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On the Structure of Graph Product von Neumann Algebras

by

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Abstract

We undertake a comprehensive study of structural properties of graph products of von Neumann algebras equipped with faithful, normal states, as well as properties of the graph products relative to subalgebras coming from induced subgraphs. Among the technical contributions in this paper is a complete bimodule calculation for subalgebras arising from subgraphs. As an application, we obtain a complete classification of when two subalgebras coming from induced subgraphs can be amenable relative to each other. We also give complete characterizations of when the graph product can be full, diffuse, or a factor. Our results are obtained in a broad generality, and we emphasize that they are new even in the tracial setting. They also allow us to deduce new results about when graph products of groups can be amenable relative to each other.

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Contents

§0. Introduction

Graph products of operator algebras have recently emerged as a subject of intense interest, providing an interpolation between the free product and the tensor product. The term comes from the group setting where they were introduced by Green [Gre90]; in the operator algebra setting they have been reintroduced and studied under various names by Młotkowski [Mło04], by Speicher and Wysoczański [SW16], and by Caspers and Fima [CF17]. The mixture of classical and free independence provides a powerful framework for proving results in deformation/rigidity theory [BoCa24, CKE24, Cas20, CdSS18, CDD25a, CDD25b], the theory of operator space approximation properties of operator algebras [Atk20, CF17], and free probability [CC21, CdSH+25, Mło04, SW16]. Graph products of groups are also of significant current interest in group theory [Ago13, AM15, HW08, Kob12, KK15, KK13, MO15]. Several structural properties of graph product von Neumann algebras – the Haagerup property, exactness, Connes embeddability, the rapid decay property, absence of Cartan subalgebras, strong solidity, modular theory, and proper proximality – have also been investigated [Atk20, BoCo24, Cas16, Cas20, CF17, CKE24, DKE24]. Altogether, this makes graph products a natural object to study using tools from geometric group theory, approximation properties, deformation/rigidity theory, free probability, and random matrices.

Given a finite simple graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a family of groups or of von Neumann algebras associated to the vertices, their graph product is a group or a von Neumann algebra generated by copies of the input objects with the pairwise relations determined by the graph: two objects connected by an edge should be in direct or tensor product position; two objects not connected by an edge should be in free position. The relations of higher order must be given as well; we defer the precise definition for von Neumann algebras to Section 1.1, and refer to Green for the precise definition for groups [Gre90].

In this paper, we undertake a systematic study of precisely when certain natural structural properties of graph products of von Neumann algebras hold. Moreover, for applications to positions of subalgebras it is natural to consider algebras corresponding to induced subgraphs and ask when these properties hold "relative" to another. We provide a complete classification for relative amenability, fullness, factoriality, and diffuseness (we also completely settle "relative diffuseness" i.e. lack of intertwining, in the tracial setting).

Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $U \subseteq \mathcal{V}$, the subgraph induced by U is the graph $\mathcal{G}|_U$ whose vertex set is U and whose edge set is $\mathcal{E} \cap (U \times U)$. When we denote $(M, \varphi) := \ast_{v\in\mathcal{G}}(M_v, \varphi_v)$ and when it is not ambiguous, we will let (M_U, φ_U) denote the graph product of $\{(M_v, \varphi_v) : v \in U\}$ with respect to the graph $\mathcal{G}|_U$.

We now state our results characterizing relative amenability. In the greatest generality our results apply to von Neumann algebras equipped with faithful normal states which are not necessarily tracial (hereafter referred to as *statial* von Neumann algebras). However, our results specialize slightly to the setting of both tracial von Neumann algebras and group von Neumann algebras and therefore to groups; in these more restrictive settings our conditions for relative amenability become slightly nicer to state. Although we state many of our results in the statial setting, they are still new even with an added assumption of traciality.

Main Theorem 0.1. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph, let $\{(M_v, \tau_v) : v \in \mathcal{V}\}$ be a family of tracial von Neumann algebras, and let $(M, \tau) = \ast_{v\in\mathcal{G}}(M_v, \tau_v)$. Assume that M_v has a trace zero unitary for every $v \in \mathcal{V}$. For $V_1, V_2 \subseteq \mathcal{V}$, M_{V_1} is amenable relative to M_{V_2} in M if and only if the following occur:*

(1) *M_v is amenable for each $v \in V_1 \setminus V_2$.*

(2) *For each $v \in V_1 \setminus V_2$ and $w \in V_1$ with $v \neq w$, either v and w are adjacent or both the following occur:*

   (a) $\dim(M_v) = \dim(M_w) = 2$,

   (b) *v and w are adjacent to all vertices in $V_1 \setminus \{v, w\}$.*

Note that Theorem 0.1 is new even in the case where $M_v = L(\Gamma_v)$ for a family discrete groups $\{\Gamma_v : v \in \mathcal{V}\}$. As such, we obtain a complete classification of when subgroups corresponding to induced subgraphs can be amenable relative to each other for graph products of groups, which follows immediately from Theorem 0.1.

Corollary 0.2. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph and $\{\Gamma_v : v \in \mathcal{V}\}$ be a family of groups. Let Γ be the graph product of $\{\Gamma_v : v \in \mathcal{V}\}$ with respect to $\mathcal{G}$, and for $U \subseteq \mathcal{V}$ let Γ_U be the graph product of $\{\Gamma_v : v \in U\}$ with respect to $\mathcal{G}|_U$. Then for $V_1, V_2 \subseteq V$ we have that Γ_{V_1} is amenable relative to Γ_{V_2} inside Γ if and only if both of the following occur:*

(1) *Γ_v is amenable for each $v \in V_1 \setminus V_2$.*

(2) *For each $v \in V_1 \setminus V_2$ and $w \in V_1$ with $v \neq w$, either $(v, w) \in \mathcal{E}$ or both of the following occur:*

   (a) *$\Gamma_v \cong \Gamma_w \cong \mathbb{Z}/2\mathbb{Z}$,*

   (b) *v and w are adjacent to all vertices in $V_1 \setminus \{v, w\}$.*

We also have a complete result in the statial setting that generalizes Theorem 0.1. This also yields a complete characterization of when the graph product von Neumann algebra is amenable (see Proposition 6.3).

Main Theorem 0.3. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph, let $\{(M_v, \varphi_v) : v \in \mathcal{V}\}$ be a family of statial von Neumann algebras, and let $(M, \varphi) = \mathop{\ast}_{v\in\mathcal{G}} (M_v, \varphi_v)$. Assume $M_v^{\varphi_v}$ has a state zero unitary for every $v \in \mathcal{V}$. For $V_1, V_2 \subseteq \mathcal{V}$, M_{V_1} is amenable relative to M_{V_2} inside M if and only if both of the following occur:*

(1) *M_v is amenable for each $v \in V_1 \setminus V_2$.*

(2) *For each $v \in V_1 \setminus V_2$ and $w \in V_1$ with $v \neq w$, either v and w are adjacent or both of the following occur:*

   (a) *$M_{\{v,w\}} = M_v * M_w$ is either amenable if $w \notin V_2$ or is amenable relative to M_w if $w \in V_2$,*

   (b) *v and w are adjacent to all vertices in $V_1 \setminus \{v, w\}$.*

Our assumption that the centralizer subalgebra $M_v^{\varphi_v}$ admits a state zero unitary is mild (see Appendix A for a characterization in terms of minimal central projections). Indeed, in the tracial setting this holds for any non-trivial group von Neumann algebra, any diffuse algebra, and any finite factor but $\mathbb{C}$. Moreover, this assumption already appears in foundational works in the non-tracial setting [Bar95, Shl97]. While this was removed in the free product setting through the work of Ueda [Ued11], doing so in our setting is likely to involve significant effort which we leave for future investigation.

A significant tool for our study of relative amenability is to apply work of [BMO20] (building upon prior work of [AD95, Haa93]), which states that there is a (not assumed to be normal) conditional expectation $\langle M, e_Q\rangle \to N$ when $N, Q \leq M$ are with expectation if and only if

$$ {}_N L^2(M)_N \prec_N L^2(\langle M, e_Q\rangle)_N. $$

Strictly speaking, the existence of such a conditional expectation is different from N being amenable relative to Q inside M, but this turns out to be not a problem. Since we may view $L^2(\langle M, e_Q\rangle)$ as a relative tensor product, it thus makes sense to address relative amenability via understanding the bimodule structure of ${}_{M_{V_1}} L^2(M)_{M_{V_2}}$ for V_1, V_2 subsets of the vertices, as well as the fusion rules for such bimodules. We obtain a complete description of such bimodules and their fusion rules in terms of the combinatorial structure of the graph, and the dimensions of the algebras attached to the vertices.

Main Theorem 0.4. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph, let $\{(M_v, \varphi_v) : v \in \mathcal{V}\}$ be a family of statial von Neumann algebras, and let $(M, \varphi) = \mathop{\ast}_{v\in\mathcal{G}}(M_v, \varphi_v)$. For $V_1, V_2 \subseteq \mathcal{V}$ one has*

$$ {}_{M_{V_1}} L^2(M,\varphi)_{M_{V_2}} \cong \bigoplus_{U\subseteq V_1\cap V_2} \left({}_{M_{V_1}} L^2(M_{V_1}, \varphi_{V_1}) \underset{M_U}{\otimes} L^2(M_{V_2}, \varphi_{V_2})_{M_{V_2}}\right)^{\oplus k_{\mathcal{G}}(V_1,V_2,U)}, \tag{0.1} $$

where $k_{\mathcal{G}}(V_1, V_2, U)$ is explicitly determined in terms of the graph structure and dimension of the vertex algebras (see Theorem 5.4 for the precise description). Moreover, we have the following fusion rules. For $V_1, V_2 \subseteq \mathcal{V}$ and $U \subseteq V_1 \cap V_2$, let

$$ \mathscr{H}_U(V_1, V_2) = {}_{M_{V_1}} L^2(M_{V_1}, \varphi_{V_1}) \underset{M_U}{\otimes} L^2(M_{V_2}, \varphi_{V_2})_{M_{V_2}}. $$

Then for $U_1 \subseteq V_1 \cap V_2$ and $U_2 \subseteq V_2 \cap V_3$,

$$ \mathscr{H}_{U_1}(V_1, V_2) \otimes_{M_{V_2}} \mathscr{H}_{U_2}(V_2, V_3) \cong \bigoplus_{W\subseteq U_1\cap U_2} \mathscr{H}_W(V_1, V_3)^{\oplus k_{\mathcal{G}_2}(U_1,U_2,W)}, $$

where $\mathcal{G}_2$ is the subgraph of $\mathcal{G}$ induced by V_2.

Theorem 0.4 is proved in two parts in the body of the paper: in Theorem 5.4 and Proposition 5.5. The utility of such a precise computation can be seen from Theorem 5.6, which provides a very easy to check characterization of when certain bimodules are weakly coarse.

We also give a complete characterization of fullness, factoriality, and diffuseness.

Main Theorem 0.5. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph, let $\{(M_v, \varphi_v) : v \in \mathcal{V}\}$ be a family of statial von Neumann algebras, and let $(M, \varphi) = \mathop{\ast}_{v\in\mathcal{G}}(M_v, \varphi_v)$. Assume $M_v^{\varphi_v}$ has a state zero unitary for every $v \in \mathcal{V}$.*

(1) *M is diffuse if and only if either* (a) *some M_v is diffuse, or* (b) *$\mathcal{G}$ is not a complete graph.*
(2) *M is a factor if and only if both* (a) *whenever a vertex v is adjacent to all other vertices of $\mathcal{G}$, then M_v is a factor, and* (b) *if v and w are not adjacent to each other but are adjacent to all other vertices of $\mathcal{G}$, then* $\max(\dim M_v, \dim M_w) \geq 3$.
(3) *M is full if and only if both* (a) *whenever a vertex v is adjacent to all other vertices of $\mathcal{G}$, then M_v is full, and* (b) *if v and w are not adjacent to each other but are adjacent to all other vertices of $\mathcal{G}$, then* $\max(\dim M_v, \dim M_w) \geq 3$.

For tracial algebras, we also provide a complete characterization of relative diffuseness (or lack of intertwining) of M_{V_1} relative to M_{V_2} analogous to Theorem 0.5(1). We refer the reader to Proposition 4.2 for the relevant statement, which amounts to in (a) requiring that a diffuse algebra be attached to a vertex in $V_1 \setminus V_2$, and replacing the "non-completeness" in (b) with the lack of edge between a vertex in $V_1 \setminus V_2$ with a vertex in $V_1 \cap V_2$.

§1. Preliminaries

§1.1. $\mathcal{G}$-independence and graph products

Throughout, a *graph* is a pair $(\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is a finite set of *vertices* and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of *edges* such that $(u, v) \in \mathcal{E}$ if and only if $(v, u) \in \mathcal{E}$; we also insist that $(u, u) \notin \mathcal{E}$ for all $u \in \mathcal{V}$. In other words, our graphs are finite and simple (undirected, and without self-loops). We write $v \sim w$ (respectively, $v \not\sim w$) whenever $(v, w) \in \mathcal{E}$ (respectively, $(v, w) \notin \mathcal{E}$); we make the dependence on the graph implicit. For a given $v \in \mathcal{V}$, we denote the sphere centered at v by $S(v) = \{w \in \mathcal{V} : w \sim v\}$, and the ball centered at v by $B(v) := S(v) \cup \{v\}$.

A word $v_1 \cdots v_n$ in the alphabet $\mathcal{V}$ is said to be $\mathcal{G}$-*reduced* if whenever $i < k$ with $v_i = v_k$, there is some $i < j < k$ so that $(v_i, v_j) \notin \mathcal{E}$. (By repeatedly applying this condition, we could further assume that $v_i \neq v_j$.)

Suppose that $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a graph and (M, φ) is a statial von Neumann algebra. For each $v \in \mathcal{V}$, let $1 \in M_v \subseteq M$ be a unital $*$-subalgebra. Then the family $\{M_v : v \in \mathcal{V}\}$ is said to be $\mathcal{G}$-*independent* if whenever $v_1 \cdots v_n$ is a $\mathcal{G}$-reduced word and $x_1, \ldots, x_n \in M$ with $x_i \in \ker(\varphi) \cap M_{v_i}$, we have

$$\varphi(x_1 \cdots x_n) = 0.$$

On the other hand, given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a family of statial von Neumann algebras $\{(M_v, \varphi_v) : v \in \mathcal{V}\}$, there is up to isomorphism a unique statial von Neumann algebra (M, φ) and state-preserving inclusions $M_v \hookrightarrow M$ so that the images of the M_v are $\mathcal{G}$-independent and generate M. We refer to this algebra (M, φ) as the *graph product* of the family $\{(M_v, \varphi_v) : v \in \mathcal{V}\}$ and write

$$(M, \varphi) = \mathop{\Large\ast}_{v \in \mathcal{G}} (M_v, \varphi_v).$$

The existence and uniqueness of the graph product was shown by Młotkowski and also by Caspers and Fima; moreover, if each φ_v is tracial then so is the state on the graph product [Mło04, CF17].

§1.2. Structural properties of von Neumann algebras

We recall the definitions of the structural properties appearing in the theorems in the introduction of the paper.

A von Neumann algebra M is said to be *full* if whenever a bounded net $(x_i)_{i \in I} \subset M$ satisfies $\|\varphi([x_i, \,\cdot\,])\| \to 0$ for all $\varphi \in M_*$ then there exists a net of scalars $(\lambda_i)_{i \in I} \subset \mathbb{C}$ such that $(x_i - \lambda_i) \to 0$ strongly. This notion was introduced for von Neumann algebras with separable preduals by Connes [Con74], where he showed it was equivalent to $\mathrm{Inn}(M)$ being closed in $\mathrm{Aut}(M)$ under the point norm topology [Con74, Thm. 3.5]. [HMV19, AH14] considered this notion in the more general σ-finite case, where they showed it was equivalent to $M' \cap M^\omega = \mathbb{C}$ [AH14, Prop. 4.35, Thm. 5.2], [HMV19, Cor. 3.7]. Here, M^ω denotes the Ocneanu ultrapower (see Appendix B), and in this paper we will always verify fullness by proving $M' \cap M^\omega = \mathbb{C}$. We note that the proof of this implication can be found in [AH14, Prop. 4.35], and in fact it is an exercise from [Con74, Prop. 2.8].

Let $A, B \leq M$ be inclusions of von Neumann algebras with conditional expectations E_A, E_B. Let $\langle M, e_B \rangle$ denote the basic construction associated to the inclusion $(B \subset M, E_B)$. We say that A is *amenable relative to B inside M* if there exists a conditional expectation $\Phi \colon \langle M, e_B \rangle \to A$ such that $\Phi|_M$ is normal [Pop86] (see also [Pop99] and [MP03, Def. 4]).

§2. Diffuseness, factoriality, and fullness

In this section we classify when a graph product W*-algebra has various properties (diffuseness, amenability, factoriality, fullness) based on the input algebras M_v (see [CF17, Cor. 2.29] for a partial result in this direction).

We will use the graph join operation to produce a tensor product decomposition for the graph product, thereby reducing the study of various properties of the graph product over $\mathcal{G}$ to the properties of the subgraphs $\mathcal{G}_1, \dots, \mathcal{G}_n$. Given graphs $\mathcal{G}_j = (\mathcal{V}_j, \mathcal{E}_j)$ for $j = 1, \ldots, n$, the *graph join* $\mathcal{G}_1 + \mathcal{G}_2 + \cdots + \mathcal{G}_n$ is the graph obtained from the disjoint union of $\mathcal{G}_1, \ldots, \mathcal{G}_n$ by adding edges from every vertex of $\mathcal{G}_i$ to every vertex of $\mathcal{G}_j$ for $i \neq j$. We say that $\mathcal{G}$ is *join-irreducible* if it is non-empty and cannot be decomposed as a graph join of two non-empty graphs. By [Cun82, Thm. 1], every graph $\mathcal{G}$ has a unique (up to permutation) decomposition as $\mathcal{G}_1 + \cdots + \mathcal{G}_n$, where $\mathcal{G}_1, \ldots, \mathcal{G}_n$ are join-irreducible (here we allow a single vertex to be considered as a join-irreducible graph). The next proposition follows immediately from the definition of the graph product for statial von Neumann algebras.

Proposition 2.1. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph and let $\{(M_v, \varphi_v) : v \in \mathcal{V}\}$ be a family of statial von Neumann algebras. If $\mathcal{G} = \mathcal{G}_1 + \cdots + \mathcal{G}_n$ for graphs $\mathcal{G}_j = (\mathcal{V}_j, \mathcal{E}_j)$, $j = 1, \ldots, n$, then*

$$\bigstar_{v \in \mathcal{G}} (M_v, \varphi_v) = \overline{\bigotimes_{1 \leq j \leq n}} \bigstar_{v \in \mathcal{G}_j} (M_v, \varphi_v).$$

Since it is known that diffuseness, factoriality, and fullness of a tensor product can be characterized in terms of the corresponding properties for the tensor factors (see the proof of Theorem 0.5 in Section 2.1 below), the above proposition allows us to reduce our analysis to the join-irreducible case. The general outline of the argument is as follows. By the foregoing argument, we reduce to the case when $\mathcal{G}$ is join-irreducible, then further divide into cases based on whether the number of vertices of $\mathcal{G}$ is 1, 2, or greater than 2, and decide diffuseness, amenability, factoriality, or fullness in each case. Of course, if $\mathcal{G}$ consists of a single vertex v, then this is simply the diffuseness, amenability, factoriality, or fullness of the input algebra M_v, and so we will only address the cases of $|\mathcal{V}| = 2$ and $|\mathcal{V}| \geq 3$ below. If $\mathcal{G}$ has two vertices, then these two vertices must not be connected by an edge, because otherwise $\mathcal{G}$ would decompose as the graph join of the two vertices. Hence, $\bigstar_{v \in \mathcal{G}} (M_v, \varphi_v)$ is the free product $(M_1, \varphi_1) * (M_2, \varphi_2)$ of the two input algebras. Now, if we assume that $M_1^{\varphi_1}$ and $M_2^{\varphi_2}$ each contain state zero unitaries u_1 and u_2, then by free independence the product $u_1 u_2$ will be a Haar unitary in $(M_1, \varphi_1) * (M_2, \varphi_2)$, and hence $(M_1, \varphi_1) * (M_2, \varphi_2)$ is diffuse. If $M_1 \cong M_2 \cong \mathbb{C} \oplus \mathbb{C}$ with equal weight on each of the two summands, then $M_1 * M_2$ is amenable and not a factor, and in all other cases (under the assumption that $M_1^{\varphi_1}$ and $M_2^{\varphi_2}$ admit state zero unitaries), it is a full factor by results of Ueda [Ued11]. The remaining case is then when $\mathcal{G}$ has at least three vertices, which we will handle separately as a general argument.

