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Bicomplex Hyperfunctions and Bicomplex Microfunctions

by

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Abstract

In this paper, we study bicomplex hyperfunctions introduced by Colombo et al. (Ann. Mat. Pura Appl. 190 (2011), 247–261) with functorial techniques and prove the idempotent representation theorem for them. Using the method of this paper, we can easily reconstruct the theory of bicomplex hyperfunctions and develop it into the theory of bicomplex microfunctions.

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§1. Introduction

The notion of hyperfunctions was introduced by Mikio Sato [10, 11] as a generalization of functions. Let $V \subset \mathbb{R}^n$ be an open set and $\Omega \subset \mathbb{C}^n$ a complex neighborhood of V satisfying $V = \mathbb{R}^n \cap \Omega$. Let $\mathcal{O}_{\mathbb{C}^n}$ denote the sheaf of complex holomorphic functions on $\mathbb{C}^n$. Sato proved that the relative cohomology group $H^p_V(\Omega; \mathcal{O}_{\mathbb{C}^n})$ of $\mathcal{O}_{\mathbb{C}^n}$ supported in V vanishes if $p \neq n$ and defined a hyperfunction as an element of the relative cohomology group

$$\mathcal{B}_{\mathbb{C}^n}(V) = H^n_V(\Omega; \mathcal{O}_{\mathbb{C}^n}).$$

Intuitively, a hyperfunction is represented as a finite sum of boundary values of complex holomorphic functions. He also proved that $\mathcal{B}_{\mathbb{C}^n}$ is a flabby sheaf on $\mathbb{R}^n$. It is well known that hyperfunctions are more natural and useful than distributions in studying linear partial differential equations with real analytic coefficients. In particular, it is important to consider singularities of hyperfunctions in the cotangent bundle $T^*\mathbb{R}^n$. The notion of microfunctions has been introduced in order to

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realize it and the theory of hyperfunctions and microfunctions has vastly developed as algebraic analysis [12].

Bicomplex algebra was introduced by Segre, inspired by the work of Hamilton and Clifford on quaternions. It is defined by

$$\mathbb{BC} = \{Z = z_1 + z_2 j \mid z_1, z_2 \in \mathbb{C}\},$$

where j is another imaginary unit commuting with the imaginary unit i of the field of complex numbers $\mathbb{C}$. Since $\mathbb{BC}$ is a commutative ring but has zero divisors, it is more difficult to study bicomplex functions than complex functions. Nevertheless, we can define the notion of holomorphicity of bicomplex functions similarly to that of complex functions. The study of bicomplex holomorphic functions is called bicomplex analysis. References [3] and [7] are fundamental textbooks for this subject. See Sections 2.1 and 2.2 for a quick review. See [5] and [6] for the author's recent works in this area.

In bicomplex analysis, the idempotent representation plays an important role. Setting

$$\mathbf{e} = \frac{1+ij}{2}, \quad \mathbf{e}^\dagger = \frac{1-ij}{2},$$

$\mathbf{e}$ and $\mathbf{e}^\dagger$ are the non-complex idempotent elements satisfying $\mathbf{e}\mathbf{e}^\dagger = 0$. Then any bicomplex number $Z = z_1 + z_2 j \in \mathbb{BC}$ has the idempotent representation

$$Z = Z_{\mathbf{e}}\mathbf{e} + Z_{\mathbf{e}^\dagger}\mathbf{e}^\dagger,$$

where we set $Z_{\mathbf{e}} = z_1 - z_2 i$, $Z_{\mathbf{e}^\dagger} = z_1 + z_2 i \in \mathbb{C}$. By the idempotent representation, for any bicomplex holomorphic function F, there exist locally two complex holomorphic functions $F_{\mathbf{e}}$ and $F_{\mathbf{e}^\dagger}$ with respect to only the variables $Z_{\mathbf{e}}$ and $Z_{\mathbf{e}^\dagger}$ respectively such that we have

$$F(Z_{\mathbf{e}}\mathbf{e} + Z_{\mathbf{e}^\dagger}\mathbf{e}^\dagger) = F_{\mathbf{e}}(Z_{\mathbf{e}})\mathbf{e} + F_{\mathbf{e}^\dagger}(Z_{\mathbf{e}^\dagger})\mathbf{e}^\dagger. \tag{1.1}$$

The idempotent representation (1.1) is one of the strongest properties of bicomplex holomorphic functions. After the idempotent representation (1.1), we can immediately obtain several fundamental properties of bicomplex holomorphic functions by applying properties of complex holomorphic functions to coefficient functions $F_{\mathbf{e}}$ and $F_{\mathbf{e}^\dagger}$.

The notion of bicomplex hyperfunctions was introduced by Colombo et al. [1] as a natural generalization of classical hyperfunctions to bicomplex analysis. Let $V \subset \mathbb{R}^n$ be an open set and $\Omega \subset \mathbb{BC}^n$ a bicomplex neighborhood of V satisfying $V = \mathbb{R}^n \cap \Omega$. Let $\mathcal{O}_{\mathbb{BC}^n}$ denote the sheaf of bicomplex holomorphic functions on $\mathbb{BC}^n$. Colombo et al. proved that the relative cohomology group $H^p_V(\Omega; \mathcal{O}_{\mathbb{BC}^n})$ of

$\mathcal{O}_{\mathbb{BC}^n}$ supported in V vanishes if $p \neq 3n$ and defined a bicomplex hyperfunction as an element of the relative cohomology group

$$\mathcal{B}_{\mathbb{BC}^n}(V) = H^{3n}_V(\Omega; \mathcal{O}_{\mathbb{BC}^n}).$$

They also proved that $\mathcal{B}_{\mathbb{BC}^n}$ is a flabby sheaf on $\mathbb{R}^n$ and the duality theorem. See also [9, 14] for further developments.

In this paper, we prove the idempotent representation theorem for bicomplex hyperfunctions. In other words, we prove that a bicomplex hyperfunction corresponds to a pair of two complex hyperfunctions. As a corollary of this principle, we can immediately obtain several fundamental properties of bicomplex hyperfunctions, such as flabbiness and some fundamental operations. Note that via the Čech cohomology it is not easy to compare two cohomology groups $H^{3n}_V(\Omega; \mathcal{O}_{\mathbb{BC}^n})$ and $H^n_V(\Omega; \mathcal{O}_{\mathbb{C}^n})$ directly. The proof is based on the idempotent representation (1.1) of bicomplex holomorphic functions and functorial techniques in the theory of sheaves. By our method, we can easily reconstruct the theory of bicomplex hyperfunctions in [1] and develop it into the theory of bicomplex microfunctions. See Sections 3, 4 and 5 for the details.

Similarly, we can also develop the results into the theory of multicomplex hyperfunctions introduced in [14], and furthermore that on real analytic manifolds. We can define the notion of multicomplex hyperfunctions of several variables, prove the idempotent representation theorem for them and develop it into the theory of multicomplex microfunctions. See [4] for the details.

§2. Preliminaries

§2.1. Bicomplex numbers

In this subsection, we review the definition and fundamental properties of bicomplex numbers. See [3, 7] for more details.

Let $\mathbb{C}$ be the field of complex numbers with the imaginary unit i. We define the set of bicomplex numbers by

$$\begin{aligned}\mathbb{BC} &= \{Z = z_1 + z_2 j \mid z_1, z_2 \in \mathbb{C}\}\\ &= \{Z = x_1 + y_1 i + x_2 j + y_2 ij \mid x_1, y_1, x_2, y_2 \in \mathbb{R}\},\end{aligned}$$

where j is another imaginary unit independent of and commuting with i:

$$j \notin \mathbb{C}, \quad ij = ji, \quad i^2 = j^2 = -1.$$

By defining addition and multiplication naturally, $\mathbb{BC}$ has a structure of a commutative ring. The ring $\mathbb{BC}$ is neither a field nor an integral domain. The set of zero divisors of $\mathbb{BC}$ with 0 is described as

$$\mathfrak{S}_0 = \{Z = z_1 + z_2 j \in \mathbb{BC} \mid z_1^2 + z_2^2 = 0\}.$$

Note that in $\mathbb{BC}$ a zero divisor is equivalent to a non-unit element. Setting

$$\mathbf{e} = \frac{1+ij}{2}, \quad \mathbf{e}^\dagger = \frac{1-ij}{2},$$

$\mathbf{e}$ and $\mathbf{e}^\dagger$ are the non-complex idempotent elements satisfying $\mathbf{e}\mathbf{e}^\dagger = 0$.

