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Analogues of Hyperlogarithm Functions on Affine Complex Curves

by

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Abstract

For C a smooth affine complex curve, there is a unique minimal unital subalgebra A_C of the algebra $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ of holomorphic functions on its universal cover $\widetilde{C}$, which is stable under all the operations $f \mapsto \int f\omega$, for ω in the space $\Omega(C)$ of regular differentials on C. We identify A_C with the image of the iterated integration map $I_{x_0}\colon \mathrm{Sh}(\Omega(C)) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ based at any point x_0 of $\widetilde{C}$ (here $\mathrm{Sh}(-)$ denotes the shuffle algebra of a vector space), as well as with the unipotent part, with respect to the action of $\mathrm{Aut}(\widetilde{C}/C)$, of a subalgebra of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ of moderate growth functions. We show that any regular Maurer–Cartan (MC) element J on C with values in the topologically free Lie algebra over $\mathrm{H}^1_{\mathrm{dR}}(C)^*$ gives rise to an isomorphism of A_C with $\mathcal{O}(C)\otimes\mathrm{Sh}(\mathrm{H}^1_{\mathrm{dR}}(C))$, where $\mathcal{O}(C)$ is the algebra of regular functions on C, leading to the assignment of a subalgebra $\mathcal{H}_C(J)$ of A_C (isomorphic to $\mathrm{Sh}(\mathrm{H}^1_{\mathrm{dR}}(C))$) to any MC element. We also associate an MC element J_σ to each section σ of the projection $\Omega(C) \to \mathrm{H}^1_{\mathrm{dR}}(C)$; when C has genus zero, we exhibit a particular section σ_0 for which $\mathcal{H}_C(J_{\sigma_0})$ is the algebra of hyperlogarithm functions (Poincaré, Lappo-Danilevsky).

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§1. Introduction

§1.1. The context

To an inclusion $\mathcal{O}\subset\widetilde{\mathcal{O}}$ of unital complex commutative algebras and a derivation ∂ of $\widetilde{\mathcal{O}}$ which is both surjective and with $\ker(\partial)=\mathbb{C}$, one may associate the smallest subalgebra of $\widetilde{\mathcal{O}}$ which contains $\mathcal{O}$ and is stable under the antiderivation operation $\widetilde{\mathcal{O}}\ni f\mapsto\partial^{-1}(f)\subset\widetilde{\mathcal{O}}$. Two instances of this construction were studied in detail in the literature:

- $\widetilde{\mathcal{O}}=\mathcal{O}_{\mathrm{hol}}(\mathfrak{H})$ is the algebra of holomorphic functions on the complex upper half-plane $\mathfrak{H}=\{\tau\in\mathbb{C}\mid\Im(\tau)>0\}$, $\mathcal{O}=\mathrm{QM}_*$ is the algebra of quasi-modular forms for $\mathrm{SL}_2(\mathbb{Z})$, and $\partial=d/d\tau$ (see [Ma]);
- $\mathcal{O}=\mathbb{C}[z,1/(z-s),s\in S_\infty]=:\mathcal{O}(\mathbb{P}^1_{\mathbb{C}}\smallsetminus S)$, where $S\subset\mathbb{P}^1_{\mathbb{C}}$ is a finite subset with $S\ni\infty$, S_∞ denotes $S\smallsetminus\{\infty\}$, $\widetilde{\mathcal{O}}$ is the algebra of holomorphic functions on a universal cover of $\mathbb{P}^1_{\mathbb{C}}\smallsetminus S$, and $\partial=d/dz$ (see [Br]).

In both cases, precise results were obtained on the structure of the said smallest subalgebra. Let us describe the results of the second case in more detail. In that case, the conditions on the looked-for algebra are equivalent to requiring it to be both unital and stable under all the operations $f\mapsto\int_{z_0}f\omega:=(z\mapsto\int_{z_0}^z f\omega)$, where ω runs over all the regular differentials on $\mathbb{P}^1_{\mathbb{C}}\smallsetminus S$; indeed, the latter condition implies that the algebra contains the functions $\int_{z_0}df$, where f runs over $\mathcal{O}(\mathbb{P}^1_{\mathbb{C}}\smallsetminus S)$.

Such an algebra necessarily contains all the iterated integrals of the differentials $d\log(z-s)$, $s\in S_\infty$, which are the hyperlogarithm (HL) functions L_w

indexed by $w \in \hat{S}_\infty^*$ (where $\hat{S}_\infty^* := \bigsqcup_{n \geq 0} \hat{S}_\infty^n$ is the set of words in $\hat{S}_\infty$, which is S_∞ viewed as an abstract set). The generating series $\mathbf{L} := \sum_w L_w \cdot w$ is a multivalued holomorphic function on $\mathbb{P}^1_\mathbb{C} \smallsetminus S$ with values in the group of group-like elements of the algebra of noncommutative formal series $\mathbb{C}\langle\langle \hat{S}_\infty \rangle\rangle$, such that $d\mathbf{L}(z) = \mathbf{L}(z) \cdot \sum_s (\hat{s} \cdot d\log(z-s))$. It was proven in [Br, Cor. 5.6] that

- the algebra $A_{\mathbb{P}^1_\mathbb{C} \smallsetminus S} := \mathcal{O}(\mathbb{P}^1_\mathbb{C} \smallsetminus S)[L_w, w \in \hat{S}_\infty^*]$ is stable under antiderivation, so that $A_{\mathbb{P}^1_\mathbb{C} \smallsetminus S}$ is the smallest (for the inclusion) extension of $\mathcal{O}(\mathbb{P}^1_\mathbb{C} \smallsetminus S)$ with this property;
- the map $\mathcal{O}(\mathbb{P}^1_\mathbb{C} \smallsetminus S) \otimes \mathrm{Sh}(\mathbb{C}\hat{S}_\infty) \to A_{\mathbb{P}^1_\mathbb{C} \smallsetminus S}$, $f \otimes w \mapsto f \cdot L_w$ is an algebra isomorphism, where $\mathrm{Sh}(V)$ is the shuffle algebra associated with a vector space V; in particular, the family $(L_w)_w$ is linearly independent over $\mathcal{O}(\mathbb{P}^1_\mathbb{C} \smallsetminus S)$ (this was also proved in [DDMS]).

The HL functions, and hence all the functions of $A_{\mathbb{P}^1_\mathbb{C} \smallsetminus S}$, have unipotent monodromies along the paths encircling the points of S, and one can show that $A_{\mathbb{P}^1_\mathbb{C} \smallsetminus S}$ is a union of unipotent modules (i.e. iterated extensions of the trivial module) over $\pi_1(\mathbb{P}^1_\mathbb{C} \smallsetminus S)$.

The HL functions were introduced in [Po], motivated by monodromy computations. They were later applied in [LD] to the Riemann–Hilbert problem, and subsequently in [Br] to the identification of a set of periods arising from the moduli space of marked stable genus-zero curves with the set of multiple zeta values (Goncharov–Manin conjecture). The HL techniques of [Br] led in [Pa] to an algorithm which can be used to express, in physics, a large class of Feynman integrals[1] in terms of HLs; this was implemented in the software program `HyperInt`.

Similar questions were studied replacing $\mathbb{P}^1_\mathbb{C}$ by a curve of genus one. To an elliptic curve $\mathcal{E}$, one attaches an algebra $\mathcal{A}_3$ containing the function field of $\mathcal{E}$ (see [BDDT1], three lines before (3.35)) using iterated integration. In [BDDT1, §6], it is proved that $\mathcal{A}_3$ is stable under $f \mapsto \int_{z_0} f\omega_0$, where ω_0 is a fixed nonzero regular differential over $\mathcal{E}$ and z_0 is any point in $\mathcal{E}$. One can derive from this the construction, for any finite subset S of $\mathcal{E}$, of an algebra containing the algebra of regular functions on $\mathcal{E} \smallsetminus S$, which is stable under $f \mapsto \int_{z_0} f\omega_0$; this algebra is therefore stable under the operations $f \mapsto \int_{z_0} f\omega$, where ω runs over all the regular differentials on $\mathcal{E} \smallsetminus S$. Similarly to the genus-zero case, the functions from $\mathcal{A}_3$ arise naturally in the computation of Feynman integrals (see [BDDT2]).

It is a natural question to construct analogues of the HL functions associated to an arbitrary affine curve C. Such functions are likely to find an application in

[1]Feynman integrals are a useful tool to obtain approximations of scattering amplitudes, which predict in quantum field theory the probability of interactions of elementary particles.

physics also when the genus of the curve is higher than one, such as for instance to compute hyperelliptic Feynman integrals (see [MMPPW]), or the genus-two contribution to string theory amplitudes (see [DGP]).

In order to treat this problem, we fix a universal cover $p\colon \widetilde{C} \to C$ and introduce the notion of a *minimal stable subalgebra* (MSSA) of the algebra of holomorphic functions $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ of $\widetilde{C}$ as follows: we call a *stable subalgebra* (SSA) of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ a unital subalgebra A of the algebra $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ such that for any $f \in A$, regular differential ω on C, and $z_0 \in \widetilde{C}$, the function $\int_{z_0} f\omega$ belongs to A. The intersection A_C of all SSA of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ is again an SSA which is minimal for the inclusion, and which we call the MSSA of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$.

The present paper is devoted to the study of A_C. We introduce the notion of a Maurer–Cartan element associated with the curve C, and show each such element gives rise via iterated integration to an algebra isomorphism $A_C \simeq \mathcal{O}(C) \otimes \mathrm{Sh}(\mathrm{H}^1_{\mathrm{dR}}(C))$. We also show that A_C is a union of unipotent $\pi_1(C)$-modules, contained in an algebra of moderate growth functions over $\widetilde{C}$, and is maximal with respect to this property. All this shows that the properties of A_C are generalizations of those of $A_{\mathbb{P}^1_{\mathbb{C}} \smallsetminus S}$; the isomorphism $A_C \simeq \mathcal{O}(C) \otimes \mathrm{Sh}(\mathrm{H}^1_{\mathrm{dR}}(C))$ is also an analogue of the main result of [Ma]. In the companion paper [EZ], we make A_C explicit when $C = \mathcal{E} \smallsetminus S$, with S a finite subset of an elliptic curve $\mathcal{E}$, and we explicitly relate A_C with the algebra $\mathcal{A}_3$ from [BDDT1]. The recent work [DHS], which introduces nonholomorphic variants of HL functions over one-punctured curves C of arbitrary genus, could hopefully be related to the present work.

§1.2. The main results

1.2.1. Conventions. The following conventions will be adopted throughout the paper. The base field of all the algebraic structures (vector spaces, Lie, Hopf, or associative algebras, etc.) is $\mathbb{C}$. We denote[2] by C a smooth complex affine algebraic curve, as well as the underlying Riemann surface, by $p\colon \widetilde{C} \to C$ a universal cover, and by $\mathcal{O}(C)$ the algebra of regular functions on C. Then $p^*\colon \mathcal{O}(C) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ is an injective algebra morphism. We denote by $\Omega(C)$ the space of regular differentials on C, and we set $\mathrm{H}_C := \Omega(C)/d\mathcal{O}(C)$ $(= \mathrm{H}^1_{\mathrm{dR}}(C)$ as C is affine).

1.2.2. Maurer–Cartan elements and the associated isomorphisms. Denote by $\mathfrak{g} := \mathbb{L}(\mathrm{H}_C^*)$ the free Lie algebra generated by H_C^*; it is graded by the condition that H_C^* has degree 1, and we denote by $\hat{\mathfrak{g}}$ its degree completion.

[2]Except in Remark 1.4(a), in Section 2.4, and in the second half of Section 5.5, where C takes a particular value.

Definition 1.1. Consider the following definitions:

(a) A *Maurer–Cartan (MC) element* for C is an element $J \in \Omega(C) \mathbin{\widehat{\otimes}} \hat{\mathfrak{g}}$.

(b) J is *nondegenerate* if and only if $\mathrm{im}(J \in \Omega(C) \mathbin{\widehat{\otimes}} \hat{\mathfrak{g}} \to \mathrm{H}_C \otimes \mathrm{H}_C^*) = \mathrm{id}$, the map being given by the tensor product of the canonical projections.

(c) $\mathrm{MC}(C)$ is the set of all MC elements for C, and $\mathrm{MC}_{\mathrm{nd}}(C)$ is the subset of all nondegenerate elements.

Let $(J, x_0) \in \mathrm{MC}_{\mathrm{nd}}(C) \times \widetilde{C}$. One proves that there is a unique smooth function $\mathbf{L}_{J,x_0} \colon \widetilde{C} \to \exp(\hat{\mathfrak{g}}) := \mathcal{G}((U\mathfrak{g})^\wedge)$ (where $\mathcal{G}$ stands for the group of group-like elements of a topological Hopf algebra, and $(U\mathfrak{g})^\wedge$ is the degree completion of the universal enveloping algebra of $\mathfrak{g}$) such that $d\mathbf{L}_{J,x_0} = \mathbf{L}_{J,x_0} \cdot J$ and $\mathbf{L}_{J,x_0}(x_0) = 1$, which turns out to be holomorphic (see Proposition 2.14).

Define then $\tilde{f}_{J,x_0} \colon \mathrm{Sh}(\mathrm{H}_C) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ to be the map taking a to the function

$$\tilde{f}_{J,x_0}(a) := (\widetilde{C} \ni x \mapsto \langle a, \mathbf{L}_{J,x_0}(x)\rangle \in \mathbb{C}), \tag{1.1}$$

where $\langle , \rangle$ is the pairing $\mathrm{Sh}(\mathrm{H}_C) \times (U\mathfrak{g})^\wedge \to \mathbb{C}$ induced by the composition $\mathrm{Sh}(\mathrm{H}_C) = \bigoplus_{n\geq 0}(U\mathfrak{g})[n]^* \to (\prod_{n\geq 0} U\mathfrak{g}[n])^* = ((U\mathfrak{g})^\wedge)^*$.

Similarly to the case of classical hyperlogarithms, $\mathbf{L}_{J,x_0}$ may be viewed as an element of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C}) \mathbin{\widehat{\otimes}} (U\mathfrak{g})^\wedge$. Hence $\mathbf{L}_{J,x_0}$ is a generating series of the image by $\tilde{f}_{J,x_0}$ of a basis of $\mathrm{Sh}(\mathrm{H}_C)$, which are multivalued functions on C defined by *iterated integrals*. By Lemma-Definition 2.1, if $x \in \widetilde{C}$ and $\omega_1, \dots, \omega_k \in \Omega(C)$, then the iterated integral $\int_\gamma \omega_1 \cdots \omega_k := \int_{0\leq t_1 \leq \cdots \leq t_k \leq 1} \gamma^*\omega_1(t_1) \wedge \cdots \wedge \gamma^*\omega_k(t_k)$ is independent of a path γ from x_0 to x, and denoted by $\int_{x_0}^x \omega_1 \cdots \omega_k$.

Definition 1.2. Consider the following definitions:

(a) Σ_C denotes the set of sections $\sigma \colon \mathrm{H}_C \to \Omega(C)$ of the canonical projection.

(b) $\sigma \mapsto J_\sigma$ is the map $\Sigma_C \to \mathrm{MC}_{\mathrm{nd}}(C)$ such that $\sigma \mapsto J_\sigma := \sum_i \sigma(h_i) \otimes h^i$, where $(h_i)_i$ is a basis of H_C and $(h^i)_i$ is the dual basis of H_C^*.

Lemma 1.3 (See Lemma 2.13). *Let $(J, x_0) \in \mathrm{MC}_{\mathrm{nd}}(C) \times \widetilde{C}$.*

(a) *The map $\tilde{f}_{J,x_0} \colon \mathrm{Sh}(\mathrm{H}_C) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ is a morphism of algebras.*

(b) *If $\sigma \in \Sigma_C$, $\tilde{f}_{J_\sigma,x_0}([h_1|\cdots|h_k]) = (x \mapsto \int_{x_0}^x \sigma(h_1)\cdots\sigma(h_k))$, where $[h_1|\cdots|h_k] \in \mathrm{Sh}(\mathrm{H}_C)$ is the element corresponding to $h_1 \otimes \cdots \otimes h_k \in \mathrm{H}_C^{\otimes k}$.*

Theorem A (See Section 5.4). *The following statements hold true:*

(a) *For any $J \in \mathrm{MC}_{\mathrm{nd}}(C)$, the image $\mathrm{im}(\tilde{f}_{J,x_0} \colon \mathrm{Sh}(\mathrm{H}_C) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C}))$ is independent of $x_0 \in \widetilde{C}$; it will be denoted $\mathcal{H}_C(J)$.*

(b) *Let* $(J, x_0) \in \mathrm{MC}_{\mathrm{nd}}(C) \times \widetilde{C}$. *The map* $f_{J,x_0}\colon \mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$, $a \otimes f \mapsto p^*(f) \cdot \tilde{f}_{J,x_0}(a)$ *induces an algebra isomorphism* $f_{J,x_0}\colon \mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C) \to A_C$.

In particular, any $\mathbb{C}$-basis of $\mathcal{H}_C(J)$ (for example, the family $(\tilde{f}_{J,x_0}(w))_w$, where w runs over a basis of $\mathrm{Sh}(\mathrm{H}_C)$) is linearly independent over $\mathcal{O}(C)$ and forms a basis of A_C as an $\mathcal{O}(C)$-module.

Remark 1.4. Note the following facts:

(a) If $C = \mathbb{P}^1_{\mathbb{C}} \smallsetminus S$, then $\mathrm{H}_C \simeq \mathbb{C}\hat{S}_\infty$. A particular element of Σ_C is σ_0 given by $\mathbb{C}\hat{S}_\infty \ni \hat{s} \mapsto d\log(z-s) \in \Omega(\mathbb{P}^1_{\mathbb{C}} \smallsetminus S)$. Then $\mathcal{H}_{\mathbb{P}^1_{\mathbb{C}} \smallsetminus S}(J_{\sigma_0})$ is equal to $\mathbb{C}[L_w, w \in \hat{S}^*_\infty]$ (see Section 2.4). It follows that the algebras $\mathcal{H}_C(J)$, where $J \in \mathrm{MC}_{\mathrm{nd}}(C)$, are generalizations of the algebra of HL functions.

(b) While $\mathcal{H}_C(J)$ varies with J, Theorem A(b) says that the product $\mathcal{O}(C)\cdot\mathcal{H}_C(J)$ does not and is equal to A_C.

1.2.3. Group aspects of the isomorphisms associated with the MC elements. Let $\Gamma_C := \mathrm{Aut}(\widetilde{C}/C)$ be the automorphism group of the cover $p\colon \widetilde{C} \to C$; it is equal to the fundamental group of C, therefore is free with $2h + |S| - 1$ generators.

Definition 1.5. For Γ a group, define $(\mathbb{C}\Gamma)'$ to be the subset of $(\mathbb{C}\Gamma)^*$ of all linear forms which vanish on the union $\bigcup_{n\geq 0}(\mathbb{C}\Gamma)_+^{n+1}$, where $(\mathbb{C}\Gamma)_+$ is the augmentation ideal of the group algebra $\mathbb{C}\Gamma$.

If Γ is finitely generated, it follows from Lemma A.6 that $(\mathbb{C}\Gamma)'$ is a commutative Hopf algebra, which may be identified with the function algebra of the prounipotent completion Γ^{un} (see Appendix D.2). This is in particular the case if $\Gamma = \Gamma_C$. The algebra $\mathrm{Sh}(\mathrm{H}_C)$ is also equipped with a commutative Hopf algebra structure, the coproduct $\Delta_{\mathrm{Sh}(\mathrm{H}_C)}$ being given by deconcatenation.

Lemma 1.6 (See Lemma 4.6 and Proposition 4.11). *For any* $(J, x_0) \in \mathrm{MC}_{\mathrm{nd}}(C) \times \widetilde{C}$, *the map*

$$p_{J,x_0}\colon \mathbb{C}\Gamma_C \times \mathrm{Sh}(\mathrm{H}_C) \to \mathbb{C}, \quad \gamma \otimes a \mapsto \langle a, \mathbf{L}_{J,x_0}(\gamma x_0)\rangle$$

is a Hopf algebra pairing. It gives rise to an isomorphism of commutative Hopf algebras

$$\nu(p_{J,x_0})\colon\ \mathrm{Sh}(\mathrm{H}_C) \to (\mathbb{C}\Gamma_C)'.$$

The relation of this result with Chen's "π_1 de Rham theorem" is discussed in Remark 4.12.

The group of $\mathbb{C}$-points of the spectrum of a commutative Hopf algebra O is $\mathrm{Spec}(O)(\mathbb{C}) := \mathrm{Hom}(O, \mathbb{C})$. Then[3] $\mathrm{Spec}(\mathrm{Sh}(\mathrm{H}_C))(\mathbb{C}) = \mathcal{G}((U\mathfrak{g})^\wedge) = \exp(\hat{\mathfrak{g}})$, and $\mathrm{Spec}((\mathbb{C}\Gamma_C)')(\mathbb{C}) = \mathcal{G}((\mathbb{C}\Gamma_C)^\wedge) = \Gamma_C^{\mathrm{un}}(\mathbb{C})$ (see [BGF, Thm. 3.224] and Appendix D.2). The group isomorphism corresponding to $\nu(p_{J,x_0})$ is

$$\Gamma_C^{\mathrm{un}}(\mathbb{C}) \to \exp(\hat{\mathfrak{g}}), \quad \gamma \to \mathbf{L}_{J,x_0}(\gamma x_0). \tag{1.2}$$

Definition 1.7 (See Definition B.1). A *Hopf algebra with a comodule-algebra* (HACA) is a pair (O, A), with O a Hopf algebra and A an associative algebra, equipped with an algebra morphism $\Delta_A \colon A \to O \otimes A$, which turns A into a comodule over the coalgebra O.

The action of an algebraic group G on a variety V gives rise to a HACA $(\mathcal{O}(G), \mathcal{O}(V))$. For any pair $(O, \mathbf{a})$ of a commutative Hopf algebra O and a commutative algebra $\mathbf{a}$, the pair $(O, O \otimes \mathbf{a})$ is a HACA, with $\Delta_{O\otimes\mathbf{a}} := \Delta_O \otimes \mathrm{id}_{\mathbf{a}}$; it corresponds to the action of $G := \mathrm{Spec}(O)$ on $V := \mathrm{Spec}(O) \times \mathrm{Spec}(\mathbf{a})$. In particular, the pair $(\mathrm{Sh}(\mathrm{H}_C), \mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C))$ is a HACA.

Lemma 1.8 (See Lemma 2.6, Lemma-Definition 5.8, and Lemma B.10[4]). *The right action $(f, \gamma) \mapsto f_{|\gamma} := (x \mapsto f(\gamma x))$ of Γ_C on $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ induces a HACA structure on $((\mathbb{C}\Gamma_C)', A_C)$.*

Theorem B (See Section 5.4). *Let $(J, x_0) \in \mathrm{MC}(C) \times \widetilde{C}$. The pair*

$$(\nu(p_{J,x_0}), f_{J,x_0}) \colon (\mathrm{Sh}(\mathrm{H}_C), \mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C)) \to ((\mathbb{C}\Gamma_C)', A_C)$$

is a HACA isomorphism.

To the HACA structure $(\mathrm{Sh}(\mathrm{H}_C), \mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C))$ (resp. $((\mathbb{C}\Gamma_C)', A_C)$), one associates an action of the group $\exp(\hat{\mathfrak{g}})$ (resp. $\Gamma^{\mathrm{un}}(\mathbb{C})$) on $\mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C)$ (resp. on A_C). Theorem B can then be translated into an equivariance statement: the algebra isomorphism f_{J,x_0} is compatible with the group isomorphism (1.2) and with the action of its source and target on the target and source of f_{J,x_0}.

One can also introduce the notion of a connection over a HACA, which generalizes the notion of connection over a principal bundle in the case of the HACA $(\mathcal{O}(G), \mathcal{O}(V))$, with V a principal G-bundle (see Section 6). We construct a connection ∇ on the HACA $((\mathbb{C}\Gamma_C)', A_C)$ and compute its pull-back by $(\nu(p_{J,x_0}), f_{J,x_0})$, which is independent of x_0, for any $(J, x_0) \in \mathrm{MC}_{\mathrm{nd}}(C) \times \widetilde{C}$.

[3]One checks that the bijection $\mathrm{Hom}_{\mathbb{C}\text{-vec}}(\mathrm{Sh}(\mathrm{H}_C), \mathbb{C}) \simeq \widehat{T}(\mathrm{H}_C^*) = (U\mathfrak{g})^\wedge$ induces a bijection $\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathrm{Sh}(\mathrm{H}_C), \mathbb{C}) \simeq \mathcal{G}((U\mathfrak{g})^\wedge)$.

[4]The combination of Lemma 2.6 and Lemma-Definition 5.8 implies that A_C is a Γ_C-stable subalgebra of $F_\infty\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ (the notation F_∞ being as in Lemma B.10), and therefore that $(\mathbb{C}\Gamma_C, A_C)$ is an object in $\mathbf{HAMA}_{\mathrm{fd}}$ (see Definition B.9); Lemma 1.8 then follows from Lemma B.10.

1.2.4. Isomorphisms of filtrations. Let $\overline{C}$ be the smooth compactification of C, and $S := \overline{C} \smallsetminus C$ be the complement of C in $\overline{C}$, which is a finite set; then $\widetilde{C}$ is a universal cover of $\widetilde{\overline{C}} \smallsetminus p^{-1}(S)$. A function of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ is called *moderate growth* if its growth at the neighborhood of $p^{-1}(S)$ is moderate in the sense of [Ph, §IX.1] (see Definition 3.9).

Lemma 1.9 (See Proposition 3.10). *The subset $\mathcal{O}_{\mathrm{mod}}(\widetilde{C}) \subset \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ of moderate growth functions is a subalgebra, which is stable under the action of Γ_C.*

One attaches a filtration of $\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$ to the action of Γ_C on $\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$ as follows.

Definition 1.10. For $n \geq 0$, we set $F_n\mathcal{O}_{\mathrm{mod}}(\widetilde{C}) := \{f \in \mathcal{O}_{\mathrm{mod}}(\widetilde{C}) \mid f_{|(\mathbb{C}\Gamma_C)_+^{n+1}} = 0\}$.

Lemma 1.11 (See Proposition 3.12 and Lemma 4.16). *The collection of subspaces $F_\bullet\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$ is an increasing algebra filtration of $\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$ with $F_0\mathcal{O}_{\mathrm{mod}}(\widetilde{C}) = \mathcal{O}(C)$, stable under the action of Γ_C.*

Inspired by [Ch], we also define two "differential" filtrations of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$.

Definition 1.12. Consider the following definitions:

(a) $F_0^\delta\mathcal{O}_{\mathrm{hol}}(\widetilde{C}) := \mathbb{C}$.
(b) For $n \geq 0$, $F_{n+1}^\delta\mathcal{O}_{\mathrm{hol}}(\widetilde{C}) := \{f \in \mathcal{O}_{\mathrm{hol}}(\widetilde{C}) \mid d(f) \in \Omega(C) \cdot F_n^\delta\mathcal{O}_{\mathrm{hol}}(\widetilde{C})\}$.
(c) For $n \geq 0$, $F_n^\mu\mathcal{O}_{\mathrm{hol}}(\widetilde{C}) := \mathcal{O}(C) \cdot F_n^\delta\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$.

Lemma 1.13 (See Proposition 5.5). *Set $F_\bullet^{\delta/\mu} := F_\bullet^{\delta/\mu}\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$.*

(a) *$F_\bullet^\delta$ and $F_\bullet^\mu$ are increasing algebra filtrations of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$.*
(b) *$F_0^\delta \subset F_0^\mu \subset F_1^\delta \subset F_1^\mu \subset \cdots$.*

Moreover, let us set $F_n \operatorname{Sh}(V) := \bigoplus_{k \leq n} \operatorname{Sh}_n(V)$ for $n \geq 0$ and V any vector space, and let us remark that, for $x_0 \in \widetilde{C}$, the map $I_{x_0}\colon \operatorname{Sh}(\Omega(C)) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ given by $[\omega_1|\cdots|\omega_n] \mapsto (x \mapsto \int_{x_0}^x \omega_1 \cdots \omega_n)$ is an algebra morphism (see Lemma 2.2). Then one has the following.

Lemma 1.14 (See Proposition 5.4). *The collection of subspaces $F_\bullet \operatorname{Sh}(\Omega(C))$ is an algebra filtration of $\operatorname{Sh}(\Omega(C))$ and $I_{x_0}(F_\bullet \operatorname{Sh}(\Omega(C)))$ is an algebra filtration of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$, which is independent of x_0.*

These various algebra filtrations can be compared as follows.

Theorem C (See Section 5.4). *Let $(J, x_0) \in \mathrm{MC}_{\mathrm{nd}}(C) \times \widetilde{C}$.*

(a) *One has the following equalities of algebra filtrations of* $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$*:*

$$F_\bullet\mathcal{O}_{\mathrm{mod}}(\widetilde{C}) = f_{J,x_0}(F_\bullet\,\mathrm{Sh}(\mathrm{H}_C)\otimes\mathcal{O}(C)) = F^\mu_\bullet\mathcal{O}_{\mathrm{hol}}(\widetilde{C}), \tag{1.3}$$

$$\begin{aligned} I_{x_0}\big(F_\bullet\,\mathrm{Sh}(\Omega(C))\big) &= F^\delta_\bullet\mathcal{O}_{\mathrm{hol}}(\widetilde{C}) \\ &= f_{J,x_0}(F_\bullet\,\mathrm{Sh}(\mathrm{H}_C)\otimes\mathbb{C} + F_{\bullet-1}\,\mathrm{Sh}(\mathrm{H}_C)\otimes\mathcal{O}(C)). \end{aligned} \tag{1.4}$$

(b) *One has the equalities*

$$\begin{aligned} A_C = I_{x_0}\big(\mathrm{Sh}(\Omega(C))\big) &= F^\delta_\infty\mathcal{O}_{\mathrm{hol}}(\widetilde{C}) = F^\mu_\infty\mathcal{O}_{\mathrm{hol}}(\widetilde{C}) \\ &= f_{J,x_0}(\mathrm{Sh}(\mathrm{H}_C)\otimes\mathcal{O}(C)) = F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C}) \end{aligned} \tag{1.5}$$

of subalgebras of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$*, where* $F_\infty X := \bigcup_{n\geq 0} F_n X$ *for* $F_\bullet X$ *a filtration on a vector space* X*.*

1.2.5. Filtered formality for HACA structures. Theorem B leads to the following extension of the notion of filtered formality ([SW1]) to the setting of HACAs. Any Hopf algebra O is equipped with a Hopf algebra filtration $F_\bullet O$, given when $O = (\mathbb{C}\Gamma)'$ by $F_n O = ((\mathbb{C}\Gamma)_+^{n+1})^\perp$ for $n \geq 0$ (see Lemma A.2, [BGF, §3.3.2], and [Fr, §7.2]), which gives rise to a graded Hopf algebra $\mathrm{gr}(O)$. We say that O is *filtered formal* if there exists an isomorphism of filtered Hopf algebras $O \to \mathrm{gr}(O)$, compatible with the filtrations and whose associated graded is the identity (see Definition C.2); we show that if a finitely generated group Γ is filtered formal in the sense of [SW1], then the Hopf algebra $(\mathbb{C}\Gamma)'$ is filtered formal (see Proposition D.18). These notions extend to HACAs as follows. Any HACA (O, A) is equipped with a HACA filtration $(F_\bullet O, F_\bullet A)$, whose first term is the above filtration of O (see Lemma B.3), and therefore gives rise to a graded HACA $(\mathrm{gr}(O), \mathrm{gr}(A))$ (see Lemma C.3). We say that the HACA (O, A) is *filtered formal* if there is an isomorphism of filtered HACAs $(O, A) \to (\mathrm{gr}(O), \mathrm{gr}(A))$, whose associated graded is the identity (see Def. C.4).

Proposition 1.15 (See Proposition 4.18). *The HACA* $((\mathbb{C}\Gamma)', F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C}))$ *is filtered formal; the associated graded HACA is* $(\mathrm{gr}((\mathbb{C}\Gamma)'), \mathrm{gr}(\mathcal{O}_{\mathrm{mod}}(\widetilde{C})))$*, where the components are the graded spaces associated to the filtrations of* $(\mathbb{C}\Gamma)'$ *and* $F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$*.*

§1.3. Organization of the paper

In Part I we prove the results announced in Section 1.2. In Section 2 (resp. Section 3), we introduce the necessary material on iterated integrals, Maurer–Cartan elements, and hyperlogarithm functions (resp. on moderate growth functions). In Section 4 we prove that f_{J,x_0} is an isomorphism of filtered algebras; the argument relies on techniques of HACAs, which are explained in the appendices. In Section 5 we prove results on the filtrations on $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ and their relation with the minimal stable subalgebra A_C, which lead to the proofs of Theorems A, B, and C.

Part II is devoted to complementary results. In Section 6 we introduce and study the notion of connections on HACAs. In Section 7 we provide more information on the local behavior of elements of A_C, which may be viewed, since $A_C \subset \mathcal{O}_{\mathrm{mod}}(\widetilde{C})$, as functions on a universal cover of $\widetilde{\overline{C}} \smallsetminus p^{-1}(S)$ with moderate growth near each element of $p^{-1}(S)$ (see Proposition 7.2); in particular, the germs of the functions of A_C near each such an element are *Nilsson-class functions* (in the sense of [Ph, p. 154]) of a particular kind. In Section 8 we relate A_C to the minimal acyclic extension of the differential graded algebra (dga) $\Omega^\bullet(C) \coloneqq (\mathcal{O}(C) \oplus \Omega(C), d)$ (see Proposition 8.5). In Section 9 we identify $\ker(I_{x_0})$ with the image of an explicit map (see Theorem 9.7(a)), in the spirit of bar-complex theory (see [H]), and we associate to each section $\sigma \colon \mathrm{H}_C \to \Omega(C)$ a complement Sub_σ of $\ker(I_{x_0})$ in $\mathrm{Sh}(\Omega(C))$ (Theorem 9.7(b)).

Part III is divided into four appendices, dealing with Hopf algebras and module or comodule algebras (HAMAs/HACAs).

Part I. Theorems A, B, C and their proofs

§2. Iterated integrals, Maurer–Cartan elements, and hyperlogarithm functions

In Section 2.1 we introduce the iterated integral morphism I_{x_0} attached to a point $x_0 \in \widetilde{C}$ and review its standard properties. In Section 2.2 we attach to each $J \in \mathrm{MC}(C)$ an algebra morphism $J_* \colon \mathrm{Sh}(\mathrm{H}_C) \to \mathrm{Sh}(\Omega(C))$. In Section 2.3 (see Proposition 2.15), we show that it gives rise to the morphism $\tilde{f}_{J,x_0} \colon \mathrm{Sh}(\mathrm{H}_C) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ from Section 1.2. We show that the image of $\tilde{f}_{J,x_0}$ is independent of x_0, and denote by $\mathcal{H}_C(J)$ its image. When $\sigma \in \Sigma_C$ (see Definition 1.2), we set $\mathcal{H}_C(\sigma) \coloneqq \mathcal{H}_C(J_\sigma)$. In Section 2.4 we consider the case where $C = \mathbb{P}^1_{\mathbb{C}} \smallsetminus S$, and we exhibit a section σ_0 such that $\mathcal{H}_{\mathbb{P}^1_{\mathbb{C}} \smallsetminus S}(\sigma_0)$ coincides with the algebra of hyperlogarithm functions (Proposition 2.19).

§2.1. Iterated integrals

Recall that the shuffle algebra $(\mathrm{Sh}(V), ⧢)$ associated with a vector space V is isomorphic to the tensor algebra $\bigoplus_{d\geq 0} V^{\otimes d}$ as a vector space and that $⧢$ is commutative. It has the decomposition $\mathrm{Sh}(V) = \bigoplus_{d\geq 0} \mathrm{Sh}_d(V)$, where $\mathrm{Sh}_d(V)$ is the degree d component $V^{\otimes d}$, which is an algebra grading; an element $v_1 \otimes \cdots \otimes v_d \in V^{\otimes d}$ will be denoted $[v_1|\cdots|v_d] \in \mathrm{Sh}_d(V)$.

Lemma-Definition 2.1. *Let $x_0 \in \widetilde{C}$.*

(a) *For $x \in \widetilde{C}$ and $\omega_1, \dots, \omega_k \in \Omega(C)$, the iterated integral*

$$\int_\gamma \omega_1 \cdots \omega_k := \int_{0 \le t_1 \le \cdots \le t_k \le 1} \gamma^* \omega_1(t_1) \wedge \cdots \wedge \gamma^* \omega_k(t_k)$$

is independent of a path γ from x_0 to x; it will be denoted $\int_{x_0}^x \omega_1 \cdots \omega_k$.

(b) *For any $n \ge 0$, there exists a unique linear map $I_{x_0}^{(n)} \colon \mathrm{Sh}_n(\Omega(C)) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ such that $I_{x_0}^{(0)}(1) = 1$, and for any $\omega_1, \dots, \omega_n \in \Omega(C)$, one has*

$$I_{x_0}^{(n)}([\omega_1 | \cdots | \omega_n]) = \Big(x \mapsto \int_{x_0}^x \omega_1 \cdots \omega_n\Big).$$

(c) *The linear map $I_{x_0} \colon \mathrm{Sh}(\Omega(C)) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ is the direct sum $\bigoplus_{n \ge 0} I_{x_0}^{(n)}$.*

Proof. Let $\Omega_{\mathrm{hol}}(\widetilde{C})$ be the space of holomorphic differentials on $\widetilde{C}$ and let $\mathrm{int}_{x_0} \colon \Omega_{\mathrm{hol}}(\widetilde{C}) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ be the linear map $\omega \mapsto (x \mapsto \int_{x_0}^x \omega)$, which is well defined since $\widetilde{C}$ is simply connected and since the elements of $\Omega_{\mathrm{hol}}(\widetilde{C})$ are closed. Statement (a) follows from the equality $\int_\gamma \omega_1 \cdots \omega_k = \mathrm{int}_{x_0}(\omega_k \cdot \mathrm{int}_{x_0}(\omega_{k-1} \cdots \omega_2 \cdot \mathrm{int}_{x_0}(\omega_1)))(x)$. Statements (b) and (c) are obvious. $\square$

One has therefore, for $n \ge 1$ and $\omega_1, \dots, \omega_n \in \Omega(C)$, the equality (in $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$)

$$I_{x_0}([\omega_1 | \cdots | \omega_n]) = \mathrm{int}_{x_0}(I_{x_0}([\omega_1 | \cdots | \omega_{n-1}]) \cdot p^* \omega_n). \tag{2.1}$$

Lemma 2.2. *For any $x_0 \in \widetilde{C}$, $I_{x_0} \colon \mathrm{Sh}(\Omega(C)) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ is an algebra morphism.*

Proof. See for example [BGF, Thm. 3.19, (3.22)]. $\square$

Lemma 2.3. *For any $x_0, x_1 \in \widetilde{C}$, $\gamma \in \Gamma_C$, and $a \in \mathrm{Sh}(\Omega(C))$, $I_{x_0}(a)(x_1) = I_{\gamma x_0}(a)(\gamma x_1)$.*

Proof. It suffices to prove this for a homogeneous. One argues by induction on $\deg(a)$. The identity is obvious for $\deg(a) = 0$, and the identity for degree n follows from the identity for degree $n-1$, (2.1), and the invariance of $p^*\omega$ for $\omega \in \Omega(C)$. $\square$

Definition 2.4. If V is a vector space, then $\Delta_{\mathrm{Sh}(V)}$ is the deconcatenation coproduct on $\mathrm{Sh}(V)$, defined by $[v_1 | \cdots | v_n] \mapsto \sum_{k=0}^n [v_1 | \cdots | v_k] \otimes [v_{k+1} | \cdots | v_n]$ for $v_1, \dots, v_n \in V$; it equips $\mathrm{Sh}(V)$ with a commutative Hopf algebra structure.

Lemma 2.5. *For $x_0, x_1 \in \widetilde{C}$ and $a \in \mathrm{Sh}(\Omega(C))$, one has*

$$I_{x_0}(a) = I_{x_0}(a^{(1)})(x_1) I_{x_1}(a^{(2)}), \tag{2.2}$$

where $a^{(1)} \otimes a^{(2)}$ is Sweedler's notation for $\Delta_{\mathrm{Sh}(\Omega(C))}(a)$.

Proof. Let us prove (2.2) by induction on the degree of a homogeneous element a in $\mathrm{Sh}(\Omega(C))$. Equation (2.2) is obvious if a has degree 0. Assume that (2.2) is proved for any a of degree $< n$ and let us prove it in degree n. Let $\omega_1, \dots, \omega_n \in \Omega(C)$. Then

$$\begin{aligned} dI_{x_0}([\omega_1|\cdots|\omega_n]) &= I_{x_0}([\omega_1|\cdots|\omega_{n-1}]) \cdot p^*\omega_n \\ &= \sum_{k=0}^{n-1} I_{x_0}([\omega_1|\cdots|\omega_k])(x_1) I_{x_1}([\omega_{k+1}|\cdots|\omega_{n-1}]) \cdot p^*\omega_n \\ &= \sum_{k=0}^{n-1} I_{x_0}([\omega_1|\cdots|\omega_k])(x_1) dI_{x_1}([\omega_{k+1}|\cdots|\omega_n]) \\ &= d\Bigg(\sum_{k=0}^{n-1} I_{x_0}([\omega_1|\cdots|\omega_k])(x_1) I_{x_1}([\omega_{k+1}|\cdots|\omega_n])\Bigg), \end{aligned}$$

where the first and third equalities follow from (2.1), and the second equality from (2.2) in degree $n-1$.

It follows that

$$I_{x_0}([\omega_1|\cdots|\omega_n]) - \sum_{k=0}^{n-1} I_{x_0}([\omega_1|\cdots|\omega_k])(x_1) I_{x_1}([\omega_{k+1}|\cdots|\omega_n]) \in \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$$

is a constant function. Its value at x_1 is $I_{x_0}([\omega_1|\cdots|\omega_n])(x_1)$, therefore

$$I_{x_0}([\omega_1|\cdots|\omega_n]) = \sum_{k=0}^{n} I_{x_0}([\omega_1|\cdots|\omega_k])(x_1) I_{x_1}([\omega_{k+1}|\cdots|\omega_n]),$$

proving (2.2) in degree n. □

Lemma 2.6. *If $x_0 \in \widetilde{C}$, then $I_{x_0}(\mathrm{Sh}(\Omega(C)))$ is a Γ_C-stable subalgebra of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$, and for any $f \in I_{x_0}(\mathrm{Sh}(\Omega(C)))$ there exists $n \geq 0$ such that $f_{|(\mathbb{C}\Gamma_C)_+^{n+1}} = 0$.*

Proof. It follows from Lemma 2.5 that the map $a \otimes \gamma \mapsto a_{|\gamma} := I_{x_0}(a^{(1)})(\gamma \cdot x_0)a^{(2)}$ defines a right action of Γ_C on the algebra $\mathrm{Sh}(\Omega(C))$, and that the algebra morphism $I_{x_0} \colon \mathrm{Sh}(\Omega(C)) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ is Γ_C-equivariant. This implies that $I_{x_0}(\mathrm{Sh}(\Omega(C)))$ is a Γ_C-stable subalgebra of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$. The second statement follows from the equivariance of I_{x_0} and from $a_{|(\mathbb{C}\Gamma_C)_+^{n+1}} = 0$ for any $a \in \mathrm{Sh}_n(\Omega(C))$. □

§2.2. Maurer–Cartan elements and associated morphisms

Let O be a Hopf algebra with coproduct Δ_O. For $a \geq 1$, let $\Delta_O^{(a)} \colon O \to O^{\otimes a}$ be the morphism obtained by iteration of Δ_O. Let $\mathrm{pr}_O \colon O \to O/\mathbb{C}$ be the canonical projection.

Definition 2.7 (See [Q1, Appx. B3], [BGF, §3.3.2], or [Fr, §7.2]). For $n \geq 0$, we define $F_n O := \ker(\mathrm{pr}_O^{\otimes n+1} \circ \Delta_O^{(n+1)})$.

