

Bounds for eigenvalue sums of Schrödinger operators with complex radial potentials

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Abstract. We consider eigenvalue sums of Schrödinger operators $-\Delta + V$ on $L^2(\mathbb{R}^d)$ with complex radial potentials $V \in L^q(\mathbb{R}^d)$, $q < d$. We prove quantitative bounds on the distribution of the eigenvalues in terms of the L^q norm of V . A consequence of our bounds is that, if the eigenvalues (z_j) accumulate to a point in $(0, \infty)$, then $(\operatorname{Im} z_j)$ is summable. The key technical tools are resolvent estimates in Schatten spaces. We show that these resolvent estimates follow from spectral measure estimates by an epsilon removal argument.

1. Introduction and main results

We consider Schrödinger operators $H = -\Delta + V$ on $L^2(\mathbb{R}^d)$ with complex-valued potentials $V \in L^q(\mathbb{R}^d)$, with $q < \infty$. The spectrum of H consists of $[0, \infty)$ together with a discrete set of eigenvalues z_j . We are interested in quantitative bounds on the z_j that depend only on an L^q norm of the potential.

1.1. Bounds for single eigenvalues

Laptev and Safronov [22] conjectured that, in $d \geq 2$ dimensions, any non-positive eigenvalue z of H satisfies the bound

$$|z|^\gamma \leq D_{\gamma,d} \int_{\mathbb{R}^d} |V(x)|^{\gamma+d/2} dx \quad (1.1)$$

for $0 < \gamma \leq d/2$, and with $D_{\gamma,d}$ independent of V and z . Prior to this, Abramov, Aslanyan, and Davies [1] had shown that (1.1) holds if $d = 1$ and $\gamma = 1/2$. For $d \geq 2$, the Laptev–Safronov conjecture was proved for $0 < \gamma \leq 1/2$ by Frank [15] and disproved for $\gamma > 1/2$ by Bögli and the first author [4]. The conjecture is true for *radial* potentials in the range $0 < \gamma < d/2$, as proved by Frank and Simon [20],

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and it fails for $\gamma \geq d/2$ by a counterexample of Bögli [3]. There are some further refinements and generalisations of (1.1) (see, e.g., [8] and references therein), and there is by now a more-or-less complete picture of bounds of the type (1.1) for a single eigenvalue z .

1.2. Bounds for eigenvalue sums

The situation for eigenvalue sums is considerably less well understood. The starting point is the celebrated Lieb–Thirring inequality for *real-valued potentials*,

$$\sum_j |z_j|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} |V|^{\gamma+d/2} dx, \quad (1.2)$$

which holds for $\gamma \geq 1/2$ if $d = 1$ and $\gamma > 0$ if $d \geq 2$. Frank, Laptev, Lieb, and Seiringer [18] proved that, for $\gamma \geq 1$, inequality (1.2) holds for all eigenvalues outside a fixed cone, $|\operatorname{Im} z_j| \geq \kappa \operatorname{Re} z_j$, with a constant that blows up as $\kappa \rightarrow 0$ at a rate $\kappa^{-\gamma-d/2}$. In $d = 1$, the blow up rate was shown to be optimal by Bögli [3]. Averaging the bound of Frank, Laptev, Lieb, and Seiringer with respect to κ , Demuth, Hansmann, and Katriel [11] proved that

$$\sum_j |z_j|^\gamma \left(\frac{\delta(z_j)}{|z_j|} \right)^{\gamma+d/2+\varepsilon} \leq C_{\gamma,d,\varepsilon} \int_{\mathbb{R}^d} |V|^{\gamma+d/2} dx, \quad (1.3)$$

where $\delta(z) := \operatorname{dist}(z, [0, \infty))$, $\gamma \geq 1$ and $\varepsilon > 0$. They also posed the question [13] whether (1.3) is true for $\varepsilon = 0$. For $d = 1$, Bögli and Štampach [5] answered this question in the negative.

A rather different set of results concerning eigenvalue sums have been established by Frank and Sabin [19] and Frank [16]. These bounds are of the (scale-invariant) form

$$\left(\sum_j |z_j|^\alpha \left(\frac{\delta(z_j)}{|z_j|} \right)^\beta \right)^{\gamma/\alpha} \leq C_{\alpha,\beta,\gamma,d} \int_{\mathbb{R}^d} |V|^{\gamma+d/2} dx. \quad (1.4)$$

It would be too technical for this introduction to state the precise values of α, β, γ, d for which (1.4) holds, so we will only remark some general features of these bounds.¹

- If $0 < \gamma \leq 1/2$ ($\gamma = 1/2$ if $d = 1$), then $\beta = 1$. This means that if the eigenvalues accumulate in $(0, \infty)$ then they are summable, i.e., $(\operatorname{Im} z_j) \in \ell^1$.