For

$$Z = z_1 + z_2 j = x_1 + y_1 i + x_2 j + y_2 ij \in \mathbb{BC},$$

we define the surjective ring homomorphisms $\Phi_{\mathbf{e}} \colon \mathbb{BC} \to \mathbb{C}$ and $\Phi_{\mathbf{e}^\dagger} : \mathbb{BC} \to \mathbb{C}$ by

$$\Phi_{\mathbf{e}}(Z) = z_1 - z_2 i = (x_1 + y_2) + (y_1 - x_2)i,$$
$$\Phi_{\mathbf{e}^\dagger}(Z) = z_1 + z_2 i = (x_1 - y_2) + (y_1 + x_2)i$$

respectively. Also, setting $Z_{\mathbf{e}} = \Phi_{\mathbf{e}}(Z)$ and $Z_{\mathbf{e}^\dagger} = \Phi_{\mathbf{e}^\dagger}(Z)$, any bicomplex number Z has the idempotent representation

$$Z = \Phi_{\mathbf{e}}(Z)\mathbf{e} + \Phi_{\mathbf{e}^\dagger}(Z)\mathbf{e}^\dagger = Z_{\mathbf{e}}\mathbf{e} + Z_{\mathbf{e}^\dagger}\mathbf{e}^\dagger.$$

By the idempotent representation, the set of zero divisors with 0 is represented by

$$\mathfrak{S}_0 = \{Z = Z_{\mathbf{e}}\mathbf{e} + Z_{\mathbf{e}^\dagger}\mathbf{e}^\dagger \in \mathbb{BC} \mid Z_{\mathbf{e}} Z_{\mathbf{e}^\dagger} = 0\} = \mathbb{C}\mathbf{e} \cup \mathbb{C}\mathbf{e}^\dagger.$$

In the case of several bicomplex variables, for $Z = (Z_1, \ldots, Z_n) \in \mathbb{BC}^n$, we also define the maps $\Phi_{\mathbf{e}} \colon \mathbb{BC}^n \to \mathbb{C}^n$ and $\Phi_{\mathbf{e}^\dagger} : \mathbb{BC}^n \to \mathbb{C}^n$ by

$$\Phi_{\mathbf{e}}(Z) = (\Phi_{\mathbf{e}}(Z_1), \ldots, \Phi_{\mathbf{e}}(Z_n)), \quad \Phi_{\mathbf{e}^\dagger}(Z) = (\Phi_{\mathbf{e}^\dagger}(Z_1), \ldots, \Phi_{\mathbf{e}^\dagger}(Z_n))$$

respectively. Then we have the idempotent representation of $Z \in \mathbb{BC}^n$ as

$$Z = \Phi_{\mathbf{e}}(Z)\mathbf{e} + \Phi_{\mathbf{e}^\dagger}(Z)\mathbf{e}^\dagger.$$

In order to emphasize components, we may identify an image of a point by $\Phi_{\mathbf{e}}$ (resp. $\Phi_{\mathbf{e}^\dagger}$) with a point of the $\mathbf{e}$-axis $\mathbb{C}^n\mathbf{e}$ (resp. the $\mathbf{e}^\dagger$-axis $\mathbb{C}^n\mathbf{e}^\dagger$) in $\mathbb{BC}^n$. Namely, we may use the notation $\Phi_{\mathbf{e}}(\mathbb{BC}^n) = \mathbb{C}^n\mathbf{e}$, $\Phi_{\mathbf{e}^\dagger}(\mathbb{BC}^n) = \mathbb{C}^n\mathbf{e}^\dagger$ and so on.

§2.2. Bicomplex holomorphic functions

In this subsection, we review some definitions and fundamental results in bicomplex analysis. See [3, 7, 8] for more details.

For any bicomplex number $Z = z_1 + z_2 j \in \mathbb{BC}$, we define the norm $\|Z\|$ of Z by

$$\|Z\| = \sqrt{|z_1|^2 + |z_2|^2}.$$

It induces a topology on $\mathbb{BC}$, which is isomorphic to the Euclidean space $\mathbb{C}^2$. Moreover, the maps $\Phi_{\mathbf{e}}$ and $\Phi_{\mathbf{e}^\dagger}$ are continuous and open.

Let $\Omega \subset \mathbb{BC}$ be an open set, $F\colon \Omega \to \mathbb{BC}$ a bicomplex function on Ω and $Z_0 \in \Omega$. We say that F is bicomplex differentiable at Z_0 if the limit

$$\lim_{\substack{Z \to Z_0 \\ Z - Z_0 \notin \mathfrak{S}_0}} \frac{F(Z) - F(Z_0)}{Z - Z_0}$$

exists, which is denoted by $F'(Z_0)$. We also say that F is bicomplex holomorphic on Ω if F is bicomplex differentiable at any point of Ω. We denote the set of bicomplex holomorphic functions on Ω by $\mathcal{O}_{\mathbb{BC}}(\Omega)$. Then $\mathcal{O}_{\mathbb{BC}}$ has a sheaf structure on $\mathbb{BC}$.

The idempotent representation of bicomplex holomorphic functions plays an important role in bicomplex analysis. Let F be a bicomplex function on Ω. By the idempotent representation, we have

$$\begin{aligned} F(Z) &= F_{\mathbf{e}}(Z)\mathbf{e} + F_{\mathbf{e}^\dagger}(Z)\mathbf{e}^\dagger \\ &= F_{\mathbf{e}}(Z_{\mathbf{e}}, Z_{\mathbf{e}^\dagger})\mathbf{e} + F_{\mathbf{e}^\dagger}(Z_{\mathbf{e}}, Z_{\mathbf{e}^\dagger})\mathbf{e}^\dagger, \end{aligned}$$

where $F_{\mathbf{e}}, F_{\mathbf{e}^\dagger} : \Omega \to \mathbb{C}$ are considered as complex-valued functions of two complex variables. The following theorem, called the Ringleb theorem, is one of the most important results in bicomplex analysis.

Theorem 2.1. *Assume that each fiber of $\Phi_{\mathbf{e}}|_\Omega$ and $\Phi_{\mathbf{e}^\dagger}|_\Omega$ is connected. Then F is bicomplex holomorphic on Ω if and only if $F_{\mathbf{e}}$ (resp. $F_{\mathbf{e}^\dagger}$) is complex holomorphic on $\Phi_{\mathbf{e}}(\Omega)$ (resp. $\Phi_{\mathbf{e}^\dagger}(\Omega)$) of one variable with respect to $Z_{\mathbf{e}}$ (resp. $Z_{\mathbf{e}^\dagger}$). Moreover, any bicomplex holomorphic function F on Ω is analytically continued to $\Phi_{\mathbf{e}}^{-1}(\Phi_{\mathbf{e}}(\Omega)) \cap \Phi_{\mathbf{e}^\dagger}^{-1}(\Phi_{\mathbf{e}^\dagger}(\Omega))$ and we have*

$$F(Z) = F_{\mathbf{e}}(Z_{\mathbf{e}})\mathbf{e} + F_{\mathbf{e}^\dagger}(Z_{\mathbf{e}^\dagger})\mathbf{e}^\dagger.$$

By Theorem 2.1, we can immediately generalize fundamental properties of complex holomorphic functions of one variable such as Taylor's theorem, the principle of analytic continuation and so on to those of bicomplex holomorphic functions.

Let $\mathcal{O}_{\mathbb{C}}$ denote the sheaf of complex holomorphic functions of one variable. In the terminology of sheaves, Theorem 2.1 can be described as an isomorphism of sheaves on $\mathbb{BC}$:

$$\mathcal{O}_{\mathbb{BC}} \simeq \Phi_{\mathbf{e}}^{-1}\mathcal{O}_{\mathbb{C}\mathbf{e}}\mathbf{e} \oplus \Phi_{\mathbf{e}^\dagger}^{-1}\mathcal{O}_{\mathbb{C}\mathbf{e}^\dagger}\mathbf{e}^\dagger.$$

In the case of several bicomplex variables, let $\Omega \subset \mathbb{BC}^n$ be an open set, $F\colon \Omega \to \mathbb{BC}$ a bicomplex function on Ω. We say that F is bicomplex holomorphic on Ω if and only if F is partially holomorphic in each variable on Ω. We denote the sheaf of bicomplex holomorphic functions of several variables by $\mathcal{O}_{\mathbb{BC}^n}$. Then we also have an isomorphism of sheaves on $\mathbb{BC}^n$:

$$\mathcal{O}_{\mathbb{BC}^n} \simeq \Phi_{\mathbf{e}}^{-1}\mathcal{O}_{\mathbb{C}^n_{\mathbf{e}}}\mathbf{e} \oplus \Phi_{\mathbf{e}^\dagger}^{-1}\mathcal{O}_{\mathbb{C}^n_{\mathbf{e}^\dagger}}\mathbf{e}^\dagger, \tag{2.1}$$

where $\mathcal{O}_{\mathbb{C}^n}$ denotes the sheaf of complex holomorphic functions of several variables.