We also set $F_{-1}O = \{0\}$. Note that $F_0 O = \mathbb{C}1$.

Lemma 2.8. *Let O be a Hopf algebra such that $O = F_\infty O$, with $F_\bullet O$ as in Definition 2.7, let V be a vector space, and let $\mu\colon O \to V$ be a linear map which is a derivation with respect to the counit ϵ of O, i.e. satisfying the identity $\mu(fg) = \mu(f)\epsilon(g) + \epsilon(f)\mu(g)$. Then the map $\mu_*\colon O \to \mathrm{Sh}(V)$ given by $f \mapsto \sum_{r\geq 0}[\mu(f^{(1)})|\cdots|\mu(f^{(r)})]$ is well defined (using Sweedler's notation for the iterated coproduct of O, which is the counit if $r = 0$), and is a Hopf algebra morphism.*

Proof. Since $\mu(1) = 0$, the map $f \mapsto \sum_{r\geq 0}[\mu(f^{(1)})|\cdots|\mu(f^{(r)})]$ takes $F_n O$ to $F_n \mathrm{Sh}(V)$ for any $n \geq 0$, which implies that μ_* is well defined. For $f \in O$, and denoting by Δ_X the coproduct of X for X any of the Hopf algebras O and $\mathrm{Sh}(V)$, one has

$$\begin{aligned}
\mu_*^{\otimes 2} \circ \Delta_O(f) &= \mu_*(f^{(1)}) \otimes \mu_*(f^{(2)}) \\
&= \sum_{r,s\geq 0} [\mu(f^{(1)(1)})|\cdots|\mu(f^{(1)(r)})] \otimes [\mu(f^{(2)(1)})|\cdots|\mu(f^{(2)(s)})] \\
&= \sum_{r,s\geq 0} [\mu(f^{(1)})|\cdots|\mu(f^{(r)})] \otimes [\mu(f^{(r+1)})|\cdots|\mu(f^{(r+s)})] \\
&= \Delta_{\mathrm{Sh}(V)} \circ \mu_*(f).
\end{aligned}$$

For $f, g \in O$, one has

$$\begin{aligned}
\mu_*(fg) &= \sum_{n\geq 0}[\mu(f^{(1)}g^{(1)})|\cdots|\mu(f^{(n)}g^{(n)})] \\
&= \sum_{n\geq 0}[\epsilon(f^{(1)})\mu(g^{(1)}) + \mu(f^{(1)})\epsilon(g^{(1)})|\cdots|\epsilon(f^{(n)})\mu(g^{(n)}) + \mu(f^{(n)})\epsilon(g^{(n)})] \\
&= \sum_{n\geq 0} \sum_{n=k+l} \sum_{\substack{K,L||K|=l,|L|=l \\ K\sqcup L=[\![1,n]\!]}} \prod_{a=1}^{k} \mu(f^{(i_a)})^{i_a} \prod_{b=1}^{l} \mu(g^{(j_b)})^{j_b} \prod_{a=1}^{k} \epsilon(g^{(i_a)}) \prod_{b=1}^{l} \epsilon(f^{(j_b)}) \\
&= \sum_{n\geq 0} \sum_{n=k+l} \sum_{\substack{K,L||K|=l,|L|=l \\ K\sqcup L=[\![1,n]\!]}} [\mu(f^{(1)})|\cdots|\mu(f^{(k)})]^K \cdot [\mu(g^{(1)})|\cdots|\mu(g^{(l)})]^L \\
&= \sum_{n\geq 0} \sum_{n=k+l} [\mu(f^{(1)})|\cdots|\mu(f^{(k)})] \sqcup\!\sqcup [\mu(g^{(1)})|\cdots|\mu(g^{(l)})] = \mu_*(f)\mu_*(g),
\end{aligned}$$

where the transport to $\mathrm{Sh}(V)$ of the product in the tensor algebra $T(V)$ is denoted by the first two product signs in the third line and by $\cdot$ in the fourth line; in the third line, we denote by $v \mapsto v^i$ the map

$$V \to \mathrm{Sh}(V), \quad v \mapsto [\underbrace{1 \cdots 1}_{i-1}|v|\underbrace{1 \cdots 1}_{n-i}],$$

and by $(i_1, \ldots, i_k)$ and $(j_1, \ldots, j_l)$ the increasing sequences such that $K = \{i_1, \ldots, i_k\}$ and $L = \{j_1, \ldots, j_l\}$; in the fourth line, $x \mapsto x^K$ is the map $\mathrm{Sh}_k(V) \to \mathrm{Sh}(V)$, $[v_1|\ldots,|v_k] \mapsto \prod_{a=1}^k v_a^{i_a}$ and $y \mapsto y^L$ has a similar meaning.

This proves that μ_* is compatible with the products and coproducts; one checks that it is compatible with the other aspects of the Hopf algebra structure (unit, counit, antipode). $\square$

Lemma 2.9. *For V a vector space, $\mathrm{Sh}(V) = F_\infty(\mathrm{Sh}(V))$, where $F_\bullet \mathrm{Sh}(V)$ is*[5] *as in Definition 2.7.*

Proof. Since $\mathrm{Sh}(V)$ is a connected graded Hopf algebra, this follows from Proposition A.2(d). $\square$

Let $J \in \mathrm{MC}(C)$ (see Definition 1.1). Let $\mu_J \colon \mathrm{Sh}(\mathrm{H}_C) \to \Omega(C)$ be the composition $\mathrm{Sh}(\mathrm{H}_C) \simeq \bigoplus_{n\geq 0} T(\mathrm{H}_C^*)[n]^* \to \bigoplus_{n\geq 0} \mathbb{L}(\mathrm{H}_C^*)[n]^* \to \Omega(C)$, where the second map is dual to the inclusion $\mathbb{L}(\mathrm{H}_C^*)[n] \subset T(\mathrm{H}_C^*)[n]$ and the last map is induced by J.

Lemma 2.10. *Let $J \in \mathrm{MC}(C)$. Then the linear map $\mu_J \colon \mathrm{Sh}(\mathrm{H}_C) \to \Omega(C)$ is a derivation with respect to the counit of $\mathrm{Sh}(\mathrm{H}_C)$.*

Proof. For $f, g \in \mathrm{Sh}(\mathrm{H}_C)$, one has

$$\begin{aligned}\mu_J(fg) = (fg \otimes \mathrm{id})(J) &= (f \otimes g \otimes \mathrm{id})((\Delta_{\widehat{T}(\mathrm{H}_C^*)} \otimes \mathrm{id})(J)) \\ &= (f \otimes g \otimes \mathrm{id})(J^{13} + J^{23}) = \mu_J(f)\epsilon(g) + \epsilon(f)\mu_J(g),\end{aligned}$$

where the third equality follows from the primitiveness of the first component of J, and $x \mapsto x^{13}, x^{23}$ are the maps $\widehat{T}(\mathrm{H}_C^*) \mathbin{\widehat{\otimes}} \Omega(C) \to \widehat{T}(\mathrm{H}_C^*)^{\widehat{\otimes} 2} \mathbin{\widehat{\otimes}} \Omega(C)$ induced by $t \otimes \omega \mapsto t \otimes 1 \otimes \omega$, $1 \otimes t \otimes \omega$. $\square$

Corollary 2.11. *If $J \in \mathrm{MC}(C)$, then the map*

$$J_* \colon \mathrm{Sh}(\mathrm{H}_C) \to \mathrm{Sh}(\Omega(C)), \quad \xi \mapsto \sum_{r\geq 0} [\mu_J(\xi^{(1)})|\cdots|\mu_J(\xi^{(r)})] \tag{2.3}$$

is well defined, and is an algebra morphism.

Proof. This follows from Lemmas 2.8, 2.9 with $V = \mathrm{H}_C$, and 2.10. $\square$

[5] This "Hopf algebra" filtration of $\mathrm{Sh}(V)$ can be shown to coincide with the "degree" filtration, which a posteriori justifies denoting them in the same way.

Lemma-Definition 2.12. *For $J \in \mathrm{MC}_{\mathrm{nd}}(C)$, there is a unique $\sigma \in \Sigma_C$ such that the degree 1 component of J (for the degree of $\widehat{\mathbb{L}}(\mathrm{H}_C)^*$) is equal to J_σ. This defines a map $\mathrm{MC}_{\mathrm{nd}}(C) \to \Sigma_C$, $J \mapsto \sigma_J$. It satisfies the identity $\sigma_{J_\sigma} = \sigma$.*

Proof. Obvious. □

Lemma 2.13. *The following statements hold true:*

(a) *For $\sigma \in \Sigma_C$, the morphism $(J_\sigma)_*\colon \mathrm{Sh}(\mathrm{H}_C) \to \mathrm{Sh}(\Omega(C))$ (see Definition 1.2) is graded, and coincides with the morphism σ_* functorially induced by the linear map σ.*

(b) *For $J \in \mathrm{MC}_{\mathrm{nd}}(C)$, the morphism $J_*\colon \mathrm{Sh}(\mathrm{H}_C) \to \mathrm{Sh}(\Omega(C))$ is compatible with the filtrations $F_\bullet$ of both sides, and the associated graded morphism coincides with the morphism attached to σ_J so $\mathrm{gr}(J_*) = (\sigma_J)_*$.*

Proof. (a) is obvious. For (b), let $n \geq 1$, $h_1, \dots, h_n \in \mathrm{H}_C$, and $\xi := [h_1|\cdots|h_n] \in \mathrm{Sh}(\mathrm{H}_C)$. Then for $r \geq n$,

$$\xi^{(1)} \otimes \cdots \otimes \xi^{(r)} \in \delta_{r,n} h_1 \otimes \cdots \otimes h_n + \bigoplus_{i=1}^{r} \mathrm{Sh}(\mathrm{H}_C)^{\otimes i-1} \otimes 1 \otimes \mathrm{Sh}(\mathrm{H}_C)^{\otimes r-i},$$

which since $\mu_J(1) = 0$ (see Lemma 2.10) and $\mu_J(h) = \sigma_J(h)$ for $h \in \mathrm{H}_C$ implies $[\mu_J(\xi^{(1)})|\cdots|\mu_J(\xi^{(r)})] = \delta_{r,n}[\sigma_J(h_1)|\cdots|\sigma_J(h_n)]$. The statement follows by combining this with (2.3). □

§2.3. The algebras $\mathcal{H}_C(J)$, $\mathcal{H}_C(\sigma)$

Let $J \in \mathrm{MC}_{\mathrm{nd}}(C)$, $d := \dim(\mathrm{H}_C)$, and $(h_i)_{i\in[\![1,d]\!]}$ be[6] a basis of H_C. Then

$$(I_{x_0} \circ J_*([h_{i_1}|\cdots|h_{i_k}]))_{k\geq 0, i_1,\dots,i_k \in [\![1,d]\!]}$$

is a family of elements of $\mathcal{H}_C(J)$. When $J = J_\sigma$ with $\sigma \in \Sigma_C$, this family is equal to

$$\big(I_{x_0}([\sigma(h_{i_1})|\cdots|\sigma(h_{i_k})])\big)_{k\geq 0, i_1,\dots,i_k \in [\![1,d]\!]}.$$

The following statement is a generalization of [Br, Cor. 5.6].

Proposition 2.14. *For $(J, x_0) \in \mathrm{MC}_{\mathrm{nd}}(C) \times \widetilde{C}$, the element*

$$\mathbf{L}_{J,x_0} := \sum_{\substack{k\geq 0\\ i_1,\dots,i_k \in [\![1,d]\!]}} I_{x_0} \circ J_*([h_{i_1}|\cdots|h_{i_k}]) \otimes (h^{i_1} \otimes \cdots \otimes h^{i_k}) \in \mathcal{O}_{\mathrm{hol}}(\widetilde{C}) \mathbin{\widehat{\otimes}} \widehat{T}((\mathrm{H}_C)^*),$$

where $(h^i)_{i\in[\![1,d]\!]}$ is the basis of $(\mathrm{H}_C)^*$ dual to $(h_i)_{i\in[1,d]}$, satisfies the equality

$$(d \otimes \mathrm{id})(\mathbf{L}_{J,x_0}) = \mathbf{L}_{J,x_0} \cdot J \tag{2.4}$$

[6] We set $[\![1,d]\!] := \{1, 2, \dots, d\}$.

(equality in $\Omega_{\mathrm{hol}}(\widetilde{C}) \mathbin{\widehat{\otimes}} \widehat{T}((\mathrm{H}_C)^*)$), as well as $\mathbf{L}_{J,x_0}(x_0) = 1$; it is the only element of the algebra $\mathcal{O}_{\mathrm{hol}}(\widetilde{C}) \mathbin{\widehat{\otimes}} \widehat{T}((\mathrm{H}_C)^*)$ satisfying these conditions.

Proof. It follows from the definition of μ_J that

$$J = \sum_{r \geq 0} \sum_{(i_1,\ldots,i_r) \in [\![1,d]\!]^r} \mu_J([h_{i_1}|\cdots|h_{i_r}]) \otimes h^{i_1} \otimes \cdots \otimes h^{i_r}$$

(equality in $\Omega(C) \mathbin{\widehat{\otimes}} \widehat{T}(\mathrm{H}_C^*)$), and therefore, expanding J as a sum $\sum_\alpha \omega_\alpha \otimes x^\alpha$ (convergent for the topology of $\Omega(C) \mathbin{\widehat{\otimes}} \widehat{T}(\mathrm{H}_C^*)$), that

$$\mathbf{L}_{J,x_0} = \sum_{r \geq 0} \sum_{\alpha_1,\ldots,\alpha_r} I_{x_0}([\omega_{\alpha_1}|\cdots|\omega_{\alpha_r}]) \otimes x^{\alpha_1} \cdots x^{\alpha_r}.$$

Then

$$\begin{aligned}(d \otimes \mathrm{id})\mathbf{L}_{J,x_0} &= \sum_{r \geq 1} \sum_{\alpha_1,\ldots,\alpha_r} I_{x_0}([\omega_{\alpha_1}|\cdots|\omega_{\alpha_{r-1}}])\omega_{\alpha_r} \otimes x^{\alpha_1} \cdots x^{\alpha_r} \\ &= \mathbf{L}_{J,x_0} \cdot \sum_\alpha \omega_\alpha \otimes x^\alpha = \mathbf{L}_{J,x_0} \cdot J.\end{aligned}$$

The second statement follows from the fact that $t \mapsto I_{x_0}(t)(x_0)$ is the augmentation map $\mathrm{Sh}(\Omega(C)) \to \mathbb{C}$. The uniqueness follows from the fact the ratio of two solutions must be equal to 1 at x_0 and be killed by d, hence be 1. $\square$

Equation (2.4) is a generalization of [Br, eq. (5.1)], and $\mathbf{L}_{J,x_0}$ is a generalization of the solution $L(z)$ of this equation constructed in [Br, Prop. 5.1].

Proposition 2.15. *For any $(J, x_0) \in \mathrm{MC}_{\mathrm{nd}}(C) \times \widetilde{C}$, the algebra morphism $I_{x_0} \circ J_*\colon \mathrm{Sh}(H_C) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ is such that for any $a \in \mathrm{Sh}(\mathrm{H}_C)$,*

$$I_{x_0} \circ J_*(a) = \langle \mathrm{id} \otimes a, \mathbf{L}_{J,x_0}\rangle.$$

Therefore, $I_{x_0} \circ J_* = \tilde{f}_{J,x_0}$ (see (1.1)); in particular $\tilde{f}_{J,x_0}\colon \mathrm{Sh}(\mathrm{H}_C) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ is an algebra morphism.

Proof. It follows from the fact that $(k, i_1, \ldots, i_k) \mapsto [h_{i_1}|\cdots|h_{i_k}]$ and $(k, i_1, \ldots, i_k) \mapsto h^{i_1} \otimes \cdots \otimes h^{i_k}$ are dual bases of $\mathrm{Sh}(\mathrm{H}_C)$ and $T(\mathrm{H}_C^*)$. $\square$

Lemma 2.16. *Let $J \in \mathrm{MC}_{\mathrm{nd}}(C)$. The image $\tilde{f}_{J,x_0}(\mathrm{Sh}(\mathrm{H}_C))$ is independent of $x_0 \in \widetilde{C}$. We denote this subalgebra by $\mathcal{H}_C(J) \subset \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ and set $\mathcal{H}_C(\sigma) := \mathcal{H}_C(J_\sigma)$ for any $\sigma \in \Sigma_C$.*

Proof. Let $x_0, x_1 \in \widetilde{C}$. Let $\mu_{x_0}^{x_1}\colon \mathrm{Sh}(\mathrm{H}_C) \to \mathbb{C}$ be the map $t \mapsto I_{x_0}(J_*(t))(x_1)$. This is an algebra morphism, as it is the composition of the algebra morphism $\tilde{f}_{J,x_0}$ and the morphism $\mathcal{O}_{\mathrm{mod}}(\widetilde{C}) \xrightarrow{\mathrm{ev}_{x_1}} \mathbb{C}$ of evaluation at x_1. Set $a_{x_0}^{x_1} := (\mu_{x_0}^{x_1} \otimes$

id) $\circ\, \Delta_{\mathrm{Sh}(\mathrm{H}_C)}$; this is an algebra endomorphism of $\mathrm{Sh}(\mathrm{H}_C)$ as it is a composition of algebra morphisms. It is also a vector space automorphism of $\mathrm{Sh}(\mathrm{H}_C)$ since it is compatible with the filtration and the associated graded endomorphism of $\mathrm{gr}(\mathrm{Sh}(\mathrm{H}_C))$ is the identity. Therefore, $a_{x_0}^{x_1}$ is an algebra automorphism of $\mathrm{Sh}(\mathrm{H}_C)$.

Equation (2.2) implies the identity $I_{x_1}(J_*(t)) = I_{x_1}(J_*(t^{(1)}))(x_0) I_{x_0}(J_*(t^{(2)}))$ (equality in $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$) for any $t \in \mathrm{Sh}(\mathrm{H}_C)$, therefore the equality $I_{x_1} \circ J_* = I_{x_0} \circ J_* \circ a_{x_0}^{x_1}$ (equality of maps $\mathrm{Sh}(\mathrm{H}_C) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$), i.e. $\tilde{f}_{J,x_1} = \tilde{f}_{J,x_0} \circ a_{x_0}^{x_1}$, which together with the bijectivity of $a_{x_0}^{x_1}$ implies the statement. □

§2.4. The algebra $\mathcal{H}_C(\sigma)$ in the genus-zero case

Let $S \subset \mathbb{P}^1_{\mathbb{C}}$ be a finite set containing 0 and ∞, let $C := \mathbb{P}^1_{\mathbb{C}} \smallsetminus S$. Recall the linear isomorphism $\mathrm{H}_C \simeq \mathbb{C}\hat{S}_\infty$, where $S_\infty = S \smallsetminus \{\infty\}$ and let $\sigma_0 \in \Sigma_C$ be such that for any $s \in S_\infty$, $\sigma_0(\hat{s}) = \mathrm{dlog}(z-s)$. By Lemma 2.16, one attaches to it the subalgebra $\mathcal{H}_{\mathbb{P}^1_{\mathbb{C}} \smallsetminus S}(\sigma_0) \subset \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$.

Set $\mathrm{H}_C^{(0)} := \bigoplus_{s \in S_\infty \smallsetminus \{0\}} \mathbb{C}\hat{s} \subset \mathrm{H}_C$ and $\mathrm{Sh}^*(\mathrm{H}_C) := \mathbb{C} \oplus [\mathrm{H}_C^{(0)} | \mathrm{Sh}(\mathrm{H}_C)]$ (recall that $[-|-]$ denotes the concatenation in $\mathrm{Sh}(\mathrm{H}_C)$). Then $\mathrm{Sh}^*(\mathrm{H}_C)$ is a subalgebra of $\mathrm{Sh}(\mathrm{H}_C)$.

Lemma 2.17 (See [Pa, §3.3]). *The following statements hold true:*

(a) *Denote by* $\mathrm{Sh}^*(\mathrm{H}_C)[X]$ *the polynomial algebra in one variable over the algebra* $\mathrm{Sh}^*(\mathrm{H}_C)$. *The combination of the canonical injection* $\mathrm{Sh}^*(\mathrm{H}_C) \hookrightarrow \mathrm{Sh}(\mathrm{H}_C)$ *and of the assignment* $X \mapsto [\hat{0}]$ *gives rise to an algebra isomorphism* $\mathrm{Sh}^*(\mathrm{H}_C)[X] \to \mathrm{Sh}(\mathrm{H}_C)$.

(b) *Let* $\delta > 0$ *be such that* $]0,\delta[\subset C$. *Fix a connected component* K *of* $p^{-1}(]0,\delta[)$. *The restriction of* p *is a bijection* $K \to]0,\delta[$*; denote by* $q_K \colon]0,\delta[\to K$ *the inverse bijection. For* $t \in \mathrm{Sh}^*(\mathrm{H}_C)$ *and* $z \in \widetilde{C}$*, the limit*

$$\lim_{\epsilon \to 0} I_{q_K(\epsilon)}((\sigma_0)_*(t))(z)$$

exists; the function

$$\tilde{f}^*_{\sigma_0,0}(t) := \Big(z \mapsto \lim_{\epsilon \to 0} I_{q_K(\epsilon)}((\sigma_0)_*(t))(z)\Big)$$

belongs to $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$*; the map* $\tilde{f}^*_{\sigma_0,0} \colon \mathrm{Sh}^*(\mathrm{H}_C) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$, $t \mapsto \tilde{f}^*_{\sigma_0,0}(t)$ *is an algebra morphism.*

Definition 2.18 (See [Pa, §3.3]). We denote by $\tilde{f}_{\sigma_0,0} \colon \mathrm{Sh}(\mathrm{H}_C) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ the composition with the inverse of the isomorphism of Lemma 2.17(a) of the morphism $\mathrm{Sh}^*(\mathrm{H}_C)[X] \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ extending $\tilde{f}^*_{\sigma_0,0}$ by $X \mapsto \log$.

The map $\tilde f_{\sigma_0,0}$ is an algebra morphism. For $w \in \hat S^*_\infty \subset \mathrm{Sh}(\mathrm{H}_C)$, the element $\tilde f_{\sigma_0,0}(w) \in \mathcal{O}_{\mathrm{hol}}(\widetilde C)$ is denoted L_w in [Pa, §3.3] and called the hyperlogarithm function associated to w.

Proposition 2.19. *One has* $\tilde f_{\sigma_0,0}(\mathrm{Sh}(\mathrm{H}_C)) = \mathbb{C}[L_w | w \in \hat S^*_\infty] = \mathcal{H}_{\mathbb{P}^1_{\mathbb{C}} \smallsetminus S}(\sigma_0)$.

Proof. Fix $z_0 \in \widetilde C$. The subalgebra $\mathrm{Sh}^*(\mathrm{H}_C)$ of $\mathrm{Sh}(\mathrm{H}_C)$ is a right coideal for the coalgebra structure, so that $\Delta_{\mathrm{Sh}(\mathrm{H}_C)}$ induces an algebra morphism $\Delta_{\mathrm{Sh}^*(\mathrm{H}_C)} : \mathrm{Sh}^*(\mathrm{H}_C) \to \mathrm{Sh}^*(\mathrm{H}_C) \otimes \mathrm{Sh}(\mathrm{H}_C)$. Composing with the tensor product of the composition

$$\mathrm{Sh}^*(\mathrm{H}_C) \xrightarrow{\tilde f^*_{\sigma_0,0}} \mathcal{O}_{\mathrm{hol}}(\widetilde C) \xrightarrow{\mathrm{ev}_{z_0}} \mathbb{C}$$

with the identity, one gets an algebra morphism $((\mathrm{ev}_{z_0} \circ \tilde f^*_{\sigma_0,0}) \otimes \mathrm{id}) \circ \Delta_{\mathrm{Sh}^*(\mathrm{H}_C)}$: $\mathrm{Sh}^*(\mathrm{H}_C) \to \mathrm{Sh}(\mathrm{H}_C)$. By Lemma 2.17, there exists a unique algebra endomorphism $a_0^{z_0}$ of $\mathrm{Sh}(\mathrm{H}_C)$, whose restriction to $\mathrm{Sh}^*(\mathrm{H}_C)$ coincides with $((\mathrm{ev}_{z_0} \circ \tilde f^*_{\sigma_0,0}) \otimes \mathrm{id}) \circ \Delta_{\mathrm{Sh}^*(\mathrm{H}_C)}$ and such that $[\hat 0] \mapsto [\hat 0] + \log(z_0)$. One checks that $a_0^{z_0}$ is compatible with the filtration of $\mathrm{Sh}(\mathrm{H}_C)$ and that the associated graded endomorphism of $\mathrm{gr}(\mathrm{Sh}(\mathrm{H}_C))$ is the identity, so that $a_0^{z_0}$ is an algebra automorphism of $\mathrm{Sh}(\mathrm{H}_C)$.

Specializing (2.2) for $t \in (\sigma_0)_*(\mathrm{Sh}^*(\mathrm{H}_C))$ and taking its limit for $z \to 0$, one obtains the equality $\tilde f^*_{\sigma_0,0}(t)(z'') = \tilde f^*_{\sigma_0,0}(t^{(1)})(z') I_{z'}(\sigma_0(t^{(2)}))(z'')$ for $t \in \mathrm{Sh}^*(\mathrm{H}_C)$ and any $z', z'' \in \widetilde C$. Setting $z' := z_0$ in this identity and viewing both sides as a function of z'', one obtains the identity

$$\tilde f^*_{\sigma_0,0}(t) = \tilde f^*_{\sigma_0,0}(t^{(1)})(z_0) I_{z_0}((\sigma_0)_*(t^{(2)}))$$

(in $\mathcal{O}_{\mathrm{hol}}(\widetilde C)$) which is equivalent to the statement that the restrictions to $\mathrm{Sh}^*(\mathrm{H}_C)$ of $\tilde f_{\sigma_0,0}$ and $I_{z_0} \circ (\sigma_0)_* \circ a_0^{z_0}$, which are algebra morphisms $\mathrm{Sh}(\mathrm{H}_C) \to \mathcal{O}_{\mathrm{hol}}(\widetilde C)$ are equal. The images by these morphisms of $[\hat 0] \in \mathrm{Sh}(\mathrm{H}_C)$ are also equal, since $\tilde f_{\sigma_0,0}([\hat 0]) = (z \mapsto \log(z))$ while $a_0^{z_0}([\hat 0]) = [\hat 0] + \log(z_0)$ and $I_{z_0} \circ (\sigma_0)_*([\hat 0]) = (z \mapsto \log(z) - \log(z_0))$. As $\mathrm{Sh}(\mathrm{H}_C)$ is generated by $\mathrm{Sh}^*(\mathrm{H}_C)$ and $[\hat 0]$, the algebra morphism status of both $\tilde f_{\sigma_0,0}$ and $I_{z_0} \circ (\sigma_0)_* \circ a_0^{z_0}$ implies that

$$\tilde f_{\sigma_0,0} = I_{z_0} \circ (\sigma_0)_* \circ a_0^{z_0} = \tilde f_{\sigma_0,0} \circ a_0^{z_0} \tag{2.5}$$

(equality of algebra morphisms $\mathrm{Sh}(\mathrm{H}_C) \to \mathcal{O}_{\mathrm{hol}}(\widetilde C)$).

One then has $\tilde f_{\sigma_0,0}(\mathrm{Sh}(\mathrm{H}_C)) = \tilde f_{\sigma_0,z_0} \circ a_0^{z_0}(\mathrm{Sh}(\mathrm{H}_C)) = \tilde f_{\sigma_0,z_0}(\mathrm{Sh}(\mathrm{H}_C)) = \mathcal{H}_{\mathbb{P}^1_{\mathbb{C}} \smallsetminus S}(\sigma_0)$, where the first equality follows from (2.5), the second follows from the automorphism status of $a_0^{z_0}$, and the last from Lemma 2.16.

One also has $\mathbb{C}[L_w | w \in \hat S^*_\infty] = \tilde f_{\sigma_0,0}(\mathbb{C}[w | w \in \hat S^*_\infty]) = \tilde f_{\sigma_0,0}(\mathrm{Sh}(\mathrm{H}_C))$. □

Remark 2.20. The element $J_{\sigma_0} = \sum_{s \in S_\infty} d\log(z - s) \otimes h^s$ is related to the Knizhnik–Zamolodchikov (KZ) connection as follows. Recall that this connection, denoted $\nabla_{\mathrm{KZ}} = d + A_{\mathrm{KZ}}$, is an $\exp(\mathfrak{t}_n)$-connection over the configuration space $C_n(\mathbb{C})$ of n points in $\mathbb{C}$, where $\mathfrak{t}_n$ is the topological Lie algebra with generators t_{ij}, $i \neq j \in [\![1, n]\!]$ and relations $t_{ji} = t_{ij}$ for $|\{i,j\}| = 2$, $[t_{ik} + t_{jk}, t_{ij}] = 0$ for $|\{i,j,k\}| = 3$ and $[t_{ij}, t_{kl}] = 0$ for $|\{i,j,k,l\}| = 4$, and that $A_{\mathrm{KZ}} = \sum_{i\neq j} d\log(z_i - z_j) \otimes t_{ij}$. Let $(s_1, \dots, s_n) \in C_n(\mathbb{C})$ and $S_\infty := \{s_1, \dots, s_n\}$; then $\mathbb{C} \smallsetminus S_\infty$ is the preimage of $(s_1, \dots, s_n)$ by the projection $C_{n+1}(\mathbb{C}) \to C_n(\mathbb{C})$. Let $\mathrm{inj}_{S_\infty} \colon \mathbb{C} \smallsetminus S_\infty \hookrightarrow C_{n+1}(\mathbb{C})$ be the canonical injection, and let $\iota \colon \mathbb{L}((\mathrm{H}_C)^*) \to \mathfrak{t}_{n+1}$ be the Lie algebra morphism induced by $h^i \mapsto t_{i,n+1}$ for $i = 1, \dots, n$. Then

$$\iota_*(J_{\sigma_0}) = (\mathrm{inj}_{S_\infty})^*(A_{\mathrm{KZ}}).$$

§3. Moderate growth functions

In Section 3.1 we introduce the algebra $\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$ of moderate growth functions on $\widetilde{C}$, and in Section 3.2 the space of moderate growth differentials $\Omega_{\mathrm{mod}}(\widetilde{C})$, which is a module over it. In Section 3.3 we study the relations of the iterated integral morphism I_{x_0} with moderate growth functions.

§3.1. Moderate growth functions on $\widetilde{C}$

3.1.1. Moderate growth functions on a disc. Set $D := \{z \in \mathbb{C} \mid |z| < 1\}$, $D^\times := D \smallsetminus \{0\}$, and $\widetilde{D}^\times := \{u \in \mathbb{C} \mid \Im(u) > 0\}$. Let $e \colon \widetilde{D}^\times \to D^\times$ be the map defined by $e(u) := \exp(2\pi \mathrm{i} u)$ (we set $\mathrm{i} := \sqrt{-1}$), and let θ be the automorphism of $\widetilde{D}^\times$ given by $\theta(u) = u + 1$. For $f \colon M \to N$ a morphism of complex manifolds, we denote by $f^* \colon \mathcal{O}_{\mathrm{hol}}(N) \to \mathcal{O}_{\mathrm{hol}}(M)$ the induced morphism between algebras of holomorphic functions. For $F = D, D^\times, \widetilde{D}^\times$, we denote by $\mathcal{O}_{\mathrm{hol}}(F)$ the algebra of holomorphic functions on F.

Definition 3.1 (See [Ph, Chap. VIII, Def. 1.2]). Define $\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times) \subset \mathcal{O}_{\mathrm{hol}}(\widetilde{D}^\times)$ as the set of functions f such that there exist an integer $n > 0$ and a function $\{(a,b) \in \mathbb{R}^2 \mid a \leq b\} \ni (a,b) \mapsto C_{a,b} \in \mathbb{R}_+$ such that $|f(x + \mathrm{i}y)| \leq C_{a,b} e^{2\pi n y}$ for $(x,y) \in [a,b] \times \mathbb{R}_+$.

Remark 3.2. If f satisfies these conditions for the pair $(n, (a,b) \mapsto C_{a,b})$, then it satisfies it also for $(n+1, (a,b) \mapsto C_{a,b})$.

Lemma 3.3. *The subspace $\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$ is a subalgebra of $\mathcal{O}_{\mathrm{hol}}(\widetilde{D}^\times)$, equipped with an action of $\mathbb{Z}$ where 1 acts by θ^*.*

Proof. The constant function c satisfies the conditions from Definition 3.1 with $n = 0$ and $(a,b) \mapsto C_{a,b} = |c|$; if the function f (resp. g) satisfies them with $(n, (a,b) \mapsto C_{a,b})$ (resp. $(m, (a,b) \mapsto D_{a,b})$), then the function $f + g$ satisfies them with $(\max(n,m),(a,b)\mapsto C_{a,b}+D_{a,b})$, the function fg satisfies them with $(n+m,(a,b)\mapsto C_{a,b}D_{a,b})$, and the function θ^*f satisfies them with $(n,(a,b)\mapsto C_{a+1,b+1})$. $\square$

Definition 3.4. Set $\mathcal{O}(D^\times):=\{f\in\mathcal{O}_{\mathrm{hol}}(D^\times)\mid \exists n\geq 0,\ z^nf\in\mathcal{O}_{\mathrm{hol}}(D)\}$.

Then $\mathcal{O}(D^\times)$ is the algebra of meromorphic functions on D with the only possible poles at 0.

Lemma 3.5. *$\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)^{\mathbb{Z}}=\mathcal{O}(D^\times)$.*

Proof. The inclusion $\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)^{\mathbb{Z}}\supset\mathcal{O}(D^\times)$ is evident; let us show the opposite inclusion. Let $f\in\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)^{\mathbb{Z}}$. Then there exist $n>0$ and $C_{0,1}\in\mathbb{R}_+$, such that $|f(x+\mathrm{i}y)|\leq C_{0,1}e^{2\pi ny}$ for any $(x,y)\in[0,1]\times\mathbb{R}_+$. Since f is $\mathbb{Z}$-invariant, $f\in\mathcal{O}_{\mathrm{hol}}(D^\times)$, therefore $z^nf\in\mathcal{O}_{\mathrm{hol}}(D^\times)$. For $z\in D^\times$, there exists a unique $(x,y)\in[0,1[\times\mathbb{R}_+$ such that $z=e(x+\mathrm{i}y)$. Then $|z^nf(z)|\leq C_{0,1}e^{-2\pi ny}e^{2\pi ny}=C_{0,1}$. By the removable singularity theorem (see [L, Thm. 3.1]), z^nf is the restriction to $D^\times$ of a function of $\mathcal{O}_{\mathrm{hol}}(D)$, therefore $f\in\mathcal{O}(D^\times)$. $\square$

3.1.2. The algebra $\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$ of moderate growth functions.

Definition 3.6. Consider the following definitions:

(a) $(\overline{C},S)$ is the pair of a nonsingular projective algebraic curve $\overline{C}$ and a finite set S of complex points of C such that $C=\overline{C}\smallsetminus S$.

(b) For $s\in S$, $\varphi_s\colon D\to\overline{C}$ is an injective holomorphic map, such that $0\mapsto s$ and $\varphi_s(D)\cap S=\{s\}$; we set $U_s:=\varphi_s(D)\subset\overline{C}$, so that φ_s corestricts to a biholomorphic map $D\xrightarrow{\simeq}U_s$.

(c) For $s\in S$, $\varphi_s^\times\colon D^\times\to C$ is the injective holomorphic map obtained by restriction of φ_s; we set $U_s^\times:=U_s\smallsetminus\{s\}$, so that $\varphi_s^\times$ corestricts to a biholomorphic map $D^\times\xrightarrow{\simeq}U_s^\times$.

(d) For $s\in S$, we set $\widetilde{U}_s^\times:=p^{-1}(U_s^\times)$ and $X_s:=\pi_0(\widetilde{U}_s^\times)$; for $x\in X_s$, we define $\widetilde{U}_{s,x}^\times$ as the connected component of $\widetilde{U}_s^\times$ corresponding to x.

(e) For $s\in S$ and $x\in X_s$, $\tilde{\varphi}_{s,x}^\times\colon\widetilde{D}^\times\to\widetilde{C}$ is a holomorphic map with image contained in $\widetilde{U}_{s,x}^\times$, such that $p\circ\tilde{\varphi}_{s,x}^\times=\varphi_s^\times\circ e$. Then $\tilde{\varphi}_{s,x}^\times$ corestricts to a biholomorphic map $\widetilde{D}^\times\to\widetilde{U}_{s,x}^\times$.

The choice of $(\tilde{\varphi}_{s,x}^\times)_{s\in S,x\in X_s}$ is not unique, but any two choices are related by $\tilde{\psi}_{s,x}^\times=\tilde{\varphi}_{s,x}^\times\circ\theta^{a_{s,x}}$, where $(a_{s,x})_{s\in S,x\in X_s}$ is in $\bigoplus_{s\in S}\mathbb{Z}^{X_s}$.

For any $s\in S$, the group Γ_C acts on $\widetilde{U}_s^\times$, and therefore also on the set X_s. The latter action is transitive, and the stabilizer of any element $x\in X_s$ is a cyclic group, generated by an element $\theta_{s,x} \in \Gamma_C$ which restricts to an automorphism of $\widetilde{U}^\times_{s,x}$, equal to the conjugation of θ by the corestriction of $\tilde{\varphi}^\times_{s,x}$ to an isomorphism $\widetilde{D}^\times \to \widetilde{U}^\times_{s,x}$.

Lemma 3.7. *There exists a map $c\colon \Gamma_C \times X_s \to \mathbb{Z}$ satisfying the identity $c(\gamma'\gamma, x) = c(\gamma', \gamma x) + c(\gamma, x)$, such that for any $(\gamma, x) \in \Gamma_C \times X_s$, one has $\tilde{\varphi}^\times_{s,\gamma x} \circ \theta^{c(\gamma,x)} = \gamma \circ \tilde{\varphi}^\times_{s,x}$ (equality of holomorphic maps $\widetilde{D}^\times \to \widetilde{C}$).*

Proof. The existence of a map c satisfying the identity $\tilde{\varphi}^\times_{s,\gamma x} \circ \theta^{c(\gamma,x)} = \gamma \circ \tilde{\varphi}^\times_{s,x}$ follows from the fact that for any pair of holomorphic maps $\alpha, \beta\colon \widetilde{D}^\times \to \widetilde{C}$ such that $p \circ \alpha = \varphi^\times_s \circ e = p \circ \beta$, there exists $n \in \mathbb{Z}$ such that $\beta = \alpha \circ \theta^n$. Then $\tilde{\varphi}^\times_{s,\gamma'\gamma x} \circ \theta^{c(\gamma'\gamma,x)} = \gamma'\gamma \circ \tilde{\varphi}^\times_{s,x} = \gamma' \circ \tilde{\varphi}^\times_{s,\gamma x} \circ \theta^{c(\gamma,x)} = \tilde{\varphi}^\times_{s,\gamma'\gamma x} \circ \theta^{c(\gamma',\gamma x)} \circ \theta^{c(\gamma,x)}$ which implies the identity satisfied by c. □

Lemma 3.8. *The following statements hold true:*

(a) *For any $s \in S$, the space $X_s \times \widetilde{D}^\times$ is equipped with an action of Γ_C given by $\gamma \cdot (x, d) := (\gamma x, \theta^{c(\gamma,x)} d)$. The induced right action of Γ_C on $\prod_{x \in X_s} \mathcal{O}_{\mathrm{hol}}(\widetilde{D}^\times)$ is given by $(f_x)_{x \in X_s} \cdot \gamma = (g_x)_{x \in X_s}$, where $g_x := (\theta^{c(\gamma,x)})^*(f_{\gamma x})$.*

(b) *The map $\coprod_{s \in S} X_s \times \widetilde{D}^\times \to \widetilde{C}$, $(s, x, d) \mapsto \tilde{\varphi}^\times_{s,x}(d)$ is Γ_C-equivariant, its source being equipped with the direct sum of the actions of Γ_C defined in* (a)*. It induces a Γ_C-equivariant algebra morphism*

$$\bigoplus_{s \in S} \prod_{x \in X_s} (\tilde{\varphi}^\times_{s,x})^*\colon \mathcal{O}_{\mathrm{hol}}(\widetilde{C}) \to \bigoplus_{s \in S} \prod_{x \in X_s} \mathcal{O}_{\mathrm{hol}}(\widetilde{D}^\times), \tag{3.1}$$

in which the target is equipped with the direct sum over $s \in S$ of the right actions from (a).

Proof. This follows from Lemma 3.7. □

Definition 3.9. Denote by $\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$ the subset of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ of all functions f such that, for any $s \in S$ and $x \in X_s$, one has $(\tilde{\varphi}^\times_{s,x})^*(f) \in \mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$.

Proposition 3.10. *The subset $\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$ is a subalgebra of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$, stable under the action of Γ_C.*

Proof. This follows from the equality of $\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$ with the preimage of the set $\bigoplus_{s \in S} \prod_{x \in X_s} \mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$ under (3.1), and from the fact that this is a Γ_C-stable subalgebra of the target of (3.1). □

3.1.3. Computation of $\mathcal{O}_{\mathrm{mod}}(\widetilde{C})^{\Gamma_C}$.

Lemma 3.11. *For any $s \in S$, the diagonal embedding $\mathcal{O}(D^\times) \hookrightarrow \prod_{x \in X_s} \mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$ gives rise to an isomorphism $\mathcal{O}(D^\times) \simeq (\prod_{x \in X_s} \mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times))^{\Gamma_C}$.*

Proof. The fact that the image of the diagonal embedding is included in the set $(\prod_{x\in X_s}\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times))^{\Gamma_C}$ is evident; let us prove the opposite inclusion. Let $y\in X_s$. The stabilizer of y for the action of Γ_C on X_s is a cyclic group, generated by an element $\theta_y\in\Gamma_C$ such that $c(\theta_y,y)=1$. If now $\underline{f}:=(f_x)_{x\in X_s}\in(\prod_{x\in X_s}\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times))^{\Gamma_C}$, then $\underline{f}=\underline{f}\cdot\theta_y$, therefore for any $x\in X_s$, one has $f_x=(\theta^{c(\theta_y,x)})^*(f_{\theta_y x})$, which for $x=y$ implies $f_x=\theta^* f_x$. By Lemma 3.5(b), this implies $f_x\in\mathcal{O}(D^\times)$. For any $\gamma\in\Gamma$, one has $\underline{f}=\underline{f}\cdot\gamma$, which given the θ-invariance of each f_x, $x\in X_s$, implies $f_x=f_{\gamma x}$ for any $x\in X_s$. Since the action of Γ_C on X_s is transitive, this implies that the map $x\mapsto f_x$ is constant. □

Proposition 3.12. *$\mathcal{O}_{\mathrm{mod}}(\widetilde{C})^{\Gamma_C}=\mathcal{O}(C)$.*

Proof. Let $n\geq 0$. It follows from Proposition 3.10 that (3.1) induces a linear and Γ_C-equivariant algebra morphism

$$\mathcal{O}_{\mathrm{mod}}(\widetilde{C})\to\bigoplus_{s\in S}\prod_{x\in X_s}\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times). \tag{3.2}$$

This map restricts to a linear map $\mathcal{O}_{\mathrm{mod}}(\widetilde{C})^{\Gamma_C}\to(\bigoplus_{s\in S}\prod_{x\in X_s}\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times))^{\Gamma_C}$. The target is equal to $\bigoplus_{s\in S}(\prod_{x\in X_s}\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times))^{\Gamma_C}$, which by Lemma 3.11 is equal to $\bigoplus_{s\in S}\mathcal{O}(D^\times)$.

