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Boundary spectral estimates for semiclassical Gevrey operators

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Abstract. We obtain the spectral and resolvent estimates for semiclassical pseudodifferential operators with symbol of Gevrey-s regularity, near the boundary of the range of the principal symbol. We prove that the boundary spectrum free region is of size $\mathcal{O}(h^{1-\frac{1}{s}})$ where the resolvent is at most fractional exponentially large in h, as the semiclassical parameter $h \to 0^+$. This is a natural Gevrey analogue of a result by N. Dencker, J. Sjöstrand, and M. Zworski in the C^∞ and analytic cases.

1. Introduction and statement of results

In this work, we study the spectrum of a non-self-adjoint semiclassical Gevrey pseudodifferential operator, as the semiclassical parameter $h \to 0^+$. Unlike self-adjoint operators, it is well known that the spectrum of a non-self-adjoint operator may lie deep inside the range of its leading symbol as $h \to 0^+$. For instance, the complex harmonic oscillator: $-h^2 \frac{d^2}{dx^2} + ix^2$ on $L^2(\mathbb{R})$, which was used by Davies [5] as an inspiring example of non-normal differential operators, has purely discrete spectrum $\{e^{i\pi/4}h(2k+1) : k \in \mathbb{N}\}$; while the range of its symbol $\xi^2 + ix^2$ on $(x,\xi) \in T^*\mathbb{R}$ is the sector $\{z \in \mathbb{C} : 0 \le \arg z \le \pi/2\}$.

More generally, Dencker, Sjöstrand, and Zworski [6] considered spectral estimates for quantizations of bounded functions, with all derivatives bounded,

$$p \in C_b^\infty(\mathbb{R}^{2n}) := \{u \in C^\infty(\mathbb{R}^{2n}) : \partial^\alpha u \in L^\infty(\mathbb{R}^{2n}) \text{ for all } \alpha \in \mathbb{N}^{2n}\}.$$

It has been pointed out in [6] that the case of functions whose values avoid a point in $\mathbb{C}$ and tend to infinity as $(x,\xi) \to \infty$ can be reduced to this case. Let us denote the closure of range of p by $\Sigma(p) := \overline{p(T^*\mathbb{R}^n)}$, and denote by $\Sigma_\infty(p)$ the set of accumulation points of p at infinity. Let $z_0 \in \partial\Sigma(p)$ and suppose the principal-type condition:

$$dp(x,\xi) \neq 0, \quad \text{if } p(x,\xi) = z_0,\ (x,\xi) \in T^*\mathbb{R}^n. \tag{1.1}$$

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For every $\rho \in p^{-1}(z_0)$ with $z_0 \in \partial\Sigma(p)$, let $\theta = \theta(\rho) \in \mathbb{R}$ be such that $e^{-i\theta} dp$ is real at ρ. Let us also assume a non-trapping condition on p and z_0 (see [6] or [30]):

$$\begin{aligned}&\text{for every } \rho \in p^{-1}(z_0), \text{ the complete trajectory of } H_{\mathrm{Re}(e^{-i\theta(\rho)}p)}\\ &\text{that passes through } \rho \text{ is not contained in } p^{-1}(z_0),\end{aligned} \tag{1.2}$$

where $H_f(\rho) = (f'_\xi(\rho), -f'_x(\rho))$, $\rho \in T^*\mathbb{R}^n$, is the Hamilton vector field of f.

Under these conditions, it has been proved in [6] that for a semiclassical operator $P(h)$ whose principal part is given by the Weyl quantization of $p \in C_b^\infty(\mathbb{R}^{2n})$:

$$p^w(x, hD)u(x) = \frac{1}{(2\pi h)^n} \int\limits_{\mathbb{R}^{2n}} e^{\frac{i}{h}(x-y)\cdot\theta} p\Big(\frac{x+y}{2}, \theta\Big) u(y)\, dy\, d\theta, \quad u \in \mathcal{S}(\mathbb{R}^n), \tag{1.3}$$

if $z_0 \in \partial\Sigma(p) \setminus \Sigma_\infty(p)$ satisfies (1.1) and (1.2), then for any $M > 0$, there exists $h_0(M) > 0$ such that

$$\Big\{z \in \mathbb{C} : |z - z_0| < Mh \log\Big(\frac{1}{h}\Big)\Big\} \cap \sigma(P(h)) = \emptyset, \quad 0 < h < h_0(M), \tag{1.4}$$

where $\sigma(P(h))$ denotes the spectrum of $P(h)$.

Furthermore, in the case where p is a bounded holomorphic function in a tubular neighborhood of $\mathbb{R}^{2n} \subset \mathbb{C}^{2n}$, then there exist $\delta_0, h_0 > 0$ such that

$$\{z \in \mathbb{C} : |z - z_0| < \delta_0\} \cap \sigma(P(h)) = \emptyset, \quad 0 < h < h_0. \tag{1.5}$$

If we compare the size of the spectrum free region near the boundary of $\Sigma(p)$ for bounded smooth symbols p in (1.4) with that for bounded holomorphic symbols in (1.5), we observe that the size improves from $\mathcal{O}(h \log(1/h))$ to $\mathcal{O}(1)$. Motivated by that, the purpose of this work is to explore the spectrum free region near the boundary of $\Sigma(p)$ for bounded Gevrey symbols p (see the definition below), which can be viewed as an interpolating case between the bounded smooth and holomorphic symbols.

The consideration of Gevrey (pseudo)differential operators has a long-standing tradition in the theory of PDEs, beginning with the seminal work [2], see also [20,25]. Gevrey regularity problems have been studied in various contexts, including quantum theory [1,7,15,26,27], FBI transform in Gevrey classes [8,19], propagation of Gevrey singularities [17, 18, 31], and Gevrey pseudodifferential operators in the complex domain [11, 12].

Let $s > 1$. The bounded (global) Gevrey-s class, denoted by $\mathcal{G}_b^s(\mathbb{R}^d)$, consists of all functions $u \in C^\infty(\mathbb{R}^d)$ such that there exists $C > 0$ such that

$$|\partial^\alpha u(x)| \le C^{1+|\alpha|} \alpha!^s, \quad \text{for all } \alpha \in \mathbb{N}^d,\ x \in \mathbb{R}^d.$$

The Denjoy–Carleman theorem [14, Theorem 1.3.8] implies that the Gevrey-s class is non-quasianalytic, i.e., $\mathscr{G}^s_{\mathrm{c}}(\mathbb{R}^d) := \mathscr{G}^s_{\mathrm{b}}(\mathbb{R}^d) \cap C^\infty_{\mathrm{c}}(\mathbb{R}^n) \neq \{0\}$, when $s > 1$. Therefore, there are $\mathscr{G}^s$ partitions of unity.

To study semiclassical Gevrey operators, we define Gevrey symbols as functions in $\mathscr{G}^s_{\mathrm{b}}(\mathbb{R}^{2n})$ that may depend on the semiclassical parameter $h \in (0, 1]$. More precisely, we write $a(\cdot; h) \in \mathscr{G}^s_{\mathrm{b}}(\mathbb{R}^{2n})$ if and only if for some $C > 0$ uniformly in $h \in (0, 1]$ we have

$$|\partial_x^\alpha \partial_\theta^\beta a(x, \theta; h)| \leq C^{1+|\alpha|+|\beta|} \alpha!^s \beta!^s \quad \text{for all } l\alpha, \beta \in \mathbb{N}^n,\ (x, \theta) \in \mathbb{R}^{2n}.$$

Let us then introduce semiclassical Gevrey pseudo-differential operators, which are semiclassical Weyl quantizations of $a(\cdot; h) \in \mathscr{G}^s_{\mathrm{b}}(\mathbb{R}^{2n})$ acting on $u \in \mathcal{S}(\mathbb{R}^n)$,

$$a^w(x, hD; h)u(x) = \frac{1}{(2\pi h)^n} \int\limits_{\mathbb{R}^{2n}} e^{\frac{i}{h}(x-y)\cdot\theta} a\Big(\frac{x+y}{2}, \theta; h\Big) u(y)\, dy\, d\theta.$$

We recall that $a^w(x, hD; h)$ extends to a bounded operator on $L^2(\mathbb{R}^d)$ uniformly in $h \in (0, 1]$, see for instance [32, Section 4.5].

We say that an h-independent function $a_0 \in \mathscr{G}^s_{\mathrm{b}}(\mathbb{R}^{2n})$ is the *principal symbol* of the semiclassical symbol $a(\cdot; h) \in \mathscr{G}^s_{\mathrm{b}}(\mathbb{R}^{2n})$ if there exists $r(\cdot; h) \in \mathscr{G}^s_{\mathrm{b}}(\mathbb{R}^{2n})$ such that

$$a(x, \theta; h) = a_0(x, \theta) + hr(x, \theta; h) \quad \text{for all } (x, \theta) \in \mathbb{R}^{2n},\ h \in (0, 1].$$

Moreover, $a_0^w(x, hD)$ is called the principal part of $a^w(x, hD; h)$.

