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On Some Congruence Conjectures Involving Binary Quadratic Forms

by

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Abstract

In this paper, we mainly prove some congruence conjectures of Z.-H. Sun, for example, let p be a prime such that $p \equiv 1 \pmod 6$ and $p = x^2 + 3y^2$. Then $p\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(3k+1)16^k} \equiv 4x^2 - 2p - \frac{p^2}{4x^2} \pmod{p^3}$.

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§1. Introduction

Let k be an integer not less than 3 and p be a prime $\equiv 1 \pmod k$. Define the integer f by $p = kf + 1$. For $0 < s < r < k$, Yeung [27] studied binomial coefficients of the form $\binom{rf}{sf}$ modulo p^2. Some of them have been determined modulo p in terms of representation of p by certain binary quadratic forms. Gauss proved that if $p = 4f + 1 = a^2 + b^2$ with $a \equiv 1 \pmod 4$, then

$$\binom{2f}{f} \equiv 2a \pmod p.$$

Later, Jacobi proved that if $p = 3f + 1$ and $4p = c^2 + 27d^2$, then

$$\binom{2f}{f} \equiv -c \pmod p.$$

After Gauss and Jacobi, many congruences of this kind were proved by Dickson, E. Lehmer, Stern and Whiteman. But the first congruence of this type modulo p^2

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was obtained by Chowla, Dwork and Evans [6]: they proved that

$$\binom{2f}{f} \equiv \Big(1+\frac{1}{2}pq_p(2)\Big)\Big(2a-\frac{p}{2a}\Big) \pmod{p^2},$$

where a and p are as above, and $q_p(a)=(a^{p-1}-1)/p$ is the Fermat quotient. This congruence was conjectured by Beukers [4], and later in 2012, Pan [15] re-proved this congruence by an elementary method.

Three years after Chowla, Dework and Evans [6], Yeung [27] studied congruences modulo p^2 for binomial coefficients by using the p-adic gamma function and the Gross–Koblitz formula. He determined completely all binomial coefficients of the form $\binom{rf}{sf}$ modulo p^2 in the cases of $k=3,4,6$.

In 2010, Cosgrave and Dilcher [7] obtained the first congruence of this type modulo p^3,

$$\begin{aligned}\binom{2f}{f} \equiv &\Big(2a-\frac{p}{2a}-\frac{p^2}{8a^3}\Big)\\ &\times\Big(1+\frac{1}{2}pq_p(2)+\frac{1}{8}p^2(2E_{p-3}-q_p^2(2))\Big) \pmod{p^3},\end{aligned}$$

where a and p are as above, and $\{E_n\}$ are the Euler numbers defined by

$$E_0=1 \quad \text{and} \quad E_{2n}=-\sum_{k=1}^{n}\binom{2n}{2k}E_{2n-2k} \quad (n\geq 1).$$

They also obtained another congruence which extends one of Yeung's result. For integers r, s, prime $p=6f+1$, $4p=r^2+3s^2$ with $r\equiv 1 \pmod 3$ and $s\equiv 0 \pmod 3$,

$$\binom{4f}{2f} \equiv \Big(-r+\frac{p}{r}+\frac{p^2}{r^3}\Big)\Big(1+\frac{1}{6}p^2B_{p-2}\Big(\frac{1}{3}\Big)\Big) \pmod{p^3},$$

where $\{B_n(x)\}$ are the Bernoulli polynomials given by

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k} \quad (n=0,1,2,\ldots)$$

and $\{B_n\}$ are Bernoulli numbers given by

$$B_0=1, \quad \sum_{k=0}^{n-1}\binom{n}{k}B_k=0 \quad (n\geq 2).$$

In this paper, our first goal is to prove a conjecture of Z. H. Sun's [23, Conj. 2.1].

Theorem 1.1. *Let p be a prime such that $p \equiv 1 \pmod 6$ and $p = x^2 + 3y^2$. Then*

$$p\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(3k+1)16^k} \equiv p\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(6k+1)16^k} \equiv 4x^2 - 2p - \frac{p^2}{4x^2} \pmod{p^3}.$$

Remark 1.1. Z. H. Sun [23, Thm. 2.2] proved the above congruence modulo p^2 and then he proposed this conjecture.

In 1987, Beukers [5] conjectured that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k} \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p = x^2+4y^2 \equiv 1 \pmod 4, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

This conjecture was proved by several authors including Van Hamme [26] for ($p \equiv 3 \pmod 4$), Ishikawa [10] for ($p \equiv 1 \pmod 4$) and Ahlgren [1]. In addition, Z. H. Sun [24] proved the following congruence: Let p be an odd prime. Then modulo p^3, we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k} \equiv \begin{cases} 4x^2-2p-\dfrac{p^2}{4x^2} & \text{if } p = x^2+y^2 \equiv 1 \pmod 4 \text{ and } 2 \nmid x, \\ -\dfrac{p^2}{4}\dbinom{\frac{p-3}{2}}{\frac{p-3}{4}}^{-2} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Rodriguez-Villegas [19] posed some conjectures on supercongruences modulo p^2 in 2013, one of which is

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k} \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p = x^2+3y^2 \equiv 1 \pmod 3, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

This conjecture has been confirmed by Mortenson [14] and Z.-W. Sun [25], and then Z.-H. Sun [22, Conj. 4.15] generalized this conjecture to the following one.