On the other hand, $\mathcal{O}_{\mathrm{mod}}(\widetilde{C})^{\Gamma_C}\subset\mathcal{O}_{\mathrm{hol}}(\widetilde{C})^{\Gamma_C}=\mathcal{O}_{\mathrm{hol}}(C)$. All this implies that $\mathcal{O}_{\mathrm{mod}}(\widetilde{C})^{\Gamma_C}$ is the preimage of $\bigoplus_{s\in S}\mathcal{O}(D^\times)$ by the map

$$\bigoplus_{s\in S}(\varphi_s^\times)^*\colon\mathcal{O}_{\mathrm{hol}}(C)\to\bigoplus_{s\in S}\mathcal{O}_{\mathrm{hol}}(D^\times),$$

which is equal to $\mathcal{O}(C)$. □

One checks that the subalgebra $\mathcal{O}_{\mathrm{mod}}(\widetilde{D})\subset\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ is independent of the choice of the family $(\varphi_s)_{s\in S}$.

§3.2. The module $\Omega_{\mathrm{mod}}(\widetilde{C})$ of moderate growth differentials

If M is a complex manifold, let $\Omega^\bullet_{\mathrm{hol}}(M)$ be the dga of holomorphic differential forms on M. Then $\Omega^0_{\mathrm{hol}}(M)=\mathcal{O}_{\mathrm{hol}}(M)$. The assignment $M\mapsto\Omega^\bullet_{\mathrm{hol}}(M)$ is a contravariant functor, so a morphism $f\colon M\to N$ of complex manifolds gives rise to a dga morphism $f^*\colon\Omega^\bullet_{\mathrm{hol}}(N)\to\Omega^\bullet_{\mathrm{hol}}(M)$. If M is 1-dimensional, we set $\Omega_{\mathrm{hol}}(M):=\Omega^1_{\mathrm{hol}}(M)$; then $\Omega_{\mathrm{hol}}(M)$ is an $\mathcal{O}_{\mathrm{hol}}(M)$-module, equipped with a derivation $d\colon\mathcal{O}_{\mathrm{hol}}(M)\to\Omega_{\mathrm{hol}}(M)$.

Lemma 3.13. *The space $\Omega_{\mathrm{hol}}(\widetilde{D}^\times)$ is a free rank 1 module over $\mathcal{O}_{\mathrm{hol}}(\widetilde{D}^\times)$ generated by $e^*(dz/z)$, so $f\mapsto f\cdot e^*(dz/z)$ gives rise to a $\mathcal{O}_{\mathrm{hol}}(\widetilde{D}^\times)$-module isomorphism $\mathcal{O}_{\mathrm{hol}}(\widetilde{D}^\times)\xrightarrow{\sim}\Omega_{\mathrm{hol}}(\widetilde{D}^\times)$.*

Proof. This follows from the fact that dz/z is an invertible differential in $\Omega_{\mathrm{hol}}(D^\times)$, which implies the same about its pull-back by $\widetilde{D}^\times \to D^\times$. $\square$

Definition 3.14. Define $\Omega_{\mathrm{mod}}(\widetilde{D}^\times) := \mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times) \cdot e^*(dz/z)$ as the image of $\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$ under the isomorphism from Lemma 3.13.

Lemma 3.15. *The space $\Omega_{\mathrm{mod}}(\widetilde{D}^\times)$ is a free rank 1 module over $\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$.*

Proof. The follows from the injectivity of the map $\mathcal{O}_{\mathrm{hol}}(\widetilde{D}^\times) \to \Omega_{\mathrm{hol}}(\widetilde{D}^\times)$, $f \mapsto f \cdot e^*(dz/z)$. $\square$

The morphism $\prod_{s\in S}\prod_{x\in X_s}\varphi_{s,x}\colon \prod_{s\in S}\prod_{x\in X_s}\widetilde{D}^\times \to \widetilde{C}$ gives rise to a morphism

$$\bigoplus_{s\in S}\prod_{x\in X_s}(\tilde{\varphi}^\times_{s,x})^*\colon \Omega_{\mathrm{hol}}(\widetilde{C}) \to \bigoplus_{s\in S}\prod_{x\in X_s}\Omega_{\mathrm{hol}}(\widetilde{D}^\times). \tag{3.3}$$

Definition 3.16. Define $\Omega_{\mathrm{mod}}(\widetilde{C})$ as the preimage of $\bigoplus_{s\in S}\prod_{x\in X_s}\Omega_{\mathrm{mod}}(\widetilde{D}^\times)$ under (3.3).

Lemma 3.17. *The space $\Omega_{\mathrm{mod}}(\widetilde{C})$ is a module over $\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$.*

Proof. If $f \in \mathcal{O}_{\mathrm{mod}}(\widetilde{C})$, $\omega \in \Omega_{\mathrm{mod}}(\widetilde{C})$, then $f\omega \in \Omega_{\mathrm{hol}}(\widetilde{C})$. If $s \in S$ and $x \in X_s$, then $i^*_{s,x}(f\omega) = i^*_{s,x}(f)i^*_{s,x}(\omega) \in \Omega_{\mathrm{mod}}(\widetilde{D}^\times)$, where the last relation follows from $i^*_{s,x}(f) \in \mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$, $i^*_{s,x}(\omega) \in \Omega_{\mathrm{mod}}(\widetilde{D}^\times)$. Therefore, $f\omega \in \Omega_{\mathrm{mod}}(\widetilde{C})$. $\square$

Since $\widetilde{D}^\times$ is simply connected, the assignment $\omega \mapsto (z \mapsto \int_{\mathrm{i}}^z \omega)$ is a well-defined linear map $\mathrm{int}_{\mathrm{i}}\colon \Omega_{\mathrm{hol}}(\widetilde{D}^\times) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{D}^\times)$.

Lemma 3.18. *The map $\mathrm{int}_{\mathrm{i}}$ takes $\Omega_{\mathrm{mod}}(\widetilde{D}^\times)$ to $\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$.*

Proof. Let $\omega \in \Omega_{\mathrm{mod}}(\widetilde{D}^\times)$. Then there exists $f \in \mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$ such that $\omega = f \cdot e^*(dz/z)$. Let $n > 0$ and $(a,b) \mapsto C_{a,b}$ be the integer and function associated with f (see Definition 3.1). Then $\mathrm{int}_{\mathrm{i}}(\omega) \in \mathcal{O}_{\mathrm{hol}}(\widetilde{D}^\times)$ is the function $u \mapsto 2\pi\mathrm{i}\int_{\mathrm{i}}^u f(u')\,du'$. Let $A \geq 0$ and $u = x + \mathrm{i}y$ with $x \in [-A, A]$; set $C_A := C_{-A,A}$. Then the path of integration may be chosen as the sequence of paths $\mathrm{i} \to \mathrm{i}y \to u = x+\mathrm{i}y$; therefore $\int_{\mathrm{i}}^u f(u')\,du' = \int_{\mathrm{i}}^{\mathrm{i}y} f(u')\,du' + \int_{\mathrm{i}y}^{x+\mathrm{i}y} f(u')\,du' = \mathrm{i}\int_1^y f(\mathrm{i}t)\,dt + \int_0^x f(t+\mathrm{i}y)dt$.

One has $|f(\mathrm{i}t)| \leq C_A e^{2\pi nt}$ for $t \in [1, y]$, while $|f(t+\mathrm{i}y)| \leq C_A e^{2\pi ny}$ for $t \in [0, x]$. Therefore $|\mathrm{i}\int_1^y f(\mathrm{i}t)\,dt| \leq C_A|\int_1^y e^{2\pi nt}\,dt| = C_A|e^{2\pi ny} - e^{2\pi n}|/(2\pi n)$, and $|\int_0^x f(t+\mathrm{i}y)\,dt| \leq AC_A e^{2\pi ny}$. Therefore

$$\left|\mathrm{i}\int_{\mathrm{i}}^u f(u')\,du'\right| \leq C_A\frac{|e^{2\pi ny} - e^{2\pi n}|}{2\pi n} + AC_A e^{2\pi ny}. \tag{3.4}$$

If $y \geq 1$, the right-hand side of (3.4) is $\leq C_A(A + 1/(2\pi n))e^{2\pi ny}$ (expressing the absolute value as its argument as the latter is ≥ 0 and bounding the resulting expression from above by removing the negative term); if $y < 1$, the right-hand side of (3.4) is $\leq C_A e^{2\pi n}/(2\pi n) + AC_A e^{2\pi ny}$ (expressing the absolute value as the negative of its argument as the latter is ≤ 0 and bounding the resulting expression from above by removing the negative term) which is $\leq C_A(e^{2\pi n}/(2\pi n))e^{2\pi ny} + AC_A e^{2\pi ny}$ (as $e^{2\pi ny} \geq 1$), therefore the right-hand side of (3.4) is $\leq (A + (e^{2\pi n}/(2\pi n)))C_A e^{2\pi ny}$.

Set $D_A := (A+(e^{2\pi n}/(2\pi n)))C_A$; then one obtains $|\mathrm{i}\int_{\mathrm{i}}^u f(u')\,du'| \leq D_A e^{2\pi ny}$ for every $u \in [-A, A] + \mathrm{i}\mathbb{R}_+^\times$, and therefore $|\int_{\mathrm{i}}^u \omega| \leq 2\pi D_A e^{2\pi ny}$ for any $u \in [-A, A] + \mathrm{i}\mathbb{R}_+^\times$. This shows that $u \mapsto \int_{\mathrm{i}}^u \omega$ satisfies the condition of Definition 3.1 with the pair $(n, (a, b) \mapsto 2\pi D_{\max(|a|,|b|)})$, and therefore belongs to $\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$. □

Fix a point $x_0 \in \widetilde{C}$.

Lemma 3.19. *The map* int_{x_0} *(see Section 1.2.2) takes* $\Omega_{\mathrm{mod}}(\widetilde{C})$ *to* $\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$.

Proof. Let $\omega \in \Omega_{\mathrm{mod}}(\widetilde{C})$. Let $s \in S$ and $x \in X_s$. By additivity of the integral with respect to the composition of paths, one has

$$(\tilde{\varphi}_{s,x}^\times)^*(\mathrm{int}_{x_0}(\omega)) = \int_{x_0}^{\tilde{\varphi}_{s,x}^\times(\mathrm{i})} \omega + \mathrm{int}_{\mathrm{i}}((\tilde{\varphi}_{s,x}^\times)^*\omega) \tag{3.5}$$

(equality in $\mathcal{O}_{\mathrm{hol}}(\widetilde{D}^\times)$), where $\int_{x_0}^{\tilde{\varphi}_{s,x}^\times(\mathrm{i})} \omega$ belongs to $\mathbb{C}$. Since $\omega \in \Omega_{\mathrm{mod}}(\widetilde{C})$, one has $(\tilde{\varphi}_{s,x}^\times)^*\omega \in \Omega_{\mathrm{mod}}(\widetilde{D}^\times)$; it then follows from Lemma 3.18 that $\mathrm{int}_{\mathrm{i}}((\tilde{\varphi}_{s,x}^\times)^*\omega) \in \mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$. Since $\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$ is an algebra containing $\mathbb{C}$, it follows that the right-hand side of (3.5) belongs to $\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$. Equation (3.5) then implies the relation $(\tilde{\varphi}_{s,x}^\times)^*(\mathrm{int}_{x_0}(\omega)) \in \mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$. As this holds for any $s \in S$ and $x \in X_s$, one derives $\mathrm{int}_{x_0}(\omega) \in \mathcal{O}_{\mathrm{mod}}(\widetilde{C})$. □

§3.3. Iterated integrals and moderate growth functions

Lemma 3.20. *The following statements hold true:*

(a) *The maps* $e^*\colon \mathcal{O}(D^\times) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{D}^\times)$ *and* $e^*\colon \Omega(D^\times) \to \Omega_{\mathrm{hol}}(\widetilde{D}^\times)$ *have their images respectively contained in* $\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$ *and* $\Omega_{\mathrm{mod}}(\widetilde{D}^\times)$.

(b) *The maps* $p^*\colon \mathcal{O}(C) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ *and* $p^*\colon \Omega(C) \to \Omega_{\mathrm{hol}}(\widetilde{C})$ *have their images respectively contained in* $\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$ *and* $\Omega_{\mathrm{mod}}(\widetilde{C})$.

Proof. (a) Let $f \in \mathcal{O}(D^\times)$. Then there exist $n \geq 1$ and $g \in \mathcal{O}(D)$ such that $f = g/z^n$. Then $e^*f = e^*g/(u \mapsto e(nu))$. The function g is bounded on D, therefore there exists $C \in \mathbb{R}_+$ such that for any $u \in \widetilde{D}^\times$, $|(e^*f)(u)| \leq C|1/e(nu)| = Ce^{2\pi ny}$.

So e^*f satisfies the condition of Definition 3.1 with the pair $(n,(a,b)\mapsto C)$, so $e^*f\in\mathcal{O}_{\mathrm{mod}}(D^\times)$.

Let $\omega\in\Omega(D^\times)$. There exists $f\in\mathcal{O}(D^\times)$ such that $\omega=f\cdot(dz/z)$. Then $e^*\omega=e^*f\cdot e^*(dz/z)\in\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)\cdot e^*(dz/z)=\Omega_{\mathrm{mod}}(\widetilde{D}^\times)$ (see Definition 3.14).

(b) Let $f\in\mathcal{O}(C)$ and $s\in S$, $x\in X_t$. Then it follows from $\varphi_s^\times\circ e=p\circ\tilde{\varphi}_{s,x}^\times$ that $(\tilde{\varphi}_{s,x}^\times)^*p^*f=e^*(\varphi_s^\times)^*f$. Then $(\varphi_s^\times)^*f\in\mathcal{O}(D^\times)$, and (a) then implies $e^*(\varphi_s^\times)^*f\in\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$. It follows that $(\tilde{\varphi}_{s,x}^\times)^*p^*f\in\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$ for any pair (s,x), therefore by Definition 3.9, that $p^*f\in\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$.

Similarly, let $\omega\in\Omega(C)$ and $s\in S$, $x\in X_s$. Then $(\tilde{\varphi}_{s,x}^\times)^*p^*\omega=e^*(\varphi_s^\times)^*\omega$, and since $(\varphi_s^\times)^*\omega\in\Omega(D^\times)$, (a) implies $e^*(\varphi_s^\times)^*\omega\in\Omega_{\mathrm{mod}}(\widetilde{D}^\times)$. It follows that $(\tilde{\varphi}_{s,x}^\times)^*p^*\omega\in\Omega_{\mathrm{mod}}(\widetilde{D}^\times)$ for any pair (s,x), therefore by Definition 3.16, that $p^*f\in\Omega_{\mathrm{mod}}(\widetilde{C})$. □

Proposition 3.21. *For any $x_0\in\widetilde{C}$, the image of $I_{x_0}\colon \mathrm{Sh}(\Omega(C))\to\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ is contained in $\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$; therefore it induces an algebra morphism $I_{x_0}\colon \mathrm{Sh}(\Omega(C))\to\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$.*

Proof. We prove inductively on $n\geq 0$ that $\mathrm{im}(I_{x_0}^{(n)})\subset\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$. This is obvious if $n=0$. Assume that $\mathrm{im}(I_{x_0}^{(n-1)})\subset\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$. Let $\omega_1,\dots,\omega_n\in\Omega(C)$. By the induction hypothesis, $I_{x_0}^{(n-1)}([\omega_1|\cdots|\omega_{n-1}])\in\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$, and by Lemma 3.20(b), $p^*\omega_n\in\Omega_{\mathrm{mod}}(\widetilde{C})$; since $\Omega_{\mathrm{mod}}(\widetilde{C})$ is a module over $\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$ (see Lemma 3.17), then $I_{x_0}^{(n-1)}([\omega_1|\cdots|\omega_{n-1}])\cdot p^*\omega_n\in\Omega_{\mathrm{mod}}(\widetilde{C})$. Lemma 3.19 then implies the relation

$$\mathrm{int}_{x_0}(I_{x_0}^{(n-1)}([\omega_1|\cdots|\omega_{n-1}])\cdot p^*\omega_n)\in\mathcal{O}_{\mathrm{mod}}(\widetilde{C}).$$

By (2.1), the latter term is $I_{x_0}^{(n)}([\omega_1|\cdots|\omega_n])$, and therefore $I_{x_0}^{(n)}([\omega_1|\cdots|\omega_n])\in\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$, so $\mathrm{im}(I_{x_0}^{(n)})\subset\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$. □

§4. The isomorphism of filtered algebras $f_{J,x_0}\colon F_\bullet\,\mathrm{Sh}(\mathrm{H}_C)\otimes\mathcal{O}(C)\to F_\bullet\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$

We start this section with reminders on filtrations (Section 4.1). We compute the restricted dual $(\mathbb{C}\Gamma_C)'$ of the Hopf algebra $\mathbb{C}\Gamma_C$ in Section 4.2. We define a pairing p_{J,x_0} between this Hopf algebra and $F_\infty\,\mathrm{Sh}(\mathrm{H}_C)=\mathrm{Sh}(\mathrm{H}_C)$ in Section 4.3 and prove in Section 4.4 that the induced Hopf algebra morphism $\nu(p_{J,x_0})$ is an isomorphism (Proposition 4.11). We define a filtered algebra morphism f_{J,x_0} in Section 4.5 and use Proposition 4.11 and Proposition B.18(b) to show in Proposition 4.17 that it is an isomorphism. In Section 4.6 we use the material of this proof together with Proposition B.18(c) to prove a filtered formality statement.

§4.1. Background on filtrations

A *vector space filtration* of a $\mathbb{C}$-vector space M is an increasing collection $F_\bullet M = (F_n M)_{n\geq 0}$ of vector subspaces of M. The filtration $F_\bullet M$ is called *exhaustive* if and only if $\bigcup_{n\geq 0} F_n M = M$. If $F_\bullet M$ is a filtration of M, then $F_\infty M \coloneqq \bigcup_{n\geq 0} F_n M$ is a vector subspace of M, called the *total vector space* of the filtration; then $F_\bullet M$ is an exhaustive filtration of this vector subspace. The *associated graded* of $F_\bullet M$ is the graded $\mathbb{C}$-vector space $\mathrm{gr}_\bullet(M) \coloneqq \bigoplus_{n\geq 0} \mathrm{gr}_n(M)$, where $\mathrm{gr}_n(M) = F_n M/F_{n-1}M$ (with $F_{-1}M \coloneqq 0$).

Let $f\colon M \to N$ be a morphism of $\mathbb{C}$-vector spaces and $F_\bullet M$ be a filtration of M. Then f is said to be *compatible* with a filtration $F_\bullet N$ on N if and only if $f(F_n M) \subset F_n N$ for any $n \geq 0$. This is the case in particular if $F_\bullet N$ is the *image of* $F_\bullet M$ *by* f (denoted $f(F_\bullet M)$), defined by $F_n N \coloneqq f(F_n M)$ for any $n \geq 0$; one then has $F_\infty N = f(F_\infty M)$.

Lemma 4.1. *If M, N are filtered vector spaces and $f\colon M \to N$ is a linear map compatible with the filtrations, and such that $\mathrm{gr}_\bullet(f)\colon \mathrm{gr}_\bullet(M) \to \mathrm{gr}_\bullet(N)$ is an isomorphism of graded vector spaces, then the maps $F_n f\colon F_n M \to F_n N$ for any $n \geq 0$, as well as $F_\infty f\colon F_\infty M \to F_\infty N$, are linear isomorphisms.*

Proof. Let us prove the first statement by induction on $n \geq 0$. The map $F_0 f$: $F_0 M \to F_0 N$ is the composition of the isomorphisms $F_0 M \simeq \mathrm{gr}_0 M$, $\mathrm{gr}_0 N \simeq F_0 N$, and $\mathrm{gr}_0(f)$, which is an isomorphism, therefore $F_0 f\colon F_0 M \to F_0 N$ is an isomorphism of vector spaces. Let $n \geq 0$ and assume that $F_n f$ is an isomorphism of vector spaces. The image of $\ker(F_{n+1} f) \subset F_{n+1} M \to \mathrm{gr}_{n+1} M$ is contained in $\ker(\mathrm{gr}_{n+1} f)$ which is 0 by assumption, so this image is 0, which implies that $\ker(F_{n+1} f) \subset F_n M$; the restriction of $F_{n+1} f$ to $F_n M$ coincides with $F_n f$, which by the induction hypothesis is injective, therefore $\ker(F_{n+1} f) = 0$, so $F_{n+1} f$ is injective. For $y \in F_{n+1} N$, let $\bar{y}$ be its image in $\mathrm{gr}_{n+1} N$. By the surjectivity of $\mathrm{gr}_{n+1}(f)$, there exists $\alpha \in \mathrm{gr}_{n+1} M$ with image $\bar{y}$ by $\mathrm{gr}_{n+1}(f)$. Then if $x \in F_{n+1} M$ is any lift of α, one has $F_{n+1}(x) \equiv y \bmod F_n N$. Then $y - F_{n+1}(x) \in F_n N$. Since $F_n f\colon F_n M \to F_n N$ is surjective, there exists $x_0 \in F_n M$ such that $F_n(x_0) = y - F_{n+1}(x)$. Then $y = F_n(x + x_0)$, which implies the surjectivity of $F_{n+1} f$. It follows that $F_{n+1} f$ is an isomorphism, proving the induction.

One has $\ker(F_\infty f) = \bigcup_{n\geq 0}(\ker(F_\infty f) \cap F_n M) = \bigcup_{n\geq 0} \ker(F_n f) = 0$, where the first equality follows from $F_\infty M = \bigcup_{n\geq 0} F_n M$ and the last equality follows from the injectivity of $F_n f$ for $n \geq 0$; this shows the injectivity of $F_\infty f$.

For any $n \geq 0$, one has $\mathrm{im}(F_\infty M) \supset \mathrm{im}(F_n M)$, and $\mathrm{im}(F_n M) = F_n N$ by the surjectivity of $F_n f$. Then $\mathrm{im}(F_\infty M) \supset \bigcup_{n\geq 0} F_n N = F_\infty N$, which shows the surjectivity of $F_\infty f$. It follows that $F_\infty f$ is a linear isomorphism. □

If $F_\bullet M$ and $F_\bullet N$ are filtrations of $\mathbb{C}$-vector spaces M and N, then a filtration $F_\bullet(M \otimes N)$ of their tensor product $M \otimes N$ is defined by $F_n(M \otimes N) := \sum_{p+q=n} F_p M \otimes F_q N$; we denote it by $F_\bullet M \otimes F_\bullet N$, and we call it the *tensor product of* $F_\bullet M$ *and* $F_\bullet N$.

An *algebra filtration* of a $\mathbb{C}$-algebra A is a vector space filtration $F_\bullet A$ of A, such that $F_n A \cdot F_m A \subset F_{n+m} A$ for $n, m \geq 0$. Then $F_\infty A$ is a subalgebra of A, called the *total algebra* of the filtration; $\mathrm{gr}_\bullet(A)$ is then a graded algebra. If $f\colon A \to B$ is an algebra morphism and $F_\bullet A$ is a filtration of A, then $f(F_\bullet A)$ is an algebra filtration of B. If $F_\bullet A$ and $F_\bullet B$ are filtrations of $\mathbb{C}$-algebras A and B, then $F_\bullet A \otimes F_\bullet B$ is an algebra filtration of $A \otimes B$.

An example of a filtration of an algebra A is the *trivial filtration* $F_\bullet^{\mathrm{triv}} A$ defined by $F_n^{\mathrm{triv}} A = A$ for any $n \geq 0$. If A has a unit, another example is the *unit filtration* $F_\bullet^{\mathrm{unit}} A$ defined by $F_0^{\mathrm{unit}} A = \mathbb{C}1$ and $F_n^{\mathrm{unit}} A = A$ for any $n > 0$.

Similarly, a Hopf algebra filtration of a Hopf algebra H with coproduct Δ_H is a vector space filtration $F_\bullet H$ of H, which is an algebra filtration and such that $\Delta_H(F_n H) \subset \sum_{p+q=n} F_p H \otimes F_q H$ for any $n \geq 0$. Then $F_\infty H$ is a Hopf subalgebra of H and $\mathrm{gr}_\bullet(H)$ is a graded Hopf algebra.

§4.2. Computation of $(\mathbb{C}\boldsymbol{\Gamma}_{\boldsymbol{C}})'$

In Appendix D.1 we recall the category **CHA** of complete Hopf algebras (CHAs) and the functor $\mathbf{HA}_{\mathrm{coco}} \to \mathbf{CHA}$, $H \mapsto H^\wedge$ with source the category $\mathbf{HA}_{\mathrm{coco}}$ of cocommutative Hopf algebras.

Lemma 4.2. *If* Γ *is a free group, there is an isomorphism* $(\mathbb{C}\Gamma)^\wedge \simeq \widehat{T}(\Gamma^{\mathrm{ab}} \otimes \mathbb{C})$ *of CHAs, where* Γ^{ab} *is the abelianization of* Γ *and for* V *a vector space,* $\widehat{T}(V)$ *is the CHA defined as the degree completion of the tensor algebra of* V*, where the elements of* V *are primitive.*

Proof. By assumption, Γ is the free group over a set X. Let $(\gamma_x)_{x\in X}$ be the corresponding generating family. Then $\Gamma^{\mathrm{ab}} \otimes \mathbb{C} = \mathbb{C}X$; let $(v_x)_{x\in X}$ be the canonical generating family of $\mathbb{C}X$. The assignment $\gamma_x \mapsto \exp(v_x)$ (defined as $\sum_{n\geq 0} v_x^{\otimes n}/n!$) for $x \in X$ defines a group morphism $\Gamma \to \widehat{T}(\mathbb{C}X)^\times$, therefore an algebra morphism $\mathbb{C}\Gamma \to \widehat{T}(\mathbb{C}X)$, which is checked to be compatible with coproducts. It is compatible with augmentations, therefore gives rise to a CHA morphism $(\mathbb{C}\Gamma)^\wedge \to \widehat{T}(\mathbb{C}X)$. The assignment $v_x \mapsto \log(\gamma_x)$ (defined as $\sum_{n\geq 1} (-1)^{n+1}(\gamma_x - 1)^n/n$) for $x \in X$ defines a linear map $\mathbb{C}X \to (\mathbb{C}\Gamma)^\wedge$, therefore an algebra morphism $T(\mathbb{C}X) \to (\mathbb{C}\Gamma)^\wedge$, which is checked to be compatible with coproducts. It is compatible with augmentations, therefore giving rise to a CHA morphism $\widehat{T}(\mathbb{C}X) \to (\mathbb{C}\Gamma)^\wedge$. The two constructed CHA morphisms can be checked to be inverses of each other. $\square$

In Appendix D.1 we define a subcategory $\mathbf{HA}_{\rm fd}$ of finite-dimensional Hopf algebras of the category $\mathbf{HA}$ of Hopf algebras, and a duality functor $\mathbf{HA}_{\rm fd} \to \mathbf{HA}$, $H \mapsto H'$. When H is the group algebra of a finitely generated group, H' is as in Definition 1.5.

Lemma 4.3. *Let Γ be the free group over a finite set of generators (for example, $\Gamma = \Gamma_C$). Then $\mathbb{C}\Gamma$ is an object in $\mathbf{HA}_{\rm fd}$ and there is a Hopf algebra isomorphism $(\mathbb{C}\Gamma)' \simeq \mathrm{Sh}((\Gamma^{\rm ab} \otimes \mathbb{C})^*)$ (we denote by V^* the dual of a vector space V).*

Proof. In this proof, we set $V := \Gamma^{\rm ab} \otimes \mathbb{C}$. The first statement follows from the finite generation of Γ. It implies that $(\mathbb{C}\Gamma)^\wedge$ is an object in $\mathbf{CHA}_{\rm fd}$, and by Lemma D.7, the duals $(\mathbb{C}\Gamma)'$ and $((\mathbb{C}\Gamma)^\wedge)'$ are well-defined isomorphic objects in $\mathbf{HA}_{\rm coco}$.

By Lemma 4.2, the CHAs $(\mathbb{C}\Gamma)^\wedge$ and $\widehat{T}(V)$ are isomorphic. Since $(\mathbb{C}\Gamma)^\wedge$ is an object in $\mathbf{CHA}_{\rm fd}$, so is $\widehat{T}(V)$, so $(\mathbb{C}\Gamma)^\wedge$ and $\widehat{T}(V)$ are isomorphic objects in $\mathbf{CHA}_{\rm fd}$. By Lemma D.5(b), this gives rise to an isomorphism $((\mathbb{C}\Gamma)^\wedge)' \simeq \widehat{T}(V)'$ in $\mathbf{HA}_{\rm coco}$. Since V is finite-dimensional, there is an isomorphism $\widehat{T}(V)' \simeq \mathrm{Sh}(V^*)$.

The result follows by composition of these isomorphisms. One knows that Γ_C is a free group over a finite set of generators, which implies that it gives an example of the above statements. ☐

Remark 4.4. Lemma 4.3 is proved in [BGF, Exa. 3.229] when $|X| = 2$.

§4.3. A Hopf pairing $p_{J,x_0}\colon \mathrm{Sh}(\mathrm{H}_C) \otimes \mathbb{C}\Gamma_C \to \mathbb{C}$

Until the end of Section 4, an element $(J, x_0) \in \mathrm{MC}_{\rm nd}(C) \times \widetilde{C}$ will be fixed.

Definition 4.5. Define p_{J,x_0} as the linear map $\mathbb{C}\Gamma_C \otimes \mathrm{Sh}(\mathrm{H}_C) \to \mathbb{C}$ such that $\gamma \otimes a \mapsto I_{x_0}(J_*(a))(\gamma x_0)$ for $\gamma \in \Gamma_C$, $a \in \mathrm{Sh}(\mathrm{H}_C)$.

Lemma 4.6. *The pairing $p_{J,x_0}\colon \mathbb{C}\Gamma_C \otimes \mathrm{Sh}(\mathrm{H}_C) \to \mathbb{C}$ is a Hopf pairing (in the sense of Appendix A.3).*

Proof. Let $p_{J,x_0}\colon \mathbb{C}\Gamma_C \otimes \mathrm{Sh}(\mathrm{H}_C) \to \mathbb{C}$ be the map defined in Definition 4.6. For any $\gamma, \gamma' \in \Gamma_C$ and $a \in \mathrm{Sh}(\mathrm{H}_C)$, one has

$$\begin{aligned} p_{J,x_0}(\gamma\gamma', a) &= I_{x_0}(J_*(a))(\gamma\gamma' x_0) \\ &= I_{x_0}(J_*(a^{(1)}))(\gamma x_0) I_{\gamma x_0}(J_*(a^{(2)}))(\gamma\gamma' x_0) \\ &= I_{x_0}(J_*(a^{(1)}))(\gamma x_0) I_{x_0}(J_*(a^{(2)}))(\gamma' x_0) \\ &= p_{J,x_0}(\gamma, a^{(1)}) p_{J,x_0}(\gamma', a^{(2)}), \end{aligned} \tag{4.1}$$

where the second equality follows from Lemma 2.5 and the fact that $J_*\colon \mathrm{Sh}(\mathrm{H}_C) \to \mathrm{Sh}(\Omega(C))$ is a Hopf algebra morphism, the third equality follows from the invariance of the image of I_{x_0} by the diagonal action of Γ_C (see Lemma 2.3), and the first and last equalities follow from definitions.

Denote by $\sqcup\!\sqcup$ the product in the algebra $\mathrm{Sh}(\mathrm{H}_C)$. Let $\gamma \in \Gamma_C$ and $a, a' \in \mathrm{Sh}(\mathrm{H}_C)$. Then

$$\begin{aligned} p_{J,x_0}(\gamma, a \sqcup\!\sqcup a') &= I_{x_0}(J_*(a \sqcup\!\sqcup a'))(\gamma x_0) \\ &= I_{x_0}(J_*(a))(\gamma x_0) I_{x_0}(J_*(a'))(\gamma x_0) \\ &= p_{J,x_0}(\gamma, a) p_{J,x_0}(\gamma, a') \\ &= p_{J,x_0}(\gamma^{(1)}, a) p_{J,x_0}(\gamma^{(2)}, a'), \end{aligned} \tag{4.2}$$

where the first and third equalities follow from definitions, the second equality follows from the facts that $J_* \colon \mathrm{Sh}(\mathrm{H}_C) \to \mathrm{Sh}(\Omega(C))$ and $I_{x_0} \colon \mathrm{Sh}(\Omega(C)) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ are algebra morphisms, and the last equality follows from the group-likeness of γ for the coproduct of $\mathbb{C}\Gamma_C$.

Equalities (4.1) and (4.2) imply the statement. □

§4.4. Proof that $\nu(p_{J,x_0}) \colon F_\infty \mathrm{Sh}(\mathrm{H}_C) \to (\mathbb{C}\Gamma_C)'$ is a Hopf algebra isomorphism

By Lemma A.9, the Hopf algebra pairing p_{J,x_0} (see Lemma 4.6) gives rise to a Hopf algebra morphism $\nu(p_{J,x_0}) \colon F_\infty \mathrm{Sh}(\mathrm{H}_C) \to (\mathbb{C}\Gamma_C)'$, which we now study.

4.4.1. Construction of a Hopf algebra morphism $\mathrm{Sh}(\mathrm{H}_C) \to \mathrm{Sh}((\mathrm{H}_C^{\mathrm{B}})^*)$.

Let $\mathrm{H}_1(C, \mathbb{Z})$ be the first singular homology group of C with integer coefficients, and let us set $\mathrm{H}_C^{\mathrm{B}} := \mathrm{H}_1(C, \mathbb{Z}) \otimes \mathbb{C}$.

Lemma 4.7. *There is a Hopf algebra isomorphism* $(\mathbb{C}\Gamma_C)' \simeq \mathrm{Sh}((\mathrm{H}_C^{\mathrm{B}})^*)$.

Proof. Since C is an affine curve, the group Γ_C is free. Lemma 4.3 then implies that $(\mathbb{C}\Gamma_C)'$ is isomorphic to $\mathrm{Sh}((\Gamma_C^{\mathrm{ab}} \otimes \mathbb{C})^*)$. The choice of a point x_0 in $\widetilde{C}$ induces an isomorphism $\Gamma_C \simeq \pi_1(C, x_0)$, whose conjugation class is independent of this choice; this isomorphism induces an isomorphism $\Gamma_C^{\mathrm{ab}} \simeq \pi_1(C, x_0)^{\mathrm{ab}} = \mathrm{H}_1^{\mathrm{B}}(C, \mathbb{Z})$ also independent of this choice, from which one derives an isomorphism $\Gamma_C^{\mathrm{ab}} \otimes \mathbb{C} \simeq \mathrm{H}_C^{\mathrm{B}}$. □

Definition 4.8. Define $q_{J,x_0} \colon \mathrm{Sh}(\mathrm{H}_C) \to \mathrm{Sh}((\mathrm{H}_C^{\mathrm{B}})^*)$ as the Hopf algebra morphism obtained by composition of

(a) the Hopf algebra isomorphism $\mathrm{Sh}(\mathrm{H}_C) \xrightarrow{\sim} F_\infty \mathrm{Sh}(\mathrm{H}_C)$ (see Lemma 2.9 with $V = \mathrm{H}_C$),
(b) the Hopf algebra morphism $\nu(p_{J,x_0}) \colon F_\infty \mathrm{Sh}(\mathrm{H}_C) \to (\mathbb{C}\Gamma_C)'$,
(c) the Hopf algebra isomorphism $(\mathbb{C}\Gamma_C)' \simeq \mathrm{Sh}((\mathrm{H}_C^{\mathrm{B}})^*)$ (see Lemma 4.7).

4.4.2. A criterion for a Hopf algebra morphism $\mathrm{Sh}(V) \to \mathrm{Sh}(W)$ to be an isomorphism.

Lemma 4.9. *Let V, W be vector spaces and let $f\colon \mathrm{Sh}(V) \to \mathrm{Sh}(W)$ be a Hopf algebra morphism. Then $f(V) \subset W$, where V, W are the degree 1 subspaces of $\mathrm{Sh}(V)$, $\mathrm{Sh}(W)$. Denote by $\mathrm{gr}_1(f)\colon V \to W$ the corresponding linear map. Then f is a Hopf algebra isomorphism if and only if $\mathrm{gr}_1(f)$ is a vector space isomorphism.*

Proof. Since f is a Hopf algebra morphism, Lemma A.2(c) implies that it induces a linear map $F_nf\colon F_n\,\mathrm{Sh}(V) \to F_n\,\mathrm{Sh}(W)$ for any $n \geq 0$. When $n = 1$, F_1f is a linear map $\mathbb{C} \oplus V \to \mathbb{C} \oplus W$. The compatibility of F_1f with the units and counits on both sides implies that F_1f is the direct sum of $\mathrm{id}_{\mathbb{C}}\colon \mathbb{C} \to \mathbb{C}$ and a linear map $V \to W$, which can be identified with the associated graded of f for the filtration $F_\bullet$.

If f is a Hopf algebra morphism, then for each $n \geq 0$, $F_nf\colon F_n\,\mathrm{Sh}(V) \to F_n\,\mathrm{Sh}(W)$ is a linear isomorphism, which when $n = 1$ implies the same for $\mathrm{id}_{\mathbb{C}} \oplus \mathrm{gr}_1(f)$, which implies that $\mathrm{gr}_1(f)$ is a linear isomorphism.

Assume now that $f\colon \mathrm{Sh}(V) \to \mathrm{Sh}(W)$ is a Hopf algebra morphism such that $\mathrm{gr}_1(f)$ is a linear isomorphism. The associated graded map $\mathrm{gr}_\bullet f\colon \mathrm{gr}_\bullet\,\mathrm{Sh}(V) \to \mathrm{gr}_\bullet\,\mathrm{Sh}(W)$ can be identified, under the canonical isomorphisms $\mathrm{gr}_\bullet\,\mathrm{Sh}(X) \simeq \mathrm{Sh}(X)$ for $X = V, W$ (see Lemma 2.9), with $\mathrm{Sh}(\mathrm{gr}_1(f))$, which is an isomorphism of graded vector spaces. Since the filtrations $F_\bullet$ in the source and target are exhaustive and by Lemma 4.1, this implies that f is an isomorphism. $\square$

4.4.3. Isomorphism status of the linear map $\mathrm{gr}_1(q_{J,x_0})\colon \mathrm{H}_C \to (\mathrm{H}_C^{\mathrm{B}})^*$. For H a Hopf algebra, denote by H_+ the kernel of its counit morphism.

Lemma 4.10. *The following statements hold true:*

(a) *The pairing $(\mathbb{C}\Gamma_C)_+ \otimes \Omega(C) \to \mathbb{C}$ given by $(\gamma - 1) \otimes \omega \mapsto \int_{x_0}^{\gamma x_0} \omega$ for $\gamma \in \Gamma_C$, $\omega \in \Omega$, is independent of x_0. It factors through a pairing*

$$\mathrm{H}_C^{\mathrm{B}} \otimes \mathrm{H}_C = (\mathbb{C}\Gamma_C)_+/(\mathbb{C}\Gamma_C)_+^2 \otimes (\Omega(C)/d\mathcal{O}(C)) \to \mathbb{C}. \tag{4.3}$$

(b) *The linear map $\mathrm{H}_C \to (\mathrm{H}_C^{\mathrm{B}})^*$ induced by (4.3) is equal to $\mathrm{gr}_1(q_{J,x_0})\colon \mathrm{H}_C \to (\mathrm{H}_C^{\mathrm{B}})^*$.*

(c) *The linear map $\mathrm{gr}_1(q_{J,x_0})\colon \mathrm{H}_C \to (\mathrm{H}_C^{\mathrm{B}})^*$ is an isomorphism.*

Proof. (a) If $x_0, x_1 \in \widetilde{C}$, $\gamma \in \Gamma_C$, and $\omega \in \Omega(C)$, then one has $\int_{x_1}^{\gamma x_1} \omega - \int_{x_0}^{\gamma x_0} \omega = (\int_{x_1}^{x_0} - \int_{\gamma x_1}^{\gamma x_0})\omega = 0$ by the Γ_C-invariance of ω; this implies the claimed independence. If $\gamma, \gamma' \in \Gamma_C$ and $\omega \in \Omega(C)$, then one has

$$\Big(\int_{x_0}^{\gamma\gamma' x_0} - \int_{x_0}^{\gamma x_0} - \int_{x_0}^{\gamma' x_0}\Big)\omega = \Big(\int_{\gamma x_0}^{\gamma\gamma' x_0} - \int_{x_0}^{\gamma' x_0}\Big)\omega = 0$$

by the same reason; since the elements $(\gamma-1)(\gamma'-1)$ generate $(\mathbb{C}\Gamma_C)_+^2$, this implies the claimed factorization. The equality follows from $\mathrm{gr}^1(\mathbb{C}\Gamma_C) = \Gamma_C^{\mathrm{ab}} \otimes \mathbb{C} = \mathrm{H}_C^{\mathrm{B}}$ (see [Q2]).

(b) Let O, H be Hopf algebras with $\mathrm{gr}^1(H)$ finite-dimensional and let $p\colon O\otimes H \to \mathbb{C}$ be a Hopf algebra pairing. By Lemma A.9, p gives rise to a Hopf algebra morphism $\nu(p)\colon F_\infty O \to H'$, and by Proposition A.2(c), this morphism is compatible with the filtrations $F_\bullet$ on both sides. For any $n \geq 0$, the restriction of p to $F_nO\otimes H$ induces a pairing $F_nO \otimes (H/F^{n+1}H) \to \mathbb{C}$, which gives rise to a linear map $F_nO \to (H/F^{n+1}H)^* = F_nH^*$; composing this linear map with the identification from Lemma A.7 gives rise to a linear map $F_nO \to F_nH'$, which is equal to $F_n\nu(p)$. The morphism $\nu(p)$ is compatible with the augmentation maps ϵ_O, $\epsilon_{H'}$, and therefore gives rise to a linear map $F_nO\cap O_+ \to F_nH'\cap(H')_+$, which for $n = 1$ coincides with $\mathrm{gr}_1(\nu(p))$. It follows that $\mathrm{gr}_1(\nu(p))$ may be constructed as follows: the restriction of p to $(F_1O \cap J_O) \otimes F^1H$ induces a pairing $(F_1O \cap O_+) \otimes (F^1H/F^2H) \to \mathbb{C}$; then $\mathrm{gr}_1(\nu(p))$ is the induced map $\mathrm{gr}_1(O) = F_1O\cap O_+ \to (F^1H/F^2H)^* = \mathrm{gr}_1(H')$.

It follows that $\mathrm{gr}_1(\nu(p_{J,x_0}))\colon \mathrm{H}_C = \mathrm{gr}_1(\mathrm{Sh}(\mathrm{H}_C)) \to \mathrm{gr}_1((\mathbb{C}\Gamma_C)') = (\Gamma_C^{\mathrm{ab}} \otimes \mathbb{C})^*$ is induced by the restriction of p_{J,x_0} to $\mathrm{H}_C \otimes (\mathbb{C}\Gamma_C)_+$. This restriction coincides with the lift of (4.3) by Lemma 2.13(b), which implies the statement.

(c) The pairing (4.3) coincides with the period pairing for C, which is nondegenerate. It follows that the map $\mathrm{H}_C \to (\mathrm{H}_C^{\mathrm{B}})^*$ induced by (4.3) is an isomorphism. The statement then follows from (b). $\square$

4.4.4. Proof that $\nu(p_{J,x_0})\colon \mathrm{Sh}(\mathrm{H}_C) \to (\mathbb{C}\Gamma_C)'$ is a Hopf algebra isomorphism.