§2.3. Functorial study of classical hyperfunctions and microfunctions

In this subsection, we review a functorial study of classical hyperfunctions and microfunctions in the derived category of sheaves. A reference for derived categories is [2] and we follow the terminology in it. Moreover, [2] and [12] are fundamental references for results in this subsection. For example, for a topological space X, $\mathbf{D}^b(X)$ denotes the derived category of bounded complexes of sheaves of $\mathbb{C}_X$-modules on X. Recall that for any morphism $f\colon X \to Y$ of topological manifolds there exists a functor

$$f^!\colon \mathbf{D}^b(Y) \longrightarrow \mathbf{D}^b(X)$$

as the right adjoint functor to $Rf_!\colon \mathbf{D}^b(X) \to \mathbf{D}^b(Y)$. In the case where f is a closed embedding, we have

$$f^! \simeq f^{-1}R\Gamma_X. \tag{2.2}$$

In the case where X and Y are orientable and f is a topological submersion, we have

$$f^! \simeq f^{-1}[\dim X - \dim Y]. \tag{2.3}$$

A (complex) hyperfunction is a generalized function, which is represented as a finite formal sum of boundary values of complex holomorphic functions. We explain the functorial description of the boundary value below. In the derived category $\mathbf{D}^b(\mathbb{R}^n)$, the sheaf of (complex) hyperfunctions $\mathcal{B}_{\mathbb{C}^n}$ corresponds to the complex $R\Gamma_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{C}^n})|_{\mathbb{R}^n}[n]$. Here, $\mathcal{O}_{\mathbb{C}^n}$ denotes the sheaf of complex holomorphic functions on $\mathbb{C}^n$. The following properties are well known.

Theorem 2.2. *The following properties hold:*

(i) *The complex $R\Gamma_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{C}^n})|_{\mathbb{R}^n}$ is concentrated in degree n.*
(ii) *The sheaf $\mathcal{B}_{\mathbb{C}^n} = H^n(R\Gamma_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{C}^n})|_{\mathbb{R}^n})$ is flabby on $\mathbb{R}^n$.*

A (complex) microfunction is a microlocal object for a (complex) hyperfunction, which is described as the singularities of it in the conormal bundle $\pi_{\mathbb{C}^n}\colon T^*_{\mathbb{R}^n}\mathbb{C}^n \to \mathbb{R}^n$ to $\mathbb{R}^n$ in $\mathbb{C}^n$. Let us set $\dot{T}^*_{\mathbb{R}^n}\mathbb{C}^n = T^*_{\mathbb{R}^n}\mathbb{C}^n \setminus \mathbb{R}^n$ and denote by $\dot{\pi}_{\mathbb{C}^n}$ the natural projection $\dot{T}^*_{\mathbb{R}^n}\mathbb{C}^n \to \mathbb{R}^n$. In the derived category $\mathbf{D}^b(T^*_{\mathbb{R}^n}\mathbb{C}^n)$, the sheaf of (complex) microfunctions $\mathcal{C}_{\mathbb{C}^n}$ corresponds to the complex $\mu_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{C}^n})[n]$. Here, $\mu_{\mathbb{R}^n}\colon \mathbf{D}^b(\mathbb{C}^n) \to \mathbf{D}^b(T^*_{\mathbb{R}^n}\mathbb{C}^n)$ denotes the microlocalization functor along $\mathbb{R}^n$. The following properties are well known.

Theorem 2.3. *The following properties hold:*

(i) *The complex $\mu_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{C}^n})$ is concentrated in degree n.*
(ii) *Let us set $\mathcal{C}_{\mathbb{C}^n} = H^n(\mu_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{C}^n}))$. Then the sheaf $\mathcal{C}_{\mathbb{C}^n}|_{\dot{T}^*_{\mathbb{R}^n}\mathbb{C}^n}$ is conically flabby on $\dot{T}^*_{\mathbb{R}^n}\mathbb{C}^n$. Namely, its direct image on $\dot{T}^*_{\mathbb{R}^n}\mathbb{C}^n/\mathbb{R}^+$ is flabby.*
(iii) *There exists an exact sequence of sheaves on $\mathbb{R}^n$:*

$$0 \longrightarrow \mathcal{O}_{\mathbb{C}^n}|_{\mathbb{R}^n} \longrightarrow \mathcal{B}_{\mathbb{C}^n} \longrightarrow \dot{\pi}_{\mathbb{C}^n *}\mathcal{C}_{\mathbb{C}^n} \longrightarrow 0. \tag{2.4}$$

(iv) *There exists an isomorphism of sheaves on $\mathbb{R}^n$:*

$$\mathrm{sp}\colon \mathcal{B}_{\mathbb{C}^n} \xrightarrow{\sim} \pi_{\mathbb{C}^n *}\mathcal{C}_{\mathbb{C}^n}.$$

The exact sequence (2.4) is called Sato's fundamental exact sequence and the morphism sp is called the spectrum isomorphism.

For a detailed study of Sato's fundamental exact sequence (2.4), let us consider the normal bundle $\tau_{\mathbb{C}^n}\colon T_{\mathbb{R}^n}\mathbb{C}^n \to \mathbb{R}^n$ to $\mathbb{R}^n$ in $\mathbb{C}^n$ and the specialization functor $\nu_{\mathbb{R}^n}\colon \mathbf{D}^b(\mathbb{C}^n) \to \mathbf{D}^b(T_{\mathbb{R}^n}\mathbb{C}^n)$ along $\mathbb{R}^n$. The following property is well known.

Theorem 2.4. *Let us set $\widetilde{\mathcal{A}_{\mathbb{C}^n}} = H^0(\nu_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{C}^n}))$. Then there exists an exact sequence of sheaves on $T_{\mathbb{R}^n}\mathbb{C}^n$:*

$$0 \longrightarrow \widetilde{\mathcal{A}_{\mathbb{C}^n}} \longrightarrow \tau_{\mathbb{C}^n}^{-1}\mathcal{B}_{\mathbb{C}^n} \longrightarrow (p^+_{\mathbb{C}^n 1})_*(p^+_{\mathbb{C}^n 2})^{-1}\mathcal{C}_{\mathbb{C}^n}, \tag{2.5}$$

where $p^+_{\mathbb{C}^n 1}$ (resp. $p^+_{\mathbb{C}^n 2}$) denotes the natural projection from

$$\big\{(\eta,\xi) \in T_{\mathbb{R}^n}\mathbb{C}^n \underset{\mathbb{R}^n}{\times} T^*_{\mathbb{R}^n}\mathbb{C}^n \;\big|\; \langle \eta,\xi\rangle > 0\big\}$$

to $T_{\mathbb{R}^n}\mathbb{C}^n$ (resp. $T^*_{\mathbb{R}^n}\mathbb{C}^n$).

The distinguished triangle in $\mathbf{D}^b(T_{\mathbb{R}^n}\mathbb{C}^n)$ generalized (2.5) on the Fourier–Sato transformation is called Uchida's triangle. See [13] for the details. We call a section of the specialization $\widetilde{\mathcal{A}_{\mathbb{C}^n}}$ a holomorphic function on an infinitesimal wedge. The morphism $b\colon \widetilde{\mathcal{A}_{\mathbb{C}^n}} \to \tau_{\mathbb{C}^n}^{-1}\mathcal{B}_{\mathbb{C}^n}$ of (2.5), which is induced by $\mathrm{id} \to \tau^!_{\mathbb{C}^n} R\tau_{\mathbb{C}^n !}$, is called the boundary value morphism. By the boundary value morphism b, a hyperfunction is represented as a finite sum of boundary values of holomorphic functions on infinitesimal wedges.

§3. Functorial study of bicomplex hyperfunctions

In this section, we study bicomplex hyperfunctions by functorial techniques. Considering the classical case, it is natural to study the complex $R\Gamma_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n})$ of sections of the sheaf $\mathcal{O}_{\mathbb{BC}^n}$ supported in $\mathbb{R}^n$ in the derived category $\mathbf{D}^b(\mathbb{BC}^n)$. In order to study it, let us consider the following diagonal embedding:

$$\mathbb{R}^n \hookrightarrow \mathbb{R}^n\mathbf{e} + \mathbb{R}^n\mathbf{e}^\dagger \hookrightarrow \mathbb{C}^n\mathbf{e} + \mathbb{C}^n\mathbf{e}^\dagger = \mathbb{BC}^n$$

of the real space $\mathbb{R}^n$ into the bicomplex space $\mathbb{BC}^n$. The idempotent representation (2.1) of bicomplex holomorphic functions induces that of $R\Gamma_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n})|_{\mathbb{R}^n}$ in the derived category $\mathbf{D}^b(\mathbb{R}^n)$.

