algebra morphism $\mathcal{O}_{L_s} \rightarrow \mathcal{O}_E/\alpha_E^{>m}$ and thus, there is a K_s -injection $L_s \rightarrow E$. As $N_s \subseteq E$, this can be extended to a K_s -injection $M_s \rightarrow E$, and since the extension M_s/K_s is Galois, one obtains an N_s -injection $M_s \rightarrow E$ by precomposing with a suitable automorphism of M_s . ■

Theorem 5.15. *Let*

$$\alpha = \lfloor M_0 \rfloor + 1 = \left\lfloor \log_p \left(\max \left\{ \frac{ip}{p-1}, \frac{(i-1)e}{p-1} \right\} \right) \right\rfloor + 1.$$

Then

- (1) $v_K(\mathcal{D}_{L/K}) < 1 + e\alpha + \frac{iep}{p^\alpha(p-1)} - \frac{1}{p^\alpha}$.
- (2) *For any μ satisfying*

$$\mu > 1 + e\alpha + \max \left\{ \frac{iep}{p^\alpha(p-1)} - \frac{1}{p^\alpha}, \frac{e}{p-1} \right\},$$

$G_K^{(\mu)}$ *acts trivially on* T .

Proof. We may set $s = \alpha$ as the condition $s > M_0$ is then satisfied. Propositions 5.3 and 5.14 then imply that $v_K(\mathcal{D}_{L_s/K_s}) < a/p^s$ (where $a = iep/(p-1)$) and thus

$$v_K(\mathcal{D}_{L_s/K}) = v_K(\mathcal{D}_{K_s/K}) + v_K(\mathcal{D}_{L_s/K_s}) < 1 + es - \frac{1}{p^s} + \frac{a}{p^s} = 1 + e\alpha + \frac{a-1}{p^\alpha}.$$

Similarly, we have $v_K(\mathcal{D}_{L/K}) = v_K(\mathcal{D}_{L_s/K}) - v_K(\mathcal{D}_{L_s/L}) \leq v_K(\mathcal{D}_{L_s/K})$, and the claim (1) thus follows.

To prove (2), let M_s and N_s be as in Proposition 5.14. The fields N_s and $M_s = LN_s$ are both Galois over K , hence Lemma 5.2 applies and we thus have

$$\mu_{M_s/K} = \max \{ \mu_{N_s/K}, \phi_{N_s/K}(\mu_{M_s/N_s}) \}.$$

By [21, Remark 5.5], we have

$$\mu_{N_s/K} = 1 + es + \frac{e}{p-1}.$$

As for the second argument, Proposition 5.3 gives the estimate

$$\mu_{M_s/N_s} \leq e_{N_s/K} m = \frac{e_{N_s/K}}{p^s} a.$$

The function $\phi_{N_s/K}(t)$ is concave and has a constant slope $1/e_{N_s/K}$ beyond $t = \lambda_{N_s/K}$, where it attains the value $\phi_{N_s/K}(\lambda_{N_s/K}) = \mu_{N_s/K} = 1 + es + e/(p-1)$. Thus, $\phi_{N_s/K}(t)$ can be estimated linearly from above as follows:

$$\phi_{N_s/K}(t) \leq 1 + es + \frac{e}{p-1} + \frac{1}{e_{N_s/K}}(t - \lambda_{N_s/K}) = 1 + es + \frac{t}{e_{N_s/K}} - \frac{\lambda_{N_s/K}}{e_{N_s/K}} + \frac{e}{p-1}.$$

There is an automorphism $\sigma \in \text{Gal}(N_s/K)$ with $\sigma(\pi_s) = \zeta_p \pi_s$. That is, $v_K(\sigma(\pi_s) - \pi_s) = e/(p-1) + 1/p^s$, showing that

$$\lambda_{N_s/K} \geq e_{N_s/K} \left(\frac{e}{p-1} + \frac{1}{p^s} \right),$$

and combining this with the estimate of $\phi_{N_s/K}(t)$, we obtain

$$\phi_{N_s/K}(t) \leq 1 + es + \frac{t}{e_{N_s/K}} - \frac{1}{p^s}.$$

Plugging in the estimate for μ_{M_s/N_s} then yields

$$\phi_{N_s/K}(\mu_{M_s/N_s}) \leq 1 + es + \frac{a}{p^s} - \frac{1}{p^s} = 1 + es + \frac{\frac{iep}{p-1} - 1}{p^s}.$$