¹We are indebted to Rupert Frank for communicating these remarks very clearly during an online IAMP seminar [17].

- In all known bounds, we have $\gamma/\alpha < 1$. In other words, a sum is bounded by a *power* of an integral strictly greater than 1. This means that there is a loss of *locality*, which is a crucial feature of the Lieb–Thirring inequality (1.2).
- One particular bound from [16] for $d = 1$ states that

$$\left(\sum_j \delta(z_j)^\beta\right)^{1/(2\beta)} \leq C_\beta \int_{\mathbb{R}^d} |V(x)| dx \quad (1.5)$$

holds with $\beta = 1$. The first-named author showed in [9] that the inequality fails for $\beta < 1$, so the bound (1.5) is optimal.

The example of [9] also shows that one can have unexpectedly many eigenvalues compared to the number of resonances. In higher dimensions, it is an open problem whether there are Schrödinger operators with complex potentials that have significantly more eigenvalues than their real counterparts.

1.3. Main results

Our main contribution is to improve the results of Frank [16], concerning bounds of the type (1.4), under the assumption that V is *radial*. As we already mentioned, radial potentials satisfy better estimates in the case of individual eigenvalues. To the best of our knowledge, such an effect has not been observed so far for sums of eigenvalues.

Theorem 1.1. *Let $d \geq 2$, $q \in ((d+1)/2, d)$, $p > (d-1)q/(d-q)$. Then for radial $V \in L^q(\mathbb{R}^d)$, the eigenvalues z_j of H satisfy*

$$\left(\sum_j \delta(z_j) |z_j|^{p(1-d/(2q))-1}\right)^{q/p} \leq C_{p,q} \int_{\mathbb{R}^d} |V|^q dx. \quad (1.6)$$

In particular, if (z_j) accumulates to a point in $(0, \infty)$, then $(\operatorname{Im} z_j) \in \ell^1$.

Remark 1. (i) Frank and Sabin [19] proved the case $q \in (d/2, (d+1)/2]$ with the same accumulation rate.

(ii) Frank [16] proved similar bounds for $q > (d+1)/2$, but with a slower accumulation rate. Moreover, Frank's bounds distinguish between eigenvalues lying in a disk around the origin and eigenvalues lying outside this disk.

(iii) The overall powers of $|z_j|$ in the bounds of Frank and Sabin [19] and Frank [16] are always negative, whereas ours are positive.

(iv) Since $q/p < (d-q)/(d-1) < 1$, the bound (1.6) exhibits the same non-locality as those in [16, 19]. We conjecture that the exponent q/p in (1.6) cannot be

increased, i.e., that the inequality

$$v\left(\sum_j \delta(z_j)^\beta |z_j|^{\beta p(1-d/(2q))-1}\right)^{q/(\beta p)} \leq C_{p,q,\beta} \int_{\mathbb{R}^d} |V|^q dx$$

fails for $\beta < 1$. We leave it as an open problem to find a counterexample.

(v) Another interesting question is whether one can dispense with the radiality assumption and instead replace the right-hand side of (1.6) by a constant multiple of the mixed norm

$$\int_0^\infty \sup_{\omega \in S^{d-1}} |V(r\omega)|^q r^{d-1} dr.$$

Such generalizations for bounds of single eigenvalues appear in [20].

The proof of Theorem 1.1 follows the same general strategy as in [16, 19]. It is based on the identification of eigenvalues of the Schrödinger operator H with zeroes of an analytic function (a regularised determinant). This method was pioneered by Demuth and Katriel [14] and Borichev, Golinskii, and Kupin [6] and extended by Demuth, Hansmann, and Katriel [11, 12]. The method rests upon a remarkable generalization of Jensen's inequality for analytic functions, due to [6], for functions that blow up at some points of the boundary. We will appeal to the quantitative version of Frank [16, Theorem 3.1], but the special case $q = 0$ there is essentially contained in the proof of [19, Theorem 16]. It corresponds to a blow up at a single point ($z = 0$). The main new technical ingredient in our proof is a uniform resolvent bound in trace ideals.

Theorem 1.2. *Let $d/2 \leq q < d$ and $p > (d-1)q/(d-q)$. Then, for all $z \in \mathbb{C} \setminus [0, \infty)$ and for all radial functions $W_1, W_2 \in L^q(\mathbb{R}^d)$,*

$$\|W_1(-\Delta - z)^{-1}W_2\|_{\mathfrak{S}^p(L^2(\mathbb{R}^d))} \leq C_{p,q,d}|z|^{d/(2q)-1} \|W_1\|_{L^{2q}} \|W_2\|_{L^{2q}}. \quad (1.7)$$

Remark 2. (i) Theorem 1.2 extends the uniform resolvent estimates of Frank and Sabin [19, Theorem 12] from $q \leq (d+1)/2$ to $q < d$ under the assumption of radial symmetry.