Theorem 3.1. *We have an isomorphism*

$$\begin{aligned} R\Gamma_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n})|_{\mathbb{R}^n} \simeq{}& (\Phi_{\mathbf{e}}|_{\mathbb{R}^n})^{-1} R\Gamma_{\mathbb{R}^n\mathbf{e}}(\mathcal{O}_{\mathbb{C}^n\mathbf{e}})|_{\mathbb{R}^n\mathbf{e}}\mathbf{e}[-2n] \\ &\oplus (\Phi_{\mathbf{e}^\dagger}|_{\mathbb{R}^n})^{-1} R\Gamma_{\mathbb{R}^n\mathbf{e}^\dagger}(\mathcal{O}_{\mathbb{C}^n\mathbf{e}^\dagger})|_{\mathbb{R}^n\mathbf{e}^\dagger}\mathbf{e}^\dagger[-2n] \end{aligned} \tag{3.1}$$

in $\mathbf{D}^b(\mathbb{R}^n)$.

Proof. By the idempotent representation (2.1), we have

$$R\Gamma_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n})|_{\mathbb{R}^n} \simeq R\Gamma_{\mathbb{R}^n}(\Phi_{\mathbf{e}}^{-1}\mathcal{O}_{\mathbb{C}^n\mathbf{e}})|_{\mathbb{R}^n}\mathbf{e} \oplus R\Gamma_{\mathbb{R}^n}(\Phi_{\mathbf{e}^\dagger}^{-1}\mathcal{O}_{\mathbb{C}^n\mathbf{e}^\dagger})|_{\mathbb{R}^n}\mathbf{e}^\dagger.$$

Let us consider the following commutative diagram where the square is Cartesian:

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, hook, "d"] \arrow[dr, "\Phi_{\mathbf{e}}|_{\mathbb{R}^n}"'] & \mathbb{R}^n\mathbf{e} + \mathbb{C}^n\mathbf{e}^\dagger \arrow[r, hook, "r"] \arrow[d, "\Phi_{\mathbf{e}}"] \arrow[dr, phantom, "\square"] & \mathbb{BC}^n \arrow[d, "\Phi_{\mathbf{e}}"] \\ & \mathbb{R}^n\mathbf{e} \arrow[r, hook, "r|_{\mathbb{R}^n\mathbf{e}}"] & \mathbb{C}^n\mathbf{e}. \end{tikzcd}$$

Here, $d\colon \mathbb{R}^n \to \mathbb{R}^n\mathbf{e}+\mathbb{C}^n\mathbf{e}^\dagger$ is the diagonal embedding and $r\colon \mathbb{R}^n\mathbf{e}+\mathbb{C}^n\mathbf{e}^\dagger \to \mathbb{BC}^n$ is the natural embedding. Note that $\Phi_{\mathbf{e}}|_{\mathbb{R}^n}$ is an isomorphism. Since $\Phi_{\mathbf{e}}$ is a projection with fiber dimension $2n$, we have the following isomorphisms:

$$\begin{aligned} R\Gamma_{\mathbb{R}^n}(\Phi_{\mathbf{e}}^{-1}\mathcal{O}_{\mathbb{C}^n\mathbf{e}})|_{\mathbb{R}^n} &\simeq (r\circ d)^! \Phi_{\mathbf{e}}^! \mathcal{O}_{\mathbb{C}^n\mathbf{e}}[-2n] \\ &\simeq (\Phi_{\mathbf{e}}|_{\mathbb{R}^n})^! (r|_{\mathbb{R}^n\mathbf{e}})^! \mathcal{O}_{\mathbb{C}^n\mathbf{e}}[-2n] \\ &\simeq (\Phi_{\mathbf{e}}|_{\mathbb{R}^n})^{-1} R\Gamma_{\mathbb{R}^n\mathbf{e}}(\mathcal{O}_{\mathbb{C}^n\mathbf{e}})|_{\mathbb{R}^n\mathbf{e}}[-2n] \end{aligned}$$

by (2.2) and (2.3). In the same way, we have

$$R\Gamma_{\mathbb{R}^n}(\Phi_{\mathbf{e}^\dagger}^{-1}\mathcal{O}_{\mathbb{C}^n\mathbf{e}^\dagger})|_{\mathbb{R}^n} \simeq (\Phi_{\mathbf{e}^\dagger}|_{\mathbb{R}^n})^{-1} R\Gamma_{\mathbb{R}^n\mathbf{e}^\dagger}(\mathcal{O}_{\mathbb{C}^n\mathbf{e}^\dagger})|_{\mathbb{R}^n\mathbf{e}^\dagger}[-2n].$$

Therefore we obtain (3.1). □

By Theorems 2.2(i) and 3.1, we can re-prove the vanishing theorem of the complex $R\Gamma_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n})|_{\mathbb{R}^n}$ first proved in [1].

Theorem 3.2. *The complex $R\Gamma_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n})|_{\mathbb{R}^n}$ is concentrated in degree $3n$.*

By Theorem 3.2, we can redefine the notion of bicomplex hyperfunctions, first introduced in [1].

Definition 3.3. We define the sheaf of bicomplex hyperfunctions on $\mathbb{R}^n$ by

$$\mathcal{B}_{\mathbb{BC}^n} = H^{3n}(R\Gamma_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n})|_{\mathbb{R}^n}).$$

We call a section of $\mathcal{B}_{\mathbb{BC}^n}$ a bicomplex hyperfunction.

By Theorems 3.1 and 3.2, we obtain the idempotent representation theorem for bicomplex hyperfunctions.

Theorem 3.4. *We obtain an isomorphism of sheaves on $\mathbb{R}^n$:*

$$\mathcal{B}_{\mathbb{BC}^n} \simeq (\Phi_{\mathbf{e}}|_{\mathbb{R}^n})^{-1}\mathcal{B}_{\mathbb{C}^n{}_{\mathbf{e}}}\mathbf{e} \oplus (\Phi_{\mathbf{e}^\dagger}|_{\mathbb{R}^n})^{-1}\mathcal{B}_{\mathbb{C}^n{}_{\mathbf{e}^\dagger}}\mathbf{e}^\dagger. \tag{3.2}$$

Proof. By taking the $3n$-th cohomology of (3.1), we obtain (3.2). □

Theorem 3.4 gives us a new characterization of bicomplex hyperfunctions. It says that any bicomplex hyperfunction is described as a linear combination of $\mathbf{e}$ and $\mathbf{e}^\dagger$ with classical complex hyperfunction coefficients. In other words, a bicomplex hyperfunction corresponds to a pair of two classical complex hyperfunctions. As a corollary of this principle, we immediately obtain several fundamental properties of bicomplex hyperfunctions by applying properties of complex hyperfunctions to coefficients of $\mathbf{e}$ and $\mathbf{e}^\dagger$. For example, we can re-prove the flabbiness of the sheaf $\mathcal{B}_{\mathbb{BC}^n}$, first proved in [1], by that of the sheaf $\mathcal{B}_{\mathbb{C}^n}$.

Theorem 3.5. *The sheaf $\mathcal{B}_{\mathbb{BC}^n}$ is flabby on $\mathbb{R}^n$.*

Moreover, we can define several fundamental operations of bicomplex hyperfunctions such as linear differential operators with bicomplex real analytic coefficients, substitution, integration along fibers, products and so on. We omit detailed studies of them here, but they will be studied in future works.

Note that, although via the Čech cohomology it is not easy to compare two cohomology groups $H^{3n}_{\mathbb{R}^n}(\mathbb{BC}^n;\mathcal{O}_{\mathbb{BC}^n})$ and $H^n_{\mathbb{R}^n}(\mathbb{C}^n;\mathcal{O}_{\mathbb{C}^n})$ directly, thanks to a functorial argument we can obtain the idempotent representation theorem for bicomplex hyperfunctions.

Remark 3.6. By Kashiwara's vanishing result [12] and in the same way as Theorem 3.1, we can obtain the following result. Let $G \subset \mathbb{C}^n \subset \mathbb{BC}^n$ be a closed convex subset and $Z_0 \in G$. If there exists no complex affine linear subspace L of dimension d through Z_0, such that $L \cap G$ is a neighborhood of Z_0 in L, then we have $H^k_G(\mathcal{O}_{\mathbb{BC}^n})_{Z_0} = 0$ for $k \leq 3n - d$.