The following is the main result of this work.

Theorem 1. *Let $p \in \mathscr{G}^s_b(\mathbb{R}^{2n})$, $s > 1$, and let $p^w(x, hD)$ be the principal part of a semiclassical Gevrey operator $P(h) = P(x, hD; h)$. If p and $z_0 \in \partial\Sigma(p) \setminus \Sigma_\infty(p)$ satisfy conditions (1.1) and (1.2), then there exist $h_0 > 0$ and $C > 0$ such that*

$$\{z \in \mathbb{C} : |z - z_0| < C^{-1} h^{1-\frac{1}{s}}\} \cap \sigma(P(h)) = \emptyset, \quad 0 < h < h_0.$$

Furthermore, for $z \in \mathbb{C}$ with $|z - z_0| < C^{-1} h^{1-\frac{1}{s}}$ we have the resolvent estimate

$$(P(h) - z)^{-1} = \mathcal{O}(1) \exp(\mathcal{O}(1) h^{-\frac{1}{s}}) \colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad 0 < h < h_0. \tag{1.6}$$

Remark 1. In the context of resonances, it has been shown in [26, 27] that semiclassical Schrödinger operators with $\mathscr{G}^s$ potentials which are dilation analytic near infinity have a resonance free region of size $\mathcal{O}(h^{1-\frac{1}{s}})$ near a non-trapping energy level in the semiclassical limit $h \to 0^+$ and that the exponent $1 - 1/s$ is optimal by constructing a $\mathscr{G}^s$ potential such that there exist resonances E near a non-trapping energy level $E_0 > 0$ with $\operatorname{Im} E \approx -Ch^{1-\frac{1}{s}}$, $C > 0$, for h sufficiently small [26]. We can therefore infer that the exponent $1 - 1/s$ for the spectrum free region in Theorem 1 is optimal.

Let us also highlight a special case of Theorem 1, which is commonly considered for evolution equations or semigroups $\exp(-tP/h)$, i.e., the case where

$$\operatorname{Re} p \geq 0 \quad \text{near } p^{-1}(0). \tag{1.7}$$

The principal-type condition (1.1) in this case implies that

$$d \operatorname{Im} p \neq 0, \quad d \operatorname{Re} p = 0, \quad \text{on } p^{-1}(0). \tag{1.8}$$

In view of the non-trapping condition (1.2), we assume in this case that

$$\begin{aligned} \text{for all } \rho \in p^{-1}(0), \quad & \text{the maximal trajectory of } H_{\operatorname{Im} p} \text{ passing through } \rho \\ & \text{contains a point where } \operatorname{Re} p > 0. \end{aligned} \tag{1.9}$$

Under these conditions, we have the following.

Theorem 2. *Let $p \in \mathscr{G}_b^s(\mathbb{R}^{2n})$, $s > 1$, and let $p^w(x, hD)$ be the principal part of a semiclassical Gevrey operator $P(h) = P(x, hD; h)$. Suppose that $0 \notin \Sigma_\infty(p)$ and that p satisfies (1.7)–(1.9). Then there exist $h_0 > 0$ and $C > 0$ such that*

$$|z| < C^{-1} \textit{ and } \operatorname{Re} z < C^{-1} h^{1-\frac{1}{s}} \implies z \notin \sigma(P(h)), \quad 0 < h < h_0.$$

Moreover, the estimate (1.6) holds for $z \in \mathbb{C}$ with $|z| < C^{-1}$ and $\operatorname{Re} z < C^{-1} h^{1-\frac{1}{s}}$.

In the case of analytic symbols, Dencker et al [6] proved (1.5) by studying the action of $P(h)$ on microlocally weighted spaces associated to a family of complex $I\mathrm{R}$ manifolds distorted from the phase space $T^*\mathbb{R}^n$, see [9] for the original method and [29, Chapter 12] for a detailed presentation. For a broader context, we also mention what we call the “Martinez’ method” [22]: one can use a non-holomorphic FBI transform $T\colon L^2(\mathbb{R}^n) \to L^2(T^*\mathbb{R}^n)$ and impose exponential weights e^{g/h^α} directly on $T^*\mathbb{R}^n$. This approach was used to study the tunneling effects, for analytic operators [21, 23], and for Gevrey operators [15]. In this paper, we adopt an approach different from those mentioned above: one modifies the exponential weights for the Bargmann space, working with a holomorphic FBI transform. A key ingredient in our proof is a Toeplitz identity that connects the action of semiclassical operators on the complex domain to the multiplication by the principal symbols. Such a result is essentially well known, see [28] for the analytic case, [10] for the smooth case, and [27] for the Gevrey case. Thanks to the techniques recently developed in [12], we will use a straightforward argument to establish a more general Toeplitz identity than that in [27], see Remark 3.

The paper is organized as follows. In Section 2, we review and introduce some essential tools for semiclassical pseudodifferential operators with Gevrey symbols, including a Toeplitz identity in the complex domain and a composition formula in the real domain. Section 3 is devoted to the proof of Theorem 2 by introducing small complex deformations of $\mathbb{R}^{2n}$ and working on the FBI transform side. In Section 4, we introduce a Gevrey multiplier using a version of the Malgrange preparation theorem for Gevrey functions, which allows us to reduce Theorem 1 to the more special Theorem 2, thus completing the proof of our main theorem.

2. Review of semiclassical Gevrey operators in the complex domain

Let Φ_0 be a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$ and let us set

$$\Lambda_{\Phi_0} = \Big\{\Big(x, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x)\Big),\ x \in \mathbb{C}^n\Big\} \subset \mathbb{C}^{2n}.$$

Let us also introduce the Bargmann space

$$H_{\Phi_0}(\mathbb{C}^n) = \mathrm{Hol}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n, e^{-2\Phi_0/h} L(dx)), \tag{2.1}$$

where $L(dx)$ is the Lebesgue measure on $\mathbb{C}^n$ and $0 < h \leq 1$ is the semiclassical parameter. Using the projection map $\pi_x \colon \Lambda_{\Phi_0} \ni (x,\xi) \mapsto x \in \mathbb{C}^n_x \cong \mathbb{R}^{2n}$, we identify Λ_{Φ_0} with $\mathbb{C}^n_x$ and define the Gevrey spaces $\mathscr{G}^s_{\mathrm{b}}(\Lambda_{\Phi_0})$, $\mathscr{G}^s_{\mathrm{c}}(\Lambda_{\Phi_0})$. Let $a \in \mathscr{G}^s_b(\Lambda_{\Phi_0})$ be an h-independent symbol, for some $s > 1$, and let $u \in \mathrm{Hol}(\mathbb{C}^n)$ be such that

$$u(x) = \mathcal{O}_{h,N}(1)\langle x\rangle^{-N} e^{\Phi_0(x)/h},$$

for all $N \in \mathbb{N}$. We introduce the semiclassical Weyl quantization of a acting on u,

$$a^w_\Gamma(x, hD_x)u(x) = \frac{1}{(2\pi h)^n} \iint\limits_{\Gamma(x)} e^{\frac{i}{h}(x-y)\cdot\theta} a\Big(\frac{x+y}{2}, \theta\Big) u(y)\, dy \wedge d\theta. \tag{2.2}$$

Here $\Gamma(x) \subset \mathbb{C}^{2n}_{y,\theta}$ is the natural integration contour given by

$$\theta = \frac{2}{i}\frac{\partial \Phi_0}{\partial x}\Big(\frac{x+y}{2}\Big), \quad y \in \mathbb{C}^n.$$

Let next $\Phi_1 \in C^{1,1}(\mathbb{C}^n; \mathbb{R})$ be such that

$$\|\nabla^k(\Phi_1 - \Phi_0)\|_{L^\infty(\mathbb{C}^n)} \leq C^{-1} h^{1-\frac{1}{s}}, \quad k = 0, 1, 2, \tag{2.3}$$

for some $C > 0$ sufficiently large. We set $\omega = h^{1-\frac{1}{s}}$ and introduce the following $2n$-dimensional Lipschitz contour for $x \in \mathbb{C}^n$:

$$\Gamma^{\Phi_1}_\omega(x): \quad \theta = \frac{2}{i}\frac{\partial \Phi_1}{\partial x}\Big(\frac{x+y}{2}\Big) + i f_\omega(x-y), \quad y \in \mathbb{C}^n, \tag{2.4}$$

where

$$f_\omega(z) = \begin{cases} \bar{z}, & |z| \leq \omega, \\ \omega |z|^{-1} \bar{z}, & |z| > \omega. \end{cases} \tag{2.5}$$