Before proceeding in this way, we first observe a combinatorial condition that follows from a lack of graph join decomposition.

Lemma 2.2. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a join-irreducible graph. Then either $\mathcal{G}$ is disconnected or for every vertex $v_0 \in \mathcal{V}$, there exist $v_1, v_2 \in \mathcal{V} \setminus \{v_0\}$ such that*

$$v_0 \sim v_1, \quad v_0 \not\sim v_2, \quad v_1 \not\sim v_2.$$

Proof. We proceed by contrapositive. Suppose that $\mathcal{G}$ is connected and that there exists a vertex v_0 such that for all $v_1, v_2 \in \mathcal{V} \setminus \{v_0\}$, if $v_1 \sim v_0$ and $v_2 \not\sim v_0$, then $v_1 \sim v_2$. Fix such a v_0. Let $S = S(v_0)$. We claim that every vertex in S is adjacent to every vertex in S^c. Let $v \in S$ and $w \in S^c$. If $w = v_0$, then $w \sim v$ by definition of $S(v_0)$. If $w \neq v_0$, then because $w \not\sim v_0$ and $v \sim v_0$, we have $v \sim w$. Since every vertex in S is connected to every vertex in S^c, we can decompose $\mathcal{G}$ as the graph join of the two induced subgraphs with vertex sets S and S^c. $\square$

Remark 2.3. The converse of this lemma does not hold. In fact, suppose that we take graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ which both satisfy that for every $v_0 \in \mathcal{V}$, there exist $v_1, v_2 \in \mathcal{V} \setminus \{v_0\}$ such that $v_0 \sim v_1$, $v_0 \not\sim v_2$, $v_1 \not\sim v_2$. Then $\mathcal{G}_1 + \mathcal{G}_2$ also satisfies this condition. More generally, if $\mathcal{V}$ is expressed as a union of subsets V_j, and the subgraphs induced by V_j have this property, then the whole graph has this property.

The following is a special case of Theorem 0.5 for join-irreducible graphs, which will be used in the general proof in conjunction with strategy outlined after Proposition 2.1. The proof makes use of Ocneanu ultrapowers and some related lemmas which are detailed in Appendix B. It also uses the fact that subalgebras M_U corresponding to induced subgraphs admit unique state-preserving, faithful, normal, conditional expectations $E_{M_U} : M \to M_U$ (see [CF17, Rem. 2.14]). The uniqueness implies, in particular, that $M_{V_1 \cap V_2}$, M_{V_1}, M_{V_2}, M form a commuting square for any subsets $V_1, V_2 \subset \mathcal{V}$.

Theorem 2.4. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a join-irreducible graph. Let $\{(M_v, \varphi_v) : v \in \mathcal{V}\}$ be a family of statial von Neumann algebras and let $(M, \varphi) = \mathop{\ast}_{v \in \mathcal{G}} (M_v, \varphi_v)$. Assume $M_v^{\varphi_v}$ has a state zero unitary for every $v \in \mathcal{V}$.*

- *If $|\mathcal{V}| = 2$ with $\mathcal{V} = \{v, w\}$ and $\dim(M_v) = \dim(M_w) = 2$, then M is diffuse but not a factor.*
- *If $|\mathcal{V}| = 2$, and $\max(\dim(M_v), \dim(M_w)) \geq 3$, then M is a diffuse full factor.*
- *If $|\mathcal{V}| \geq 3$, then M is a diffuse full factor.*

Proof. First, suppose $\mathcal{V} = \{v_1, v_2\}$ and recall that join-irreducibility of $\mathcal{G}$ implies $(M, \varphi) = (M_{v_1}, \varphi_{v_1}) * (M_{v_2}, \varphi_{v_2})$. If one of M_{v_1} or M_{v_2} has dimension at least 3, then M is diffuse and a full factor by [Ued11, Thm. 4.1 and Rem. 4.2]. If $\dim(M_{v_1}) = \dim(M_{v_2}) = 2$ so that $M_{v_i} \cong \mathbb{C} \oplus \mathbb{C}$ for $i = 1, 2$, then φ_{v_1}, φ_{v_2}

are necessarily tracial and our assumption on the existence of trace zero unitaries forces these traces to put equal weight on each factor of $\mathbb{C}$. Hence, M is diffuse but is not a factor by [Dyk93, Thm. 1.1].

We now assume that $|\mathcal{V}| \geq 3$. Note that if a von Neumann algebra P has a normal conditional expectation onto a diffuse subalgebra, then P is diffuse (this follows from restricting such a conditional expectation to the maximal purely atomic direct summand of P and applying [Bla06, Thm. IV.2.2.3]). Since we already have normal conditional expectations onto subalgebras corresponding to induced subgraphs, it follows from the above paragraph that M is diffuse in this case. So we only focus on proving M is a full factor. By Lemma 2.2, it suffices to prove the theorem under the weaker condition that either $\mathcal{G}$ is disconnected or for every $v_0 \in \mathcal{V}$, there exist $v_1, v_2 \in \mathcal{V} \setminus \{v_0\}$ such that $v_0 \sim v_1$, $v_0 \not\sim v_2$, $v_1 \not\sim v_2$.

Suppose $\mathcal{G}$ is disconnected and $|\mathcal{V}| \geq 3$. Then there exist a vertex v_0 and two other vertices v_1 and v_2 that are not in the same connected component as v_0. Let $V_0 \subset \mathcal{V}$ be the vertices in the connected component of $\mathcal{G}$ containing v_0. Then

$$M = M_{V_0} * M_{\mathcal{V}\setminus V_0}.$$

Let u_0, u_1, and u_2 be state zero unitaries in M_{v_0}, M_{v_1}, and M_{v_2} respectively. We have $\varphi(u_1^* u_2) = \varphi(u_1^*)\varphi(u_2) = 0$ in both cases $v_1 \sim v_2$ and $v_1 \not\sim v_2$. Thus, the unitaries satisfy the hypotheses of Lemma B.1 with $B = \mathbb{C}$. It follows that for every cofinal ultrafilter ω on a directed set, we have $M' \cap M^\omega \subseteq \mathbb{C}^\omega = \mathbb{C}$, so that M is full.

Now consider the case where for every $v_0 \in \mathcal{V}$, there exist $v_1, v_2 \in \mathcal{V}\setminus\{v_0\}$ such that $v_0 \sim v_1$, $v_0 \not\sim v_2$, $v_1 \not\sim v_2$. (In this case automatically $|\mathcal{V}| \geq 3$.) Fix a cofinal ultrafilter ω on a direct set, and a vertex v_0. Note that by [CF17, Thm. 2.26],

$$M = M_{B(v_0)} *_{M_{S(v_0)}} M_{\mathcal{V}\setminus\{v_0\}}.$$

Let v_1 and v_2 be vertices with $v_1 \sim v_0$, $v_2 \not\sim v_0$, $v_1 \not\sim v_2$. Let u_0, u_1, and u_2 be trace zero unitaries from $M_{v_0}^{\varphi_{v_0}}$, $M_{v_1}^{\varphi_{v_1}}$, and $M_{v_2}^{\varphi_{v_2}}$ respectively. We want to apply Lemma B.1 to the unitaries u_0, u_2, and $u_1^* u_2 u_1$. Note that the words v_0, v_2, $v_1 v_2 v_1$, and $v_1 v_2 v_1 v_2$ are reduced and each have some element not in $S(v_0)$; therefore, by the alternating expectation condition defining free independence with amalgamation,

$$E_{M_{S(v_0)}}[u_0] = E_{M_{S(v_0)}}[u_2] = E_{M_{S(v_0)}}[u_1 u_2 u_1^*] = E_{M_{S(v_0)}}[(u_1^* u_2 u_1)^* u_2] = 0.$$

Moreover, $u_1 u_2^* u_1$ is in the centralizer of $M_{\mathcal{V}\setminus\{v_0\}}$. Therefore, by Lemma B.1,

$$M' \cap M^\omega \subseteq (M_{S(v_0)})^\omega.$$

Now the vertex v_0 was arbitrary, and therefore, by Lemma B.2,

$$M' \cap M^\omega \subseteq \bigcap_{v_0 \in V} M^\omega_{S(v_0)} = \left(\bigcap_{v_0 \in V} M_{S(v_0)} \right)^\omega .$$

By [CF17, Prop. 2.25],

$$\bigcap_{v_0 \in V} M_{S(v_0)} = M_{\bigcap_{v_0 \in V} S(v_0)}.$$

Because $v_0 \notin S(v_0)$ by definition, we have $\bigcap_{v_0 \in V} S(v_0) = \varnothing$. Hence, $M' \cap M^\omega \subseteq \mathbb{C}$, so that M is full. $\square$

Remark 2.5. In particular, suppose that the graph $\mathcal{G}$ has diameter at least 3, meaning that there exist two vertices v and w with distance at least 3 in the graph. Then $\mathcal{G}$ is join-irreducible because in a graph join any two vertices have distance at most 2. Therefore, the theorem implies that $\mathop{\ast}_{v \in \mathcal{G}} (M_v, \varphi_v)$ is a full factor provided that each $M_v^{\varphi_v}$ contains a state zero unitary.

Consider a non-join-irreducible graph $\mathcal{G}$ and suppose $\mathcal{G} = \mathcal{G}_1 + \cdots + \mathcal{G}_n$ is its graph join decomposition for graphs $\mathcal{G}_j = (\mathcal{V}_j, \mathcal{E}_j)$. Since diffuseness, factoriality, and fullness are all automatic for graph products over $\mathcal{G}_j$ when $|\mathcal{V}_j| \geq 3$, to understand these properties for graph products over $\mathcal{G}$ it is not necessary to compute its entire graph join decomposition. We merely need to be able to locate the $\mathcal{G}_j$ that have 1 or 2 vertices. For this purpose, we record the following observation.

Lemma 2.6. *Let v be a vertex of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Then v comprises one of the components in the graph join decomposition of $\mathcal{G}$ if and only if v is adjacent to all the other vertices of $\mathcal{G}$.*

Similarly, let v and w be distinct vertices of $\mathcal{G}$. Then $\{v, w\}$ comprises one of the components in the graph join decomposition of $\mathcal{G}$ if and only if v and w are not adjacent to each other but are adjacent to all the other vertices in $\mathcal{G}$.

We remark that detecting components in the graph join decomposition of $\mathcal{G}$ with one or two vertices is algorithmically much simpler than finding the full graph join decomposition (it can be done in polynomial time in the number of vertices).

§2.1. Proof of Theorem 0.5

Let $\mathcal{G} = \mathcal{G}_1 + \cdots + \mathcal{G}_n$ be the graph join decomposition for graphs $\mathcal{G}_j = (\mathcal{V}_j, \mathcal{E}_j)$, $j = 1, \ldots, n$. Denote $(N_j, \psi_j) := \mathop{\ast}_{v \in \mathcal{G}_j} (M_v, \varphi_v)$ for each $j = 1, \ldots, n$, so that

$$(M, \varphi) \cong (N_1, \psi_1) \mathbin{\bar{\otimes}} \cdots \mathbin{\bar{\otimes}} (N_n, \psi_n)$$

by Proposition 2.1.

(1). M is diffuse if and only if N_j is diffuse for some j. If $\mathcal{G}_j$ has at least two vertices, then N_j is diffuse by Theorem 2.4. Thus, the only way M can fail to be diffuse is if all the $\mathcal{G}_j$ are singletons (that is, $\mathcal{G}$ is a complete graph), and none of the M_v are diffuse.

(2). M is a factor if and only if N_j is a factor for each $j = 1, \ldots, n$. If $\mathcal{G}_j$ has at least three vertices, then N_j is automatically a factor by Theorem 2.4. So for M to be a factor it is necessary and sufficient that N_j is a factor whenever $|\mathcal{V}_j| \leq 2$. For $\mathcal{V}_j = \{v\}$, this reduces to M_v being a factor, and from the characterization of singleton components in Lemma 2.6 this yields condition (a). For $\mathcal{V}_j = \{v, w\}$, N_j is a factor if and only if $\max(\dim(M_v), \dim(M_w)) \geq 3$ by Theorem 2.4, and from the characterization of two-element components in Lemma 2.6 this yields condition (b).

(3). M is full if and only if N_j is full for each $j = 1, \ldots, n$ by [Con76, Cor. 2.3], [HMV19, Cor. B]. Noting that the characterization of fullness coincides with that of factoriality for join-irreducible graphs in Theorem 2.4, the same argument used in the previous part completes the proof.

Remark 2.7. Observe that under our standard assumption that $M_v^{\varphi_v}$ admits a state zero unitary, the graph product over $\mathcal{G}$ gives a non-full factor if and only if there exists $v \in \mathcal{V}$ adjacent to every other vertex with M_v a non-full factor. Indeed, using the notation of the above proof, M is a non-full factor if and only if each N_j is a factor and at least one, say N_{j_0}, is non-full. According to Theorem 2.4, this is only possible if $\mathcal{V}_{j_0}$ consists of a single vertex and the algebra over that vertex is a non-full factor.

§3. Relatively reduced words and conditional expectations

Młotkowski [Mło04] and Caspers–Fima [CF17] used reduced words to describe how the standard form of a graph product is analogous to a Fock space. From their description, one can build an orthonormal basis for L^2 of the graph product using an orthonormal basis of the vertex algebras. In Section 5 we will have to describe the standard form of the graph product as a bimodule over two subalgebras coming from subgraphs. In order to investigate relative amenability in Section 6, we will also have to describe the fusion rules. In this bimodule situation it is natural to look for (an analogue of) a *Pimsner–Popa basis* instead of an *orthonormal basis*. As we will show in Section 5, this can be done by modifying the consideration of *reduced words* to be reduced "relative" to a pair of subgraphs as in [BoCa24, Lem. 1.7]. This is similar to considering double-cosets relative to a pair of subgroups coming from subgraphs in a graph product of groups. We define this notion of relatively reduced words in this section. In order to later show they give something akin to a Pimsner–Popa basis and compute the fusion rules, we will also need to compute some conditional expectations coming from relatively reduced words, which we also do in this section. These formulas for conditional expectation will also be used to investigate *relative diffuseness* (i.e. lack of intertwining) in Section 4.

§3.1. $\mathcal{G}$-reduced words

Definition 3.1. We define the following kinds of operations on words in the alphabet $\mathcal{V}$:

- An *admissible swap* switches two consecutive letters w_i and w_{i+1} that are adjacent vertices in $\mathcal{G}$.
- A *splitting* replaces one occurrence of a letter w_i by two copies of w_i. (For example, 1231 could be transformed to 12231 by splitting the second letter.)
- A *merge* replaces two consecutive occurrences of the same letter by one occurrence of the letter.

Two words are said to be *equivalent* if one can be transformed into the other by a sequence of these three types of operations. We denote this by $w \approx \widehat{w}$.

It is easy to see that this is indeed an equivalence relation. It is reflexive and transitive by construction. It is symmetric because a swap operation is reversed by another swap, and the splitting and merge operations are inverse to each other. Moreover, every word is equivalent to some reduced word through a sequence of admissible swaps and merges (see [CF17, Lem. 1.3(1)]).

In the sequel, we will use the following characterization of when two reduced words are equivalent.

Proposition 3.2. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph. Let $w = w_1 \cdots w_m$ and $\widehat{w} = \widehat{w}_1 \cdots \widehat{w}_n$ be two words in the alphabet $\mathcal{V}$. Let $w = w_1 \cdots w_m$ and $\widehat{w} = \widehat{w}_1 \cdots \widehat{w}_n$ be two $\mathcal{G}$-reduced words. Then the following are equivalent:*

(i) *w and $\widehat{w}$ are equivalent;*

(ii) *w can be transformed into $\widehat{w}$ by a sequence of admissible swaps;*

(iii) *$m = n$ and there is a permutation $\sigma\colon [m] \to [m]$ such that*

- *$\widehat{w}_{\sigma(i)} = w_i$;*
- *if $i < j$ and w_i is not adjacent to w_j, then $\sigma(i) < \sigma(j)$.*

This proposition is a strengthening of [CF17, Lem. 1.3]. For instance, [CF17, Lem. 1.3] showed that if w and $\widehat{w}$ are equivalent, then $m = n$ and there is some permutation matching the letters of w and $\widehat{w}$, but did not characterize the exact properties this permutation should have in order to get the reverse implication. Moreover, they expressed condition (ii) as "Type II equivalence" and stopped short of showing it is the same as equivalence in the case of reduced words.

For the proof, (ii) $\Rightarrow$ (i) is immediate and (iii) $\Rightarrow$ (ii) follows by induction. The implication (i) $\Rightarrow$ (iii) or (ii) is non-trivial since it involves reasoning about non-reduced words in intermediate stages of the sequence of transformations. We first take care of (iii) $\Rightarrow$ (ii).

Lemma 3.3. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph. Let $w = w_1 \cdots w_m$ and $\widehat{w} = \widehat{w}_1 \cdots \widehat{w}_n$ be two words in the alphabet V, and suppose $\sigma\colon [m] \to [m]$ is a permutation with $\widehat{w}_{\sigma(i)} = w_i$ such that if $i < j$ and w_i is not adjacent to w_j, then $\sigma(i) < \sigma(j)$. Then w and $\widehat{w}$ are equivalent by swaps.*

Proof. We proceed by induction on m. If $\sigma(1) = 1$, then σ restricts to a permutation of $\{2, \ldots, m-1\}$ and we can apply our inductive hypothesis. Otherwise, $i = \sigma^{-1}(1) > 1$, and $\widehat{w}_i$ must be adjacent to $\widehat{w}_1, \ldots, \widehat{w}_{i-1}$. Therefore, by successive swaps, we may move $w_1 = \widehat{w}_i$ to the left past $\widehat{w}_1, \ldots, \widehat{w}_{i-1}$. Then note that σ restricts to a permutation of $m-1$ elements satisfying the original hypotheses for the words $w' = w_2 \cdots w_m$ to $\widehat{w}' = \widehat{w}_1 \cdots \widehat{w}_{\sigma(1)-1} \widehat{w}_{\sigma(1)+1} \cdots \widehat{w}_m$. By the inductive hypothesis, w' and $\widehat{w}'$ are equivalent by a sequence of swaps, and hence w and $\widehat{w}$ are equivalent by a sequence of swaps as desired. □

For (i) $\Rightarrow$ (iii), we have to produce a permutation out of the sequence of operations. It is easy to see that an admissible swap corresponds to a transposition permutation satisfying the monotonicity condition in (iii). However, if we perform a split or a merge operation, then naturally two indices are mapped to one or vice versa, so in that setting, we need to replace the permutation (i.e. bijective function) by a relation from $[m]$ to $[n]$.

Recall that a *relation* $R\colon A \to B$ between two sets A and B is a subset of $R \subseteq A \times B$. Given relations $R\colon A \to B$ and $S\colon B \to C$, the composition $S \circ R$ is defined by

$$S \circ R = \big\{(a, c) \in A \times C : \text{there exists } b \in B \text{ with } (a, b) \in R \text{ and } (b, c) \in S\big\}.$$

Note that this definition extends the composition of functions.

Definition 3.4. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph. Let $w = w_1 \cdots w_m$ and $\widehat{w} = \widehat{w}_1 \cdots \widehat{w}_n$ be two words in the alphabet V. A *$\mathcal{G}$-monotone matching* from w to $\widehat{w}$ is a relation $R\colon [m] \to [n]$ satisfying the following conditions:

(1) For every $i \in [m]$, there is some $j \in [n]$ with $(i, j) \in R$.

