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Extensions of Co-Compact Gabor Frames on Locally Compact Abelian Groups and Applications

by

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Abstract

This paper addresses the extensions of co-compact Gabor Bessel sequences to tight frames and dual pairs on locally compact abelian (LCA) groups, and its applications to the $L^2(\mathbb{R}^d)$-setting. Firstly, we present a method to construct co-compact Gabor frames on LCA groups under a mild condition. This condition is optimal in the sense that it reduces to the usual density condition for lattice-based Gabor frames in $L^2(\mathbb{R}^d)$. Secondly, we obtain an extension theorem of a co-compact Gabor Bessel sequence (a pair of co-compact Gabor Bessel sequences) to a tight co-compact Gabor frame (a pair of dual co-compact Gabor frames). Finally, as an application, we derive a strategy to obtain co-compact Gabor frames for $L^2(\mathbb{R}^d)$, and establish an extension theorem of dual co-compact Gabor frames for $L^2(\mathbb{R}^d)$ with $C_c^\infty(\mathbb{R}^d)$-window functions. An example is also provided. It demonstrates that, for general co-compact (i.e., at least one of time and frequency translations is not a lattice) Gabor frames, the product of the sizes of time and frequency translations can take an arbitrary positive number.

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§1. Introduction

Due to unifying a number of different results into a general framework with a concise and elegant notation, the frame theory on locally compact abelian (LCA) groups has attracted much attention from mathematicians in the last two decades. The LCA group approach enables us to visualize hidden relationships between the different components of the theory. Recall that the LCA group approach applies to signals on all groups of the form $\mathbb{R}^s \times \mathbb{Z}^d \times \mathbb{T}^q \times \mathbb{Z}_p$ which include multichannel video signals based on the group $\mathbb{Z}^d \times \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the finite group of the integers modulo p. In practice, one can also encounter signals on LCA groups other than $\mathbb{R}^s \times \mathbb{Z}^d \times \mathbb{T}^q \times \mathbb{Z}_p$ ([2, 6, 5, 3, 11]). Gabor analysis on LCA groups dates back to the works [14, 24] by Gröchenig, Kaniuth and Kutyniok. Gabor frames are closely related to shift- (translation-)invariant systems. Uniform lattice-based shift-invariant systems and Gabor frames on LCA groups were studied by many authors including Cabrelli, Christensena, Feichtinger, Goh, Kamyabi Gol, Kazarian, Kozek, Kutyniok, Labate, Luef, Mohammadian, Paternostro, Raisi Tousi, Tabatabaie and Jokar; cf. [2, 23, 6, 12, 25, 7, 8, 22, 26, 30, 33]. Recall that not all LCA groups possess uniform lattices, for example the group $\mathbb{Q}_p$ of p-adic numbers, but every LCA group has co-compact subgroups. Co-compact translation-invariant systems were recently studied by some authors including Bownik, Ross, Gumber, Shukla, Jakobsen and Lemvig; cf. [1, 15, 21, 20].

For frame extension, Casazza and Leonhard [4] showed that every Bessel sequence in a finite-dimensional space can be extended to a tight frame. Li and Sun [27] generalized this result to general Hilbert spaces, and proved that every Gabor Bessel sequence in $L^2(\mathbb{R})$ can be extended to a tight frame by adding one Gabor system with the same time-frequency parameters. Christensen, Kim and Kim [9] obtained similar results in the setting of dual frames. In particular, they proved that every pair of Gabor Bessel sequences in $L^2(\mathbb{R})$ can be extended to a pair of dual frames by adding one extra Gabor system to each Bessel sequence, and that the generators of added Gabor systems can be chosen to be compactly supported if the generators of initial Gabor Bessel sequences have compact support in addition.

Motivated by the above works, in this paper we investigate the extension of co-compact Gabor frames on second countable LCA groups and their applications. For a group, we will denote the neutral element by 0, the group operation by "+" and its inverse operation by "−". Let G be a second countable LCA group. We denote its *dual group* by $\widehat{G}$ which consists of all characters, i.e., all continuous homomorphisms from G into the torus $\mathbb{T}$. When equipped with the weak* topology and group operation pointwise multiplication, $\widehat{G}$ is also an LCA group. A closed subgroup H of G is said to be *co-compact* if G/H is compact, and is said to be a (*uniform*) *lattice* if H is discrete and co-compact. For a closed subgroup H of G, its *annihilator* $H^\perp$ is defined by $H^\perp := \{\gamma \in \widehat{G} : \gamma(x) = 1,\ \forall x \in H\}$. Then $H^\perp$ is a closed subgroup of $\widehat{G}$. We denote by μ_G the *Haar measure* on G (it is unique up to a positive constant), by $L^p(G)$ the Banach space of all p-integral functions on G for $1 \leq p < \infty$, and by $L^\infty(G)$ the Banach space of all essentially bounded functions on G with respect to μ_G. In particular, $L^2(G)$ is a separable Hilbert space. We define the *Fourier transform* by

$$\mathcal{F}f(\gamma) = \hat{f}(\gamma) = \int_G f(x)\overline{\gamma(x)}\, d\mu_G(x), \quad \gamma \in \widehat{G}$$

for $f \in L^1(G)$ and extend it to $L^2(G)$ in the usual way. For convenience, we require that the measures μ_G and $\mu_{\widehat{G}}$ are normalized so that the *Plancherel theorem* holds, i.e.,

$$\int_G f(x)\overline{g(x)}\, d\mu_G(x) = \int_{\widehat{G}} \hat{f}(\gamma)\overline{\hat{g}(\gamma)}\, d\mu_{\widehat{G}}(\gamma) \quad \text{for } f,\, g \in L^2(G).$$

In this case, we say that μ_G and $\mu_{\widehat{G}}$ are *dual measures*. We mention *Weil's theorem* in [31].

Proposition 1.1. *Let H denote a closed subgroup of the LCA group G. Then the following hold:*

(i) *Taking any Haar measures on two of the LCA groups G, H, and G/H, the third Haar measure can be normalized such that for all $f \in L^1(G)$,*

$$\int_G f(x)\, d\mu_G(x) = \int_{G/H}\int_H f(x+h)\, d\mu_H(h)\, d\mu_{G/H}(\dot{x}). \tag{1.1}$$

(ii) *If the measures on G, H, and G/H, are chosen such that (1.1) holds, then the corresponding dual measures on the dual groups $\widehat{G}$, $\widehat{H} = \widehat{G}/H^\perp$ and $\widehat{G/H} = H^\perp$ satisfy that for all $F \in L^1(G)$,*

$$\int_{\widehat{G}} F(\gamma)\, d\mu_{\widehat{G}}(\gamma) = \int_{\widehat{G}/H^\perp}\int_{H^\perp} F(\gamma+\omega)\, d\mu_{H^\perp}(\omega)\, d\mu_{\widehat{G}/H^\perp}(\dot{\gamma}).$$

From Proposition 1.1, if two of the measures on G, H, G/H, $\widehat{G}$, $H^\perp$ and $\widehat{G}/H^\perp$ are given, and these two are not dual measures, then all others are uniquely determined by requiring Plancherel's theorem and Weil's theorem. Throughout this paper, all measures are assumed to satisfy Plancherel's theorem and Weil's theorem. A *Borel section* or a *fundamental domain* of a closed subgroup H in G is a Borel measurable subset Ξ of G which meets each coset G/H once. By [29, Lem. 1.1] or [13], an arbitrary closed subgroup H of G has a Borel section. The *size* of H is defined by

$$s(H) = \int_{G/H} d\mu_{G/H}(\dot{x}).$$

Then $s(H)$ is finite if and only if H is co-compact by [5]. In particular, if H is discrete in addition, then the mapping $T\colon \Xi \to G/H$ defined by

$$T\xi = \xi + H \quad \text{for } \xi \in \Xi$$

is measure preserving by [1, Prop. 3.2], which implies that $s(H) = \mu_G(\Xi)$. And also by [1, Prop. 3.2], a co-compact subgroup H is a lattice if and only if $s(H^{\perp}) < \infty$. We refer to [17, 18, 32] for the basics on LCA groups.

Similarly to the case of Gabor analysis on $L^2(\mathbb{R})$, we begin our analysis on $L^2(G)$ with relevant operators. Define the *translation operator* $T_s\colon L^2(G) \to L^2(G)$ with $s \in G$ and the *modulation operator* $E_t\colon L^2(G) \to L^2(G)$ with $t \in \widehat{G}$ by

$$T_s f(\cdot) = f(\cdot - s) \quad \text{and} \quad E_t f(\cdot) = t(\cdot) f(\cdot)$$

for $f \in L^2(G)$, respectively. Given co-compact subgroups Λ and Γ of G and $\widehat{G}$ respectively, the *Gabor system* generated by a finite family $\{g_l : 1 \le l \le L\}$ in $L^2(G)$ is defined by

$$\{E_\eta T_\lambda g_l\}_{1 \le l \le L, \eta \in \Gamma, \lambda \in \Lambda}.$$

We say that it is a *frame* for $L^2(G)$ if there exist constants $A, B > 0$ such that

$$\begin{aligned} A\|f\|^2 &\le \sum_{l=1}^{L} \int_\Gamma \int_\Lambda |\langle f, E_\eta T_\lambda g_l\rangle|^2 \, d\mu_\Lambda(\lambda) \, d\mu_\Gamma(\eta) \\ &\le B\|f\|^2 \quad \text{for all } f \in L^2(G), \end{aligned} \tag{1.2}$$

where A and B are called the lower and upper *frame bounds* respectively. It is called a *tight frame* if $A = B$ in (1.2), and it is called a *Bessel sequence* in $L^2(G)$ if at least the upper inequality in (1.2) is satisfied. Given $g \in L^2(G)$ and a measurable function $c(\lambda, \eta)$ on $\Lambda \times \Gamma$, $f = \int_\Gamma \int_\Lambda c(\lambda, \eta) E_\eta T_\lambda g \, d\mu_\Lambda(\lambda) \, d\mu_\Gamma(\eta)$ means that

$$\mathcal{A}\colon L^2(G) \to \mathbb{C} \quad \text{by} \quad \mathcal{A}h = \int_\Gamma \int_\Lambda \overline{c(\lambda, \eta)} \langle h, E_\eta T_\lambda g\rangle \, d\mu_\Lambda(\lambda) \, d\mu_\Gamma(\eta) \quad \text{for } h \in L^2(G)$$

defines a bounded linear functional on $L^2(G)$, and f is exactly the unique element in $L^2(G)$ such that $\mathcal{A}h = \langle h, f\rangle$ for $h \in L^2(G)$. Recall that if $c(\lambda, \eta) \in L^2(\Lambda \times \Gamma)$ and $\{E_\eta T_\lambda g\}_{\eta \in \Gamma, \lambda \in \Lambda}$ is a Bessel sequence in $L^2(G)$, then