Proposition 4.11. *The map $\nu(p_{J,x_0})\colon \mathrm{Sh}(\mathrm{H}_C) \to (\mathbb{C}\Gamma_C)'$ is a Hopf algebra isomorphism.*

Proof. By Lemma 4.10(c) and Lemma 4.9, q_{J,x_0} is a Hopf algebra isomorphism. By Definition 4.8, $\nu(p_{J,x_0})$ is obtained from q_{J,x_0} by pre- and post-composition with Hopf algebra isomorphisms, which implies that it is a Hopf algebra isomorphism. $\square$

Remark 4.12. Proposition 4.11 may be related to Chen's π_1 theorem as follows. Let $(\mathrm{H}^\bullet_{\mathrm{dR}}(C), \varepsilon_{\mathrm{triv}})$ be the augmented dga with $\mathrm{H}^\bullet_{\mathrm{dR}}(C) := \mathbb{C}\oplus\mathrm{H}_C$ with zero differential and $\varepsilon_{\mathrm{triv}}\colon \mathrm{H}^\bullet_{\mathrm{dR}}(C) \to \mathbb{C}$ be the projection in degree 0, and let $(\mathcal{E}^\bullet(C), \varepsilon_{x_0})$ be the augmented dga of smooth differential forms on C, with ε_{x_0} given by evaluation at x_0. Then the dga morphism $\mathrm{H}^\bullet_{\mathrm{dR}}(C) \to \mathcal{E}^\bullet(C)$ induced by J is compatible with the augmentations, therefore it induces an isomorphism of commutative Hopf algebras $H^0(B(\mathrm{H}^\bullet_{\mathrm{dR}}(C), \varepsilon_{\mathrm{triv}})) \to H^0(B(\mathcal{E}^\bullet(C), \varepsilon_{x_0}))$, where $H^0(B(-))$ is the zeroth cohomology of the bar-construction of an augmented dga. One easily constructs a Hopf algebra isomorphism $H^0(B(\mathrm{H}^\bullet_{\mathrm{dR}}(C), \varepsilon_{\mathrm{triv}})) \simeq \mathrm{Sh}(\mathrm{H}_C)$. The combination of these isomorphisms with the Hopf algebra isomorphism $H^0(B(\mathcal{E}^\bullet(C), \varepsilon_{x_0})) \simeq (\mathbb{C}\Gamma_C)'$ from Chen's "π_1 de Rham theorem" ([BGF, Thm. 3.264]) is the Hopf algebra isomorphism from Proposition 4.11.

§4.5. The isomorphism of filtered algebras $f_{J,x_0} \colon F_\bullet \mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C) \to F_\bullet \mathcal{O}_{\mathrm{mod}}(\widetilde{C})$

Lemma-Definition 4.13. *There is a unique linear map $f_{J,x_0} \colon \mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C) \to \mathcal{O}_{\mathrm{mod}}(\widetilde{C})$ such that $f_{J,x_0}(a \otimes f) := I_{x_0} \circ J_*(a) \cdot p^* f$; it is an algebra morphism.*

Proof. The fact that f_{J,x_0} is well defined as a linear map follows from Proposition 3.21 and from the inclusion $\mathcal{O}(C) \subset \mathcal{O}_{\mathrm{mod}}(\widetilde{C})$, which follows from Proposition 3.12. The fact that it is an algebra morphism follows from the decomposition of f_{J,x_0} as

$$\mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C) \xrightarrow{(I_{x_0} \circ J_*) \otimes p^*} \mathcal{O}_{\mathrm{mod}}(\widetilde{C})^{\otimes 2} \xrightarrow{m_{\mathcal{O}_{\mathrm{mod}}(\widetilde{C})}} \mathcal{O}_{\mathrm{mod}}(\widetilde{C}),$$

where $m_{\mathcal{O}_{\mathrm{mod}}(\widetilde{C})}$ is the product map of $\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$, and from the algebra morphism status of I_{x_0}, p^*, J_* and $m_{\mathcal{O}_{\mathrm{mod}}(\widetilde{C})}$ (the latter coming from the commutativity of $\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$). $\square$

In Appendix B, we introduce the notions of Hopf algebra with comodule algebra (HACA) and Hopf algebra with module algebra (HAMA). Then, by Proposition B.18(a), the algebra $\mathcal{O}(C)$ and the Hopf algebra $\mathrm{Sh}(\mathrm{H}_C)$ give rise to a HACA structure $(\mathrm{Sh}(\mathrm{H}_C), \mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C))$; on the other hand, a HAMA structure is constructed as follows.

Lemma 4.14. *The pair $(\mathcal{O}_{\mathrm{mod}}(\widetilde{C}), \mathbb{C}\Gamma_C)$ is equipped with a HAMA structure.*

Proof. The HAMA structure is induced by the right Γ_C-action on $\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$ (see Proposition 3.10 and Definition B.6). $\square$

In Appendix B (see Definition B.12), we also introduce the notion of a pairing-morphism from a HACA (O, A) to a HAMA (B, H), and denote by $\mathbf{PM}((O, A), (B, H))$ the set of such structures.

Recall the Hopf algebra pairing $p_{J,x_0} \colon \mathrm{Sh}(\mathrm{H}_C) \otimes \mathbb{C}\Gamma_C \to \mathbb{C}$ (see Lemma 4.6).

Lemma 4.15. $(p_{J,x_0}, f_{J,x_0}) \in \mathbf{PM}((\mathrm{Sh}(\mathrm{H}_C), \mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C)), (\mathcal{O}_{\mathrm{mod}}(\widetilde{C}), \mathbb{C}\Gamma_C))$.

Proof. By Definition B.12, the identity to check is

$$\big(I_{x_0}(J_*(a)) p^*(f)\big)_{|\gamma} = I_{x_0}(J_*(a^{(2)})) p^*(f) p_{J,x_0}(\gamma \otimes a^{(1)})$$

for any $a \in \mathrm{Sh}(\mathrm{H}_C)$, $f \in \mathcal{O}(C)$, and $\gamma \in \Gamma_C$, using the notation $\Delta_{\mathrm{Sh}(\mathrm{H}_C)}(a) = a^{(1)} \otimes a^{(2)}$. This follows from the invariance of $p^* f$ under the action of Γ_C and from the identity $I_{x_0}(J_*(a))_{|\gamma} = I_{x_0}(J_*(a^{(2)}))p_{J,x_0}(\gamma \otimes a^{(1)})$ which is proved as follows: for any $x \in \widetilde{C}$, one has

$$\begin{aligned} I_{x_0}(J_*(a))_{|\gamma}(x) = I_{x_0}(J_*(a))(\gamma x) &= I_{x_0}(J_*(a^{(1)}))(\gamma x_0) I_{\gamma x_0}(J_*(a^{(2)}))(\gamma x) \\ &= p_{J,x_0}(\gamma \otimes a^{(1)}) I_{x_0}(J_*(a^{(2)}))(x), \end{aligned}$$

where the second identity follows from Lemma 2.5 and the third identity follows from the definition of p_{J,x_0} and Lemma 2.3. □

Lemma 4.16. *The HAMA structure $(\mathcal{O}_{\mathrm{mod}}(\widetilde{C}), \mathbb{C}\Gamma_C)$ gives rise to an algebra filtration $F_\bullet \mathcal{O}_{\mathrm{mod}}(\widetilde{C})$ of $\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$, which fits in a HACA $((\mathbb{C}\Gamma_C)', F_\infty \mathcal{O}_{\mathrm{mod}}(\widetilde{C}))$.*

Proof. The construction of the said HACA from this HAMA follows from Lemma B.8 and the fact that $\mathbb{C}\Gamma_C$ is an object in $\mathbf{HA}_{\mathrm{fd}}$ (see the proof of Lemma 4.3). The filtration $F_\bullet \mathcal{O}_{\mathrm{mod}}(\widetilde{C})$ is then as in the introduction (see Definition 1.10). □

Proposition 4.17. *The following statements hold true:*

(a) $(\nu(p_{J,x_0}), f_{J,x_0})$ *induces an isomorphism of HACAs*

$$(\mathrm{Sh}(\mathrm{H}_C), \mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C)) \to ((\mathbb{C}\Gamma_C)', F_\infty \mathcal{O}_{\mathrm{mod}}(\widetilde{C})).$$

(b) f_{J,x_0} *induces an isomorphism of algebra filtrations* $F_\bullet \mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C) \to F_\bullet \mathcal{O}_{\mathrm{mod}}(\widetilde{C})$.

Proof. Let $\mathbf{a} := \mathcal{O}(C)$, $O := \mathrm{Sh}(\mathrm{H}_C)$, $B := \mathcal{O}_{\mathrm{mod}}(\widetilde{C})$, $H := \mathbb{C}\Gamma_C$, (B, H) be the HAMA structure induced by the action of Γ_C on $\widetilde{C}$; it is an object in $\mathbf{HAMA}_{\mathrm{fd}}$ (see Definition B.9) since Γ_C is finitely generated. Set $p := p_{J,x_0} \in \mathbf{Pair}(O, H)$ (see Definition A.8), $f := f_{J,x_0}$. By Lemma 4.15, $(p, f) \in \mathbf{PM}((O, O \otimes a), (B, H))$. By Proposition 4.11, $\nu(p_{J,x_0})\colon \mathrm{Sh}(\mathrm{H}_C) \to (\mathbb{C}\Gamma_C)'$ is a Hopf algebra isomorphism and by Proposition 3.12, f_{J,x_0} induces an algebra isomorphism $\mathbb{C} \otimes \mathcal{O}(C) \to \mathcal{O}_{\mathrm{mod}}(\widetilde{C})^{\mathbb{C}\Gamma_C}$. The assumptions of Proposition B.18(b) are therefore satisfied; the result is then a consequence of this statement. □

§4.6. Filtered formality of the HACA $((\mathbb{C}\Gamma_C)', F_\infty \mathcal{O}_{\mathrm{mod}}(\widetilde{C}))$

In Appendix C.2 we introduce the definition of a filtered formal HACA.

Proposition 4.18. *The pair $((\mathbb{C}\Gamma_C)', F_\infty \mathcal{O}_{\mathrm{mod}}(\widetilde{C}))$ is filtered formal.*

Proof. It follows from the proof of Lemma 4.17 that the data

$$(O, \mathbf{a}) := (\mathrm{Sh}(\mathrm{H}_C), \mathcal{O}(C)), \quad (B, H) = (\mathcal{O}_{\mathrm{mod}}(\widetilde{C}), \mathbb{C}\Gamma_C), \quad (p, f) = (p_{J,x_0}, f_{J,x_0}) \tag{4.4}$$

satisfy the hypotheses of Proposition B.18(b),(c). The statement is then a consequence of Proposition C.5. □

§5. Filtrations on $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$, and the minimal stable subalgebra A_C

In Section 5.1 we study the filtration of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ given by the image by I_{x_0} of the filtration $F_\bullet(\mathrm{Sh}(\Omega(C)))$ of $\mathrm{Sh}(\Omega(C))$, and identify it with the image by f_{J,x_0} of the filtration $F_\bullet\,\mathrm{Sh}(\mathrm{H}_C) \otimes F_\bullet^{\mathrm{unit}}\mathcal{O}(C)$ of $\mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C)$ (Proposition 5.3). In Section 5.2 we introduce and study the filtrations $F_\bullet^\delta\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ and $F_\bullet^\mu\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ inspired by [Ch] and identify the latter with the image by f_{J,x_0} of the filtration $F_\bullet\,\mathrm{Sh}(\mathrm{H}_C) \otimes F_\bullet^{\mathrm{triv}}\mathcal{O}(C)$ of $\mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C)$ (Proposition 5.6). We study the relation of the total space of these filtrations with the MSSA A_C of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ in Section 5.3. In Section 5.4 we prove Theorems A, B, and C, and in Section 5.5 we draw consequences of Theorem A on the algebras $\mathcal{H}_C(J)$ constructed in Section 2.3, namely we show that each such algebra is a free $\mathcal{O}(C)$-module with an explicit basis. In Section 5.6 we discuss the relation of this material with the study in [Ch] of Picard–Vessiot extensions of the function algebra of a smooth manifold.

§5.1. An algebra filtration of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ defined by I_{x_0}

In the present Section 5.1, a point $x_0 \in \widetilde{C}$ is fixed. Recall the algebra morphism $I_{x_0}\colon \mathrm{Sh}(\Omega(C)) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ (Lemma-Definition 2.1 and Lemma 2.2) and the algebra filtration $F_\bullet\,\mathrm{Sh}(V)$ for an arbitrary vector space V (see Section 2.1). By Section 4.1, these data give rise to an algebra filtration $I_{x_0}(F_\bullet\,\mathrm{Sh}(\Omega(C)))$ of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$, which we study in Proposition 5.3.

Lemma 5.1. *One has*

$$\forall\, p, q > 0, \quad I_{x_0}\big([\mathrm{Sh}_p(\Omega(C))|d\mathcal{O}(C)|\,\mathrm{Sh}_q(\Omega(C))]\big) \subset I_{x_0}\big(\mathrm{Sh}_{p+q}(\Omega(C))\big), \tag{5.1}$$

$$\forall\, n > 0, \quad I_{x_0}\big([d\mathcal{O}(C)|\,\mathrm{Sh}_n(\Omega(C))]\big) \subset I_{x_0}\big(\mathrm{Sh}_n(\Omega(C))\big), \tag{5.2}$$

$$\forall\, n > 0, \quad I_{x_0}\big([\mathrm{Sh}_n(\Omega(C))|d\mathcal{O}(C)]\big) \subset p^*\mathcal{O}(C) \cdot I_{x_0}\big(\mathrm{Sh}_n(\Omega(C))\big). \tag{5.3}$$

Proof. Let us prove (5.1). The space $[\mathrm{Sh}_p(\Omega(C))|d\mathcal{O}(C)|\,\mathrm{Sh}_q(\Omega(C))]$ is linearly spanned by the elements $[\alpha_1|\cdots|\alpha_p|df|\beta_1|\cdots|\beta_q]$, where $\alpha_i, \beta_j \in \Omega(C)$ for any i, j and $f \in \mathcal{O}(C)$. Then

$$\begin{aligned} &I_{x_0}([\alpha_1|\cdots|\alpha_p|df|\beta_1|\cdots|\beta_q]) \\ &\quad = I_{x_0}([\alpha_1|\cdots|\alpha_p|f\cdot\beta_1|\cdots|\beta_q] - [\alpha_1|\cdots|\alpha_p\cdot f|\beta_1|\cdots|\beta_q]) \in I_{x_0}\big(\mathrm{Sh}_{p+q}(\Omega(C))\big). \end{aligned}$$

Let us prove (5.2). The space $[d\mathcal{O}(C)|\operatorname{Sh}_n(\Omega(C))]$ is linearly spanned by the elements $[df|\alpha_1|\cdots|\alpha_n]$, where $f\in\mathcal{O}(C)$ and $\alpha_i\in\Omega(C)$ for any i. Then

$$\begin{aligned}&I_{x_0}([df|\alpha_1|\cdots|\alpha_n])\\&\quad=I_{x_0}([f\cdot\alpha_1|\cdots|\alpha_n])-f(x_0)\cdot I_{x_0}([\alpha_1|\cdots|\alpha_n])\in I_{x_0}\big(\operatorname{Sh}_n(\Omega(C))\big).\end{aligned}$$

Equation (5.3) similarly follows from

$$\begin{aligned}&I_{x_0}([\alpha_1|\cdots|\alpha_n|df])\\&\quad=p^*f\cdot I_{x_0}([\alpha_1|\cdots|\alpha_n])-I_{x_0}([\alpha_1|\cdots|\alpha_n\cdot f])\in p^*\mathcal{O}(C)\cdot I_{x_0}\big(\operatorname{Sh}_n(\Omega(C))\big)\square\end{aligned}$$

Lemma 5.2. *For any $\sigma\in\Sigma_C$ and any $n\geq 0$, one has the inclusion*

$$I_{x_0}\big(F_n\operatorname{Sh}(\Omega(C))\big)\subset f_{J_\sigma,x_0}(F_n\operatorname{Sh}(\mathrm{H}_C)\otimes\mathbb{C}1+F_{n-1}\operatorname{Sh}(\mathrm{H}_C)\otimes\mathcal{O}(C)).$$

Proof. By induction on $n\geq 0$. For $n=0$, the equality is obvious as both sides are equal to $\mathbb{C}$. Let $n>0$, assume the equality for all steps $\leq n-1$, and let us prove it at step n. By the induction hypothesis, it suffices to prove the inclusion $I_{x_0}(\operatorname{Sh}_n(\Omega(C)))\subset f_{J_\sigma,x_0}(F_n\operatorname{Sh}(\mathrm{H}_C)\otimes\mathbb{C}1+F_{n-1}\operatorname{Sh}(\mathrm{H}_C)\otimes\mathcal{O}(C))$, i.e.

$$I_{x_0}\big(\operatorname{Sh}_n(\Omega(C))\big)\subset I_{x_0}\big(F_n\operatorname{Sh}(\sigma(\mathrm{H}_C))\big)+p^*\mathcal{O}(C)\cdot I_{x_0}(F_{n-1}\operatorname{Sh}(\mathrm{H}_C)).\tag{5.4}$$

The space $\operatorname{Sh}_n(\Omega(C))$ is linearly spanned by the elements $[\omega_1|\cdots|\omega_n]$ where ω_1, $\ldots,\omega_n$ belong to $\Omega(C)$. For all i, let $h_i\in\mathrm{H}_C$ be the projection of ω_i and choose $f_i\in\mathcal{O}(C)$ such that $\omega_i=\sigma(h_i)+df_i$. Then

$$\begin{aligned}&[\omega_1|\cdots|\omega_n]\in[\sigma(h_1)|\cdots|\sigma(h_n)]+[df_1|\operatorname{Sh}_{n-1}(\Omega(C))]+[\operatorname{Sh}_{n-1}(\Omega(C))|df_{n-1}]\\&\quad+\sum_{i=1}^{n-1}[\operatorname{Sh}_{i-1}(\Omega(C))|df_i|\operatorname{Sh}_{n-i}(\Omega(C))]\subset\operatorname{Sh}_n(\sigma(\mathrm{H}_C))+[d\mathcal{O}(C)|\operatorname{Sh}_{n-1}(\Omega(C))]\\&\quad+[\operatorname{Sh}_{n-1}(\Omega(C))|d\mathcal{O}(C)]+\sum_{i=1}^{n-1}[\operatorname{Sh}_{i-1}(\Omega(C))|d\mathcal{O}(C)|\operatorname{Sh}_{n-i}(\Omega(C))].\end{aligned}$$

Lemma 5.1 then implies that

$$\begin{aligned}I_{x_0}([\omega_1|\cdots|\omega_n])&\in I_{x_0}\big(\operatorname{Sh}_n(\sigma(\mathrm{H}_C))\big)+p^*\mathcal{O}(C)\cdot I_{x_0}\big(\operatorname{Sh}_{n-1}(\Omega(C))\big)\\&\qquad+p^*\mathcal{O}(C)\cdot I_{x_0}\big(\operatorname{Sh}_{n-1}(\Omega(C))\big)+\sum_{i=1}^{n-1}I_{x_0}\big(\operatorname{Sh}_{n-1}(\Omega(C))\big)\\&=I_{x_0}\big(\operatorname{Sh}_n(\sigma(\mathrm{H}_C))\big)+p^*\mathcal{O}(C)\cdot I_{x_0}\big(\operatorname{Sh}_{n-1}(\Omega(C))\big).\end{aligned}\tag{5.5}$$

Moreover,

$$\begin{aligned}p^*\mathcal{O}(C)\cdot I_{x_0}\big(\operatorname{Sh}_{n-1}(\Omega(C))\big)&\subset p^*\mathcal{O}(C)\cdot I_{x_0}\big(F_{n-1}\operatorname{Sh}(\Omega(C))\big)\\&\subset p^*\mathcal{O}(C)\cdot I_{x_0}\big(F_{n-1}\operatorname{Sh}(\sigma(\mathrm{H}_C))\big),\end{aligned}$$

where the first inclusion follows from $\mathrm{Sh}_{n-1}(\Omega(C)) \subset F_{n-1}\,\mathrm{Sh}(\Omega(C))$ and the second inclusion from the induction hypothesis, i.e. (5.4) at step $n-1$ by multiplication by $p^*\mathcal{O}(C)$. Combining this inclusion with (5.5), one obtains

$$I_{x_0}([\omega_1|\cdots|\omega_n]) \in I_{x_0}\big(\mathrm{Sh}_n(\sigma(\mathrm{H}_C))\big) + p^*\mathcal{O}(C) \cdot I_{x_0}\big(F_{n-1}\,\mathrm{Sh}(\sigma(\mathrm{H}_C))\big),$$

which is (5.4) at step n. □

Proposition 5.3. *For any $J \in \mathrm{MC}_{\mathrm{nd}}(C)$, one has the equality*

$$I_{x_0}\big(F_\bullet\,\mathrm{Sh}(\Omega(C))\big) = f_{J,x_0}(F_\bullet\,\mathrm{Sh}(\mathrm{H}_C) \otimes \mathbb{C}1 + F_{\bullet-1}\,\mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C))$$

of filtrations of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$, where f_{J,x_0} is the algebra morphism from Lemma-Definition 4.13.

Proof. For $n \geq 0$, the space $f_{J,x_0}(F_n\,\mathrm{Sh}(\mathrm{H}_C) \otimes \mathbb{C}1 + F_{n-1}\,\mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C))$ is equal to $I_{x_0}(J_*(F_n\,\mathrm{Sh}(\mathrm{H}_C))) + p^*(\mathcal{O}(C)) \cdot I_{x_0}(J_*(F_{n-1}\,\mathrm{Sh}(\mathrm{H}_C)))$, which is also equal to $I_{x_0}(J_*(F_n\,\mathrm{Sh}(\mathrm{H}_C)) + d\mathcal{O}(C) \sqcup J_*(F_{n-1}\,\mathrm{Sh}(\mathrm{H}_C)))$ since $p^*(\mathcal{O}(C)) = \mathbb{C} + I_{x_0}([d\mathcal{O}(C)])$. This implies the equality

$$\begin{aligned} &f_{J,x_0}(F_n\,\mathrm{Sh}(\mathrm{H}_C) \otimes \mathbb{C}1 + F_{n-1}\,\mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C)) \\ &\qquad = I_{x_0}\big(J_*(F_n\,\mathrm{Sh}(\mathrm{H}_C)) + d\mathcal{O}(C) \sqcup J_*(F_{n-1}\,\mathrm{Sh}(\mathrm{H}_C))\big). \end{aligned} \tag{5.6}$$

Let us now prove

$$\begin{aligned} \forall n \geq 0,\quad & I_{x_0}\big(F_n\,\mathrm{Sh}(\Omega(C))\big) \\ &\qquad = I_{x_0}\big(J_*(F_n\,\mathrm{Sh}(\mathrm{H}_C)) + d\mathcal{O}(C) \sqcup J_*(F_{n-1}\,\mathrm{Sh}(\mathrm{H}_C))\big), \end{aligned} \tag{5.7}$$

The argument of I_{x_0} in the right-hand side of (5.7) is contained in $F_n\,\mathrm{Sh}(\Omega(C))$, which implies the inclusion (left-hand side of (5.7)) $\supset$ (right-hand side of (5.7)).

We now prove the opposite inclusion. There is a sequence of inclusions (in $\mathrm{Sh}(\Omega(C))$)

$$\begin{aligned} (\sigma_J)_*(F_n\,\mathrm{Sh}(\mathrm{H}_C)) + d\mathcal{O}(C) \sqcup (\sigma_J)_*(F_{n-1}\,\mathrm{Sh}(\mathrm{H}_C)) &\subset J_*(F_n\,\mathrm{Sh}(\mathrm{H}_C)) \\ + F_{n-1}\,\mathrm{Sh}(\Omega(C)) + d\mathcal{O}(C) \sqcup J_*(F_{n-1}\,\mathrm{Sh}(\mathrm{H}_C)) &+ d\mathcal{O}(C) \sqcup F_{n-2}\,\mathrm{Sh}(\Omega(C)) \\ = J_*(F_n\,\mathrm{Sh}(\mathrm{H}_C)) + d\mathcal{O}(C) \sqcup J_*(F_{n-1}\,\mathrm{Sh}(\mathrm{H}_C)) &+ F_{n-1}\,\mathrm{Sh}(\Omega(C)), \end{aligned}$$

σ_J being as in Lemma-Definition 2.12, where the first inclusion follows from Lemma 2.13(b) and the second inclusion follows from $d\mathcal{O}(C) \sqcup F_{n-2}\,\mathrm{Sh}(\Omega(C)) \subset F_{n-1}\,\mathrm{Sh}(\Omega(C))$. One has therefore

$$\begin{aligned} \forall n \geq 0,\quad & (\sigma_J)_*(F_n\,\mathrm{Sh}(\mathrm{H}_C)) + d\mathcal{O}(C) \sqcup (\sigma_J)_*(F_{n-1}\,\mathrm{Sh}(\mathrm{H}_C)) \\ &\subset J_*(F_n\,\mathrm{Sh}(\mathrm{H}_C)) + d\mathcal{O}(C) \sqcup J_*(F_{n-1}\,\mathrm{Sh}(\mathrm{H}_C)) + F_{n-1}\,\mathrm{Sh}(\Omega(C)). \end{aligned} \tag{5.8}$$

For any $n \geq 0$, one then has

$$\begin{aligned}
I_{x_0}\big(F_n\big(\mathrm{Sh}(\Omega(C))\big)\big) &\subset f_{J_{\sigma_J},x_0}(F_n \mathrm{Sh}(\mathrm{H}_C) \otimes \mathbb{C}1 + F_{n-1} \mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C)) \\
&= I_{x_0}\big((\sigma_J)_*(F_n \mathrm{Sh}(\mathrm{H}_C)) + d\mathcal{O}(C) ⧢ (\sigma_J)_*(F_{n-1} \mathrm{Sh}(\mathrm{H}_C))\big) \\
&\subset I_{x_0}\big(J_*(F_n \mathrm{Sh}(\mathrm{H}_C)) + d\mathcal{O}(C) ⧢ J_*(F_{n-1} \mathrm{Sh}(\mathrm{H}_C)) + F_{n-1} \mathrm{Sh}(\Omega(C))\big) \\
&= I_{x_0}\big(J_*(F_n \mathrm{Sh}(\mathrm{H}_C)) + d\mathcal{O}(C) ⧢ J_*(F_{n-1} \mathrm{Sh}(\mathrm{H}_C))\big) + I_{x_0}\big(F_{n-1} \mathrm{Sh}(\Omega(C))\big) \\
&= f_{J,x_0}(F_n \mathrm{Sh}(\mathrm{H}_C) \otimes \mathbb{C}1 + F_{n-1} \mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C)) + I_{x_0}\big(F_{n-1} \mathrm{Sh}(\Omega(C))\big),
\end{aligned}$$

where the first relation follows from Lemma 5.2, the second relation follows from (5.6) applied to J_{σ_J}, the third relation follows from (5.8), and the last relation follows from the second relation follows from (5.6) applied to J. The relation

$$\forall\, n \geq 0, \quad I_{x_0}\big(F_n\big(\mathrm{Sh}(\Omega(C))\big)\big) \subset f_{J,x_0}(F_n \mathrm{Sh}(\mathrm{H}_C) \otimes \mathbb{C}1 + F_{n-1} \mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C))$$

then follows by induction. Therefore (left-hand side of (5.7)) $\subset$ (right-hand side of (5.7)), which ends the proof of (5.7).

The result then follows from the combination of (5.7) and (5.6). □

§5.2. The filtrations $F^\delta_\bullet \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ and $F^\mu_\bullet \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$

In Definition 1.12, we defined $F^\delta_\bullet \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$, $F^\mu_\bullet \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$, $F^\delta_\infty \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$, and $F^\mu_\infty \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ (see Theorem C).

Proposition 5.4. *For any $x_0 \in \widetilde{C}$, one has the equality*

$$F^\delta_\bullet \mathcal{O}_{\mathrm{hol}}(\widetilde{C}) = I_{x_0}\big(F_\bullet \mathrm{Sh}(\Omega(C))\big)$$

of filtrations of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$.

Proof. Let us prove

$$\forall\, n \geq 0, \quad I_{x_0}\big(F_n \mathrm{Sh}(\Omega(C))\big) = F^\delta_n \mathcal{O}_{\mathrm{hol}}(\widetilde{C}) \tag{5.9}$$

by induction on n. For $n = 0$, the equality holds since both sides are equal to $\mathbb{C}$. Assume the equality at step $n \geq 0$ and let us show it at step $n + 1$.

Let us first show the inclusion $I_{x_0}(F_{n+1} \mathrm{Sh}(\Omega(C))) \subset F^\delta_{n+1}\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$. For this, in view of the induction hypothesis, it suffices to prove $I_{x_0}(\mathrm{Sh}_{n+1}(\Omega(C))) \subset F^\delta_{n+1}\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$. The space is linearly spanned by the elements $[\omega_1|\cdots|\omega_{n+1}]$, where $\omega_1, \ldots, \omega_{n+1} \in \Omega(C)$. Then $d(I_{x_0}([\omega_1|\cdots|\omega_{n+1}])) = I_{x_0}([\omega_1|\cdots|\omega_n]) \cdot \omega_{n+1}$, and $I_{x_0}([\omega_1|\cdots|\omega_n]) \in I_{x_0}(\mathrm{Sh}_n(\Omega(C))) \subset F^\delta_n \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$, where the last inclusion follows from the induction hypothesis. This shows that $I_{x_0}([\omega_1|\cdots|\omega_{n+1}]) \in F^\delta_{n+1}\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$, therefore $I_{x_0}(\mathrm{Sh}_{n+1}(\Omega(C))) \subset F^\delta_{n+1}\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ as wanted.

Let us now show the inclusion $F^{\delta}_{n+1}\mathcal{O}_{\mathrm{hol}}(\widetilde{C}) \subset I_{x_0}(F_{n+1}\,\mathrm{Sh}(\Omega(C)))$. Let $f \in F^{\delta}_{n+1}\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$; then there exist elements $f_1, \dots, f_k \in F^{\delta}_{n}\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ and $\omega_1, \dots, \omega_k \in \Omega(C)$ such that $df = \sum_i f_i \cdot p^*\omega_i$. By the induction hypothesis, there exist $t_1, \dots, t_k \in F_n\,\mathrm{Sh}(\Omega(C))$, such that $f_i = I_{x_0}(t_i)$ for any i. Then $df = \sum_i I_{x_0}(t_i) \cdot p^*\omega_i$. Integration gives

$$f = f(x_0) + \sum_i I_{x_0}([t_i|\omega_i]) = I_{x_0}\Big(f(x_0) + \sum_i [t_i|\omega_i]\Big) \in I_{x_0}\big(F_{n+1}\big(\mathrm{Sh}(\Omega(C))\big)\big),$$

which proves the claimed inclusion. □

Proposition 5.5. *The following statements hold true:*

(a) *Both $F^{\delta}_{\bullet}\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ and $F^{\mu}_{\bullet}\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ are algebra filtrations of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$.*

(b) *For any $n \geq 0$, one has $F^{\delta}_{n}\mathcal{O}_{\mathrm{hol}}(\widetilde{C}) \subset F^{\mu}_{n}\mathcal{O}_{\mathrm{hol}}(\widetilde{C}) \subset F^{\delta}_{n+1}\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$.*

(c) *One has $F^{\delta}_{\infty}\mathcal{O}_{\mathrm{hol}}(\widetilde{C}) = F^{\mu}_{\infty}\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ (equality of subalgebras of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$).*

Proof. Recall the shorthand $F^{\delta/\mu}_{\bullet} := F^{\delta/\mu}_{\bullet}\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ (see Lemma 1.13). By Proposition 5.4, $F^{\delta}_{\bullet}$ is the image of the increasing algebra filtration $F_{\bullet}\,\mathrm{Sh}(\Omega(C))$ by the morphism $I_{x_0}\colon \mathrm{Sh}(\Omega(C)) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$, which implies that $F^{\delta}_{\bullet}$ is an increasing algebra filtration of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$. The fact that $F^{\mu}_{\bullet}$ is the product of $F^{\delta}_{\bullet}$ with the fixed subalgebra $\mathcal{O}(C)$ of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ implies that $F^{\mu}_{\bullet}$ is an increasing algebra filtration of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ as well. This proves (a).

For any $f \in \mathcal{O}(C)$, $df \in \Omega(C) = \Omega(C) \cdot F^{\delta}_{0}$, which implies $\mathcal{O}(C) \subset F^{1}_{\delta}$. For $n \geq 0$, one then has $F^{\mu}_{n} = \mathcal{O}(C) \cdot F^{\delta}_{n} \subset F^{\delta}_{1} \cdot F^{\delta}_{n} \subset F^{\delta}_{n+1}$. For $n \geq 0$, one clearly also has $F^{\delta}_{n} \subset F^{\mu}_{n}$, which implies (b). Statement (c) follows from (b). □

Proposition 5.6. *For any $(J, x_0) \in \mathrm{MC}(C) \times \widetilde{C}$, one has the equality*

$$F^{\mu}_{\bullet}\mathcal{O}_{\mathrm{hol}}(\widetilde{C}) = f_{J,x_0}(F_{\bullet}\,\mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C))$$

of filtrations of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$.

Proof. Let $n \geq 0$. In Proposition 5.3, we proved the equality

$$I_{x_0}\big(F_n\,\mathrm{Sh}(\Omega(C))\big) = I_{x_0}\big(J_*(F_n\,\mathrm{Sh}(\mathrm{H}_C))\big) + p^*\mathcal{O}(C) \cdot I_{x_0}\big(J_*(F_{n-1}\,\mathrm{Sh}(\mathrm{H}_C))\big).$$

Multiplying it by $p^*\mathcal{O}(C)$, we obtain $p^*\mathcal{O}(C) \cdot I_{x_0}(F_n\,\mathrm{Sh}(\Omega(C))) = p^*\mathcal{O}(C) \cdot I_{x_0}(J_*(F_n\,\mathrm{Sh}(\mathrm{H}_C)))$. By Proposition 5.4, $I_{x_0}(F_n\,\mathrm{Sh}(\Omega(C))) = F^{\delta}_{n}\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$, and so $p^*\mathcal{O}(C) \cdot I_{x_0}(F_n\,\mathrm{Sh}(\Omega(C))) = p^*\mathcal{O}(C) \cdot F^{\delta}_{n}\mathcal{O}_{\mathrm{hol}}(\widetilde{C}) = F^{\mu}_{n}\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$. It follows that $F^{\mu}_{n}\mathcal{O}_{\mathrm{hol}}(\widetilde{C}) = p^*\mathcal{O}(C) \cdot I_{x_0}(J_*(F_n\,\mathrm{Sh}(\mathrm{H}_C)))$; the right-hand side of this equality is equal to $f_{J,x_0}(F_n\,\mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C))$. □

§5.3. Relation with A_C

Recall the following from Section 1.1.

Definition 5.7. Consider the following definitions:

(a) A *stable subalgebra* (SSA) of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ is a unital subalgebra A of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$, such that for any $f \in A$ and $\omega \in \Omega(C)$, one has $\mathrm{int}_{x_0}(f \cdot p^*\omega) = (x \mapsto \int_{x_0}^{x} f \cdot p^*\omega) \in A$.

(b) $A_C \coloneqq \bigcap_{A \text{ an SSA of } \mathcal{O}_{\mathrm{hol}}(\widetilde{C})} A$.

Lemma-Definition 5.8. *The subspace A_C is an SSA of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$, contained in any SSA of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$; we therefore call A_C the minimal stable subalgebra (MSSA) of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$.*

Proof. This follows from the fact that an intersection of two SSAs of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ is an SSA of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$. □

Proposition 5.9. *For any $x_0 \in \widetilde{C}$, one has $A_C = I_{x_0}(\mathrm{Sh}(\Omega(C)))$.*

Proof. For $\omega \in \Omega(C)$, let prim_ω be the vector space endomorphism of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ given by $f \mapsto (x \mapsto \int_{x_0}^{x} f \cdot p^*\omega)$. Also, let r_ω be the linear endomorphism of $\mathrm{Sh}(\Omega(C))$ given by right concatenation with ω; explicitly, $r_\omega([\omega_1|\cdots|\omega_k]) =$ for any $\omega_1, \ldots, \omega_k \in \Omega(C)$. Then one checks the identity

$$\mathrm{prim}_\omega \circ I_{x_0} = I_{x_0} \circ R_\omega. \tag{5.10}$$

It follows that $\mathrm{prim}_\omega(I_{x_0}(t)) = I_{x_0}(R_\omega(t))$ for any $t \in \mathrm{Sh}(\Omega(C))$, which together with unitality implies that $I_{x_0}(\mathrm{Sh}(\Omega(C)))$ is a stable subalgebra of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$, hence $A_C \subset I_{x_0}(\mathrm{Sh}(\Omega(C)))$.

Now let A be a stable subalgebra of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$. Let $n \geq 0$. For any $\omega_1, \ldots, \omega_n \in \Omega(C)$, the element $\mathrm{prim}_{\omega_n} \circ \cdots \circ \mathrm{prim}_{\omega_1}(1)$ belongs to A since $1 \in A$ and by the stability of A. It follows from (5.10) that this element is equal to $I_{x_0}([\omega_1|\cdots|\omega_n])$, and therefore A contains $I_{x_0}(\mathrm{Sh}_n(\Omega(C)))$. This implies that A contains $I_{x_0}(\mathrm{Sh}(\Omega(C)))$, thus concluding the proof. □

§5.4. Proof of Theorems A, B, C

5.4.1. Proof of Theorem C. In (1.3), the first (resp. second) equation follows from Proposition 4.17(b) (resp. Proposition 5.6). In (1.4), the first (resp. second) equation follows from Proposition 5.4 (resp. Proposition 5.3). In (1.5), the first (resp. second, third, fourth, fifth) equality follows from Proposition 5.9 (resp. Proposition 5.4 at infinity, Proposition 5.5(c), Proposition 5.6 at infinity, Proposition 4.17(b) at infinity).

5.4.2. Proof of Theorem A. Theorem A(a) is proved in Lemma 2.16. It follows from Proposition 4.17 that the algebra morphism f_{J,x_0} induces an algebra isomorphism $f_{J,x_0}\colon \mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C) \to F_\infty \mathcal{O}_{\mathrm{mod}}(\widetilde{C})$. By Theorem C(b), one has $F_\infty \mathcal{O}_{\mathrm{mod}}(\widetilde{C}) = A_C$, which implies Theorem A(b).

5.4.3. Proof of Theorem B. Theorem B follows from the combination of Proposition 4.17(a) and from the equality $F_\infty \mathcal{O}_{\mathrm{mod}}(\widetilde{C}) = A_C$, which follows from Theorem C(b).

§5.5. Consequences for hyperlogarithm functions

Proposition 5.10. *Let $J \in \mathrm{MC}_{\mathrm{nd}}(C)$, $x_0 \in \widetilde{C}$ and $(h_i)_{i\in[\![1,d]\!]}$ be a basis of H_C.*

(a) *The family $(\tilde{f}_{J,x_0}([h_{i_1}|\cdots|h_{i_k}]))_{k\geq 0, i_1,\ldots,i_k\in[\![1,d]\!]}$ is a basis of the vector space $\mathcal{H}_C(J)$.*

(b) *The family in (a) is linearly independent over $\mathcal{O}(C)$, i.e. for any family $(\phi^{i_1,\ldots,i_k})_{k\geq 0, i_1,\ldots,i_k\in[\![1,d]\!]}$ in $\mathcal{O}(C)$, the relation*

$$\sum_{k\geq 0, i_1,\ldots,i_k\in[\![1,d]\!]} p^*(\phi^{i_1,\ldots,i_k})\tilde{f}_{J,x_0}([h_{i_1}|\cdots|h_{i_k}]) = 0$$

implies the vanishing of $(\phi^{i_1,\ldots,i_k})_{k\geq 0, i_1,\ldots,i_k\in[\![1,d]\!]}$.

Proof. It follows from Proposition 4.17(a) that the algebra morphism $\tilde{f}_{J,x_0}\colon \mathrm{Sh}(\mathrm{H}_C) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ is injective, which then implies (a). (b) follows from (a) and Proposition 4.17(a). □

Now let C be as in Section 2.4, so $C = \mathbb{P}^1_{\mathbb{C}} \smallsetminus S$, with S a finite set containing 0 and ∞. Let $\tilde{f}_{\sigma_0,0}\colon \mathrm{Sh}(\mathrm{H}_C) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ be as in Definition 2.18.

Define an algebra morphism $f_{\sigma_0,0}\colon \mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ by $t \otimes f \mapsto p^*(f)\tilde{f}_{\sigma_0,0}(t)$.

Lemma 5.11. *The following statements hold true:*

(a) *The algebra morphism $f_{\sigma_0,0}$ is injective.*

(b) *The map $\tilde{f}_{\sigma_0,0}$ is injective.*

Proof. (a) Choose $z_0 \in \widetilde{C}$. Since $f_{\sigma_0,0} = m \circ (\tilde{f}_{\sigma_0,0} \otimes p^*)$ and $f_{\sigma_0,z_0} = m \circ ((I_{z_0} \circ \sigma_0) \otimes p^*)$ (where $m\colon \mathcal{O}_{\mathrm{hol}}(\widetilde{C})^{\otimes 2} \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ is the product map), one has

$$f_{\sigma_0,0} = f_{\sigma_0,z_0} \circ (a_0^{z_0} \otimes \mathrm{id}) \tag{5.11}$$

(equality of algebra morphisms $\mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$). The statement then follows from (5.11), together with the injectivity of f_{σ_0,z_0} (see Proposition 4.17(a)) and the automorphism status of $a_0^{z_0}$. Statement (b) follows from (a), as $\tilde{f}_{\sigma_0,0}$ is the composition of $f_{\sigma_0,0}$ with the canonical injection $-\otimes 1\colon \mathrm{Sh}(\mathrm{H}_C) \to \mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C)$. $\square$

Proposition 5.12 (See also [Br, Cor. 5.6]). *The following statements hold true:*

(a) *The family $(L_{[s_1|\cdots|s_k]})_{k\geq 0, s_1,\ldots,s_k\in S_\infty}$ of functions of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ (see notation after Definition 2.18) is a basis of $\mathcal{H}_{\mathbb{P}^1_{\mathbb{C}}\smallsetminus S}(\sigma_0)$.*

(b) *The family of functions in* (a) *is linearly independent over $\mathcal{O}(C)$.*

Proof. Statement (a) follows from Lemma 5.11(b) and the second part of Proposition 2.19. Statement (b) follows from (a) and Lemma 5.11. $\square$

§5.6. Relation with Chen's work

In the Introduction to [Ch], Chen defines an algebra filtration $\widetilde{F}_\bullet C^\infty(\widetilde{M})$ of $C^\infty(\widetilde{M})$ for any smooth manifold M, where $\widetilde{M} \to M$ is the universal cover; it has the additional property that $\widetilde{F}_n C^\infty(\widetilde{M})$ is a subalgebra of $C^\infty(\widetilde{M})$ for any $n \geq 0$. If X is a nonsingular complex algebraic variety, and $\widetilde{X}$ is its universal cover, equipped with its natural structure of complex manifold, one can similarly define a filtration $\widetilde{F}_\bullet\mathcal{O}_{\mathrm{hol}}(\widetilde{X})$ of the algebra $\mathcal{O}_{\mathrm{hol}}(\widetilde{X})$, by replacing in the definition of [Ch] the spaces of smooth functions and 1-forms (denoted there $\Lambda^0(M)$ and $\Lambda^1(M)$) by the spaces of regular functions and differentials on X. When $X = C$, the explicit definition of $\widetilde{F}_\bullet\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ is as follows: $\widetilde{F}_0\mathcal{O}_{\mathrm{hol}}(\widetilde{C}) := p^*\mathcal{O}(C)$ and $\widetilde{F}_{r+1}\mathcal{O}_{\mathrm{hol}}(\widetilde{C}) := \mathbb{C}[f, \mathrm{int}_{x_0}(g\cdot p^*\omega) \mid f,g \in \widetilde{F}_r\mathcal{O}_{\mathrm{hol}}(\widetilde{C}),\ \omega\in\Omega(C)]$ for $r \geq 0$, where $\mathbb{C}[-]$ means the subalgebra generated by a family.