(2) For every $j \in [n]$, there is some $i \in [m]$ with $(i, j) \in R$.

(3) If $(i,j) \in R$, then $w_i = \widehat{w}_j$.

(4) If $(i,j) \in R$ and $(i',j') \in R$ and w_i is not adjacent to $w_{i'}$ in $\mathcal{G}$, then $i \leq i'$ if and only if $j \leq j'$.

Note in the case that the relation R is a bijective function, then (1) and (2) of Definition 3.4 hold, while (3) and (4) reduce to the conditions on the permutation σ in Proposition 3.2(iii).

Lemma 3.5. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph. Let $w = w_1 \cdots w_m$ and $\widehat{w} = \widehat{w}_1 \cdots \widehat{w}_n$ be two words in the alphabet V. If w and $\widehat{w}$ are equivalent, then there exists a $\mathcal{G}$-monotone matching from w to $\widehat{w}$.*

Proof. It suffices to show (a) that each of the operations leads to a $\mathcal{G}$-monotone matching and (b) that a $\mathcal{G}$-monotone matching from w and $\widehat{w}$ and a $\mathcal{G}$-monotone matching from $\widehat{w}$ to $\widetilde{w}$ compose to form a $\mathcal{G}$-monotone matching from w to $\widetilde{w}$. For (a), we note the following:

- If $\widehat{w}$ is obtained from w by swapping i and $i+1$, where w_i and w_{i+1} are adjacent, then a $\mathcal{G}$-monotone matching $R\colon [m] \to [m]$ is given by the relation $R = \{(j,j) : j \neq i,\ i+1\} \cup \{(i,i+1),(i+1,i)\}$.
- If $\widehat{w}$ is obtained from w by merging i and $i+1$, where $w_i = w_{i+1}$, then the $\mathcal{G}$-monotone matching $R\colon [m] \to [m-1]$ is given by the relation $R = \{(1,1),\dots,(i,i)\} \cup \{(i+1,i),\dots,(m,m-1)\}$.
- If $\widehat{w}$ is obtained from w by splitting the index i into i and $i+1$, then the $\mathcal{G}$-monotone matching is given by $R = \{(1,1),\dots,(i,i)\} \cup \{(i,i+1),\dots,(m,m+1)\}$.

For (b), suppose $\widetilde{w} = \widetilde{w}_1,\dots,\widetilde{w}_o$ is another word, suppose R is a $\mathcal{G}$-monotone matching from w to $\widehat{w}$, and S is a $\mathcal{G}$-monotone matching from $\widehat{w}$ to $\widetilde{w}$. One can check that $S \circ R$ is a $\mathcal{G}$-monotone matching from w to $\widetilde{w}$ by verifying each condition directly:

(1) Given $i \in [m]$, there exists some $j \in [n]$ with $(i,j) \in R$, and then there exists some $k \in [o]$ with $(j,k) \in S$, and hence $(i,k) \in S \circ R$.

(2) The second condition is checked in a symmetrical way.

(3) If $(i,k) \in S \circ R$, then there exists some $j \in [n]$ with $(i,j) \in R$ and $(j,k) \in S$. Hence, $w_i = \widehat{w}_j = \widetilde{w}_k$ by condition (3) applied to R and S.

(4) Let $(i,k),(i',k') \in S \circ R$. Suppose w_i and $w_{i'}$ are not adjacent. Pick j and $j' \in [n]$ with $(i,j),(i',j') \in R$ and $(j,k),(j',k') \in S$. Recall that $w_i = \widehat{w}_j = \widetilde{w}_k$ and $w_{i'} = \widehat{w}_{j'} = \widetilde{w}_{k'}$ by (3). Hence, $i \leq i'$ if and only if $j \leq j'$ if and only if $k \leq k'$ by condition (4) applied to R and S. $\square$

Lemma 3.6. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph. Let $w = w_1 \cdots w_m$ and $\widehat{w} = \widehat{w}_1 \cdots \widehat{w}_n$ be two reduced words in the alphabet $\mathcal{V}$. If w and $\widehat{w}$ are equivalent, then $m = n$ and there is a permutation $\sigma\colon [m] \to [m]$ with $\widehat{w}_{\sigma(i)} = w_i$ such that if $i < i'$ and w_i is not adjacent to $w_{i'}$, then $\sigma(i) < \sigma(i')$.*

Proof. By the previous lemma, there exists a $\mathcal{G}$-monotone matching R from w to $\widehat{w}$. We claim that R defines a bijection.

For each $i \in [m]$, we know that there exists some $j \in [n]$ with $(i, j) \in R$. We claim that this j is unique. Suppose that $(i, j) \in R$ and $(i, j') \in R$ with $j < j'$. Since $\widehat{w}$ is reduced, there exists some ℓ strictly between j and j' such that $\widehat{w}_\ell$ is not equal or adjacent to $\widehat{w}_j$. Moreover, there exists some $k \in [m]$ with $(k, \ell) \in R$. Then condition (4) of $\mathcal{G}$-monotonicity tells us that $j \leq \ell \leq j'$ implies that $i \leq k \leq i$, hence $k = i$. However, this contradicts that $w_i = \widehat{w}_j \neq \widehat{w}_\ell = w_k$.

A symmetrical argument shows that for every $j \in [n]$, there is a unique $i \in [m]$ with $(i, j) \in R$. Thus, R defines a bijection as desired, so that $m = n$ and R has the form $R = \{(i, \sigma(i)) : i \in [m]\}$ for some permutation σ. By Definition 3.4, we see that if $i < i'$ and w_i is not adjacent to $w_{i'}$, then $\sigma(i) < \sigma(i')$. □

This lemma completes the proof of (i) ⇒ (iii) in Proposition 3.2.

Remark 3.7. If w and $\widehat{w}$ are equivalent $\mathcal{G}$-reduced words, note that the permutation σ is uniquely determined by the property that for each $v \in V$, σ maps $\{i : w_i = v\}$ onto $\{j : \widehat{w}_j = v\}$ monotonically. In particular, the permutation in Proposition 3.2(iii) is unique.

Remark 3.8. The method of proof more generally shows that arbitrary words w and $\widehat{w}$ are equivalent if and only if there exists a $\mathcal{G}$-monotone matching from w to $\widehat{w}$. Indeed, Lemma 3.5 shows that equivalence of w and $\widehat{w}$ implies the existence of a $\mathcal{G}$-monotone matching. On the other hand, suppose there is a $\mathcal{G}$-monotone matching from w to $\widehat{w}$. Note w and $\widehat{w}$ are equivalent to some reduced words w' and $\widehat{w}'$, and hence there are $\mathcal{G}$-monotone matchings from w' to w, from w to $\widehat{w}$, and from $\widehat{w}$ to $\widehat{w}'$. The composition yields a $\mathcal{G}$-monotone matching from w' to $\widehat{w}'$, so by Proposition 3.2, w' and $\widehat{w}'$ are equivalent by swaps, hence also w and $\widehat{w}$ are equivalent.

§3.2. Relatively $\mathcal{G}$-reduced words

In order to compute conditional expectations and study relative properties of subalgebras, we use a relative notion of reduced word as in [BoCa24, Lem. 1.7].

Definition 3.9. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph and V_1, $V_2 \subseteq \mathcal{V}$. Let w be a word in the alphabet $\mathcal{V}$.

(1) w is *$\mathcal{G}$-reduced relative to V_1 on the left* if $v_1 w$ is $\mathcal{G}$-reduced for every letter $v_1 \in V_1$.

(2) w is *$\mathcal{G}$-reduced relative to V_2 on the right* if wv_2 is $\mathcal{G}$-reduced for every $v_2 \in V_2$.

(3) w is *$\mathcal{G}$-reduced relative to* (V_1, V_2) if both (1) and (2) hold.

Remark 3.10. In the case of $\varnothing \subset \mathcal{V}$, we take w being $\mathcal{G}$-reduced relative to $\varnothing$ on the left or right to just mean that w is $\mathcal{G}$-reduced. Consequently, w is $\mathcal{G}$-reduced relative to V_1 on the left if and only if w is $\mathcal{G}$-reduced relative to $(V_1, \varnothing)$. Similarly, w is $\mathcal{G}$-reduced relative to V_2 on the right if and only if w is $\mathcal{G}$-reduced relative to $(\varnothing, V_2)$. We also note that all relatively $\mathcal{G}$-reduced words are, in particular, $\mathcal{G}$-reduced words.

The next three lemmas show existence and uniqueness of a certain factorization of reduced words based on the vertex sets V_1 and V_2. This will be useful in Section 5 when we compute the fusion rules for bimodules arising from subgraphs.

Lemma 3.11. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph and V_1, $V_2 \subseteq \mathcal{V}$. Suppose that $w = w^{(1)} \cdot w^{(2)} \cdot w^{(3)}$, where*

(1) *$w^{(1)}$ is a $\mathcal{G}$-reduced word in the alphabet V_1,*

(2) *$w^{(2)}$ is $\mathcal{G}$-reduced relative to (V_1, V_2),*

(3) *$w^{(3)}$ is a word in the alphabet V_2 that is $\mathcal{G}$-reduced relative to $(U, \varnothing)$, where U is the set of vertices in $V_1 \cap V_2$ that are adjacent to* all *the letters in $w^{(2)}$.*

Then w is $\mathcal{G}$-reduced.

Proof. Denote $w = w_1 \cdots w_n$ and suppose that $i < j$ with $w_i = w_j$. We must find some $i < k < j$ such that w_k is not adjacent to $w_i = w_j$. We proceed in cases:

(A) If w_i and w_j are both from $w^{(1)}$, the claim follows because $w^{(1)}$ is reduced. Similarly for $w^{(2)}$ and $w^{(3)}$.

(B) Suppose that w_i comes from $w^{(1)}$ and w_j comes from $w^{(2)}$. Because $w_i \in V_1$ and $w^{(2)}$ is $\mathcal{G}$-reduced relative to (V_1, V_2), the word $w_i \cdot w^{(2)}$ is $\mathcal{G}$-reduced, and hence there exists some index $k < j$, within $w^{(2)}$, such that w_k is not equal or adjacent to $w_i = w_j$.

(C) Suppose that w_i comes from $w^{(2)}$ and w_j comes from $w^{(3)}$. Using that $w_j \in V_2$ and thus $w^{(2)} \cdot w_j$ is $\mathcal{G}$-reduced, we can argue analogously to the previous case.

(D) Finally, suppose that w_i is from $w^{(1)}$ and w_j is from $w^{(3)}$. Note that $w_i = w_j$ must be in $V_1 \cap V_2$. Then there are two subcases: (a) $w_i \notin U$ and (b) $w_i \in U$. For (a), the definition of U implies there exists some index k from $w^{(2)}$ such that w_k is not adjacent to w_i. Since k is from $w^{(2)}$, we have $i < k < j$, so we are done. For (b), because $w^{(3)}$ is $\mathcal{G}$-reduced relative to $(U, \varnothing)$, we know $w_i \cdot w^{(3)}$ is $\mathcal{G}$-reduced, and so there is some index $k < j$ from $w^{(3)}$ such that w_k is not adjacent to w_i. □

Lemma 3.12. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph and V_1, $V_2 \subseteq \mathcal{V}$. Every word w is equivalent to a word of the form $w^{(1)} \cdot w^{(2)} \cdot w^{(3)}$ satisfying the conditions in Lemma 3.11.*

Proof. Since every word is equivalent to a $\mathcal{G}$-reduced word, we may assume without loss of generality that w is $\mathcal{G}$-reduced. If all the letters of w are in V_1, then the desired decomposition is $w^{(1)} = w$; $w^{(2)}$ and $w^{(3)}$ are the empty word.

So without loss of generality, we may assume that $\{j : w_j \notin V_1\} \neq \varnothing$. Set $a(w) = \min\{j : w_j \notin V_1\}$. Let C be the set of $\mathcal{G}$-reduced words equivalent to w. Let w' be an element in C which maximizes $a(w')$:

$$b(w') = \begin{cases} a(w') - 1 & \text{if } \{j \geq a(w') : w'_j \notin V_2\} = \varnothing, \\ \max\{j : w'_j \notin V_2\} & \text{if } \{j \geq a(w') : w'_j \notin V_2\} \neq \varnothing. \end{cases}$$

Let w'' be an element of C that minimizes $b(w'')$ subject to the constraint that $a(w'') = a(w')$. Write $a = a(w') = a(w'')$ and $b = b(w'')$. Write $w'' = w^{(1)} \cdot w^{(2)} \cdot w^{(3)}$, where

$$\begin{aligned} w^{(1)} &= w''_1 \cdots w''_{a-1}, \\ w^{(2)} &= w''_a \cdots w''_b, \\ w^{(3)} &= w''_{b+1} \cdots w''_\ell, \end{aligned}$$

where ℓ is the length of w''. In the special case where $\{j \geq a(w') : w'_j \notin V_2\} = \varnothing$, meaning that all the letters starting at index a are in V_2, then $b = a - 1$, and so $w^{(2)}$ is the empty word. In all cases, $w^{(1)}$ is a word in the alphabet in V_1 and $w^{(3)}$ is a word in the alphabet V_2. Moreover, $w^{(1)}$, $w^{(2)}$, and $w^{(3)}$ are all $\mathcal{G}$-reduced since they are subwords of the $\mathcal{G}$-reduced word w''. We will complete the proof via a series of claims.

Claim 1. $w^{(2)} \cdot w^{(3)}$ is $\mathcal{G}$-reduced relative to $(V_1, \varnothing)$.

Fix $v \in V_1$ and let $i < j$ be two indices in $v \cdot w^{(2)} \cdot w^{(3)}$ labeled with the same vertex. We must show there is some index in between labeled by a non-adjacent vertex. If the two indices i and j are both from $w^{(2)} \cdot w^{(3)}$, then it suffices to note that $w^{(2)} \cdot w^{(3)}$ is $\mathcal{G}$-reduced since it is a subword of w'', which is reduced because it is equivalent by admissible swaps to w. Otherwise, i corresponds to the first letter v in $v \cdot w^{(2)} \cdot w^{(3)}$. Suppose for contradiction that there does not exist some index k between i and j such that w''_k is not adjacent to w''_j. Then all the letters between v and w''_j in $v \cdot w^{(2)} \cdot w^{(3)}$ are adjacent to v, and hence w''_j can be moved past them to the left by repeated swaps, so that it comes to the left-hand side of $w^{(2)} \cdot w^{(3)}$. Thus, by grouping the letter w''_j with $w^{(1)}$ instead of $w^{(2)} \cdot w^{(3)}$, we obtain a contradiction to the assumption that $a(w'')$ is maximal.

Claim 2. $w^{(2)}$ is $\mathcal{G}$-reduced relative to $(\varnothing, V_2)$.

In the case where $b = a - 1$ or equivalently $w^{(2)}$ is the empty word, there is nothing to prove. So assume $w^{(2)}$ is non-empty. Fix $v \in V_2$ and let $i < j$ be two indices in $w^{(2)} \cdot v$ labeled by the same vertex. Note that $w^{(2)}$ is $\mathcal{G}$-reduced, so if i and j are both from $w^{(2)}$ then we are done. Otherwise, j corresponds to the last letter v in $w^{(2)} \cdot v$. If w''_k is adjacent to w''_i for all $k > i$ among the indices of $w^{(2)}$, then arguing as in Claim 1 we would contradict minimality of $b(w'')$.

Observe that the combination of Claims 1 and 2 gives that $w^{(2)}$ is $\mathcal{G}$-reduced relative to (V_1, V_2). It remains to show that $w^{(3)}$ is $\mathcal{G}$-reduced relative to $(U, \varnothing)$, where U is the set of vertices in $V_1 \cap V_2$ that are adjacent to all the letters in $w^{(2)}$ (of course, in the case where $w^{(2)}$ is the empty word, we have $U = V_1 \cap V_2$).

Claim 3. $w^{(3)}$ is $\mathcal{G}$-reduced relative to $(U, \varnothing)$.

Fix $v \in U$ and let $i < j$ be two indices in $v \cdot w^{(3)}$ labeled by the same vertex. Since $w^{(3)}$ is $\mathcal{G}$-reduced, if i and j are both from $w^{(3)}$ then we are done. So suppose i corresponds to v and that there is some index j in $w^{(3)}$ with $w''_j = v$. Since $w^{(2)} \cdot w^{(3)}$ is $\mathcal{G}$-reduced relative to $(V_1, \varnothing)$ by Claim 1 and $U \subseteq V_1$, there must be some index $k < j$ in $w^{(2)} \cdot w^{(3)}$ with w''_k not adjacent to $v = w''_j$. By definition of U, v is adjacent to all the letters in $w^{(2)}$. Hence, the index k must have come from $w^{(3)}$. Thus, w_k occurs as a letter in $v \cdot w^{(3)}$ between v and w''_j, and w''_k is not adjacent to v. □

Lemma 3.13. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph and V_1, $V_2 \subseteq \mathcal{V}$. Let $w = w^{(1)} \cdot w^{(2)} \cdot w^{(3)}$ and $\widehat{w} = \widehat{w}^{(1)} \cdot \widehat{w}^{(2)} \cdot \widehat{w}^{(3)}$ satisfy the conditions in Lemma 3.11 (here, in the third condition for w and $\widehat{w}$, we use respectively U and $\widehat{U}$, where U and $\widehat{U}$ are the sets of vertices in $V_1 \cap V_2$ that are adjacent to all letters in $w^{(2)}$ and $\widehat{w}^{(2)}$ respectively). If $w \approx \widehat{w}$, then $w^{(1)} \approx \widehat{w}^{(1)}$, $w^{(2)} \approx \widehat{w}^{(2)}$, and $w^{(3)} \approx \widehat{w}^{(3)}$.*

Proof. First, observe that $w^{(2)} \cdot w^{(3)}$ and $\widehat{w}^{(2)} \cdot \widehat{w}^{(3)}$ are both $\mathcal{G}$-reduced relative to $(V_1, \varnothing)$, by applying Lemma 3.11 to $v \cdot w^{(2)} \cdot w^{(3)}$ and $\widehat{w}^{(2)} \cdot \widehat{w}^{(3)}$ to $v \in V_1$.

Now, since $w \approx \widehat{w}$, Proposition 3.2 shows that there is a permutation σ with $w_i = \widehat{w}_{\sigma(i)}$, such that if $i < j$ and w_i is not adjacent to w_j, then $\sigma(i) < \sigma(j)$. We claim that σ maps the indices of $w^{(2)} \cdot w^{(3)}$ into the letters of $\widehat{w}^{(2)} \cdot \widehat{w}^{(3)}$. We proceed by induction on the indices of $w^{(2)} \cdot w^{(3)}$, from left to right. Let i be one of these indices and suppose the claim is known for all indices to its left in $w^{(2)} \cdot w^{(3)}$. There are now two cases:

- Suppose $w_i \notin V_1$. Then $\widehat{w}_{\sigma(i)} = w_i$ is not in V_1 and hence $\sigma(i)$ cannot be one of the indices in $\widehat{w}^{(1)}$, so it must be one of the indices in $\widehat{w}^{(2)} \cdot \widehat{w}^{(3)}$.
- Suppose that $w_i \in V_1$. Then because $w^{(2)} \cdot w^{(3)}$ is $\mathcal{G}$-reduced relative to $(V_1, \varnothing)$, we know $w_i \cdot w^{(2)} \cdot w^{(3)}$ is $\mathcal{G}$-reduced, so there must exist some index $j < i$ in $w^{(2)} \cdot w^{(3)}$ such that w_j is not adjacent to w_i in $\mathcal{G}$. By the induction hypothesis, $\sigma(j)$ is one of the indices in $\widehat{w}^{(2)} \cdot \widehat{w}^{(3)}$. By Lemma 3.5, we must have $\sigma(i) > \sigma(j)$ and hence $\sigma(i)$ is one of the indices in $\widehat{w}^{(2)} \cdot \widehat{w}^{(3)}$, as desired.