$$f = \int_\Gamma \int_\Lambda c(\lambda, \eta) E_\eta T_\lambda g \, d\mu_\Lambda(\lambda) \, d\mu_\Gamma(\eta)$$

is well defined, and by a simple computation,

$$\mathcal{B}f = \int_\Gamma \int_\Lambda c(\lambda,\eta) E_\eta T_\lambda \mathcal{B} g \, d\mu_\Lambda(\lambda)\, d\mu_\Gamma(\eta)$$

for each bounded linear operator $\mathcal{B}$ which commutes $E_\eta T_\lambda$ for all $\lambda \in \Lambda$, $\eta \in \Gamma$. Given a Bessel sequence $\{E_\eta T_\lambda g\}_{\eta\in\Gamma,\lambda\in\Lambda}$ in $L^2(G)$, the *synthesis operator*, *analysis operator* and *frame operator* are respectively defined by

$$U\colon L^2(\Lambda\times\Gamma)\to L^2(G),\quad Uc = \int_\Gamma \int_\Lambda c(\lambda,\eta) E_\eta T_\lambda g \, d\mu_\Lambda(\lambda)\, d\mu_\Gamma(\eta) \ \text{ for } c\in L^2(\Lambda\times\Gamma),$$

$$U^*\colon L^2(G)\to L^2(\Lambda\times\Gamma),\ U^* f = \big\{\langle f, E_\eta T_\lambda g\rangle\big\}_{\lambda\in\Lambda,\eta\in\Gamma} \ \text{ for } f\in L^2(G),$$

$$S\colon L^2(G)\to L^2(G),\qquad S = UU^*.$$

Two frames $\{E_\eta T_\lambda g_l\}_{1\le l\le L,\eta\in\Gamma,\lambda\in\Lambda}$ and $\{E_\eta T_\lambda h_l\}_{1\le l\le L,\eta\in\Gamma,\lambda\in\Lambda}$ forming a pair of *dual frames* for $L^2(G)$ means that

$$f = \sum_{l=1}^{L} \int_\Gamma \int_\Lambda \langle f, E_\eta T_\lambda g_l\rangle E_\eta T_\lambda h_l \, d\mu_\Lambda(\lambda)\, d\mu_\Gamma(\eta) \quad \text{for all } f\in L^2(G). \tag{1.3}$$

In particular, (1.3) converges unconditionally if Λ and Γ are lattices. By a standard argument, two Gabor systems $\{E_\eta T_\lambda g_l\}_{1\le l\le L,\eta\in\Gamma,\lambda\in\Lambda}$ and $\{E_\eta T_\lambda h_l\}_{1\le l\le L,\eta\in\Gamma,\lambda\in\Lambda}$ form a pair of dual frames for $L^2(G)$ if and only if they are Bessel sequences in $L^2(G)$ satisfying (1.3). By the proof of [1, Thm. 5.1] and an argument before [20, Cor. 6.8], there exists no Riesz sequence of the form $\{E_\eta T_\lambda g\}_{\eta\in\Gamma,\lambda\in\Lambda}$ in $L^2(G)$ if either Λ or Γ is not discrete.

Now we come back to our theme. Let Λ and Γ be respective co-compact subgroups of G and $\widehat{G}$, $Q\subset G$ and $\Omega\subset G$ be respective Borel sections for Λ and $\Gamma^\perp$. This paper addresses the extension from a co-compact Gabor Bessel sequence (a pair of co-compact Gabor Bessel sequences) to a tight co-compact Gabor frame (a pair of dual co-compact Gabor frames) under the assumption that

$$\operatorname*{ess\,inf}_{x\in Q} \mu_\Lambda[(K-x)\cap\Lambda] > 0 \quad \text{for some measurable } K\subset\Omega. \tag{1.4}$$

To the best of our knowledge, this assumption does not appear in the existing literature. Let $\mathbb{R}^d$ be the LCA group equipped with the usual addition, topology and Lebesgue measure as its Haar measure. The following Example 1.1 shows that, to some extent, (1.4) is a substitution of density condition for lattice-based Gabor frames in $L^2(\mathbb{R}^d)$. In addition, Example 1.2 gives a class of examples for which (1.4) is necessary for the existence of complete co-compact Gabor systems $\{E_\eta T_\lambda g\}_{\eta\in\Gamma,\lambda\in\Lambda}$ in $L^2(\mathbb{R}^d)$ with $\operatorname{supp}(g)\subset\Omega$. This partially explains why (1.4) is required through this paper. For readability, the proof of Example 1.2 will be given in Section 3.

Example 1.1. Let $\Lambda = \mathcal{A}\mathbb{Z}^d$ and $\Gamma = \mathcal{B}\mathbb{Z}^d$ in $\mathbb{R}^d$ with $\mathcal{A}$ and $\mathcal{B}$ being $d \times d$ invertible real matrices, and take $Q = \mathcal{A}([0,1)^d + y^{(1)})$ and $K = \Omega = \mathcal{B}^\sharp([0,1)^d + y^{(2)})$ for some $y^{(1)}$, $y^{(2)} \in \mathbb{R}^d$, where $\mathcal{B}^\sharp$ is the inverse of the transpose of $\mathcal{B}$, i.e., $\mathcal{B}^\sharp = (\mathcal{B}^\mathsf{T})^{-1}$. Then (1.4) holds if and only if

$$(\mathcal{B}^\sharp([0,1)^d + y^{(2)}) - x) \cap \mathcal{A}\mathbb{Z}^d \neq \emptyset \quad \text{for a.e. } x \in \mathcal{A}([0,1)^d + y^{(1)}),$$

equivalently,

$$[0,1)^d \subset \bigcup_{k \in \mathbb{Z}^d} (\mathcal{A}^{-1}\mathcal{B}^\sharp[0,1)^d + k) + z \quad \text{with } z = \mathcal{A}^{-1}\mathcal{B}^\sharp y^{(2)} - y^{(1)}.$$

It is in turn equivalent to

$$\mathbb{R}^d = \bigcup_{k \in \mathbb{Z}^d} (\mathcal{A}^{-1}\mathcal{B}^\sharp[0,1)^d + k).$$

This implies that $|\det \mathcal{A}|\,|\det \mathcal{B}| \le 1$ by [28, Lem. 2.1], and in particular, (1.4) is equivalent to $|\mathcal{A}|\,|\mathcal{B}| \le 1$ when $d = 1$. Recall from [16, Thm. 1.3] that the existence of frames of the form $\{E_\eta T_\lambda g\}_{\eta \in \Gamma, \lambda \in \Lambda}$ is equivalent to $|\det \mathcal{A}|\,|\det \mathcal{B}| \le 1$.

Example 1.2. Given $0 < s < d$, let $\mathcal{A}$ and $\mathcal{B}$ be $d \times d$ invertible real matrices having the following forms:

$$\mathcal{A} = \begin{pmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} B_{1,1} & 0 \\ 0 & B_{2,2} \end{pmatrix}, \tag{1.5}$$

where $A_{2,2}$ and $B_{2,2}$ are $(d-s) \times (d-s)$ matrices satisfying

$$A_{2,2}^{-1} B_{2,2}^\sharp = \operatorname{diag}(\zeta_1, \zeta_2, \ldots, \zeta_{d-s}) \quad \text{with } 0 < \zeta_l < 1 \text{ for } 1 \le l \le d-s, \tag{1.6}$$

and $\operatorname{rank}(A_{1,2}) = s$. Define

$$\Lambda = \mathcal{A}(\mathbb{R}^s \times \mathbb{Z}^{d-s}) \quad \text{and} \quad \Gamma = \mathcal{B}(\mathbb{R}^s \times \mathbb{Z}^{d-s}). \tag{1.7}$$

Take $Q = \mathcal{A}(\{0\}^s \times [0,1)^{d-s})$ and $\Omega = \mathcal{B}^\sharp(\mathbb{R}^s \times [0,1)^{d-s})$. Then

$$\operatorname*{ess\,inf}_{x \in Q} \mu_\Lambda[(\Omega - x) \cap \Lambda] = 0, \tag{1.8}$$

and there exists no $g \in L^2(\mathbb{R}^d)$ with $\operatorname{supp}(g) \subset \Omega$ such that $\{E_\eta T_\lambda g\}_{\eta \in \Gamma, \lambda \in \Lambda}$ is complete in $L^2(\mathbb{R}^d)$.

This paper is organized as follows. In Section 2, under the assumption (1.4), we first give a method to construct co-compact Gabor frames whose canonical dual windows have the same support with initial windows (see Theorem 2.1). Then we derive the extension theorems from co-compact Gabor Bessel sequences to tight co-compact Gabor frames, and from a pair of co-compact Gabor Bessel sequences to a pair of dual co-compact Gabor frames (see Theorems 2.2 and 2.3). In Section 3, by normalizing Haar measures on $\mathbb{R}^d$, its co-compact subgroups and the corresponding quotient groups in Weil's theorem, we give an application to co-compact Gabor systems in $L^2(\mathbb{R}^d)$. Applying Theorem 2.1, we first give a method to construct co-compact Gabor frames for $L^2(\mathbb{R}^d)$ with $C_c^\infty(\mathbb{R}^d)$-window functions, whose canonical dual window functions belong to $C_c^\infty(\mathbb{R}^d)$ if initial window functions are real valued in addition (see Theorem 3.1). Then we obtain an extension theorem from a pair of co-compact Gabor Bessel sequences to a pair of dual co-compact Gabor frames, where both initial and adding generators belong to $C_c^\infty(\mathbb{R}^d)$ (see Theorem 3.2). It is worth noting that for general co-compact (i.e., at least one of Λ and Γ is not a lattice) Gabor frames $\{E_\eta T_\lambda g\}_{\eta\in\Gamma,\lambda\in\Lambda}$ in $L^2(\mathbb{R}^d)$, $s(\Lambda)s(\Gamma)$ can take an arbitrary positive number (see Example 3.1).

§2. Extension theorems

This section focuses on the extension from a co-compact Gabor Bessel sequence (a pair of co-compact Gabor Bessel sequences) to a tight co-compact Gabor frame (a pair of dual co-compact Gabor frames) on LCA groups. For this purpose, we first present some lemmas. The following result, Lemma 2.1, is an extension of [21, Thm. 4.1] or [5, Cor. 21.8.1] to the multi-window case. We can prove it by a line-by-line procedure as in [21, Thm. 4.1].