Let $\tilde{a} \in \mathscr{G}_b^s(\mathbb{C}^{2n})$ be an almost holomorphic extension of a such that $\operatorname{supp} \tilde{a} \subset \Lambda_{\Phi_0} + B_{\mathbb{C}^{2n}}(0, C_0)$, for some $C_0 > 0$. We remark that the existence of such an almost holomorphic extension whose Gevrey order is the same as that of a is due to Carleson [4] (see also [8]). It has been established in [12, Theorem 1.1] that for $1 < s \leq 2$ (the complementary range $s > 2$ will be discussed later),

$$\begin{gathered} a^w_\Gamma(x, hD_x) - \tilde{a}^w_{\Gamma^{\Phi_1}_\omega}(x, hD_x) = \mathcal{O}(1) \exp(-C^{-1} h^{-\frac{1}{s}}) : \\ H_{\Phi_1}(\mathbb{C}^n) \to L^2(\mathbb{C}^n, e^{-2\Phi_1/h} L(dx)), \end{gathered} \tag{2.6}$$

where $C > 0$ is a constant and we have set, similarly to (2.1),

$$H_{\Phi_1}(\mathbb{C}^n) = \mathrm{Hol}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n, e^{-2\Phi_1/h} L(dx)).$$

The realization

$$\tilde{a}^w_{\Gamma^{\Phi_1}_\omega}(x, hD_x) u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma^{\Phi_1}_\omega(x)} e^{\frac{i}{h}(x-y)\cdot\theta} \tilde{a}\Big(\frac{x+y}{2}, \theta\Big) u(y)\, dy \wedge d\theta$$

satisfies (see [12, Theorem. 1.1 and 1.2])

$$\tilde{a}^w_{\Gamma^{\Phi_1}_\omega}(x, hD_x) = \mathcal{O}(1) : H_{\Phi_1}(\mathbb{C}^n) \to L^2(\mathbb{C}^n, e^{-2\Phi_1/h} L(dx)).$$

Let us also recall the following version of the Fourier inversion formula in the complex domain, see for instance [13]. Let $u \in \mathrm{Hol}(\mathbb{C}^n)$ be such that

$$u(x) = \mathcal{O}_h(1) \langle x \rangle^{N_0} e^{\Phi_0(x)/h},$$

for some $N_0 \geq 0$. Then,

$$u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma^{\Phi_1}_\omega(x)} e^{\frac{i}{h}(x-y)\cdot\theta} u(y)\, dy \wedge d\theta. \tag{2.7}$$

In particular, (2.7) holds for $u \in H_{\Phi_1}(\mathbb{C}^n) = H_{\Phi_0}(\mathbb{C}^n)$ (they are equal as linear spaces). Similarly, for such functions, we find by Stokes' formula, writing $D_{x_j} := i^{-1}\partial_{x_j}$,

$$hD_{x_j} u(x) = \frac{1}{(2\pi h)^n} \iint_{\Gamma^{\Phi_1}_\omega(x)} e^{\frac{i}{h}(x-y)\cdot\theta} \theta_j u(y)\, dy \wedge d\theta, \quad 1 \leq j \leq n. \tag{2.8}$$

Setting $\xi_1(x) = \frac{2}{i}\frac{\partial \Phi_1}{\partial x}(x)$, we get by a Taylor expansion, when $(y,\theta) \in \Gamma_\omega^{\Phi_1}(x)$,

$$\tilde{a}\Big(\frac{x+y}{2},\theta\Big) = \tilde{a}(\rho_0) + \partial_x \tilde{a}(\rho_0)\cdot \Delta y + \partial_\theta \tilde{a}(\rho_0)\cdot \Delta\theta + r(x,y,\theta), \tag{2.9}$$

with more compact notation: $\rho_0 = (x, \xi_1(x))$, $\Delta y = \frac{y-x}{2}$, $\Delta\theta = \theta - \xi_1(x)$, and

$$r(x,y,\theta) := \partial_{\bar{x}}\tilde{a}(\rho_0)\cdot \overline{\Delta y} + \partial_{\bar{\theta}}\tilde{a}(\rho_0)\cdot\overline{\Delta\theta} + \int_0^1 (1-t)\,\tilde{a}_t^{(2)}(x,y,\theta)\,dt. \tag{2.10}$$

Here $\tilde{a}_t^{(2)}(x,y,\theta) = \mathcal{O}(1)|x-y|^2$, since, along the contour $\Gamma_\omega^{\Phi_1}(x)$, we have

$$|\Delta\theta| = |\theta - \xi_1(x)| \le \|\nabla^2\Phi_1\|_{L^\infty(\mathbb{C}^n)}|x-y| + |f_\omega(x-y)| \le \mathcal{O}(|x-y|).$$

Let us recall from [8, Remark 1.7] that there exists $C > 0$ such that

$$|\bar{\partial}\tilde{a}(x,\xi)| \le C\exp\big(-C^{-1}\mathrm{dist}((x,\xi),\Lambda_{\Phi_0})^{-\frac{1}{s-1}}\big), \quad (x,\xi)\in\mathbb{C}^{2n}. \tag{2.11}$$

We also note that

$$\begin{aligned}\mathrm{dist}(\rho_0,\Lambda_{\Phi_0}) &\le \Big|\frac{2}{i}\frac{\partial\Phi_1}{\partial x}(x) - \frac{2}{i}\frac{\partial\Phi_0}{\partial x}(x)\Big| \\ &\le 2\|\nabla(\Phi_1-\Phi_0)\|_{L^\infty(\mathbb{C}^n)} \le \mathcal{O}(1)h^{1-\frac{1}{s}}.\end{aligned} \tag{2.12}$$

Combing (2.10)–(2.12) we conclude

$$r(x,y,\theta) = \mathcal{O}(1)|x-y|^2 + \mathcal{O}(1)\exp\big(-C^{-1}h^{-\frac{1}{s}}\big), \quad C>0. \tag{2.13}$$

Let us set

$$Ru(x) = \frac{1}{(2\pi h)^n}\iint_{\Gamma_\omega^{\Phi_1}(x)} e^{\frac{i}{h}(x-y)\cdot\theta} r(x,y,\theta)\,u(y)dy\wedge d\theta,$$

we shall next check that

$$R = \mathcal{O}(h)\colon L^2(\mathbb{C}^n, e^{-2\Phi_1/h}L(dx)) \to L^2(\mathbb{C}^n, e^{-2\Phi_1/h}L(dx)). \tag{2.14}$$

To this end, we consider the distribution kernel of R, writing

$$Ru(x) = \int k(x,y;h)u(y)\,L(dy),$$

we infer from the proof of [12, Theorem 3.3] together with (2.13) that

$$e^{-\frac{\Phi_1(x)}{h}}|k(x,y;h)|e^{\frac{\Phi_1(y)}{h}} \le \mathcal{O}(1)h^{-n}(|x-y|^2 + e^{-C^{-1}h^{-\frac{1}{s}}})e^{-\frac{F_\omega(x-y)}{2h}}$$

provided that the constant in (2.3) is sufficiently large. Here, following [12], we set

$$0 \le F_\omega(z) = \mathrm{Re}(z \cdot f_\omega(z)) = \begin{cases} |z|^2, & |z| \le \omega, \\ \omega|z|, & |z| > \omega. \end{cases}$$

In view of Schur's lemma, we only have to control the L^1 norm

$$\begin{aligned} h^{-n} \int |x|^2 e^{-\frac{F_\omega(x)}{2h}} L(dx) &= h^{-n} \int_{|x|\le\omega} |x|^2 e^{-\frac{|x|^2}{2h}} L(dx) + h^{-n} \int_{|x|\ge\omega} |x|^2 e^{-\frac{\omega|x|}{2h}} L(dx) \\ &\le \mathcal{O}(1)h + \mathcal{O}(1)h\frac{h^{n+1}}{\omega^{2n+2}} = \mathcal{O}(h), \end{aligned}$$

since $\frac{h}{\omega^2} = h^{\frac{2}{s}-1} \le 1$ if $1 < s \le 2$. The estimate (2.14) therefore follows, and combining it with (2.6), (2.9), (2.7), and (2.8), we get for $u \in H_{\Phi_1}(\mathbb{C}^n)$,

$$a^w(x, hD_x)u(x) = \tilde{a}(x, \xi_1(x))u(x) + \partial_\theta \tilde{a}(x, \xi_1(x)) \cdot (hD_x - \xi_1(x))u(x) + \widetilde{R}u, \tag{2.15}$$

with

$$\widetilde{R} = \mathcal{O}(h)\colon H_{\Phi_1}(\mathbb{C}^n) \to L^2(\mathbb{C}^n, e^{-2\Phi_1/h} L(dx)).$$

The discussion above, developed in the case $1 < s \le 2$, extends to the complementary range $s > 2$. Indeed, in this case, an application of [12, Theorem 1.2] yields

$$\begin{aligned} a_\Gamma^w(x, hD_x) - \tilde{a}^w_{\Gamma^{\Phi_1}_{h^{1/2}}}(x, hD_x) &= \mathcal{O}(1) \exp(-C^{-1} h^{-\frac{1}{2s-2}}), \\ H_{\Phi_1}(\mathbb{C}^n) &\to L^2(\mathbb{C}^n, e^{-2\Phi_1/h} L(dx)), \quad C > 0. \end{aligned}$$

Here $\Gamma^{\Phi_1}_{h^{1/2}}(x)$ is the $2n$-dimensional Lipschitz contour defined as in (2.4) and (2.5), with ω replaced by $h^{1/2} \ge \omega$. We have

$$\tilde{a}^w_{\Gamma^{\Phi_1}_{h^{1/2}}}(x, hD_x) = \mathcal{O}(1)\colon H_{\Phi_1}(\mathbb{C}^n) \to L^2(\mathbb{C}^n, e^{-2\Phi_1/h} L(dx)).$$

It is then easy to see that we still get (2.15) for $s > 2$.