Thus, we have

$$\mu_{L/K} \leq \mu_{M_s/K} \leq 1 + e\alpha + \max \left\{ \frac{iep}{p^\alpha(p-1)} - \frac{1}{p^\alpha}, \frac{e}{p-1} \right\},$$

which finishes the proof of part (2). ■

5.2. Comparisons of bounds

Finally, let us compare the bounds obtained in Theorem 5.15 with other results from the literature. These are summarized in the table below.

	$\mu_{L/K} \leq \dots$
Theorem 5.15	$1 + e(\lfloor \log_p(\max\{\frac{ip}{p-1}, \frac{(i-1)e}{p-1}\}) \rfloor + 1) + \max\{\beta, \frac{e}{p-1}\}, \quad \beta < \min(e, 2p)^{12}$
Caruso–Liu [10]	$1 + e(\lfloor \log_p(\frac{ip}{p-1}) \rfloor + 1) + \max\{\beta, \frac{e}{p-1}\}, \quad \beta < e^{13}$
Caruso [9]	$1 + c_0(K) + e(s_0(K) + \log_p(ip)) + \frac{e}{p-1}$
Hattori [21]	$\begin{cases} 1 + e + \frac{e}{p-1}, & i = 1, \\ 1 + e + \frac{ei}{p-1} - \frac{1}{p}, & i > 1, \end{cases} \quad \text{under } ie < p-1$
Abrashkin [1], Fontaine [17]	$1 + \frac{i}{p-1}, \quad \text{under } e = 1, i < p-1$

Table 1. Comparisons of estimates of $\mu_{L/K}$.

¹²More precisely: When $i = 1$, it is easy to see that $\beta = (iep/(p-1) - 1)/p^\alpha$ is smaller than $e/(p-1)$, and hence does not have any effect. When $i > 1$, one can easily show using $p^\alpha > ip/(p-1)$, $p^\alpha > (i-1)e/(p-1)$ that $\beta < e$ and $\beta < pi/(i-1) \leq 2p$.

¹³The number β here has different meaning than the number β of [10, Theorem 1.1].

Comparison with [21]. If we assume $ie < p - 1$, then the first maximum in the estimate of $\mu_{L/K}$ is realized by $ip/(p - 1) \in (1, p)$; that is, in Theorem 5.15 one has $\alpha = 1$ and thus,

$$\mu_{L/K} \leq 1 + e + \max \left\{ \frac{ei}{p-1} - \frac{1}{p}, \frac{e}{p-1} \right\},$$

which agrees precisely with the estimate [21].

Comparison with [1, 17]. Specializing to $e = 1$ in the previous case, the bound becomes

$$\mu_{L/K} \leq \begin{cases} 2 + \frac{1}{p-1}, & i = 1, \\ 2 - \frac{1}{p} + \frac{i}{p-1} & i > 1. \end{cases}$$

This is clearly a slightly worse bound than that of [17] and [1] (by 1 and $(p - 1)/p$, respectively).

Comparison with [10]. From the shape of the bounds it is clear that the bounds are equivalent when $(i - 1)e \leq ip$, that is, when $e \leq p$ and some “extra” cases that include the case when $i = 1$ (more precisely, these extra cases are when $e > p$ and $i \leq e/(e - p)$), and in fact, the terms β in such situation agree. In the remaining case when $(i - 1)e > ip$, our estimate becomes gradually worse compared to [10].

Remarks 5.16. Several explanations regarding this comparison are in order.