Lemma 2.1. *Let Λ and Γ be co-compact subgroups of G and $\widehat{G}$ respectively, $\{E_\eta T_\lambda g_l\}_{1\le l\le L,\eta\in\Gamma,\lambda\in\Lambda}$ and $\{E_\eta T_\lambda h_l\}_{1\le l\le L,\eta\in\Gamma,\lambda\in\Lambda}$ be Bessel sequences in $L^2(G)$. Then the following statements are equivalent:*

(i) *$\{E_\eta T_\lambda g_l\}_{1\le l\le L,\eta\in\Gamma,\lambda\in\Lambda}$ and $\{E_\eta T_\lambda h_l\}_{1\le l\le L,\eta\in\Gamma,\lambda\in\Lambda}$ are a pair of dual frames for $L^2(G)$.*

(ii) *For each $\alpha\in\Lambda^\perp$,*

$$\frac{1}{s(\Lambda)}\sum_{l=1}^{L}\int_\Gamma \overline{\widehat{g_l}(\gamma+\eta)}\widehat{h_l}(\gamma+\eta+\alpha)\,d\mu_\Gamma(\eta)=\delta_{\alpha,0}\quad \text{for a.e. } \gamma\in\widehat{G}.$$

(iii) *For each $\beta\in\Gamma^\perp$,*

$$\frac{1}{s(\Gamma)}\sum_{l=1}^{L}\int_\Lambda \overline{g_l(x+\lambda)}h_l(x+\lambda+\beta)\,d\mu_\Lambda(\lambda)=\delta_{\beta,0}\quad \text{for a.e. } x\in G.$$

The following lemma gives sufficient conditions for co-compact Gabor systems to be Bessel sequences and frames in $L^2(G)$, respectively.

Lemma 2.2. *Let Λ and Γ be co-compact subgroups of G and $\widehat{G}$ respectively. Consider the Gabor system $\{E_\eta T_\lambda g_l\}_{1\le l\le L,\eta\in\Gamma,\lambda\in\Lambda}$ in $L^2(G)$; then the following hold:*

(i) *If*

$$B := \operatorname*{ess\,sup}_{x\in G} \frac{1}{s(\Gamma)} \sum_{l=1}^{L} \int_\Lambda \sum_{\alpha\in\Gamma^\perp} |g_l(x+\lambda)g_l(x+\lambda+\alpha)|\, d\mu_\Lambda(\lambda) < \infty,$$

then $\{E_\eta T_\lambda g_l\}_{1\le l\le L,\eta\in\Gamma,\lambda\in\Lambda}$ is a Bessel sequence in $L^2(G)$ with bound B.

(ii) *Furthermore, if also*

$$A := \operatorname*{ess\,inf}_{x\in G} \frac{1}{s(\Gamma)} \Bigg(\sum_{l=1}^{L} \int_\Lambda |g_l(x+\lambda)|^2\, d\mu_\Lambda(\lambda) - \sum_{l=1}^{L} \int_\Lambda \sum_{\alpha\in\Gamma^\perp\setminus\{0\}} |g_l(x+\lambda)g_l(x+\lambda+\alpha)|\, d\mu_\Lambda(\lambda) \Bigg) > 0,$$

then $\{E_\eta T_\lambda g_l\}_{1\le l\le L,\eta\in\Gamma,\lambda\in\Lambda}$ is a frame for $L^2(G)$ with bounds A and B.

Proof. By the unitarity of the Fourier transform, $\{E_\eta T_\lambda g_l\}_{1\le l\le L,\eta\in\Gamma,\lambda\in\Lambda}$ is a Bessel sequence (frame) for $L^2(G)$ if and only if $\{T_\eta E_{-\lambda} \hat{g}_l\}_{1\le l\le L,\eta\in\Gamma,\lambda\in\Lambda}$ is a Bessel sequence (frame) for $L^2(\widehat{G})$. Applying [19, Prop. IV.1] to the Gabor system $\{T_\eta E_{-\lambda} \hat{g}_l\}_{1\le l\le L,\eta\in\Gamma,\lambda\in\Lambda}$ gives the lemma. □

Remark 2.1. Lemma 2.2(i) demonstrates that if $\{g_l : 1\le l\le L\}$ is a finite family in $L^\infty(G)$ with each g_l being of compact support, then $\{E_\eta T_\lambda g_l\}_{1\le l\le L,\eta\in\Gamma,\lambda\in\Lambda}$ is a Bessel sequence in $L^2(G)$ for arbitrary co-compact subgroups Λ and Γ of G and $\widehat{G}$ respectively.

The following lemma is auxiliary to Theorems 2.1 and 2.2.

Lemma 2.3. *Let Γ be a closed subgroup of $\widehat{G}$. Suppose $f_1, f_2 \in \mathcal{D}$ and $g, h \in L^2(G)$, where*

$$\mathcal{D} = \big\{ f \in L^2(G) : f \in L^\infty(G) \text{ and } \operatorname{supp}(f) \text{ is compact} \big\}. \tag{2.1}$$

Then for all $\lambda \in G$,

$$\int_\Gamma \langle f_1, E_\eta T_\lambda g\rangle \langle E_\eta T_\lambda h, f_2\rangle\, d\mu_\Gamma(\eta) = \frac{1}{s(\Gamma)} \int_G \sum_{\alpha\in\Gamma^\perp} f_1(x)\overline{f_2(x+\alpha)}\,\overline{g(x+\lambda)} h(x+\lambda+\alpha)\, d\mu_G(x).$$

Proof. Arbitrarily fix $\lambda \in G$. By Plancherel's theorem, $\langle f, E_\eta T_\lambda g\rangle = \langle \hat{f}, T_\eta E_{-\lambda}\hat{g}\rangle$ for $\eta \in \Gamma$ and $f, g \in L^2(G)$. Applying [21, Lem. 2.2] to $E_{-\lambda}\hat{g}$ and $E_{-\lambda}\hat{h}$ gives the lemma. $\square$

The following theorem gives a method to construct co-compact Gabor frames for $L^2(G)$.

Theorem 2.1. *Given co-compact subgroups Λ of G and Γ of $\widehat{G}$, let $Q \subset G$ and $\Omega \subset G$ be Borel sections for Λ and $\Gamma^\perp$ respectively, and let K be a measurable set satisfying (1.4). Suppose $g \in L^\infty(G) \cap L^2(G)$ is such that $K \subset \mathrm{supp}(g) \subset \Omega$:*

$$\operatorname*{ess\,sup}_{x\in Q} \mu_\Lambda[(\mathrm{supp}(g) - x) \cap \Lambda] < \infty \quad \textit{and} \quad \operatorname*{ess\,inf}_{x\in K} |g(x)| > 0. \tag{2.2}$$

Then $\{E_\eta T_\lambda g\}_{\eta\in\Gamma,\lambda\in\Lambda}$ is a frame for $L^2(G)$, and

$$Sf = \frac{\widetilde{G}}{s(\Gamma)} f, \quad S^{-1} f = \frac{s(\Gamma)}{\widetilde{G}} f \quad \textit{for } f \in L^2(G), \tag{2.3}$$

where S is the frame operator for $\{E_\eta T_\lambda g\}_{\eta\in\Gamma,\lambda\in\Lambda}$, and $\widetilde{G}(y) = \int_\Lambda |g(y+\lambda)|^2\, d\mu_\Lambda(\lambda)$ for a.e. $y \in G$.

Proof. Since $\mathrm{supp}(g) \subset \Omega$, we have $\sum_{\alpha\in\Gamma^\perp} |g(\cdot + \lambda) g(\cdot + \lambda + \alpha)| = |g(\cdot + \lambda)|^2$. Then, by Lemma 2.2, we only need to prove that $\int_\Lambda |g(\cdot + \lambda)|^2\, d\mu_\Lambda(\lambda)$ has positive lower and upper bounds on Q. Observe that

$$\int_\Lambda |g(x+\lambda)|^2\, d\mu_\Lambda(\lambda) = \int_{\Lambda\cap(\mathrm{supp}(g)-x)} |g(x+\lambda)|^2\, d\mu_\Lambda(\lambda)$$

for $x \in Q$. It implies that

$$\int_\Lambda |g(x+\lambda)|^2\, d\mu_\Lambda(\lambda) \geq \operatorname*{ess\,inf}_{x\in Q} \mu_\Lambda[(K - x)\cap\Lambda] \cdot \operatorname*{ess\,inf}_{x\in K} |g(x)|^2,$$

and

$$\int_\Lambda |g(x+\lambda)|^2\, d\mu_\Lambda(\lambda) \leq \|g\|_\infty^2 \cdot \operatorname*{ess\,sup}_{x\in Q} \mu_\Lambda[(\mathrm{supp}(g) - x)\cap\Lambda]$$

by (1.4) and (2.2). Thus $\{E_\eta T_\lambda g\}_{\eta\in\Gamma,\lambda\in\Lambda}$ is a frame for $L^2(G)$. On the other hand,

$$\begin{aligned}\int_\Gamma |\langle f, E_\eta T_\lambda g\rangle|^2\, d\mu_\Gamma(\eta) &= \frac{1}{s(\Gamma)} \int_G \sum_{\alpha\in\Gamma^\perp} f(y)\overline{f(y+\alpha)}\,\overline{g(y+\lambda)} g(y+\lambda+\alpha)\, d\mu_G(y)\\ &= \frac{1}{s(\Gamma)} \int_G |f(y)|^2 |g(y+\lambda)|^2\, d\mu_G(y) \qquad (2.4)\end{aligned}$$

for $f \in \mathcal{D}$ by Lemma 2.3 and the fact that $\operatorname{supp}(g) \subset \Omega$, where $\mathcal{D}$ is as in (2.1). Integrating the two sides of (2.4) over Λ gives

$$\begin{aligned}
&\int_\Lambda \int_\Gamma |\langle f, E_\eta T_\lambda g\rangle|^2 \, d\mu_\Gamma(\eta)\, d\mu_\Lambda(\lambda) \\
&= \frac{1}{s(\Gamma)} \int_G \left(\int_\Lambda |g(y+\lambda)|^2 \, d\mu_\Lambda(\lambda) \right) |f(y)|^2 \, d\mu_G(y)
\end{aligned} \tag{2.5}$$

for $f \in \mathcal{D}$. Obviously, (2.5) can be rewritten as

$$\langle Sf, f\rangle = \frac{1}{s(\Gamma)} \langle \widetilde{G}(\cdot) f(\cdot), f(\cdot)\rangle \quad \text{for } f \in \mathcal{D}.$$

This leads to (2.3) due to $\mathcal{D}$ being dense in $L^2(G)$. □

Remark 2.2. We have the following supplementary explanations for Theorem 2.1:

(i) Obviously, $S^{-1}g$, $S^{-\frac{1}{2}}g \in L^\infty(G)$, and $\{E_\eta T_\lambda S^{-\frac{1}{2}} g\}_{\eta\in\Gamma,\lambda\in\Lambda}$ is a Parseval frame for $L^2(G)$ under the assumptions of Theorem 2.1. If we make a further convention that $\operatorname{supp}(g)$ is compact in Theorem 2.1, then $S^{-1}g$ and $S^{-\frac{1}{2}}g$ also have compact support.

(ii) The idea of Theorem 2.1 closely resembles the classical painless nonorthogonal expansions by Daubechies, Grossmann and Meyer [10]. Also, in view of Example 1.2, Theorem 2.1 can be regarded as a generalized version of painless nonorthogonal expansions.

The following theorem gives a tight co-compact Gabor frame extension which preserves the support property.