Lemma 5.13. *Let $x_0 \in \widetilde{C}$.*

(a) *For any $r \geq 0$, $\widetilde{F}_r\mathcal{O}_{\mathrm{hol}}(\widetilde{C}) \subset I_{x_0}(\mathrm{Sh}(\Omega(C)))$.*

(b) *For any $r \geq 0$, $I_{x_0}(\mathrm{Sh}_r(\Omega(C))) \subset \widetilde{F}_r\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$.*

(c) *One has $\widetilde{F}_\infty\mathcal{O}_{\mathrm{hol}}(\widetilde{C}) = I_{x_0}(\mathrm{Sh}(\Omega(C)))$ (equality of subalgebras of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$).*

Proof. (a) $\widetilde{F}_0\mathcal{O}_{\mathrm{hol}}(\widetilde{C}) = p^*\mathcal{O}(C) = I_{x_0}(\mathbb{C}\oplus[d\mathcal{O}(C)]) \subset I_{x_0}(\mathrm{Sh}(\Omega(C)))$. Let $r \geq 0$ and assume that $\widetilde{F}_r\mathcal{O}_{\mathrm{hol}}(\widetilde{C}) \subset I_{x_0}(\mathrm{Sh}(\Omega(C)))$. Let $g \in \widetilde{F}_r\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$, $\omega\in\Omega(C)$. One knows that for some $a \in \mathrm{Sh}(\Omega(C))$, $\omega = I_{x_0}(a)$. Then $\mathrm{int}_{x_0}(g\cdot p^*\omega) = \mathrm{int}_{x_0}(I_{x_0}(a)\cdot p^*\omega) = I_{x_0}([a|\omega]) \in I_{x_0}(\mathrm{Sh}(\Omega(C)))$. Then $\widetilde{F}_{r+1}\mathcal{O}_{\mathrm{hol}}(\widetilde{C}) = \mathbb{C}[f, \mathrm{int}_{x_0}(g\cdot p^*\omega) \mid f,g \in \widetilde{F}_r\mathcal{O}_{\mathrm{hol}}(\widetilde{C}),\ \omega\in\Omega(C)] \subset I_{x_0}(\mathrm{Sh}(\Omega(C)))$ by $\widetilde{F}_r\mathcal{O}_{\mathrm{hol}}(\widetilde{C}) \subset I_{x_0}(\mathrm{Sh}(\Omega(C)))$ and the fact that $I_{x_0}(\mathrm{Sh}(\Omega(C)))$ is an algebra.

(b) $I_{x_0}(\mathrm{Sh}_0(\Omega(C))) = \mathbb{C} \subset \widetilde{F}_0\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$. Let $r \geq 0$ and assume the inclusion $I_{x_0}(\mathrm{Sh}_r(\Omega(C))) \subset \widetilde{F}_r\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$. Let $a \in \mathrm{Sh}_{r+1}(\Omega(C))$. Then there exist elements $(a_i)_{i=1,\ldots,k}$, $(\omega_i)_{i=1,\ldots,k}$ where $a_i \in \mathrm{Sh}_r(\Omega(C))$, $\omega_i\in\Omega(C)$ such that $a = \sum_{i=1}^k [a_i|\omega_i]$.

Then

$$I_{x_0}(a) = \sum_{i=1}^{k} I_{x_0}([a_i|\omega_i]) = \sum_{i=1}^{k} \mathrm{int}_{x_0}(I_{x_0}(a_i) \cdot p^*\omega_i).$$

One has $I_{x_0}(a_i) \in \widetilde{F}_r\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ by assumption, therefore the final term of this equality belongs to $\widetilde{F}_{r+1}\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$. Therefore, $I_{x_0}(\mathrm{Sh}_{r+1}(\Omega(C))) \subset \widetilde{F}_{r+1}\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$. The statement follows by induction.

(c) By (a), we know that $\widetilde{F}_\infty\mathcal{O}_{\mathrm{hol}}(\widetilde{C}) \subset I_{x_0}(\mathrm{Sh}(\Omega(C)))$. Moreover, (b) implies that, for any $r \geq 0$, $I_{x_0}(\mathrm{Sh}_r(\Omega(C))) \subset \widetilde{F}_\infty\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$, and so $I_{x_0}(\mathrm{Sh}(\Omega(C))) \subset \widetilde{F}_\infty\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$. □

Remark 5.14. [Ch, §2.3] contains the definition of another filtration $F_\bullet C^\infty(\widetilde{M})$. This definition is both an analogue of that of $F^\mu_\bullet\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ (as both definitions give analogous values for the degree 0 term of the filtration) and of $F^\delta_\bullet\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ (as both definitions share the same induction step). However, the statement "$F_rC^\infty(\widetilde{M}) \cdot F_sC^\infty(\widetilde{M}) \subset F_{r+s}C^\infty(\widetilde{M})$ for any $r, s \geq 0$" from [Ch, Prop. 2.3.1] appears to be wrong. Indeed, if $r = 0$, $s = 1$, $F_0C^\infty(\widetilde{M}) = p^*C^\infty(M)$ while $F_1C^\infty(\widetilde{M}) = \{f \in C^\infty(\widetilde{M}) \mid df \in F_0C^\infty(\widetilde{M}) \cdot p^*\Lambda^1(M)\}$; since $F_0C^\infty(\widetilde{M}) \cdot p^*\Lambda^1(M) = p^*\Lambda^1(M)$, the set $F_1C^\infty(\widetilde{M})$ is the set of functions on $\widetilde{M}$ of the form $x \mapsto c + \int_{x_0}^x p^*\omega$, where $c \in \mathbb{C}$ and $\omega \in \Lambda^1(M)$; this set is not stable under multiplication by $p^*C^\infty(M)$ (the mistake can be traced to the proof of Proposition 2.3.1, which overlooks the fact that the inclusion $dF_rC^\infty(\widetilde{M}) \subset F_{r-1}C^\infty(\widetilde{M}) \cdot p^*\Lambda^1(M)$ is valid in general only if one introduces $F_{-1}C^\infty(\widetilde{M}) = \mathbb{C}$).

Part II. Complementary results

§6. Connections for HACAs

We introduce the notion of connection on a HACA in Section 6.1. In Section 6.2 we construct a natural connection on the HACA $((\mathbb{C}\Gamma_C)', F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C}))$. We compute its pull-back under the HACA isomorphism $(\nu(p_{J,x_0}), f_{J,x_0})$ from Proposition 4.17 in Section 6.3.

§6.1. Connections for HACAs

Let $\mathbf{a}$ be a commutative algebra. Recall that the $\mathbf{a}$-module of Kähler differentials of $\mathbf{a}$ is the quotient $\Omega_{\mathbf{a}} := \ker(m)/\ker(m)^2$, where m is the product map $\mathbf{a}\otimes\mathbf{a} \to \mathbf{a}$; the derivation $d\colon \mathbf{a} \to \Omega_{\mathbf{a}}$ is defined by $d(a) = a \otimes 1 - 1 \otimes a + \ker(m)^2$. One has $\Omega_{\mathcal{O}(C)} = \Omega(C)$.

Let (O, A) be a HACA with coaction morphism $\Delta_A \colon A \to O \otimes A$. One has $A^O = \{a \in A \mid \Delta_A(a) = 1 \otimes a\} = F_0 A$. If A^O is a commutative algebra, then $A \otimes_{A^O} \Omega_{A^O}$ is a left A-module.

Definition 6.1. Let (O, A) be a HACA such that A^O is central in A, so that $A \otimes_{A^O} \Omega_{A^O}$ is a right A-module. A connection for (O, A) is a map $\nabla_A \colon A \to A \otimes_{A^O} \Omega_{A^O}$, which

(a) is a derivation, i.e. $\nabla_A(aa') = a\nabla_A(a') + \nabla_A(a)a'$ for any $a, a' \in A$;

(b) is O-equivariant, i.e. $(\Delta_A \otimes \mathrm{id}_{\Omega_{A^O}}) \circ \nabla_A = (\mathrm{id}_O \otimes \nabla_A) \circ \Delta_A$;

(c) is such that $\nabla_A(a) = 1 \otimes da$ for $a \in A^O$.

If (O, A) is a HACA with connection ∇_A, then one defines the pull-back of ∇_A by a HACA isomorphism $(O', A') \to (O, A)$, which is a connection for (O', A').

Remark 6.2. If G is an algebraic group and P is a principal G-bundle over an affine base, and (O, A) is the pair of regular functions on these spaces, then a connection for (O, A) is an algebraic version of a G-invariant Ehresmann connection on the bundle $P \to P/G$.

§6.2. A connection for $((\mathbb{C}\Gamma_C)', F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C}))$

Proposition 6.3. *The following statements hold true:*

(a) *The map $F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C}) \otimes_{\mathcal{O}(C)} \Omega(C) \to \Omega_{\mathrm{hol}}(\widetilde{C})$ given by $f \otimes \omega \mapsto f \cdot p^*\omega$ is injective.*

(b) *There exists a unique map $\nabla \colon F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C}) \to F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C}) \otimes_{\mathcal{O}(C)} \Omega(C)$ such that the diagram*

$$\begin{tikzcd} F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C}) \arrow[r, "\nabla"] \arrow[d, hook] & F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C}) \otimes_{\mathcal{O}(C)} \Omega(C) \arrow[d, hook] \\ \mathcal{O}_{\mathrm{hol}}(\widetilde{C}) \arrow[r, "d"'] & \Omega_{\mathrm{hol}}(\widetilde{C}) \end{tikzcd} \tag{6.1}$$

commutes.

(c) *∇ is a connection for the HACA $((\mathbb{C}\Gamma_C)', F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C}))$ (see Lemma 4.16).*

Proof. (a) By Lemma 4.17, $f_{J_\sigma,x_0} \colon \mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C) \to F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$ is an isomorphism of filtered $\mathcal{O}(C)$-modules. Its image by the functor $- \otimes_{\mathcal{O}(C)} \Omega(C)$ is an isomorphism of filtered vector spaces $\varphi_{\sigma,x_0} \colon \mathrm{Sh}(\mathrm{H}_C) \otimes \Omega(C) \to F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C}) \otimes_{\mathcal{O}(C)} \Omega(C)$. The natural morphism $\mathrm{can} \colon F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C}) \otimes_{\mathcal{O}(C)} \Omega(C) \to \Omega_{\mathrm{hol}}(\widetilde{C})$ is Γ_C-equivariant, therefore by Lemma B.5(b) is compatible with the filtrations induced by the action of Γ_C. The composed morphism $\mathrm{can}\circ\varphi_{\sigma,x_0}\colon \mathrm{Sh}(\mathrm{H}_C)\otimes\Omega(C)\to\Omega_{\mathrm{hol}}(\widetilde{C})$, given by $\varphi_{\sigma,x_0}(a\otimes\omega)=I_{x_0}(\sigma(a))\cdot p^*(\omega)$, is therefore compatible with the filtrations $F_\bullet\,\mathrm{Sh}(\mathrm{H}_C)\otimes\Omega(C)$ of the source and $F_\bullet\Omega_{\mathrm{hol}}(\widetilde{C})$ of the target (see Definition B.4), therefore it gives rise to an associated graded map $\mathrm{gr}_\bullet(\mathrm{can}\circ\varphi_{\sigma,x_0})\colon \mathrm{Sh}_\bullet(\mathrm{H}_C)\otimes\Omega(C)\to\mathrm{gr}_\bullet\Omega_{\mathrm{hol}}(\widetilde{C})$. By Lemma B.5(c), one attaches to the Γ_C-module $\Omega_{\mathrm{hol}}(\widetilde{C})$ the injective graded map

$$\mathrm{gr}_\bullet\Omega_{\mathrm{hol}}(\widetilde{C})\hookrightarrow\bigoplus_{n\geq 0}\mathrm{Hom}_{\mathbb{C}\text{-vec}}(\mathrm{gr}^n(\mathbb{C}\Gamma_C),\Omega_{\mathrm{hol}}(\widetilde{C})^{\Gamma_C}),$$

where $\Omega_{\mathrm{hol}}(\widetilde{C})^{\Gamma_C}$ can be identified with the space of holomorphic differentials on C; it contains $\Omega(C)$ as a subspace.

For any $n\geq 0$, the composition of the inclusion

$$\mathrm{gr}_n\Omega_{\mathrm{hol}}(\widetilde{C})\hookrightarrow\mathrm{Hom}_{\mathbb{C}\text{-vec}}(\mathrm{gr}^n(\mathbb{C}\Gamma_C),\Omega_{\mathrm{hol}}(\widetilde{C})^{\Gamma_C})$$

with $\mathrm{gr}_n(\mathrm{can}\circ\varphi_{\sigma,x_0})$ is given by the composition of the isomorphism

$$\mathrm{Hom}_{\mathbb{C}\text{-vec}}(\mathrm{gr}^n(\mathbb{C}\Gamma_C),\mathbb{C})\otimes\Omega_{\mathrm{hol}}(\widetilde{C})^{\Gamma_C}\simeq\mathrm{Hom}_{\mathbb{C}\text{-vec}}(\mathrm{gr}^n(\mathbb{C}\Gamma_C),\Omega_{\mathrm{hol}}(\widetilde{C})^{\Gamma_C})$$

(due to the finite-dimensionality of $\mathrm{gr}^n(\mathbb{C}\Gamma_C)$) with the tensor product of the injection $\Omega(C)\hookrightarrow\Omega_{\mathrm{hol}}(\widetilde{C})^{\Gamma_C}$ with the map $\mathrm{Sh}_n(\mathrm{H}_C)\to\mathrm{Hom}_{\mathbb{C}\text{-vec}}(\mathrm{gr}^n(\mathbb{C}\Gamma_C),\mathbb{C})$, which is injective by Lemma 4.7. It follows that $\mathrm{gr}_\bullet(\mathrm{can}\circ\varphi_{\sigma,x_0})$ is injective, which, as the filtration of the source of $\mathrm{can}\circ\varphi_{\sigma,x_0}$ is exhaustive, implies the injectivity of $\mathrm{can}\circ\varphi_{\sigma,x_0}$, which as φ_{σ,x_0} is an isomorphism implies the injectivity of can.

(b) In this proof, we abbreviate $F^\delta_\bullet\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ into $F^\delta_\bullet$. For any $n\geq 0$, the inclusion $F^\delta_n\subset F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$ gives rise to the inclusion of subspaces

$$\mathrm{im}(F^\delta_n\otimes\Omega(C)\to\Omega_{\mathrm{hol}}(\widetilde{C}))\subset\mathrm{im}(F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C})\otimes\Omega(C)\to\Omega_{\mathrm{hol}}(\widetilde{C}))\subset\Omega_{\mathrm{hol}}(\widetilde{C}).$$

By the definition of $F^\delta_\bullet$, one has $d(F^\delta_{n+1})\subset\mathrm{im}(F^\delta_n\otimes\Omega(C)\to\Omega_{\mathrm{hol}}(\widetilde{C}))$, therefore $d(F^\delta_{n+1})\subset\mathrm{im}(F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C})\otimes\Omega(C)\to\Omega_{\mathrm{hol}}(\widetilde{C}))$. This holds for any $n\geq 0$ and $F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C})=\bigcup_{n\geq 0}F^\delta_n$ (see Propositions 4.17, 5.5, and 5.6), and therefore $d(F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C}))\subset\mathrm{im}(F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C})\otimes\Omega(C)\to\Omega_{\mathrm{hol}}(\widetilde{C}))$. The linear map $F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C})\otimes\Omega(C)\to\Omega_{\mathrm{hol}}(\widetilde{C})$ admits a factorization $F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C})\otimes\Omega(C)\twoheadrightarrow F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C})\otimes_{\mathcal{O}(C)}\Omega(C)\to\Omega_{\mathrm{hol}}(\widetilde{C})$, where the first map is surjective, therefore $\mathrm{im}(F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C})\otimes\Omega(C)\to\Omega_{\mathrm{hol}}(\widetilde{C}))=\mathrm{im}(F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C})\otimes_{\mathcal{O}(C)}\Omega(C)\to\Omega_{\mathrm{hol}}(\widetilde{C}))$, which implies

$$dF_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C})\subset\mathrm{im}(F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C})\otimes_{\mathcal{O}(C)}\Omega(C)\to\Omega_{\mathrm{hol}}(\widetilde{C})).$$

The claim then follows from (a).

(c) The derivation property of ∇ follows from the derivation property of d and of the injectivity of the right vertical map of (6.1). The equivariance of ∇ follows from the same injectivity and from the Γ_C-equivariance of d; the identity $\nabla(f) = 1 \otimes df$ for $f \in \mathcal{O}(C)$ follows from the same injectivity. $\square$

§6.3. A connection for $(\mathrm{Sh}(\mathrm{H}_C), \mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C))$

Lemma 6.4. *The following statements hold true:*

(a) *Let O be a Hopf algebra and $\mathbf{a}$ be a commutative algebra. Then $(O, O \otimes \mathbf{a})$, equipped with the coaction morphism $O \otimes \mathbf{a} \xrightarrow{\Delta_O \otimes \mathrm{id}} O \otimes (O \otimes \mathbf{a})$, is a HACA satisfying the assumptions of Definition 6.1.*

(b) *Any linear map $\mu \colon O \to \mathbf{a}$ such that $\mu(fg) = \epsilon(f)\mu(g) + \mu(f)\epsilon(g)$ gives rise to a connection on $(O, O \otimes \mathbf{a})$, where $\nabla \colon O \otimes \mathbf{a} \to O \otimes \Omega(\mathbf{a})$ is given by $f \otimes a \mapsto f \otimes d(a) + f^{(1)} \otimes \mu(f^{(2)})a$.*

Proof. (a) The fact that $(O, O \otimes \mathbf{a})$ is a HACA follows from Proposition B.18. The fact that it satisfies the assumptions of Definition 6.1 follows from $A^O = \mathbf{a}$.

(b) The statement follows from the axioms, and from the fact that the left and right action of $\mathbf{a}$ on $\Omega^1_{\mathbf{a}}$ coincide, as $\mathbf{a}$ is commutative. The axioms of invariance and restriction to $(O \otimes \mathbf{a})^O = \mathbf{a}$ are immediate. $\square$

Proposition 6.5. *Let $J \in \mathrm{MC}(C)$.*

(a) *The map $\nabla_J \colon \mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C) \to \mathrm{Sh}(\mathrm{H}_C) \otimes \Omega(C)$ given by $a \otimes f \mapsto a \otimes df + a^{(1)} \otimes \mu_J(a^{(2)})f$, where μ_J is as in Section 2.2, defines a connection for the HACA $(\mathrm{Sh}(\mathrm{H}_C), \mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C))$.*

(b) *The connection ∇_J is the pull-back of ∇ under the HACA isomorphism $(\nu(p_{J,x_0}), f_{J,x_0})$ (see Proposition 4.17).*

Proof. (a) follows from Lemmas 2.10 and 6.4.

(b) For $a \in \mathrm{Sh}(\mathrm{H}_C)$, one has

$$\begin{aligned}\nabla(f_{J,x_0}(a \otimes 1)) &= d\big(I_{x_0}(J_*(a))\big) = I_{x_0}(J_*(a)^{(1)})\pi_{\mathrm{Sh}(\Omega(C))}(J_*(a)^{(2)}) \\ &= I_{x_0}(J_*(a^{(1)}))\pi_{\mathrm{Sh}(\Omega(C))}(J_*(a^{(2)})) = I_{x_0}(J_*(a^{(1)}))\mu_J(a^{(2)}) \\ &= (f_{J,x_0} \otimes_{\mathcal{O}(C)} \mathrm{id}_{\Omega(C)}) \circ \nabla_J(a \otimes 1),\end{aligned}$$

where $\pi_{\mathrm{Sh}(\Omega(C))} \colon \mathrm{Sh}(\Omega(C)) \to \Omega(C)$ is as in Section 2.1, the second equality follows from (2.1), the third equality follows from $(\mathrm{id} \otimes \pi_{\mathrm{Sh}(\Omega(C))}) \circ \Delta_{\mathrm{Sh}(\Omega(C))} \circ J_* = (J_* \otimes \mu_J) \circ \Delta_{\mathrm{Sh}(\mathrm{H}_C)}$ (equality of linear maps $\mathrm{Sh}(\mathrm{H}_C) \to \mathrm{Sh}(\Omega(C) \otimes \Omega(C))$), which is proved as follows: for any $k \geq 0$ and $\xi_1, \ldots, \xi_k \in \mathrm{H}_C$, one has

$$\begin{aligned}
&(\mathrm{id} \otimes \pi_{\mathrm{Sh}(\Omega(C))}) \circ \Delta_{\mathrm{Sh}(\Omega(C))} \circ J_*([\xi_1|\cdots|\xi_k]) \\
&\quad = \sum_{\substack{s\geq 1 \\ 1\leq k_1\leq\cdots\leq k_s\leq k}} [\mu_J([\xi_1|\cdots|\xi_{k_1}])|\cdots|\mu_J([\xi_{k_{s-1}+1}|\cdots|\xi_{k_s}])] \otimes \mu_J([\xi_{k_s+1}|\cdots|\xi_k]) \\
&\quad = \sum_{1\leq l\leq k} J_*([\xi_1|\cdots|\xi_{l-1}]) \otimes \mu_J([\xi_1|\cdots|\xi_k]) = (J_* \otimes \mu_J) \circ \Delta_{\mathrm{Sh}(\mathrm{H}_C)}([\xi_1|\cdots|\xi_k]),
\end{aligned}$$

the fourth equality follows from the equality $\mu_J(a) = \pi_{\mathrm{Sh}(\Omega(C))}(J_*(a))$ for any $a \in \mathrm{Sh}(\mathrm{H}_C)$, which follows from the definition of μ_J, and the fifth equality follows from the definitions of ∇_J and f_{J,x_0}. One derives the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C) \arrow[r, "f_{J,x_0}"] \arrow[d, "\nabla_J"'] & F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C}) \arrow[d, "\nabla"] \\
\mathrm{Sh}(\mathrm{H}_C) \otimes_{\mathcal{O}(C)} \Omega(C) \arrow[r, "f_{J,x_0}\otimes_{\mathcal{O}(C)}\mathrm{id}_{\Omega(C)}"'] & F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C}) \otimes_{\mathcal{O}(C)} \Omega(C).
\end{tikzcd} \qquad \square$$

Remark 6.6. The HACA $(\mathrm{Sh}(\mathrm{H}_C), \mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C))$ corresponds to the trivial principal bundle over C with group $\mathrm{Spec}(\mathrm{Sh}(\mathrm{H}_C))$, and ∇_J is the flat connection $d + J$ over it; one has $\mathbb{L}((\mathrm{H}_C)^*) = \mathrm{Lie}\,\mathrm{Spec}(\mathrm{Sh}(\mathrm{H}_C))$. When $C = \mathbb{P}^1 \smallsetminus S$ (see Section 2.4) and $J = J_{\sigma_0}$ (see Remark 2.20), ∇_{J_0} is the map

$$\mathrm{Sh}(\mathbb{C}\hat{S}_\infty) \otimes \mathbb{C}[z, 1/(z-s), s \in S_\infty] \to \mathrm{Sh}(\mathbb{C}\hat{S}_\infty) \otimes \mathbb{C}[z, 1/(z-s), s \in S_\infty] \cdot dz$$

given by

$$[a_1|\cdots|a_k] \otimes f \mapsto [a_1|\cdots|a_k] \otimes df + \sum_{s\in\hat{S}_\infty} (a_k)_s [a_1|\cdots|a_{k-1}] \otimes f \cdot dz/(z-s),$$

where for $a \in \mathbb{C}\hat{S}_\infty$, $a = \sum_{s\in S_\infty} a_s \cdot \hat{s}$.

§7. Local expansion of the elements of $F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$

By Section 3.1.1, one may view u, z as elements of $\mathcal{O}_{\mathrm{hol}}(\widetilde{D}^\times)$ and $\mathcal{O}_{\mathrm{hol}}(D^\times)$ respectively, such that $e^*z = e(u)$. We therefore use the notation $e^* \log z$ for $2\pi i u$. Recall the action of $\mathbb{Z}$ on $\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$, where 1 acts by θ^* (cf. Lemma 3.3). Let us denote by $F_\bullet\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$ the algebra filtration of $\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$ induced by this action according to Definition B.7 and Lemma B.8.

Lemma 7.1. *The algebra morphism $\mathcal{O}(D^\times)[X] \to \mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$ given by $f \mapsto f$ for $f \in \mathcal{O}(D^\times)$ and $X \mapsto e^* \log z$ induces an isomorphism between the algebra* *filtrations* $F_\bullet\mathcal{O}(D^\times)[X]$ *of* $\mathcal{O}(D^\times)[X]$ *given by* $F_n\mathcal{O}(D^\times)[X] := \mathcal{O}(D^\times)[X]_{\leq n}$ *for* $n \geq 0$ *and* $F_\bullet\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$ *of* $\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$. *In particular, one has* $F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times) = \mathcal{O}(D^\times)[e^*\log z]$ *(equality of subalgebras of* $\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$*).*

Proof. Let us denote by $\mathrm{can}\colon \mathcal{O}(D^\times)[X] \to \mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$ the algebra morphism from the statement. The image of $\mathcal{O}(D^\times)$ by can is contained in $F_0\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$ and the image of $e^*\log z$ by can is contained in $F_1\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$ since $(\theta^*-1)^2(e^*\log z) = (\theta^*-1)(1) = 0$; therefore, for any $n \geq 0$, $\mathrm{can}(F_n\mathcal{O}(D^\times)[X]) = \mathrm{can}(\mathcal{O}(D^\times)[X]_{\leq n}) \subset F_n\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$. Therefore can is compatible with the algebra filtrations in its source and target. Let

$$\mathrm{gr}(\mathrm{can})\colon \mathcal{O}(D^\times)[X] \to \mathrm{gr}(\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times))$$

be the corresponding graded algebra morphism. Its restriction to the degree 0 part of its source is given by $\mathcal{O}(D^\times) \ni f \mapsto f \in \mathcal{O}(D^\times) = F_0\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times) = \mathrm{gr}_0\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$ and it is such that $\mathcal{O}(D^\times)[X] \ni X \mapsto [e^*\log z] \in \mathrm{gr}_1\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$ (degree 1 elements).

Let $n \geq 0$. By Lemma B.8(b), the linear map $\mu_n\colon F_n\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times) \to \mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$ given by $f \mapsto (\theta^*-1)^n(f)/n!$ has its image contained in $F_0\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$, whereas the image of the subspace $F_{n-1}\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$ by this map is zero.

Under the identification $\mathbb{CZ} \simeq \mathbb{C}[X, X^{-1}]$ with X group-like, the ideal $F_n\mathbb{CZ}$ is identified with $((X-1)^n)$, therefore $\ker(\mu_n) = \mathrm{Ann}((\theta^*-1)^n) = \mathrm{Ann}(F_n\mathbb{CZ}) = F_{n-1}\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$. It follows that μ_n induces an injective linear map

$$\mu'_n\colon \mathrm{gr}_n(\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)) \to F_0\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times),$$

where by Lemma 3.5, the target space is equal to $\mathcal{O}(D^\times)$.

Define a graded linear map

$$\mu\colon \mathrm{gr}(\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)) \to \mathcal{O}(D^\times)[X]$$

by $\mu(a) := \mu'_n(a)X^n$ for $a \in \mathrm{gr}_n(\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times))$ and $n \geq 0$. As μ is a direct sum of injective maps, μ is injective.

Let us show that for $n, m \geq 0$, the diagram

$$\begin{tikzcd} F_n\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times) \otimes F_m\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times) \arrow[r, "\mu'_n\otimes\mu'_m"] \arrow[d] & \mathcal{O}(D^\times)\otimes\mathcal{O}(D^\times) \arrow[d] \\ F_n\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times) \arrow[r, "\mu'_{n+m}"'] & \mathcal{O}(D^\times) \end{tikzcd} \tag{7.1}$$

is commutative, where the vertical maps are given by multiplication. The relation

$$\frac{(X\otimes X-1)^{n+m}}{(n+m)!}\in\frac{(X-1)^n}{n!}\otimes\frac{(X-1)^m}{m!}+((X-1)^{n+1})\otimes\mathbb{C}[X,X^{-1}]$$
$$+\mathbb{C}[X,X^{-1}]\otimes((X-1)^{m+1}) \tag{7.2}$$

in $\mathbb{C}[X,X^{-1}]^{\otimes 2}$ is a consequence of the relation

$$\begin{aligned}\frac{(XY-1)^{n+m}}{(n+m)!}&=\frac{((X-1)Y+(Y-1))^{n+m}}{(n+m)!}\\ &\in\frac{((X-1)Y)^nY^m}{n!m!}+I=\frac{(X-1)^n}{n!}\frac{(Y-1)^m}{m!}+I\end{aligned}$$

in $\mathbb{C}[X^{\pm1},Y^{\pm1}]$, where $I:=((X-1)^{n+1})+((Y-1)^{m+1})$. Then $f\in F_n\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$, $g\in F_m\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$, one has

$$\begin{aligned}\mu'_{n+m}(fg)&=\frac{(\theta^*-1)^{n+m}(fg)}{(n+m)!}\\ &=\frac{(X-1)^{n+m}}{(n+m)!}\cdot(f\cdot g)=\Delta\Big(\frac{(X-1)^{n+m}}{(n+m)!}\Big)\cdot(f\otimes g)\\ &\in\frac{(X-1)^n}{n!}\otimes\frac{(X-1)^m}{m!}+((X-1)^{n+1})\otimes\mathbb{C}[X,X^{-1}]\\ &\quad+\mathbb{C}[X,X^{-1}]\otimes((X-1)^{m+1})\cdot(f\otimes g)\\ &=\frac{(X-1)^n}{n!}\cdot f\otimes\frac{(X-1)^m}{m!}\cdot g=\mu'_n(f)\cdot\mu'_m(g),\end{aligned}$$

where the third equality follows from the Hopf algebra action properties, the inclusion relation follows from (7.2) and the group-likeness of X, and the last equality follows from $(\theta^*-1)^{n+1}f=(\theta^*-1)^{m+1}g=0$ as $f\in F_n\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$, $g\in F_m\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times)$. The commutativity of (7.1) implies that $\mu\colon\mathrm{gr}(\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^\times))\to\mathcal{O}(D^\times)[X]$ is a morphism of graded algebras.

Then $\mu\circ\mathrm{gr}(\mathrm{can})$ is a graded algebra endomorphism of $\mathcal{O}(D^\times)[X]$. For $f\in\mathcal{O}(D^\times)$, one has $\mu\circ\mathrm{gr}(\mathrm{can})(f)=\mu(f)=f$. One also has $\mu\circ\mathrm{gr}(\mathrm{can})(X)=\mu([e^*\log z])=\mu'_1([e^*\log z])X=X$ as $\mu'_1([e^*\log z])=\mu_1(e^*\log z)=1$. It follows that $\mu\circ\mathrm{gr}(\mathrm{can})$ is the identity of $\mathcal{O}(D^\times)[X]$, therefore that μ is surjective. It is therefore an isomorphism, which, using $\mu\circ\mathrm{gr}(\mathrm{can})=\mathrm{id}$ again, implies that $\mathrm{gr}(\mathrm{can})$ is an isomorphism. Lemma 4.1 then implies the statement. $\square$

Proposition 7.2. *The algebra morphism $\bigoplus_{s\in S}\prod_{s\in X_s}(\tilde\varphi^\times_{s,x})^*$ (see (3.1)) is such that*

$$\Big(\bigoplus_{s\in S}\prod_{x\in X_s}(\tilde\varphi^\times_{s,x})^*\Big)(F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C}))\subset\bigoplus_{s\in S}\prod_{x\in X_s}\mathcal{O}(D^\times)[e^*\log z].$$

Proof. It follows from the proof of Proposition 3.12 that (3.2) is a Γ_C-equivariant algebra morphism, where the action on the target is the direct sum over $s \in S$ of the actions of Lemma 3.8(a). For any $s \in S$, the composition of (3.2) with the canonical projection is a Γ_C-equivariant algebra morphism $\bigoplus_{x\in X_s}(\tilde{\varphi}^{\times}_{s,x})^*\colon \mathcal{O}_{\mathrm{mod}}(\widetilde{C}) \to \bigoplus_{x\in X_s}\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^{\times})$. For $x \in X_s$, the decomposition of the target as $\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^{\times}) \oplus (\bigoplus_{x'\in X_s\smallsetminus\{x\}}\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^{\times}))$ is preserved by the action of the stabilizer subgroup $\mathrm{Stab}_{\Gamma_C}(x) \subset \Gamma_C$ of $x \in X_s$ under the action of Γ_C. The map

$$(\tilde{\varphi}^{\times}_{s,x})^*\colon \mathcal{O}_{\mathrm{mod}}(\widetilde{C}) \to \mathcal{O}_{\mathrm{mod}}(\widetilde{D}^{\times}) \tag{7.3}$$

is therefore equivariant under the action of $\mathrm{Stab}_{\Gamma_C}(x)$.

By Section 3.1.2, there is a group isomorphism $\mathbb{Z} \simeq \mathrm{Stab}_{\Gamma_C}(x)$ given by $1 \mapsto \theta_{s,x}$, and (7.3) is $\mathbb{Z}$-equivariant, the action of $\mathbb{Z}$ on the target being as in Lemma 7.1. Then

$$\begin{aligned} F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C}) &= F_\infty^{\mathbb{C}\Gamma_C}\mathcal{O}_{\mathrm{mod}}(\widetilde{C}) \subset F_\infty^{\mathbb{C}\mathbb{Z}}\mathcal{O}_{\mathrm{mod}}(\widetilde{C}) \subset (\varphi^*_{s,x})^{-1}(F_\infty^{\mathbb{C}\mathbb{Z}}\mathcal{O}_{\mathrm{mod}}(\widetilde{D}^{\times})) \\ &= (\varphi^*_{s,x})^{-1}(\mathcal{O}_{\mathrm{mod}}(D^{\times})[e^*\log z]), \end{aligned}$$

where we use the notation of Lemma B.11, the first equality is the definition of $F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$, the first inclusion follows from Lemma B.11(a), the second inclusion follows from Lemma B.11(b), and the last equality follows from Lemma 7.1. □

Remark 7.3. Proposition 7.2 implies that the elements of $F_\infty\mathcal{O}_{\mathrm{mod}}(\widetilde{C})$ are Nilsson class functions on C in the sense of [Ph, p. 154]. Indeed, for any $f \in \mathcal{O}(D^{\times})[e^*\log z]$, there exists $\alpha_0 \in \mathbb{Z}$ such that $f = z^{\alpha_0}P_0(\log z)$ with P_0 as in [Ph, eqn. (1.4), p. 151]. The Nilsson class functions obtained in this way are not of the most general form, as the class in $\mathbb{C}/\mathbb{Z}$ of the α_i in their expansion from [Ph, eqn. (1.4), p. 151] is always 0.

§8. Relation of A_C with minimal acyclic extensions of dgas

Recall the definition of the dga $(\Omega^\bullet_{\mathrm{hol}}(\widetilde{C}), d)$ from Section 3.2. Denote by $(\Omega^\bullet(C), d)$ the dga of algebraic differential forms on C; it is concentrated in degrees 0 and 1, with $\Omega^0(C) = \mathcal{O}(C)$, $\Omega^1(C) = \Omega(C)$, algebra structure given by the algebra structure of $\mathcal{O}(C)$ and the module structure of $\Omega(C)$ over it, and differential given by d. The pull-back of $p\colon \widetilde{C} \to C$ gives rise to an injective dga morphism $p^*\colon (\Omega^\bullet(C), d) \hookrightarrow (\Omega^\bullet_{\mathrm{hol}}(\widetilde{C}), d)$. One has $\mathrm{H}^1(\Omega^\bullet(C)) \simeq \mathrm{H}_C$, while $\mathrm{H}^1(\Omega^\bullet_{\mathrm{hol}}(\widetilde{C})) = 0$.

Definition 8.1. Consider the following definitions:

(a) An acyclic extension (AE) of $\Omega^\bullet(C)$ is a dga $(E^\bullet, d)$ with $\Omega^\bullet(C) \subset E^\bullet \subset \Omega^\bullet_{\mathrm{hol}}(\widetilde{C})$ and $\mathrm{H}^1(E^\bullet) = 0$.

(b) $E_C^\bullet \coloneqq \bigcap_{E^\bullet \text{ an AE of } \Omega^\bullet(C)} E^\bullet$.

Lemma-Definition 8.2. *The dga $E_C^\bullet$ is an AE of $\Omega^\bullet(C)$, which is contained in any AE of $\Omega^\bullet(C)$; we therefore call it the* minimal AE *of $\Omega^\bullet(C)$.*

Proof. Let us show that if $E^\bullet$, $F^\bullet$ are AEs of $\Omega^\bullet(C)$, then so is $E^\bullet \cap F^\bullet$. The intersection $E^\bullet \cap F^\bullet$ is obviously a dga containing $\Omega^\bullet(C)$. If $x \in E^1 \cap F^1$, there exists $e \in E^0$ such that $x = d(e)$ since $\mathrm{H}^1(E^\bullet) = 0$, and $f \in E^0$ such that $x = d(f)$ since $\mathrm{H}^1(F^\bullet) = 0$. Then $d(e-f) = 0$, therefore $e - f \in \mathbb{C}$, therefore $f \in e + \mathbb{C} \subset E^0 + \mathbb{C} = E^0$, so $f \in E^0 \cap F^0$. So $E^1 \cap F^1 \subset d(E^0 \cap F^0)$, which implies $\mathrm{H}^1(E^\bullet \cap F^\bullet) = 0$. It follows that $E^\bullet \cap F^\bullet$ is an AE of $\Omega^\bullet(C)$. This fact implies the statement. □

Lemma 8.3. *The maps $E^\bullet \mapsto E^0$ and $A \mapsto (A \oplus dA, d)$ define inverse bijections between the sets* $\{$AEs of $\Omega^\bullet(C)\}$ *and*

$$\mathrm{Alg}_C \coloneqq \big\{\text{algebras } A \text{ with } \mathcal{O}(C) \subset A \subset \mathcal{O}_{\mathrm{hol}}(\widetilde{C}) \text{ such that } d(A) \supset \Omega(C) \\ \text{and } A \cdot d(A) = dA \text{ (equality of subspaces of } \Omega_{\mathrm{hol}}(\widetilde{C}))\big\}.$$

Proof. Let $E^\bullet$ be an AE of $\Omega^\bullet(C)$. Then $\mathcal{O}(C) \subset E^0 \subset \Omega_{\mathrm{hol}}(\widetilde{C})$ since $\Omega^\bullet(C) \subset E^\bullet \subset \Omega^\bullet_{\mathrm{hol}}(\widetilde{C})$. Since $\mathrm{H}^1(E^\bullet) = 0$, one has $E^1 = d(E^0)$ and since $\Omega^\bullet(C) \subset E^\bullet$, one has $E^1 \supset \Omega(C)$, therefore $d(E^0) \supset \Omega(C)$. The equality $E^1 = d(E^0)$ implies the two extreme equalities in $E^0 \cdot d(E^0) = E^0 \cdot E^1 = E^1 = d(E^0)$, while the middle equality follows from the fact that $E^\bullet$ is a dga with unit. It follows that E^0 belongs to Alg_C.

Let A belong to Alg_C. Since $A \cdot d(A) = d(A)$, the pair $(A \oplus d(A), d)$ is a sub-dga of $\Omega^\bullet_{\mathrm{hol}}(\widetilde{C})$, and since $d(A) \supset \Omega(C)$ and $A \supset \mathcal{O}(C)$, the dga $(A \oplus d(A), d)$ contains $\Omega^\bullet(C)$ as a sub-dga. One has clearly $\mathrm{H}^1(A \oplus dA, d) = 0$. It follows that $(A \oplus d(A), d)$ is an AE of $\Omega^\bullet(C)$.

The composed map $\mathrm{Alg}_C \to \{\text{AEs of } \Omega^\bullet(C)\} \to \mathrm{Alg}_C$ is obviously the identity, and the fact that the composed map $\{\text{AEs of } \Omega^\bullet(C)\} \to \mathrm{Alg}_C \to \{\text{AEs of } \Omega^\bullet(C)\}$ is the identity follows from the fact that if $E^\bullet$ is an AE of $\Omega^\bullet(C)$, then $E^1 = d(E^0)$ due to $\mathrm{H}^1(E^\bullet) = 0$. □

Lemma 8.4. *The following statements hold true:*

(a) $\mathrm{Alg}_C \subset \{\text{SSAs of } \mathcal{O}_{\mathrm{hol}}(\widetilde{C})\}$.

(b) $A_C \in \mathrm{Alg}_C$.

(c) $\bigcap_{A \in \mathrm{Alg}_C} A = A_C$.

Proof. (a) Let $A \in \mathrm{Alg}_C$. Then A is unital. Let $f \in A$, $\omega \in \Omega(C)$ and set $F \coloneqq (z \mapsto \int_{x_0}^z f \cdot p^*\omega)$. Since $\Omega(C) \subset d(A)$ (by $A \in \mathrm{Alg}_C$), there exists $g \in A$ with

$dg = \omega$. Then $dF = f \cdot p^*\omega = f \cdot dg \in A \cdot dA = dA$, where the last equality follows from $A \in \mathrm{Alg}_C$. It follows that $F \in A$, therefore A is an SSA of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$.

(b) By Lemma 5.8, A_C is an SSA of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$, therefore it contains 1 and the functions $z \mapsto \int_{x_0}^z df$ for any $f \in \mathcal{O}(C)$, therefore A_C is a subalgebra of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ containing $\mathcal{O}(C)$. Let $\omega \in \Omega(C)$ and set $F_\omega := (z \mapsto \int_{x_0}^z p^*\omega) \in \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$. Then $F_\omega \in A_C$ by the stability properties of A_C. Then $d(A_C) \ni d(F_\omega) = \omega$, which implies $d(A_C) \supset \Omega(C)$.

Let us show that $A_C \cdot dA_C = dA_C$. It suffices to prove that, for any $f, g \in A_C$, the element $h := (z \mapsto \int_{x_0}^z f \cdot dg) \in \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ belongs to A_C. By Proposition 5.9 and $F_\infty^\delta \mathcal{O}_{\mathrm{hol}}(\widetilde{C}) = \bigcup_{n \geq 0} F_n^\delta \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$, there exist $n, m \geq 0$ such that $f \in F_n^\delta \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ and $g \in F_m^\delta \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$. Then there exists a finite set I and maps $I \to \Omega(C)$, $I \to F_{m-1}^\delta \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ denoted $i \mapsto \omega_i$, $i \mapsto k_i$ such that $dg = \sum_{i \in I} k_i \cdot p^*\omega_i$. Then $dh = \sum_{i \in I} f k_i \cdot p^*\omega_i$. For any $i \in I$, $f k_i \in F_n^\delta \mathcal{O}_{\mathrm{hol}}(\widetilde{C}) \cdot F_{m-1}^\delta \mathcal{O}_{\mathrm{hol}}(\widetilde{C}) \subset F_{n+m-1}^\delta \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$, where the last inclusion follows from Proposition 5.5(a). It follows that $h \in F_{n+m}^\delta \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$, and therefore $h \in A_C$ by $F_\infty^\delta \mathcal{O}_{\mathrm{hol}}(\widetilde{C}) = \bigcup_{n \geq 0} F_n^\delta \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ and Proposition 5.9.

(c) By (a), one has $\bigcap_{A \in \mathrm{Alg}_C} A \supset \bigcap_{A \text{ an SSA of } \mathcal{O}(C)} A = A_C$. On the other hand, $\bigcap_{A \in \mathrm{Alg}_C} A \subset A$ for any $A \in \mathrm{Alg}_C$, so by (b), $\bigcap_{A \in \mathrm{Alg}_C} A \subset A_C$. □

Proposition 8.5. $E_C^\bullet = (A_C \oplus d(A_C), d)$.

Proof. Set $X := \bigcap_{A \in \mathrm{Alg}_C} A$. One has

$$E_C^\bullet = \bigcap_{E^\bullet \text{ an AE of } \Omega^\bullet(C)} E^\bullet = \bigcap_{A \in \mathrm{Alg}_C} (A \oplus d(A), d) = (X \oplus d(X), d) = (A_C \oplus d(A_C), d),$$

where the first equality follows from the definition of $E_C^\bullet$, the second equality follows from Lemma 8.3, the third equality follows from $(A \oplus d(A)) \cap (B \oplus d(B)) = ((A \cap B) \oplus d(A \cap B))$ (equality of subspaces of $\Omega^\bullet(\widetilde{C})$) for A, B any pair of subspaces of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$ containing 1, and the last equality follows from Lemma 8.4(c). □

§9. Computation of $\ker(I_{x_0})$

In this section we fix $x_0 \in \widetilde{C}$. The main result of this section is Theorem 9.7, where we compute $\ker(I_{x_0})$ and exhibit a complement of this space in $\mathrm{Sh}(\Omega(C))$. These results are expressed in terms of an element $\sigma \in \Sigma_C$, which is fixed in the whole section. We also set $f_{\sigma,x_0} := f_{J_\sigma,x_0}$.