§4. Bicomplex microfunctions

Let us consider microlocally the study in Section 3. In this section, we consider $\mathbb{BC}^n$ as a real analytic manifold and the conormal bundle $\pi\colon T^*_{\mathbb{R}^n}\mathbb{BC}^n \to \mathbb{R}^n$ to $\mathbb{R}^n$ in $\mathbb{BC}^n$. We set $\dot{T}^*_{\mathbb{R}^n}\mathbb{BC}^n = T^*_{\mathbb{R}^n}\mathbb{BC}^n \backslash \mathbb{R}^n$ and denote the natural projection $\dot{T}^*_{\mathbb{R}^n}\mathbb{BC}^n \to \mathbb{R}^n$ by $\dot{\pi}$. Considering the classical case, it is natural to study the microlocalization $\mu_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n})$ of $\mathcal{O}_{\mathbb{BC}^n}$ along $\mathbb{R}^n$ in the derived category $\mathbf{D}^b(T^*_{\mathbb{R}^n}\mathbb{BC}^n)$. In order to study it, let us also consider the diagonal embedding

$$\mathbb{R}^n \longrightarrow \mathbb{R}^n\mathbf{e} + \mathbb{R}^n\mathbf{e}^\dagger \longrightarrow \mathbb{C}^n\mathbf{e} + \mathbb{C}^n\mathbf{e}^\dagger = \mathbb{BC}^n$$

of the real space $\mathbb{R}^n$ into the bicomplex space $\mathbb{BC}^n$ and the following morphisms:

$$\begin{aligned} &T^*_{\mathbb{R}^n}\mathbb{BC}^n \xleftarrow{{}^t\Phi'_{\mathbf{e}}} \mathbb{R}^n \underset{\mathbb{R}^n\mathbf{e}}{\times} T^*_{\mathbb{R}^n\mathbf{e}}\mathbb{C}^n\mathbf{e} \xrightarrow{\Phi_{\mathbf{e}\pi}} T^*_{\mathbb{R}^n\mathbf{e}}\mathbb{C}^n\mathbf{e}, \\ &T^*_{\mathbb{R}^n}\mathbb{BC}^n \xleftarrow{{}^t\Phi'_{\mathbf{e}^\dagger}} \mathbb{R}^n \underset{\mathbb{R}^n\mathbf{e}^\dagger}{\times} T^*_{\mathbb{R}^n\mathbf{e}^\dagger}\mathbb{C}^n\mathbf{e}^\dagger \xrightarrow{\Phi_{\mathbf{e}^\dagger\pi}} T^*_{\mathbb{R}^n\mathbf{e}^\dagger}\mathbb{C}^n\mathbf{e}^\dagger, \end{aligned}$$

induced by the maps $\Phi_{\mathbf{e}}$ and $\Phi_{\mathbf{e}^\dagger}$ respectively. The explicit descriptions of them associated with two coordinates

$$Z = (\widetilde{x_1} + \widetilde{y_1}i)\mathbf{e} + (\widetilde{x_2} + \widetilde{y_2}i)\mathbf{e}^\dagger = x_1 + y_1 i + x_2 j + y_2 ij$$

of $\mathbb{BC}^n$ are given by

$$\begin{aligned} {}^t\Phi'_{\mathbf{e}}(x, (x\mathbf{e}; \xi\, d\widetilde{y_1})) &= (x\mathbf{e} + x\mathbf{e}^\dagger; \xi\, d\widetilde{y_1}) = (x; \xi\, dy_1 - \xi\, dx_2), \\ \Phi_{\mathbf{e}\pi}(x, (x\mathbf{e}; \xi\, d\widetilde{y_1})) &= (x\mathbf{e}; \xi\, d\widetilde{y_1}), \\ {}^t\Phi'_{\mathbf{e}^\dagger}(x, (x\mathbf{e}^\dagger; \xi\, d\widetilde{y_2})) &= (x\mathbf{e} + x\mathbf{e}^\dagger; \xi\, d\widetilde{y_2}) = (x; \xi\, dy_1 + \xi\, dx_2), \\ \Phi_{\mathbf{e}^\dagger\pi}(x, (x\mathbf{e}^\dagger; \xi\, d\widetilde{y_2})) &= (x\mathbf{e}^\dagger; \xi\, d\widetilde{y_2}) \end{aligned}$$

for $x, \xi \in \mathbb{R}^n$. Note that $\Phi_{\mathbf{e}\pi}$ and $\Phi_{\mathbf{e}^\dagger\pi}$ are isomorphisms and ${}^t\Phi'_{\mathbf{e}}$ and ${}^t\Phi'_{\mathbf{e}^\dagger}$ are closed embeddings. Then the idempotent representation (2.1) of bicomplex holomorphic functions induces that of $\mu_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n})$ in the derived category $\mathbf{D}^b(T^*_{\mathbb{R}^n}\mathbb{BC}^n)$.

Theorem 4.1. *We have an isomorphism*

$$\mu_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n}) \simeq {}^t\Phi'_{\mathbf{e}*}\Phi_{\mathbf{e}\pi}^{-1}\mu_{\mathbb{R}^n\mathbf{e}}(\mathcal{O}_{\mathbb{C}^n\mathbf{e}})[-2n]\mathbf{e} \oplus {}^t\Phi'_{\mathbf{e}^\dagger *}\Phi_{\mathbf{e}^\dagger\pi}^{-1}\mu_{\mathbb{R}^n\mathbf{e}^\dagger}(\mathcal{O}_{\mathbb{C}^n\mathbf{e}^\dagger})[-2n]\mathbf{e}^\dagger \tag{4.1}$$

in $\mathbf{D}^b(T^*_{\mathbb{R}^n}\mathbb{BC}^n)$.

Proof. By the idempotent representation (2.1), we have

$$\mu_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n}) \simeq \mu_{\mathbb{R}^n}(\Phi_{\mathbf{e}}^{-1}\mathcal{O}_{\mathbb{C}^n\mathbf{e}})\mathbf{e} \oplus \mu_{\mathbb{R}^n}(\Phi_{\mathbf{e}^\dagger}^{-1}\mathcal{O}_{\mathbb{C}^n\mathbf{e}^\dagger})\mathbf{e}^\dagger.$$

Since $\Phi_{\mathbf{e}}$ is a projection with fiber dimension $2n$, $\Phi_{\mathbf{e}\pi}$ is an isomorphism and ${}^t\Phi'_{\mathbf{e}}$ is a closed embedding, we have the following isomorphisms:

$$\begin{aligned}\mu_{\mathbb{R}^n}(\Phi_{\mathbf{e}}^{-1}\mathcal{O}_{\mathbb{C}^n\mathbf{e}}) &\simeq \mu_{\mathbb{R}^n}(\Phi_{\mathbf{e}}^{!}\mathcal{O}_{\mathbb{C}^n\mathbf{e}}[-2n]) \\ &\simeq {}^t\Phi'_{\mathbf{e}*}\Phi_{\mathbf{e}\pi}^{!}\mu_{\mathbb{R}^n\mathbf{e}}(\mathcal{O}_{\mathbb{C}^n\mathbf{e}})[-2n] \\ &\simeq {}^t\Phi'_{\mathbf{e}*}\Phi_{\mathbf{e}\pi}^{-1}\mu_{\mathbb{R}^n\mathbf{e}}(\mathcal{O}_{\mathbb{C}^n\mathbf{e}})[-2n]\end{aligned}$$

by (2.3) and [2, Prop. 4.3.5]. In the same way, we have

$$\mu_{\mathbb{R}^n}(\Phi_{\mathbf{e}^\dagger}^{-1}\mathcal{O}_{\mathbb{C}^n\mathbf{e}^\dagger}) \simeq {}^t\Phi'_{\mathbf{e}^\dagger *}\Phi_{\mathbf{e}^\dagger\pi}^{-1}\mu_{\mathbb{R}^n\mathbf{e}^\dagger}(\mathcal{O}_{\mathbb{C}^n\mathbf{e}^\dagger})[-2n].$$

Therefore we obtain (4.1). □

By Theorems 2.3(i) and 4.1, we can prove the vanishing theorem of the complex $\mu_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n})$.

Theorem 4.2. *The complex* $\mu_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n})$ *is concentrated in degree* $3n$*.*

By Theorem 4.2, we can define the notion of bicomplex microfunctions.

Definition 4.3. We define the sheaf of bicomplex microfunctions on $T^*_{\mathbb{R}^n}\mathbb{BC}^n$ by

$$\mathcal{C}_{\mathbb{BC}^n} = H^{3n}(\mu_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n})).$$

We call a section of $\mathcal{C}_{\mathbb{BC}^n}$ a bicomplex microfunction.

By Theorems 4.1 and 4.2, we obtain the idempotent representation theorem for bicomplex microfunctions.