Theorem 2.2. *Given co-compact subgroups Λ of G and Γ of $\widehat{G}$, let $Q \subset G$ and $\Omega \subset G$ be Borel sections for Λ and $\Gamma^\perp$, respectively, and let K be a measurable set satisfying (1.4). Assume that $\{E_\eta T_\lambda g_1\}_{\eta\in\Gamma,\lambda\in\Lambda}$ is a Bessel sequence in $L^2(G)$ with bound B. Then, for each $\beta \geq B$, there exists $g_2 \in L^2(G)$ such that*

$$\{E_\eta T_\lambda g_1\}_{\eta\in\Gamma,\lambda\in\Lambda} \cup \{E_\eta T_\lambda g_2\}_{\eta\in\Gamma,\lambda\in\Lambda}$$

is a tight frame for $L^2(G)$ with bound β. Furthermore, g_2 can be chosen to have compact support if $\operatorname{supp}(g_1)$ *is compact and* $\operatorname{supp}(g_1) \subset \Omega$ *in addition.*

Proof. Let S_1 denote the frame operator for $\{E_\eta T_\lambda g_1\}_{\eta\in\Gamma,\lambda\in\Lambda}$. Then

$$\langle S_1 f, f\rangle = \int_\Gamma \int_\Lambda |\langle f, E_\eta T_\lambda g_1\rangle|^2 \, d\mu_\Lambda(\lambda)\, d\mu_\Gamma(\eta) \leq B\|f\|^2 \quad \text{for } f \in L^2(G). \tag{2.6}$$

It follows that $\beta I - S_1$ is a positive operator. Now let us express $(\beta I - S_1)f$. Choose a Parseval frame $\{E_\eta T_\lambda g\}_{\eta\in\Gamma,\lambda\in\Lambda}$ for $L^2(G)$ (this can be done by Theorem 2.1 and

Remark 2.2(i)). By [20, Lem. 5.1], $S_1 E_\eta T_\lambda = E_\eta T_\lambda S_1$ for $\eta \in \Gamma$, $\lambda \in \Lambda$. This implies that

$$(\beta I - S_1)^{\frac{1}{2}} E_\eta T_\lambda = E_\eta T_\lambda (\beta I - S_1)^{\frac{1}{2}}$$

for $\eta \in \Gamma$, $\lambda \in \Lambda$. Thus

$$\begin{aligned}(\beta I - S_1)^{\frac{1}{2}} f &= \int_\Gamma \int_\Lambda \langle (\beta I - S_1)^{\frac{1}{2}} f, E_\eta T_\lambda g\rangle E_\eta T_\lambda g \, d\mu_\Lambda(\lambda)\, d\mu_\Gamma(\eta) \\ &= \int_\Gamma \int_\Lambda \langle f, (\beta I - S_1)^{\frac{1}{2}} E_\eta T_\lambda g\rangle E_\eta T_\lambda g \, d\mu_\Lambda(\lambda)\, d\mu_\Gamma(\eta) \\ &= \int_\Gamma \int_\Lambda \langle f, E_\eta T_\lambda (\beta I - S_1)^{\frac{1}{2}} g\rangle E_\eta T_\lambda g \, d\mu_\Lambda(\lambda)\, d\mu_\Gamma(\eta)\end{aligned}$$

for $f \in L^2(G)$. It follows that

$$(\beta I - S_1) f = \int_\Gamma \int_\Lambda \langle f, E_\eta T_\lambda (\beta I - S_1)^{\frac{1}{2}} g\rangle E_\eta T_\lambda (\beta I - S_1)^{\frac{1}{2}} g \, d\mu_\Lambda(\lambda)\, d\mu_\Gamma(\eta) \tag{2.7}$$

for $f \in L^2(G)$. Let

$$g_2 = (\beta I - S_1)^{\frac{1}{2}} g. \tag{2.8}$$

Then $\{E_\eta T_\lambda g_2\}_{\eta\in\Gamma,\lambda\in\Lambda}$ is a Bessel sequence since $\{E_\eta T_\lambda g\}_{\eta\in\Gamma,\lambda\in\Lambda}$ is a frame, and

$$\begin{aligned}\beta\|f\|^2 &= \langle S_1 f, f\rangle + \langle (\beta I - S_1) f, f\rangle \\ &= \int_\Gamma \int_\Lambda |\langle f, E_\eta T_\lambda g_1\rangle|^2 \, d\mu_\Lambda(\lambda)\, d\mu_\Gamma(\eta) \\ &\quad + \int_\Gamma \int_\Lambda \langle f, E_\eta T_\lambda g_2\rangle E_\eta T_\lambda g_2 \, d\mu_\Lambda(\lambda)\, d\mu_\Gamma(\eta)\end{aligned}$$

for $f \in L^2(G)$ by (2.6) and (2.7). Therefore, $\{E_\eta T_\lambda g_1\}_{\eta\in\Gamma,\lambda\in\Lambda} \cup \{E_\eta T_\lambda g_2\}_{\eta\in\Gamma,\lambda\in\Lambda}$ is a tight frame for $L^2(G)$ with bound β.

Next we prove that g_2 can be chosen to have compact support if $\mathrm{supp}(g_1)$ is compact and $\mathrm{supp}(g_1) \subset \Omega$. Choose g in (2.8) such that $\mathrm{supp}(g)$ is compact (this can be done by Theorem 2.1 and Remark 2.2(i)). Then

$$S_1 f(x) = \left(\frac{1}{s(\Gamma)} \int_\Lambda |g_1(x-\lambda)|^2 \, d\mu_\Lambda(\lambda)\right) f(x) \quad \text{for } f \in L^2(G)$$

by the same procedure as in Theorem 2.1. This leads to

$$g_2(x) = \left(\beta - \frac{1}{s(\Gamma)} \int_\Lambda |g_1(x-\lambda)|^2 \, d\mu_\Lambda(\lambda)\right)^{\frac{1}{2}} g(x)$$

by (2.8), and thus g_2 is compactly supported. □

In Theorem 2.2, $\mathrm{supp}(g_1) \subset \Omega$ is required to guarantee $\mathrm{supp}(g_2)$ being compact. In the following dual extension theorem, Theorem 2.3, $\mathrm{supp}(g_1)$ being compactly supported (not necessarily contained in Ω) is enough to guarantee added generators having compact support. Therefore, the dual extension enjoys more freedom than the tight extension.

Theorem 2.3. *Given co-compact subgroups Λ of G and Γ of $\widehat{G}$, let $Q \subset G$ and $\Omega \subset G$ be Borel sections for Λ and $\Gamma^{\perp}$, respectively, and let K be a measurable set satisfying (1.4). Assume that $\{E_\eta T_\lambda g_1\}_{\eta\in\Gamma,\lambda\in\Lambda}$ and $\{E_\eta T_\lambda h_1\}_{\eta\in\Gamma,\lambda\in\Lambda}$ are Bessel sequences in $L^2(G)$. Then there exist $g_2, h_2 \in L^2(G)$ such that*

$$\{E_\eta T_\lambda g_1\}_{\eta\in\Gamma,\lambda\in\Lambda} \cup \{E_\eta T_\lambda g_2\}_{\eta\in\Gamma,\lambda\in\Lambda}$$

and

$$\{E_\eta T_\lambda h_1\}_{\eta\in\Gamma,\lambda\in\Lambda} \cup \{E_\eta T_\lambda h_2\}_{\eta\in\Gamma,\lambda\in\Lambda}$$

form a pair of dual frames for $L^2(G)$. Furthermore, g_2 and h_2 can be chosen to have compact support if $\mathrm{supp}(g_1)$ *and* $\mathrm{supp}(h_1)$ *are compact.*

Proof. Let T and U denote the synthesis operators for $\{E_\eta T_\lambda g_1\}_{\eta\in\Gamma,\lambda\in\Lambda}$ and $\{E_\eta T_\lambda h_1\}_{\eta\in\Gamma,\lambda\in\Lambda}$ respectively. Then

$$TU^*f = \int_\Gamma \int_\Lambda \langle f, E_\eta T_\lambda h_1\rangle E_\eta T_\lambda g_1 \, d\mu_\Lambda(\lambda)\, d\mu_\Gamma(\eta) \quad \text{for } f \in L^2(G). \tag{2.9}$$

Next we will express $(I - TU^*)f$. Choose a pair of dual frames $(\{E_\eta T_\lambda \gamma_1\}_{\eta\in\Gamma,\lambda\in\Lambda}, \{E_\eta T_\lambda \gamma_2\}_{\eta\in\Gamma,\lambda\in\Lambda})$ for $L^2(G)$ (this can be done by Theorem 2.1 and Remark 2.2(i)). Then

$$\begin{aligned}(I - TU^*)f &= \int_\Gamma \int_\Lambda \langle (I - TU^*)f, E_\eta T_\lambda \gamma_1\rangle E_\eta T_\lambda \gamma_2 \, d\mu_\Lambda(\lambda)\, d\mu_\Gamma(\eta)\\ &= \int_\Gamma \int_\Lambda \langle f, (I - UT^*)E_\eta T_\lambda \gamma_1\rangle E_\eta T_\lambda \gamma_2 \, d\mu_\Lambda(\lambda)\, d\mu_\Gamma(\eta)\\ &= \int_\Gamma \int_\Lambda \langle f, E_\eta T_\lambda (I - UT^*)\gamma_1\rangle E_\eta T_\lambda \gamma_2 \, d\mu_\Lambda(\lambda)\, d\mu_\Gamma(\eta) \qquad (2.10)\end{aligned}$$

for $f \in L^2(G)$ by [20, Lem. 5.1]. Take

$$h_2 = (I - UT^*)\gamma_1, g_2 = \gamma_2. \tag{2.11}$$

Observe that $\{E_\eta T_\lambda \gamma_1\}_{\eta\in\Gamma,\lambda\in\Lambda}$ and $\{E_\eta T_\lambda \gamma_2\}_{\eta\in\Gamma,\lambda\in\Lambda}$ are both Bessel sequences in $L^2(G)$ and that $(I - UT^*)E_\eta T_\lambda \gamma_1 = E_\eta T_\lambda h_2$ for $\eta \in \Gamma$, $\lambda \in \Lambda$ by [20, Lem. 5.1]. It follows that $\{E_\eta T_\lambda h_2\}_{\eta\in\Gamma,\lambda\in\Lambda}$ and $\{E_\eta T_\lambda g_2\}_{\eta\in\Gamma,\lambda\in\Lambda}$ are Bessel sequences in

$L^2(G)$. Collecting (2.9) and (2.10) gives

$$\begin{aligned} f &= TU^*f + (I - TU^*)f \\ &= \int_\Gamma \int_\Lambda \langle f, E_\eta T_\lambda h_1 \rangle E_\eta T_\lambda g_1 \, d\mu_\Lambda(\lambda) \, d\mu_\Gamma(\eta) \\ &\quad + \int_\Gamma \int_\Lambda \langle f, E_\eta T_\lambda h_2 \rangle E_\eta T_\lambda g_2 \, d\mu_\Lambda(\lambda) \, d\mu_\Gamma(\eta) \end{aligned}$$

for $f \in L^2(G)$. Therefore,

$$\{E_\eta T_\lambda g_1\}_{\eta\in\Gamma,\lambda\in\Lambda} \cup \{E_\eta T_\lambda g_2\}_{\eta\in\Gamma,\lambda\in\Lambda} \quad \text{and} \quad \{E_\eta T_\lambda h_1\}_{\eta\in\Gamma,\lambda\in\Lambda} \cup \{E_\eta T_\lambda h_2\}_{\eta\in\Gamma,\lambda\in\Lambda}$$

are a pair of dual frames for $L^2(G)$.