- (1) It should be noted that the bounds from [10] do not necessarily apply to our situation as it is not clear when $H_{\text{ét}}^i(\mathcal{X}_{\bar{\eta}}, \mathbb{Z}/p\mathbb{Z})$ (or rather their duals) can be obtained as a quotient of two lattices in a semistable representation with Hodge–Tate weights in $[0, i]$. To our knowledge the only result along these lines is [13, Theorem 1.3.1] that states that this is indeed the case when $i = 1$ (and X is a proper smooth variety over K with semistable reduction). Interestingly, in this case the bound from Theorem 5.15 always agrees with the one from [10].
- (2) Let us point out that the verbatim reading of the bound from [10] as described in Theorem 1.1 of *loc. cit.* would have the term $\lceil \log_p(ip/(p - 1)) \rceil$ (i.e., upper integer part) instead of the term $\lfloor \log_p(ip/(p - 1)) \rfloor + 1$ as in Table 1, but we believe this version to be correct. Indeed, the proof of Theorem 1.1 in [10] (in the case $n = 1$) ultimately relies on the objects $J_{1,a}^{(s),E}(\mathfrak{M})$ that are analogous to $J_a^{(s),E}(M_{\text{BK}})$, where $s = \lceil \log_p(ip/(p - 1)) \rceil$. In particular, Lemma 4.2.3 of *loc. cit.* needs to be applied with $c = a$, and the implicitly used fact that the ring $\mathcal{O}_E/\alpha_E^{>a/p^s}$ is a k -algebra (i.e., of characteristic p) relies on the *strict* inequality $e > a/p^s$, equivalently $s > \log_p(ip/(p - 1))$. In the case that $ip/(p - 1)$ happens to be equal to p^t for some integer t , one therefore needs to take $s = t + 1$ rather than $s = t$. This precisely corresponds to the indicated change.

Comparison with [9]. Let us explain the constants $s_0(K)$, $c_0(K)$ that appear in the estimate. The integer $s_0(K)$ is the smallest integer s such that $1 + p^s\mathbb{Z}_p \subseteq \chi(\text{Gal}(K_{p^\infty}/K))$ where χ denotes the cyclotomic character. The rational number $c_0(K) \geq 0$ is the small-

est constant c such that $\psi_{K/K'}(1+t) \geq 1 + et - c$ (this exists since the last slope of $\psi_{K/K'}(t)$ is e).¹⁴

In the case when K/K' is tamely ramified, the estimate from [9] becomes

$$\mu_{L/K} \leq 1 + e(\log_p(ip) + 1) + \frac{e}{p-1},$$

which is fairly equivalent to the bound from Theorem 5.15 when $e < p$ (and again also in some extra cases, e.g. when $i = 1$ for any e and p), with the difference of estimates being approximately

$$e\left(\log_p\left(\frac{p}{p-1}\right) - \frac{1}{p-1}\right) \in \left(-\frac{e}{4\sqrt{p}}, 0\right).$$

When e is big and coprime to p , the bound in [9] becomes gradually better unless e.g. $i = 1$.

When K has relatively large wild absolute ramification, we expect that the bound from Theorem 5.15 generally becomes stronger, especially if K contains p^n -th roots of unity for large n , as can be seen in the following examples (where we assume $i > 1$; for $i = 1$, our estimate retains the shape of the tame ramification case and hence the difference between the estimates becomes even larger).

Example 5.17. Let $i > 0$ be an integer. Consider the base field K as follows.

- (1) When $K = \mathbb{Q}_p(\zeta_{p^n})$ for $n \geq 2$, one has $e = (p-1)p^{n-1}$, $s_0(K) = n$ and from the classical computation of $\psi_{K/\mathbb{Q}_p}$ (e.g. as in [31, IV Section 4]), one obtains

$$c_0(K) = [(n-1)(p-1) - 1]p^{n-1} + 1.$$

The difference between the two estimates is thus approximately

$$ne - p^{n-1} + 1 > (n-1)e.$$

- (2) When $K = \mathbb{Q}_p(p^{1/p^n})$ for $n \geq 3$, one has $e = p^n$ and $s_0(K) = 1$. The description of $\psi_{K/\mathbb{Q}_p}$ in [10, Section 4.3] implies that $c_0(K) = np^n = ne$. The difference between the two estimates is thus approximately

$$e(1 + \log_p(i) - \log_p(i-1) + \log_p(p-1)) \approx 2e.$$

(In the cases $n = 1, 2$, one can check that the difference is still positive and bigger than p .)

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¹⁴To make sense of this in general, one needs to extend the definition of the functions $\psi_{L/M}, \varphi_{L/M}$ to the case when the extension L/M is not necessarily Galois. This is done e.g. in [9, Section 2.2.1].

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