Using (2.15) and arguing as in [10, 28], we get the following result.

Proposition 2. *Let $a \in \mathcal{G}^s_b(\Lambda_{\Phi_0})$, $s > 1$, and let $\Phi_1 \in C^{1,1}(\mathbb{C}^n)$ be such that (2.3) holds. Let $\tilde{a} \in \mathcal{G}^s_b(\mathbb{C}^{2n})$ be an almost holomorphic extension of a such that $\operatorname{supp} \tilde{a} \subset \Lambda_{\Phi_0} + B_{\mathbb{C}^{2n}}(0, C_0)$, for some $C_0 > 0$. Let $\psi \in W^{1,\infty}(\mathbb{C}^n) \iff \psi \in L^\infty(\mathbb{C}^n)$, $\nabla\psi \in L^\infty(\mathbb{C}^n)$. We have for $u, v \in H_{\Phi_1}(\mathbb{C}^n)$,*

$$\begin{aligned} (\psi\, a^w(x, hD_x)u, v)_{H_{\Phi_1}} = &\int \psi(x)\tilde{a}(x, \xi_1(x))u(x)\overline{v(x)} e^{-2\Phi_1(x)/h} L(dx) \\ &+ \mathcal{O}(h)\|u\|_{H_{\Phi_1}} \|v\|_{H_{\Phi_1}}. \end{aligned} \tag{2.16}$$

Let $b \in \mathscr{G}_b^s(\Lambda_{\Phi_0})$, $s > 1$, and let us apply Proposition 2 with $b^w(x, hD_x)v$ replacing v, for some $v \in H_{\Phi_1}(\mathbb{C}^n)$, using also (2.16) for $b^w(x, hD_x)$. We obtain

$$\begin{aligned}\int \psi(x)\tilde{a}(x, \xi_1(x))\overline{\tilde{b}(x, \xi_1(x))}u(x)\overline{v(x)}e^{-2\Phi_1(x)/h}\,L(dx) \\ = (\psi\, a^w(x, hD_x)u, b^w(x, hD_x)v)_{H_{\Phi_1}} + \mathcal{O}(h)\|u\|_{H_{\Phi_1}}\|v\|_{H_{\Phi_1}}.\end{aligned} \tag{2.17}$$

Here $\tilde{b} \in \mathscr{G}_b^s(\mathbb{C}^{2n})$ is an almost holomorphic extension of b, as above, and we have also used the fact that $b^w(x, hD_x) = \mathcal{O}(1)\colon H_{\Phi_1}(\mathbb{C}^n) \to H_{\Phi_1}(\mathbb{C}^n)$, see [11, 12].

Remark 3. The Toeplitz identity (Proposition 2) we derived is more general than [27, Proposition 4.1]. Notably, the weight Φ_1 is only required to be $C^{1,1}$-close to Φ_0, rather than close in $\mathscr{G}^s$ This relaxation allows us to use just a C^∞ escape function (see Lemma 6) instead of a $\mathscr{G}^s$ escape function. More importantly, the cutoff ψ is only assumed to be $W^{1,\infty}$ instead of C_c^∞ as required in [27]. This weaker assumption makes it possible to derive elliptic estimates near infinity, such as Proposition 4.

As an application of Proposition 2 and (2.17), we derive an elliptic estimate for future reference. Let us make an assumption on $a \in \mathscr{G}_{\mathrm{b}}^s(\Lambda_{\Phi_0})$ that there exists a bounded open subset $U \subset \mathbb{C}^n$ with a constant $C > 0$ such that

$$|a(x, \xi)| \geq 2/C, \quad (x, \xi) \in \Lambda_{\Phi_0},\ \ x \in \mathbb{C}^n \setminus U. \tag{2.18}$$

Under this assumption, we have the following.

Proposition 4. *Suppose $a \in \mathscr{G}_b^s(\Lambda_{\Phi_0})$ satisfies (2.18) for some bounded open set $U \subset \mathbb{C}^n$ and some $C > 0$. Let $\tilde{a} \in \mathscr{G}_b^s(\mathbb{C}^{2n})$ be an almost holomorphic extension of a as in Proposition 2, and let $\Phi_1 \in C^{1,1}(\mathbb{C}^n)$ be such that (2.3) holds. Then there exists $h_0 > 0$ such that for all $0 < h < h_0$ and $u \in H_{\Phi_1}$ we have*

$$\int_{\mathbb{C}^n \setminus U} |u(x)|^2 e^{-\frac{2\Phi_1(x)}{h}}\,L(dx) \leq \mathcal{O}(1)\|a^w(x, hD_x)u\|^2_{H_{\Phi_1}(\mathbb{C}^n)} + \mathcal{O}(h)\|u\|^2_{H_{\Phi_1}(\mathbb{C}^n)}. \tag{2.19}$$

Proof. , It follows from Proposition 2 and more specifically, equation (2.17) that for $u \in H_{\Phi_1}(\mathbb{C}^n)$,

$$\begin{aligned}\int |\tilde{a}(x, \xi_1(x))|^2|u(x)|^2 e^{-\frac{2\Phi_1(x)}{h}}\,L(dx) \\ \leq \|a^w(x, hD_x)u\|^2_{H_{\Phi_1}(\mathbb{C}^n)} + \mathcal{O}(h)\|u\|^2_{H_{\Phi_1}(\mathbb{C}^n)}.\end{aligned} \tag{2.20}$$

Recalling from (2.12) that $\operatorname{dist}((x, \xi_1(x)), \Lambda_{\Phi_0}) = \mathcal{O}(1)h^{1-\frac{1}{s}}$, then for h sufficiently small we have, in view of (2.18) and the fact that $\tilde{a} \in \mathscr{G}_{\mathrm{b}}^s(\mathbb{C}^{2n})$, $\tilde{a}|_{\Lambda_{\Phi_0}} = a$,

$$|\tilde{a}(x, \xi_1(x))| \geq 1/C, \quad x \in \mathbb{C}^n \setminus U.$$

Combining this with (2.20) we obtain (2.19). ∎

We finish this section by discussing the composition of semiclassical Gevrey operators in the real domain. It has been proved in [17, 19] that for $a, b \in \mathscr{G}_{\mathrm{b}}^s(\mathbb{R}^{2n})$ one has $a^w(x, hD) \circ b^w(x, hD) = c^w(x, hD; h)$ where $c(\cdot; h) = a \# b \in \mathscr{G}_{\mathrm{b}}^s(\mathbb{R}^{2n})$. An alternative proof of this result has been provided in [12, Section 3.3] using contour deformations. For future reference, we note a slightly finer characterization of the composed symbol $c = a \# b$ than $c \in \mathscr{G}_{\mathrm{b}}^s(\mathbb{R}^{2n})$ as follows.

Proposition 5. *Let $a, b \in \mathscr{G}_b^s(\mathbb{R}^{2n})$ be h-independent symbols for some $s > 1$, and let $c^w(x, hD; h) = a^w(x, hD) \circ b^w(x, hD)$. Then the symbol c satisfies*

$$c(x, \theta; h) = a(x, \theta) b(x, \theta) + h r(x, \theta; h), \quad (x, \theta) \in \mathbb{R}^{2n}, \tag{2.21}$$

for some $r(\cdot; h) \in \mathscr{G}_b^s(\mathbb{R}^{2n})$.