Lemma-Definition 9.1. *The direct sum of the map $\sigma_* \colon \mathrm{Sh}(\mathrm{H}_C) \to \mathrm{Sh}(\Omega(C))$ and of its concatenation with the canonical inclusion $d\mathcal{O}(C) \hookrightarrow \Omega(C)$ is an injective* *map* $\mathrm{Sh}(\mathrm{H}_C) \oplus [\mathrm{Sh}(\mathrm{H}_C)|d\mathcal{O}(C)] \to \mathrm{Sh}(\Omega(C))$. *We define*

$$\mathrm{Sub}_\sigma := \sigma_*(\mathrm{Sh}(\mathrm{H}_C)) \oplus [\sigma_*(\mathrm{Sh}(\mathrm{H}_C))|d\mathcal{O}(C)] \hookrightarrow \mathrm{Sh}(\Omega(C)) \tag{9.1}$$

to be the image of this map.

Proof. The statement follows from $\Omega(C) = \sigma(\mathrm{H}_C) \oplus d\mathcal{O}(C)$. $\square$

Define a vector space filtration $F_\bullet\mathrm{Sub}_\sigma$ by

$$F_n\mathrm{Sub}_\sigma := \sigma_*(F_n\,\mathrm{Sh}(\mathrm{H}_C)) \oplus [\sigma_*(F_n\,\mathrm{Sh}(\mathrm{H}_C))|d\mathcal{O}(C)]$$

for $n \geq 0$.

Lemma 9.2. *One has for any* $n \geq 0$,

$$\begin{aligned} I_{x_0}(F_n\mathrm{Sub}_\sigma) + I_{x_0}\big(F_n\,\mathrm{Sh}(\Omega(C))\big) = f_{\sigma,x_0}(F_n\,\mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C)) \\ + I_{x_0}\big(F_n\,\mathrm{Sh}(\Omega(C))\big), \end{aligned} \tag{9.2}$$

$$f_{\sigma,x_0}(F_n\,\mathrm{Sh}(\mathrm{H}_C) \otimes \mathbb{C}1) \subset I_{x_0}(F_n\mathrm{Sub}_\sigma). \tag{9.3}$$

Proof. Equation (9.2) follows from the inclusion $F_{n-1}\mathrm{Sub}_\sigma \subset F_n\,\mathrm{Sh}(\Omega(C))$, from the fact that $\mathrm{gr}_n\mathrm{Sub}_\sigma$ is linearly spanned by the elements $[\sigma(h_1)|\cdots|\sigma(h_n)|df] \oplus 0$ and $0 \oplus [\sigma(h_1)|\cdots|\sigma(h_n)]$, where $h_1, \ldots, h_n \in \mathrm{H}_C$ and $f \in \mathcal{O}(C)$, and from the identities

$$\begin{aligned} I_{x_0}([\sigma(h_1)|\cdots|\sigma(h_n)|df] \oplus 0) = f_{\sigma,x_0}([h_1|\cdots|h_n] \otimes f) \\ - I_{x_0}([\sigma(h_1)|\cdots|\sigma(h_{n-1})|\sigma(h_n)p^*(f)]), \\ I_{x_0}(0 \oplus [\sigma(h_1)|\cdots|\sigma(h_n)]) = f_{\sigma,x_0}([h_1|\cdots|h_n] \otimes 1). \end{aligned} \tag{9.4}$$

Equation (9.3) follows from the equality of the maps $F_n\,\mathrm{Sh}(\mathrm{H}_C) \to \mathcal{O}_{\mathrm{hol}}(\widetilde{C})$, $t \mapsto f_{\sigma,x_0}(t \otimes 1)$, and $t \mapsto I_{x_0}(\sigma(t) \oplus 0)$. $\square$

Lemma 9.3. *One has, for any* $n \geq 0$,

$$f_{\sigma,x_0}(F_n\,\mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C)) = I_{x_0}(F_n\mathrm{Sub}_\sigma). \tag{9.5}$$

Proof. For any $n \geq 0$, one has

$$\begin{aligned} f_{\sigma,x_0}(F_n\,\mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C)) &\subset I_{x_0}(F_n\mathrm{Sub}_\sigma) + I_{x_0}\big(F_n\,\mathrm{Sh}(\Omega(C))\big) \\ &= I_{x_0}(F_n\mathrm{Sub}_\sigma) + f_{\sigma,x_0}(F_n\,\mathrm{Sh}(\mathrm{H}_C) \otimes \mathbb{C}1 + F_{n-1}\,\mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C)) \\ &= I_{x_0}(F_n\mathrm{Sub}_\sigma) + f_{\sigma,x_0}(F_n\,\mathrm{Sh}(\mathrm{H}_C) \otimes \mathbb{C}1) + f_{\sigma,x_0}(F_{n-1}\,\mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C)) \\ &= I_{x_0}(F_n\mathrm{Sub}_\sigma) + f_{\sigma,x_0}(F_{n-1}\,\mathrm{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C)) \end{aligned}$$

where the first inclusion follows from (9.2), the first equality follows from Proposition 5.3, and the third equality follows from the inclusion (9.3) applied to the two first summands of its left-hand side. Therefore, one finds that

$$f_{\sigma,x_0}(F_n \operatorname{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C)) \subset I_{x_0}(F_n\mathrm{Sub}_\sigma) + f_{\sigma,x_0}(F_{n-1} \operatorname{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C)).$$

Based on this inclusion, one proves inductively that, for any $n \geq 0$, one has

$$f_{\sigma,x_0}(F_n \operatorname{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C)) \subset I_{x_0}(F_n\mathrm{Sub}_\sigma). \tag{9.6}$$

On the other hand, for any $n \geq 0$, one has

$$\begin{aligned} I_{x_0}(F_n\mathrm{Sub}_\sigma) &\subset f_{\sigma,x_0}(F_n \operatorname{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C)) + I_{x_0}\big(F_n \operatorname{Sh}(\Omega(C))\big) \\ &= f_{\sigma,x_0}(F_n \operatorname{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C)) + f_{\sigma,x_0}(F_n \operatorname{Sh}(\mathrm{H}_C) \otimes \mathbb{C}1 + F_{n-1} \operatorname{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C)) \\ &= f_{\sigma,x_0}(F_n \operatorname{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C)), \end{aligned}$$

where the inclusion follows from (9.2), the first equality follows from Proposition 5.3, and the last equality follows from the inclusion of the second summand of its left-hand side in its first one. Therefore, one finds that

$$I_{x_0}(F_n\mathrm{Sub}_\sigma) \subset f_{\sigma,x_0}(F_n \operatorname{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C)),$$

which together with (9.6) implies the statement. □

It follows from Proposition 4.17(a) that the corestriction of the map f_{σ,x_0} defines a linear isomorphism $\underline{f}_{\sigma,x_0} \colon F_n \operatorname{Sh}(\mathrm{H}_C)\otimes\mathcal{O}(C) \to f_{\sigma,x_0}(F_n \operatorname{Sh}(\mathrm{H}_C)\otimes\mathcal{O}(C))$. Define

$$\mathrm{map}_{\sigma,x_0} \colon \mathrm{Sub}_\sigma \to \operatorname{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C)$$

to be the composition

$$\mathrm{Sub}_\sigma \xrightarrow{I_{x_0}} I_{x_0}(\mathrm{Sub}_\sigma) = f_{\sigma,x_0}(\operatorname{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C)) \xrightarrow{(\underline{f}_{\sigma,x_0})^{-1}} \operatorname{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C), \tag{9.7}$$

where the equality (equality of subspaces of $\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$) follows from the collection of all equalities (9.5) for $n \geq 0$.

Lemma 9.4. *The map* $\mathrm{map}_{\sigma,x_0}$ *is an isomorphism of vector spaces.*

Proof. For any $n \geq 0$, the composition (9.7) restricts to a composition

$$F_n\mathrm{Sub}_\sigma \xrightarrow{I_{x_0}} I_{x_0}(F_n\mathrm{Sub}_\sigma) = f_{\sigma,x_0}(F_n \operatorname{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C)) \xrightarrow{(\underline{f}_{\sigma,x_0})^{-1}} F_n \operatorname{Sh}(\mathrm{H}_C) \otimes \mathcal{O}(C),$$

where the equality follows from (9.5).

It follows that the map $\mathrm{map}_{\sigma,x_0}$ is compatible with the filtrations on both sides, and therefore induces, for any $n \geq 0$, a linear map

$$F_n\mathrm{map}_{\sigma,x_0}\colon F_n\mathrm{Sub}_\sigma \to F_n\,\mathrm{Sh}(\mathrm{H}_C)\otimes \mathcal{O}(C).$$

The composition of the associated graded map with the canonical isomorphisms is then

$$\begin{aligned}\mathrm{map}_n\colon \sigma_*(\mathrm{Sh}_n(\mathrm{H}_C)) \oplus [\sigma_*(\mathrm{Sh}_n(\mathrm{H}_C))|d\mathcal{O}(C)] &\simeq \mathrm{gr}_n\mathrm{Sub}_\sigma \\ \xrightarrow{\mathrm{gr}_n\mathrm{map}_{\sigma,x_0}} \mathrm{gr}_n\,\mathrm{Sh}(\mathrm{H}_C)\otimes\mathcal{O}(C) &\simeq \mathrm{Sh}_n(\mathrm{H}_C)\otimes\mathcal{O}(C). \end{aligned} \tag{9.8}$$

Recall that a 2-step filtration of a vector space is the same as a vector subspace. The source and target of map_n are equipped with the 2-step filtrations associated respectively with the subspaces $\sigma_*(\mathrm{Sh}_n(\mathrm{H}_C))$ and $\mathrm{Sh}_n(\mathrm{H}_C)\otimes\mathbb{C}$.

The composition (9.7) restricts to a composition

$$\sigma_*(\mathrm{Sh}(\mathrm{H}_C)) \xrightarrow{I_{x_0}} I_{x_0}\big(\sigma_*(\mathrm{Sh}_n(\mathrm{H}_C))\big) = f_{\sigma,x_0}(\mathrm{Sh}(\mathrm{H}_C)\otimes\mathbb{C}) \xrightarrow{(\underline{f}_{\sigma,x_0})^{-1}} \mathrm{Sh}(\mathrm{H}_C)\otimes\mathbb{C},$$

where the middle equality follows from the equality of maps $\mathrm{Sh}(\mathrm{H}_C)\to\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$, $t\mapsto I_{x_0}(\sigma_*(t))$ and $t\mapsto f_{\sigma,x_0}(t\otimes 1)$. Therefore $\mathrm{map}_{\sigma,x_0}$ restricts to the isomorphism $\sigma_*(\mathrm{Sh}(\mathrm{H}_C))\to\mathrm{Sh}(\mathrm{H}_C)\otimes\mathbb{C}$ given by $\sigma_*(t)\mapsto t\otimes 1$. This map is compatible with the filtrations, therefore it induces an isomorphism $\sigma_*(F_n\,\mathrm{Sh}(\mathrm{H}_C))\to F_n\,\mathrm{Sh}(\mathrm{H}_C)\otimes\mathbb{C}$ for any $n\geq 0$, which by passing to the associated graded implies that map_n is compatible to the 2-step filtrations on both sides. Its associated graded for this filtration is then a map

$$\bigoplus_{i=0}^{1}\mathrm{gr}_i\mathrm{map}_n\colon \sigma_*(\mathrm{Sh}_n(\mathrm{H}_C))\oplus[\sigma_*(\mathrm{Sh}_n(\mathrm{H}_C))|d\mathcal{O}(C)] \to \mathrm{Sh}_n(\mathrm{H}_C)\otimes\big(\mathbb{C}\oplus(\mathcal{O}(C)/\mathbb{C})\big).$$

It follows from the fact that $\mathrm{map}_{\sigma,x_0}$ restricts to the isomorphism $\sigma_*(\mathrm{Sh}(\mathrm{H}_C))\to\mathrm{Sh}(\mathrm{H}_C)\otimes\mathbb{C}$ given by $\sigma_*(t)\mapsto t\otimes 1$ that $\mathrm{gr}_0\mathrm{map}_n$ is the map $\sigma_*(\mathrm{Sh}_n(\mathrm{H}_C))\to F_n\,\mathrm{Sh}(\mathrm{H}_C)\otimes\mathbb{C}$, $\sigma_*(t)\mapsto t\otimes 1$. Moreover, (9.4) implies

$$\begin{aligned} I_{x_0}([\sigma(h_1)|\cdots|\sigma(h_n)|df]\oplus 0) &\in f_{\sigma,x_0}([h_1|\cdots|h_n]\otimes f) + I_{x_0}\big(F_n\,\mathrm{Sh}(\Omega(C))\big) \\ &\subset f_{\sigma,x_0}([h_1|\cdots|h_n]\otimes f) + f_{\sigma,x_0}(F_\bullet\,\mathrm{Sh}(\mathrm{H}_C)\otimes\mathbb{C}1 + F_{\bullet-1}\,\mathrm{Sh}(\mathrm{H}_C)\otimes\mathcal{O}(C)), \end{aligned}$$

where the first relation follows from (9.4) and the second follows from Proposition 5.3, thus implying that $\mathrm{gr}_1\mathrm{map}_n$ is the map $[\sigma_*(\mathrm{Sh}_n(\mathrm{H}_C))|d\mathcal{O}(C)]\to\mathrm{Sh}_n(\mathrm{H}_C)\otimes(\mathcal{O}(C)/\mathbb{C})$, $[\sigma_*(t)|df]\mapsto t\otimes[f]$.

The maps $\mathrm{gr}_i\mathrm{map}_n$ are isomorphisms for $i=0,1$ and any $n\geq 0$, which implies that map_n is an isomorphism for any $n\geq 0$. This implies that $\mathrm{gr}_n\mathrm{map}_{\sigma,x_0}$ is an isomorphism for any $n\geq 0$; since the filtrations on the source and target of $\mathrm{map}_{\sigma,x_0}$ are complete, one concludes that this map is an isomorphism. □

Lemma 9.5. *The restriction of I_{x_0} to Sub_σ is injective.*

Proof. This follows from the equality of this restriction with the composition $f_{\sigma,x_0}\circ \mathrm{map}_{\sigma,x_0}$, which is injective by Proposition 4.17(a) and Lemma 9.4. $\square$

Proposition 9.6. *One has* $\mathrm{Sh}(\Omega(C)) = \mathrm{Sub}_\sigma + \mathrm{im}(D_{x_0})$.

Proof. Let us prove the inclusion

$$\mathrm{Sh}_n(\Omega(C)) \subset \mathrm{Sub}_\sigma + \mathrm{im}(D_{x_0}) + F_{n-1}\,\mathrm{Sh}(\Omega(C)) \tag{9.9}$$

for any $n \geq 0$. One has $\mathbb{C}1 \in \mathrm{Sub}_\sigma$, which proves (9.9) for $n = 0$. One has $\Omega(C) = \sigma(\mathrm{H}_C) \oplus d\mathcal{O}(C)$, therefore $\mathrm{Sh}_1(\Omega(C)) \subset \mathrm{Sub}_\sigma$; this proves (9.9) for $n = 1$. Let $n \geq 2$. It follows from $\Omega(C) = d\mathcal{O}(C) + \sigma(\mathrm{H}_C)$ that

$$\begin{aligned}\mathrm{Sh}_n(\Omega(C)) = {}&[\sigma_*(\mathrm{Sh}_{n-1}(\mathrm{H}_C))|\Omega(C)]\\ &+\sum_{k=0}^{n-1}[\mathrm{Sh}_{k-1}(\Omega(C))|d\mathcal{O}(C)|\,\mathrm{Sh}_{n-k}(\Omega(C))].\end{aligned} \tag{9.10}$$

One has

$$[\sigma_*(\mathrm{Sh}_n(\mathrm{H}_C))|\Omega(C)] = \sigma_*(\mathrm{Sh}_{n-1}(\mathrm{H}_C)) \oplus [\sigma_*(\mathrm{Sh}_{n-1}(\mathrm{H}_C))|d\mathcal{O}(C)] \subset \mathrm{Sub}_\sigma. \tag{9.11}$$

Moreover, for $k \in [\![0, n-1]\!]$, $f \in \mathcal{O}(C)$, and $\omega_i \in \Omega(C)$, $i \in [\![0, n]\!] \smallsetminus \{k\}$, one has

$$\begin{aligned}[\omega_1|\cdots|\omega_{k-1}|df|\omega_{k+1}|\cdots|\omega_n] &= D_{x_0}([\omega_1|\cdots|\omega_{k-1}] \otimes f \otimes [\omega_{k+1}|\cdots|\omega_n])\\ &\quad + [\omega_1|\cdots|\omega_{k-1}|p^*(f)\omega_{k+1}|\cdots|\omega_n]\\ &\quad - [\omega_1|\cdots|\omega_{k-1}p^*(f)|\omega_{k+1}|\cdots|\omega_n]\\ &\in \mathrm{im}(D_{x_0}) + F_{n-1}\,\mathrm{Sh}(\Omega(C))\end{aligned}$$

if $k > 0$ and

$$\begin{aligned}[df|\omega_2|\cdots|\omega_n] &= D_{x_0}(1 \otimes f \otimes [\omega_2|\cdots|\omega_n]) + [p^*(f)\omega_2|\cdots|\omega_n] - f(x_0)[\omega_2|\cdots|\omega_n]\\ &\in \mathrm{im}(D_{x_0}) + F_{n-1}\,\mathrm{Sh}(\Omega(C))\end{aligned}$$

if $k = 0$, which implies

$$[\mathrm{Sh}_{k-1}(\Omega(C))|d\mathcal{O}(C)|\,\mathrm{Sh}_{n-k}(\Omega(C))] \subset \mathrm{im}(D_{x_0}) + F_{n-1}\,\mathrm{Sh}(\Omega(C)). \tag{9.12}$$

Then (9.10), (9.11), and (9.12) imply (9.9), which in its turn can be shown to imply the statement by induction on n. $\square$

Let $\mathrm{Sh}_+(\Omega(C))$ be the augmentation ideal of $\mathrm{Sh}(\Omega(C))$. A left module structure of $\mathrm{Sh}_+(\Omega(C))$ over $\mathcal{O}(C)$ is defined by $f \cdot [\omega_1|\cdots|\omega_k] := [f\omega_1|\cdots|\omega_k]$ and a right module structure of $\mathrm{Sh}(\Omega(C))$ over $\mathcal{O}(C)$ is defined by $[\omega_1|\cdots|\omega_k]\cdot f := [\omega_1|\cdots|\omega_k f]$ if $k>0$ and $1\cdot f := f(x_0)1$. Define a map

$$D_{x_0}\colon \mathrm{Sh}(\Omega(C))\otimes\mathcal{O}(C)\otimes\mathrm{Sh}_+(\Omega(C))\to\mathrm{Sh}(\Omega(C)),$$
$$s\otimes f\otimes s'\mapsto [s|df|s']-[s|f\cdot s']+[s\cdot f|s'].$$

Theorem 9.7. *The following statements hold true:*

(a) *The sequence of maps*

$$\mathrm{Sh}(\Omega(C))\otimes\mathcal{O}(C)\otimes\mathrm{Sh}_+(\Omega(C))\xrightarrow{D_{x_0}}\mathrm{Sh}(\Omega(C))\xrightarrow{I_{x_0}}\mathcal{O}_{\mathrm{hol}}(\widetilde{C})$$

is an exact complex, so that $\ker(I_{x_0})=\mathrm{im}(D_{x_0})$.

(b) *There is a direct sum decomposition* $\mathrm{Sh}(\Omega(C))=\mathrm{Sub}_\sigma\oplus\ker(I_{x_0})$.

Proof. (a) One checks that $I_{x_0}\circ D_{x_0}=0$, therefore $\mathrm{im}(D_{x_0})\subset\ker(I_{x_0})$. Let us prove the opposite inclusion. The subspace $\mathrm{Sub}_\sigma\subset\mathrm{Sh}(\Omega(C))$ (see (9.1)) is such that (i) the restriction of I_{x_0} to Sub_σ is injective (see Lemma 9.5), (ii) $\mathrm{Sh}(\Omega(C))=\mathrm{Sub}_\sigma+\mathrm{im}(D_{x_0})$ (see Proposition 9.6). Then (i) implies $\mathrm{Sub}_\sigma\cap\ker(I_{x_0})=0$, and therefore $\mathrm{Sub}_\sigma\cap\mathrm{im}(D_{x_0})=0$. Then (ii) implies that $\mathrm{Sh}(\Omega(C))=\mathrm{Sub}_\sigma\oplus\mathrm{im}(D_{x_0})$. The restrictions of I_{x_0} to the two summands are then respectively injective (by Lemma 9.5) and zero (by $I_{x_0}\circ D_{x_0}=0$), which implies $\mathrm{im}(D_{x_0})=\ker(I_{x_0})$.

(b) This follows from combining the already proved equalities $\mathrm{Sh}(\Omega(C))=\mathrm{Sub}_\sigma\oplus\mathrm{im}(D_{x_0})$ and $\mathrm{im}(D_{x_0})=\ker(I_{x_0})$. □

Part III. Appendices

Appendix A. Background on Hopf algebras

This section is devoted to constructions on Hopf algebras. In Appendix A.1 we define an endofunctor $O\mapsto F_\infty O$ of the category of Hopf algebras $\mathbf{HA}$, and in Appendix A.2 a duality functor $\mathbf{HA}\supset\mathbf{HA}_{\mathrm{fd}}\to\mathbf{HA}_{\mathrm{comm}}$, $H\mapsto H'$. In Appendix A.3 we show that a Hopf algebra pairing $p\colon O\otimes H\to\mathbb{C}$ gives rise, under a finite-dimensionality assumption, to a Hopf algebra morphism $\nu(p)\colon F_\infty O\to H'$ (see Lemma A.9).

Appendix A.1. An endofunctor $O\mapsto F_\infty O$ of HA

Let O be a Hopf algebra with coproduct Δ_O. Recall that for $n\geq 0$, one defines $F_nO:=\ker(\mathrm{pr}_O^{\otimes n+1}\circ\Delta_O^{(n+1)})\subset O$, where $\Delta_O^{(n)}\colon O\to O^{\otimes n}$ is the morphism obtained by iteration of Δ_O and $\mathrm{pr}_O\colon O\to O/\mathbb{C}$ is the canonical projection (see Definition 2.7).

Lemma A.1 (See also [Fr, §7.2.15]). *The following statements hold true:*

(a) *For $n \geq 0$, $F_nO \subset F_{n+1}O$.*

(b) *For $n \geq 0$ and $k \in [\![0, n+1]\!]$, one has $\Delta_O(F_nO) \subset F_{k-1}O \otimes O + O \otimes F_{n-k}O$.*

(c) *For $n \geq 0$, one has $\Delta_O(F_nO) \subset \sum_{k=0}^{n} F_kO \otimes F_{n-k}O$.*

(d) *For $n, m \geq 0$, one has $(F_nO) \cdot (F_mO) \subset F_{n+m}O$.*

Proof. (a) Let η_O, ϵ_O be the unit and counit maps of O. One checks that $F_nO = \ker((\mathrm{id} - \eta_O\epsilon_O)^{\otimes n+1} \circ \Delta_O^{(n+1)})$. One has $(\mathrm{id} - \eta_O\epsilon_O)^{\otimes 2} \circ \Delta_O = \Delta_O \circ (\mathrm{id} - \eta_O\epsilon_O)$, which implies that

$$(\mathrm{id} - \eta_O\epsilon_O)^{\otimes n+2} \circ \Delta_O^{(n+2)} = (\Delta_O \otimes \mathrm{id}_O^{\otimes n}) \circ (\mathrm{id} - \eta_O\epsilon_O)^{\otimes n+1} \circ \Delta_O^{(n+1)}$$

(equality of linear maps $O \to O^{\otimes n+2}$). Therefore,

$$F_nO = \ker((\mathrm{id} - \eta_O\epsilon_O)^{\otimes n+1} \circ \Delta_O^{(n+1)}) \subset \ker((\mathrm{id} - \eta_O\epsilon_O)^{\otimes n+2} \circ \Delta_O^{(n+2)}) = F_{n+1}O.$$

(b) Assume that $k \in [\![1, n]\!]$. One has $\Delta_O^{(n+1)} = (\Delta_O^{(k)} \otimes \Delta_O^{(n-k+1)}) \circ \Delta_O$, and so $\mathrm{pr}_O^{\otimes n+1} \circ \Delta_O^{(n+1)} = ((\mathrm{pr}_O^{\otimes k} \circ \Delta_O^{(k)}) \otimes (\mathrm{pr}_O^{\otimes n-k+1} \circ \Delta_O^{(n-k+1)})) \circ \Delta_O$. Therefore,

$$\begin{aligned}\Delta_O(F_nO) &\subset \ker((\mathrm{pr}_O^{\otimes k} \circ \Delta_O^{(k)}) \otimes (\mathrm{pr}_O^{\otimes n-k+1} \circ \Delta_O^{(n-k+1)}))\\ &= \ker(\mathrm{pr}_O^{\otimes k} \circ \Delta_O^{(k)}) \otimes O + O \otimes \ker(\mathrm{pr}_O^{\otimes n-k+1} \circ \Delta_O^{(n-k+1)})\\ &= F_{k-1}O \otimes O + O \otimes F_{n-k}O.\end{aligned}$$

Assume that $k = 0$. It follows from the statement with $k = 1$ that $\Delta_O(F_nO) \subset \mathbb{C} \otimes O + O \otimes F_{n-1}O$. It follows that for $f \in F_nO$, there exists $a \in O$ with $\Delta_O(f) \in 1 \otimes a + O \otimes F_{n-1}O$. Applying $\epsilon_O \otimes \mathrm{id}$, one derives $f \in a + F_{n-1}O$, hence $a \in f + F_{n-1}O \subset F_nO$. Therefore, $\Delta_O(F_nO) \subset \mathbb{C} \otimes F_nO + O \otimes F_{n-1}O$. It follows that $\Delta_O(F_nO) \subset O \otimes F_nO$.

For $k = n+1$, the proof of the statement $\Delta_O(F_nO) \subset F_nO \otimes O$ is similar, based on the statement for $k = n$.

(c) It follows from the statements $\Delta_O(F_nO) \subset O \otimes F_nO$ and $\Delta_O(F_nO) \subset F_nO \otimes O$ ((b) for $k = 0, n$) that $\Delta_O(F_nO) \subset F_nO \otimes F_nO$. Together with statement (b) for $k \in [\![1, n]\!]$, this implies that $\Delta_O(F_nO) \subset F_{k-1}O \otimes F_nO + F_nO \otimes F_{n-k}O$ for any $k \in [\![1, n]\!]$. The statement then follows from

$$\bigcap_{k=1}^{n} (F_{k-1}O \otimes F_nO + F_nO \otimes F_{n-k}O) = \sum_{k=0}^{n} F_kO \otimes F_{n-k}O, \tag{A.1}$$

which we now prove. For $i \in [\![0,n]\!]$, let A_i be a complement of F_iO in $F_{i-1}O$. One has then $F_nO = \bigoplus_{i=0}^n A_i$. Then

$$F_{k-1}O \otimes F_nO + F_nO \otimes F_{n-k}O = \bigoplus_{\substack{(i,j)\in[\![0,n]\!]^2 | \\ i\le k-1 \text{ or } j \le n-k}} A_i \otimes A_j.$$

It follows that $\bigcap_{k=1}^n (F_{k-1}O \otimes F_nO + F_nO \otimes F_{n-k}O) = \sum_{(i,j)\in S} A_i \otimes A_j$, where $S := \{(i,j) \in [\![0,n]\!]^2 \mid \forall\, k \in [\![1,n]\!],$ one has $i \le k-1$ or $j \le n-k\}$. One checks that $S = \{(i,j) \in [\![0,n]\!]^2 \mid i+j \le n\}$, therefore the left-hand side of (A.1) is equal to $\bigoplus_{(i,j)\in[\![0,n]\!]^2|i+j\le n} A_i \otimes A_j \sum_{k=0}^n F_kO \otimes F_{n-k}O$, proving (A.1).

(d) It follows from (c) that

$$\Delta^{(n+m+1)}(F_nO) \subset \sum_{\substack{(i_1,\dots,i_{n+m+1})\in\mathbb{Z}_{\ge0}^{n+m+1} | \\ i_1+\cdots+i_{n+m+1}=n}} F_{i_1}O \otimes \cdots \otimes F_{i_{n+m+1}}O. \tag{A.2}$$

For $a \ge 1$ and $L \subset [\![1,a]\!]$, define $\varphi_L \colon [\![1,a]\!] \to \{0,1\}$ by $\varphi_L(x) = 0$ if $x \in L$ and $\varphi_L(x) = 1$ otherwise. Then set

$$O_L^{(a)} := \bigotimes_{i=1}^a F^{\text{unit}}_{\varphi_L(i)}O \subset O^{\otimes a}, \tag{A.3}$$

where we recall $F_0^{\text{unit}}O = \mathbb{C}1$, $F_1^{\text{unit}}O = O$.

Then for any $(i_1,\dots,i_{n+m+1}) \in \mathbb{Z}_{\ge0}^{n+m+1}$, one has $F_{i_1}O \otimes \cdots \otimes F_{i_{n+m+1}}O \subset O^{(n+m+1)}_{\{j|i_j=0\}}$. This and (A.2), together with the fact that $|\{j \mid i_j = 0\}| \ge m+1$ if $(i_1,\dots,i_{n+m+1}) \in \mathbb{Z}_{\ge0}^{n+m+1}$ is such that $i_1+\cdots+i_{n+m+1} \ge m+1$, imply that

$$\Delta^{(n+m+1)}(F_nO) \subset \sum_{\substack{J\subset[\![1,n+m+1]\!] | \\ |J|\ge m+1}} O_J^{(n+m+1)}. \tag{A.4}$$

Then

$$\begin{aligned}
\Delta^{(n+m+1)}((F_nO)\cdot(F_mO)) &\subset \Delta^{(n+m+1)}(F_nO)\cdot\Delta^{(n+m+1)}(F_mO) \\
&\subset \sum_{\substack{J,K\subset[\![1,n+m+1]\!] | \\ |J|\ge m+1, |K|\ge n+1}} O_J^{(n+m+1)} \cdot O_K^{(n+m+1)} \subset \sum_{\substack{J,K\subset[\![1,n+m+1]\!] | \\ |J|\ge m+1, |K|\ge n+1}} O_{J\cap K}^{(n+m+1)} \\
&\subset \sum_{\substack{L\subset[\![1,n+m+1]\!] | \\ L\ne\emptyset}} O_L^{(n+m+1)} \subset \ker(\mathrm{pr}_O^{\otimes n+m+1}),
\end{aligned}$$

where the first inclusion follows from the fact that $\Delta^{(n+m+1)}$ is an algebra morphism, the second inclusion follows from (A.4), the third inclusion follows from

$O_J^{(n+m+1)} \cdot O_K^{(n+m+1)} \subset O_{J\cap K}^{(n+m+1)}$ for $J, K \subset [\![1, n+m+1]\!]$, the fourth inclusion follows from the fact that if $J, K \subset [\![1, n+m+1]\!]$ are such that $|J| \geq m+1, |K| \geq n+1$, then $J \cap K \neq \emptyset$, and the last inclusion follows from the vanishing of the restriction of $\mathrm{pr}_O^{\otimes n+m+1}$ to any O_L where $\emptyset \neq L \subset [\![1, n+m+1]\!]$. The resulting inclusion $\Delta^{(n+m+1)}((F_nO) \cdot (F_mO)) \subset \ker(\mathrm{pr}_O^{\otimes n+m+1})$ implies the statement. □

Proposition A.2. *The following statements hold true:*

(a) *$F_\bullet O$ defines a Hopf algebra filtration of O.*

(b) *$F_\infty O$ is a Hopf subalgebra of O. The assignment $O \mapsto F_\infty O$ is an endofunctor of the category* **HA** *of Hopf algebras.*

(c) *If $f: O_1 \to O_2$ is a morphism in* **HA**, *then f is compatible with the filtrations $F_\bullet$ on both sides.*

(d) *If O is a $\mathbb{Z}_{\geq 0}$-graded connected Hopf algebra, then $O = F_\infty O$.*

Proof. (a) follows from Lemma A.1. (b) follows from the fact that $F_\infty O$ is the total space of $F_\bullet O$ and from (a). The functoriality statement is obvious. (c) follows from $f^{\otimes n+1} \circ (\mathrm{id} - \eta_{O_1}\epsilon_{O_1})^{\otimes n+1} \circ \Delta_{O_1}^{(n+1)} = (\mathrm{id} - \eta_{O_2}\epsilon_{O_2})^{\otimes n+1} \circ \Delta_{O_2}^{(n+1)} \circ f$ for any $n \geq 0$. (d) If $n \geq 0$, then $\Delta_0^{(n+1)}(O[n])$ is contained in the sum of $O[k_1] \otimes \cdot \otimes O[k_{n+1}]$, where $(k_1, \ldots, k_{n+1})$ is such that $k_1 + \cdots + k_{n+1} = n$. If $(k_1, \ldots, k_{n+1})$ is such a tuple, then there exists $i \in [\![1, n+1]\!]$ such that $k_i = 0$, which by the connectedness of O implies that the corresponding summand is contained in $O^{\otimes i-1} \otimes \mathbb{C} \otimes O^{\otimes n-i}$. Therefore, $\mathrm{pr}_O^{\otimes n+1} \circ \Delta_O^{(n+1)}(O[n]) = 0$, hence $O[n] \subset F_nO$. Therefore, $O = F_\infty O$. □

Appendix A.2. A duality functor $\mathbf{HA_{fd}} \to \mathbf{HA^{op}}$, $H \mapsto H'$

Let H be a Hopf algebra with coproduct Δ_H. Recall that H_+ is the augmentation ideal of H, and by H_+^n the n-th power of this ideal. Set $F^nH := H$ for $n = 0$, $F^nH := H_+^n$ for $n \geq 1$.

Lemma A.3 (See also [Fr, §8.1.1]). *For $n, m \geq 0$, one has $F^nH \cdot F^mH \subset F^{n+m}H$ and $\Delta_H(F^nH) \subset \sum_{n'+n''=n} F^{n'}H \otimes F^{n''}H$.*

Proof. The decreasing character of $(F^nH)_{n\in\mathbb{Z}}$ is obvious. The inclusion $F^nH \cdot F^mH \subset F^{n+m}H$ follows from definitions. The last statement follows from $\Delta_H(H_+) \subset H \otimes H_+ + H_+ \otimes H$, which is itself a consequence of the compatibility of Δ_H with the augmentation of H. □

The coalgebra structure of H induces an algebra structure on H^*. For $n \geq 0$, set $F_nH^* := (F^{n+1}H)^\perp$.

Lemma-Definition A.4. *The collection of subspaces $F_\bullet H^*$ is an algebra filtration of H^*, and*

$$H' := \bigcup_{n\geq 0} F_n H^*$$

is a subalgebra of H^*.

Proof. Let $n, m \geq 0$, $\alpha \in (F^{n+1}H)^\perp$, $\beta \in (F^{m+1}H)^\perp$. Then if $h \in F^{n+m+1}H$, one has $\Delta_H(h) \in F^n H \otimes H + H \otimes F^m H$ by the second statement of Lemma A.3, therefore $(\alpha \cdot \beta)(h) = (\alpha \otimes \beta)(\Delta_H(h)) = 0$, therefore $\alpha \cdot \beta \in (F^{n+m+1}H)^\perp$. This proves the first statement. The second statement follows from the first, as H' is the total subspace of an algebra filtration. $\square$

Definition A.5. Define $\mathbf{HA}_{\mathrm{fd}}$ as the full subcategory of $\mathbf{HA}$ of Hopf algebras H such that $\mathrm{gr}^1 H := F^1H/F^2H$ is finite-dimensional.

Lemma A.6. *If H is an object of $\mathbf{HA}_{\mathrm{fd}}$, then H' is equipped with a linear map $\Delta_{H'} \colon H' \to H' \otimes H'$, uniquely determined by the identity $\Delta_{H'}(\alpha)(h \otimes h') = \alpha(hh')$ for $\alpha \in H'$ and $h, h' \in H$. Then $(H', \Delta_{H'})$ is a Hopf algebra. The assignment $H \mapsto H'$ is a functor $\mathbf{HA}_{\mathrm{fd}} \to \mathbf{HA}^{\mathrm{op}}$.*

Proof. It follows from Lemma A.3 that $F^\bullet H$ is a decreasing algebra filtration of H. The associated graded algebra $\mathrm{gr}^\bullet H$ is such that $\mathrm{gr}^n H = F^nH/F^{n+1}H$ for any $n \geq 0$. Then $\mathrm{gr}^0 H = \mathbb{C}$, and $\mathrm{gr}^\bullet H$ is generated by $\mathrm{gr}^1 H$. As this space is finite-dimensional, so is $\mathrm{gr}^k H$ for any $k \geq 0$. It follows that for any $n \geq 0$, $\bigoplus_{i=0}^{n-1} \mathrm{gr}^i H$ is finite-dimensional. As this space is noncanonically isomorphic to the quotient space H/F^nH, this quotient is finite-dimensional as well.

It also follows from Lemma A.3 that for any $n \geq 0$, F^nH is a two-sided ideal of H, therefore H/F^nH is an algebra, and $H/F^{n+1}H \to H/F^nH$ is an algebra morphism.

Since H/F^nH is finite-dimensional, its associative algebra structure gives rise to a coassociative coalgebra structure on its dual $(H/F^nH)^* = F_{n-1}H^*$ (with $F_{-1}H^* := 0$). It follows from the algebra morphism status of $H/F^{n+1}H \to H/F^nH$ that the canonical inclusion $i_{n,n+1} \colon F_{n-1}H^* \subset F_nH^*$ is a coalgebra morphism.

For $n \geq 0$, define then an algebra morphism $\Delta_{H',n} \colon F_nH^* \to (H')^{\otimes 2}$ to be the composition $i_n^{\otimes 2} \circ \Delta_{F_nH^*}$, where $\Delta_{F_nH^*}$ is the coproduct of the coalgebra structure of F_nH^* and $i_n \colon F_nH^* \to H'$ is the canonical inclusion. One has

$$\Delta_{H',n+1} \circ i_{n,n+1} = \Delta_{H',n}. \tag{A.5}$$

Indeed,

$$\begin{aligned}\Delta_{H',n+1}\circ i_{n,n+1} &= i_{n+1}^{\otimes 2}\circ\Delta_{F_{n+1}H^*}\circ i_{n,n+1}\\ &= i_{n+1}^{\otimes 2}\circ i_{n,n+1}^{\otimes 2}\circ\Delta_{F_nH^*}\\ &= i_n^{\otimes 2}\circ\Delta_{F_nH^*}=\Delta_{H',n},\end{aligned}$$

where the first and last equalities follow from the definitions of $\Delta_{H',n+1}$ and $\Delta_{H',n+1}$, the second equality follows from the coalgebra morphism status of $i_{n,n+1}$, and the third equality follows from $i_{n+1}\circ i_{n,n+1}=i_n$. It follows from (A.5) and from $H'=\bigcup_{n\geq 0}F_nH^*$ that there is a unique linear map $\Delta_{H'}\colon H'\to(H')^{\otimes 2}$, such that $\Delta_{H'}\circ i_n=\Delta_{H',n}$ for any $n\geq 0$. The identities relating $\Delta_{H',n}$ with the product implies that it satisfies the announced identity. The uniqueness statement follows from the fact that the annihilator of $H\otimes H$ in $F_nH^*\otimes F_mH^*$ is zero for any $n,m\geq 0$. One checks that $\Delta_{H'}$ satisfies the Hopf algebra axioms as well as the functoriality statement. □

If H is an object of $\mathbf{HA}_{\mathrm{fd}}$, it follows from Lemmas A.6 and A.2 that H' is equipped with a Hopf algebra filtration $F_\bullet H'$; it is also equipped with the vector space filtration $F_\bullet H^*$ used to define it, given by $F_nH^*=(F^{n+1}H)^\perp$ for any $n\geq 0$. One has the following lemma.

Lemma A.7. *If H is an object of $\mathbf{HA}_{\mathrm{fd}}$, then $F_\bullet H'=F_\bullet H^*$. One has $F_\infty H'=H'$.*

Proof. Let $n\geq 0$ and let us show that $F_nH'=F_nH^*$. Recall that $F_nH'=\ker((\mathrm{id}-\eta_{H'}\epsilon_{H'})^{\otimes n+1}\circ\Delta_{H'}^{(n+1)})$, where $\Delta_{H'}$, $\eta_{H'}$, and $\epsilon_{H'}$ are the coproduct, unit, and counit maps of H'. Recall that $H'=\bigcup_{m\geq 0}F_mH^*\subset H^*$ and that for each $m\geq 0$, F_mH^* is a sub-coalgebra of H'; denote by $\Delta_{F_mH^*}\colon F_mH^*\to(F_mH^*)^{\otimes 2}$ the corresponding coproduct. Let $\epsilon_{F_mH^*}\colon F_mH^*\to\mathbb{C}$ be the composition of $\epsilon_{H'}$ with the inclusion $F_mH^*\subset H'$. The unit of H' corresponds to the counit map of H, which as it vanishes on $F^{m+1}H$ defines an element in $(H/F^{m+1}H)^*=F_mH^*$; let $\eta_{F_mH^*}$ be the corresponding map $\mathbb{C}\to F_mH^*$. Then

$$F_nH'\cap F_mH^* = \ker\Big(F_mH^*\xrightarrow{\Delta_{F_mH^*}^{(n+1)}}(F_mH^*)^{\otimes n+1}\xrightarrow{(\mathrm{id}-\eta_{F_mH^*}\epsilon_{F_mH^*})^{\otimes n+1}}(F_mH^*)^{\otimes n+1}\Big)$$

(a subspace of F_mH^*). Using that if $f\colon E\to F$ is a linear map, then $\ker(f^*\colon F^*\to E^*)=\mathrm{im}(f)^\perp$, that the duals of $\Delta_{F_mH^*}\colon F_mH^*\to(F_mH^*)^{\otimes 2}$, $\eta_{F_mH^*}\colon\mathbb{C}\to F_mH^*$, and $\epsilon_{F_mH^*}\colon F_mH^*\to\mathbb{C}$ are respectively the product map $m_{H/F^{m+1}H}\colon(H/F^{m+1}H)^{\otimes 2}\to H/F^{m+1}H$, the map $\epsilon_{H/F^{m+1}}\colon H/F^{m+1}\to\mathbb{C}$ induced by the counit of H, and the map $\eta_{H/F^{m+1}H}\colon \mathbb{C} \to H/F^{m+1}H$ induced by the unit of H, we see that $F_nH' \cap F_mH^*$ is the annihilator (in $(H/F^{m+1}H)^*$) of the image of the map

$$(H/F^{m+1}H)^{\otimes n+1} \xrightarrow{(\mathrm{id}-\eta_{H/F^{m+1}H}\epsilon_{H/F^{m+1}H})^{\otimes n+1}} (H/F^{m+1})^{\otimes n+1} \xrightarrow{m^{(n+1)}_{H/F^{m+1}H}} H/F^{m+1}H.$$

Since the image of $\mathrm{id} - \eta_H\epsilon_H\colon H \to H$ is F^1H, and since the image of $(F^1H)^{\otimes n+1}$ by $m_H^{(n+1)}\colon H^{\otimes n+1} \to H$ is $F^{n+1}H$, the said image is $(F^{n+1}H + F^{m+1}H)/F^{m+1}H$. Therefore, the subspace $F_nH' \cap F_mH^*$ of $(H/F^{m+1}H)^*$ is the annihilator of the set $(F^{n+1}H + F^{m+1}H)/F^{m+1}H$. If $m \geq n$, this subspace is the annihilator of $F^{n+1}H/F^{m+1}H$, which is the kernel of the canonical map $(H/F^{m+1}H)^* \to (F^{n+1}H/F^{m+1}H)^*$, which is the image of the injection $F_nH^* \hookrightarrow F_mH^*$. Therefore, $F_nH' = F_nH^*$.