Conjecture 1.1. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k} \equiv \begin{cases} 4x^2-2p-\dfrac{p^2}{4x^2} \pmod{p^3} & \textit{if } p = x^2+3y^2 \equiv 1 \pmod 6, \\ -\dfrac{p^2}{2}\dbinom{\frac{p-1}{2}}{\frac{p-5}{6}}^{-2} \pmod{p^3} & \textit{if } p \equiv 5 \pmod 6. \end{cases}$$

Our last goal is to prove the above conjecture, and partially prove [22, Conjs 4.11 and 4.18].

Theorem 1.2. *Firstly, Conjecture 1.1 is true. Secondly, let $p>7$ be a prime and $p\neq 71$. If $p=x^2+2y^2\equiv 1 \pmod 8$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3} \tag{1.1}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\equiv\begin{cases}\dfrac{p^2}{3}\dbinom{\frac{p-1}{4}}{\frac{p-5}{8}}^{-2}\pmod{p^3} & \text{if } p\equiv 5\pmod 8,\\[2ex] -\dfrac{3p^2}{2}\dbinom{\frac{p-3}{4}}{\frac{p-7}{8}}^{-2}\pmod{p^3} & \text{if } p\equiv 7\pmod 8.\end{cases} \tag{1.2}$$

Lastly, let p be an odd prime. If $p=x^2+2y^2\equiv 1 \pmod 8$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}\equiv(-1)^{\frac{p-1}{2}}\left(4x^2-2p-\frac{p^2}{4x^2}\right)\pmod{p^3} \tag{1.3}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}\equiv\begin{cases}\dfrac{p^2}{3}\dbinom{\frac{p-1}{4}}{\frac{p-5}{8}}^{-2}\pmod{p^3} & \text{if } p\equiv 5\pmod 8,\\[2ex] \dfrac{3p^2}{2}\dbinom{\frac{p-3}{4}}{\frac{p-7}{8}}^{-2}\pmod{p^3} & \text{if } p\equiv 7\pmod 8.\end{cases} \tag{1.4}$$

We are going to prove Theorem 1.1 in Section 2. Section 3 is devoted to proving Theorem 1.2. Our proofs make use of the p-adic Gamma function, hypergeometric functions. Throughout this paper, p denotes an odd prime and $\mathbb{Z}_p$ denotes the ring of p-adic integers.

§2. Proof of Theorem 1.1

To prove Theorem 1.1, we need the following identity from [8, (3.100)]:

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\frac{(-1)^k}{3k+1}=\frac{1}{3}\frac{(\frac{2}{3})_n}{(\frac{1}{3})_{n+1}}. \tag{2.1}$$

Here and in what follows, $(a)_n=a(a+1)\cdots(a+n-1)$ for any positive integer n and $(a)_0=1$ denotes the Pochhammer symbol.

For $n,m\in\{1,2,3,\ldots\}$, define

$$H_n^{(m)}:=\sum_{1\le k\le n}\frac{1}{k^m},\quad H_0^{(m)}:=0;$$

these numbers with $m=1$ are called the classic harmonic numbers.

Lemma 2.1 ([20, 21]). *Let $p>5$ be a prime. Then*

$$H^{(2)}_{p-1}\equiv 0 \pmod p,\quad H^{(2)}_{\frac{p-1}{2}}\equiv 0 \pmod p,$$
$$\frac{1}{5}H^{(2)}_{\lfloor\frac{p}{6}\rfloor}\equiv H^{(2)}_{\lfloor\frac{p}{3}\rfloor}\equiv \frac{1}{2}\Big(\frac{p}{3}\Big)B_{p-2}\Big(\frac{1}{3}\Big) \pmod p,$$

and modulo p^2,

$$\begin{aligned}
H_{\frac{p-1}{2}}&\equiv -2q_p(2)+pq_p^2(2),\quad H_{p-1}\equiv 0,\\
H_{\lfloor\frac{p}{3}\rfloor}&\equiv -\frac{3}{2}q_p(3)+\frac{3p}{4}q_p^2(3)-\frac{p}{6}\Big(\frac{p}{3}\Big)B_{p-2}\Big(\frac{1}{3}\Big),\\
H_{\lfloor\frac{2p}{3}\rfloor}&\equiv -\frac{3}{2}q_p(3)+\frac{3p}{4}q_p^2(3)+\frac{p}{3}\Big(\frac{p}{3}\Big)B_{p-2}\Big(\frac{1}{3}\Big),\\
H_{\lfloor\frac{p}{6}\rfloor}&\equiv -2q_p(2)-\frac{3}{2}q_p(3)+pq_p^2(2)+\frac{3p}{4}q_p^2(3)-\frac{5p}{12}\Big(\frac{p}{3}\Big)B_{p-2}\Big(\frac{1}{3}\Big).
\end{aligned}$$