By symmetrical reasoning, σ^{-1} must map the indices of $\widehat{w}^{(2)} \cdot \widehat{w}^{(3)}$ into the indices of $w^{(2)} \cdot w^{(3)}$. Therefore, σ restricts to $\mathcal{G}$-monotone matchings from $w^{(1)}$ to $\widehat{w}^{(1)}$ and from $w^{(2)} \cdot w^{(3)}$ to $\widehat{w}^{(2)} \cdot \widehat{w}^{(3)}$. That is, $w^{(1)} \approx \widehat{w}^{(1)}$ and $w^{(2)} \cdot w^{(3)} \approx \widehat{w}^{(2)} \cdot \widehat{w}^{(3)}$.

Finally, we argue σ as above maps the indices of $w^{(2)}$ into the indices of $\widehat{w}^{(2)}$. We again proceed by induction on the indices of $w^{(2)}$, this time from right to left. Let i be an index in $w^{(2)}$ and the claim is already known for all indices j to its right. We again have two cases:

- If $w_i \notin V_2$, then $\sigma(i)$ must be an index in $\widehat{w}^{(2)}$.
- If $w_i \in V_2$, then since $w^{(2)}$ is $\mathcal{G}$-reduced relative to (V_1, V_2), then there is some index $j > i$ in $w^{(2)}$ such that w_j is not adjacent to w_i. Then $\sigma(i) < \sigma(j)$, which is by the induction hypothesis an index in $\widehat{w}^{(2)}$. Thus, $\sigma(i)$ is an index in $\widehat{w}^{(2)}$.

Symmetrically, σ^{-1} maps the indices of $\widehat{w}^{(2)}$ into the indices of $w^{(2)}$. Thus, as above, we have $w^{(2)} \approx \widehat{w}^{(2)}$ and $w^{(3)} \approx \widehat{w}^{(3)}$. $\square$

§3.3. Computation of conditional expectation

We recall the following facts which follow from the Fock space description of L^2 of the graph product in [CF17, Sect. 2.1].

Lemma 3.14 ([CF17, Rem. 2.7]). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph, let $\{(M_v, \varphi_v) : v \in \mathcal{V}\}$ be a family of statial von Neumann algebras, and $(M, \varphi) = \star_{v\in\mathcal{G}}(M_v, \varphi_v)$. The $*$-subalgebra generated by $(M_v)_{v\in\mathcal{V}}$ is spanned by 1 and elements of the form $x_1 \cdots x_m$, where $x_j \in M_{w_j}$ with $\varphi(x_j) = 0$ for some $\mathcal{G}$-reduced word $w = w_1 \cdots w_m$.*

Lemma 3.15 (Comments following [CF17, Rem. 2.11]). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph and $(M, \varphi) = \star_{v\in\mathcal{G}}(M_v, \varphi_v)$. Let $w = w_1 \cdots w_m$ and $\widetilde{w} = \widetilde{w}_1 \cdots \widetilde{w}_n$ be $\mathcal{G}$-reduced words. Let $x_j \in M_{w_j} \cap \ker(\varphi)$ and $\tilde{x}_j \in M_{\widetilde{w}_j} \cap \ker(\varphi)$.*

(i) *If w and $\widetilde{w}$ are not equivalent, then $\varphi((x_1 \cdots x_m)^*(\tilde{x}_1 \cdots \tilde{x}_n)) = 0$.*

(ii) *If w and $\widetilde{w}$ are equivalent, then*

$$\varphi((x_1 \cdots x_m)^*(\tilde{x}_1 \cdots \tilde{x}_m)) = \varphi(x_1^* \tilde{x}_{\sigma(1)}) \cdots \varphi(x_m^* \tilde{x}_{\sigma(m)}^*),$$

where the permutation $\sigma\colon [m] \to [m]$ is the $\mathcal{G}$-monotone matching from w to $\widetilde{w}$ guaranteed by Lemma 3.6, which also gives $m = n$.

Lemma 3.16 ([CF17, Rem. 2.14]). *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph, let $\{(M_v, \varphi_v) : v \in \mathcal{V}\}$ be a family of statial von Neumann algebras, let $(M, \varphi) = \ast_{v\in\mathcal{G}}(M_v, \varphi_v)$, and let $V_0 \subseteq \mathcal{V}$. For a $\mathcal{G}$-reduced word $w = w_1 \cdots w_m$, if $x_j \in M_{w_j}$ for each $j = 1, \ldots, m$ then $E_{M_{V_0}}[x_1 \cdots x_m] = 0$ unless $w_1, \ldots, w_n \in V_0$.*

Our goal is to prove a conditional analogue of Lemma 3.15.

Lemma 3.17. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph, let $\{(M_v, \varphi_v) : v \in \mathcal{V}\}$ be a family of statial von Neumann algebras, and $V_1, V_2 \subseteq \mathcal{V}$. Let $w = w_1 \cdots w_m$ and $\widetilde{w} = \widetilde{w}_1 \cdots \widetilde{w}_n$ be $\mathcal{G}$-reduced words relative to (V_1, V_2). Let $x_j \in M_{w_j} \cap \ker(\varphi)$ and $\tilde{x}_j \in M_{\widetilde{w}_j} \cap \ker(\varphi)$, and write*

$$x = x_1 \cdots x_m, \quad \tilde{x} = \tilde{x}_1 \cdots \tilde{x}_n.$$

(In the case that w or $\widetilde{w}$ is empty, that is, $m = 0$ or $n = 0$, we take by convention $x = 1$ or $\tilde{x} = 1$ respectively.) Let U be the set of vertices in $V_1 \cap V_2$ that are adjacent to all *letters of w (note: if w is the empty word, then $U = V_1 \cap V_2$, by convention). Then*

$$E_{M_{V_2}}(\tilde{x}^* y x) = \varphi(\tilde{x}^* x) E_{M_U}(y), \quad \forall y \in M_{V_1}. \tag{3.1}$$

In particular, $E_{M_{V_2}}(\tilde{x}^* y x) = 0$ for all $y \in M_{V_1}$ if w and $\widetilde{w}$ are not equivalent.

Proof. It suffices to show that for all $y \in M_{V_1}$ and $z \in M_{V_2}$, we have

$$\varphi(\tilde{x}^* y x z) = \varphi(\tilde{x}^* x)\varphi(E_{M_U}(y) z). \tag{3.2}$$

By Lemma 3.14, it further suffices to prove the claim when

$$z = z_1 \cdots z_\ell, \quad z_j \in M_{a_j} \cap \ker(\varphi), \tag{3.3}$$

where $a = a_1 \cdots a_\ell$ is a $\mathcal{G}$-reduced word in the alphabet V_2. Additionally, by Proposition 3.2(ii) and Lemma 3.12, we can assume without loss of generality that $a = a^{(1)} \cdot a^{(2)}$, where $a^{(1)}$ is a $\mathcal{G}$-reduced word in U and $a^{(2)}$ is $\mathcal{G}$-reduced relative to $(U, \varnothing)$. This results in a corresponding factorization $z = z^{(1)} z^{(2)}$ with $z^{(1)} \in M_U$. Then

$$\varphi(\tilde{x}^* y x z^{(1)} z^{(2)}) = \varphi(\tilde{x}^* y z^{(1)} x z^{(2)}).$$

Thus, it suffices to prove the claim with y replaced by $y z^{(1)}$ and z replaced by $z^{(2)}$.

In other words, we can assume without loss of generality that z is given by (3.3) where a is $\mathcal{G}$-reduced relative to $(U, \varnothing)$. Furthermore, again by Lemma 3.14, it suffices to consider the case where

$$y = y_1 \cdots y_k, \quad y_j \in M_{b_j} \cap \ker(\varphi),$$

where $b = b_1 \cdots b_k$ is a $\mathcal{G}$-reduced word in V_1. By Lemma 3.11, $b \cdot w \cdot a$ is $\mathcal{G}$-reduced. Moreover, by Lemma 3.13, the only way for $\widetilde{w}$ and $b \cdot w \cdot a$ to be equivalent is if $\widetilde{w} \approx w$ and $\varnothing \approx b$ and $\varnothing \approx a$ (hence a and b are empty). Similarly, the only way for $\widetilde{w}$ and $b \cdot w$ to be equivalent is if $w \approx \widetilde{w}$ and $b = \varnothing$. Thus, the claim can be checked in several cases:

- In the case $a = b = \varnothing$, so then $y = z = 1$, we have

$$\varphi(\tilde{x}^* yxz) = \varphi(\tilde{x}^* x) = \varphi(\tilde{x}^* x)\varphi(E_{M_U}[1]1) = \varphi(\tilde{x}^* x)\varphi(E_{M_U}[y]z).$$

- In the case $a = \varnothing$ and $b \neq \varnothing$, then since $b \cdot w$ is not equivalent to $\widetilde{w}$, we get $\varphi(\tilde{x}^* yx) = 0$ by Lemma 3.15, hence the left-hand side of (3.1) is zero. Meanwhile, $\varphi(y) = 0$ by definition of the graph product, so the right-hand side of (3.1) is $\varphi(E_{M_U}[y]) = \varphi(y) = 0$.
- In the case $a \neq \varnothing$, then again $b \cdot w \cdot a$ is not equivalent to $\widetilde{w}$, and hence the left-hand side of (3.2) is zero, by Lemma 3.15. Meanwhile, since the word a is $\mathcal{G}$-reduced relative to $(U, \varnothing)$, the element z is orthogonal M_U by Lemma 3.16, hence $\varphi(E_{M_U}[y]z) = 0$, so the right-hand side of (3.2) is zero. □

§4. Non-intertwining

Let (M, τ) be a tracial von Neumann algebra and $B, N \leq M$. We say that B *intertwines into* N *inside* M if there exist non-zero projections $p_0 \in B$, $q_0 \in N$, and a normal unital $*$-homomorphism $\theta\colon p_0 P p_0 \to q_0 Q q_0$, together with a non-zero partial isometry $v \in q_0 M p_0$ such that $v^* v = p_0$, $vv^* = q_0$, and $\theta(x)v = vx$ for all $x \in p_0 P p_0$. In this case one writes $N \preceq_M B$.

Theorem 4.1 ([Pop06, Sect. 2]). *Let (M, τ) be a tracial von Neumann algebra, $p, q \in \mathcal{P}(M)$ projections, and $B \leq pMp$, $N \leq qMq$. Then the following are equivalent:*

(i) $N \not\preceq_M B$;

(ii) *there is a net $(u_n)_{n \in I}$ in $\mathcal{U}(N)$ with $\|E_B(xu_n y)\|_2 \to 0$ for all $x, y \in M$;*

(iii) *for any subgroup $G \leq \mathcal{U}(N)$ with $N = W^*(G)$ there is a net $(u_n)_{n \in I}$ in G satisfying $\|E_B(xu_n y)\|_2 \to 0$ for all $x, y \in M$;*

(iv) *any P–Q-subbimodule K of $pL^2(M)q$ satisfies* $\dim(K_Q) = +\infty$.

In this section we completely characterize when two subalgebras corresponding to induced subgraphs do not intertwine into each other (partial results were previously obtained in [CF17, Lem. 2.27]). We will say "N is diffuse relative to B in M" to mean $N \not\preceq_M B$. This is motivated by the case $B = \mathbb{C}$, since $N \not\preceq_M \mathbb{C}$ means precisely that N is diffuse. This also provides intuition for our main result in this section, since in our setting N being diffuse relative to B in M will be equivalent to a combination of conditions which either require that a vertex algebra is diffuse or a lack of edges between subgraphs (i.e. some "free independence outside the subgraph") in a manner analogous to Theorem 0.5(1).

Proposition 4.2. *Suppose that $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a graph, and for each $v \in V$ let (M_v, τ_v) be a tracial von Neumann algebra such that M_v contains a trace zero unitary. Let $(M, \tau) = \mathop{\ast}_{v\in V}(M_v, \tau_v)$. For $V_1, V_2 \subseteq \mathcal{V}$, the following are equivalent:*

(i) *M_{V_1} is diffuse relative to M_{V_2} in M;*

(ii) *M_{V_1} is diffuse relative to $M_{V_1\cap V_2}$ in M;*

(iii) *at least one of the following holds:*

  (a) *there are $v \in V_1 \setminus V_2$ and $v' \in V_1 \cap V_2$ with $v \not\sim v'$; or*

  (b) *$M_{V_1\setminus V_2}$ is diffuse;*

(iv) *at least one of the following holds:*

  (a) *there are $v \in V_1 \setminus V_2$ and $v' \in V_1$ with $v \neq v'$ and $v \not\sim v'$; or*

  (b) *there is a $v \in V_1 \setminus V_2$ for which M_v is diffuse.*

Proof. (i) $\Rightarrow$ (ii). Using the characterization from Theorem 4.1(ii), this follows from the identity $E_A = E_A \circ E_B$ for von Neumann subalgebras $A \subset B \subset M$ and the fact that the trace-preserving conditional expectation is contractive with respect to the L^2 norm.

(ii) $\Rightarrow$ (iii). We proceed by contrapositive and assume (iiia) and (iiib) are false. It follows that

$$M_{V_1} = M_{V_1\setminus V_2} \mathbin{\bar{\otimes}} M_{V_1\cap V_2},$$

and $M_{V_1\setminus V_2}$ is not diffuse. Let $z \in M_{V_1\setminus V_2}$ be a central projection such that $zM_{V_1\setminus V_2} \cong M_d(\mathbb{C})$ for some $d \in \mathbb{N}$. Suppose

$$u = (u_{i,j})_{i,j=1}^d \in M_d(M_{V_1\cap V_2}) \cong (z \otimes 1)M_{V_1}$$

is a unitary. Observe that

$$1 = \frac{1}{d}\sum_{i,j=1}^d \|u_{i,j}\|_2^2 = \frac{1}{d}\sum_{i,j=1}^d \|E_{M_{V_1\cap V_2}}(e_{ii}ue_{ji})\|_2^2.$$

Hence, $(z\otimes 1)M_{V_1}(z\otimes 1) \preceq_{(z\otimes 1)M(z\otimes 1)} M_{V_1\cap V_2}$, and consequently $M_{V_1} \preceq_M M_{V_1\cap V_2}$.

(iii) $\Rightarrow$ (iv). We again proceed by contrapositive and assume (iva) and (ivb) are false. Then every $v \in V_1 \setminus V_2$ must be adjacent to every $v' \in V_1$, so in particular (iiia) fails. Moreover, any two vertices in $V_1 \setminus V_2$ are adjacent, that is, $V_1 \setminus V_2$ is a complete graph. Since (ivb) fails, we know that for every $v \in V_1 \setminus V_2$ there is a minimal projection p_v in M_v. In particular, $p = \bigotimes_{v\in V_1\setminus V_2} p_v$ is a minimal projection in $M_{V_1\setminus V_2}$, and thus $M_{V_1\setminus V_2}$ is not diffuse and hence (iiib) fails.

(iva) $\Rightarrow$ (i). Let v, v' be as in (iva). Let u_0 be a trace zero unitary in $M_{v'}$ and let u_1 be a trace zero unitary in M_v. We claim that for $x, y \in M$, we have

$$\lim_{k\to\infty} \|E_{M_{V_2}}[x(u_0u_1)^k y]\|_2 = 0. \tag{4.1}$$

It suffices to show this for a set of x and y that have dense linear span. Hence, by Lemma 3.14, we may assume that $x = x_1 \cdots x_m$ with $x_j \in M_{v_j}$ for a $\mathcal{G}$-reduced word $w_1 \cdots w_m$ and with $\varphi(x_j) = 0$ (in the case that $x = 1$, we take w to be the empty word). Similarly, assume that $y = y_1 \cdots y_n$ with $\varphi(y_j) = 0$ and $y_j \in M_{\widehat{w}_j}$ with $\widehat{w}$ a reduced word.

By Lemma 3.12, w is equivalent to $w^{(1)} \cdot w^{(2)} \cdot w^{(3)}$, where $w^{(1)}$ is a $\mathcal{G}$-reduced word in V_2, $w^{(3)}$ is $\mathcal{G}$-reduced word in $\{v, v'\}$, and $w^{(2)}$ is a $\mathcal{G}$-reduced word relative to $(V_2, \{v, v'\})$. By swapping the x_j according to the swaps to transform w into $w^{(1)} \cdot w^{(2)} \cdot w^{(3)}$, we then obtain a factorization $x = x^{(1)}x^{(2)}x^{(3)}$, where $x^{(j)}$ is a product of centered elements indexed by the word $w^{(j)}$. Similarly, $\widehat{w}$ is equivalent to $\widehat{w}^{(1)} \cdot \widehat{w}^{(2)} \cdot \widehat{w}^{(3)}$, where $\widehat{w}^{(1)}$ is a reduced word in $\{v, v'\}$, $\widehat{w}^{(3)}$ is a reduced word in V_2, and $\widehat{w}^{(2)}$ is $\mathcal{G}$-reduced relative to $(\{v, v'\}, V_2)$. Write $y = y^{(1)}y^{(2)}y^{(3)}$ in an analogous way.

Since $x^{(1)}$ and $y^{(3)}$ are in M_{V_2}, we have

$$E_{M_{V_2}}[x^{(1)}x^{(2)}x^{(3)}(u_0u_1)^k y^{(1)}y^{(2)}y^{(3)}] = x^{(1)}E_{M_{V_2}}[x^{(2)}x^{(3)}(u_0u_1)^k y^{(1)}y^{(2)}]y^{(3)}.$$

Next, by Lemma 3.17, since $w^{(2)}$ is $(V_2, \{v, v'\})$-reduced and $\widehat{w}^{(2)}$ is $(\{v, v'\}, V_2)$-reduced, this equals

$$x^{(1)}E_{M_{V_2}}[x^{(2)}x^{(3)}(u_0u_1)^k y^{(1)}y^{(2)}]y^{(3)} = \varphi(x^{(2)}y^{(2)})x^{(1)}E_{M_U}[x^{(3)}(u_0u_1)^k y^{(1)}]y^{(3)},$$

where U is the set of vertices in $V_2 \cap \{v, v'\}$ that are adjacent to all the letters in $w^{(2)}$. Hence, in order to prove (4.1) and hence finish (iva) $\Rightarrow$ (i), it suffices to show that

$$\lim_{k\to\infty} \|E_{M_U}[x^{(3)}(u_0u_1)^k y^{(1)}]\|_2 = 0.$$

Since v' is not in V_2, then U must equal $\varnothing$ or $\{v\}$. Hence, since $\varphi(x^{(3)}(u_0u_1)^k y^{(1)}) = \varphi\circ E_{M_v}[x^{(3)}(u_0u_1)^k y^{(1)}]$, it suffices to show that

$$\lim_{k\to\infty} \|E_{M_v}[x^{(3)}(u_0u_1)^k y^{(1)}]\|_2 = 0.$$

However, for such $x^{(3)}$ and $y^{(1)}$ the above sequence is zero for sufficiently large k by free independence (see also [GEPT25, Proof of Prop. 3.16]).

(ivb) $\Rightarrow$ (i). Suppose that M_v is diffuse for some $v\in V_1\setminus V_2$, and thus there exists a Haar unitary $u\in M_v$ (i.e. a unitary so that $\tau(u^k)=0$ for any $k\in\mathbb{Z}\setminus\{0\}$). We claim that for $x,y\in M$, we have

$$\lim_{k\to\infty} \|E_{M_{V_2}}[xu^k y]\|_2 = 0. \tag{4.2}$$

As in the previous case, it suffices to consider x and y which are products of centered elements according to words w and $\widehat{w}$ respectively. And again, we take a decomposition $w\approx w^{(1)}\cdot w^{(2)}\cdot w^{(3)}$ as in Lemma 3.12 with respect to $(V_2,\{v\})$ and a decomposition $\widehat{w}\approx \widehat{w}^{(1)}\cdot\widehat{w}^{(2)}\cdot\widehat{w}^{(3)}$ with respect to $(\{v\},V_2)$. Let $x=x^{(1)}x^{(2)}x^{(3)}$ and $y=y^{(1)}y^{(2)}y^{(3)}$ be the resulting factorizations of x and y. Then

$$\begin{aligned} E_{M_{V_2}}[x^{(1)}x^{(2)}x^{(3)}u^k y^{(1)}y^{(2)}y^{(3)}] &= x^{(1)}E_{M_{V_2}}[x^{(2)}x^{(3)}u^k y^{(1)}y^{(2)}]y^{(3)} \\ &= x^{(1)}\varphi(x^{(2)}y^{(2)})\varphi(x^{(3)}u^k y^{(1)})y^{(3)}, \end{aligned}$$

where the second equality follows from Lemma 3.17. Here, the set U is empty since $\{v\}\cap V_2=\varnothing$. Because u is a Haar unitary, we have $u^k\to 0$ weakly as $k\to\infty$ and thus $\varphi(x^{(3)}u^k y^{(1)})\to 0$. This completes the proof of (4.2) and hence the proposition. □

§5. Bimodules from subgraphs and their fusion rules

Let $U\subseteq W$. We want to understand the basic construction of M_U inside M_W. Hence we want to understand $L^2(M_U,\varphi_U)$ as an M_W–M_W-bimodule. More generally, for $V_1,V_2\subseteq W$, we want to understand M_W as an M_{V_1}–M_{V_2}-bimodule. We first recall a few facts about standard forms and Connes fusion of bimodules.