Proof. Let us first recall the following oscillatory integral representation of the composed symbol $c = a \# b$, see for instance [32, Chapter 4]:

$$\begin{aligned} &c(x, \theta; h) \\ &\quad = \frac{1}{(\pi h)^{2n}} \int_{\mathbb{R}^{4n}} e^{-\frac{2i}{h}\sigma(y_1, \eta_1; y_2, \eta_2)} a(x + y_1, \theta + \eta_1) b(x + y_2, \theta + \eta_2)\, dy_1 d\eta_1 dy_2 d\eta_2. \end{aligned} \tag{2.22}$$

Here σ is the standard symplectic form on $\mathbb{R}^{2n}$. Let $\chi \in \mathscr{G}_{\mathrm{c}}^s(\mathbb{R}^{4n})$ be a Gevrey cutoff function such that $\chi(Y) = 1$ for $|Y| \le 1$, $Y \in \mathbb{R}^{4n}$, with $\operatorname{supp}\chi \subset B_{\mathbb{R}^{4n}}(0, 2)$, and let

$$\begin{aligned} r_\chi(x, \theta; h) = \frac{1}{(\pi h)^{2n}} \int_{\mathbb{R}^{4n}} & e^{-\frac{2i}{h}\sigma(y_1, \eta_1; y_2, \eta_2)} (1 - \chi(y_1, \eta_1, y_2, \eta_2)) \\ & a(x + y_1, \theta + \eta_1) b(x + y_2, \theta + \eta_2)\, dy_1 d\eta_1 dy_2 d\eta_2. \end{aligned} \tag{2.23}$$

It has been established in [12, Proposition 3.8] that for some $C > 0$ uniformly in $h \in (0, 1]$ we have for all $\alpha, \beta \in \mathbb{N}^n$ and $(x, \theta) \in \mathbb{R}^{2n}$,

$$|\partial_x^\alpha \partial_\theta^\beta r_\chi(x, \theta; h)| \le C^{1+|\alpha|+|\beta|} \alpha!^s \beta!^s \exp\Big(-\frac{1}{\mathcal{O}(1)} h^{-\frac{1}{s}}\Big). \tag{2.24}$$

To analyze the term $c - r_\chi$, we consider the following more general integral:

$$I_\chi(x; h) = h^{-N/2} \int_{\mathbb{R}^N} e^{iq(y)/h} \chi(y) a(x + y)\, dy,$$

where $q(y) = \frac{1}{2} Ay \cdot y$ is a real non-degenerate quadratic form on $\mathbb{R}^N$, $a \in \mathscr{G}_{\mathrm{b}}^s(\mathbb{R}^N)$ and $\chi \in \mathscr{G}_{\mathrm{c}}^s(\mathbb{R}^N)$ satisfies $\chi(y) = 1$ for $|y| \le 1$.

By Parseval's formula, writing $\tau_x a(y) = a(x+y)$ we have

$$I_\chi(x;h) = C_A \int_{\mathbb{R}^N} e^{-\frac{ih}{2}A^{-1}\eta\cdot\eta}\widehat{\chi\tau_x a}(\eta)\,d\eta, \quad C_A = \frac{(2\pi)^{-N/2}e^{i\frac{\pi}{4}\operatorname{sgn} A}}{|\det A|^{1/2}}$$

where $\operatorname{sgn} A$ is the signature of A, $\widehat{\chi\tau_x a}(\eta) = \int_{\mathbb{R}^N} e^{-iy\cdot\eta}\chi(y)a(x+y)\,dy$ is the Fourier transform. Using $e^{i\sigma} = 1 + i\sigma\int_0^1 e^{it\sigma}dt$, $\sigma \in \mathbb{R}$, we get by $\int_{\mathbb{R}^N}\hat{u}(\eta)\,d\eta = (2\pi)^N u(0)$,

$$I_\chi(x;h) = C_A((2\pi)^N a(x) + hI_{\chi,1}(x;h)), \tag{2.25}$$

where $I_{\chi,1}(x;h) = -\frac{i}{2}\int_0^1\int_{\mathbb{R}^N} e^{-\frac{ith}{2}A^{-1}\eta\cdot\eta}(A^{-1}\eta\cdot\eta)\widehat{\chi\tau_x a}(\eta)\,d\eta\,dt$.

To derive Gevrey estimates for $I_{\chi,1}(x;h)$, we observe that for every $\alpha \in \mathbb{N}^N$,

$$\partial_x^\alpha I_{\chi,1}(x;h) = \frac{i}{2}\int_0^1\int_{\mathbb{R}^N} e^{-\frac{ith}{2}A^{-1}\eta\cdot\eta}\widehat{u_{x,\alpha}}(\eta)\,d\eta\,dt, \tag{2.26}$$

with

$$u_{x,\alpha}(y) = (A^{-1}\partial_y\cdot\partial_y)(\chi(y)\partial^\alpha a(x+y)).$$

It follows that $|\partial_x^\alpha I_{\chi,1}(x;h)| \le \frac{1}{2}\|\widehat{u_{x,\alpha}}\|_{L^1(\mathbb{R}^N)}$. By the Cauchy–Schwarz inequality,

$$\begin{aligned}\int_{\mathbb{R}^N}|\hat{u}(\eta)|\,d\eta &\le \Big(\int_{\mathbb{R}^N}|\hat{u}(\eta)|^2(1+|\eta|^2)^k d\eta\Big)^{1/2}\Big(\int_{\mathbb{R}^N}(1+|\eta|^2)^{-k}d\eta\Big)^{1/2}\\ &= C_k\|u\|_{H^k(\mathbb{R}^N)}\end{aligned}$$

for any $k > N/2$, $k \in \mathbb{N}$. We obtain therefore $|\partial_x^\alpha I_{\chi,1}(x;h)| \le C\|u_{x,\alpha}\|_{H^k(\mathbb{R}^N)}$ for some $k \in \mathbb{N}$ and $C > 0$ depending only on the dimension N. Recalling the definition of $u_{x,\alpha}$ given in (2.26) and the assumptions that $\chi \in \mathcal{G}_\mathrm{c}^s$, $a \in \mathcal{G}_\mathrm{b}^s$, we conclude that there exists $C > 0$ uniformly in $h \in (0,1]$ such that

$$|\partial_x^\alpha I_{\chi,1}(x;h)| \le C^{1+|\alpha|}\alpha!^s \quad \text{for all } \alpha \in \mathbb{N}^N,\ x \in \mathbb{R}^N. \tag{2.27}$$

Combining (2.25) and (2.27), and applying the result to the integral representation of $c - r_\chi$, we obtain via a direct computation that

$$c(x,\theta;h) - r_\chi(x,\theta;h) = a(x,\theta)b(x,\theta) + hr_1(x,\theta;h), \quad (x,\theta) \in \mathbb{R}^{2n},$$

for some $r_1(\cdot;h) \in \mathcal{G}_\mathrm{b}^s(\mathbb{R}^{2n})$. This and (2.24), together with equations (2.22) and (2.23), imply the desired result (2.21). ∎

3. Complex deformations and Proof of Theorem 2

Let $p \in \mathcal{G}_{\mathrm{b}}^{s}(\mathbb{R}^{2n})$, $s > 1$. Let $z_0 \in \partial\Sigma(p) \setminus \Sigma_\infty(p)$ satisfy conditions (1.7)–(1.9). We first recall from [6, Lemma 4.2] the existence of an escape function.

Lemma 6. *Let p and z_0 be given as in Theorem 2 such that (1.7)–(1.9) hold. Then there exists $G \in C_c^\infty(\mathbb{R}^{2n}; \mathbb{R})$ such that for some constant $c > 0$ we have*

$$H_{\operatorname{Im} p} G(\rho) < -c < 0, \quad \rho \in p^{-1}(z_0). \tag{3.1}$$

We now introduce small compactly supported deformations of the real space $\mathbb{R}^{2n}$. Let $H_G(\rho) = (G'_\xi(\rho), -G'_x(\rho))$ be the Hamilton vector field of G, and set

$$\Lambda_{tG} := \{\rho + itH_G(\rho); \rho \in T^*\mathbb{R}^n\} \subset \mathbb{C}^{2n}, \quad t \in \mathbb{R},\ |t| \text{ small}. \tag{3.2}$$

We note that Λ_{tG} is I-Lagrangian and R-symplectic, in the sense that $\sigma|_{\Lambda_{tG}}$ is real and non-degenerate, where $\sigma = d\xi \wedge dx$ is the complex symplectic form on $\mathbb{C}^n_x \times \mathbb{C}^n_\xi$. Let $\widetilde{p} \in \mathcal{G}_b^s(\mathbb{C}^{2n})$ be an almost holomorphic extension of p, which is supported in a bounded tubular neighborhood of $\mathbb{R}^{2n} \subset \mathbb{C}^{2n}$. We shall explore the behavior of $\widetilde{p}|_{\Lambda_{tG}}$ near some $\rho \in \operatorname{neigh}(p^{-1}(z_0); \mathbb{R}^{2n})$.