One then has $F_\infty H' = \bigcup_{n\geq 0}(F_nH)^\perp = H'$, where the first equality follows from the previous statement and the second equality follows from the definition of H'. $\square$

Appendix A.3. Hopf algebra pairings and Hopf algebra morphisms

Recall that if O, H are Hopf algebras with coproducts Δ_O, Δ_H and counits ϵ_O, ϵ_H, then a Hopf algebra pairing between O and H is a linear map $p\colon O \otimes H \to \mathbb{C}$, such that

$$p(oo' \otimes h) = (p \otimes p) \circ \tau_2(o \otimes o' \otimes \Delta_H(h)),$$

$$p(o \otimes h') = (p \otimes p) \circ \tau_2(\Delta_O(o) \otimes h \otimes h'),$$

$$p(1 \otimes h) = \epsilon_H(h), \quad p(o \otimes 1) = \epsilon_O(o)$$

for $o, o' \in O$ and $h, h' \in H$ (for any $n \geq 2$, τ_n is the canonical map $O^{\otimes n} \otimes H^{\otimes n} \to (O \otimes H)^{\otimes n}$).

Definition A.8. For O, H Hopf algebras, we denote by $\mathbf{Pair}(O, H)$ the set of Hopf algebra pairings between O and H.

Lemma A.9. *For O an object of $\mathbf{HA}$ and H an object of $\mathbf{HA}_{\mathrm{fd}}$, there is a map*

$$\nu\colon \mathbf{Pair}(O, H) \to \mathbf{HA}(F_\infty O, H').$$

For $p \in \mathbf{Pair}(O, H)$, $\nu(p)$ is such that the diagram

$$\begin{tikzcd} F_\infty O \otimes H \arrow[r, "\nu(p)\otimes \mathrm{id}_H"] \arrow[d, "i_O\otimes \mathrm{id}_H"'] & H' \otimes H \arrow[d, "p_H"] \\ O \otimes H \arrow[r, "p"'] & \mathbb{C} \end{tikzcd}$$

commutes, $i_O\colon F_\infty O \to O$ being the canonical injection and $p_H\colon H \otimes H' \to \mathbb{C}$ being the composition $H \otimes H' \hookrightarrow H \otimes H^* \to \mathbb{C}$.

Proof. Let $p \in \mathbf{Pair}(O, H)$. Let $n \geq 0$; then

$$\begin{aligned} p(F_nO \otimes F^{n+1}H) &= p(F_nO \otimes H_+^{n+1}) \subset p^{\otimes n+1} \circ \tau_{n+1}(\Delta_O^{(n+1)}(F_nO) \otimes H_+^{\otimes n+1}) \\ &\subset \sum_{\emptyset \neq L \subset [\![1,n+1]\!]} p^{\otimes n+1} \circ \tau_{n+1}(O_L^{(n+1)} \otimes H_+^{\otimes n+1}) = 0, \end{aligned}$$

where the first inclusion follows from the behavior of p with respect to coproducts, the second inclusion follows from $\Delta_O^{(n+1)}(F_nO) \subset \sum_{\emptyset \neq L \subset [\![1,n+1]\!]} O_L^{(n+1)}$, where $O_L^{(n+1)}$ is as in (A.3) (a consequence of the definition of F_nO), and the last inclusion follows from $p(1 \otimes H_+) = 0$, itself a consequence of the behavior of p with respect to counits. It follows that $p(F_nO \otimes F^{n+1}H) = 0$. This implies that the restriction $p_{|F_nO\otimes H}$ of p to $F_nO \otimes H$ induces a linear map $p_n\colon F_nO \otimes (H/F^{n+1}H) \to \mathbb{C}$. As $H/F^{n+1}H$ is finite-dimensional, this gives rise to a linear map $\nu(p)_n\colon F_nO \to (H/F^{n+1}H)^* = F_nH^*$. One checks that $i_{n,n+1}^{H'} \circ \nu(p)_n = \nu(p)_{n+1} \circ i_{n,n+1}^O$, where $i_{n,n+1}^O$, $i_{n,n+1}^{H'}$ are the canonical maps $F_nO \to F_{n+1}O$ and $F_nH^* \to F_{n+1}H^*$. It follows that there exists a unique linear map $\nu(p)\colon F_\infty O \to H'$, such that $\nu(p) \circ i_n^O = i_n^{H'} \circ \nu(p)_n$ for any $n \geq 0$, where i_n^O, $i_n^{H'}$ are the canonical maps $F_nO \to F_\infty O$ and $F_nH^* \to H'$. One checks that $\nu(p)$ defines a Hopf algebra morphism as well as the announced commutative diagram. □

Appendix B. Background on Hopf algebras with (co)actions on algebras

In Appendix B.1 we introduce a category **HACA** of Hopf algebras with comodule algebra (HACAs) and an endofunctor of that category extending that of Appendix A.1. In Appendix B.2 we introduce a category **HAMA** of Hopf algebras with module algebras (HAMAs), together with a diagram $\mathbf{HAMA} \supset \mathbf{HAMA}_{\mathrm{fd}} \to \mathbf{HACA}$ extending that of Appendix A.2. In Appendix B.3 we introduce the notion of a pairing-morphism from a HAMA to a HACA and show that it gives rise, under a finite-dimensionality assumption, to a HACA morphism extending the Hopf algebra morphism of Appendix A.3. In Appendix B.4 we make use of the natural filtration attached to each HACA to construct two functors $\mathrm{gr}, \Phi\colon \mathbf{HACA} \to \mathbf{Alg}_{\mathrm{gr}}$ and a natural transformation $\mathrm{nat}\colon \mathrm{gr} \Rightarrow \Phi$ between these functors. Appendix B.5 is devoted to the main technical tool of this paper (Proposition B.18), based on nat and giving a criterion for the HACA morphisms arising from pairing-morphisms of a certain type by the construction of Appendix B.3 to be isomorphisms.

Appendix B.1. An endofunctor $(O, A) \mapsto (F_\infty O, F_\infty A)$ of HACA

Definition B.1. A Hopf algebra with comodule algebra (HACA) is a pair (O, A) where O is a Hopf algebra and A is an algebra, equipped with a left coaction of O; the coproduct map of O being denoted Δ_O and the coaction by $\Delta_A\colon A \to O \otimes A$, in particular Δ_A is an algebra morphism and $(\Delta_O \otimes \mathrm{id}_A) \circ \Delta_A = (\mathrm{id}_O \otimes \Delta_A) \circ \Delta_A$. If (O, A) and (O', A') are HACAs, a morphism from (O, A) to (O', A') is the pair of (f_A, f_O) of an algebra morphism $f_A\colon A \to A'$ and a Hopf algebra morphism $f_O\colon O \to O'$, such that $(f_O \otimes f_A) \circ \Delta_A = \Delta_{A'} \circ f_A$. Denote the resulting category by **HACA**.

Let (O, A) be a HACA.

Definition B.2. For $n \geq 0$, define $F_n A$ to be the preimage of $F_n O \otimes A$ by the map $\Delta_A\colon A \to O \otimes A$.

Lemma B.3. *The following statements hold true:*

(a) *$F_\bullet A$ is an algebra filtration of A.*

(b) *For $n \geq 0$, $\Delta_A(F_n A) \subset F_n O \otimes F_n A$.*

(c) *For $n \geq 0$, $\Delta_A(F_n A) \subset \sum_{p+q=n} F_p O \otimes F_q A$.*

(d) *The restriction of Δ_A to $F_\infty A$ corestricts to $F_\infty O \otimes F_\infty A$. Together with the structures of algebra of $F_\infty A$ and of Hopf algebra of $F_\infty O$, the resulting map $\Delta_{F_\infty A}\colon F_\infty A \to F_\infty O \otimes F_\infty A$ equips $(F_\infty O, F_\infty A)$ with a HACA structure. The assignment $(O, A) \mapsto (F_\infty O, F_\infty A)$ is an endofunctor of* **HACA**.

Proof. (a) $A \otimes F_\bullet O$ is an algebra filtration of the algebra $O \otimes A$; as $\Delta_A\colon A \to O \otimes A$ is an algebra morphism, the preimage of this filtration is an algebra filtration of the source; as this preimage is $F_\bullet A$, the latter is an algebra filtration.

(b) Let $a \in F_n A$; then $\Delta_A(a) \in F_n O \otimes A$. There exist families $(o_s)_{s\in S}$, $(a_s)_{s\in S}$ of elements of O and A, indexed by a finite set S, such that $(o_s)_{s\in S}$ is free and $\Delta_A(a) = \sum_s o_s \otimes a_s$. Since $\Delta_O(F_n O) \subset (F_n O)^{\otimes 2}$, one has $(\Delta_O \otimes \mathrm{id}_A) \circ \Delta_A(a) \in (F_n O)^{\otimes 2} \otimes A$. By the coassociativity of the coaction, this term is equal to $(\mathrm{id}_O \otimes \Delta_A) \circ \Delta_A(a)$, so $(\mathrm{id}_O \otimes \Delta_A) \circ \Delta_A(a) \in (F_n O)^{\otimes 2} \otimes A$. Therefore, $\sum_{s\in S} o_s \otimes \Delta_A(a_s) \in$

$(F_nO)^{\otimes 2}\otimes A$. As $(o_s)_{s\in S}$ is free, this implies $\Delta_A(a_s)\in F_nO\otimes A$ for any s, therefore $a_s\in F_nA$. The claim follows.

(c) Let $a\in F_nA$; then by (b) $\Delta_A(a)\in F_nO\otimes F_nA$. Let U_i be a complement of $F_{i-1}O$ in F_iO, so $F_iO=U_1\oplus\cdots\oplus U_i$. Then

$$F_nO\otimes F_nA=(U_0\otimes F_nA)\oplus\cdots\oplus(U_n\otimes F_nA),$$

giving rise to a decomposition $\Delta_A(a)=z_0+\cdots+z_n$.

Since $\Delta_A(a)\in F_nO\otimes F_nA$, Lemma A.1(c) implies that $(\Delta_O\otimes\mathrm{id}_A)\circ\Delta_A(a)\in\sum_{p+q=n}F_pO\otimes F_qO\otimes F_nA\subset(F_nO)^{\otimes 2}\otimes F_nA$. By the coassociativity identity, this term is equal to $(\mathrm{id}_O\otimes\Delta_A)\circ\Delta_A(a)=\sum_{i=0}^n(\mathrm{id}_O\otimes\Delta_A)(z_i)$. It follows that $\sum_{i=0}^n(\mathrm{id}_O\otimes\Delta_A)(z_i)\in\sum_{p+q=n}F_pO\otimes F_qO\otimes F_nA$. There is a direct sum decomposition $(F_nO)^{\otimes 2}\otimes F_nA=\bigoplus_{i,j\in[\![0,n]\!]}U_i\otimes U_j\otimes F_nA$; then

$$\sum_{p+q=n}F_pO\otimes F_qO\otimes F_nA=\bigoplus_{i,j\in[\![0,n]\!],i+j\le n}U_i\otimes U_j\otimes F_nA,$$

and

$$(\mathrm{id}_O\otimes\Delta_A)(z_i)\in U_i\otimes F_nO\otimes F_nA=\bigoplus_{j\in[\![0,n]\!]}U_i\otimes U_j\otimes F_nA.$$

Decomposing $(\mathrm{id}_O\otimes\Delta_A)(z_i)=\sum_{j\in[\![0,n]\!]}t_{ij}$ for any $i\in[\![0,n]\!]$, one obtains $\sum_{i,j\in[\![0,n]\!]}t_{ij}\in\bigoplus_{i,j\in[\![0,n]\!],i+j\le n}U_i\otimes U_j\otimes F_nA$, therefore $t_{ij}=0$ if $i+j>n$, therefore for any $i\in[\![0,n]\!]$, one has

$$(\mathrm{id}_O\otimes\Delta_A)(z_i)\in\bigoplus_{j=0}^{n-i}U_i\otimes U_j\otimes F_nA=U_i\otimes F_{n-i}O\otimes F_nA.$$

There exist families $(o_s^i)_{s\in S_i}$, $(a_s^i)_{s\in S_i}$ of elements of U_i and F_nA, indexed by a finite set S_i, such that $(o_s^i)_{s\in S_i}$ is free and $z_i=\sum_{s\in S_i}o_s^i\otimes a_s^i$. Then $\sum_{s\in S_i}o_s^i\otimes\Delta_A(a_s^i)\in U_i\otimes F_{n-i}O\otimes F_nA$. Since $(o_s^i)_{s\in S_i}$ is free, this implies $\Delta_A(a_s^i)\in F_{n-i}O\otimes F_nA$ for any $s\in S_i$, therefore $a_s^i\in F_{n-i}A$. Therefore, $z_i\in F_iO\otimes F_{n-i}A$, proving the claim.

(d) By (b), $\Delta_A(F_nA)\subset F_\infty O\otimes F_\infty A$ for any $n\ge 0$, which implies $\Delta_A(F_\infty A)\subset F_\infty O\otimes F_\infty A$. One can then define the announced map $\Delta_{F_\infty A}\colon F_\infty A\to F_\infty O\otimes F_\infty A$ and check it to have the announced properties. The endofunctor statement is then immediate. $\square$

Appendix B.2. A category HAMA and a functor $\mathbf{HAMA_{fd}}\to\mathbf{HACA}$

For M a right module over an algebra A and $V\subset A$, we use the notation $\mathrm{Ann}(V)$ to denote $\{m\in M\mid m\cdot V=0\}$.

Definition B.4. For H a Hopf algebra, M a right H-module, define $F_nM \coloneqq \mathrm{Ann}(F^{n+1}H, M)$ for $n \geq 0$.

Note that F_0M is equal to M^H, the submodule of M of invariants of the action of H.

Lemma B.5. *Let H be a Hopf algebra.*

(a) *For M a right H-module, $F_\bullet M$ is an increasing H-module filtration of M.*
(b) *Any morphism $f\colon M \to N$ a morphism of right H-modules is compatible with the filtrations.*
(c) *If M is a right H-module and $n \geq 0$, then for any $m \in F_nM$, the map $F^nH \to M$, $h \mapsto m \cdot h$ takes its values in M^H. The resulting map $F_nM \to \mathrm{Hom}_{\mathbb{C}\text{-vec}}(F^nH, M^H)$ induces a map $\mathrm{gr}_nM \to \mathrm{Hom}_{\mathbb{C}\text{-vec}}(\mathrm{gr}^nH, M^H)$, which is injective.*

Proof. (a) follows from the decreasing character of $n \mapsto F^{n+1}H$ and from the ideal nature of $F^{n+1}H$ for $n \geq 0$. (b) is immediate.

(c) If $m \in F_nM$, $h \in F^nH$ and $h' \in F_1H$, then $hh' \in F^{n+1}H$ so $m \cdot (hh') = 0$, therefore $(m \cdot h) \cdot h' = 0$, so $m \cdot h \in M^H$. The map $F_nM \to \mathrm{Hom}_{\mathbb{C}\text{-vec}}(F^nH, M^H)$ factors through a map $F_nM \to \mathrm{Hom}_{\mathbb{C}\text{-vec}}(\mathrm{gr}^nH, M^H)$ as $m \cdot h = 0$ for any $m \in F_nM$, $h \in F^{n+1}H$. The restriction of this map to $F_{n-1}M$ is zero as $m \cdot h = 0$ for any $m \in F_{n-1}M$, $h \in F^nH$. This leads to the announced map $\mathrm{gr}_nM \to \mathrm{Hom}_{\mathbb{C}\text{-vec}}(\mathrm{gr}^nH, M^H)$. As the map $\mathrm{Hom}_{\mathbb{C}\text{-vec}}(\mathrm{gr}^nH, M^H) \to \mathrm{Hom}_{\mathbb{C}\text{-vec}}(F^nH, M^H)$ is injective, the kernel of the map $F_nM \to \mathrm{Hom}_{\mathbb{C}\text{-vec}}(\mathrm{gr}^nH, M^H)$ is equal to the kernel of the map $F_nM \to \mathrm{Hom}_{\mathbb{C}\text{-vec}}(F^nH, M^H)$, which is $\{m \in F_nM \mid m \cdot h = 0 \text{ for any } h \in F^nH\}$, and this is equal to $F^{n-1}M$. It follows that the map $\mathrm{gr}_nM \to \mathrm{Hom}_{\mathbb{C}\text{-vec}}(\mathrm{gr}^nH, M^H)$ is injective. □

Definition B.6. A Hopf algebra with module algebra (HAMA) is pair (B, H) where H is a Hopf algebra and B is an algebra equipped with a right action of H, i.e. a linear map $B \otimes H \to H$, $b \otimes h \mapsto b \cdot h$, such that $(b \cdot h) \cdot h' = b \cdot (hh')$ and $(bb') \cdot h = (b \cdot h^{(1)})(b' \cdot h^{(2)})$ for any $b, b' \in B$ and $h, h' \in H$, the coproduct of $h \in H$ being denoted $h^{(1)} \otimes h^{(2)}$ (Sweedler). If (B, H) and (B', H') are HAMAs, a morphism $(B, H) \to (B', H')$ is a pair (f_H, f_B), where $f_H\colon H' \to H$ is a Hopf algebra morphism and $f_B\colon B \to B'$ is an algebra morphism, such that $f_B(b \cdot f_H(h')) = f_B(b) \cdot h'$ for any $b \in B$ and $h' \in H'$. Morphisms are composed as follows: $(g_H, g_B) \circ (f_H, f_B) \coloneqq (f_H \circ g_H, g_B \circ f_B)$.

One checks that HAMAs build up a category, which will be denoted **HAMA**.

Definition B.7. Let (B, H) be a HAMA. For $n \geq 0$, set

$$F_n B \coloneqq \mathrm{Ann}(F^{n+1}H, B) = \{b \in B \mid b \cdot F^{n+1}H = 0\}.$$

Lemma B.8. *Let (B, H) be a HAMA.*

(a) *$F_\bullet B$ is an algebra filtration of B.*

(b) *Let $n, m \geq 0$. For $n \geq m$, then $(F_n B) \cdot (F^m H) \subset F_{n-m} B$. If $n < m$, then $(F_n B) \cdot (F^m H) = 0$.*

(c) *For any $n \geq 0$, the action map $B \otimes H \to B$ induces a linear map $F_n B \otimes H \to F_n B$, which factorizes through a linear map $F_n B \otimes (H/F^{n+1}H) \to F_n B$.*

Proof. (a) Let $n, m \geq 0$. By the second part of Lemma A.3, one has $\Delta_H(F^{n+m+1}H) \subset F^{n+1}H \otimes H + H \otimes F^{m+1}H$. Together with the Hopf algebra action axiom, this implies the inclusion in

$$((F_n B)(F_m B)) \cdot (F^{n+m+1}H) \subset ((F_n B) \cdot (F^{n+1}H))((F_m B) \cdot H) \\ + ((F_n B) \cdot H)((F_m B) \cdot (F^{m+1}H)) = 0,$$

and the equality follows from $(F_n B) \cdot (F^{n+1}H) = (F_m B) \cdot (F^{m+1}H) = 0$. The resulting vanishing of $(F_n B)(F_m B) \cdot (F^{n+m+1}H)$ implies $(F_n B)(F_m B) \subset F_{n+m} B$.

(b) Assume $n \geq m \geq 0$. Then

$$((F_n B) \cdot (F^m H)) \cdot (F^{n-m+1}H) \subset (F_n B) \cdot ((F^m H)(F^{n-m+1}H)) \\ \subset (F_n B) \cdot (F^{n+1}H) = 0,$$

where the first inclusion follows from the right action axioms, the second inclusion follows from the first part of Lemma A.3, and the equality follows from the definition of $F_n B$. This implies $(F_n B) \cdot (F^m H) \subset F_{n-m} B$, as claimed. In particular, for any $n \geq 0$, $(F_n B) \cdot (F^n H) \subset F_0 B = B^H$. Then if $m > n$, one has

$$(F_n B) \cdot (F^m H) = (F_n B) \cdot ((F^n H)(F^{m-n} H)) \subset ((F_n B) \cdot (F^n H)) \cdot (F^{m-n} H) \\ \subset B^H \cdot (F^{m-n} H) = 0,$$

where the first equality follows from $F^m H = (F^n H)(F^{m-n} H)$, the first inclusion follows from the right action axioms, the second inclusion follows from the inclusion $(F_n B) \cdot (F^n H) \subset B^H$, and the last equality follows from $B^H \cdot H_+ = 0$. This completes the proof of the claim.

(c) The first statement follows from (b) for $m = 0$. The second statement follows from $F_n B \cdot (F^{n+1}H) = 0$, which follows from the definition of $F_n B$. ☐

Definition B.9. Define $\mathbf{HAMA}_{\mathrm{fd}}$ to be the full subcategory of $\mathbf{HAMA}$ of objects (B, H) such that H is an object of $\mathbf{HA}_{\mathrm{fd}}$.

Lemma B.10. *Let (B, H) be an object of $\mathbf{HAMA}_{\mathrm{fd}}$.*

(a) *For any $n \geq 0$, the linear map $F_nB \otimes (H/F^{n+1}H) \to F_nB$ from Lemma B.8(c) induces a linear map $\Delta_{F_nB} \colon F_nB \to (H/F^{n+1}H)^* \otimes F_nB = F_nH^* \otimes F_nB$.*

(b) *There is a linear map $\Delta_{F_\infty B} \colon F_\infty B \to H' \otimes F_\infty B$, uniquely determined by the condition that $\Delta_{F_\infty B} \circ i_n^{F_\infty B} = (i_n^{H'} \otimes i_n^{F_\infty B}) \circ \Delta_{F_nB}$ for any $n \geq 0$, where Δ_{F_nB} is as in* (a) *and $i_n^{H'} \colon F_nH^* \to H'$, $i_n^{F_\infty B} \colon F_nB \to F_\infty B$ are the canonical injections.*

(c) *Together with the algebra structure of $F_\infty B$ and Hopf algebra structure of H', $\Delta_{F_\infty B}$ equips $(H', F_\infty B)$ with a HACA structure. The assignment $(B, H) \mapsto (H', F_\infty B)$ is a functor $\mathbf{HAMA}_{\mathrm{fd}} \to \mathbf{HACA}$.*

Proof. (a) follows from the finite-dimensionality of $H/F^{n+1}H$, which has been established in the proof of Lemma A.6.

(b) Let $n \geq 0$ and $i_{n,n+1}^{F_\infty B} \colon F_nB \hookrightarrow F_{n+1}B$ and $i_{n,n+1}^{H'} \colon F_nH^* \hookrightarrow F_{n+1}H^*$ be the canonical inclusions. Recall that $i_{n,n+1}^{H'}$ is the dual of the projection map $p_{n+1,n}^H \colon H/F^{n+2}H \to H/F^{n+1}H$. The diagram

$$\begin{tikzcd}
 & F_nB \otimes H \arrow[r] \arrow[dl] \arrow[d] & F_nB \otimes (H/F^{n+1}B) \arrow[dd] \\
F_{n+1}B \otimes H \arrow[r] \arrow[d] & B \otimes H \arrow[dr] & \\
F_{n+1}B \otimes (H/F^{n+2}B) \arrow[rr] & & B
\end{tikzcd}$$

is built up of two commutative squares and a commutative triangle, therefore it is commutative; it is the composition of the square whose commutativity expresses the equality $\Delta_{F_\infty B} \circ i_{n,n+1}^{F_\infty B} = (i_{n,n+1}^{F_\infty B} \otimes i_{n,n+1}^{H'}) \circ \Delta_{F_nB}$ with the morphisms $F_{n+1}B \to B$ and $F_nB \otimes H \to F_nB \otimes (H/F^{n+2}H)$, which are respectively injective and surjective. This implies the commutativity of this square, therefore the said equality, from which one derives the statement.

(c) Let us show the coassociativity of $(H', F_\infty B)$. Recall that for $\xi \in H'$ and $b \in F_\infty B$, one has $\xi \cdot b = \langle \mathrm{id} \otimes \xi, \Delta_B(b) \rangle$. Let $b \in F_\infty B$ and $\xi, \eta \in H'$. Then

$$\begin{aligned}
\xi \cdot (\eta \cdot b) &= \langle \mathrm{id} \otimes \xi, \Delta_B(\eta \cdot b) \rangle = \langle \mathrm{id} \otimes \xi, \Delta_B(\langle \mathrm{id} \otimes \eta, \Delta_B(b) \rangle) \rangle \\
&= \langle \mathrm{id} \otimes \xi \otimes \eta, (\Delta_B \otimes \mathrm{id}) \circ \Delta_B(b) \rangle = \langle \mathrm{id} \otimes \xi \otimes \eta, (\mathrm{id} \otimes \Delta_H) \circ \Delta_B(b) \rangle \\
&= \langle \mathrm{id} \otimes (\xi \cdot \eta), \Delta_B(b) \rangle = (\xi \cdot \eta) \cdot b. \qquad \square
\end{aligned}$$

Lemma B.11. *The following statements hold true:*

(a) *Let (B,H) be a HAMA and let $H' \subset H$ be a Hopf subalgebra. Let us denote by $F^H_\bullet B$, $F^{H'}_\bullet B$ the algebra filtrations of B attached to the actions of H, H'. Then $F^H_\infty B \subset F^{H'}_\infty B$.*

(b) *Let $f\colon B \to C$ be an algebra morphism; let H be a Hopf algebra right acting on B and C; assume that f is H-equivariant. Then $(f,\mathrm{id})\colon (B,H) \to (C,H)$ is a HAMA morphism, and $f(F_\infty B) \subset F_\infty C$.*

Proof. (a) For $n \geq 0$, one has $F^H_n B = \{b \in B \mid h \cdot b = 0 \text{ for } h \in F^{n+1}H\} \subset \{b \in B \mid h \cdot b = 0 \text{ for } h \in F^{n+1}H'\} = F^{H'}_n B$, where the inclusion follows from $F^{n+1}H' \subset F^{n+1}H$. This implies $F^H_\infty B \subset F^{H'}_\infty B$.

(b) For $n \geq 0$, $b \in F_n B$, and $h \in F^{n+1}H$, one has $f(b) \cdot h = f(b \cdot h) = f(0) = 0$, where the first equality follows from the equivariance of f and the second equality follows from $b \in F_n B$. Therefore, $f(b) \in F_n C$. So $f(F_n B) \subset F_n C$, hence $f(F_\infty B) \subset F_\infty C$. □

Appendix B.3. Pairing-morphisms from a HACA to HAMA and HACA morphisms

Definition B.12. Let (O,A) be a HACA and (B,H) be a HAMA. A pairing-morphism from (O,A) to (B,H) is a pair (p,f), where $p\colon O \otimes H \to \mathbb{C}$ is a Hopf algebra pairing, and $f\colon A \to B$ is an algebra morphism, such that the following diagram commutes:

(B.1)
$$\begin{array}{ccccc} A \otimes H & & \xrightarrow{f\otimes\mathrm{id}} & & B \otimes H \\ \downarrow{\scriptstyle \Delta_A\otimes\mathrm{id}} & & & & \downarrow \\ O \otimes A \otimes H & \xrightarrow[\sigma_{OA}\otimes\mathrm{id}]{} & A \otimes O \otimes H & \xrightarrow[f\otimes p]{} & B, \end{array}$$

where $\Delta_A\colon A \to O \otimes A$ is the coaction morphism of (O,A), $\sigma_{OA}\colon O \otimes A \to A \otimes O$ is the permutation isomorphism, and the right vertical map is the action map of (B,H). Equivalently, one requests the identity

$$f(a) \cdot h = f(a^{(2)})p(a^{(1)} \otimes h)$$

to be satisfied for any $a \in A$ and $h \in H$, where one denotes $\Delta_A(a) := a^{(1)} \otimes a^{(2)}$.

We denote by $\mathbf{PM}((O,A),(B,H))$ the set of pairing-morphisms from (O,A) to (B,H).

Lemma B.13. *Let (O,A) be an object in* **HACA** *and (B,H) be an object in* $\mathbf{HAMA}_{\mathrm{fd}}$. *If $(p,f) \in \mathbf{PM}((O,A),(B,H))$, then $f(F_\bullet A) \subset F_\bullet B$, so that f induces* *an algebra morphism* $F_\infty f\colon F_\infty A\to F_\infty B$. *The assignment* $(p,f)\mapsto(\nu(p),F_\infty f)$ *defines a map*

$$\tilde\nu\colon \mathbf{PM}((O,A),(B,H))\to\mathbf{HACA}((F_\infty O,F_\infty A),(H',F_\infty B)).$$

Proof. Let $n\geq 0$ and $a\in F_nA$. Then $\Delta_A(a)\in F_nO\otimes A$. So $(\Delta_O^{(n+1)}\otimes\mathrm{id}_A)\circ\Delta_A(a)\in(\sum_{\emptyset\neq L\subset[\![1,n+1]\!]}O_L^{(n+1)})\otimes A\subset A\otimes O^{\otimes n+1}$. Then for $h_1,\ldots,h_{n+1}\in H_+$, one has

$$\begin{aligned}f(a)\cdot(h_1\cdots h_{n+1})&=\langle(\mathrm{id}\otimes f)\circ\Delta_A(a),(h_1\cdots h_{n+1})\otimes\mathrm{id}\rangle\\&=\langle(\mathrm{id}^{\otimes n+1}\otimes f)\circ\Delta_A^{(n+1)}(a),(h_1\otimes\cdots\otimes h_{n+1})\otimes\mathrm{id}\rangle=0,\end{aligned}$$

where the first equality follows from Definition B.12, the second equality follows from the Hopf algebra pairing axiom, and the last equality follows from $\langle O_L^{(n+1)},H_+^{\otimes n+1}\rangle=0$, which follows from $\langle 1,H_+\rangle=0$. It follows that $f(a)\in F_nB$.

Let us prove that $(\nu(p),F_\infty f)$ is a morphism in **HACA**. Let $n\geq 0$. Since $f(F_nA)\subset F_nB$, f induces a linear map $F_nf\colon F_nA\to F_nB$.

By Lemma B.3(b), the coaction map $\Delta_A\colon A\to O\otimes A$ induces a map $\Delta_{F_nA}\colon F_nA\to F_nO\otimes F_nA$. By Lemma B.8(c), the action map $\mathrm{act}_B\colon B\otimes H\to B$ induces a map $\mathrm{act}_{F_nB}\colon F_nB\otimes H\to F_nB$. Therefore (B.1) induces a commutative diagram

$$\begin{tikzcd} F_nA\otimes H \arrow[r, "F_nf\otimes\mathrm{id}"] \arrow[d, "\Delta_{F_nA}\otimes\mathrm{id}"'] & F_nB\otimes H \arrow[d, "\mathrm{act}_{F_nB}"] \\ F_nO\otimes F_nA\otimes H \arrow[r, "(F_nf\otimes p)\circ(\sigma_{OA}\otimes\mathrm{id})"'] & F_nB. \end{tikzcd}\tag{B.2}$$

By the proof of Lemma A.9, the restriction of p to $F_nO\otimes F^{n+1}H$ is zero, which induces a pairing $p_n\colon F_nO\otimes(H/F^{n+1}H)\to\mathbb{C}$. By Lemma B.8(c), the restriction of act_{F_nB} to $F_nB\otimes(H/F^{n+1}H)$ is zero, inducing a linear map $\underline{\mathrm{act}}_{F_nB}\colon F_nB\otimes(H/F^{n+1}H)\to F_nB$. The above diagram therefore gives rise to a commutative diagram

$$\begin{tikzcd} F_nA\otimes(H/F^{n+1}H) \arrow[r, "F_nf\otimes\mathrm{id}"] \arrow[d, "\Delta_{F_nA}\otimes\mathrm{id}"'] & F_nB\otimes(H/F^{n+1}H) \arrow[d, "\underline{\mathrm{act}}_{F_nB}"] \\ F_nO\otimes F_nA\otimes(H/F^{n+1}H) \arrow[r, "(F_nf\otimes p_n)\circ(\sigma_{OA}\otimes\mathrm{id})"'] & F_nB. \end{tikzcd}$$

Since $H/F^{n+1}H$ is finite-dimensional, the map p_n gives rise to a linear map $\nu(p)_n\colon F_nO\to F_nH^*$ (see the proof of Lemma A.9) and the map $\underline{\mathrm{act}}_{F_nB}$ gives rise to the map $\Delta_{F_nB}\colon F_nB\to F_nH^*\otimes F_nB$ (see Lemma B.10(a)). The above commutative diagram then gives rise to a commutative diagram

$$\begin{tikzcd} F_nA \arrow[r, "F_nf"] \arrow[d, "\Delta_{F_nA}"'] & F_nB \arrow[d, "\Delta_{F_nB}"] \\ F_nO \otimes F_nA \arrow[r, "\nu(p)_n \otimes F_nf"'] & F_nH^* \otimes F_nB. \end{tikzcd}$$

This commutativity means that the restrictions to F_nA of the two composed maps of the diagram

$$\begin{tikzcd} F_\infty A \arrow[r, "F_\infty f"] \arrow[d, "\Delta_{F_\infty A}"'] & F_\infty B \arrow[d, "\Delta_{F_\infty B}"] \\ F_\infty O \otimes F_\infty A \arrow[r, "\nu(p) \otimes F_\infty f"'] & H' \otimes F_\infty B \end{tikzcd}$$

are equal. Since $F_\infty A = \bigcup_{n\geq 0} F_nA$, this diagram is commutative, therefore the pair $(\nu(p), F_\infty f)$ is a morphism in **HACA**. $\square$

Appendix B.4. Two functors HACA $\to$ $\mathbf{Alg}_{\mathrm{gr}}$ and a natural transformation

Definition B.14. Define $\mathbf{Alg}_{\mathrm{gr}}$ to be the category of $\mathbb{Z}_{\geq 0}$-graded associative algebras.

By Proposition A.2(a), a Hopf algebra O is naturally equipped with a filtration $F_\bullet O$. Set $\mathrm{gr}(O) := \bigoplus_{i\geq 0} \mathrm{gr}_i(O)$, where $\mathrm{gr}_i(O) := F_iO/F_{i-1}O$ for $i \geq 0$. Then $\mathrm{gr}(O)$ is a $\mathbb{Z}_{\geq 0}$-graded Hopf algebra, therefore also a $\mathbb{Z}_{\geq 0}$-graded associative algebra. Recall that an object (O, A) of **HACA** gives rise to an algebra filtration $F_\bullet A$ on A (see Lemma B.3(a)). Then $\mathrm{gr}(A) := \bigoplus_{i\geq 0} \mathrm{gr}_i(A)$, where $\mathrm{gr}_i(A) := F_iA/F_{i-1}A$ for $i \geq 0$ is a $\mathbb{Z}_{\geq 0}$-graded associative algebra.

Lemma B.15. *The following statements hold true:*

(a) *The assignment* $(O, A) \mapsto \mathrm{gr}(A)$ *defines a functor* $\mathrm{gr}\colon \mathbf{HACA} \to \mathbf{Alg}_{\mathrm{gr}}$.

(b) *For* (O, A) *an object in* **HACA**, *equip* $\mathrm{gr}(O) \otimes F_0A$ *with the tensor product* $\mathbb{Z}_{\geq 0}$*-graded associative algebra structure, where* F_0A *is concentrated in degree* 0. *Then the assignment* $(O, A) \mapsto \mathrm{gr}(O) \otimes F_0A$ *defines a functor* Φ: $\mathbf{HACA} \to \mathbf{Alg}_{\mathrm{gr}}$.

Proof. (a) The assignment $(O, A) \mapsto (A, F_\bullet A)$ defines a functor $\mathbf{HACA} \to \mathbf{Alg}_{\mathrm{fil}}$ where $\mathbf{Alg}_{\mathrm{fil}}$ is the category of filtered algebras (in the sense of Section 4.1). The said assignment is the composition of this functor with the "associated-graded functor" $\mathbf{Alg}_{\mathrm{fil}} \to \mathbf{Alg}_{\mathrm{gr}}$.

(b) The assignments $(O, A) \mapsto F_0A$ and $(O, A) \mapsto \mathrm{gr}(O)$ define the functors $\mathbf{HACA} \to \mathbf{Alg}$ and $\mathbf{HACA} \to \mathbf{Alg}_{\mathrm{gr}}$, where $\mathbf{Alg}$ is the category of associative algebras. The said assignment is the composition of the product of these functors, of the "degree-0 functor" $\mathbf{Alg} \to \mathbf{Alg}_{\mathrm{gr}}$, and of the tensor product functor $\mathbf{Alg}_{\mathrm{gr}}^2 \to \mathbf{Alg}_{\mathrm{gr}}$. □

Let (O, A) be an object in $\mathbf{HACA}$. By Lemma B.3(c), the coaction map $\Delta_A \colon A \to O \otimes A$ is compatible with the filtrations on both sides, and therefore gives rise to a morphism of graded algebras $\mathrm{gr}_\bullet(\Delta_A)\colon \mathrm{gr}_\bullet(A) \to \mathrm{gr}_\bullet(O \otimes A) = \mathrm{gr}_\bullet(O) \otimes \mathrm{gr}_\bullet(A)$. There is a unique morphism of graded algebras $\mathrm{pr}_0 \colon \mathrm{gr}_\bullet(A) \to F_0A$ given by the identity in degree 0 and 0 on all degree components of degree > 0.

Definition B.16. Define $\mathrm{nat}_{(O,A)} \colon \mathrm{gr}_\bullet(A) \to \mathrm{gr}_\bullet(O) \otimes F_0A$ to be the composition of $\mathrm{gr}_\bullet(\Delta_A)$ with $\mathrm{id} \otimes \mathrm{pr}_0$.

Lemma B.17. *The following statements hold true:*

(a) *The assignment* $(O, A) \mapsto \mathrm{nat}_{(O,A)} \in \mathbf{Alg}_{\mathrm{gr}}(\mathrm{gr}(A), \Phi(O, A))$ *is a natural transformation relating the functors* $\mathrm{gr}, \Phi \colon \mathbf{HACA} \to \mathbf{Alg}_{\mathrm{gr}}$.

(b) *For any object* (O, A) *of* $\mathbf{HACA}$, *the morphism* $\mathrm{nat}_{(O,A)}$ *is injective.*

Proof. (a) $\mathrm{nat}_{(O,A)}$ is a morphism of graded algebras as it is a composition of such morphisms. The naturality is obvious.

(b) Let us prove that for any $n \geq 0$, the degree n component $\mathrm{nat}^n_{(O,A)}$ of $\mathrm{nat}_{(O,A)}$ is injective. The double inclusion of vector spaces of $O \otimes A$,

$$F_{n-1}(O \otimes A) \subset \sum_{p=1}^{n} F_{n-p}O \otimes F_pA \subset F_n(O \otimes A),$$

where $F_k(O \otimes A) := \sum_{i=0}^{k} F_{k-i}O \otimes F_iA$, gives rise to the map

$$F_n(O \otimes A)/F_{n-1}(O \otimes A) \to F_n(O \otimes A)/\Big(\sum_{p=1}^{n} F_{n-p}O \otimes F_pA\Big),$$

which can be identified with the projection $\mathrm{id} \otimes \mathrm{pr}_0 \colon \mathrm{gr}_n(O \otimes A) \to \mathrm{gr}_n(O) \otimes F_0A$. It follows that the map $\mathrm{nat}^n_{(O,A)}$ is the vertical cokernel of the diagram

$$\begin{tikzcd} F_nA \arrow[r, "\Delta_A"] & F_n(O \otimes A) \\ F_{n-1}A \arrow[u, hook] \arrow[r] & \sum_{p=1}^{n} F_{n-p}O \otimes F_pA, \arrow[u, hook] \end{tikzcd}$$

therefore that its kernel is the image in $F_nA/F_{n-1}A$ of the preimage by $\Delta_A\colon F_nA\to F_n(O\otimes A)$ of the subspace $\sum_{p=1}^n F_{n-p}O\otimes F_pA$ of $F_n(O\otimes A)$. This subspace is contained $F_{n-1}O\otimes A$, which together with Definition B.2 implies that this preimage is contained in $F_{n-1}A$. This implies the vanishing of the kernel of $\mathrm{nat}^n_{(O,A)}$ and therefore the injectivity of $\mathrm{nat}^n_{(O,A)}$. □

Appendix B.5. Isomorphisms in HACA

Proposition B.18. *The following statements hold true:*

(a) *A pair $(O,\mathbf{a})$ of a Hopf algebra O and an associative algebra $\mathbf{a}$ gives rise to an object $(O,O\otimes\mathbf{a})$ of* **HACA**, *with coaction morphism given by $\Delta_{O\otimes\mathbf{a}}:=\Delta_O\otimes\mathrm{id}_{\mathbf{a}}$ (where Δ_O is the coproduct of O). Its image by the endofunctor of* **HACA** *from Lemma B.3(d) is the pair $(F_\infty O,F_\infty O\otimes\mathbf{a})$, where the coaction morphism is $\Delta_{F_\infty O\otimes\mathbf{a}}:=\Delta_{F_\infty O}\otimes\mathrm{id}_{\mathbf{a}}$ (where $\Delta_{F_\infty O}$ is the coproduct of $F_\infty O$).*

(b) *If $(O,\mathbf{a})$ is a pair of a Hopf algebra and an associative algebra, if (B,H) is an object of $\mathbf{HAMA}_{\mathrm{fd}}$, and if $(p,f)\in\mathbf{PM}((O,O\otimes\mathbf{a}),(B,H))$ is a pairing-morphism such that $\nu(p)\colon F_\infty O\to H'$ is a Hopf algebra isomorphism and the algebra morphism $f\colon O\otimes\mathbf{a}\to B$ induces an algebra isomorphism $\mathbb{C}\otimes\mathbf{a}\xrightarrow{\sim}B^H$ between the subalgebras $\mathbb{C}\otimes\mathbf{a}$ and B^H of both sides, then the morphism $(\nu(p),F_\infty f)\colon(F_\infty O,F_\infty O\otimes\mathbf{a})\to(H',F_\infty B)$ in* **HACA** *is an isomorphism. In particular, $F_\infty f\colon F_\infty O\otimes\mathbf{a}\to F_\infty B$ is a filtered algebra isomorphism.*

(c) *In the situation of (b), the algebra morphism $f\colon O\otimes\mathbf{a}\to B$ is compatible with the filtrations $F_\bullet O\otimes\mathbf{a}$ and $F_\bullet B$, and induces an isomorphism $\mathrm{gr}_\bullet(O)\otimes\mathbf{a}\to\mathrm{gr}_\bullet(B)$ in $\mathbf{Alg}_{\mathrm{gr}}$.*

Proof. (a) If $o\in O$ is such that $\Delta_O(o)\in F_nO\otimes O$, then $o=(\mathrm{id}\otimes\epsilon_O)\circ\Delta_O(o)\in F_nO$, where ϵ_O is the counit of O, therefore $\{o\in O\mid\Delta_O(o)\in F_nO\otimes O\}\subset F_nO$. On the other hand, Lemma A.1(c) implies $F_nO\subset\{o\in O\mid\Delta_O(o)\in F_nO\otimes O\}$, therefore $F_nO=\{o\in O\mid\Delta_O(o)=F_nO\otimes O\}$. Then $F_n(O\otimes\mathbf{a})=\{o\in O\mid\Delta_O(o)\in F_nO\otimes O\}\otimes\mathbf{a}=F_nO\otimes\mathbf{a}$. It follows that $F_\infty(O\otimes\mathbf{a})=\bigcup_{n\geq0}F_nO\otimes\mathbf{a}=F_\infty O\otimes\mathbf{a}$. The fact that $(F_\infty O,F_\infty O\otimes\mathbf{a})\to(O,O\otimes\mathbf{a})$ is a morphism in **HACA** (see Lemma B.3(d)) implies that the coaction morphism of $F_\infty O\otimes\mathbf{a}$ has the claimed value.