Given a statial von Neumann algebra (M,φ), recall that $L^2(M,\varphi)$ is an M–M-bimodule with actions

$$x\cdot\xi\cdot y = x(J_\varphi y^* J_\varphi)\xi,$$

where J_φ is the modular conjugation operator. We let $M\ni x\mapsto \hat{x}\in L^2(M,\varphi)$ denote the embedding determined by $\langle \hat{x},\hat{y}\rangle_\varphi = \varphi(y^*x)$. We will say $x\in M$ is *φ-analytic* if the modular automorphism group $\mathbb{R}\ni t\mapsto \sigma_t^\varphi(x)$ has an extension to an entire function (such elements are dense by [Tak03, Lem. VIII.2.3]). In this case, for $z \in \mathbb{C}$ we write $\sigma_z(x)$ for the image of z under this (necessarily unique) entire extension. It follows that $\hat{y} \cdot x = (y\sigma_{-i/2}(x))\widehat{\ }$ whenever x is φ-analytic and $y \in M$.

We will also need to consider the Connes fusion of bimodules over σ-finite von Neumann algebras. We refer the reader to [OOT17, Sect. 2] for general details, but for our purposes it suffices to consider the following special case. Let (M, φ) and (N, ψ) be statial von Neumann algebras, and let $B \subset M$ be a von Neumann subalgebra admitting a φ-preserving conditional expectation $E_B \colon M \to B$. If $\mathcal{H}$ is a B–N-bimodule, then the M–N-bimodule

$$L^2(M, \varphi) \underset{B}{\otimes} \mathcal{H}$$

is formed by separation and completion of the algebraic tensor product $\widehat{M} \odot \mathcal{H}$ with respect to

$$\langle \hat{x} \otimes \xi, \hat{y} \otimes \eta \rangle \coloneqq \langle E_B(y^* x) \cdot \xi, \eta \rangle.$$

We will denote the equivalence class of $\hat{x} \otimes \xi$ by $\hat{x} \otimes_B \xi$. We also note that

$$L^2(M, \varphi) \otimes_B L^2(B, \varphi|_B) \cong L^2(B, \varphi|_B) \otimes_B L^2(M, \varphi) \cong L^2(M, \varphi).$$

That is, $L^2(B, \varphi|_B)$ is an identity element with respect to the operation $\otimes_B$.

Let us now return to the context of graph products over $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. For $V_1, V_2 \subset \mathcal{V}$, we will build a basis over M_{V_1}–M_{V_2} by using orthonormal bases for $L^2(M_v, \varphi_v) \ominus \mathbb{C}\hat{1}$. Since we are not assuming that our von Neumann algebras have separable predual, we will not a priori be able to build an orthonormal basis for $L^2(M_v, \varphi_v) \ominus \mathbb{C}\hat{1}$ using elements of M_v. For this reason, we will need to extend some of the results of Section 3.3 to vectors in $L^2(M_v, \varphi_v) \ominus \mathbb{C}\hat{1}$.

Lemma 5.1. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph, let $\{(M_v, \varphi_v) : v \in \mathcal{V}\}$ be a family of statial von Neumann algebras, and let $(M, \varphi) = \mathop{\ast}_{v \in \mathcal{G}} (M_v, \varphi_v)$.*

(i) *Let $w = w_1 \cdots w_\ell$ be a $\mathcal{G}$-reduced word. Then there is a unique continuous multilinear map*

$$m \colon \prod_{i=1}^{\ell} (L^2(M_v, \varphi_v) \ominus \mathbb{C}\hat{1}) \to L^2(M, \varphi) \ominus \mathbb{C}\hat{1},$$

such that $m(x_1, \ldots, x_\ell) = (x_1 \cdots x_\ell)\widehat{\ }$ when $x_i \in M_{w_i} \cap \ker(\varphi_{w_i})$. Moreover,

$$\|m(\xi_1, \ldots, \xi_\ell)\|_\varphi = \prod_{i=1}^{\ell} \|\xi_j\|_\varphi, \quad \xi = (\xi_1, \ldots, \xi_\ell) \in \prod_{j=1}^{\ell} (L^2(M_{w_j}, \varphi_{w_j}) \ominus \mathbb{C}\hat{1}).$$

We denote $m(\xi_1, \ldots, \xi_\ell) = \xi_1 \cdots \xi_\ell$.

(ii) *Let $w = w_1 \cdots w_m$ and $\widetilde{w} = \widetilde{w}_1 \cdots \widetilde{w}_n$ be $\mathcal{G}$-reduced words. Set $\xi = \xi_1 \cdots \xi_m$ and $\tilde{\xi} = \tilde{\xi}_1 \cdots \tilde{\xi}_n$, where $\xi_j \in L^2(M_{w_j}, \varphi_{w_j}) \ominus \mathbb{C}\hat{1}$, $j = 1, \dots, m$, and $\tilde{\xi}_j \in$* $\ker L^2(M_{\widetilde{w}_j}, \varphi_{\widetilde{w}_j}) \ominus \mathbb{C}\hat{1}$, *$j = 1, \dots, n$. If w, $\widetilde{w}$ are not equivalent, then ξ, $\tilde{\xi}$ are orthogonal. If w and $\widetilde{w}$ are equivalent, then*

$$\langle \xi, \tilde{\xi} \rangle_\varphi = \prod_{j=1}^{m} \langle \xi_j, \tilde{\xi}_{\sigma(j)} \rangle_\varphi,$$

where the permutation $\sigma\colon [m] \to [m]$ is the $\mathcal{G}$-monotone matching from w to $\widetilde{w}$ guaranteed by Lemma 3.6.

Proof. (i). The uniqueness of m follows from the density of $M_w \cap \ker(\varphi_w)$ in $L^2(M_w, \varphi_w) \ominus \mathbb{C}\hat{1}$. By Lemma 3.15, as well as the density of $M_w \cap \ker(\varphi_w)$ in $L^2(M_w, \varphi_w) \ominus \mathbb{C}\hat{1}$, it follows that there is a unique isometry

$$V\colon \bigotimes_{j=1}^{\ell} (L^2(M_{w_j}, \varphi_{w_j}) \ominus \mathbb{C}\hat{1}) \to L^2(M, \varphi) \ominus \mathbb{C}\hat{1}$$

such that $V(\hat{x}_1 \otimes \cdots \otimes \hat{x}_\ell) = (x_1 \cdots x_\ell)\widehat{\ }$. Setting $m(\xi_1, \dots, \xi_\ell) = V(\xi_1 \otimes \cdots \otimes \xi_\ell)$ completes the proof.

(ii). Observe that if

$$\xi \in m\left(\prod_{j=1}^{m} (M_{w_j} \cap \ker(\varphi_{w_j})) \right) \quad \text{and} \quad \tilde{\xi} \in m\left(\prod_{j=1}^{n} (M_{\widetilde{w}_j} \cap \ker(\varphi_{\widetilde{w}_j})) \right)$$

then the claim follows from Lemma 3.15. The norm equality in (i) implies these sets are dense in

$$m\left(\prod_{j=1}^{n} (L^2(M_{w_j}, \varphi_{w_j}) \ominus \mathbb{C}\hat{1}) \right) \quad \text{and} \quad m\left(\prod_{i=1}^{n} (L^2(M_{\widetilde{w}_i}, \varphi_{\widetilde{w}_i}) \ominus \mathbb{C}\hat{1}) \right),$$

respectively, which completes the proof. □

Remark 5.2. Using Haagerup's theory of non-commutative L^p-spaces ([Haa79]), one can also make sense of $m(\xi_1, \dots, \xi_n) = \xi_1 \cdots \xi_n$ as a product of operators affiliated with the continuous core of M. The fact that such a product remains in $L^2(M, \varphi)$ is a consequence of their relations via φ, which is determined by the graph product structure of M.

We will first analyze cyclic submodules generated by products over relatively $\mathcal{G}$-reduced words.

Lemma 5.3. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph, let $\{(M_v, \varphi_v) : v \in \mathcal{V}\}$ be a family of statial von Neumann algebras, let $(M, \varphi) = \bigstar_{v\in\mathcal{G}}(M_v, \varphi_v)$, and let $V_1, V_2 \subseteq \mathcal{V}$. For $w = w_1 \cdots w_n$ a $\mathcal{G}$-reduced word relative to (V_1, V_2), let*

$$\xi = \xi_1 \cdots \xi_n,$$

where $\xi_j \in L^2(M_{w_j}, \varphi_{w_j}) \ominus \mathbb{C}\hat{1}$ with $\|\xi_j\|_\varphi = 1$ for $j = 1, \ldots, n$. (In the case that w is empty, we take by convention $\xi = \hat{1}$.) Let H_ξ be the M_{V_1}–M_{V_2}-subbimodule of $L^2(M, \varphi)$ generated by ξ and denote

$$U \coloneqq \{v \in V_1 \cap V_2 : v \sim w_j,\ j = 1, \ldots, n\}.$$

(i) *There is a unique M_{V_1}–M_{V_2}-bimodular unitary $H_\xi \to L^2(M_{V_1}, \varphi_{V_1}) \otimes_{M_U} L^2(M_{V_2}, \varphi_{V_2})$ which sends ξ to $1 \otimes_{M_U} 1$.*

(ii) *If $\widetilde{w} = \widetilde{w}_1 \cdots \widetilde{w}_m$ is another $\mathcal{G}$-reduced word relative to (V_1, V_2) and $\tilde{\xi} = \tilde{\xi}_1 \cdots \tilde{\xi}_m$ is a corresponding vector, then $H_\xi \perp H_{\tilde{\xi}}$ unless w and $\widetilde{w}$ are equivalent and $\langle \xi, \tilde{\xi}\rangle_\varphi \neq 0$.*

Proof. (i). It suffices to show that

$$\langle a \cdot (1 \otimes_{M_U} 1) \cdot b, 1 \otimes_{M_U} 1\rangle = \langle a \cdot \xi_1 \cdots \xi_n \cdot b, \xi_1 \cdots \xi_n\rangle_\varphi, \tag{5.1}$$

for all $a \in M_{V_1}$ and all φ-analytic $b \in M_{V_2}$. By Lemma 5.1 for fixed a, b, the right-hand side is a continuous function of $(\xi_1, \ldots, \xi_n) \in \prod_{j=1}^n (L^2(M_{w_j}, \varphi_{w_j}) \ominus \mathbb{C}\hat{1})$. Thus, by density of $M_w \cap \ker(\varphi_w)$ in $L^2(M_w, \varphi_w) \ominus \mathbb{C}\hat{1}$, we may reduce to the case, where $\xi_j = x_j$ where $x_j \in M_{w_j} \cap \ker(\varphi_{w_j})$ and $\varphi(x_j^* x_j) = 1$. In this case, set $x = x_1 \cdots x_j$ so that $\xi = \hat{x}$.

The left-hand side of (5.1) is

$$\langle \widehat{a} \otimes_{M_U} (\sigma_{-i/2}(b))\widehat{\ }, 1 \otimes_{M_U} 1\rangle = \varphi(E_{M_U}(a)\sigma_{-i/2}(b)),$$

and the right-hand side of (5.1) is

$$\langle (ax\sigma_{-i/2}(b))\widehat{\ }, \widehat{x}\rangle_\varphi = \varphi(x^* a x \sigma_{-i/2}(b)) = \varphi(E_{M_{V_2}}(x^* a x)\sigma_{-i/2}(b)).$$

Thus Lemma 3.17 implies (5.1).

(ii). It is enough to show that for all $a \in M_{V_1}$ and φ-analytic $b \in M_{V_2}$ that

$$\langle a \cdot \xi \cdot b, \tilde{\xi}\rangle_\varphi = 0, \tag{5.2}$$

if w, $\widetilde{w}$ are not equivalent, and that

$$\langle a \cdot \xi \cdot b, \tilde{\xi}\rangle_\varphi = \langle \xi, \tilde{\xi}\rangle_\varphi \varphi(E_{M_U}(a)\sigma_{-i/2}(b))$$

if w, $\widetilde{w}$ are equivalent. As in (i) we may reduce to the case that $\xi = \widehat{x}$, $\tilde{\xi} = \widehat{\tilde{x}}$, where $x = x_1 \cdots x_n$, $\tilde{x} = \tilde{x}_1 \cdots \tilde{x}_m$, and $x_j \in M_{w_j} \cap \ker(\varphi_{w_j})$ and $\tilde{x}_i \in M_{\widetilde{w}_i} \cap \ker(\varphi_{\widetilde{w}_i})$. We then have

$$\langle a \cdot \widehat{x} \cdot b, \widehat{\tilde{x}} \rangle_\varphi = \varphi((\tilde{x})^* a x \sigma_{-i/2}(b)) = \varphi(E_{M_{V_2}}((\tilde{x})^* a x) \sigma_{-i/2}(b)),$$

so that our desired conclusion follows from Lemma 3.17. $\square$

Our main result in this section provides a classification of $L^2(M)$ as a bimodule over two subalgebras coming from induced subgraphs. This also yields the first part of Theorem 0.4.

Theorem 5.4. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph, let $\{(M_v, \varphi_v) : v \in \mathcal{V}\}$ be a family of statial von Neumann algebras, let $(M, \varphi) = \ast_{v \in \mathcal{G}} (M_v, \varphi_v)$, and let $V_1, V_2 \subseteq \mathcal{V}$. For each $U \subset V_1 \cap V_2$, denote by $\mathcal{W}_\mathcal{G}(V_1, V_2, U)$ the set of $\mathcal{G}$-reduced words relative to (V_1, V_2) of the form $w_1 \cdots w_\ell$ satisfying $U = \{v \in V_1 \cap V_2 : v \sim w_j,\ j = 1, \ldots, \ell\}$. Set*

$$k_\mathcal{G}(V_1, V_2, U) := \sum_{\substack{w_1 \cdots w_\ell \in \\ \mathcal{W}_\mathcal{G}(V_1, V_2, U)}} \prod_{j=1}^{\ell} \big(\dim(L^2(M_{w_j}, \varphi_{w_j})) - 1\big).$$

Then one has

$$ {}_{M_{V_1}} L^2(M, \varphi)_{M_{V_2}} \cong \bigoplus_{U \subseteq V_1 \cap V_2} \big({}_{M_{V_1}} L^2(M_{V_1}, \varphi_{V_1}) \underset{M_U}{\otimes} L^2(M_{V_2}, \varphi_{V_2})_{M_{V_2}}\big)^{\oplus k_\mathcal{G}(V_1, V_2, U)}. \tag{5.3}$$

Proof. For each $v \in \mathcal{V}$, fix an orthonormal basis $\mathcal{B}_v$ for $L^2(M_v, \varphi_v) \ominus \mathbb{C}$. By Lemma 5.3, the M_{V_1}–M_{V_2}-bimodules

$$\big\{ H_\xi : \xi = \xi_1 \cdots \xi_\ell,\ w = w_1 \cdots w_\ell \in \mathcal{W}_\mathcal{G}(V_1, V_2, U),\ \xi_j \in \mathcal{B}_{w_j} \text{ for } j = 1, \ldots, \ell \big\}$$

are mutually orthogonal and satisfy $H_\xi \cong L^2(M_{V_1}, \varphi_{V_1}) \otimes_{M_U} L^2(M_{V_2}, \varphi_{V_2})$, where $U = \{v \in V_1 \cap V_2 : v \sim w_j,\ j = 1, \ldots, \ell\}$. For each $U \subseteq V_1 \cap V_2$, the number of copies of ${}_{M_{V_1}} L^2(M_{V_1}, \varphi_{V_1}) \otimes_{M_U} L^2(M_{V_2}, \varphi_{V_2})_{M_{V_2}}$ is given by (5.3), since $\dim(L^2(M_v, \varphi_v) \ominus \mathbb{C}\hat{1}) = \dim(L^2(M_v, \varphi_v)) - 1$.

The proof of the direct sum decomposition will be complete once we verify that the bimodules H_ξ span a dense subset of $L^2(M, \varphi)$. From Lemma 3.14, we know that $L^2(M, \varphi)$ is densely spanned by $\xi_1 \cdots \xi_\ell$ for $\xi_j \in \mathcal{B}_{w_j}$ for reduced words $w_1 \cdots w_\ell$. By Lemma 3.12, an arbitrary reduced word w is equivalent to a word of the form $v \cdot w' \cdot u$ where v is a reduced word in V_1, u is a reduced word in V_2, and w' is reduced relative to (V_1, V_2). This shows that the span of the subspaces

$\mathcal{H}_\xi$ contains all $\xi_1 \cdots \xi_\ell$ for $x_j \in \mathcal{B}_{w_j}$ for reduced words $w_1 \cdots w_\ell$, and thus the bimodules $\mathcal{H}_\xi$ densely span $L^2(M, \varphi)$. □

For future applications to relative amenability, we determine the fusion rules for these bimodules in the following proposition. This also gives the rest of Theorem 0.4.

Proposition 5.5. *For $V_1, V_2 \subseteq \mathcal{V}$ and $U \subseteq V_1 \cap V_2$, denote*

$$\mathscr{H}_U(V_1, V_2) \coloneqq {}_{M_{V_1}} L^2(M_{V_1}, \varphi_{V_1}) \underset{M_U}{\otimes} L^2(M_{V_2}, \varphi_{V_2})_{M_{V_2}},$$

and denote by $\mathcal{G}_2$ the subgraph of $\mathcal{G}$ induced by V_2. Then we have the following fusion rules: for $U_1 \subseteq V_1 \cap V_2$ and $U_2 \subseteq V_2 \cap V_3$,

$$\mathscr{H}_{U_1}(V_1, V_2) \otimes_{M_{V_2}} \mathscr{H}_{U_2}(V_2, V_3) \cong \bigoplus_{W \subseteq U_1 \cap U_2} \mathscr{H}_W(V_1, V_3)^{\oplus k_{\mathcal{G}_2}(U_1, U_2, W)}.$$

Proof. First, observe that

$$\begin{aligned}
&\mathscr{H}_{U_1}(V_1, V_2) \underset{M_{V_2}}{\otimes} \mathscr{H}_{U_2}(V_2, V_3) \\
&\quad = \left({}_{M_{V_1}} L^2(M_{V_1}, \varphi_{V_1}) \underset{M_{U_1}}{\otimes} L^2(M_{V_2}, \varphi_{V_2}) \right)_{M_{V_2}} \\
&\qquad \underset{M_{V_2}}{\otimes} \left({}_{M_{V_2}} L^2(M_{V_2}, \varphi_{V_2}) \right) \underset{M_{U_2}}{\otimes} L^2(M_{V_3}, \varphi_{V_3})_{M_{V_3}} \\
&\quad \cong \left({}_{M_{V_1}} L^2(M_{V_1}, \varphi_{V_1}) \underset{M_{U_1}}{\otimes} L^2(M_{V_2}, \varphi_{V_2}) \right) \underset{M_{U_2}}{\otimes} L^2(M_{V_3}, \varphi_{V_3})_{M_{V_3}}.
\end{aligned}$$

Then

$$\begin{aligned}
{}_{M_{U_1}} L^2(M_{V_2}, \varphi_{V_2})_{M_{U_2}} \cong \bigoplus_{W \subseteq U_1 \cap U_2} \Big(&{}_{M_{U_1}} L^2(M_{U_1}, \varphi_{U_1}) \\
&\underset{M_W}{\otimes} L^2(M_{U_2}, \varphi_{U_2})_{M_{U_2}} \Big)^{\oplus k_{\mathcal{G}_2}(U_1, U_2, W)}.
\end{aligned}$$

Applying $L^2(M_{V_1}) \otimes_{M_{U_1}}$ on the left and $\otimes_{M_{U_2}} L^2(M_{V_2})$ on the right, we get

$$\begin{aligned}
&\mathscr{H}_{U_1}(V_1, V_2) \otimes_{M_{V_2}} \mathscr{H}_{U_2}(V_2, V_3) \\
&\quad \cong \bigoplus_{W \subseteq U_1 \cap U_2} \left({}_{M_{V_1}} L^2(M_{V_1}, \varphi) \underset{M_W}{\otimes} L^2(M_{V_2}, \varphi)_{M_{V_2}} \right)^{\oplus k_{\mathcal{G}_2}(U_1, U_2, W)} \\
&\quad = \bigoplus_{W \subseteq U_1 \cap U_2} \mathscr{H}_W(V_1, V_2)^{\oplus k_{\mathcal{G}_2}(U_1, U_2, W)}.
\end{aligned}$$

□

As a sample application, we characterize weak coarseness of subalgebras corresponding to induced subgraphs. This characterization of coarseness is stated in terms of amenability of certain subalgebras, which we provide a complete characterization of in Proposition 6.3.