Next we prove that g_2, h_2 can be chosen to have compact support if $\operatorname{supp}(g_1)$ and $\operatorname{supp}(h_1)$ are compact. Choose γ_1, γ_2 in (2.11) such that $\operatorname{supp}(\gamma_1)$ and $\operatorname{supp}(\gamma_2)$ are compact (this can be done by Theorem 2.1 and Remark 2.2(i)). We only need to prove that $\operatorname{supp}(h_2)$ is compact. By (2.11), we have

$$\begin{aligned} h_2(x) &= (I - UT^*)\gamma_1(x) \\ &= \gamma_1(x) - UT^*\gamma_1(x) \\ &= \gamma_1(x) - \int_\Gamma \int_\Lambda \langle \gamma_1, E_\eta T_\lambda g_1 \rangle E_\eta T_\lambda h_1(x) \, d\mu_\Lambda(\lambda) \, d\mu_\Gamma(\eta). \end{aligned}$$

Since $\operatorname{supp}(\gamma_1)$ and $\operatorname{supp}(g_1)$ are compact, there exists a compact subset $\widetilde{\Lambda}$ of Λ such that

$$\langle \gamma_1, E_\eta T_\lambda g_1 \rangle = 0 \quad \text{for } \lambda \in \Lambda \backslash \widetilde{\Lambda} \text{ and } \eta \in \Gamma.$$

This implies that

$$\begin{aligned} h_2(x) &= \gamma_1(x) - \int_{\widetilde{\Lambda}} \int_\Gamma \langle \gamma_1, E_\eta T_\lambda g_1 \rangle E_\eta T_\lambda h_1(x) \, d\mu_\Lambda(\lambda) \, d\mu_\Gamma(\eta) \\ &= \gamma_1(x) - \int_{\widetilde{\Lambda}} \left(\int_\Gamma \langle \gamma_1, E_\eta T_\lambda g_1 \rangle E_\eta(x) \, d\mu_\Gamma(\eta) \right) T_\lambda h_1(x) \, d\mu_\Lambda(\lambda) \\ &= \gamma_1(x) - \xi(x), \end{aligned} \tag{2.12}$$

where

$$\xi(x) = \int_{\widetilde{\Lambda}} \left(\int_\Gamma \langle \gamma_1, E_\eta T_\lambda g_1 \rangle E_\eta(x) \, d\mu_\Gamma(\eta) \right) T_\lambda h_1(x) \, d\mu_\Lambda(\lambda). \tag{2.13}$$

Observe that

$$\begin{aligned} \operatorname{supp}(\xi) &\subset \left\{ x \in G : x - \lambda \in \operatorname{supp}(h_1) \text{ for some } \lambda \in \widetilde{\Lambda} \right\} \\ &\subset \widetilde{\Lambda} + \operatorname{supp}(h_1). \end{aligned}$$

It follows that $\mathrm{supp}(\xi)$ is compact due to $\mathrm{supp}(h_1)$ being compact. Thus $\mathrm{supp}(h_2)$ is compact by (2.12). $\square$

§3. Applications to co-compact Gabor systems in $L^2(\mathbb{R}^d)$

This section focuses on the application of Theorems 2.1 and 2.3 to $L^2(\mathbb{R}^d)$. Applying Theorem 2.1, we derive a strategy to construct co-compact Gabor frames for $L^2(\mathbb{R}^d)$ generated by $C_c^\infty(\mathbb{R}^d)$-window functions (see Theorem 3.1). Then, with the help of Theorems 3.1 and 2.3, we establish a dual extension theorem from a pair of co-compact Gabor Bessel sequences to a pair of dual co-compact Gabor frames for $L^2(\mathbb{R}^d)$ with $C_c^\infty(\mathbb{R}^d)$-window functions (see Theorem 3.2).

Herein, $\mathbb{R}^d$ is an LCA group equipped with the usual addition, topology and Lebesgue measure as its Haar measure. Firstly, we fix related measures such that Weil's theorem holds. Recall that $\widehat{\mathbb{R}^d} = \mathbb{R}^d$, and that an arbitrary co-compact subgroup of $\mathbb{R}^d$ has the form $\mathcal{C}(\mathbb{R}^s \times \mathbb{Z}^{d-s})$ or $\mathcal{C}(\mathbb{Z}^s \times \mathbb{R}^{d-s})$ with $0 \le s \le d$ and $\mathcal{C}$ being a $d \times d$ invertible real matrix. In what follows, we always use $|E|$ to denote the Lebesgue measure of a Lebesgue measurable set E regardless of its dimension, and use μ to denote the counting measure. Let $\Theta = \mathcal{C}(\mathbb{R}^s \times \mathbb{Z}^{d-s})$ be a co-compact subgroup of $\mathbb{R}^d$. When $0 < s < d$, define $P_1 \colon \mathbb{R}^d \to \mathbb{R}^s$ and $P_2 \colon \mathbb{R}^d \to \mathbb{R}^{d-s}$ by

$$P_1 \begin{pmatrix} x \\ y \end{pmatrix} = x \quad \text{and} \quad P_2 \begin{pmatrix} x \\ y \end{pmatrix} = y \quad \text{for } x \in \mathbb{R}^s,\ y \in \mathbb{R}^{d-s}. \tag{3.1}$$

For arbitrary measurable sets $E \subset \mathbb{R}^d/\Theta$ and $F \subset \Theta$, define

$$\mu_{\mathbb{R}^d/\Theta}(E) = \begin{cases} |E| & \text{if } s = 0, \\ |\det \mathcal{C}|\mu(E) & \text{if } s = d, \\ |\det \mathcal{C}|\,|P_2(\mathcal{C}^{-1}E)| & \text{if } 0 < s < d, \end{cases} \tag{3.2}$$

and

$$\mu_\Theta(F) = \begin{cases} \mu(\mathcal{C}^{-1}F) & \text{if } s = 0, \\ |\mathcal{C}^{-1}F| & \text{if } s = d, \\ |P_1(\mathcal{C}^{-1}F)|\mu(P_2(\mathcal{C}^{-1}F)) & \text{if } 0 < s < d. \end{cases} \tag{3.3}$$

Similarly, if $\Theta = \mathcal{C}(\mathbb{Z}^s \times \mathbb{R}^{d-s})$ is a co-compact subgroup of $\mathbb{R}^d$, define

$$\mu_{\mathbb{R}^d/\Theta}(E) = \begin{cases} |\det \mathcal{C}|\mu(E) & \text{if } s = 0, \\ |E| & \text{if } s = d, \\ |\det \mathcal{C}|\,|P_1(\mathcal{C}^{-1}E)| & \text{if } 0 < s < d, \end{cases}$$

and

$$\mu_\Theta(F) = \begin{cases} |\mathcal{C}^{-1}F| & \text{if } s = 0, \\ \mu(\mathcal{C}^{-1}F) & \text{if } s = d, \\ \mu(P_1(\mathcal{C}^{-1}F))|P_2(\mathcal{C}^{-1}F)| & \text{if } 0 < s < d, \end{cases}$$

for arbitrary measurable sets $E \subset \mathbb{R}^d/\Theta$ and $F \subset \Theta$. In the above two cases, Weil's theorem holds, i.e.,

$$\int_{\mathbb{R}^d} f(z)\,dz = \int_{\mathbb{R}^d/\Theta} d\mu_{\mathbb{R}^d/\Theta}(\dot{z}) \int_\Theta f(z+\xi)\,d\mu_\Theta(\xi) \quad \text{for } f \in L^1(\mathbb{R}^d),$$

and

$$s(\Theta) = |\det \mathcal{C}| \quad \text{for } 0 \le s \le d.$$

With the above preparations, next we prove Example 1.2.

Proof of Example 1.2. By (3.2) and (3.3), for arbitrary measurable sets $F \subset \Lambda$ and $E \subset \mathbb{R}^d/\Lambda$, we have

$$\text{(3.4)} \quad \mu_\Lambda(F) = |P_1(\mathcal{A}^{-1}F)|\mu(P_2(\mathcal{A}^{-1}F)) \quad \text{and} \quad \mu_{\mathbb{R}^d/\Lambda}(E) = |\det \mathcal{A}|\,|P_2(\mathcal{A}^{-1}E)|.$$

Arbitrarily fix $0 < \delta \le 1 - \max\{\zeta_1, \zeta_2, \ldots, \zeta_{d-s}\}$. By (1.6), we have

$$A_{2,2}^{-1}B_{2,2}^\sharp[0,1)^{d-s} \cap ([1-\delta,1)^{d-s} + \mathbb{Z}^{d-s}) = \emptyset.$$

This implies that

$$P_2\big(\mathcal{A}^{-1}((\Omega - x)\cap\Lambda)\big) \subset (A_{2,2}^{-1}B_{2,2}^\sharp[0,1)^{d-s} - P_2(\mathcal{A}^{-1}x)) \cap \mathbb{Z}^{d-s} = \emptyset,$$

and thus

$$\text{(3.5)} \quad \mu_\Lambda[(\Omega - x)\cap\Lambda] = \big|P_1\big(\mathcal{A}^{-1}((\Omega - x)\cap\Lambda)\big)\big|\mu\big[P_2\big(\mathcal{A}^{-1}((\Omega - x)\cap\Lambda)\big)\big] = 0$$

for $x \in \mathcal{A}(\{0\}^s \times [1-\delta,1)^{d-s})$. Therefore, (1.8) holds due to

$$\mu_{\mathbb{R}^d/\Lambda}[\mathcal{A}(\{0\}^s \times [1-\delta,1)^{d-s})] = |\det \mathcal{A}| \cdot \delta^{d-s} > 0.$$

Next we prove that $\{E_\eta T_\lambda g\}_{\eta\in\Gamma,\lambda\in\Lambda}$ is incomplete in $L^2(\mathbb{R}^d)$ for each $g \in L^2(\mathbb{R}^d)$ with $\operatorname{supp}(g) \subset \Omega$. Arbitrarily fix $g \in L^2(\mathbb{R}^d)$ with $\operatorname{supp}(g) \subset \Omega$. Write

$$\Delta = A_{1,2}[0,1)^{d-s} \times (A_{2,2}[0,1)^{d-s} \backslash B_{2,2}^\sharp[0,1)^{d-s}).$$