Theorem 4.4. *We obtain an isomorphism of sheaves on* $T^*_{\mathbb{R}^n}\mathbb{BC}^n$*:*

$$\mathcal{C}_{\mathbb{BC}^n} \simeq {}^t\Phi'_{\mathbf{e}*}\Phi_{\mathbf{e}\pi}^{-1}\mathcal{C}_{\mathbb{C}^n\mathbf{e}}\mathbf{e} \oplus {}^t\Phi'_{\mathbf{e}^\dagger *}\Phi_{\mathbf{e}^\dagger\pi}^{-1}\mathcal{C}_{\mathbb{C}^n\mathbf{e}^\dagger}\mathbf{e}^\dagger. \tag{4.2}$$

Proof. Since $\Phi_{\mathbf{e}\pi}$ and $\Phi_{\mathbf{e}^\dagger\pi}$ are isomorphisms and ${}^t\Phi'_{\mathbf{e}}$ and ${}^t\Phi'_{\mathbf{e}^\dagger}$ are closed embeddings, by taking the $3n$-th cohomology of (4.1), we obtain (4.2). □

Theorem 4.4 says that any bicomplex microfunction is described as a linear combination of $\mathbf{e}$ and $\mathbf{e}^\dagger$ with classical complex microfunction coefficients. In other words, a bicomplex microfunction corresponds to a pair of two classical complex microfunctions. As a corollary of this principle, we immediately obtain several fundamental properties of bicomplex microfunctions by applying properties of complex microfunctions to coefficients of $\mathbf{e}$ and $\mathbf{e}^\dagger$. For example, we can prove the flabbiness of the sheaf $\mathcal{C}_{\mathbb{BC}^n}$ by that of the sheaf $\mathcal{C}_{\mathbb{C}^n}$. In order to state it explicitly, we note that the natural action of $\mathbb{R}^+$ to $T^*_{\mathbb{R}^n}\mathbb{C}^n$ induces an action of $(\mathbb{R}^+)^2$ to $T^*_{\mathbb{R}^n}\mathbb{BC}^n$ via ${}^t\Phi'_{\mathbf{e}}$ and ${}^t\Phi'_{\mathbf{e}^\dagger}$.

Theorem 4.5. *The sheaf $\mathcal{C}_{\mathbb{BC}^n}|_{\dot{T}^*_{\mathbb{R}^n}\mathbb{BC}^n}$ is bi-conically flabby on $\dot{T}^*_{\mathbb{R}^n}\mathbb{BC}^n$.*

Proof. Since the sheaf $\mathcal{C}_{\mathbb{C}^n}|_{\dot{T}^*_{\mathbb{R}^n}\mathbb{C}^n}$ is conically flabby, the sheaves ${}^t\Phi'_{\mathbf{e}*}\Phi^{-1}_{\mathbf{e}\pi}\mathcal{C}_{\mathbb{C}^n\mathbf{e}}$ and ${}^t\Phi'_{\mathbf{e}^\dagger *}\Phi^{-1}_{\mathbf{e}^\dagger\pi}\mathcal{C}_{\mathbb{C}^n\mathbf{e}^\dagger}$ are bi-conically flabby. By (4.1), we obtain the result. □

Moreover, by Theorems 2.3 and 4.4, we also obtain the Sato-type fundamental exact sequence and the spectrum isomorphism of bicomplex hyperfunctions and microfunctions.

Theorem 4.6. *The following properties hold:*

(i) *There exists an exact sequence of sheaves on $\mathbb{R}^n$:*

$$0 \longrightarrow \mathcal{O}_{\mathbb{BC}^n}|_{\mathbb{R}^n} \longrightarrow \mathcal{B}_{\mathbb{BC}^n} \longrightarrow \dot{\pi}_*\mathcal{C}_{\mathbb{BC}^n} \longrightarrow 0. \tag{4.3}$$

(ii) *There exists the spectrum isomorphism on $\mathbb{R}^n$:*

$$\mathrm{sp}\colon \mathcal{B}_{\mathbb{BC}^n} \xrightarrow{\sim} \pi_*\mathcal{C}_{\mathbb{BC}^n}. \tag{4.4}$$

Proof. Let $\pi_{\mathbf{e}}\colon T^*_{\mathbb{R}^n\mathbf{e}}\mathbb{C}^n\mathbf{e} \to \mathbb{R}^n\mathbf{e}$ and $\pi_{\mathbf{e}^\dagger}\colon T^*_{\mathbb{R}^n\mathbf{e}^\dagger}\mathbb{C}^n\mathbf{e}^\dagger \to \mathbb{R}^n\mathbf{e}^\dagger$ be the natural projections. Since $\Phi_{\mathbf{e}}|_{\mathbb{R}^n}$, $\Phi_{\mathbf{e}^\dagger}|_{\mathbb{R}^n}$, $\Phi_{\mathbf{e}\pi}$ and $\Phi_{\mathbf{e}^\dagger\pi}$ are isomorphisms, we have

$$(\Phi_{\mathbf{e}}|_{\mathbb{R}^n})^{-1}\pi_{\mathbf{e}*} = \pi_*{}^t\Phi'_{\mathbf{e}*}\Phi^{-1}_{\mathbf{e}\pi} \quad \text{and} \quad (\Phi_{\mathbf{e}^\dagger}|_{\mathbb{R}^n})^{-1}\pi_{\mathbf{e}^\dagger *} = \pi_*{}^t\Phi'_{\mathbf{e}^\dagger *}\Phi^{-1}_{\mathbf{e}^\dagger\pi}.$$

By Theorems 2.3 and 4.4, we obtain the results. □

Note that Theorem 4.6 can be also obtained directly by functorial properties of the microlocalization functor $\mu_{\mathbb{R}^n}$ in [2]. However, we can understand the properties better by reducing to the classical complex case by using the idempotent representations. As a corollary of the spectrum isomorphism (4.4), we obtain a simpler description of the sheaf of bicomplex microfunctions.

Corollary 4.7. *Let us set*

$$\mathcal{A}^*_{\mathbb{BC}^n} = \mathrm{Ker}(\pi^{-1}\mathcal{B}_{\mathbb{BC}^n} \xrightarrow{\mathrm{sp}} \mathcal{C}_{\mathbb{BC}^n}).$$

Then we obtain an isomorphism of sheaves on $T^*_{\mathbb{R}^n}\mathbb{BC}^n$*:*

$$\mathcal{C}_{\mathbb{BC}^n} \simeq \pi^{-1}\mathcal{B}_{\mathbb{BC}^n}/\mathcal{A}^*_{\mathbb{BC}^n}.$$

Proof. By the spectrum isomorphism (4.4) and $\pi^{-1}\pi_* \to \mathrm{id}$, we have the exact sequence

$$\pi^{-1}\mathcal{B}_{\mathbb{BC}^n} \longrightarrow \mathcal{C}_{\mathbb{BC}^n} \longrightarrow 0. \tag{4.5}$$

Therefore we obtain the result. □

Furthermore, we can define several fundamental operations of bicomplex microfunctions such as linear differential operators with bicomplex real analytic coefficients, substitution, integration along fibers, products and so on. We omit detailed studies of them here, but they will be studied in future works.

Finally, we define the notion of the singularity spectrum of a bicomplex hyperfunction.

Definition 4.8. Let u be a bicomplex hyperfunction. We denote the support of its spectrum $\mathrm{sp}(u)$ in $T^*_{\mathbb{R}^n}\mathbb{BC}^n$ by $\mathrm{SS}(u)$ and call it the singularity spectrum of u. We say that u is micro-analytic at $(x;\Xi) \in T^*_{\mathbb{R}^n}\mathbb{BC}^n$ if $(x;\Xi) \notin \mathrm{SS}(u)$.

The sheaf $\mathcal{A}^*_{\mathbb{BC}^n}$ is considered as that of micro-analytic functions. By Theorem 4.4, an estimate of the singularity spectrum of a bicomplex hyperfunction is obtained.

Theorem 4.9. *The following properties hold:*

(i) *The support of the sheaf* $\mathcal{C}_{\mathbb{BC}^n}$ *is equal to*

$$^t\Phi'_{\mathbf{e}}(\Phi^{-1}_{\mathbf{e}\pi}(T^*_{\mathbb{R}^n\mathbf{e}}\mathbb{C}^n\mathbf{e})) \cup {}^t\Phi'_{\mathbf{e}^\dagger}(\Phi^{-1}_{\mathbf{e}^\dagger\pi}(T^*_{\mathbb{R}^n\mathbf{e}^\dagger}\mathbb{C}^n\mathbf{e}^\dagger))$$

in $T^*_{\mathbb{R}^n}\mathbb{BC}^n$*, which is described as*

$$\left\{(x;\eta_1\,dy_1 + \xi_2\,dx_2 + \eta_2\,dy_2) \in T^*_{\mathbb{R}^n}\mathbb{BC}^n \;\middle|\; \eta_1 = \pm\xi_2,\ \eta_2 = 0\right\}$$

associated with the coordinate $Z = x_1 + y_1 i + x_2 j + y_2 ij$ *of* $\mathbb{BC}^n$*.*