Theorem 5.6. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph, let $\{(M_v, \varphi_v) : v \in \mathcal{V}\}$ be a family of statial von Neumann algebras, and let $(M, \varphi) = \star_{v\in\mathcal{V}}(M_v, \varphi_v)$. Assume $\dim(M_v) \geq 2$ for all $v \in \mathcal{V}$. For $V_0 \subseteq \mathcal{V}$, $L^2(M, \varphi) \ominus L^2(M_{V_0}, \varphi_{V_0})$ is weakly coarse as an M_{V_0}–M_{V_0}-bimodule if and only if $M_{B(v)\cap V_0}$ is amenable for all $v \notin V_0$.*

Proof. First, suppose that $M_{B(v)\cap V_0}$ is amenable for all $v \notin V_0$. By Theorem 5.4, ${}_{M_{V_0}}L^2(M, \varphi)_{M_{V_0}}$ is a direct sum of bimodules of the form $L^2(M_{V_0}, \varphi_{V_0}) \otimes_{M_U} L^2(M_{V_0}, \varphi_{V_0})$, where $U = \{v \in V_0 : v \sim w_j,\ j = 1, \ldots, \ell\}$ for some word $w_1 \cdots w_\ell$ that is $\mathcal{G}$-reduced relative to (V_0, V_0). To obtain the orthogonal complement of $L^2(M_{V_0}, \varphi_{V_0})$, one sums over the *non-empty* words of this form with the appropriate multiplicity. Note that $U \subseteq B(w_1) \cap V_0$, and $w_1 \notin V_0$ since $w_1 \cdots w_\ell$ is non-empty and $\mathcal{G}$-reduced relative to (V_0, V_0). Thus $M_{B(w_1)\cap V_0}$ is amenable by assumption, and since there is a faithful normal conditional expectation from this algebra on M_U, we also have that M_U is amenable. Thus,

$$ {}_{M_U}L^2(M_U, \varphi_U)_{M_U} \prec {}_{M_U}L^2(M_U, \varphi_U) \otimes L^2(M_U, \varphi_U)_{M_U} $$

by [BMO20, Cor. A.2] (see also [Con76] for the separable predual case). Now we apply ${}_{M_{V_0}}L^2(M_{V_0}, \varphi_{V_0}) \otimes_{M_U}$ on the left and apply $\otimes_{M_U} L^2(M_{V_0}, \varphi_{V_0})_{M_{V_0}}$ on the right to obtain

$$ {}_{M_{V_0}}L^2(M_{V_0}, \varphi_{V_0}) \otimes_{M_U} L^2(M_{V_0}, \varphi_{V_0})_{M_{V_0}} \prec {}_{M_{V_0}}L^2(M_{V_0}, \varphi_U) \otimes L^2(M_{V_0}, \varphi_{V_0})_{M_{V_0}}, $$

where we have used the fact that weak containment is preserved under Connes fusion [Pop86, Prop. 2.2.1]. Taking the direct sum over all such non-empty words $w_1 \cdots w_\ell$ yields that $L^2(M, \varphi) \ominus L^2(M_{V_0} \varphi_{V_0})$ is weakly coarse over M_{V_0}.

Conversely, suppose there exists some vertex $v \notin V_0$ such that $M_{B(v)\cap V_0}$ is non-amenable. For ease of notation, denote $V_1 \coloneqq B(v) \cap V_0$. Fix $x \in M_v$ with $\varphi_v(x) = 0$ and $\varphi_v(x^*x) = 1$. Let $H_{\hat{x}}$ be the M_{V_0}–M_{V_0}-subbimodule of $L^2(M, \varphi)$ generated by $\hat{x}$, which we note is in $L^2(M, \varphi) \ominus L^2(M_{V_0}, \varphi_{V_0})$ since $v \notin V_0$. Applying Lemma 5.3(i) to $V_2 \coloneqq V_1$ and $w = v$ (so that $U = V_1$), we have that the M_{V_1}–M_{V_1}-bimodule generated by $\hat{x}$ is isomorphic to $L^2(M_{V_1}, \varphi_{V_1}) \otimes_{M_{V_1}} L^2(M_{V_1}, \varphi_{V_1}) \cong L^2(M_{V_1}, \varphi_{V_1})$. In particular, since M_{V_1} is not amenable, this M_{V_1}–M_{V_1}-bimodule is not weakly coarse. Since it is an M_{V_1}–M_{V_1}-subbimodule of $H_{\hat{x}}$, it follows that ${}_{M_{V_1}}(H_{\hat{x}})_{M_{V_1}}$ is not weakly coarse, and in turn ${}_{M_{V_0}}(H_{\hat{x}})_{M_{V_0}}$ is not weakly coarse. $\square$

§6. Relative amenability via bimodules

A useful implication of relative amenability is the following. By [BMO20, Sect. 2.2], we can describe the standard form of $\langle M, e_B\rangle$ via the isomorphism

$$L^2(\langle M, e_B\rangle) \cong L^2(M) \otimes_B L^2(M),$$

as M–M-bimodules. Consequently, [BMO20, Cor. A.2] tells us that A being amenable relative to B inside M implies that $L^2(M)$ is weakly contained in $L^2(\langle M, e_B\rangle)$ as A–A-bimodules. Note that – due to the conditional expectation being required to be normal on M – the converse is not a priori true. However, in our setting M will be a graph product and A, B will be subalgebras corresponding to induced subgraphs. In this case, the detailed analysis of the previous section will lead us to a complete classification of when $L^2(M)$ is weakly contained in $L^2(\langle M, e_B\rangle)$. From this classification, we will be able to directly argue that if $L^2(M)$ is not weakly contained in $L^2(\langle M, e_B\rangle)$, then A must be amenable relative to B inside M.

As the fusion rules provided in Proposition 5.5 decompose relative tensor products as direct sums, we highlight the fact that, in the factorial case, bimodules weakly contained in direct sums are necessarily weakly contained in one of the summands. Indeed, suppose that M is factor, and for a faithful normal state φ on M let J_φ be the associated modular conjugation on $L^2(M,\varphi)$. Then the induced map $\pi\colon M \otimes_{\max} M^{\mathrm{op}} \to B(L^2(M,\psi))$ satisfying $\pi(a\otimes b^{\mathrm{op}}) = aJ_\varphi b^* J_\varphi$ has trivial commutant ($\pi(M\otimes_{\max} M^{\mathrm{op}})' = M' \cap (J_\varphi M J_\varphi)' = M \cap M'$) and is thus irreducible [Tak02, Prop. I.9.20]. Hence, the state on $M\otimes_{\max} M^{\mathrm{op}}$ given by $x \mapsto \langle \pi(x)\hat{1}, \hat{1}\rangle$ is an extreme point of the state space [Tak02, Thm. I.9.22], and so a minor modification of the proof of [Fel60, Thm. 1.5] gives the following.

Lemma 6.1. *Let M be a factor and φ a faithful normal state on M. Suppose $\mathcal{H}_1, \ldots, \mathcal{H}_n$ are M–M-bimodules with*

$$L^2(M,\varphi) \prec \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$$

as M–M-bimodules. Then there is a $1 \le i \le n$ so that $L^2(M,\varphi) \prec \mathcal{H}_i$ as M–M-bimodules.

It will also be helpful to prove the following general lemma, which will ultimately reduce our work of checking when one subalgebra corresponding to an induced subgraph is amenable relative to another, to the case of smaller subgraphs.

Lemma 6.2. *Let $\mathcal{G} = (\mathcal{V},\mathcal{E})$ be a graph, let $\{(M_v,\varphi_v) : v\in\mathcal{V}\}$ be a family of statial von Neumann algebras, let $(M,\varphi) = \mathop{\ast}_{v\in\mathcal{G}}(M_v,\varphi_v)$, and let $V_1, V_2 \subseteq \mathcal{V}$. Suppose that ${}_{M_{V_1}}L^2(M,\varphi)_{M_{V_1}}$ is weakly contained in ${}_{M_{V_1}}L^2(M,\varphi)\otimes_{M_{V_2}} L^2(M,\varphi)_{M_{V_1}}$.*

(i) *M_{V_0} is amenable for all $V_0 \subseteq V_1 \setminus V_2$.*

(ii) *If M_{V_0} is factor for $V_0 \subset V_1$, then there exists $U \subseteq V_0 \cap V_2$ (possibly empty) so that M_{V_0} is amenable relative to M_U.*

Proof. We first make a preliminary observation. For $V_0 \subset \mathcal{V}$ and $A \subset V_0 \cap V_2$, we adopt the notation from Proposition 5.5 and denote

$$\mathscr{H}_A(V_0, V_2) \coloneqq {}_{M_{V_0}} L^2(M_{V_0}, \varphi_{V_0}) \underset{M_A}{\otimes} L^2(M_{V_2}, \varphi_{V_2})_{M_{V_2}}.$$

By Theorem 5.4, we have

$${}_{M_{V_0}} L^2(M, \varphi)_{M_{V_2}} \cong \bigoplus_{A \subseteq V_0 \cap V_2} \mathscr{H}_A(V_0, V_2)^{\oplus k_{\mathcal{G}}(V_0, V_2, A)} \subseteq \bigoplus_{A \subseteq V_0 \cap V_2} \mathscr{H}_A(V_0, V_2)^{\oplus \infty}.$$

Therefore, using Proposition 5.5 we have

$$\begin{aligned} {}_{M_{V_0}} L^2(M, \varphi) \underset{M_{V_2}}{\otimes} L^2(M, \varphi)_{M_{V_0}} &\subseteq \bigoplus_{A, B \subseteq V_0 \cap V_2} \Big(\mathscr{H}_A(V_0, V_2) \underset{M_{V_2}}{\otimes} \mathscr{H}_B(V_2, V_0) \Big)^{\oplus \infty} \\ &\subseteq \bigoplus_{U \subseteq V_0 \cap V_2} \mathscr{H}_U(V_0, V_0)^{\oplus \infty}. \end{aligned}$$

By assumption, ${}_{M_{V_1}} L^2(M, \varphi)_{M_{V_1}}$ is weakly contained in ${}_{M_{V_1}} L^2(M, \varphi) \otimes_{M_{V_2}} L^2(M, \varphi)_{M_{V_1}}$. If $V_0 \subset V_1$, then by restriction we have that ${}_{M_{V_0}} L^2(M, \varphi)_{M_{V_0}}$ is weakly contained in ${}_{M_{V_0}} L^2(M, \varphi) \otimes_{M_{V_2}} L^2(M, \varphi)_{M_{V_0}}$, and so the above shows that

$$ {}_{M_{V_0}} L^2(M, \varphi)_{M_{V_0}} \prec \bigoplus_{U \subseteq V_0 \cap V_2} \mathscr{H}_U(V_0, V_0)^{\oplus \infty}. \tag{6.1}$$

Now, if $V_0 \subset V_1 \setminus V_2$, then the only term in the above direct sum corresponds to $U = \varnothing$, which has $M_U = \mathbb{C}$. Thus the above gives

$$\begin{aligned} {}_{M_{V_0}} L^2(M, \varphi)_{M_{V_0}} &\prec {}_{M_{V_0}} L^2(M_{V_0}, \varphi_{V_0}) \otimes L^2(M_{V_0}, \varphi_{V_0})_{M_{V_0}} \\ &\cong {}_{M_{V_0}} L^2(M_{V_0}, \varphi_{V_0}) \otimes \overline{L^2(M_{V_0}, \varphi_{V_0})}_{M_{V_0}}. \end{aligned}$$

Note that the bimodule in the last expression is equivalent to the standard form of $B(L^2(M_{V_0}, \varphi_{V_0}))$ with respect to its trace. Hence, M_{V_0} is amenable by [BMO20, Cor. A.2] (see also [Con76] in the case of separable preduals), which proves (i).

To prove (ii), suppose M_{V_0} is a factor for $V_0 \subset V_1$. Since $\mathcal{G}$ is a finite graph, the direct sum over $U \subset V_0 \cap V_2$ in (6.1) only has finitely many terms, and hence Lemma 6.1 implies

$${}_{M_{V_0}} L^2(M, \varphi)_{M_{V_0}} \prec {}_{M_{V_0}} L^2(M_{V_0}, \varphi_{V_0}) \underset{M_U}{\otimes} L^2(M_{V_0}, \varphi_{V_0})_{M_{V_0}},$$

for some $U \subset V_0 \cap V_2$. By [BMO20, Sect. 2.2], the latter bimodule is isomorphic to the standard form for $\langle M_{V_0}, e_{M_U}\rangle$. Thus [BMO20, Cor. A.2] yields a conditional expectation $\Phi\colon \langle M_{V_0}, e_{M_U}\rangle \to M_{V_0}$ so that M_{V_0} is amenable relative to M_U. □

§6.1. Proofs of Theorems 0.1 and 0.3

Let us first reduce Theorem 0.1 to Theorem 0.3. Comparing the two theorems, this amounts to showing that if (M_i, τ_i) is a tracial von Neumann algebra admitting a trace zero unitary for $i = 1, 2$, then the following are equivalent:

(I) $\dim(M_1) = \dim(M_2) = 2$;

(II) $M_1 * M_2$ is amenable;

(III) $M_1 * M_2$ is amenable relative to M_1.

The equivalence of the first two items is well known (see, for example, [Chi73, Thm. 2]), and that (II) implies (III) follows from the definition. So now suppose (III) holds. Applying Proposition 4.2 to the graph $\mathcal{G} = (\{1,2\}, \varnothing)$ with $V_1 = \{1,2\}$ and $V_2 = \{1\}$, we see that $M_{V_1} = M_1 * M_2$ is diffuse relative to $M_{V_2} = M_1$ inside $M_1 * M_2$. That is, $M_1 * M_2$ does not intertwine into M_1 inside $M_1 * M_2$, and thus [Ioa15, Cor. 2.12] implies (II).

We now prove Theorem 0.3. First, assume that Theorem 0.3(1) and (2) hold. Let $P_1, \ldots, P_n$ be the pairs of vertices $\{v, w\}$, where $v \in V_1 \setminus V_2$, $w \in V_1$, and v and w are not adjacent. Denote $Q_1 := V_1 \setminus (V_2 \cup P_1 \cup \cdots \cup P_n)$ and $Q_2 := V_1 \cap V_2 \setminus (P_1 \cup \cdots \cup P_n)$. By (2b), all the vertices in each P_j are connected to all other vertices in $Q_1 \cup Q_2$. Moreover, each $v \in Q_1$ is connected to all vertices in V_1 by definition of Q_1. Thus,

$$M_{V_1} = \left(\overline{\bigotimes}_{j=1}^{n} M_{P_j}\right) \bar{\otimes} M_{Q_1} \bar{\otimes} M_{Q_2}$$

and

$$M_{V_1 \cap V_2} = \left(\overline{\bigotimes}_{j=1}^{n} M_{P_j \cap V_2}\right) \bar{\otimes} \mathbb{C} \bar{\otimes} M_{Q_2}.$$

By assumption (2a), M_{P_j} is amenable relative to $M_{P_j \cap V_2}$ in M_{P_j} for each $j = 1, \ldots, n$. By assumption (1) and [Con76, Thm. 6], M_{Q_1} is amenable. Thus Lemma C.3 implies that M_{V_1} is amenable relative to $M_{V_1 \cap V_2}$ (inside M_{V_1}). By Lemma C.1, this in turn implies that M_{V_1} is amenable relative to M_{V_2} in M.

Conversely, suppose that M_{V_1} is amenable relative to M_{V_2} inside M. Recall from the discussion at the beginning of Section 6 that this implies $L^2(M, \varphi)$ is weakly contained in $L^2(M, \varphi) \otimes_{M_{V_2}} L^2(M, \varphi)$ as M_1–M_1-bimodules. Thus for each $v \in V_1 \setminus V_2$ we can apply Lemma 6.2 to $V_0 = \{v\}$ to obtain that M_v is amenable. This gives Theorem 0.3(1). To prove Theorem 0.3(2), let $v \in V_1 \setminus V_2$ and $w \in V_1$

with $w \neq v$ and assume v and w are not adjacent. We will show that (2a) and (2b) must occur.

For (2a), first note that if $\dim(M_v) = \dim(M_w) = 2$, then by [Dyk93, Thm. 1.1] we have that $M_{\{v,w\}} = M_v * M_w$ is amenable. In particular, $M_{\{v,w\}}$ is also amenable relative to M_w, proving (2a) in this case. If $\max(\dim(M_v, M_w)) \geq 3$, then [Ued11, Thm. 4.1 and Rem. 4.2] implies that $M_{\{v,w\}} = M_v * M_w$ is a factor, and thus Lemma 6.2 applied to $V_0 = \{v, w\}$ yields that $M_{\{v,w\}}$ is amenable relative to M_U for some $U \subseteq V_0 \cap V_2$. Noting that $w \notin V_2$ forces $U = \varnothing$, we see that in this case $M_{\{v,w\}}$ is amenable. If $w \in V_2$, then either $U = \{w\}$ or $U = \varnothing$, but in both cases one has that $M_{\{v,w\}}$ is amenable relative to M_w. We have thus established (2a).

For (2b), consider another vertex $u \in V_1 \setminus \{v, w\}$, and suppose towards a contradiction that one of v or w is not adjacent to u. Note that this implies the subgraph $\mathcal{G}_0$ induced by $V_0 \coloneqq \{v, w, u\}$ is join-irreducible, and hence M_{V_0} is a factor by Theorem 2.4. Consequently, Lemma 6.2 implies M_{V_0} is amenable relative to M_U for some $U \subseteq V_0 \cap V_2$. Since $U \subset \{w, u\} \subset V_0$, it follows that M_{V_0} is amenable relative to $M_{\{w,u\}}$. We will show this is a contradiction by way of Lemma C.2 using the observation that

$$M_{V_0} \cong M_{\{v,u\}} *_{M_u} M_{\{w,u\}},$$

where the amalgamated free product is taken with respect to the φ-preserving conditional expectations. Let u_0 and u_2 be state zero unitaries in $M_v^{\varphi_v}$ and $M_w^{\varphi_w}$, respectively, so that $E_{M_u}[u_0] = \varphi_v[u_0] = 0$ and similarly $E_{M_u}[u_2] = 0$. Also let x be a state zero unitary in $M_u^{\varphi_u}$. If v is not adjacent to u, then $u_1 \coloneqq x u_0 x^*$ satisfies

$$E_{M_u}[u_1] = x E_{M_u}[u_0] x^* = 0,$$

and by free independence,

$$E_{M_u}[u_0^* u_1] = \varphi_v(u_0^*) x \varphi_v(u_0) x^* = 0.$$

Consequently, Lemma C.2 gives the contradiction that M_{V_0} is not amenable relative to $M_{\{w,u\}}$. If instead w is not adjacent to u, then we define $u_1 \coloneqq x u_2 x^*$ and argue as above to get $E_{M_u}[u_1] = E_{M_u}[u_2^* u_1] = 0$, which once again gives a contradiction via Lemma C.2. Thus we must have that both v and w are adjacent to u, establishing (2b).