Lemma 2.2 (Al-Shaghay and Dilcher [2, Thms 3.4, 3.6 and 3.12]). *Let p be a prime. If $p=6f+1=x^2+3y^2$ with $x\equiv -1 \pmod 3$, we have the following congruences modulo p^3:*

$$\begin{aligned}\binom{3f}{f}\equiv -\Big(2x-\frac{p}{2x}-\frac{p^2}{8x^3}\Big)\Big(1-\frac{2p}{3}q_p(2)+\frac{3p}{4}q_p(3)+\frac{5p^2}{9}q_p^2(2)\\
-\frac{p^2}{2}q_p(2)q_p(3)-\frac{3p^2}{32}q_p^2(3)+\frac{p^2}{16}\Big(\frac{p}{3}\Big)B_{p-2}\Big(\frac{1}{3}\Big)\Big),\end{aligned}\tag{2.2}$$

$$\begin{aligned}\binom{4f}{f}\equiv (-1)^{f+1}\Big(2x-\frac{p}{2x}-\frac{p^2}{8x^3}\Big)\Big(1+\frac{4p}{3}q_p(2)-\frac{3p}{4}q_p(3)+\frac{2p^2}{9}q_p^2(2)\\
-p^2q_p(2)q_p(3)+\frac{21p^2}{32}q_p^2(3)\\
+\frac{7p^2}{48}\Big(\frac{p}{3}\Big)B_{p-2}\Big(\frac{1}{3}\Big)\Big),\end{aligned}\tag{2.3}$$

$$\begin{aligned}\binom{5f}{2f}\equiv -\Big(2x-\frac{p}{2x}-\frac{p^2}{8x^3}\Big)\Big(1-\frac{2p}{3}q_p(2)-\frac{3p}{4}q_p(3)+\frac{5p^2}{9}q_p^2(2)+\frac{p^2}{2}q_p(2)q_p(3)\\
+\frac{21p^2}{32}q_p^2(3)+\frac{43p^2}{48}\Big(\frac{p}{3}\Big)B_{p-2}\Big(\frac{1}{3}\Big)\Big).\end{aligned}\tag{2.4}$$

Lemma 2.3. *Let $p=6f+1=x^2+3y^2$ be a prime with $x\equiv 1 \pmod 3$. Then*

$$\begin{aligned}\binom{3f}{f}\equiv \Big(2x-\frac{p}{2x}-\frac{p^2}{8x^3}\Big)\Big(1-\frac{2p}{3}q_p(2)+\frac{3p}{4}q_p(3)+\frac{5p^2}{9}q_p^2(2)-\frac{p^2}{2}q_p(2)q_p(3)\\
-\frac{3p^2}{32}q_p^2(3)+\frac{p^2}{16}\Big(\frac{p}{3}\Big)B_{p-2}\Big(\frac{1}{3}\Big)\Big) \pmod{p^3},\end{aligned}\tag{2.5}$$

$$\binom{4f}{f} \equiv (-1)^f\Big(2x-\frac{p}{2x}-\frac{p^2}{8x^3}\Big)\Big(1+\frac{4p}{3}q_p(2)-\frac{3p}{4}q_p(3)+\frac{2p^2}{9}q_p^2(2)-p^2q_p(2)q_p(3)$$
$$+\frac{21p^2}{32}q_p^2(3)+\frac{7p^2}{48}\Big(\frac{p}{3}\Big)B_{p-2}\Big(\frac{1}{3}\Big)\Big) \pmod{p^3}, \tag{2.6}$$
$$\binom{5f}{2f} \equiv \Big(2x-\frac{p}{2x}-\frac{p^2}{8x^3}\Big)\Big(1-\frac{2p}{3}q_p(2)-\frac{3p}{4}q_p(3)+\frac{5p^2}{9}q_p^2(2)+\frac{p^2}{2}q_p(2)q_p(3)$$
$$+\frac{21p^2}{32}q_p^2(3)+\frac{43p^2}{48}\Big(\frac{p}{3}\Big)B_{p-2}\Big(\frac{1}{3}\Big)\Big) \pmod{p^3}. \tag{2.7}$$

Proof. By replacing x with $-u$, we have $p = 6f + 1 = u^2 + 3y^2$ is a prime with $u \equiv -1 \pmod 3$, so by (2.2)–(2.4), we immediately obtain the desired results (2.5)–(2.7). □

Lemma 2.4. *Let $p > 3$ be a prime. For any p-adic integer t and s, we have*

$$\binom{\frac{p-1}{2}+pt}{\frac{p-1}{3}} \equiv \binom{\frac{p-1}{2}}{\frac{p-1}{3}}\Big(1+pt(H_{\frac{p-1}{2}}-H_{\frac{p-1}{6}})-p^2t^2H_{\frac{p-1}{2}}H_{\frac{p-1}{6}}$$
$$+\frac{p^2t^2}{2}(H^2_{\frac{p-1}{2}}-H^{(2)}_{\frac{p-1}{2}}+H^2_{\frac{p-1}{6}}+H^{(2)}_{\frac{p-1}{6}})\Big) \pmod{p^3}, \tag{2.8}$$
$$\binom{\frac{2p-2}{3}+ps}{\frac{p-1}{2}} \equiv \binom{\frac{2p-2}{3}}{\frac{p-1}{2}}\Big(1+ps(H_{\frac{2p-2}{3}}-H_{\frac{p-1}{6}})-p^2t^2H_{\frac{2p-2}{3}}H_{\frac{p-1}{6}}$$
$$+\frac{p^2t^2}{2}(H^2_{\frac{2p-2}{3}}-H^{(2)}_{\frac{2p-2}{3}}+H^2_{\frac{p-1}{6}}+H^{(2)}_{\frac{p-1}{6}})\Big) \pmod{p^3}. \tag{2.9}$$