(b) and (c). Lemma B.13 applied to the pairing-morphism (p,f) implies that $f\colon O\otimes\mathbf{a}\to B$ is a morphism of filtered algebras. By (a), there is an isomorphism $\mathrm{gr}_\bullet(O\otimes\mathbf{a})=\mathrm{gr}_\bullet(O)\otimes\mathbf{a}$. Composing it with the associated graded algebra morphism $\mathrm{gr}_\bullet(f)\colon\mathrm{gr}_\bullet(O\otimes\mathbf{a})\to\mathrm{gr}_\bullet(B)$, one obtains a morphism

$$\mathrm{gr}_\bullet(f)\colon\mathrm{gr}_\bullet(O)\otimes\mathbf{a}\to\mathrm{gr}_\bullet(B)$$

of graded algebras. Since (B, H) is an object of $\mathbf{HAMA}_{\mathrm{fd}}$, one may apply to it the functor from Lemma B.10(c) to obtain the object $(H', F_\infty B)$ of $\mathbf{HACA}$. By Lemma B.17, the latter object gives rise to an injective morphism of graded algebras

$$\begin{aligned}\mathrm{gr}_\bullet(B) = \mathrm{gr}_\bullet(F_\infty B) \xrightarrow{\mathrm{nat}_{(H', F_\infty B)}} \Phi(H', F_\infty B) &= \mathrm{gr}_\bullet(H') \otimes (F_\infty B)^H \\ &= \mathrm{gr}_\bullet(H') \otimes B^H.\end{aligned}$$

The composition of these morphisms is the morphism

$$\mathrm{nat}_{(H', F_\infty B)} \circ \mathrm{gr}_\bullet(f) \colon \mathrm{gr}_\bullet(O) \otimes \mathbf{a} \to \mathrm{gr}_\bullet(H') \otimes B^H$$

of graded algebras. As $\nu(p) \colon F_\infty O \to H'$ is a Hopf algebra isomorphism, it induces an isomorphism between the filtrations $F_\bullet$ on both sides (see Proposition A.2(c)), and an associated graded isomorphism $\mathrm{gr}_\bullet(\nu(p)) \colon \mathrm{gr}_\bullet(F_\infty O) \to \mathrm{gr}_\bullet(H')$, which we identify with its composition with the equality $\mathrm{gr}_\bullet(O) = \mathrm{gr}_\bullet(F_\infty O)$. Let us show that

$$\mathrm{nat}_{(H', F_\infty B)} \circ \mathrm{gr}_\bullet(f) = \mathrm{gr}_\bullet(\nu(p)) \otimes f_0 \tag{B.3}$$

(equality of morphisms of graded algebras $\mathrm{gr}_\bullet(O) \otimes \mathbf{a} \to \mathrm{gr}_\bullet(H') \otimes B^H$), where f_0 is the isomorphism $f_0 \colon \mathbb{C} \otimes \mathbf{a} \to B^H$. The composition

$$\mathbb{C} \otimes \mathbf{a} = F_0(O \otimes \mathbf{a}) \xrightarrow{\mathrm{gr}_0(f)} F_0 B = B^H$$

is f_0, which together with $\mathrm{gr}_0(\nu(p)) = 1$ proves the degree 0 part of (B.3). Let $n \geq 0$ and $o \in \mathrm{gr}_n(O)$. The image of o under $\mathrm{nat}_{(H', F_\infty B)} \circ \mathrm{gr}_n(f)$ is equal to the image of $\tilde{o} \otimes 1$ by the horizontal composition

$$\begin{tikzcd} & & F_n H' \otimes B \arrow[r] & \mathrm{gr}_n(H') \otimes B \\ F_n O \otimes \mathbf{a} \arrow[r, "F_n f"'] & F_n B \arrow[r, "\Delta_{F_n B}"'] & \sum_{k=0}^n F_{n-k} H' \otimes F_k B \arrow[r] \arrow[u, hook] & \mathrm{gr}_n(H') \otimes F_0 B, \arrow[u, hook] \end{tikzcd}$$

where $\tilde{o} \in F_n O$ is a lift of o. Since $F_n f$ is compatible with the coaction maps, the image of this element in $\sum_{k=0}^n F_{n-k} H' \otimes F_k B$ is equal to $\nu(p)(\tilde{o}^{(1)}) \otimes f(\tilde{o}^{(2)} \otimes 1)$, where $\tilde{o}^{(1)} \otimes \tilde{o}^{(2)} = \Delta_O(\tilde{o}) \in \sum_{k=0}^n F_{n-k} O \otimes F_k O \subset O^{\otimes 2}$. One has $\tilde{o}^{(1)} \otimes \tilde{o}^{(2)} \in \tilde{o} \otimes 1 + F_{n-1} O \otimes O$, therefore $\nu(p)(\tilde{o}^{(1)}) \otimes f(\tilde{o}^{(2)} \otimes 1) \in \nu(p)(\tilde{o}) \otimes 1 + F_{n-1} H' \otimes B$ (inclusion in $F_n H' \otimes B$). It follows that the image of $\tilde{o} \otimes 1$ in $\mathrm{gr}_n(H') \otimes B$ is $\mathrm{im}(\nu(p)(\tilde{o}) \in F_n H' \to \mathrm{gr}_n H') \otimes 1 = \mathrm{gr}_n(\nu(p))(o) \otimes 1$. This implies that both sides of (B.3) coincide when restricted to $\mathrm{gr}_n(O) \otimes \mathbb{C}$ for any n, therefore to $\mathrm{gr}_\bullet(O) \otimes \mathbb{C}$. Equation (B.3) then follows from the fact that each of its sides is an algebra morphism, and that they agree on $\mathrm{gr}_\bullet(O) \otimes \mathbb{C}$ and $\mathbb{C} \otimes \mathbf{a}$, which generate the source algebra.

It follows from (B.3) that $\mathrm{nat}_{(H',F_\infty B)} \circ \mathrm{gr}_\bullet(f)$ is an isomorphism of graded algebras. The injectivity of $\mathrm{nat}_{(H',F_\infty B)}$ then implies that both $\mathrm{nat}_{(H',F_\infty B)}$ and $\mathrm{gr}_\bullet(f)$ are isomorphisms of graded algebras. By Lemma B.13, f induces a morphism of filtered algebras $F_\infty f \colon F_\infty O \otimes \mathbf{a} \to F_\infty B$, and $\mathrm{gr}_\bullet(f) = \mathrm{gr}_\bullet(F_\infty f)$. It follows that $\mathrm{gr}_\bullet(F_\infty f)$ is an isomorphism of graded algebras, which together with the fact that the filtrations of the source and target of $F_\infty f$ are exhaustive, implies that $F_\infty f$ is an isomorphism of filtered algebras (see Lemma 4.1). $\square$

Appendix C. Filtered formality for Hopf algebras and HACAs

In Appendix C.1 we introduce a notion of filtered formality for Hopf algebras; this notion is related in Appendix D to the similar notion for discrete groups, introduced in [SW1]. We extend this to a notion of filtered formality for HACAs in Appendix C.2. The main result of this section is Proposition C.5, which shows that a HACA constructed in the context of Proposition B.18 is filtered formal.

Appendix C.1. Filtered formality for Hopf algebras

In Proposition A.2(a), we attach to a Hopf algebra O a filtration $F_\bullet O$. Let $\mathrm{gr}(O)$ be the associated graded vector space.

Lemma C.1. *The following statements hold true:*

(a) *If O is a Hopf algebra, then $\mathrm{gr}(O)$ is a graded Hopf algebra, which is commutative if O is, and such that for each $n \geq 0$, $F_n(\mathrm{gr}(O)) = \mathrm{gr}_{\leq n}(O)$.*

(b) *The assignment $O \mapsto \mathrm{gr}(O)$ defines an endofunctor of the category $\mathbf{HA}_{\mathrm{comm}}$ of commutative Hopf algebras.*

Proof. Let us show (a). The first statement follows from the fact that $F_\bullet O$ is a Hopf algebra filtration (see Proposition A.2(a)). Let us show the second statement. Let $n \geq 0$. The inclusion $F_n(\mathrm{gr}(O)) \supset \mathrm{gr}_{\leq n}(O)$ follows from the fact that $\mathrm{gr}(O)$ is a graded and connected Hopf algebra. Let us show the opposite inclusion. Since $\mathrm{gr}(O)$ is a graded Hopf algebra, $F_n(\mathrm{gr}(O))$ is the direct sum of its intersection with the homogeneous components of $\mathrm{gr}(O)$. If $k \geq 1$, the intersection $F_n(\mathrm{gr}(O)) \cap \mathrm{gr}_{n+k}(O)$ is contained in $F_n(\mathrm{gr}(O))$, which is the kernel of

$$(\mathrm{id} - \eta_{\mathrm{gr}(O)}\epsilon_{\mathrm{gr}(O)})^{\otimes n+1} \circ \Delta^{(n+1)}_{\mathrm{gr}(O)} \colon \mathrm{gr}(O) \to (\mathrm{gr}(O))^{\otimes n+1} = \mathrm{gr}(O^{\otimes n+1}),$$

therefore $F_n(\mathrm{gr}(O)) \cap \mathrm{gr}_{n+k}(O)$ is contained in the kernel of the map $\mathrm{gr}_{n+k}(O) \to \mathrm{gr}_{n+k}(O^{\otimes n+1})$ induced by $(\mathrm{id} - \eta_{\mathrm{gr}(O)}\epsilon_{\mathrm{gr}(O)})^{\otimes n+1} \circ \Delta^{(n+1)}_{\mathrm{gr}(O)}$. This map is the degree

$n+k$ part of the associated graded of the map $(\mathrm{id}-\eta_O\epsilon_O)^{\otimes n+1}\circ\Delta_O^{(n+1)}\colon O\to O^{\otimes n+1}$. It follows that if V is the preimage of $F_n(\mathrm{gr}(O))\cap\mathrm{gr}_{n+k}(O)$ in $F_{n+k}O$, one has $(\mathrm{id}-\eta_O\epsilon_O)^{\otimes n+1}\circ\Delta_O^{(n+1)}(V)\subset F_{n+k-1}(O^{\otimes n+1})$. The map $((\mathrm{id}-\eta_O\epsilon_O)^{\otimes k}\circ\Delta_O^{(k)})\otimes\mathrm{id}^{\otimes n}$ maps $F_{n+k-1}(O^{\otimes n+1})$ to $F_{n+k-1}(O^{\otimes n+k})$, therefore $(\mathrm{id}-\eta_O\epsilon_O)^{\otimes n+k}\circ\Delta_O^{(n+k)}(V)\subset F_{n+k-1}(O^{\otimes n+k})$. Now $O=\mathbb{C}1\oplus F_1O$; one has $(\mathrm{id}-\eta_O\epsilon_O)^{\otimes n+k}\circ\Delta_O^{(n+k)}(V)\subset(F_1O)^{\otimes n+k}$ while $F_{n+k-1}(O^{\otimes n+k})\subset\sum_{i=1}^{n+k}O^{\otimes i-1}\otimes\mathbb{C}1\otimes O^{\otimes n+k-i}$. As the second terms of both inclusions have zero intersection in $O^{\otimes n+k}$, one has $(\mathrm{id}-\eta_O\epsilon_O)^{\otimes n+k}\circ\Delta_O^{(n+k)}(V)=0$. Therefore $V\subset F_{n+k-1}O$, which implies $F_n(\mathrm{gr}(O))\cap\mathrm{gr}_{n+k}(O)=0$. Therefore, $F_n(\mathrm{gr}(O))=\mathrm{gr}_{\leq n}(O)$.

(b) follows from the naturality of the assignment $O\mapsto F_\bullet O$. □

Definition C.2. The Hopf algebra $\mathcal{O}$ is called *filtered formal* if there is an isomorphism of Hopf algebra $O\to\mathrm{gr}(O)$ whose associated graded for the grading gr is the identity.

This terminology is justified by Proposition D.18.

Appendix C.2. Filtered formality for HACAs

Let $\mathbf{HACA}_{\mathrm{gr}}$ be the category of $\mathbb{Z}_{\geq 0}$-graded HACAs.

Lemma C.3. *The assignment* $(O,A)\mapsto(\mathrm{gr}(O),\mathrm{gr}(A))$ *is a functor* $(\mathrm{gr},\mathrm{gr})\colon\mathbf{HACA}\to\mathbf{HACA}_{\mathrm{gr}}$.

Proof. It follows from the fact that for (O,A) a HACA, $(F_\bullet O,F_\bullet A)$ is a HACA filtration. □

Definition C.4. The HACA (O,A) is called *filtered formal* if there exists an isomorphism $(O,A)\to(\mathrm{gr}(O),\mathrm{gr}(A))$ in $\mathbf{HACA}$, which is compatible with the filtrations $(F_\bullet O,F_\bullet A)$ in the source and induced by the grading in the target, and whose image by the functor $(\mathrm{gr},\mathrm{gr})\colon\mathbf{HACA}\to\mathbf{HACA}_{\mathrm{gr}}$ is the identity endomorphism of $(\mathrm{gr}(O),\mathrm{gr}(A))$.

If the HACA (O,A) is filtered formal, then the Hopf algebra O is filtered formal in the sense of Definition C.2.

Proposition C.5. *Let* $(O,\mathbf{a}),(B,H),(p,f)$ *be as in the hypothesis of Proposition* B.18(b),(c). *Then the HACA* $(H',F_\infty B)$ *is filtered formal.*

Proof. Recall from Lemma C.3 that $(\widetilde{O},A)\mapsto(\mathrm{gr}(\widetilde{O}),\mathrm{gr}(A))$ is a functor $\mathbf{HACA}\to\mathbf{HACA}_{\mathrm{gr}}$; one checks that the same is true of $(\widetilde{O},A)\mapsto(\mathrm{gr}(\widetilde{O}),\Phi(\widetilde{O},A))$. For $(\widetilde{O},A)$ a HACA, $(\mathrm{id}_{\widetilde{O}},\mathrm{nat}_{(\widetilde{O},A)})$ is a morphism $(\mathrm{gr}(\widetilde{O}),\mathrm{gr}(A))\to(\mathrm{gr}(\widetilde{O}),\Phi(\widetilde{O},A))$ in $\mathbf{HACA}_{\mathrm{gr}}$, which is an isomorphism if and only if $\mathrm{nat}_{(\widetilde{O},A)}$ is.

The proof of Proposition B.18(c) implies that $\mathrm{nat}_{(H',F_\infty B)}$ is an isomorphism in $\mathbf{Alg}_{\mathrm{gr}}$, therefore

$$(\mathrm{id}_{\mathrm{gr}(H')}, \mathrm{nat}_{(H',F_\infty B)})\colon (\mathrm{gr}(H'), \mathrm{gr}(B)) \to (\mathrm{gr}(H'), \mathrm{gr}(H') \otimes B^H) \tag{C.1}$$

is an isomorphism in $\mathbf{HACA}_{\mathrm{gr}}$.

By the assumptions of Proposition B.18(b), $\nu(p)\colon F_\infty O \to H'$ is an isomorphism in $\mathbf{HA}$. Proposition C.1 then implies that $\mathrm{gr}(\nu(p))\colon \mathrm{gr}(O) \to \mathrm{gr}(H')$ is an isomorphism in $\mathbf{HA}_{\mathrm{gr}}$. The assumptions of Proposition B.18(b) also imply that f restricts to an algebra isomorphism $\mathbb{C} \otimes \mathbf{a} \to B^H$, we denote by f_0 the composed isomorphism $\mathbf{a} \simeq \mathbb{C} \otimes \mathbf{a} \to B^H$. Both isomorphisms $\mathrm{gr}(\nu(p))$ and f_0 induce an isomorphism

$$\big(\mathrm{gr}(\nu(p))^{-1}, \mathrm{gr}(\nu(p))^{-1} \otimes f_0^{-1}\big)\colon (\mathrm{gr}(H'), \mathrm{gr}(H') \otimes B^H) \to (\mathrm{gr}(O), \mathrm{gr}(O) \otimes \mathbf{a}) \tag{C.2}$$

in $\mathbf{HACA}_{\mathrm{gr}}$. By Proposition B.18(b) and (c), the pair

$$(\nu(p), F_\infty f)\colon (F_\infty O, F_\infty O \otimes \mathbf{a}) \to (H', F_\infty B) \tag{C.3}$$

is an isomorphism in $\mathbf{HACA}$. The composition of (C.1), (C.2), and (C.3) gives rise to an isomorphism

$$\begin{aligned}&\big(\nu(p) \circ \big(\mathrm{gr}(\nu(p))^{-1}\big), F_\infty f \circ \big(\mathrm{gr}(\nu(p))^{-1} \otimes f_0^{-1}\big) \circ \mathrm{nat}_{(\mathrm{gr}(H'),F_\infty B)}\big)\colon \\ &\qquad (\mathrm{gr}(H'), \mathrm{gr}(B)) \to (H', F_\infty B)\end{aligned} \tag{C.4}$$

in $\mathbf{HACA}$.

By Proposition A.2(c), $\nu(p)$ is compatible with the filtrations $F_\bullet$; it follows that $\nu(p) \circ \mathrm{gr}(\nu(p))^{-1}$ is compatible with these filtrations as well, and one computes $\mathrm{gr}(\nu(p) \circ \mathrm{gr}(\nu(p))^{-1}) = \mathrm{id}$.

The algebra morphisms in (C.1) and (C.2) are compatible with the filtrations since they are graded, and the algebra morphism in (C.3) is compatible with the filtrations. It follows that the algebra morphism in (C.4) is compatible with the filtrations. Its associated graded is the composed morphism

$$\mathrm{gr}(B) \xrightarrow{\mathrm{nat}_{(H',F_\infty B)}} \mathrm{gr}(H') \otimes B^H \xrightarrow{\mathrm{gr}(\nu(p))^{-1} \otimes f_0^{-1}} F_\infty O \otimes \mathbf{a} \xrightarrow{\mathrm{gr} f} \mathrm{gr}(B). \tag{C.5}$$

It follows from the proof of Proposition B.18(b) that relation (B.3) holds, and that $\mathrm{nat}_{(H',F_\infty B)}$ and $\mathrm{gr} f$ are both isomorphisms. The combination of these facts then implies that the map (C.5) is the identity. Therefore, the algebra morphism in (C.4) is compatible with the filtrations, and its associated graded is the identity. □

Appendix D. Hopf algebra duality and prounipotent completions

Appendix D.1 relates the Hopf algebraic constructions of Appendix A to complete Hopf algebras (CHAs) in the sense of [Q1]. This is applied in Appendix D.2 to obtain an expression of the function algebra of the prounipotent completion of a finitely generated group in the terms of the functors of Appendix A. The main result of Appendix D.3 is Proposition D.18, which relates the notion of filtered formality for a finitely generated group ([SW1]) with the similar notion for Hopf algebras, introduced in Appendix C.

Appendix D.1. Completion and duality of Hopf algebras

Recall from [Q1, A1 and A2] the categories **CAA** of complete augmented algebras (CAAs) and **CHA** of complete Hopf algebras (CHAs): **CAA** is the full subcategory, in the category of pairs $(A, F^\bullet A)$ of an augmented algebra A and a decreasing algebra filtration $A = F^0A \supset F^1A \supset \cdots$, of CAAs, i.e. of pairs such that A is complete and Hausdorff for the topology of $F^\bullet A$, such that F^1A is the augmentation ideal of A and such that $\mathrm{gr}(A)$ is generated by $\mathrm{gr}^1(A)$; a monoidal structure is defined on **CAA**, given at the level of objects by $((A, F^\bullet A), (B, F^\bullet B)) \mapsto (A \mathbin{\widehat{\otimes}} B, F^\bullet(A \mathbin{\widehat{\otimes}} B))$, where $A \mathbin{\widehat{\otimes}} B := \varprojlim_n (A/F^nA) \otimes (B/F^nB)$ and $F^n(A \mathbin{\widehat{\otimes}} B) = \varprojlim_m \mathrm{im}(\sum_{n'+n''=n} F^{n'}A \otimes F^{n''}B \to (A/F^mA) \otimes (B/F^mB))$; a CHA is a pair (A, Δ_A), where A is a CAA, $\Delta_A \colon A \to A \mathbin{\widehat{\otimes}} A$ is a cocommutative and coassociative algebra morphism, which admits the augmentation of A as a counit, and a morphism in **CHA** is a morphism in **CAA** which is compatible with coproducts.

Lemma D.1. *The following statements hold true:*

(a) *Let H be a cocommutative Hopf algebra with coproduct Δ_H and counit ϵ_H. Recall $H_+ = \ker(\epsilon_H)$. Set $H^\wedge := \varprojlim_m H/H_+^m$ and for $n \geq 0$, $F^nH^\wedge := \varprojlim_m F^nH/F^{\max(n,m)}H$. Then $H^\wedge$ is a complete augmented algebra. There is a unique continuous extension of Δ_H to a map $\widehat{\Delta}_H \colon H^\wedge \to H^\wedge \mathbin{\widehat{\otimes}} H^\wedge$, which equips $H^\wedge$ with the structure of a complete Hopf algebra. The assignment $H \mapsto H^\wedge$ is a functor $\mathbf{HA}_{\mathrm{coco}} \to \mathbf{CHA}$, where $\mathbf{HA}_{\mathrm{coco}}$ is the category of cocommutative Hopf algebras.*

(b) *If V is a vector space and H is the cocommutative Hopf algebra $T(V)$ with coproduct defined by the condition that the elements of V are primitive, then $T(V)^\wedge = \widehat{T}(V)$ (the degree completion of $T(V)$).*

Proof. (a) The first statement follows from [Q1, A1, Exa. 1.2]. One has the inclusion $\Delta_H(H_+) \subset H_+ \otimes H + H \otimes H_+$, which implies that for any $n \geq 0$, one has $\Delta_H(H_+^n) \subset \sum_{p,q|p+q=n} H_+^p \otimes H_+^q$. It follows that Δ_H defines a collection of maps

$H/H_+^{2n} \to (H/H_+^n)^{\otimes 2}$ indexed by $n \geq 0$, which are compatible for various n and which therefore induce a map $\widehat{\Delta}_H : H^\wedge \to H^\wedge \widehat{\otimes} H^\wedge$, which is a continuous extension of Δ_H. The uniqueness of this extension follows from the Hausdorff property of $H^\wedge$. The verification of the other properties of $\widehat{\Delta}_H$ is standard.

(b) follows from the fact that $T(V)_+^n$ is the part of $T(V)$ of degree $\geq n$, and that $\widehat{T}(V)$ is the completion with respect to the corresponding topology. □

Definition D.2. Consider the following definitions:

(a) $\mathbf{CHA}_{\mathrm{fd}}$ is the full subcategory of $\mathbf{CHA}$ of CHAs A such that $\mathrm{gr}^1 A$ is finite-dimensional.

(b) For H a Hopf algebra, set $\mathrm{gr}^1(H) := H_+/H_+^2$, where we recall $H_+ = \ker(\epsilon_H)$, and ϵ_H is the counit of H.

(c) $\mathbf{HA}_{\mathrm{coco,fd}}$ is the full subcategory of $\mathbf{HA}_{\mathrm{coco}}$ of all cocommutative Hopf algebras H such that $\mathrm{gr}^1 H$ is finite-dimensional.

(d) $\mathbf{Gp}_{\mathrm{fg}}$ is the full subcategory of $\mathbf{Gp}$ of finitely generated groups.

Lemma D.3. *The following statements hold true:*

(a) *The functor* $\mathbf{Gp} \to \mathbf{HA}_{\mathrm{coco}}$ *given by* $\Gamma \mapsto \mathbb{C}\Gamma$ *induces a functor* $\mathbf{Gp}_{\mathrm{fg}} \to \mathbf{HA}_{\mathrm{coco,fd}}$.

(b) *The functor* $\mathbf{HA}_{\mathrm{coco}} \to \mathbf{CHA}$, $H \mapsto H^\wedge$ *from Lemma D.1 induces a functor* $\mathbf{HA}_{\mathrm{coco,fd}} \to \mathbf{CHA}_{\mathrm{fd}}$.

Proof. (a) If Γ is a group, then $\mathrm{gr}^1(\mathbb{C}\Gamma) = \Gamma^{\mathrm{ab}} \otimes \mathbb{C}$ (see [Q2]). Moreover, if Γ is finitely generated, then Γ^{ab} is a finitely generated abelian group, which together with the above equality implies the finite-dimensionality of $\mathrm{gr}^1(\mathbb{C}\Gamma)$.

(b) follows from the fact that the natural map $\mathrm{gr}^1 H \to \mathrm{gr}^1(H^\wedge)$ is a linear isomorphism. □

Definition D.4. For H a CHA with associated filtration $H = F^0 H \supset \cdots$, set $H' := \bigcup_{n \geq 0} F^n H^\perp \subset H^*$.

Lemma D.5. *The following statements hold true:*

(a) *If H is a CHA, then H' has a commutative and associative algebra structure.*

(b) *If H is an object in* $\mathbf{CHA}_{\mathrm{fd}}$, *then the algebra structure on H' can be upgraded to a commutative Hopf algebra structure; the resulting assignment $H \mapsto H'$ is a contravariant functor* $\mathbf{CHA}_{\mathrm{fd}} \to \mathbf{HA}_{\mathrm{comm}}$.

Proof. (a) Let $\xi, \eta \in H'$. Let $n \geq 0$ be such that $\xi, \eta \in F^n H^\perp$. Then $\xi \otimes \eta$ is a linear form on $(H/F^n H)^{\otimes 2}$, which can be pulled back by the map $H^{\widehat{\otimes} 2} \to (H/F^n H)^{\otimes 2}$ to define a linear form on $H^{\widehat{\otimes} 2}$; it can be checked to be independent of the choice of n and will be denoted $\ell_{\xi,\eta}$. Then $\ell_{\xi,\eta}$ vanishes on $F^n H \otimes H + H \otimes F^n H$. The assignment $\xi \cdot \eta \colon h \mapsto \ell_{\xi,\eta} \circ \Delta_H(h)$ is then a linear form on H, i.e. an element of H^*. Since $\Delta_H(F^{2n} H) \subset F^n H \otimes H + H \otimes F^n H$, one has $\xi \cdot \eta \in F^{2n} H^\perp$, therefore $\xi \cdot \eta \in H'$. One checks that this defines a commutative and associative algebra structure on H'.

(b) If $n \geq 0$, then $H/F^n H$ is noncanonically isomorphic to $\bigoplus_{k=0}^n \mathrm{gr}^k(H)$, which is finite-dimensional since it is generated by $\mathrm{gr}^1 H$ and the latter space is finite-dimensional. The coproduct Δ_H defines a coproduct $\Delta_{F^n H} \colon H/F^n H \to (H/F^n H)^{\otimes 2}$ for any $n \geq 0$. Since $H/F^n H$ is finite-dimensional, this coproduct can be dualized and defines a product $m_{(F^n H)^\perp} \colon ((F^n H)^\perp)^{\otimes 2} \to (F^n H)^\perp$. As in the proof of Lemma A.6, one checks that the collection of these products is compatible, which gives rise to a product $m_{H'} \colon H'^{\otimes 2} \to H'$, which can be shown to define a Hopf structure. $\square$

The following lemma is a topological analogue of Lemma A.7.

Lemma D.6. *If H is a CHA, then for any $n \geq 0$ one has $F_n(H') = (F^{n+1} H)^\perp$ (equality of subspaces of H^*).*

Proof. Let us show that if H is a CHA and $k, n \geq 1$, then the image of the product map $(F^1 H/F^{n+k} H)^{\otimes n} \to H/F^{n+k} H$ is $F^n H/F^{n+k} H$. We proceed by induction on k. For $k = 1$, the statement follows from the surjectivity of the product map $(F^1 H/F^2 H)^{\otimes n} \to F^n H/F^{n+1} H$. Assume the statement at order k and let us show it at order $k+1$. The product map $(F^1 H/F^2 H)^{\otimes n+k} \to F^{n+k} H/F^{n+k+1} H$ is surjective, therefore $F^{n+k} H/F^{n+k+1} H$ is equal to the image of a subspace of $(F^1 H/F^{n+k+1} H)^{\otimes n}$, namely

$$(F^1 H/F^{n+k+1} H)^{\otimes n-1} \otimes \mathrm{im}((F^1 H/F^{n+k+1} H)^{\otimes k+1} \to F^1 H/F^{n+k+1} H).$$

Combining this with the statement at order k, one obtains the statement at order $k+1$ and therefore the induction. The proof of the lemma is then similar to that of Lemma A.7. $\square$

Lemma D.7. *The functors $\mathbf{HA}_{\mathrm{coco,fd}} \to \mathbf{CHA}_{\mathrm{fd}} \to \mathbf{HA}_{\mathrm{comm}}$ (see Lemma D.3(b) and Lemma D.5(b)) and $\mathbf{HA}_{\mathrm{coco,fd}} \to \mathbf{HA}_{\mathrm{comm}}$ (see Lemma A.6) are naturally equivalent.*

Proof. For each $n \geq 0$, the algebra morphism $H \to H^\wedge$ induces an algebra isomorphism $H/F^nH \xrightarrow{\simeq} H^\wedge/F^nH^\wedge$, which fits in a commutative diagram

$$\begin{tikzcd} H \arrow[r] \arrow[d] & H^\wedge \arrow[d] \\ H/F^nH \arrow[r] & H^\wedge/F^nH^\wedge. \end{tikzcd}$$

Dualizing and using the equalities $(H/F^nH)^* = F^nH^\perp$ and $(H^\wedge/F^nH^\wedge)^* = (F^nH^\wedge)^\perp$, one obtains a commutative diagram

$$\text{(D.1)} \qquad \begin{tikzcd} (F^nH^\wedge)^\perp \arrow[r, "\sim"] \arrow[d, hook] & (F^nH)^\perp \arrow[d, hook] \\ (H^\wedge)^* \arrow[r] & H^*. \end{tikzcd}$$

If V, W are filtered vector spaces and $f\colon V \to W$ is a linear map such that for any $n \geq 0$, f induces a linear isomorphism $F_nV \to F_nW$, then f induces an isomorphism $F_\infty V \to F_\infty W$. This and (D.1) imply that the map $(H^\wedge)^* \to H^*$ restricts to a linear isomorphism $(H^\wedge)' \to H'$. One then checks this isomorphism to be compatible with the Hopf algebra structures. □

Appendix D.2. Isomorphism $\mathbb{C}\Gamma' \simeq \mathcal{O}(\Gamma_{\mathrm{unip}})$

Let Γ be a group. A pro-unipotent completion of a group Γ is the pair (U, c) of a prounipotent $\mathbb{C}$-group scheme U and a morphism $\Gamma \to U(\mathbb{C})$ satisfying a universal property (see [BGF, Def. 3.217]).

If Γ is an object in $\mathbf{Gp}_{\mathrm{fg}}$, then $\mathbb{C}\Gamma$ is an object in $\mathbf{HA}_{\mathrm{fd}}$ by Lemma D.3(a), therefore $(\mathbb{C}\Gamma)'$ is a commutative Hopf algebra (see Lemma A.6). Since $\Gamma^{\mathrm{ab}} \otimes \mathbb{C}$ is finite-dimensional, one may use [BGF, Thm. 3.224] to obtain that a pro-unipotent completion exists and is unique up to isomorphism, and can be constructed as the pair $(\Gamma_{\mathrm{unip}}, c_{\mathrm{unip}})$, where Γ_{unip} is the spectrum of the commutative Hopf algebra $\mathcal{O}(\Gamma_{\mathrm{unip}}) \coloneqq ((\mathbb{C}\Gamma)^\wedge)'$, which is isomorphic to $(\mathbb{C}\Gamma)'$ by Lemma D.7, and c_{unip} is induced by the commutative algebra morphism $\mathcal{O}(\Gamma_{\mathrm{unip}}) = (\mathbb{C}\Gamma)' \hookrightarrow (\mathbb{C}\Gamma)^* = \mathbb{C}^\Gamma$, where $\mathbb{C}^\Gamma$ is the algebra of all functions $\Gamma \to \mathbb{C}$.

Appendix D.3. Relation between the filtered formalities of Γ and $(\mathbb{C}\Gamma)'$

Definition D.8 (See [Q1, §A.2]). Consider the following definitions:

(a) A Malcev Lie algebra (MLA) is a Lie algebra $\mathfrak{g}$, equipped with a decreasing Lie algebra filtration $\mathfrak{g} = F^1\mathfrak{g} \supset F^2\mathfrak{g} \supset \cdots$ (i.e. $[F^i\mathfrak{g}, F^j\mathfrak{g}] \subset F^{i+j}\mathfrak{g}$ for $i, j \geq 1$)

for which it is complete and Hausdorff, and such that the associated graded Lie algebra $\mathrm{gr}\mathfrak{g}$ is generated by $\mathrm{gr}^1\mathfrak{g}$.

(b) A morphism between two MLAs is a Lie algebra morphism which is compatible with the filtrations. We denote the category of MLAs by **MLA**.

Lemma D.9 (See [Q1, Thm. 3.3]). *The functors $\widehat{U}\colon \mathbf{MLA} \leftrightarrow \mathbf{CHA}\colon \mathcal{P}$, where $\widehat{U}(\mathfrak{g}) := \varprojlim_i U(\mathfrak{g}/F^i\mathfrak{g})$ and $\mathcal{P}$ is the "primitive-elements functor", taking (A, Δ_A) to $\mathcal{P}(A) := \{a \in A \mid \Delta_A(a) = a \otimes 1 + 1 \otimes a\}$, equipped with the filtration given by $F_n\mathcal{P}(A) := \mathcal{P}(A) \cap F_nA$ for $n \geq 0$, are quasi-inverse to one another.*

Definition D.10. The composed functor $\mathbf{Gp} \to \mathbf{CHA} \xrightarrow{\mathcal{P}} \mathbf{MLA}$, where the first functor is $\Gamma \mapsto (\mathbb{C}\Gamma)^\wedge$ (see Lemma D.1), is denoted $\Gamma \mapsto \mathrm{Lie}(\Gamma)$.

Lemma-Definition D.11. *The assignment $\mathfrak{g} \mapsto (\widehat{\mathrm{gr}}(\mathfrak{g}), F^\bullet\widehat{\mathrm{gr}}(\mathfrak{g}))$, where $\widehat{\mathrm{gr}}(\mathfrak{g}) := \prod_{i\geq 1} \mathrm{gr}^i(\mathfrak{g})$ and for $n \geq 1$, $F^n\widehat{\mathrm{gr}}(\mathfrak{g}) := \prod_{i\geq n} \mathrm{gr}^i(\mathfrak{g})$, is an endofunctor of* **MLA**. *For $\mathfrak{g}$ an object in* **MLA**, *one has $\mathrm{gr}(\widehat{\mathrm{gr}}(\mathfrak{g})) \simeq \mathrm{gr}(\mathfrak{g})$.*

Proof. Immediate. □

Definition D.12 (See [SW1]). A group Γ is called filtered formal if there exists an isomorphism $\mathrm{Lie}(\Gamma) \to \widehat{\mathrm{gr}}\,\mathrm{Lie}(\Gamma)$ in **MLA**, whose associated graded is the identity.

One checks that $A \mapsto \widehat{\mathrm{gr}}(A) := \prod_{i\geq 0} F^iA/F^{i+1}A$ is an endofunctor of **CHA**.

Lemma D.13. *The category equivalence* **CHA** $\leftrightarrow$ **MLA** *quasi-intertwines the endofunctors $\widehat{\mathrm{gr}}$ on both sides.*

Proof. It follows from the natural isomorphism of graded Lie algebras

$$\mathrm{gr}(\mathcal{P}A) \simeq \mathcal{P}(\mathrm{gr}(A)) \tag{D.2}$$

for A an object of **CHA** from [Q1, A1, Thm. 2.14]. □

Definition D.14. One defines $\mathbf{MLA}_{\mathrm{fd}}$ to be the full subcategory of **MLA** of MLAs $\mathfrak{g}$ such that $\mathrm{gr}^1\mathfrak{g}$ is finite-dimensional.

Recall the full subcategory $\mathbf{CHA}_{\mathrm{fd}}$ of **CHA** (see Definition D.2).

Lemma D.15. *The following statements hold:*

(a) *The equivalence* **CHA** $\leftrightarrow$ **MLA** *induces a category equivalence* $\mathbf{CHA}_{\mathrm{fd}} \leftrightarrow \mathbf{MLA}_{\mathrm{fd}}$.

(b) *The endofunctors $\widehat{\mathrm{gr}}$ of* **CHA** *and* **MLA** *induce endofunctors (still denoted $\widehat{\mathrm{gr}}$) of* $\mathbf{CHA}_{\mathrm{fd}}$ *and* $\mathbf{MLA}_{\mathrm{fd}}$.

(c) *The category equivalence* $\mathbf{CHA}_{\mathrm{fd}} \leftrightarrow \mathbf{MLA}_{\mathrm{fd}}$ *quasi-intertwines the endofunctors* $\widehat{\mathrm{gr}}$ *on both sides.*

Proof. (a) For A an object in $\mathbf{CHA}$, (D.2) implies the vector space isomorphism $\mathrm{gr}^1(\mathcal{P}A) \simeq \mathcal{P}(\mathrm{gr}(A)) \cap \mathrm{gr}^1 A$; moreover, $\mathrm{gr}(A)$ is a graded connected Hopf algebra, therefore $\mathrm{gr}^1(A) \subset \mathcal{P}(\mathrm{gr}(A))$; therefore $\mathrm{gr}^1(\mathcal{P}A) \simeq \mathrm{gr}^1 A$.

(b) follows from $\mathrm{gr}^1(\widehat{\mathrm{gr}}(A)) = \mathrm{gr}^1(A)$ for A a CHA and $\mathrm{gr}^1(\widehat{\mathrm{gr}}(\mathfrak{g})) = \mathrm{gr}^1(\mathfrak{g})$ for $\mathfrak{g}$ an MLA.

(c) follows from Lemma D.13. ☐

Lemma D.16. *Let* H *be an object in* $\mathbf{CHA}_{\mathrm{fd}}$*.*

(a) *There is a natural vector space isomorphism* $(\widehat{\mathrm{gr}}H)' \simeq \bigoplus_{n\geq 0} \mathrm{gr}^n(H)^*$*; it induces a natural isomorphism* $F_n((\widehat{\mathrm{gr}}H)') \simeq \bigoplus_{k\leq n} \mathrm{gr}^k(H)^*$ *for any* $n \geq 0$.

(b) *There is a natural isomorphism*

$$(\widehat{\mathrm{gr}}H)' \simeq \mathrm{gr}(H') \tag{D.3}$$

in $\mathbf{HA}$.

(c) *The composed isomorphism* $\bigoplus_{k\leq n} \mathrm{gr}^k(H)^* \simeq F_n((\widehat{\mathrm{gr}}H)') \simeq F_n(\mathrm{gr}(H')) \simeq \mathrm{gr}_{\leq n}(H')$*, where the first isomorphism arises from* (a)*, the second from the image by* F_n *of* (D.3)*, and the third isomorphism arises from Lemma* C.1(a)*, is the direct sum over* $k \leq n$ *of the isomorphisms* $\mathrm{gr}^k(H)^* \simeq \mathrm{gr}_{\leq n}(H')$ *arising from Lemma* D.6*.*

Proof. (a) $\widehat{\mathrm{gr}}(H)$ is $\prod_{n\geq 0} \mathrm{gr}^n H$, equipped with the filtration $(\prod_{i\geq n} \mathrm{gr}^i H)_{n\geq 0}$. It follows that $\widehat{\mathrm{gr}}(H)' = \bigoplus_{n\geq 0} \mathrm{gr}^n(H)^*$. One then has $F_n(\widehat{\mathrm{gr}}(H)') = (F^{n+1}\widehat{\mathrm{gr}}(H))^\perp \simeq \bigoplus_{k\leq n} \mathrm{gr}^k(H)^*$ for any $n \geq 0$, where the first equality follows from Lemma D.6.

(b) follows from the sequence of equalities

$$\begin{aligned}\mathrm{gr}(H') &= \bigoplus_{n\geq 0} F_{n+1}H'/F_nH' = \bigoplus_{n\geq 0} (F^{n+1}H)^\perp/(F^nH)^\perp = \bigoplus_{n\geq 0} (F^nH/F^{n+1}H)^* \\ &= \bigoplus_{n\geq 0} \mathrm{gr}^n(H)^* = (\widehat{\mathrm{gr}}H)',\end{aligned}$$

where the second equality follows from $F_\bullet H' = (F^{\bullet+1}H)^\perp$ (see Lemma D.6) and the last equality follows from (a).

(c) follows from the identification of the said isomorphism with the degree $\leq k$ part of the above sequence of equalities. ☐

Lemma D.17. *Let Γ be a group. If Γ is filtered formal, then there exists an isomorphism*

$$\mathrm{iso}_\Gamma : (\mathbb{C}\Gamma)^\wedge \simeq \widehat{\mathrm{gr}}((\mathbb{C}\Gamma)^\wedge) \tag{D.4}$$

in **CHA**, *such that* $\mathrm{gr}(\mathrm{iso}_\Gamma) = \mathrm{id}$.

Proof. Let Γ be a filtered formal group. There is an isomorphism

$$\mathrm{Lie}(\Gamma) \simeq \widehat{\mathrm{gr}}\,\mathrm{Lie}(\Gamma) \tag{D.5}$$

in **MLA** with associated graded the identity. There is a sequence of isomorphisms in **CHA** given by

$$(\mathbb{C}\Gamma)^\wedge \simeq \widehat{U}(\mathrm{Lie}(\Gamma)) \simeq \widehat{U}(\widehat{\mathrm{gr}}\,\mathrm{Lie}(\Gamma)) \simeq \widehat{\mathrm{gr}}\big(\widehat{U}(\mathrm{Lie}(\Gamma))\big) = \widehat{\mathrm{gr}}((\mathbb{C}\Gamma)^\wedge),$$

where the first and last isomorphisms come from Lemma D.9 and Definition D.10, the second isomorphism arises from applying the functor $\widehat{U}$ to the isomorphism (D.5), and the third isomorphism arises from Lemma D.15(c). This results in an isomorphism with the claimed properties. □

Proposition D.18. *If Γ is a filtered formal finitely generated group, then the Hopf algebra $(\mathbb{C}\Gamma)'$ is filtered formal (in the sense of Definition C.2).*

Proof. Let Γ be a filtered formal finitely generated group. Then (D.4) is an isomorphism in $\mathbf{CHA}_{\mathrm{fd}}$. There is a sequence of isomorphisms in **HA**,

$$(\mathbb{C}\Gamma)' \simeq ((\mathbb{C}\Gamma)^\wedge)' \simeq \big(\widehat{\mathrm{gr}}((\mathbb{C}\Gamma)^\wedge)\big)' \simeq \mathrm{gr}\big(((\mathbb{C}\Gamma)^\wedge)'\big) \simeq \mathrm{gr}((\mathbb{C}\Gamma)'), \tag{D.6}$$

where the first and last isomorphisms follow from Lemma D.7, the second isomorphism arises from applying the functor $\mathbf{CHA}_{\mathrm{fd}} \to \mathbf{HA}$, $H \mapsto H'$ (see Definition D.4) to (D.4), and the third isomorphism comes from (D.3). It follows from $\mathrm{gr}(\mathrm{iso}_\Gamma) = \mathrm{id}$ and Lemma D.6 that the second isomorphism in (D.6) is compatible with the filtration $F_\bullet$ and that the associated graded is the identity. The third isomorphism in (D.6) induces an isomorphism between the F_n of both sides for any $n \geq 0$. By Lemma D.16(c), the associated graded morphism coincides with the composition of natural isomorphisms $\mathrm{gr}_n((\widehat{\mathrm{gr}}((\mathbb{C}\Gamma)^\wedge))') \simeq \mathrm{gr}^n(\mathbb{C}\Gamma)^* \simeq \mathrm{gr}_n(\mathrm{gr}(((\mathbb{C}\Gamma)^\wedge)'))$. This implies that the image by gr_n of the isomorphism $(\mathbb{C}\Gamma)' \simeq \mathrm{gr}((\mathbb{C}\Gamma)')$ induced by (D.6) is the identification $\mathrm{gr}_n(\mathbb{C}\Gamma)' \simeq \mathrm{gr}_n\mathrm{gr}((\mathbb{C}\Gamma)')$ induced by Lemma C.1(a). □

Remark D.19. Lemma D.17 relates as follows to [SW2]: combining [SW2, Cors 6.2 and 2.7] one obtains the equivalence of the filtered formality of Γ with the conclusion of Lemma D.17 for any finitely generated group Γ.

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