(ii) *Let u be a bicomplex hyperfunction and $u_{\mathbf{e}}\mathbf{e} + u_{\mathbf{e}^\dagger}\mathbf{e}^\dagger$ its decomposition as in* (3.2)*. Then the singularity spectrum* $\mathrm{SS}(u)$ *of u is described as*

$$\mathrm{SS}(u) = {}^t\Phi'_{\mathbf{e}}\big(\Phi^{-1}_{\mathbf{e}\pi}(\mathrm{SS}(u_{\mathbf{e}}))\big) \cup {}^t\Phi'_{\mathbf{e}^\dagger}\big(\Phi^{-1}_{\mathbf{e}^\dagger\pi}(\mathrm{SS}(u_{\mathbf{e}^\dagger}))\big), \tag{4.6}$$

in $T^*_{\mathbb{R}^n}\mathbb{BC}^n$, where $\mathrm{SS}(u_{\mathbf{e}})$ *(resp.* $\mathrm{SS}(u_{\mathbf{e}^\dagger})$*) is the singularity spectrum of a complex hyperfunction* $u_{\mathbf{e}}$ *(resp.* $u_{\mathbf{e}^\dagger}$*). In particular, every bicomplex hyperfunction is micro-analytic at each point of the outside of*

$${}^t\Phi'_{\mathbf{e}}(\Phi^{-1}_{\mathbf{e}\pi}(T^*_{\mathbb{R}^n\mathbf{e}}\mathbb{C}^n\mathbf{e})) \cup {}^t\Phi'_{\mathbf{e}^\dagger}(\Phi^{-1}_{\mathbf{e}^\dagger\pi}(T^*_{\mathbb{R}^n\mathbf{e}^\dagger}\mathbb{C}^n\mathbf{e}^\dagger)).$$

§5. Boundary value morphism and bicomplex hyperfunctions

Let us describe the boundary value morphism for bicomplex holomorphic functions explicitly. In this section, we also consider $\mathbb{BC}^n$ as a real analytic manifold and the normal bundle $\tau\colon T_{\mathbb{R}^n}\mathbb{BC}^n \to \mathbb{R}^n$ to $\mathbb{R}^n$ in $\mathbb{BC}^n$. Considering the classical case, it is natural to study the specialization $\nu_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n})$ of $\mathcal{O}_{\mathbb{BC}^n}$ along $\mathbb{R}^n$ in the derived category $\mathbf{D}^b(T_{\mathbb{R}^n}\mathbb{BC}^n)$. In order to study it, let us also consider the diagonal embedding

$$\mathbb{R}^n \hookrightarrow \mathbb{R}^n\mathbf{e} + \mathbb{R}^n\mathbf{e}^\dagger \hookrightarrow \mathbb{C}^n\mathbf{e} + \mathbb{C}^n\mathbf{e}^\dagger = \mathbb{BC}^n$$

of the real space $\mathbb{R}^n$ into the bicomplex space $\mathbb{BC}^n$ and the following morphisms:

$$T_{\mathbb{R}^n}\Phi_{\mathbf{e}}\colon T_{\mathbb{R}^n}\mathbb{BC}^n \longrightarrow T_{\mathbb{R}^n\mathbf{e}}\mathbb{C}^n\mathbf{e},$$
$$T_{\mathbb{R}^n}\Phi_{\mathbf{e}^\dagger}\colon T_{\mathbb{R}^n}\mathbb{BC}^n \longrightarrow T_{\mathbb{R}^n\mathbf{e}^\dagger}\mathbb{C}^n\mathbf{e}^\dagger$$

induced by the maps $\Phi_{\mathbf{e}}$ and $\Phi_{\mathbf{e}^\dagger}$ respectively. Note that they are smooth. Then the idempotent representation (2.1) of bicomplex holomorphic functions induces that of $\nu_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n})$ in the derived category $\mathbf{D}^b(T_{\mathbb{R}^n}\mathbb{BC}^n)$.

Theorem 5.1. *We have an isomorphism*

$$\nu_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n}) \simeq (T_{\mathbb{R}^n}\Phi_{\mathbf{e}})^{-1}\nu_{\mathbb{R}^n\mathbf{e}}(\mathcal{O}_{\mathbb{C}^n\mathbf{e}})\mathbf{e} \oplus (T_{\mathbb{R}^n}\Phi_{\mathbf{e}^\dagger})^{-1}\nu_{\mathbb{R}^n\mathbf{e}^\dagger}(\mathcal{O}_{\mathbb{C}^n\mathbf{e}^\dagger})\mathbf{e}^\dagger \tag{5.1}$$

in $\mathbf{D}^b(T_{\mathbb{R}^n}\mathbb{BC}^n)$.

Proof. By the idempotent representation (2.1), we have

$$\nu_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n}) \simeq \nu_{\mathbb{R}^n}(\Phi^{-1}_{\mathbf{e}}\mathcal{O}_{\mathbb{C}^n\mathbf{e}})\mathbf{e} \oplus \nu_{\mathbb{R}^n}(\Phi^{-1}_{\mathbf{e}^\dagger}\mathcal{O}_{\mathbb{C}^n\mathbf{e}^\dagger})\mathbf{e}^\dagger.$$

Since $\Phi_{\mathbf{e}}$ and $\Phi_{\mathbf{e}^\dagger}$ are projections, by [2, Prop. 4.2.5] we have the following isomorphisms:

$$\nu_{\mathbb{R}^n}(\Phi^{-1}_{\mathbf{e}}\mathcal{O}_{\mathbb{C}^n\mathbf{e}}) \simeq (T_{\mathbb{R}^n}\Phi_{\mathbf{e}})^{-1}\nu_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{C}^n\mathbf{e}}),$$
$$\nu_{\mathbb{R}^n}(\Phi^{-1}_{\mathbf{e}^\dagger}\mathcal{O}_{\mathbb{C}^n\mathbf{e}^\dagger}) \simeq (T_{\mathbb{R}^n}\Phi_{\mathbf{e}^\dagger})^{-1}\nu_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{C}^n\mathbf{e}^\dagger}).$$

Therefore we obtain (5.1). □

Let us define the notion of bicomplex holomorphic functions on infinitesimal wedges.

Definition 5.2. We define the sheaf of the specialization of bicomplex holomorphic functions by

$$\widetilde{\mathcal{A}_{\mathbb{BC}^n}} = H^0(\nu_{\mathbb{R}^n}(\mathcal{O}_{\mathbb{BC}^n})).$$

We call a section of $\widetilde{\mathcal{A}_{\mathbb{BC}^n}}$ a bicomplex holomorphic function on an infinitesimal wedge.

By Theorem 5.1, we obtain the idempotent representation theorem for the specialization of bicomplex holomorphic functions.

Theorem 5.3. *We obtain an isomorphism of sheaves on $T_{\mathbb{R}^n}\mathbb{BC}^n$:*

$$\widetilde{\mathcal{A}_{\mathbb{BC}^n}} \simeq (T_{\mathbb{R}^n}\Phi_{\mathbf{e}})^{-1}\widetilde{\mathcal{A}_{\mathbb{C}^n\mathbf{e}}}\mathbf{e} \oplus (T_{\mathbb{R}^n}\Phi_{\mathbf{e}^\dagger})^{-1}\widetilde{\mathcal{A}_{\mathbb{C}^n\mathbf{e}^\dagger}}\mathbf{e}^\dagger. \tag{5.2}$$

Proof. By taking the 0-th cohomology of (5.1), we obtain (5.2). □

Theorem 5.3 says that any bicomplex holomorphic function on an infinitesimal wedge is described as a linear combination of $\mathbf{e}$ and $\mathbf{e}^\dagger$ with classical complex holomorphic functions on infinitesimal wedge coefficients. In other words, a bicomplex holomorphic function on an infinitesimal wedge corresponds to a pair of two classical complex holomorphic functions on infinitesimal wedges. As a corollary of this principle, we immediately obtain several fundamental properties of bicomplex holomorphic functions on infinitesimal wedges by applying properties of complex holomorphic functions on infinitesimal wedges to coefficients of $\mathbf{e}$ and $\mathbf{e}^\dagger$. For example, we can easily generalize the Bochner-type tube theorem in [12] for bicomplex holomorphic functions on infinitesimal wedges.

By Theorem 2.4, we obtain the Uchida-type fundamental exact sequence of the bicomplex specialization and the sheaves of bicomplex hyperfunctions and microfunctions. Let p_1^+ (resp. p_2^+) be the natural projection from

$$P^+ = \left\{(H,\Xi) \in T_{\mathbb{R}^n}\mathbb{BC}^n \underset{\mathbb{R}^n}{\times} T^*_{\mathbb{R}^n}\mathbb{BC}^n \;\middle|\; \langle H,\Xi\rangle > 0\right\}$$

to $T_{\mathbb{R}^n}\mathbb{BC}^n$ (resp. $T^*_{\mathbb{R}^n}\mathbb{BC}^n$).