Proof. We will just prove (2.8) because the proof of (2.9) is similar. Set $m = (p-1)/2$. It is easy to check that

$$\begin{aligned}
\binom{m+pt}{\frac{p-1}{3}} &= \frac{(m+pt)\cdots(1+pt)}{(\frac{p-1}{3})!(\frac{p-1}{6}+pt)\cdots(1+pt)} \\
&\equiv \frac{m!(1+ptH_m+\frac{p^2t^2}{2}(H_m^2-H_m^{(2)}))}{(\frac{p-1}{3})!(\frac{p-1}{6})!(1+ptH_{\frac{p-1}{6}}+\frac{p^2t^2}{2}(H^2_{\frac{p-1}{6}}-H^{(2)}_{\frac{p-1}{6}}))} \\
&= \binom{m}{\frac{p-1}{3}}\Big(1+pt(H_m-H_{\frac{p-1}{6}})-p^2t^2H_mH_{\frac{p-1}{6}} \\
&\qquad +\frac{p^2t^2}{2}(H_m^2-H_m^{(2)}+H^2_{\frac{p-1}{6}}+H^{(2)}_{\frac{p-1}{6}})\Big) \pmod{p^3}.
\end{aligned}$$

So Lemma 2.4 is proved. □

Proof of Theorem 1.1. It is known that $\binom{2k}{k} \equiv 0 \pmod p$ and $p \nmid (3k+1)$ for each $\frac{p+1}{2} \le k \le p-1$, and $\binom{2k}{k}^2/16^k \equiv \binom{\frac{p-1}{2}}{k}\binom{\frac{p-1}{2}+k}{k}(-1)^k \pmod{p^2}$ for each

$0 \le k \le \frac{p-1}{2}$. So by (2.1), we have

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\frac{p}{3k+1} &\equiv \sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{2k}{k}^2}{16^k}\frac{p}{3k+1}\\ &\equiv p\sum_{k=0}^{\frac{p-1}{2}}\frac{\binom{\frac{p-1}{2}}{k}\binom{\frac{p-1}{2}+k}{k}(-1)^k}{3k+1}+\frac{\binom{\frac{2p-2}{3}}{\frac{p-1}{3}}^2}{16^{\frac{p-1}{3}}}-\binom{\frac{p-1}{2}}{\frac{p-1}{3}}\binom{\frac{5p-5}{6}}{\frac{p-1}{3}}\\ &=\frac{(\frac{2}{3})_{\frac{p-1}{2}}}{(\frac{1}{3})_{\frac{p-1}{3}}(\frac{p}{3}+1)_{\frac{p-1}{6}}}+\frac{\binom{\frac{2p-2}{3}}{\frac{p-1}{3}}^2}{16^{\frac{p-1}{3}}}-\binom{\frac{p-1}{2}}{\frac{p-1}{3}}\binom{\frac{5p-5}{6}}{\frac{p-1}{3}} \pmod{p^3}.\end{aligned}$$

Since $p \equiv 1 \pmod 6$, so modulo p^3, we have

$$\left(\frac{1}{3}\right)_{\frac{p-1}{3}} \equiv \left(\frac{p-1}{3}\right)!\left(1-\frac{p}{3}H_{\frac{p-1}{3}}+\frac{p^2}{18}\left(H_{\frac{p-1}{3}}^2-H_{\frac{p-1}{3}}^{(2)}\right)\right)$$

and

$$\left(\frac{p}{3}+1\right)_{\frac{p-1}{6}} \equiv \left(\frac{p-1}{6}\right)!\left(1+\frac{p}{3}H_{\frac{p-1}{6}}+\frac{p^2}{18}\left(H_{\frac{p-1}{6}}^2-H_{\frac{p-1}{6}}^{(2)}\right)\right).$$

These, with Lemmas 2.1 and 2.4, yield that

$$\begin{aligned}&\frac{(\frac{2}{3})_{\frac{p-1}{2}}}{(\frac{1}{3})_{\frac{p-1}{3}}(\frac{p}{3}+1)_{\frac{p-1}{6}}}\\ &\quad\equiv(-1)^{\frac{p-1}{2}}\binom{\frac{p-1}{2}}{\frac{p-1}{3}}\binom{\frac{2p-2}{3}}{\frac{p-1}{2}}\Big(1-\frac{2p}{3}(H_{\frac{2p-2}{3}}-H_{\frac{p-1}{6}})+\frac{p}{3}(H_{\frac{p-1}{3}}-H_{\frac{p-1}{6}})\\ &\qquad\qquad+\frac{p^2}{18}(H_{\frac{p-1}{3}}-H_{\frac{p-1}{6}})^2+\frac{5p^2}{18}(H^{(2)}_{\frac{p-1}{6}}+H^{(2)}_{\frac{p-1}{3}})\Big)\\ &\quad\equiv(-1)^f\binom{\frac{p-1}{2}}{\frac{p-1}{3}}\binom{\frac{2p-2}{3}}{\frac{p-1}{2}}\\ &\qquad\times\left(1-\frac{2p}{3}q_p(2)+\frac{5p^2}{9}q_p^2(2)+\frac{5p^2}{12}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right)\right)\pmod{p^3}.\end{aligned}$$