§6.2. Amenability by way of relative amenability

In this section we characterize when a graph product $(M, \varphi) = \star_{v\in\mathcal{G}} (M_v, \varphi_v)$ is amenable by specializing to the case where $V_1 = \mathcal{V}$ and $V_2 = \varnothing$. The characterization can be read off from Theorem 0.3, but in fact, we claim that this characterization holds even *without* the assumption that $M_v^{\varphi_v}$ contains a state zero unitary.

Proposition 6.3. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph, let $\{(M_v, \varphi_v) : v \in \mathcal{V}\}$ be a family of statial von Neumann algebras, and let $(M, \varphi) = \mathop{\ast}_{v \in \mathcal{G}} (M_v, \varphi_v)$. Assume $\dim M_v \geq 2$. Then M is amenable if and only if the following conditions hold:*

(1) *For each $v \in \mathcal{V}$, M_v is amenable.*
(2) *If v and w are not adjacent in $\mathcal{G}$, then $\dim(M_v) = \dim(M_w) = 2$ and v and w are adjacent to all the other vertices.*

Proof. First, suppose that (1) and (2) hold. Let $\mathcal{G} = \mathcal{G}_1 + \cdots + \mathcal{G}_n$ be the graph join decomposition of $\mathcal{G}$. We claim that each $\mathcal{G}_j$ is either a single vertex or a pair of non-adjacent vertices. Indeed, if v is a vertex in $\mathcal{G}_j$ that is adjacent to all other vertices in $\mathcal{G}_j$, then it is adjacent to all vertices in $\mathcal{G}$ and hence $\mathcal{G}_j = (\{v\}, \varnothing)$. Otherwise, there exists another vertex w in $\mathcal{G}_j$ that is not adjacent to v. But then (2) implies $\mathcal{G}_j = (\{v, w\}, \varnothing)$. Writing $(N_j, \psi_j) = \mathop{\ast}_{v \in \mathcal{G}_j} (M_v, \varphi_v)$, we have

$$(M, \varphi) \cong (N_1, \psi_1) \mathbin{\bar{\otimes}} \cdots \mathbin{\bar{\otimes}} (N_n, \psi_n).$$

If $\mathcal{G}_j$ has one vertex, then (N_j, ψ_j) is amenable by (1). If $\mathcal{G}_j$ has two vertices, then $(N_j, \psi_j) = (M_v, \varphi_v) * (M_w, \varphi_w)$, where $\dim(M_v) = \dim(M_w) = 2$ by (2), and hence is amenable by [Chi73, Thm. 2] and [Ued11, Rem. 4.2]. Thus, M is amenable as a tensor product of amenable von Neumann algebras.

Conversely, suppose that M is amenable. Recall that for any $U \subseteq \mathcal{V}$, there is a faithful normal conditional expectation from M onto M_U, so that the amenability of M implies the amenability of M_U. In particular, (1) holds since $M_{\{v\}} = M_v$ is amenable for each $v \in V$. Next, consider two non-adjacent vertices v and w. Then $M_{\{v,w\}} = M_v * M_w$ is amenable, and therefore by [Chi73, Thm. 2], [Ued11, Rem. 4.2] one must have $\dim(M_v) = \dim(M_w) = 2$. Suppose towards a contradiction that there is some vertex $u \in \mathcal{V} \setminus \{v, w\}$ that is, without loss generality, not adjacent to w. Then $M_{\{u,v,w\}}$ is the free product of $M_u \vee M_v$ and M_w with respect to the appropriate states. Since $\dim(M_u \vee M_v) \geq 3$ and $\dim(M_w) \geq 2$, $M_{\{u,v,w\}}$ is not amenable by [Chi73, Thm. 2] and [Ued11, Rem. 4.2], a contradiction. Therefore, (2) holds. ☐

Appendix A. Unitaries with state zero

For many of our results, it will be convenient to assume that the statial von Neumann algebras attached to the vertices have a unitary in the centralizer algebra with state zero. We note that a related assumption has appeared in [Bar95, Thm. 2

and Lem. 3] which provides sufficient conditions for a free product of statial von Neumann algebras to be a (possibly type III) factor. In this section we give a complete characterization of when this occurs and then explore this characterization in a few examples. This characterization is likely folklore, but as we are unable to find a citation in the literature we feel that it is useful to include it for completeness. We start with the tracial case, for which we will need the following two lemmas.

Lemma A.1. *For a tracial von Neumann algebra (M, τ), there exists a $u \in \mathcal{U}(M)$ with $\tau(u) = \inf |\tau(\mathcal{U}(M))|$.*

Proof. Write $M = M_1 \oplus M_2$ with M_1 atomic and M_2 diffuse. Set $K_i = \tau(\mathcal{U}(M_i))$ for $i = 1, 2$. Since M_1 is atomic and finite, we have that $\mathcal{U}(M_1)$ is SOT-compact, so $K_1 = \tau(\mathcal{U}(M_1))$ is compact. Since M_2 is diffuse, there is an embedding of $L^\infty([0,1])$ into M_2 which pulls back $\tau|_{M_2}$ to $\tau(1_{M_2})$ times integration against Lebesgue measure. This implies that $K_2 = \tau(\mathcal{U}(M_2)) = \{z \in \mathbb{C} : |z| \leq \tau(1_{M_2})\}$, so K_2 is also compact. Thus

$$\tau(\mathcal{U}(M)) = \big\{z + w : z \in K_1,\ w \in K_2\big\}$$

is the image of the compact space $K_1 \times K_2$ under a continuous map, and so $\tau(\mathcal{U}(M))$ is compact. The lemma thus follows from continuity of the absolute value map. □

Lemma A.2. *Suppose we have tracial von Neumann algebras $(A_i, \tau_i)_{i=1}^n$ and we equip $A = A_1 \oplus \cdots \oplus A_n$ with the trace*

$$\tau((a_i)_{i=1}^n) = \sum_{i=1}^n \alpha_i \tau_i(a_i),$$

where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \geq 0$ and $\sum_{i=1}^n \alpha_i = 1$. Denote $s := \inf |\tau_1(\mathcal{U}(A_1))|$. Then $|\tau(\mathcal{U}(A))| = [(\alpha_1(1+s) - 1) \vee 0, 1]$.

Proof. Let $u \in \mathcal{U}(A)$ and denote $u = u_1 + u_2$ where $u_1 \in A_1$ and $u_2 \in \bigoplus_{i \geq 2} A_i$. Then

$$\begin{aligned}|\tau(u)| &\geq |\tau(u_1)| - |\tau(u_2)| \\ &\geq \alpha_1 s - (\alpha_2 + \cdots + \alpha_k) = \alpha_1 s - (1 - \alpha_1).\end{aligned}$$

Hence we have $|\tau(u)| \geq (\alpha_1(s+1) - 1) \vee 0$ and $|\tau(\mathcal{U}(A))| \subset [(\alpha_1(s+1) - 1) \vee 0, 1]$.

We prove the reverse inclusion by induction on n. Since $\mathcal{U}(A) = \exp(iA_{\text{s.a.}})$, where $A_{\text{s.a.}}$ are the self-adjoint elements of A, is SOT-connected, we have that $|\tau(\mathcal{U}(A))|$ is connected. The case $n = 1$ thus follows by connectedness of $|\tau(\mathcal{U}(A))|$ and Lemma A.1. We now assume the result true for $n - 1$ with $n \geq 2$. We split into cases, where in the first case we assume $\alpha_1 \geq 1/2$ and thus $\frac{1-\alpha_1}{\alpha_1} \leq 1$. Let

$u_1 \in \mathcal{U}(A_1)$ with $\tau_1(u_1) = s \vee (\frac{1-\alpha_1}{\alpha_1})$, which exists since $|\tau_1(\mathcal{U}(A_1))|$ is connected and contains 1. Let $u = (u_1, -1, \ldots, -1) \in \mathcal{U}(A)$ so that

$$\tau(u) = \alpha_1 \Big(s \vee \Big(\frac{1-\alpha_1}{\alpha_1} \Big) \Big) - (1-\alpha_1) = (\alpha_1(1+s) - 1) \vee 0.$$

Using connectedness again, the claim follows. In the second case we assume $\alpha_1 < 1/2$, which we note implies $(\alpha_1(1+s)-1) \vee 0 = 0$. Equip $\bigoplus_{i \geq 2} A_i$ with the trace $\tau'(a) = \frac{1}{1-\alpha_1}\tau(0 \oplus a)$ and denote $s' := \inf |\tau_2(\mathcal{U}(A_2))|$. Note that our inductive hypothesis implies

$$\Big|\tau\Big(\mathcal{U}\Big(\bigoplus_{i\geq 2} A_i\Big)\Big)\Big| = \Big[\Big(\frac{\alpha_2}{1-\alpha_1}(1+s') - 1\Big) \vee 0, 1\Big].$$

Observe that

$$\frac{\alpha_2}{1-\alpha_1}(1+s') - 1 \leq \frac{2\alpha_2}{1-\alpha_1} - 1 \leq \frac{2\alpha_1}{1-\alpha_1} - 1 < \frac{1}{1-\alpha_1} - 1 = \frac{\alpha_1}{1-\alpha_1} < 1,$$

with the last inequality following as $\alpha_1 < 1/2$. Thus we can find $v \in \mathcal{U}(\bigoplus_{i\geq 2} A_i)$ with $\tau'(v) = \frac{-\alpha_1}{1-\alpha_1}$. Then $\tau(1 \oplus v) = 0$. □

We now obtain a complete characterization of when a statial von Neumann algebra has a state zero unitary in its centralizer.

Corollary A.3. *Let (M, φ) be a statial von Neumann algebra.*

(i) *Suppose φ is a trace and that there exists a non-zero minimal projection $p \in M$ with $\varphi(p) > 1/2$. Then p is central and $\varphi(u) \neq 0$ for every $u \in \mathcal{U}(M)$.*
(ii) *If φ is a trace, then there is a $u \in \mathcal{U}(M)$ with $\varphi(u) = 0$ if and only if $\varphi(p) \leq 1/2$ for every minimal projection $p \in M$.*
(iii) *There exists $u \in \mathcal{U}(M^\varphi)$ with $\varphi(u) = 0$ if and only if $\varphi(p) \leq 1/2$ for every minimal projection $p \in M^\varphi$.*

Proof. (i). Suppose that M has a minimal non-zero projection p with $\varphi(p) > 1/2$. Let z be the central support of p in M. By [KR97, Prop. 6.4.3 and Cor. 6.5.3], we have that Mz is isomorphic to $M_k(\mathbb{C})$ for some k. Since p is a minimal projection and φ is a trace, it follows that $\varphi(z) = k\varphi(p)$. Since $\varphi(p) > 1/2$, this forces $k = 1$. Thus $p = z$ is central.

Since $\mathbb{C}p$ is a central summand of M, using the notation of Lemma A.2 we have $\alpha_1 > 1/2$ and $s = 1$. Thus this lemma implies $|\varphi(u)| \geq \alpha_1(1+s) - 1 > 0$ for all $u \in \mathcal{U}(M)$.

(ii). The forward implication follows from (i). For the reverse implication, suppose that M does not have a unitary of trace zero. By decomposing the center into diffuse and atomic parts, we may write

$$M = M_0 \oplus \bigoplus_{i \in I} M_i,$$

where

- I is a countable set (potentially empty),
- each M_i is a non-zero finite factor,
- M_0 is either 0 or has diffuse center.

For $j \in I \sqcup \{0\}$ let $\alpha_j := \varphi(1_{M_j})$ and $\tau_j := \frac{1}{\alpha_j}\varphi|_{M_j}$ (note that $\alpha_0 = 0$ if $M_0 = \{0\}$). Since $\sum_i \alpha_i \leq 1$ we have that either I is finite or $\alpha_i \to 0$ as $i \to \infty$ (i.e. as i escapes all finite subsets of I). Thus there is a $j_0 \in I \sqcup \{0\}$ with $\alpha_{j_0} = \max\{\alpha_i : i \in I \sqcup \{0\}\}$. Denote $s_j := \inf |\tau_j(\mathcal{U}(M_j))|$ for each $j \in I \sqcup \{0\}$. By Lemma A.2, our hypothesis implies that

$$\alpha_{j_0}(1 + s_{j_0}) > 1.$$

This inequality implies that $s_{j_0} \neq 0$. On other hand, if $M_0 \neq 0$, then $s_0 = 0$ and for $i \in I$ if M_i is a factor of dimension at least 2, then $s_i = 0$ as well. So necessarily $M_{j_0} \cong \mathbb{C}1$ and $s_{j_0} = 1$. But then the above inequality implies $\alpha_{j_0} > 1/2$ and this proves that $1_{M_{j_0}} \in M$ is a non-zero minimal projection with $\varphi(1_{M_{j_0}}) = \alpha_{j_0} > 1/2$.

(iii). This follows from (ii), since $\varphi|_{M^\varphi}$ is a trace. $\square$

We now list a few examples of algebras with state zero unitaries in the centralizer. Let us first consider the matrix algebra case. Suppose that φ is a state on $M_n(\mathbb{C})$. Then we can write

$$\varphi(x) = \mathrm{tr}(xa),$$

for some $a \in M_n(\mathbb{C})_+$, where tr is the normalized trace on $M_n(\mathbb{C})$. Let

$$a = \sum_{j=1}^{k} \lambda_j 1_{\{\lambda_j\}}(a)$$

be the spectral decomposition of a with $\lambda_i \neq \lambda_j$ for all $i \neq j$. Set $p_j = 1_{\lambda_j}(a)$. Then p_j is central in M^φ and $M^\varphi p_j \cong M_{n\,\mathrm{tr}(p_j)}(\mathbb{C})$. Suppose $e \in M^\varphi$ is a minimal projection. Then we can find a unique j so that $ep_j = e$. In this case,

$$\varphi(e) = \varphi(ep_j) = \mathrm{tr}_n(ep_j a) = \lambda\,\mathrm{tr}_n(e) = \frac{\lambda}{n}.$$

Hence, M^φ has a state zero unitary if and only if

$$\lambda \leq \frac{n}{2}$$

for every eigenvalue λ of a.

For general finite-dimensional M, we may find central projections $z_1, \ldots, z_k$ in M with $\sum_{j=1}^k z_j = 1$ and $Mz_j \cong M_{n_j}(\mathbb{C})$. Let τ_j be the normalized trace on Mz_j. Then we can find $a_j \in (Mz_j)_+$ with $\sum_j \varphi(z_j) \operatorname{tr}_j(a_j) = 1$ and

$$\varphi(x) = \sum_{j=1}^n \varphi(z_j)\tau_j(xz_ja_j).$$

In this case,

$$M^\varphi = \sum_{j=1}^n (Mz_j)^{\tau_j(\cdot a_j)}.$$

If $e \in M^\varphi$ is a minimal projection, choose j so that $ez_j = e$. Then by the above, there is a λ_j in the spectrum of a_j with

$$\varphi(e) = \varphi(z_j)\frac{\lambda_j}{n_j}.$$

Hence, M^φ has a state zero unitary if and only if for every j we have

$$\lambda\varphi(z_j) \leq \frac{n_j}{2}$$

for every eigenvalue λ of a_j viewed as an operator on $\mathbb{C}^{n_j}$.

Another example is group von Neumann algebras, equipped with their trace $\tau\colon L(G) \to \mathbb{C}$ given by $\tau(\lambda_g) = \delta_{g=e}$. In this case, any non-trivial group element satisfies the hypotheses. Another example would be if M^φ is diffuse (e.g. φ is a trace and M is diffuse). In this case, there is a state-preserving embedding of $L^\infty([0,1])$ into M^φ and so there is a state zero unitary.

Appendix B. Ocneanu ultrapowers

For a cofinal ultrafilter ω on a directed set I and a von Neumann algebra M, denote

$$I_\omega(M) := \big\{(x_i)_{i\in I} \in \ell^\infty(I, M) : \lim_{i\to\omega} x_i = 0 \text{ in the strong-}{*} \text{ topology}\big\},$$

$$\mathcal{M}^\omega(M) := \big\{(x_i)_{i\in I} \in \ell^\infty(I, M) : (x_i)_i I_\omega(M) + I_\omega(M)(x_i)_i \subseteq I_\omega(M)\big\}.$$

By [Ocn85, AH14] the quotient C^*-algebra

$$M^\omega := \mathcal{M}^\omega(M)/I_\omega(M)$$

is a von Neumann algebra, which we call the *Ocneanu ultrapower of* M. For $(x_i)_i \in \mathcal{M}^\omega(M)$ we use $(x_i)_{i\to\omega}$ for its image in M^ω. Suppose that P is a subalgebra of M and that there is a faithful normal conditional expectation $E_P\colon M \to P$. In this case, P^ω is naturally a von Neumann subalgebra of M^ω and there is a natural conditional expectation E_{P^ω} given by

$$E_{P^\omega}((x_i)_{i\to\omega}) = (E_P(x_i))_{i\to\omega}$$

(see [HI17, Sect. 2] for details). Applying this with $P = \mathbb{C}$, we see that if φ is a faithful normal state on M, then the ultraproduct state φ^ω given by

$$\varphi^\omega((x_i)_{i\to\omega}) = \lim_{i\to\omega} \varphi(x_i)$$

remains faithful. This relates to fullness, since [AH14, Thm. 5.2] and [HMV19, Cor. 3.7] show that if M is a σ-finite von Neumann algebra, then $M' \cap M^\omega = \mathbb{C}$ if and only if M is full. The following is a statial version of [Ioa15, Lem. 6.1].

Lemma B.1. *Let (B,φ) be a statial von Neumann algebra and let $B \subset M_i$ be an inclusion with expectation $E_i\colon M \to B$ for $i = 1,2$. Denote $\varphi_i := \varphi \circ E_i$ for $i = 1,2$ and consider the amalgamated free product $(M, E_B) := (M_1, E_1) *_B (M_2, E_2)$. If there exist unitary elements $u_1 \in (M_1)^{\varphi_1}$ and $u_2, u_3 \in (M_2)^{\varphi_2}$ such that*

$$E_B[u_1] = E_B[u_2] = E_B[u_3] = E_B[u_2^* u_3] = 0,$$

then $M' \cap M^\omega \subseteq B^\omega$ for any cofinal ultrafilter ω on a directed set I.