For simplicity, we may assume $z_0 = 0$ by subtracting z_0 from p. Let us use Taylor's formula to expand $\widetilde{p}$ at $\rho \in \operatorname{neigh}(p^{-1}(0); \mathbb{R}^{2n})$ and write

$$\begin{aligned}\widetilde{p}(\rho + itH_G(\rho)) &= p(\rho) + itH_G\widetilde{p}(\rho) + \mathcal{O}(t^2)\\ &= p(\rho) - itH_{\operatorname{Re} p}G(\rho) + tH_{\operatorname{Im} p}G(\rho) + \mathcal{O}(t^2),\end{aligned}$$

where we used the facts that $\widetilde{p}|_{\mathbb{R}^{2n}} = p$, $\bar{\partial}\widetilde{p}|_{\mathbb{R}^{2n}} = 0$. We obtain therefore

$$\operatorname{Re} \widetilde{p}(\rho + itH_G(\rho)) = \operatorname{Re} p(\rho) + tH_{\operatorname{Im} p}G(\rho) + \mathcal{O}(t^2).$$

Letting $\Omega \subset \mathbb{R}^{2n}$ be a sufficiently small neighborhood of $p^{-1}(0)$ which is compact in $\mathbb{R}^{2n}$ since $0 \notin \Sigma_\infty(p)$, we can infer from (1.7), (3.1), and the computation above that there exist $\gamma > 0$ and $t_0 > 0$ such that

$$\operatorname{Re} \widetilde{p}(\rho + itH_G(\rho)) \geq \gamma|t|, \quad \rho \in \Omega,\ -t_0 < t \leq 0. \tag{3.3}$$

Let us now move to the FBI transform side. Let $\phi(x, y)$ be a holomorphic quadratic form on $\mathbb{C}^n_x \times \mathbb{C}^n_y$ with $\operatorname{Im} \phi''_{yy} > 0$, $\det \phi''_{xy} \neq 0$. To ϕ we associate the complex linear canonical transformation

$$\kappa_\phi \colon \mathbb{C}^{2n} \ni (y, -\phi'_y(x, y)) \mapsto (x, \phi'_x(x, y)) \in \mathbb{C}^{2n}. \tag{3.4}$$

Recalling for instance [32, Theorem 13.5], we have

$$\kappa_\phi(\mathbb{R}^{2n}) = \Lambda_{\Phi_0}, \quad \Lambda_{\Phi_0} := \left\{\left(x, \frac{2}{i}\frac{\partial\Phi_0}{\partial x}(x)\right); x \in \mathbb{C}^n\right\} \subset \mathbb{C}^{2n},$$

where

$$\Phi_0(x) = \max_{y\in\mathbb{R}^n} -\operatorname{Im}\phi(x,y) \tag{3.5}$$

is a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$.

Let us also recall from [30, Section 2] that

$$\kappa_\phi(\Lambda_{tG}) = \Lambda_{\Phi_t} = \Big\{\Big(x, \frac{2}{i}\frac{\partial\Phi_t}{\partial x}(x)\Big); x\in\mathbb{C}^n\Big\} \subset \mathbb{C}^{2n}, \tag{3.6}$$

with

$$\Phi_t(x) = \text{v.c.}_{(y,\eta)\in\mathbb{C}^n\times\mathbb{R}^n}(-\operatorname{Im}\phi(x,y) - \eta\cdot\operatorname{Im} y + tG(\operatorname{Re} y,\eta)). \tag{3.7}$$

Computing the critical value in (3.7) as a perturbation of $t=0$, we obtain

$$\Phi_t(x) = \Phi_0(x) + tG\Big(\kappa_\phi^{-1}\Big(x, \frac{2}{i}\frac{\partial\Phi_0}{\partial x}(x)\Big)\Big) + \mathcal{O}(t^2). \tag{3.8}$$

To see this, we first observe by comparing (3.5) and (3.7) that $\Phi_t(x)|_{t=0} = \Phi_0(x)$ since

$$\text{v.c.}_{(y,\eta)\in\mathbb{C}^n\times\mathbb{R}^n}(-\operatorname{Im}\phi(x,y) - \eta\cdot\operatorname{Im} y) = \max_{y\in\mathbb{R}^n} -\operatorname{Im}\phi(x,y),$$

and that there is a single critical point $(y_0,\eta_0) = \kappa_\phi^{-1}\big(x, \frac{2}{i}\frac{\partial\Phi_0}{\partial x}(x)\big) \in \mathbb{R}^{2n}$ due to the non-degeneracy of $\operatorname{Im}\phi''_{yy}$. It follows that the critical value in (3.7) is also evaluated at a single critical point when t is small. Let us then differentiate the critical value $\Phi_t(x)$ in the parameter t; in view of (3.7), we get

$$(\partial_t\Phi_t(x))|_{t=0} = G(\operatorname{Re} y_0,\eta_0) = G\Big(\kappa_\phi^{-1}\Big(x, \frac{2}{i}\frac{\partial\Phi_0}{\partial x}(x)\Big)\Big),$$

where we noticed that $y_0\in\mathbb{R}^n$. We obtain therefore (3.8) by Taylor's formula.

Let us introduce the FBI-Bargmann transform associated to ϕ:

$$Tu(x;h) = C_n h^{-3n/4}\int_{\mathbb{R}^n} e^{i\phi(x,y)/h}u(y)dy, \tag{3.9}$$

where C_n is chosen such that $T\colon L^2(\mathbb{R}^n)\to H_{\Phi_0}(\mathbb{C}^n)$ is unitary, see [32, Theorem 13.7]. Let $a = p\circ\kappa_\phi^{-1}\in\mathscr{G}_b^s(\Lambda_{\Phi_0})$, then $\tilde a := \tilde p\circ\kappa_\phi^{-1}\in\mathscr{G}_b^s(\mathbb{C}^{2n})$ is an almost holomorphic extension of a such that $\operatorname{supp}\tilde a\subset\Lambda_{\Phi_0} + B_{\mathbb{C}^{2n}}(0,C_0)$ for some $C_0>0$. The exact Egorov theorem (see for example [32, Theorem 13.9]) implies that

$$a^w(x,hD_x)\circ T = T\circ p^w(x,hD), \tag{3.10}$$

where $a^w(x,hD_x)$ is the semiclassical Weyl quantization of $a\in\mathscr{G}_{\mathrm{b}}^s(\Lambda_{\Phi_0})$ given by (2.2) while $p^w(x,hD)$ is the semiclassical Weyl quantization of $p\in\mathscr{G}_{\mathrm{b}}^s(\mathbb{R}^{2n})$ defined in (1.3).

Let us set $t = -\epsilon h^{1-\frac{1}{s}}$ for $\epsilon > 0$ small but independent of h. In view of (3.8), we see that (2.3) holds with Φ_t replacing Φ_1, if $\epsilon > 0$ is small enough. Applying (2.16) with $\psi \equiv 1$ and the weight Φ_t, we get

$$\begin{aligned} &\mathrm{Re}(a^w(x, hD_x)u, u)_{H_{\Phi_t}} \\ &\quad = \int \mathrm{Re}\,\tilde{a}\Big(x, \frac{2}{i}\frac{\partial \Phi_t}{\partial x}(x)\Big)|u(x)|^2 e^{-2\Phi_t(x)/h}\, L(dx) + \mathcal{O}(h)\|u\|^2_{H_{\Phi_t}}. \end{aligned} \tag{3.11}$$

Let $\Omega \subset \mathbb{R}^{2n}$ be an open neighborhood of $p^{-1}(0)$ such that (3.3) holds. We set

$$U = \pi_x(\kappa_\phi(\{\rho + itH_G(\rho) : \rho \in \Omega\})) \subset \mathbb{C}^n, \tag{3.12}$$

where $\pi_x\colon \Lambda_{\Phi_t} \ni (x, \xi) \mapsto x \in \mathbb{C}^n$ is the projection map, and κ_ϕ is the canonical transform defined in (3.4). We note that U is open and bounded since $p^{-1}(0) \subset \mathbb{R}^{2n}$ is compact, which is due to the assumption $0 \notin \Sigma_\infty(p)$. Combing (3.3) with (3.2), (3.6), and recalling $\tilde{a} = \tilde{p} \circ \kappa_\phi^{-1}$, we obtain for some $\gamma > 0$, $t_0 > 0$ as in (3.3),

$$\mathrm{Re}\,\tilde{a}\Big(x, \frac{2}{i}\frac{\partial \Phi_t}{\partial x}(x)\Big) \geq \gamma|t|, \quad x \in U, -t_0 < t \leq 0. \tag{3.13}$$

We shall next show that $a = p \circ \kappa_\phi^{-1}$ satisfies the condition (2.18) with U given in (3.12). To this end, we first note that

$$K_0 := \{x \in \mathbb{C}^n : a(x, \xi) = 0,\ (x, \xi) \in \Lambda_{\Phi_0}\} = \pi_x(\kappa_\phi(p^{-1}(0))) \subset \mathbb{C}^n \text{ is compact.}$$