Then, by Lemmas 2.1 and 2.4, we have

$$\begin{aligned}\frac{\binom{\frac{2p-2}{3}}{\frac{p-1}{3}}^2}{16^{\frac{p-1}{3}}}=\binom{-\frac{1}{2}}{\frac{p-1}{3}}^2\equiv\binom{\frac{p-1}{2}}{\frac{p-1}{3}}^2\Big(1-\frac{3p}{2}q_p(3)+\frac{15p^2}{8}q_p^2(3)\\ +\frac{5p^2}{24}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right)\Big)\pmod{p^3}.\end{aligned}$$

In view of (2.2)–(2.7), $p=6f+1=x^2+3y^2$ with either $x\equiv -1 \pmod 3$ or $x\equiv 1 \pmod 3$, we have

$$(-1)^f\binom{\frac{p-1}{2}}{\frac{p-1}{3}}\binom{\frac{2p-2}{3}}{\frac{p-1}{2}} \equiv \Big(4x^2-2p-\frac{p^2}{4x^2}\Big)\Big(1+\frac{2p}{3}q_p(2)-\frac{p^2}{9}q_p^2(2) + \frac{5p^2}{24}\Big(\frac{p}{3}\Big)B_{p-2}\Big(\frac{1}{3}\Big)\Big) \pmod{p^3},$$

$$\binom{\frac{p-1}{2}}{\frac{p-1}{3}}^2 \equiv \Big(4x^2-2p-\frac{p^2}{4x^2}\Big)\Big(1-\frac{4p}{3}q_p(2)+\frac{3p}{2}q_p(3)+\frac{14p^2}{9}q_p^2(2) - 2p^2q_p(2)q_p(3)+\frac{3p^2}{8}q_p^2(3) + \frac{p^2}{8}\Big(\frac{p}{3}\Big)B_{p-2}\Big(\frac{1}{3}\Big)\Big) \pmod{p^3}$$

and

$$\binom{\frac{p-1}{2}}{\frac{p-1}{3}}\binom{\frac{5p-5}{6}}{\frac{p-1}{3}} \equiv \Big(4x^2-2p-\frac{p^2}{4x^2}\Big)\Big(1-\frac{4p}{3}q_p(2)+\frac{14p^2}{9}q_p^2(2) + \frac{23p^2}{24}\Big(\frac{p}{3}\Big)B_{p-2}\Big(\frac{1}{3}\Big)\Big) \pmod{p^3}.$$

So we just need to verify that

$$\begin{aligned}
&\Big(1+\frac{2p}{3}q_p(2)-\frac{p^2}{9}q_p^2(2)+\frac{5p^2}{24}\Big(\frac{p}{3}\Big)B_{p-2}\Big(\frac{1}{3}\Big)\Big)\\
&\times\Big(1-\frac{2p}{3}q_p(2)+\frac{5p^2}{9}q_p^2(2)+\frac{5p^2}{12}\Big(\frac{p}{3}\Big)B_{p-2}\Big(\frac{1}{3}\Big)\Big)\\
&+\Big(1-\frac{4p}{3}q_p(2)+\frac{3p}{2}q_p(3)+\frac{14p^2}{9}q_p^2(2)-2p^2q_p(2)q_p(3)+\frac{3p^2}{8}q_p^2(3)\\
&\quad+\frac{p^2}{8}\Big(\frac{p}{3}\Big)B_{p-2}\Big(\frac{1}{3}\Big)\Big)\\
&\times\Big(1-\frac{3p}{2}q_p(3)+\frac{15p^2}{8}q_p^2(3)+\frac{5p^2}{24}\Big(\frac{p}{3}\Big)B_{p-2}\Big(\frac{1}{3}\Big)\Big)\\
&-\Big(1-\frac{4p}{3}q_p(2)+\frac{14p^2}{9}q_p^2(2)+\frac{23p^2}{24}\Big(\frac{p}{3}\Big)B_{p-2}\Big(\frac{1}{3}\Big)\Big)\equiv 1 \pmod{p^3}.
\end{aligned}$$

It is easy to check that this congruence is true when we regroup the factors of powers of p, so we immediately obtain the desired result

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\frac{p}{3k+1}\equiv 4x^2-2p-\frac{p^2}{4x^2} \pmod{p^3}.$$

The congruence

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} \frac{p}{6k+1} \equiv 4x^2 - 2p - \frac{p^2}{4x^2} \pmod{p^3}$$

can be deduced in the same way as the above; it also needs to use the six congruences (2.2)–(2.7) and some results which can be obtained in the same way that Lemma 2.4 was obtained. We omit the details of the proof. Therefore, the proof of Theorem 1.1 is complete. $\square$

§3. Proof of Theorem 1.2

For each $\alpha \in \mathbb{Z}_p$, define the p-adic order $\nu_p(\alpha) := \max\{n \in \mathbb{N} : p^n \mid \alpha\}$ and the p-adic norm $|\alpha|_p := p^{-\nu_p(\alpha)}$. Define the p-adic gamma function $\Gamma_p(\cdot)$ by

$$\Gamma_p(n) = (-1)^n \prod_{\substack{1 \le k < n \\ (k,p)=1}} k, \quad n = 1, 2, 3, \ldots,$$

$$\Gamma_p(\alpha) = \lim_{\substack{|\alpha - n|_p \to 0 \\ n \in \mathbb{N}}} \Gamma_p(n), \quad \alpha \in \mathbb{Z}_p.$$

In particular, we set $\Gamma_p(0) = 1$. In the following, we need to use the most basic properties of Γ_p, and all of them can be found in [17, 18].