Then

$$|\Delta| = |A_{1,2}[0,1)^{d-s}|\,|\det A_{2,2}|\,|[0,1)^{d-s} \backslash A_{2,2}^{-1}B_{2,2}^\sharp[0,1)^{d-s}| > 0$$

due to $\operatorname{rank}(A_{1,2}) = s$ and (1.6). Observe that

$$\big(([0,1)^{d-s} \backslash A_{2,2}^{-1}B_{2,2}^\sharp[0,1)^{d-s}) + k\big) \cap A_{2,2}^{-1}B_{2,2}^\sharp[0,1)^{d-s} = \emptyset \quad \text{for each } k \in \mathbb{Z}^{d-s}$$

by (1.6). It follows that

$$(\Delta+\lambda)\cap\Omega=\emptyset \quad \text{for each } \lambda\in\Lambda,$$

and thus

$$\langle E_\eta T_\lambda g,\chi_\Delta\rangle=0 \quad \text{for each } \eta\in\Gamma \text{ and } \lambda\in\Lambda.$$

Therefore, $\{E_\eta T_\lambda g\}_{\eta\in\Gamma,\lambda\in\Lambda}$ is incomplete in $L^2(\mathbb{R}^d)$. The proof is completed. $\square$

Now we turn to the application of Theorems 2.1 and 2.3 to $L^2(\mathbb{R}^d)$. Given an LCA group G, by Theorem 2.1 and Remark 2.2(i), we can construct co-compact frames $\{E_\eta T_\lambda g\}_{\eta\in\Gamma,\lambda\in\Lambda}$ for $L^2(G)$ with $g\in L^\infty(G)$ and $\mathrm{supp}(g)$ being compact. For $G=\mathbb{R}^d$, the following Theorem 3.1 presents a method to construct co-compact Gabor frames with $C_c^\infty(\mathbb{R}^d)$-window functions by choosing special $K\subset\Omega$ satisfying (1.4). Specifically, given co-compact subgroups $\Lambda=\mathcal{A}(\mathbb{R}^{s_1}\times\mathbb{Z}^{d-s_1})$ and $\Gamma=\mathcal{B}(\mathbb{R}^{s_2}\times\mathbb{Z}^{d-s_2})$ ($\Gamma=\mathcal{B}(\mathbb{Z}^{s_2}\times\mathbb{R}^{d-s_2})$) of $\mathbb{R}^d$ with $0\le s_1,s_2\le d$ and $\mathcal{A}$, $\mathcal{B}$ being $d\times d$ invertible real matrices, and let $Q=\mathcal{A}(\{0\}^{s_1}\times([0,1]^{d-s_1}+x^{(1)}))$ and $\Omega=\mathcal{B}^\sharp(\mathbb{R}^{s_2}\times((0,1)^{d-s_2}+x^{(2)}))$ ($\Omega=\mathcal{B}^\sharp(((0,1)^{s_2}+x^{(2)})\times\mathbb{R}^{d-s_2})$) be Borel sections for Λ and $\Gamma^\perp$, respectively. Take $K=\mathcal{A}(E_{s_1}\times([0,1]^{d-s_1}+x^{(1)}))\subset\Omega$ for some compact set E_{s_1} in $\mathbb{R}^{s_1}$ with positive measure. Then

$$\operatorname*{ess\,inf}_{x\in Q}\mu_\Lambda[(K-x)\cap\Lambda]=\begin{cases}1 & \text{if } s_1=0,\\ |E_{s_1}| & \text{if } 0<s_1\le d.\end{cases} \tag{3.6}$$

We also claim that

$$\operatorname*{ess\,sup}_{x\in Q}\mu_\Lambda[(W-x)\cap\Lambda]<\infty \quad \text{for bounded and measurable } W\subset\mathbb{R}^d. \tag{3.7}$$

Indeed, without loss of generality, we assume that

$$W\subset\mathcal{A}([a_1,b_1]^{s_1}\times([a_2,b_2]^{d-s_1}+x^{(1)}))$$

for some $a_1,b_1,a_2,b_2\in\mathbb{R}$. Then

$$W-x\subset\mathcal{A}([a_1,b_1]^{s_1}\times[a_2-1,b_2]^{d-s_1}) \quad \text{for } x\in Q. \tag{3.8}$$

By a simple computation, we have

$$\mu_\Lambda(\mathcal{A}([a_1,b_1]^{s_1}\times[a_2-1,b_2]^{d-s_1})\cap\Lambda)\le(b_1-a_1)^{s_1}(b_2-a_2+2)^{d-s_1}. \tag{3.9}$$

This together with (3.8) leads to (3.7). Since K is a compact subset of the open set Ω, we can always choose $g\in C_c^\infty(\mathbb{R}^d)$ such that

$$K\subset\mathrm{supp}(g)\subset\Omega \quad \text{and} \quad g|_K=1.$$

Thus (1.4) and (2.2) hold by (3.6) and (3.7). Applying Theorem 2.1, we have the following theorem.

Theorem 3.1. *Given co-compact subgroups*

$$\Lambda = \mathcal{A}(\mathbb{R}^{s_1} \times \mathbb{Z}^{d-s_1}) \quad \textit{and} \quad \Gamma = \mathcal{B}(\mathbb{R}^{s_2} \times \mathbb{Z}^{d-s_2}) \quad (\Gamma = \mathcal{B}(\mathbb{Z}^{s_2} \times \mathbb{R}^{d-s_2}))$$

of $\mathbb{R}^d$ *with* $0 \leq s_1, s_2 \leq d$ *and* $\mathcal{A}$, $\mathcal{B}$ *being* $d \times d$ *invertible real matrices, let* $E_{s_1} \times ([0,1]^{d-s_1} + x^{(1)})$ *be a compact set in* $\mathbb{R}^d$ *with positive measure satisfying*

$$\mathcal{A}(E_{s_1} \times ([0,1]^{d-s_1} + x^{(1)})) \subset \mathcal{B}^\sharp\big(\mathbb{R}^{s_2} \times ((0,1)^{d-s_2} + x^{(2)})\big) \tag{3.10}$$

$$\big(\mathcal{A}(E_{s_1} \times ([0,1]^{d-s_1} + x^{(1)})) \subset \mathcal{B}^\sharp(((0,1)^{s_2} + x^{(2)}) \times \mathbb{R}^{d-s_2})\big) \tag{3.11}$$

for some $x^{(2)} \in \mathbb{R}^{d-s_2}$ $(x^{(2)} \in \mathbb{R}^{s_2})$*. Choose* $g \in C_c^\infty(\mathbb{R}^d)$ *such that*

$$\mathcal{A}(E_{s_1} \times ([0,1]^{d-s_1} + x^{(1)})) \subset \operatorname{supp}(g) \subset \mathcal{B}^\sharp\big(\mathbb{R}^{s_2} \times ((0,1)^{d-s_2} + x^{(2)})\big)$$
$$\big(\mathcal{A}(E_{s_1} \times ([0,1]^{d-s_1} + x^{(1)})) \subset \operatorname{supp}(g) \subset \mathcal{B}^\sharp\big(((0,1)^{s_2} + x^{(2)}) \times \mathbb{R}^{d-s_2}\big)\big)$$

and

$$g(\cdot) = 1 \quad \textit{on } \mathcal{A}(E_{s_1} \times ([0,1]^{d-s_1} + x^{(1)})).$$

Then $\{E_\eta T_\lambda g\}_{\eta\in\Gamma,\lambda\in\Lambda}$ *is a frame for* $L^2(\mathbb{R}^d)$*, and the frame operator and its inverse are given by*

$$Sf = \frac{\widetilde{G}}{|\det \mathcal{B}|} f, \quad S^{-1} f = \frac{|\det \mathcal{B}|}{\widetilde{G}} f \quad \textit{for } f \in L^2(\mathbb{R}^d), \tag{3.12}$$

where

$$\widetilde{G}(y) = \begin{cases} \displaystyle\sum_{k\in\mathbb{Z}^{d-s_1}} \int_{\mathbb{R}^{s_1}} \left| g\left(y + \mathcal{A}\begin{pmatrix}\lambda \\ k\end{pmatrix}\right)\right|^2 d\lambda & \textit{if } 0 < s_1 < d, \\ \displaystyle\sum_{k\in\mathbb{Z}^d} |g(y+\mathcal{A}k)|^2 & \textit{if } s_1 = 0, \\ \displaystyle\int_{\mathbb{R}^d} |g(y+\mathcal{A}\lambda)|^2 d\lambda & \textit{if } s_1 = d, \end{cases}$$

for a.e. $y \in \mathbb{R}^d$.

Remark 3.1. We have the following supplementary explanations for Theorem 3.1:

(i) If g in Theorem 3.1 is required to be real valued in addition, then $\widetilde{G} \in C^\infty(\mathbb{R}^d)$ by a standard argument. Thus $S^{-1}g \in C_c^\infty(\mathbb{R}^d)$ by (3.12), and

$$(\{E_\eta T_\lambda g\}_{\eta\in\Gamma,\lambda\in\Lambda}, \{E_\eta T_\lambda S^{-1} g\}_{\eta\in\Gamma,\lambda\in\Lambda})$$

is a pair of dual frames for $L^2(\mathbb{R}^d)$ with g, $S^{-1}g \in C_c^\infty(\mathbb{R}^d)$.

(ii) Theorem 3.1 can be adjusted to the case of (Λ, Γ) such that $\Lambda = \mathcal{A}(\mathbb{Z}^{d-s_1} \times \mathbb{R}^{s_1})$ and $\Gamma = \mathcal{B}(\mathbb{Z}^{d-s_2} \times \mathbb{R}^{s_2})$ $(\Gamma = \mathcal{B}(\mathbb{R}^{d-s_2} \times \mathbb{Z}^{s_2}))$ with $0 \leq s_1, s_2 \leq d$ and $\mathcal{A}$, $\mathcal{B}$

being $d \times d$ invertible real matrices. Indeed, for $0 \le s \le d$, define the $d \times d$ permutation matrix $\mathcal{P}_s$ by

$$\mathcal{P}_s = \begin{pmatrix} 0 & I_{d-s} \\ I_s & 0 \end{pmatrix}, \tag{3.13}$$

where I_r denotes the $r \times r$ identity matrix. We can do this if $\mathcal{A}$ and $\mathcal{B}$ in Theorem 3.1 are replaced by $\mathcal{A} = \mathcal{A}\mathcal{P}_{s_1}$ and $\mathcal{B} = \mathcal{B}\mathcal{P}_{s_2}$.