Since $\pi_x\colon \Lambda_{\Phi_t} \to \mathbb{C}^n$, $-t_0 < t \leq 0$, $\kappa_\phi\colon \mathbb{R}^{2n} \to \Lambda_{\Phi_0}$ or $\Lambda_{tG} \to \Lambda_{\Phi_t}$, and $g_t\colon \mathbb{R}^{2n} \ni \rho \mapsto \rho + itH_G(\rho) \in \Lambda_{tG}$ are all diffeomorphisms between corresponding submanifolds of $\mathbb{C}^{2n}$, we get that $U_0 := \pi_x(\kappa_\phi(\Omega)) \subset \mathbb{C}^n$ is an open neighborhood of K_0 and that U defined in (3.12) is open. To see $K_0 \subset U$, it suffices to show that $\mathrm{dist}(\partial U, K_0) > 0$ (note that U is bounded). Let $x_t \in \partial U$, $x_t = \pi_x(\kappa_\phi(\rho + itH_G(\rho)))$ for some $\rho \in \partial\Omega$. Since $x = \pi_x(\kappa_\phi(\rho)) \in \partial U_0$ and $K_0 \subset U_0$, we have $\mathrm{dist}(x, K_0) \geq \mathrm{dist}(\partial U_0, K_0) =: \delta_0 > 0$. Noting that $|x_t - x| = \mathcal{O}(|t|)$, we get therefore $\mathrm{dist}(x_t, K_0) \geq \delta_0/2$ for all $-t_0 < t \leq 0$ by taking a smaller t_0 if necessary. We conclude from the above discussion that

$$|a(x, \xi)| \geq \frac{2}{C}, \quad (x, \xi) \in \Lambda_{\Phi_0},\ x \in \mathbb{C}^n \setminus U, \quad -t_0 < t \leq 0, \tag{3.14}$$

for some $C > 0$ uniform in $-t_0 < t \leq 0$ (noting that U depends on t).

We now let $t = -\epsilon h^{1-\frac{1}{s}}$ with $\epsilon > 0$ small but fixed, and let $z \in \mathbb{C}$ be such that

$$-C^{-1} < \mathrm{Re}\, z \leq \Big(\frac{\gamma\epsilon}{2}\Big)h^{1-\frac{1}{s}}, \quad |\mathrm{Im}\, z| < C^{-1}, \quad C > 0 \text{ sufficiently large.} \tag{3.15}$$

Combining (3.11), (3.13), and (3.15), we get

$$\begin{aligned}&\operatorname{Re}((a^w(x,hD_x)-z)u,u)_{H_{\Phi_t}}\\&\quad\geq\Big(\frac{\gamma\epsilon}{2}\Big)h^{1-\frac{1}{s}}\|u\|^2_{H_{\Phi_t}}-\mathcal{O}(h)\|u\|^2_{H_{\Phi_t}}-\mathcal{O}(1)\int\limits_{\mathbb{C}^n\setminus U}|u(x)|^2e^{-2\Phi_t(x)/h}\,L(dx).\end{aligned}$$

In view of (3.14) and (3.15), we can apply Proposition 4 to the symbol $a-z$ with Φ_t in place of Φ_1, therefore, for $0<h<h_0$ we have

$$\begin{aligned}&\operatorname{Re}((a^w(x,hD_x)-z)u,u)_{H_{\Phi_t}}+\mathcal{O}(1)\|(a^w(x,hD_x)-z)u\|^2_{H_{\Phi_t}}\\&\quad\geq\Big(\Big(\frac{\gamma\epsilon}{2}\Big)h^{1-\frac{1}{s}}-\mathcal{O}(h)\Big)\|u\|^2_{H_{\Phi_t}}\geq\delta h^{1-\frac{1}{s}}\|u\|^2_{H_{\Phi_t}}.\end{aligned}$$

Here h_0, $\delta>0$ are small constants. By the Peter–Paul inequality,

$$\begin{aligned}&\operatorname{Re}((a^w(x,hD_x)-z)u,u)_{H_{\Phi_t}}\\&\quad\leq\frac{\delta}{2}h^{1-\frac{1}{s}}\|u\|^2_{H_{\Phi_t}}+\frac{1}{2\delta}h^{\frac{1}{s}-1}\|(a^w(x,hD_x)-z)u\|^2_{H_{\Phi_t}}.\end{aligned}$$

Combining the estimates above, we conclude that for some $C>0$,

$$\|(a^w(x,hD_x)-z)u\|_{H_{\Phi_t}}\geq C^{-1}h^{1-\frac{1}{s}}\|u\|_{H_{\Phi_t}},\quad u\in H_{\Phi_t},\ 0<h<h_0. \tag{3.16}$$

Writing $P(h)=p^w(x,hD)+hp_1^w(x,hD;h)$ for some $p_1(\cdot;h)\in\mathcal{G}^s_{\mathrm{b}}(\mathbb{R}^{2n})$, we get

$$TP(h)=(a^w(x,hD_x)+ha_1^w(x,hD_x;h))T\colon L^2(\mathbb{R}^n)\to H_{\Phi_0}$$

with $a_1(\cdot;h)\in\mathcal{G}^s_{\mathrm{b}}(\Lambda_{\Phi_0})$. We can infer from the proofs in [12] that

$$a_1(\cdot;h)\in\mathcal{G}^s_{\mathrm{b}}(\Lambda_{\Phi_0})\implies a_1^w(x,hD_x;h)=\mathcal{O}(1):H_{\Phi_t}\to H_{\Phi_t}.$$

Therefore, by (3.16), there exists $C>0$ uniformly in $0<h<h_0$ such that

$$\|T(P(h)-z)v\|_{H_{\Phi_t}}\geq(2C)^{-1}h^{1-\frac{1}{s}}\|Tv\|_{H_{\Phi_t}},\quad v\in L^2(\mathbb{R}^n). \tag{3.17}$$

Here we note that $H_{\Phi_t}=H_{\Phi_0}$ as linear spaces. Since $\|\Phi_t-\Phi_0\|_{L^\infty(\mathbb{C}^n)}\leq C^{-1}h^{1-\frac{1}{s}}$, we can change the norm $\|\cdot\|_{H_{\Phi_t}}$ to $\|\cdot\|_{H_{\Phi_0}}$ in (3.17) and use the fact that the operator $T\colon L^2(\mathbb{R}^n)\to H_{\Phi_0}$ is unitary, to obtain

$$\|(P(h)-z)v\|_{L^2(\mathbb{R}^n)}\geq C^{-1}h^{1-\frac{1}{s}}e^{-\mathcal{O}(1)h^{-1/s}}\|v\|_{L^2(\mathbb{R}^n)},\quad v\in L^2(\mathbb{R}^n).$$

This implies that $P(h)-z\colon L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)$ is injective. Since $0\notin\Sigma_\infty(p)$, it has been shown in the proof of [6, Proposition 3.3] that $P(h)-z\colon L^2\to L^2$ is a Fredholm operator with index 0 for z in an $\mathcal{O}(1)$-neighborhood of 0 and h sufficiently small. We can therefore conclude that for some $h_0 > 0$ and for all z satisfying (3.15),

$$(P(h) - z)^{-1} = \mathcal{O}(1)e^{\mathcal{O}(1)h^{-1/s}} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad 0 < h < h_0.$$

This completes the proof of Theorem 2.

Remark 7. It is worth noting that Theorem 2 can also be proved using Martinez's method [22], together with the main results of [15, Theorems 1 and 1′]. However, we choose an alternative approach that makes use of the escape function in a more geometric manner, involving complex I-Lagrangian manifolds and modification of the exponential weights for the Bargmann space. Additionally, our proof highlights the strength of the generalized Toeplitz-type identity (Proposition 2), which is expected to be of independent interest and have further applications.

4. Proof of Theorem 1

We start with an adaption of [6, Lemma 4.1] to the Gevrey class, which can be achieved by following the proof in [6] and using a division theorem for Gevrey functions given in [3]. In fact, we only need a special case of [3, Theorem 5] (when $m = 1$).

Proposition 8. *Let Ω be a domain in $\mathbb{R}^d$ and let $f \in \mathcal{G}^s(\mathbb{R} \times \Omega; \mathbb{C})$. Let $P(t,x) = t + a(x)$ with $a \in \mathcal{G}^s(\Omega; \mathbb{C})$. Then there exist $Q \in \mathcal{G}^s(\mathbb{R} \times \Omega; \mathbb{C})$ and $R \in \mathcal{G}^s(\Omega; \mathbb{C})$ such that*

$$f(t,x) = Q(t,x)P(t,x) + R(x), \quad (t,x) \in \mathbb{R} \times \Omega.$$

Following an argument in [24], we can deduce a preparation theorem for Gevrey functions from the division theorem above.

Proposition 9. *Let $f(t,x) \in \mathcal{G}^s(\mathbb{R} \times \mathbb{R}^d; \mathbb{C})$. Suppose that $f(0,0) = 0$, $\frac{\partial f}{\partial t}(0,0) \neq 0$. Then in an open neighborhood of $(0,0) \in \mathbb{R} \times \mathbb{R}^d$ we have the factorization*

$$f(t,x) = q(t,x)(t + \lambda(x))$$

where q and λ are $\mathcal{G}^s$ complex-valued functions with $q(0,0) \neq 0$, $\lambda(0) = 0$.