For example, we know that

$$\frac{\Gamma_p(x+1)}{\Gamma_p(x)} = \begin{cases} -x & \text{if } |x|_p = 1, \\ -1 & \text{if } |x|_p < 1. \end{cases} \tag{3.1}$$

$$\Gamma_p(1-x)\Gamma_p(x) = (-1)^{a_0(x)}, \tag{3.2}$$

where $a_0(x) \in \{1, 2, \ldots, p\}$ such that $x \equiv a_0(x) \pmod p$. Among the properties we need here is the fact that for any positive integer n,

$$z_1 \equiv z_2 \pmod{p^n} \quad \text{implies} \quad \Gamma_p(z_1) \equiv \Gamma_p(z_2) \pmod{p^n}. \tag{3.3}$$

The *truncated generalized hypergeometric function* is defined by

$${}_rF_s\left[\begin{matrix}\alpha_1\ \alpha_2 \ldots \alpha_r \\ \beta_1 \ldots \beta_s\end{matrix}\middle| z\right]_n := \sum_{k=0}^{n} \frac{(\alpha_1)_k \cdots (\alpha_r)_k}{(\beta_1)_k \cdots (\beta_s)_k} \cdot \frac{z^k}{k!},$$

where the parameters $\beta_1, \beta_2, \ldots, \beta_s \notin \{\ldots, -3, -2, -1, 0\}$. For the study of congruences of truncated hypergeometric functions, the readers may see [11, 12, 13].

Definition 1. Let χ be a character and ψ be an additive character of $\mathbb{F}_q$, the Gauss sum associated to χ and ψ is defined by

$$G(\chi,\psi) = -\sum_{x\in\mathbb{F}_q^*} \chi(x)\psi(x).$$

If χ and λ are two characters of $\mathbb{F}_q$, then the Jacobi sum associated to χ and λ is defined by

$$J(\chi,\lambda) = -\sum_{x\in\mathbb{F}_q^*} \chi(x)\lambda(x).$$

The additive character ψ would usually be fixed and so we write $G(\chi)$ for $G(\chi,\psi)$. We also need the following property of the Jacobi sum [9]: If $\chi\lambda$ is nontrivial, then

$$J(\chi,\lambda) = \frac{G(\chi)G(\lambda)}{G(\chi\lambda)}. \tag{3.4}$$

Suppose $p = kf+1$ is an odd prime; then χ is the character mod p of order k satisfying

$$\bar{\chi}(n) \equiv n^f \pmod p \quad \text{for all } n\in\mathbb{Z}, \tag{3.5}$$

and suppose integers r, s satisfy $r+s<k$. Then

$$J(\bar{\chi}^r,\bar{\chi}^s) \equiv 0 \pmod p. \tag{3.6}$$

These two congruences can be found in [27, (7), (8)].

Lemma 3.1. *Let $p = 6f+1 = x^2+3y^2$ be a prime with $x\equiv 1 \pmod 3$. Then*

$$J(\chi^2,\chi^3) \equiv 2x - \frac{p}{2x} - \frac{p^2}{8x^3} \pmod{p^3}.$$

Proof. In view of [3, Thm. 2.3], we have

$$J(\chi^2,\chi^3) = K(\chi^2).$$

So

$$J(\chi^2,\chi^3) = K(\chi^2) = x + y\sqrt{-3}.$$

By (3.6), we know

$$J(\bar{\chi}^2,\bar{\chi}^3) = x - y\sqrt{-3} \equiv 0 \pmod p.$$

Thus,

$$(x-y\sqrt{-3})^3 \equiv 0 \pmod{p^3}.$$

Therefore, with the fact that $x^2+3y^2=p$, we have

$$y\sqrt{-3}\equiv\frac{x(4x^2-3p)}{4x^2-p}\equiv\frac{(4x^2-3p)(16x^4+4px^2+p^2)}{64x^5}$$
$$\equiv x-\frac{p}{2x}-\frac{p^2}{8x^3}\pmod{p^3}.$$

So we immediately get the desired result

$$J(\chi^2,\chi^3)=x+y\sqrt{-3}\equiv 2x-\frac{p}{2x}-\frac{p^2}{8x^3}\pmod{p^3}.$$

Now the proof of Lemma 3.1 is complete. □

In view of [27, (26)], we have, for $p=kf+1$ and integers r,s with $r+s<k$,

$$J(\chi^r,\chi^s)=\frac{\Gamma_p(\frac{r}{k})\Gamma_p(\frac{s}{k})}{\Gamma_p(\frac{(r+s)}{k})}, \tag{3.7}$$

which was a corollary of the Gross–Koblitz formula.

This, with Lemma 3.1, implies that if $p=x^2+3y^2\equiv 1\pmod 6$ with $x\equiv 1\pmod 3$,

$$\frac{\Gamma_p(\frac{1}{2})\Gamma_p(\frac{1}{3})}{\Gamma_p(\frac{5}{6})}=J(\chi^2,\chi^3)\equiv 2x-\frac{p}{2x}-\frac{p^2}{8x^3}\pmod{p^3}. \tag{3.8}$$

Denote the symbol $\langle x\rangle_p$ to be the least nonnegative residue of x modulo p, i.e., $\langle x\rangle_p\in\{0,1,\dots,p-1\}$ and $x\equiv\langle x\rangle_p\pmod p$, and $\alpha_p^*:=\frac{\alpha+\langle-\alpha\rangle_p}{p}$.