For the lattice case of $\Lambda = \mathcal{A}\mathbb{Z}^d$ and $\Gamma = \mathcal{B}\mathbb{Z}^d$, [16, Thm. 1.3] shows that the existence of Riesz bases (frames) of the form $\{E_\eta T_\lambda g\}_{\eta\in\Gamma,\lambda\in\Lambda}$ is equivalent to $|\det\mathcal{A}|\,|\det\mathcal{B}| = 1$ ($|\det\mathcal{A}|\,|\det\mathcal{B}| \le 1$), i.e., $s(\Lambda)s(\Gamma) = 1$ ($s(\Lambda)s(\Gamma) \le 1$). It is easy to check that $|\det\mathcal{A}|\,|\det\mathcal{B}| < 1$ if $\Lambda = \mathcal{A}\mathbb{Z}^d$ and $\Gamma = \mathcal{B}\mathbb{Z}^d$ in Theorem 3.1. Thus Theorem 3.1 gives a method to construct redundant frames $\{E_\eta T_\lambda g\}_{\eta\in\Gamma,\lambda\in\Lambda}$ for $L^2(\mathbb{R}^d)$ with $g \in C_c^\infty(\mathbb{R}^d)$. The following example shows that $s(\Lambda)s(\Gamma)$ (i.e., $|\det\mathcal{A}|\,|\det\mathcal{B}|$) can take an arbitrary positive number for general co-compact (i.e., at least one of Λ and Γ is not a lattice) Gabor frames in $L^2(\mathbb{R}^d)$, and in this case the window functions can be chosen in $C_c^\infty(\mathbb{R}^d)$. This demonstrates that there exist essential differences between lattice-based Gabor frames and general co-compact Gabor frames. For convenience, write

$$\begin{aligned}
A_1(s_1,s_2) &= \big\{(\Lambda,\Gamma) : \Lambda = \mathcal{A}(\mathbb{R}^{s_1}\times\mathbb{Z}^{d-s_1}),\ \Gamma = \mathcal{B}(\mathbb{R}^{s_2}\times\mathbb{Z}^{d-s_2})\big\},\\
A_2(s_1,s_2) &= \big\{(\Lambda,\Gamma) : \Lambda = \mathcal{A}(\mathbb{R}^{s_1}\times\mathbb{Z}^{d-s_1}),\ \Gamma = \mathcal{B}(\mathbb{Z}^{s_2}\times\mathbb{R}^{d-s_2})\big\},\\
A_3(s_1,s_2) &= \big\{(\Lambda,\Gamma) : \Lambda = \mathcal{A}(\mathbb{Z}^{s_1}\times\mathbb{R}^{d-s_1}),\ \Gamma = \mathcal{B}(\mathbb{R}^{s_2}\times\mathbb{Z}^{d-s_2})\big\},\\
A_4(s_1,s_2) &= \big\{(\Lambda,\Gamma) : \Lambda = \mathcal{A}(\mathbb{Z}^{s_1}\times\mathbb{R}^{d-s_1}),\ \Gamma = \mathcal{B}(\mathbb{Z}^{s_2}\times\mathbb{R}^{d-s_2})\big\}
\end{aligned}$$

for $0 \le s_1, s_2 \le d$.

Example 3.1. Given an arbitrary positive constant ϱ, and $0 \le s_1, s_2 \le d$ satisfying $(s_1,s_2) \ne (0,0)$ ($(0,d),(d,0),(d,d)$), there exist $g \in C_c^\infty(\mathbb{R}^d)$ and

$$(\Lambda,\Gamma) \in A_1(s_1,s_2) \quad (A_2(s_1,s_2),\ A_3(s_1,s_2),\ A_4(s_1,s_2)),$$

such that $s(\Lambda)s(\Gamma) = \varrho$ and $\{E_\eta T_\lambda g\}_{\eta\in\Gamma,\lambda\in\Lambda}$ is a frame for $L^2(\mathbb{R}^d)$.

Proof. We only treat the cases of $(s_1,s_2) \ne (0,0)$ and $(s_1,s_2) \ne (0,d)$. The others can be proved similarly. By Theorem 3.1, it is enough to show the existence of $\mathcal{A}$, $\mathcal{B}$ and E_{s_1} satisfying $|\det\mathcal{A}|\,|\det\mathcal{B}| = \varrho$ and (3.10) ((3.11)) with $x^{(2)} = 0$.

Case 1*:* $(s_1,s_2) \ne (0,0)$. If $s_1 = 0$ and $s_2 = d$, then (3.10) holds for all $\mathcal{A}$ and $\mathcal{B}$ satisfying $|\det\mathcal{A}|\,|\det\mathcal{B}| = \varrho$ (i.e., $s(\Lambda)s(\Gamma) = \varrho$). If $s_1 = 0$ and $0 < s_2 < d$, take $x^{(1)}$ such that its every component is positive, and choose $a_{s_2+1}, \dots, a_d$ small enough that

$$\begin{pmatrix} a_{s_2+1} & & \\ & \ddots & \\ & & a_d \end{pmatrix} ([0,1]^{d-s_2} + x^{(1)}) \subset (0,1)^{d-s_2}.$$

Choose $a_1, \ldots, a_{s_2} > 0$, and $\mathcal{A}$, $\mathcal{B}$ such that

$$\mathcal{B}^{\mathsf{T}}\mathcal{A} = \operatorname{diag}(a_1 \cdots a_{s_2} a_{s_2+1} \cdots a_d) \quad \text{and} \quad a_1 a_2 \cdots a_d = \varrho.$$

Then (3.10) holds and $|\det \mathcal{A}|\,|\det \mathcal{B}| = \varrho$ (i.e., $s(\Lambda)s(\Gamma) = \varrho$). If $s_1 = d$, choose $\mathcal{A}$, $\mathcal{B}$ satisfying $|\det \mathcal{A}|\,|\det \mathcal{B}| = \varrho$ (i.e., $s(\Lambda)s(\Gamma) = \varrho$) and a compact set E_d in $\mathbb{R}^d$ such that $\mathcal{B}^{\mathsf{T}}\mathcal{A}E_d \subset (0,1)^d$. Then

$$\mathcal{A}E_d \subset \mathcal{B}^{\sharp}(0,1)^d \subset \mathcal{B}^{\sharp}(\mathbb{R}^{s_2} \times (0,1)^{d-s_2}).$$

Thus (3.10) holds and $|\det \mathcal{A}|\,|\det \mathcal{B}| = \varrho$ (i.e., $s(\Lambda)s(\Gamma) = \varrho$). If $0 < s_1 < d$, take $x^{(1)}$ such that its every component is positive, and choose $a_{s_1+1}, \ldots, a_d$ small enough that

$$\begin{pmatrix} a_{s_1+1} & & \\ & \ddots & \\ & & a_d \end{pmatrix} ([0,1]^{d-s_1} + x^{(1)}) \subset (0,1)^{d-s_1}.$$

Choose $a_1, \ldots, a_{s_1} > 0$, and $\mathcal{A}$, $\mathcal{B}$ such that

$$\mathcal{B}^{\mathsf{T}}\mathcal{A} = \operatorname{diag}(a_1 \cdots a_{s_1} a_{s_1+1} \cdots a_d) \quad \text{and} \quad a_1 a_2 \cdots a_d = \varrho.$$

Take a compact set E_{s_1} in $\mathbb{R}^{s_1}$ such that $\operatorname{diag}(a_1 \cdots a_{s_1})E_{s_1} \subset (0,1)^{s_1}$. Then

$$\mathcal{A}(E_{s_1} \times ([0,1]^{d-s_1} + x^{(1)})) \subset \mathcal{B}^{\sharp}(0,1)^d \subset \mathcal{B}^{\sharp}(\mathbb{R}^{s_2} \times (0,1)^{d-s_2}).$$

Thus (3.10) holds and $|\det \mathcal{A}|\,|\det \mathcal{B}| = \varrho$ (i.e., $s(\Lambda)s(\Gamma) = \varrho$).

Case 2: $(s_1, s_2) \neq (0, d)$. If $0 < s_1 \le d$, then (3.11) holds when choosing $\mathcal{A}$, $\mathcal{B}$ and E_{s_1} as in the "$0 < s_1 \le d$" case of Case 1. If $s_1 = s_2 = 0$, then (3.11) holds for all $\mathcal{A}$ and $\mathcal{B}$ satisfying $|\det \mathcal{A}|\,|\det \mathcal{B}| = \varrho$ (i.e., $s(\Lambda)s(\Gamma) = \varrho$). If $s_1 = 0$ and $0 < s_2 < d$, take $x^{(1)}$ such that its every component is positive, and choose $a_1, \ldots, a_{s_2}$ small enough that

$$\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_{s_2} \end{pmatrix} ([0,1]^{s_2} + x^{(1)}) \subset (0,1)^{s_2}.$$

Choose $a_{s_2+1}, \ldots, a_d > 0$, and $\mathcal{A}$, $\mathcal{B}$ such that

$$\mathcal{B}^{\mathsf{T}}\mathcal{A} = \operatorname{diag}(a_1 \cdots a_{s_2} a_{s_2+1} \cdots a_d) \quad \text{and} \quad a_1 a_2 \cdots a_d = \varrho.$$

Then (3.11) holds and $|\det \mathcal{A}||\det \mathcal{B}| = \varrho$ (i.e., $s(\Lambda)s(\Gamma) = \varrho$). The proof is completed. $\square$

Given an LCA group G, Theorem 2.3 shows that, under the hypothesis of (1.4), a pair of co-compact Gabor Bessel sequences in $L^2(G)$ can be extended to a pair of dual co-compact Gabor frames for $L^2(G)$, and simultaneously, the added window functions can be chosen to have compact support if the initial ones are of compact support. For $G = \mathbb{R}^d$, as an application of Theorem 2.3, the following theorem shows that a pair of $C_c^\infty(\mathbb{R}^d)$-window-function-generated co-compact Gabor Bessel sequences in $L^2(\mathbb{R}^d)$ can be extended to a pair of dual co-compact Gabor frames for $L^2(\mathbb{R}^d)$ with the added window functions belonging to $C_c^\infty(\mathbb{R}^d)$.