Proof. For $i = 1,2$ let $L^2_0(M_i) := L^2(M_i,\varphi_i) \ominus L^2(B,\varphi)$, and observe that this is the closure of $\{x \in M_i : E_i(x) = 0\}$ in $L^2(M_i,\varphi_i)$. By definition,

$$L^2(M, \varphi\circ E_B) = L^2(B,\varphi) \oplus \bigoplus_{d=1}^{\infty}\Bigg(\bigoplus_{\substack{i_1\neq i_2,\ldots,\\ i_{d-1}\neq i_d}} L^2_0(M_{i_1}) \otimes_B \cdots \otimes_B L^2_0(M_{i_d})\Bigg).$$

Let P_i be the orthogonal projection onto

$$\mathcal{H}_i := \bigoplus_{d=1}^{\infty}\Bigg(\bigoplus_{\substack{i=i_1\neq i_2,\ldots,\\ i_{d-1}\neq i_d}} L^2_0(M_{i_1}) \otimes_B \cdots \otimes_B L^2_0(M_{i_d})\Bigg).$$

Note that $u_1, u_2, u_3 \in M^{\varphi\circ E_B}$ since the modular automorphism group of $\varphi \circ E_B$ restricts to that of φ_i on M_i for each $i = 1,2$. Thus,

$$\|x u_i^*\|_2 = \|x\|_2$$

for all $x \in M$. So right multiplication by u_i^* extends to a bounded operator on $L^2(M, \varphi \circ E_B)$, and we will continue to write ξu_i^* for the image of $\xi \in L^2(M, \varphi \circ E_B)$ under this operator. As in [Ioa15, Lem. 6.1], we have

$$u_1 \mathcal{H}_2 u_1^* \subseteq \mathcal{H}_1, \quad u_2 \mathcal{H}_1 u_2^* \subseteq \mathcal{H}_2, \quad u_3 \mathcal{H}_1 u_3^* \subseteq \mathcal{H}_2,$$

and

$$u_2 \mathcal{H}_1 u_2^* \perp (\mathcal{H}_1 + u_3 \mathcal{H}_1 u_3^*).$$

Let P_i be the orthogonal projection onto $\mathcal{H}_i$, $i = 1, 2$. Note that if $\mathcal{K} \subseteq L^2(M)$ is a closed linear subspace, and $P_{\mathcal{K}}$ is the orthogonal projection onto $\mathcal{K}$, then

$$P_{u_i \mathcal{K} u_i^*}(\cdot) = u_i P_{\mathcal{K}}(u_i^* \cdot u_i) u_i^*.$$

Hence we can argue as in [Ioa15, Lem. 6.1] to see that

$$\|P_2(u_1 \xi u_1^*)\|_2 \leq \|P_1(\xi)\|_2 \quad \text{and} \quad \|P_1(u_2 \xi u_2^*)\|_2^2 + \|P_1(u_3 \xi u_3^*)\|_2^2 \leq \|P_2(\xi)\|_2^2,$$

for all $\xi \in L^2(M, \varphi \circ E_B)$. Now let $(x_i)_{i \to \omega} \in M' \cap M^\omega$. Since $\varphi \circ E_B$ is faithful, the strong*-topology on the unit ball of M coincides with convergence with respect to $\|x\|_2 + \|x^*\|_2$ (see [Tak02, Prop. III.5.3]). We now argue exactly as in [Ioa15, Lem. 6.1] to obtain the estimates

$$\lim_{i \to \omega} \|P_2(x_i)\|_2 \leq \lim_{i \to \omega} \|P_1(x_i)\|_2 \leq \frac{1}{\sqrt{2}} \lim_{i \to \omega} \|P_2(x_i)\|_2,$$

so that $\lim_{i \to \omega} \|P_j(x_i)\|_2 = 0$ for $j = 1, 2$. Since $L^2(M, \varphi \circ E_B) = L^2(B, \varphi) \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$, we obtain that $\lim_{i \to \omega} \|x_i - E_B(x_i)\|_2 = 0$. By the same argument $\lim_{i \to \omega} \|x_i^* - E_B(x_i^*)\|_2 = 0$, and hence $(x_i)_{i \to \omega} \in B^\omega$. $\square$

In order for intersections to commute with ultrapowers, it is sufficient to have commuting square inclusions of algebras, as we now show. This is a folklore result, but we give the proof for completeness.

Lemma B.2. *Suppose that (M, φ) is a statial von Neumann algebra and that M_1, M_2 are von Neumann subalgebras with φ-preserving normal conditional expectations $E_i \colon M \to M_i$. Suppose further that $E_1 \circ E_2 = E_2 \circ E_1$ so that*

$$\begin{tikzcd} M_1 \arrow[d] & M \arrow[l] \arrow[d] \\ M_1 \cap M_2 & M_2 \arrow[l] \end{tikzcd}$$

forms a commuting square. Then $(M_1 \cap M_2)^\omega = M_1^\omega \cap M_2^\omega$.

Proof. Since $E_{M_i^\omega} = (E_i)^\omega$, $i = 1, 2$, and similarly $E_{(M_1\cap M_2)^\omega} = (E_{M_1\cap M_2})^\omega$, it is enough to show that $E_{M_1}|_{M_2} = E_{M_1\cap M_2}$. But this follows from the fact that $E_{M_1} \circ E_{M_2} = E_{M_2} \circ E_{M_1}$. □

Appendix C. Relative amenability

The following result of Monod–Popa ([MP03, Rem. 3]) allows one to restrict to certain subalgebras when checking relative amenability. We reproduce the well-known proof here.

Lemma C.1. *Suppose that (M, φ) is a statial von Neumann algebra and that M_1, M_2 are von Neumann subalgebras with φ-preserving normal conditional expectations $E_i\colon M \to M_i$. Suppose further that $E_1 \circ E_2 = E_2 \circ E_1$ so that*

$$\begin{tikzcd} M_1 \arrow[d] & M \arrow[l] \arrow[d] \\ M_1 \cap M_2 & M_2 \arrow[l] \end{tikzcd}$$

forms a commuting square. If M_1 is amenable relative to $M_1 \cap M_2$ (inside M_1), then M_1 is amenable relative to M_2 inside M.

Proof. Let $F\colon \langle M_1, e_{M_1\cap M_2}\rangle \to M_1$ be a conditional expectation. We will first construct a conditional expectation $E\colon \langle M, e_{M_1\cap M_2}\rangle \to M_1$ that is normal on M, so that M_1 is amenable relative to $M_1 \cap M_2$ inside M.

Identify $L^2(M_1, \varphi|_{M_1})$ as a subspace of $L^2(M, \varphi)$ so that the projection onto it is the Jones projection e_{M_1} for the inclusion $(M_1 \subset M, E_1)$. Similarly, the Jones projection $e_{M_1\cap M_2}$ is given by the projection onto the identification of $L^2(M_1 \cap M_2, \varphi|_{M_1\cap M_2})$ as a subspace of $L^2(M, \varphi)$. Recall that the basic construction for this inclusion satisfies

$$\langle M, e_{M_1\cap M_2}\rangle = (J_\varphi(M_1 \cap M_2)J_\varphi)' \cap B(L^2(M, \varphi)). \tag{C.1}$$

Consequently, $e_{M_1} \in \langle M, e_{M_1\cap M_2}\rangle$ since $E_{M_1\cap M_2} = E_1 \circ E_2$ is φ-preserving. Additionally, if we define $\Upsilon\colon B(L^2(M, \varphi)) \to B(L^2(M_1, \varphi|_{M_1}))$ by $\Upsilon(T) := e_{M_1} T e_{M_1}$, then $\Upsilon(\langle M, e_{M_1\cap M_2}\rangle) = \langle M_1, e_{M_1\cap M_2}\rangle$.[1] Indeed, it is a general fact that if $Q \le B(\mathcal{H})$ is a von Neumann algebra and $p \in Q' \cap B(\mathcal{H})$ is a projection, then $(Qp)' \cap B(p\mathcal{H}) = p(Q' \cap B(\mathcal{H}))p$. Applying this to $Q = J_\varphi(M_1 \cap M_2)J_\varphi$ and

[1] Here we are abusing notation to let $e_{M_1\cap M_2}$ in the second instance also denote the Jones projection for the inclusion $M_1 \cap M_2 \subset M_1$. But under the identification $B(L^2(M_1, \varphi|_{M_1})) = e_{M_1} B(L^2(M, \varphi))$ one does have $e_{M_1\cap M_2} e_{M_1} = e_{M_1\cap M_2}$.

$p = e_{M_1}$ gives

$$\begin{aligned}\Upsilon(\langle M, e_{M_1\cap M_2}\rangle) &= \Upsilon(J_\varphi(M_1\cap M_2)J_\varphi)' \cap B(L^2(M_1, \varphi|_{M_1}))\\ &= (J_{\varphi|_{M_1}}\Upsilon(M_1\cap M_2)J_{\varphi|_{M_1}})' \cap B(L^2(M_1, \varphi|_{M_1}))\\ &= (J_{\varphi|_{M_1}}(M_1\cap M_2)J_{\varphi|_{M_1}})' \cap B(L^2(M_1, \varphi|_{M_1}))\\ &= \langle M_1, e_{M_1\cap M_2}\rangle.\end{aligned}$$

Now, let $F\colon \langle M_1, e_{M_1\cap M_2}\rangle \to M_1$ be a conditional expectation with $F|_{M_1}$ normal, which is guaranteed by M_1 being amenable relative to $M_1\cap M_2$ inside M_1. Define $E\colon \langle M, e_{M_1\cap M_2}\rangle \to M_1$ by $E := F\circ\Upsilon$. For $x\in M$ we have $E(x) = F(\Upsilon(x)) = \Upsilon(x)$, and if $x\in M_1$ then one further has $E(x) = x$. Thus E is a conditional expectation onto M_1 with $E|_M$ normal.

To complete the proof of the lemma, identify $L^2(M_2, \varphi|_{M_2})$ as a subspace of $L^2(M,\varphi)$ and let e_{M_2} be the associated Jones projection for the inclusion $(M_2\subset M, E_2)$. Then (C.1) implies $\langle M, e_{M_2}\rangle \leq \langle M, e_{M_1\cap M_2}\rangle$, and so considering the restriction of E to this subalgebra gives that M_1 is amenable relative to M_2 inside M. □

The next result provides a sufficient condition for preventing amalgamated free products from being amenable relative to either of the factors; see e.g. [Oza06, DKEP23] for similar arguments.

Lemma C.2. *Let (B,φ) be a statial von Neumann algebra and let $B\subset M_i$ be an inclusion with expectation $E_i\colon M\to B$ for $i=1,2$. Denote $\varphi_i := \varphi\circ E_i$ for $i=1,2$ and consider the amalgamated free product $(M,E_B) := (M_1,E_1) *_B (M_2,E_2)$. If there exist unitary elements $u_0\in(M_1)^{\varphi_1}$, $u_1\in M_1$, and $u_2\in(M_2)^{\varphi_2}$ such that*

$$E_B[u_0] = E_B[u_1] = E_B[u_0^* u_1] = E_B[u_2] = 0,$$

then M is not amenable relative to M_i for $i=1,2$.

Proof. Define $\psi := \varphi\circ E_B$. For each $j=1,2$, denote $H_j := L^2(M_j,\varphi_j)$, which we identify as a subspace of $L^2(M,\psi)$. Also identify $L^2(B,\varphi)$ as a subspace of $L^2(M,\psi)$ and denote $H_j^\circ := H_j\ominus L^2(B,\varphi)$, $j=1,2$. Recall that

$$L^2(M,\psi) = \bigoplus_{d\in\mathbb{N}}\bigoplus_{i_1\neq\cdots\neq i_d} H_{i_1}^\circ\otimes_B\cdots\otimes_B H_{i_d}^\circ.$$

Denote

$$K := \bigoplus_{d\in\mathbb{N}}(H_1^\circ\otimes_B H_2^\circ)^{\otimes_B d}\otimes_B H_1,$$

so that as right M_1-modules we have

$$L^2(M,\psi)_{M_1} \cong K_{M_1} \oplus (H_2^\circ \otimes_B K)_{M_1}.$$

In particular, we can identify $K^\perp$ with $H_2^\circ \otimes_B K$. By assumption, $u_0, u_1 \in H_1^\circ$ so that $u_0 K^\perp, u_1 K^\perp \leq K$, and since $u_0^* u_1 \in H_1^\circ$ we further have that $u_0 K^\perp \perp u_1 K^\perp$. Thus if $P_K, P_{K^\perp} \in B(L^2(M,\psi))$ denote the projections onto K and $K^\perp$, respectively, then

$$u_0 P_{K^\perp} u_0^* + u_1 P_{K^\perp} u_1^* \leq P_K.$$

Additionally, $u_2 \in H_2^\circ$ implies $u_2 K \leq K^\perp$ so that

$$u_2 P_K u_2^* \leq P_{K^\perp}. \tag{C.2}$$

Now, suppose, towards a contradiction, that there exists a conditional expectation $\Phi\colon \langle M, e_{M_1}\rangle \to M$. Note that $P_K, P_{K^\perp} \in \langle M, e_{M_1}\rangle$ since K and $K^\perp$ are invariant for $J_\psi M_1 J_\psi$, and so the above inequalities imply

$$\psi(u_0 \Phi(P_{K^\perp}) u_0^*) + \psi(u_1 \Phi(P_{K^\perp}) u_1^*) \leq \psi(\Phi(P_K))$$

and

$$\psi(u_2 \Phi(P_K) u_2^*) \leq \psi(\Phi(P_{K^\perp})).$$

Recall that $u_0 \in (M_1)^{\varphi_1}$ and $u_2 \in (M_2)^{\varphi_2}$ so that $u_0, u_2 \in M^\psi$ and hence

$$\begin{aligned}
\psi(\Phi(P_{K^\perp})) + \psi(u_1 \Phi(P_{K^\perp}) u_1^*) &= \psi(u_0 \Phi(P_{K^\perp}) u_0^*) + \psi(u_1 \Phi(P_{K^\perp}) u_1^*) \\
&\leq \psi(\Phi(P_K)) \\
&= \psi(u_2 \Phi(P_k) u_2^*) \\
&\leq \psi(\Phi(P_{K^\perp})).
\end{aligned}$$

Hence, $\psi(u_1 \Phi(P_{K^\perp}) u_1^*) = 0$, and therefore $\Phi(P_{K^\perp}) = 0$. Since (C.2) implies

$$u_2 \Phi(P_K) u_2^* \leq \Phi(P_{K^\perp}) = 0,$$

we also have $\Phi(P_K) = 0$. But this leads to the contradiction

$$\Phi(1) = \Phi(P_K) + \Phi(P_{K^\perp}) = 0.$$

Thus M is not amenable relative to M_1.

To see that M is not amenable relative to M_2, denote

$$L := \bigoplus_{d\in\mathbb{N}} (H_2^\circ \otimes_B H_1^\circ)^{\otimes_B d} \otimes_B H_2$$

so that as right M_2-modules we have

$$L^2(M,\psi)_{M_2} \cong L_{M_2} \oplus (H_1^\circ \otimes_B L)_{M_2}.$$

Then u_0L and u_1L are orthogonal subspaces in $L^\perp = H_1^\circ \otimes_B L$, and $u_2L^\perp \leq L$. So if one assumes there exists a conditional expectation from $\langle M, e_{M_2}\rangle$ to M, then we can proceed as above to obtain a contradiction. □

In contrast to the previous lemma, the next result shows that tensoring relatively amenable inclusions yields a relatively amenable inclusion.

Lemma C.3. *For $i = 1, 2$, let $N_i \leq M_i$ be an inclusion of von Neumann algebras admitting faithful normal conditional expectations $E_i\colon M_i \to N_i$. If M_i is amenable relative to N_i for each $i = 1, 2$, then $M_1 \bar{\otimes} M_2$ is amenable relative to $N_1 \bar{\otimes} N_2$.*

Proof. By Tomita's commutation theorem [Tak02, Thm. IV.5.9], we have a canonical isomorphism

$$\langle M_1 \bar{\otimes} M_2, e_{N_1\bar{\otimes}N_2}\rangle \cong \langle M_1, e_{N_1}\rangle \bar{\otimes} \langle M_2, e_{N_2}\rangle$$

satisfying

$$(x_1 \otimes x_2)e_{N_1\bar{\otimes}N_2}(x_2 \otimes y_2) \mapsto (x_1e_{N_1}y_1) \otimes (x_2e_{N_2}y_2).$$

By assumption, there are conditional expectations $\Phi_i\colon \langle M_i, e_{N_i}\rangle \to M_i$ for $i = 1, 2$. We would like to obtain a conditional expectation $\langle M_1 \bar{\otimes} M_2, e_{N_1\bar{\otimes}N_2}\rangle \to M_1 \bar{\otimes} M_2$ as $(\Phi_1 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Phi_2)$, but since the expectations Φ_1 and Φ_2 are not normal, it is not immediately clear how to extend $\Phi_1 \otimes \mathrm{id}$ and $\mathrm{id} \otimes \Phi_2$ from the algebraic tensor product $\langle M_1, e_{N_1}\rangle \odot \langle M_2, e_{N_2}\rangle$ to $\langle M_1 \bar{\otimes} M_2, e_{N_1\bar{\otimes}N_2}\rangle$. However, one can accomplish something similar using the following more abstract claim.

Claim. Let $B \leq S$ be von Neumann algebras and let $\mathcal{H}$ be any Hilbert space. If $\Phi\colon S \to B$ is a conditional expectation, then there is a conditional expectation $\widetilde{\Phi}\colon S \bar{\otimes} B(\mathcal{H}) \to B \bar{\otimes} B(\mathcal{H})$. Moreover, if $T \leq B(\mathcal{H})$ is a von Neumann algebra, then $\widetilde{\Phi}|_{S\bar{\otimes}T}$ is a conditional expectation onto $B \bar{\otimes} T$.

To prove the claim, view $S \subseteq B(\mathcal{K})$ and let $(e_i)_{i\in I}$ be an orthonormal basis for $\mathcal{H}$. Let $\omega_{i,j} \in B(\mathcal{H})$ denote the rank-one operator $\omega_{i,j}(\xi) := \langle \xi, e_j\rangle e_i$. For $A \in S \bar{\otimes} B(\mathcal{H})$, we wish to write $A = \sum_{i,j} A_{i,j} \otimes \omega_{i,j}$ and then to define $\widetilde{\Phi}(A)$ as

$$\sum_{i,j\in I} \Phi(A_{i,j}) \otimes \omega_{i,j}.$$

More precisely, $\sum_{i,j} A_{i,j} \otimes \omega_{i,j}$ should be interpreted as

$$\lim_{F\Subset I} \sum_{i,j\in F} A_{i,j} \otimes \omega_{i,j},$$

where the limit is over the directed system of finite subsets of I, which converges in SOT to A since the projections $p_F := \sum_{i\in F} 1 \otimes \omega_{i,i}$ converge to 1 in SOT.

To show convergence of the sum for defining $\widetilde{\Phi}(A)$, observe that

$$\Big\| \sum_{i,j\in F} \Phi(A_{ij}) \otimes \omega_{i,j} \Big\| \leq \Big\| \sum_{i,j\in F} A_{ij} \otimes \omega_{i,j} \Big\| = \Big\| p_F\Big(\sum_{i,j\in I} A_{ij} \otimes \omega_{i,j} \Big) p_F \Big\| \\ \leq \Big\| \sum_{i,j\in I} A_{ij} \otimes \omega_{i,j} \Big\|,$$

where the first inequality follows from Φ being completely bounded with $\|\Phi\|_{\mathrm{cb}} = 1$. Thus the sum defining $\widetilde{\Phi}(A)$ converges in the strong operator topology since the net $(\sum_{i,j\in F} \Phi(A_{ij}) \otimes \omega_{i,j})_{F\Subset I}$ converges pointwise on the dense subspace $\mathrm{span}\{\xi \otimes e_i : \xi \in \mathcal{K},\ i \in I\}$ and is uniformly bounded in norm. Now that $\widetilde{\Phi}(A)$ is well defined, it is easy to check that it is unital, completely positive, and B–B-bimodular from the corresponding properties of Φ.

For the second part of the claim, let $y \in T'$. Then since $1 \bar{\otimes} B(\mathcal{H})$ is in the multiplicative domain of $\widetilde{\Phi}$, we have that $1 \otimes y \in \widetilde{\Phi}(S \bar{\otimes} T)'$. Tomita's commutation theorem thus shows that

$$\widetilde{\Phi}(S \bar{\otimes} T) \subseteq (1 \otimes T')' \cap (B \bar{\otimes} B(\mathcal{H})) = B \bar{\otimes} T.$$

This proves the claim.

Applying the claim first to $\Phi_1 \colon \langle M_1, e_{N_1}\rangle \to M_1$ and $T = M_2$ yields a conditional expectation

$$\widetilde{\Phi}_1 \colon \langle M_1, e_{N_1}\rangle \bar{\otimes} M_2 \to M_1 \bar{\otimes} M_2.$$

Next, applying the claim (with the order of the tensorands flipped) to $\Phi_2 \colon \langle M_2, e_{N_2}\rangle \to M_2$ and $T = \langle M_1, e_{N_1}\rangle$ yields a conditional expectation

$$\widetilde{\Phi}_2 \colon \langle M_1, e_{N_1}\rangle \bar{\otimes} \langle M_2, e_{N_2}\rangle \to \langle M_1, e_{N_1}\rangle \bar{\otimes} M_2.$$

Hence, $\widetilde{\Phi}_1 \circ \widetilde{\Phi}_2$ is a conditional expectation that witnesses the relative amenability of $M_1 \bar{\otimes} M_2$ to $N_1 \bar{\otimes} N_2$. $\square$

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