Lemma 3.2 ([16, Thm. 6.1]). *Let p be an odd prime and let $\alpha\in\mathbb{Z}_p$. If $\langle-\alpha\rangle_p$ is even, then*

$${}_3F_2\left[\begin{matrix}\alpha & 1-\alpha & \frac{1}{2}\\ & 1 & 1\end{matrix}\middle|\,1\right]_{p-1}\equiv\frac{\Gamma_p(\frac{1}{2})^2}{\Gamma_p(1-\frac{1}{2}\alpha)^2\Gamma_p(\frac{1}{2}+\frac{1}{2}\alpha)^2}\pmod{p^3}. \tag{3.9}$$

If $\langle-\alpha\rangle_p$ is odd, then modulo p^3 we have

$${}_3F_2\left[\begin{matrix}\alpha & 1-\alpha & \frac{1}{2}\\ & 1 & 1\end{matrix}\middle|\,1\right]_{p-1}\equiv\frac{p^2\alpha_p^*(\alpha_p^*-1)}{4}\frac{\Gamma_p(\frac{1}{2})^2}{\Gamma_p(1-\frac{1}{2}\alpha)^2\Gamma_p(\frac{1}{2}+\frac{1}{2}\alpha)^2}. \tag{3.10}$$

Proof of Conjecture 1.1. Case $p\equiv 1\pmod 6$ with $p=x^2+3y^2$ and $x\equiv 1\pmod 3$. Here we set χ as a character $\pmod p$ of order 6. It is easy to see that by (3.9) with $\alpha=\frac{1}{3}$, (3.2) and (3.8), we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k}={}_3F_2\left[\begin{matrix}\frac{1}{2} & \frac{1}{3} & \frac{2}{3}\\ & 1 & 1\end{matrix}\middle|\,1\right]_{p-1}\equiv\frac{\Gamma_p(\frac{1}{2})^2}{\Gamma_p(\frac{5}{6})^2\Gamma_p(\frac{2}{3})^2}=\frac{\Gamma_p(\frac{1}{2})^2\Gamma_p(\frac{1}{3})^2}{\Gamma_p(\frac{5}{6})^2}$$
$$=J(\chi^2,\chi^3)^2\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3}.$$

For $p=x^2+3y^2\equiv 1 \pmod 6$ with $x\equiv -1 \pmod 3$, we can get the desired result by replacing x with $-x$.

Case $p\equiv 5 \pmod 6$. It is easy to check that by (3.10) with $\alpha=\frac{1}{3}$, (3.2) and (3.3), we have the following modulo p^3:

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k}={}_3F_2\left[\begin{matrix}\frac{1}{2} & \frac{1}{3} & \frac{2}{3}\\ & 1 & 1\end{matrix}\middle| 1\right]_{p-1}\equiv -\frac{p^2}{18}\frac{\Gamma_p(\frac{1}{2})^2}{\Gamma_p(\frac{5}{6})^2\Gamma_p(\frac{2}{3})^2}$$
$$=-\frac{p^2}{18}\frac{\Gamma_p(\frac{1}{6})^2\Gamma_p(\frac{1}{3})^2}{\Gamma_p(\frac{1}{2})^2}\equiv -\frac{p^2}{18}\frac{\Gamma_p(\frac{p+1}{6})^2\Gamma_p(\frac{p+1}{3})^2}{\Gamma_p(\frac{p+1}{2})^2}\equiv -\frac{p^2}{2}\binom{\frac{p-1}{2}}{\frac{p-5}{6}}^{-2}.$$

Lemma 3.3. *Let $p=x^2+2y^2\equiv 1 \pmod 8$ with $x\equiv -1 \pmod 4$ be a prime and let χ be a character $(\mathrm{mod}\, p)$ of order 8. Then*

$$J(\chi,\chi^4)\equiv 2x-\frac{p}{2x}-\frac{p^2}{8x^3} \pmod{p^3}.$$

Proof. In view of [3, Thm. 3.12], we have

$$J(\chi,\chi^4)=K(\chi)=x+y\sqrt{-2}.$$

By (3.6) we have

$$J(\bar{\chi},\bar{\chi}^4)=x-y\sqrt{-2}\equiv 0 \pmod p.$$

So

$$(x-y\sqrt{-2})^3\equiv 0 \pmod{p^3}.$$

This, with $p=x^2+2y^2$, yields that

$$y\sqrt{-2}\equiv\frac{x^3-6xy^2}{3x^2-2y^2}=\frac{4x^3-3px}{4x^2-p}\equiv x-\frac{p}{2x}-\frac{p^2}{8x^3} \pmod{p^3}.$$

So we immediately get the desired result

$$J(\chi,\chi^4)\equiv x+y\sqrt{-2}\equiv 2x-\frac{p}{2x}-\frac{p^2}{8x^3} \pmod{p^3}.$$