Theorem 3.2. *Given $d > 1$, and co-compact subgroups $\Lambda = \mathcal{A}(\mathbb{R}^{s_1} \times \mathbb{Z}^{d-s_1})$ ($\Lambda = \mathcal{A}(\mathbb{Z}^{s_1} \times \mathbb{R}^{d-s_1})$) and $\Gamma = \mathcal{B}(\mathbb{R}^{s_2} \times \mathbb{Z}^{d-s_2})$ of $\mathbb{R}^d$ with $0 \le s_1, s_2 \le d$ and $\mathcal{A}$, $\mathcal{B}$ being $d \times d$ invertible real matrices, let*

$$\mathcal{A}(E_{s_1} \times ([0,1]^{d-s_1} + x^{(1)})) \subset \mathcal{B}^\sharp\big(\mathbb{R}^{s_2} \times ((0,1)^{d-s_2} + x^{(2)})\big)$$
$$\big(\mathcal{A}(([0,1]^{s_1} + x^{(1)}) \times E_{d-s_1}) \subset \mathcal{B}^\sharp\big(\mathbb{R}^{s_2} \times ((0,1)^{d-s_2} + x^{(2)})\big)\big)$$

for some $x^{(2)} \in \mathbb{R}^{d-s_2}$ and compact set $E_{s_1} \times ([0,1]^{d-s_1} + x^{(1)})$ $(([0,1]^{s_1} + x^{(1)}) \times E_{d-s_1})$ in $\mathbb{R}^d$ with positive measure. Assume that $\{E_\eta T_\lambda g_1\}_{\eta\in\Gamma,\lambda\in\Lambda}$ and $\{E_\eta T_\lambda h_1\}_{\eta\in\Gamma,\lambda\in\Lambda}$ are Bessel sequences in $L^2(\mathbb{R}^d)$, and that $g_1, h_1 \in C_c^\infty(\mathbb{R}^d)$. Then there exist $g_2, h_2 \in C_c^\infty(\mathbb{R}^d)$ such that

$$\{E_\eta T_\lambda g_1\}_{\eta\in\Gamma,\lambda\in\Lambda} \cup \{E_\eta T_\lambda g_2\}_{\eta\in\Gamma,\lambda\in\Lambda}$$

and

$$\{E_\eta T_\lambda h_1\}_{\eta\in\Gamma,\lambda\in\Lambda} \cup \{E_\eta T_\lambda h_2\}_{\eta\in\Gamma,\lambda\in\Lambda}$$

are a pair of dual frames for $L^2(\mathbb{R}^d)$.

Proof. Choose a pair of dual frames $(\{E_\eta T_\lambda \gamma_1\}_{\eta\in\Gamma,\lambda\in\Lambda}, \{E_\eta T_\lambda \gamma_2\}_{\eta\in\Gamma,\lambda\in\Lambda})$ for $L^2(\mathbb{R}^d)$ with $\gamma_1, \gamma_2 \in C_c^\infty(\mathbb{R}^d)$ in Theorem 2.3. This can be done by Remark 3.1(i). Define g_2 and h_2 as in (2.11). Then, by Theorem 2.3,

$$\{E_\eta T_\lambda g_1\}_{\eta\in\Gamma,\lambda\in\Lambda} \cup \{E_\eta T_\lambda g_2\}_{\eta\in\Gamma,\lambda\in\Lambda} \quad \text{and} \quad \{E_\eta T_\lambda h_1\}_{\eta\in\Gamma,\lambda\in\Lambda} \cup \{E_\eta T_\lambda h_2\}_{\eta\in\Gamma,\lambda\in\Lambda}$$

form a pair of dual frames for $L^2(\mathbb{R}^d)$ with $g_2 \in C_c^\infty(\mathbb{R}^d)$ and h_2 being compactly supported. Next we will prove that $h_2 \in C^\infty(\mathbb{R}^d)$ to finish the proof. Since $\gamma_1 \in C^\infty(\mathbb{R}^d)$, by (2.12) and (2.13), we only need to prove that $\xi \in C^\infty(\mathbb{R}^d)$, where

$$\xi(x) = \int_{\widetilde{\Lambda}} \bigg(\int_\Gamma \langle \gamma_1, E_\eta T_\lambda g_1\rangle E_\eta(x)\, d\mu_\Gamma(\eta)\bigg) T_\lambda h_1(x)\, d\mu_\Lambda(\lambda) \tag{3.14}$$

for some compact subset $\widetilde{\Lambda}$ of Λ. By a simple computation, we have

$$D^{\alpha}E_{\mathcal{B}\eta}T_{\lambda}h_1(x) = \sum_{0\leq l\leq\alpha} C_{\alpha}^{l}(2\pi i)^{|l|}(\mathcal{B}\eta)^{l}E_{\mathcal{B}\eta}(x)T_{\lambda}D^{\alpha-l}h_1(x) \tag{3.15}$$

for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{Z}_+^d$, where $l = (l_1, l_2, \ldots, l_d)$, $|l| = l_1 + l_2 + \cdots + l_d$,

$$\begin{aligned} C_{\alpha}^{l} &= C_{\alpha_1}^{l_1}C_{\alpha_2}^{l_2}\cdots C_{\alpha_d}^{l_d} \\ &= \frac{\alpha_1\cdots(\alpha_1-l_1+1)}{l_1!}\frac{\alpha_2\cdots(\alpha_2-l_2+1)}{l_2!}\cdots\frac{\alpha_d\cdots(\alpha_d-l_d+1)}{l_d!}, \quad 0\leq l\leq\alpha, \end{aligned}$$

means that $0 \leq l_i \leq \alpha_i$ with $1 \leq i \leq d$ and $z^l = z_1^{l_1}z_2^{l_2}\cdots z_d^{l_d}$ for $z = (z_1, z_2, \ldots, z_d)^{\mathsf{T}} \in \mathbb{R}^d$. Observe that

$$|(\mathcal{B}\eta)^l| \leq (1 + |\mathcal{B}\eta|)^{|\alpha|}$$

and

$$|E_{\mathcal{B}\eta}(x)T_{\lambda}D^{\alpha-l}h_1(x)| \leq \max_{0\leq l\leq\alpha} \|D^{\alpha-l}h_1\|_{\infty}$$

for $0 \leq l \leq \alpha$, where $|\xi| = |\xi_1| + |\xi_2| + \cdots + |\xi_d|$ for $\xi \in \mathbb{R}^d$. It follows that

$$\|D^{\alpha}E_{\mathcal{B}\eta}T_{\lambda}h_1\|_{\infty} \leq M_1(1 + |\mathcal{B}\eta|)^{|\alpha|} \tag{3.16}$$

for $\lambda \in \widetilde{\Lambda}$ and $\eta \in \mathbb{R}^{s_2} \times \mathbb{Z}^{d-s_2}$ by a standard argument, where $M_1 = (2\pi + 1)^{|\alpha|}\max_{0\leq l\leq\alpha}\|D^{\alpha-l}h_1\|_{\infty}$. Now let us estimate $\langle\gamma_1, E_{\mathcal{B}\eta}T_{\lambda}g_1\rangle$. Observe that $T_{\lambda}\overline{g_1} = (E_{-\lambda}\widehat{\overline{g_1}})^{\vee}$ and $\gamma_1 = (\widehat{\gamma_1})^{\vee}$. It follows that

$$\gamma_1 T_{\lambda}\overline{g_1} = (\widehat{\gamma_1} * E_{-\lambda}\widehat{\overline{g_1}})^{\vee}.$$

This implies that

$$\begin{aligned} \langle\gamma_1, E_{\mathcal{B}\eta}T_{\lambda}g_1\rangle &= (\gamma_1 T_{\lambda}\overline{g_1})^{\wedge}(\mathcal{B}\eta) \\ &= \widehat{\gamma_1} * E_{-\lambda}\widehat{\overline{g_1}}(\mathcal{B}\eta), \end{aligned}$$

and thus

$$|\langle\gamma_1, E_{\mathcal{B}\eta}T_{\lambda}g_1\rangle| \leq |\widehat{\gamma_1}| * |\widehat{\overline{g_1}}|(\mathcal{B}\eta).$$

Also, observing that $\widehat{\gamma_1}, \widehat{\overline{g_1}} \in \mathcal{S}(\mathbb{R}^d)$ leads to the fact that to every $\tau \in \mathbb{Z}_+$ there corresponds a constant C_{τ} such that

$$|\langle\gamma_1, E_{\mathcal{B}\eta}T_{\lambda}g_1\rangle| \leq \frac{C_{\tau}}{(1 + |\mathcal{B}\eta|)^{\tau}} \quad \text{for } \eta \in \mathbb{R}^d. \tag{3.17}$$

Since $\mathcal{B}$ is invertible, there exists a constant $a > 0$ such that

$$|\mathcal{B}x| \geq a|x| \quad \text{for } x \in \mathbb{R}^d. \tag{3.18}$$

Choose $\tau = |\alpha| + d + 1$ in (3.17). Then collecting (3.16)–(3.18) leads to

$$\begin{aligned}
&\sum_{\substack{\eta_{s_2+1},\dots,\\ \eta_d\in\mathbb{Z}}} \int_{\mathbb{R}^{s_2}} \|\langle \gamma_1, E_{\mathcal{B}\eta}T_\lambda g_1\rangle D^\alpha E_{\mathcal{B}\eta}T_\lambda h_1(\cdot)\|_\infty \, d\eta_1\cdots d\eta_{s_2} \\
&\quad\le M_1 C_\tau \sum_{\substack{\eta_{s_2+1},\dots,\\ \eta_d\in\mathbb{Z}}} \int_{\mathbb{R}^{s_2}} \frac{d\eta_1\cdots d\eta_{s_2}}{(1+a|\eta|)^{\tau-|\alpha|}} \\
&\quad\le M_1 C_\tau \sum_{\substack{\eta_{s_2+1},\dots,\\ \eta_d\in\mathbb{Z}}} \frac{1}{(1+a(|\eta_{s_2+1}|+\cdots+|\eta_d|))^{d-s_2+\frac12}} \\
&\qquad\qquad \times \int_{\mathbb{R}^{s_2}} \frac{d\eta_1\cdots d\eta_{s_2}}{(1+a(|\eta_1|+\cdots+|\eta_{s_2}|))^{s_2+\frac12}} \\
&\quad< \infty. \qquad (3.19)
\end{aligned}$$

This implies that

$$\int_{\widetilde{\Lambda}} \sum_{\substack{\eta_{s_2+1},\dots,\\ \eta_d\in\mathbb{Z}}} \int_{\mathbb{R}^{s_2}} \|\langle \gamma_1, E_{\mathcal{B}\eta}T_\lambda g_1\rangle D^\alpha E_{\mathcal{B}\eta}T_\lambda h_1(\cdot)\|_\infty \, d\eta_1\cdots d\eta_{s_2}\, d\mu_\Lambda(\lambda) < \infty \tag{3.20}$$

due to $\widetilde{\Lambda}$ being compact. Also observe that $D^\alpha E_{\mathcal{B}\eta}T_\lambda h_1$ is continuous for an arbitrary $\alpha \in \mathbb{Z}_+^d$. It follows that $\xi \in C^\infty(\mathbb{R}^d)$. The proof is completed. $\square$

Remark 3.2. Let $\mathcal{P}_s$ be as in (3.13). If $\mathcal{B}$ in Theorem 3.2 is replaced by $\mathcal{B}\mathcal{P}_{s_2}$, Theorem 3.2 can be adjusted to the case of $(\Lambda,\,\Gamma)$ that $\Lambda = \mathcal{A}(\mathbb{R}^{s_1}\times\mathbb{Z}^{d-s_1})$ ($\Lambda = \mathcal{A}(\mathbb{Z}^{s_1}\times\mathbb{R}^{d-s_1})$) and $\Gamma = \mathcal{B}(\mathbb{Z}^{d-s_2}\times\mathbb{R}^{s_2})$ with $0\le s_1, s_2\le d$, $\mathcal{A}$ and $\mathcal{B}$ being $d\times d$ invertible real matrices.

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