Proof. Let us introduce a generic complex variable $\lambda \in \mathbb{C}$. We note that $F(t,x,\lambda) := f(t,x) \in \mathcal{G}^s(\mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{C}_\lambda)$, $P(t,x,\lambda) := t + \lambda$, then by Proposition 8, we have

$$f(t,x) = Q(t,x,\lambda)(t + \lambda) + R(x,\lambda), \tag{4.1}$$

for some $Q \in \mathcal{G}^s(\mathbb{R}_t \times \mathbb{R}^d_x \times \mathbb{C}_\lambda)$, $R \in \mathcal{G}^s(\mathbb{R}^d_x \times \mathbb{C}_\lambda)$. Since $f(0,0) = 0$ we must have $R(0,0) = 0$, thus $\frac{\partial f}{\partial t}(0,0) \neq 0$ implies $Q(0,0,0) \neq 0$. Moreover, applying $\partial_{\bar{\lambda}}$

to (4.1) and taking $(t,x,\lambda)=(0,0,0)$, we get $\frac{\partial R}{\partial \bar\lambda}(0,0)=0$, thus $\frac{\partial \bar R}{\partial \lambda}(0,0)=0$. Suppose $\frac{\partial R}{\partial \lambda}(0,0)=0$, we deduce from (4.1) by applying ∂_λ and taking $(t,x,\lambda)=(0,0,0)$ that $Q(0,0,0)=0$, a contradiction. Therefore, we have $\frac{\partial R}{\partial \lambda}(0,0)\neq 0$. We conclude from the discussion above that the Jacobian matrix $\frac{\partial(R,\bar R)}{\partial(\lambda,\bar\lambda)}(0,0)\neq 0$. It follows from the implicit function theorem in the Gevrey-s class by [16] that there exists an open neighborhood $U\subset\mathbb{R}^d$ of 0 and $\lambda(x)\in\mathscr{G}^s(U;\mathbb{C})$ such that $\lambda(0)=0$, and $R(x,\lambda(x))=0$ for all $x\in U$. The desired factorization follows by letting $q(t,x):=Q(t,x,\lambda(x))\in\mathscr{G}^s(\mathbb{R}_t\times U_x)$. ∎

Following the proof of [6, Lemma 4.1] while replacing the Malgrange preparation theorem used there by Proposition 9, and using a $\mathscr{G}^s$ (Gevrey-s) partition of unity, we obtain the following result.

Lemma 10. *Let $p\in\mathscr{G}^s_b(\mathbb{R}^{2n})$ and $z_0\in\partial\Sigma(p)$ such that (1.1) and (1.2) hold. Then there exists $q\in\mathscr{G}^s_b(\mathbb{R}^{2n})$ such that $q\neq 0$ on $p^{-1}(z_0)$ and*

$$\operatorname{Im}(q(p-z_0))\geq 0 \quad \textit{near } p^{-1}(z_0); \qquad |d\operatorname{Re}(q(p-z_0))|\geq c>0 \quad \textit{on } p^{-1}(z_0). \tag{4.2}$$

We are now ready to prove Theorem 1. Let $p\in\mathscr{G}^s_b(\mathbb{R}^{2n})$, $s>1$, and let $p^w(x,hD)$ be the principal part of the semiclassical Gevrey operator $P(h)$. Suppose $z_0\in\partial\Sigma(p)\setminus\Sigma_\infty(p)$ satisfies conditions (1.1) and (1.2). Let $q\in\mathscr{G}^s_b(\mathbb{R}^{2n})$ be given in Lemma 10. We can further assume that $|q|=1$ near $p^{-1}(z_0)$, as the inequalities in (4.2) still hold with q replaced by $q/|q|$. Since $z_0\in\partial\Sigma(p)$, there exists a continuous branch of $\arg q(\rho)$, $\rho\in\operatorname{neigh}(p^{-1}(z_0);\mathbb{R}^{2n})$, i.e., $\arg q\in(\theta_0,\theta_0+2\pi)$ for some $\theta_0\in\mathbb{R}$. Otherwise, we can construct a simple closed curve $\gamma\subset\operatorname{neigh}(p^{-1}(z_0);\mathbb{R}^{2n})\setminus p^{-1}(z_0)$ such that the winding number of $q(\gamma)$ around $0\in\mathbb{C}$ is non-zero, which implies that the winding number of $p(\gamma)$ around z_0 is non-zero since $\operatorname{Im}(q(p-z_0))\geq 0$, in contradiction with the assumption that $z_0\in\partial\Sigma(p)$. Therefore, we have $q=e^{i\theta}$ with $\theta\in\mathscr{G}^s(U;\mathbb{R})$, $U=\operatorname{neigh}(p^{-1}(z_0);\mathbb{R}^{2n})$. Let $V\subset\mathbb{R}^{2n}$ be an open subset satisfying $p^{-1}(z_0)\subset V\Subset U$ and let $\chi\in\mathscr{G}^s_c(\mathbb{R}^{2n})$ be such that $\chi=1$ on $\bar V$ and $\operatorname{supp}\chi\subset U$, we then set

$$q(x,\xi)=\exp(i\chi(x,\xi)\theta(x,\xi))\in\mathscr{G}^s_b(\mathbb{R}^{2n};\mathbb{C}) \quad \text{satisfies (4.2).} \tag{4.3}$$

Fixing z_0 as above, let us consider the semiclassical Gevrey operator

$$P_1(h):=i^{-1}q^w(x,hD)(P(h)-z_0) \tag{4.4}$$

whose principal part is $p_1^w(x,hD)$ with $p_1:=i^{-1}q(p-z_0)$ by Proposition 5. We next verify that p_1 satisfies conditions (1.7)–(1.9). Noting that $p_1^{-1}(0)=p^{-1}(z_0)$ by (4.3), it then follows from (4.2) that

$$\operatorname{Re} p_1\geq 0 \quad \text{near } p_1^{-1}(0); \qquad d\ \operatorname{Im} p_1\neq 0 \quad \text{on } p_1^{-1}(0). \tag{4.5}$$

Let $\rho_0 \in p_1^{-1}(0) = p^{-1}(z_0)$. By a direct computation, we see that one can obtain a trajectory of $H_{\operatorname{Re}(e^{-i\theta(\rho_0)}p)}$ passing through ρ_0 that is contained in $p^{-1}(z_0)$ from a trajectory of $H_{\operatorname{Im} p_1}$ passing through ρ_0 that is contained in $p_1^{-1}(0)$ simply by reparametrization. Therefore, condition (1.9) must hold for p_1 and 0, i.e.,

$$\text{for all } \rho \in p_1^{-1}(0), \quad \text{the maximal trajectory of } H_{\operatorname{Im} p_1} \text{ passing through } \rho \text{ contains a point where } \operatorname{Re} p_1 > 0. \tag{4.6}$$

Otherwise, condition (1.2) on p and z_0 would be contradicted. In view of (4.5) and (4.6), repeating the steps to derive (3.17), we can conclude that there exist $h_0 > 0$ and $C > 0$ such that (with the unitary operator T defined in (3.9))

$$\|TP_1(h)v\|_{H_{\Phi_t}} \geq C^{-1}h^{1-\frac{1}{s}}\|Tv\|_{H_{\Phi_t}}, \quad v \in L^2(\mathbb{R}^n),\ 0 < h < h_0. \tag{4.7}$$

Recalling (see (3.10)) that $Tq^w(x, hD) = b^w(x, hD_x)T$ for $b = q \circ \kappa_\phi^{-1} \in \mathscr{G}_b^s(\Lambda_{\Phi_0})$, and noting that $b^w(x, hD_x) = \mathcal{O}(1)\colon H_{\Phi_t} \to H_{\Phi_t}$ by [12, Theorems 1.1 and 1.2], we get from (4.4) and (4.7) that for $0 < h < h_0$,

$$\|T(P(h) - z_0)v\|_{H_{\Phi_t}} \geq C^{-1}h^{1-\frac{1}{s}}\|Tv\|_{H_{\Phi_t}}, \quad v \in L^2(\mathbb{R}^n).$$

It follows that for $z \in \mathbb{C}$ such that $|z - z_0| < (2C)^{-1}h^{1-\frac{1}{s}}$ we have for $0 < h < h_0$,

$$\|T(P(h) - z)v\|_{H_{\Phi_t}} \geq (2C)^{-1}h^{1-\frac{1}{s}}\|Tv\|_{H_{\Phi_t}}, \quad v \in L^2(\mathbb{R}^n).$$

Arguing as in the proof of Theorem 2, we conclude from the above estimate that

$$(P(h) - z)^{-1} = \mathcal{O}(1)e^{\mathcal{O}(1)h^{-1/s}} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad |z - z_0| < C^{-1}h^{1-\frac{1}{s}}.$$

This completes the proof of Theorem 1.

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