Now we have finished the proof of Lemma 3.3. □

Proof of (1.1). Firstly, if $p=x^2+2y^2\equiv 1 \pmod 8$ with $x\equiv -1 \pmod 4$, then by (3.9) with $\alpha=\frac{1}{4}$, (3.2), (3.7) and Lemma 3.3, we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}={}_3F_2\left[\begin{matrix}\frac{1}{2} & \frac{1}{4} & \frac{3}{4}\\ & 1 & 1\end{matrix}\middle| 1\right]_{p-1}\equiv\frac{\Gamma_p(\frac{1}{2})^2}{\Gamma_p(\frac{7}{8})^2\Gamma_p(\frac{5}{8})^2}=\frac{\Gamma_p(\frac{1}{2})^2\Gamma_p(\frac{1}{8})^2}{\Gamma_p(\frac{5}{8})^2}$$
$$=J(\chi,\chi^4)^2\equiv 4x^2-2p-\frac{p^2}{4x^2} \pmod{p^3}.$$

For the case $p=x^2+2y^2\equiv 1 \pmod 8$ with $x\equiv 1 \pmod 4$, we can obtain the desired result by replacing x with $-x$.

Proof of (1.2). *Case* $p\equiv 5 \pmod 8$. It is easy to see that by (3.10) with $\alpha=\frac14$, (3.2) and (3.3), we have

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}&={}_3F_2\left[\begin{matrix}\frac12&\frac14&\frac34\\&1&1\end{matrix}\middle|\,1\right]_{p-1}\equiv-\frac{3p^2}{64}\frac{\Gamma_p(\frac12)^2}{\Gamma_p(\frac78)^2\Gamma_p(\frac58)^2}\\&=-\frac{3p^2}{64}\frac{\Gamma_p(\frac18)^2\Gamma_p(\frac38)^2}{\Gamma_p(\frac12)^2}\equiv-\frac{3p^2}{64}\frac{\Gamma_p(\frac{3p+1}{8})^2\Gamma_p(\frac{p+3}{8})^2}{\Gamma_p(\frac{p+1}{2})^2}\equiv-\frac{p^2}{3}\binom{\frac{p-1}{2}}{\frac{p-5}{8}}^{-2}\\&\equiv-\frac{p^2}{3}\frac{(-4)^{\frac{p-5}{4}}}{\binom{\frac{p-5}{4}}{\frac{p-5}{8}}^2}\equiv-\frac{p^2}{3}\frac{2^{\frac{p-5}{2}}}{\frac14\binom{\frac{p-1}{4}}{\frac{p-5}{8}}^2}=\frac{p^2}{3}\binom{\frac{p-1}{4}}{\frac{p-5}{8}}^{-2}\pmod{p^3}.\end{aligned}$$

Case $p\equiv 7 \pmod 8$. It is easy to check that by (3.10) with $\alpha=\frac14$, (3.2) and (3.3), we have

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}&={}_3F_2\left[\begin{matrix}\frac12&\frac14&\frac34\\&1&1\end{matrix}\middle|\,1\right]_{p-1}\equiv-\frac{3p^2}{64}\frac{\Gamma_p(\frac12)^2}{\Gamma_p(\frac78)^2\Gamma_p(\frac58)^2}\\&=-\frac{3p^2}{64}\frac{\Gamma_p(\frac18)^2\Gamma_p(\frac38)^2}{\Gamma_p(\frac12)^2}\equiv-\frac{3p^2}{64}\frac{\Gamma_p(\frac{p+1}{8})^2\Gamma_p(\frac{3p+3}{8})^2}{\Gamma_p(\frac{p+1}{2})^2}\equiv-\frac{p^2}{3}\binom{\frac{p-1}{2}}{\frac{p-7}{8}}^{-2}\\&\equiv-\frac{p^2}{3}\frac{(-4)^{\frac{p-7}{4}}}{\binom{\frac{p-7}{4}}{\frac{p-7}{8}}^2}\equiv-\frac{p^2}{3}\frac{2^{\frac{p-7}{2}}}{\frac{1}{36}\binom{\frac{p-3}{4}}{\frac{p-7}{8}}^2}=-\frac{3p^2}{2}\binom{\frac{p-3}{4}}{\frac{p-7}{8}}^{-2}\pmod{p^3}.\end{aligned}$$

Proof of (1.3) *and* (1.4). In view of [16, (8.2)], we have

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}&={}_3F_2\left[\begin{matrix}\frac12&\frac12&\frac12\\&1&1\end{matrix}\middle|-1\right]_{p-1}\equiv(-1)^{\frac{p-1}{2}}{}_3F_2\left[\begin{matrix}\frac12&\frac14&\frac34\\&1&1\end{matrix}\middle|\,1\right]_{p-1}\\&=(-1)^{\frac{p-1}{2}}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\pmod{p^3}.\end{aligned}$$

So we immediately get the desired results with the help of (1.1) and (1.2).

Now the proof of Theorem 1.2 is complete. □

Remark 3.1. Similarly, we could prove the following congruences by Lemma 3.2 and [3, Thms 3.9 and 3.19]: If $p\equiv 1 \pmod{12}$ with $p=x^2+y^2$ and $2\nmid x$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{k}}{12^{3k}}\equiv 4x^2-2p-\frac{p^2}{4x^2}\pmod{p^3},$$

and if $p \equiv 3 \pmod 4$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{k}}{12^{3k}} \equiv -\frac{5}{36}p^2 \binom{\frac{p-3}{2}}{\lfloor \frac{p}{12} \rfloor}^{-2} \pmod{p^3}.$$

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