Theorem 5.4. *There exists an exact sequence of sheaves on $T_{\mathbb{R}^n}\mathbb{BC}^n$:*

$$0 \longrightarrow \widetilde{A_{\mathbb{BC}^n}} \longrightarrow \tau^{-1}\mathcal{B}_{\mathbb{BC}^n} \longrightarrow (p_1^+)_*(p_2^+)^{-1}\mathcal{C}_{\mathbb{BC}^n}. \tag{5.3}$$

Proof. Let $\tau_{\mathbf{e}}\colon T_{\mathbb{R}^n\mathbf{e}}\mathbb{C}^n\mathbf{e} \to \mathbb{R}^n\mathbf{e}$ be the natural projection and $p_{\mathbf{e}1}^+$ (resp. $p_{\mathbf{e}2}^+$) the natural projection from $P_{\mathbf{e}}^+ = \{(\eta,\xi) \in T_{\mathbb{R}^n\mathbf{e}}\mathbb{C}^n\mathbf{e} \underset{\mathbb{R}^n\mathbf{e}}{\times} T^*_{\mathbb{R}^n\mathbf{e}}\mathbb{C}^n\mathbf{e} \mid \langle\eta,\xi\rangle > 0\}$ to $T_{\mathbb{R}^n\mathbf{e}}\mathbb{C}^n\mathbf{e}$ (resp. $T^*_{\mathbb{R}^n\mathbf{e}}\mathbb{C}^n\mathbf{e}$). By Theorem 2.4, we have the exact sequence

$$\begin{aligned} 0 \longrightarrow (T_{\mathbb{R}^n}\Phi_{\mathbf{e}})^{-1}\widetilde{\mathcal{A}_{\mathbb{C}^n\mathbf{e}}} &\longrightarrow (T_{\mathbb{R}^n}\Phi_{\mathbf{e}})^{-1}\tau_{\mathbf{e}}^{-1}(\Phi_{\mathbf{e}}|_{\mathbb{R}^n})^{-1}\mathcal{B}_{\mathbb{C}^n\mathbf{e}} \\ &\longrightarrow (T_{\mathbb{R}^n}\Phi_{\mathbf{e}})^{-1}(p_{\mathbf{e}1}^+)_*(p_{\mathbf{e}2}^+)^{-1}\mathcal{C}_{\mathbb{C}^n\mathbf{e}}. \end{aligned} \tag{5.4}$$

Let us consider the following commutative diagram where the two squares are Cartesian:

$$
\begin{tikzcd}
T^*_{\mathbb{R}^n}\mathbb{B}\mathbb{C}^n & T^*_{\mathbb{R}^n\mathbf{e}}\mathbb{C}^n\mathbf{e} \arrow[l, "{}^t\Phi'_{\mathbf{e}}"'] & \\
P^+ \arrow[u, "p_2^+"] \arrow[dr, "p_1^+"'] \arrow[ur, phantom, "\square"] & \widetilde{P^+} \arrow[l, "q_1"'] \arrow[r, "q_2"'] \arrow[u, "q_3"'] \arrow[d, "q_4"] \arrow[dr, phantom, "\square"] & P^+_{\mathbf{e}} \arrow[ul, "p^+_{\mathbf{e}2}"'] \arrow[d, "p^+_{\mathbf{e}1}"] \\
& T_{\mathbb{R}^n}\mathbb{B}\mathbb{C}^n \arrow[r, "T_{\mathbb{R}^n}\Phi_{\mathbf{e}}"] & T_{\mathbb{R}^n\mathbf{e}}\mathbb{C}^n\mathbf{e}.
\end{tikzcd}
$$

Here, $\mathbb{R}^n \underset{\mathbb{R}^n\mathbf{e}}{\times} T^*_{\mathbb{R}^n\mathbf{e}}\mathbb{C}^n\mathbf{e}$ is identified with $T^*_{\mathbb{R}^n\mathbf{e}}\mathbb{C}^n\mathbf{e}$ via the isomorphism $\Phi_{\mathbf{e}\pi}$ for the sake of simplicity. Furthermore, let us set

$$\widetilde{P^+} = P^+ \underset{T^*_{\mathbb{R}^n}\mathbb{B}\mathbb{C}^n}{\times} T^*_{\mathbb{R}^n\mathbf{e}}\mathbb{C}^n\mathbf{e} = T_{\mathbb{R}^n}\mathbb{B}\mathbb{C}^n \underset{T_{\mathbb{R}^n\mathbf{e}}\mathbb{C}^n\mathbf{e}}{\times} P^+_{\mathbf{e}}$$

and let q_1, q_2, q_3, q_4 denote the natural projections from $\widetilde{P^+}$ to P^+, $P^+_{\mathbf{e}}$, $T^*_{\mathbb{R}^n\mathbf{e}}\mathbb{C}^n\mathbf{e}$, $T_{\mathbb{R}^n}\mathbb{B}\mathbb{C}^n$ respectively. Since $T_{\mathbb{R}^n}\Phi_{\mathbf{e}}$, $p^+_{\mathbf{e}2}$ and p_2^+ are topological submersions with fiber dimension $2n$, n and $3n$ respectively and ${}^t\Phi'_{\mathbf{e}}$ is a closed embedding, we have the following isomorphisms in $\mathbf{D}^b(T_{\mathbb{R}^n}\mathbb{B}\mathbb{C}^n)$:

$$
\begin{aligned}
(T_{\mathbb{R}^n}\Phi_{\mathbf{e}})^{-1}R(p^+_{\mathbf{e}1})_*(p^+_{\mathbf{e}2})^{-1}\mu_{\mathbb{R}^n\mathbf{e}}(\mathcal{O}_{\mathbb{C}^n\mathbf{e}}) &\simeq (T_{\mathbb{R}^n}\Phi_{\mathbf{e}})^! R(p^+_{\mathbf{e}1})_*(p^+_{\mathbf{e}2})^!\mu_{\mathbb{R}^n\mathbf{e}}(\mathcal{O}_{\mathbb{C}^n\mathbf{e}})[-3n] \\
&\simeq Rq_{4*}q_2^!(p^+_{\mathbf{e}2})^!\mu_{\mathbb{R}^n\mathbf{e}}(\mathcal{O}_{\mathbb{C}^n\mathbf{e}})[-3n] \\
&\simeq R(p_1^+)_*Rq_{1*}q_3^!\mu_{\mathbb{R}^n\mathbf{e}}(\mathcal{O}_{\mathbb{C}^n\mathbf{e}})[-3n] \\
&\simeq R(p_1^+)_*(p_2^+)^!({}^t\Phi'_{\mathbf{e}})_*\mu_{\mathbb{R}^n\mathbf{e}}(\mathcal{O}_{\mathbb{C}^n\mathbf{e}})[-3n] \\
&\simeq R(p_1^+)_*(p_2^+)^{-1}({}^t\Phi'_{\mathbf{e}})_*\mu_{\mathbb{R}^n\mathbf{e}}(\mathcal{O}_{\mathbb{C}^n\mathbf{e}})
\end{aligned}
$$

by [2, Prop. 3.1.9]. By (5.4), we have the exact sequence

$$0 \longrightarrow (T_{\mathbb{R}^n}\Phi_{\mathbf{e}})^{-1}\widetilde{\mathcal{A}_{\mathbb{C}^n\mathbf{e}}} \longrightarrow \tau^{-1}(\Phi_{\mathbf{e}}|_{\mathbb{R}^n})^{-1}\mathcal{B}_{\mathbb{C}^n\mathbf{e}} \longrightarrow (p_1^+)_*(p_2^+)^{-1}({}^t\Phi'_{\mathbf{e}})_*\mathcal{C}_{\mathbb{C}^n\mathbf{e}}.$$

In the same way, we have

$$0 \longrightarrow (T_{\mathbb{R}^n}\Phi_{\mathbf{e}^\dagger})^{-1}\widetilde{\mathcal{A}_{\mathbb{C}^n\mathbf{e}^\dagger}} \longrightarrow \tau^{-1}(\Phi_{\mathbf{e}^\dagger}|_{\mathbb{R}^n})^{-1}\mathcal{B}_{\mathbb{C}^n\mathbf{e}^\dagger} \longrightarrow (p_1^+)_*(p_2^+)^{-1}({}^t\Phi'_{\mathbf{e}^\dagger})_*\mathcal{C}_{\mathbb{C}^n\mathbf{e}^\dagger}.$$

By Theorems 3.4, 4.4 and 5.3, we obtain (5.3). □

Note that the exact sequence (5.3) can also be obtained directly by the abstract Uchida triangle in [13]. However, we can understand the property better by reducing to the classical complex case by using the idempotent representations. We call the morphism $b\colon \widetilde{\mathcal{A}_{\mathbb{B}\mathbb{C}^n}} \to \tau^{-1}\mathcal{B}_{\mathbb{B}\mathbb{C}^n}$ of (5.3), which is induced by

$\mathrm{id} \to \tau^! R\tau_!$, the boundary value morphism. By the boundary value morphism b, a bicomplex hyperfunction is represented as a finite sum of boundary values of bicomplex holomorphic functions on infinitesimal wedges.

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