(ii) Our Schatten exponent p coincides with that of [19] and is optimal in the range $q \leq (d+1)/2$, as shown in [19, Theorem 6]. The resolvent bounds of Frank and Sabin hold for arbitrary potentials, not just radial potentials. However, their counterexample is radial and the argument remains valid for $q > (d+1)/2$, so the same example shows optimality of the Schatten exponent p in Theorem 1.2.

(iii) The Schatten norm bound in Theorem 1.2 is crucial for applications involving sums of eigenvalues. For bounds involving only individual eigenvalues, a weaker

bound for the operator norm would suffice. By Hölder's inequality, the operator norm bound is equivalent to an $L^p \rightarrow L^{p'}$ bound for $(-\Delta - z)^{-1}$. In the range $d/2 \leq q \leq (d+1)/2$, this is a special case of the uniform resolvent bounds of Kenig, Ruiz, and Sogge [21]. The idea of using uniform resolvent bounds in the context of eigenvalue bounds for Schrödinger operators with complex potentials is due to Frank [15].

(iv) For radial potentials, Frank and Simon proved uniform $L^p \rightarrow L^{p'}$ bounds (equivalently, bounds with the operator norm instead of a Schatten norm in the left-hand side of (1.7)) in the optimal range $d/2 \leq q < d$. Thus, one of our main contributions is to upgrade their bounds to stronger trace ideal bounds.

Notation

We write $A \lesssim B$ for two non-negative quantities $A, B \geq 0$ to indicate that there is a constant $C > 0$ such that $A \leq CB$. The dependence of the constant on fixed parameters like d and q is usually omitted.

If $K: \mathcal{H} \rightarrow \mathcal{H}'$ is a compact operator between Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, we denote its singular values by $s_n(K)$. For $1 \leq p < \infty$, the Schatten norm and weak Schatten norm of K order p are defined by $\|K\|_{\mathcal{S}^p(\mathcal{H}, \mathcal{H}')} := (\sum_{n \in \mathbb{N}} s_n(K)^p)^{1/p}$ and $\|K\|_{\mathcal{S}_w^p(\mathcal{H}, \mathcal{H}')} := \sup_{n \in \mathbb{N}} n^{1/p} s_n(K)$, respectively. The latter defines a quasinorm, and for $2 < p < \infty$, an equivalent norm (hence satisfying the triangle inequality) is given by $\|K\|_{\mathcal{S}_w^p(\mathcal{H}, \mathcal{H}')} := \sup_{N \in \mathbb{N}} N^{-1+1/p} \sum_{n \leq N} s_n(K)$. We abbreviate $(\mathcal{H}, \mathcal{H}')$ by $\mathcal{H}$ when $\mathcal{H}' = \mathcal{H}$ and omit the notation entirely when the underlying spaces are clear from context.

The indicator function of a set $E \subset \mathbb{R}^d$ is denoted by $\mathbf{1}_E$. For $1 \leq p \leq \infty$, we denote its Hölder conjugate by $p' = (1 - 1/p)^{-1}$. An arbitrary ball of radius R will be denoted by B_R , without specifying its center. We use the convention $\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx$ for the Fourier transform of f .

2. Spectral measure estimate

Let $E(\lambda) = \mathbf{1}_{[0, \lambda]}(\sqrt{-\Delta})$. The spectral function can be factorised as

$$dE(\lambda)/d\lambda = c_d \lambda^{d-2} \mathcal{E}(\lambda) \mathcal{E}(\lambda)^*, \quad (2.1)$$

where $\mathcal{E}(\lambda): L^1(\mathbb{S}^{d-1}) \rightarrow L^\infty(\mathbb{R}^d)$ is the Fourier extension operator

$$\mathcal{E}(\lambda)g(x) = \int_{\mathbb{S}^{d-1}} e^{i\lambda x \cdot \xi} g(\xi) dS(\xi), \quad x \in \mathbb{R}^d, \lambda > 0,$$

and where dS denotes induced Lebesgue measure on the unit sphere $\mathbb{S}^{d-1}$. If $\lambda = 1$, we will write $\mathcal{E} := \mathcal{E}(1)$.

The main result of this section is the following precursor to Theorem 1.2.

Theorem 2.1. *Let $1 \leq q < d$ and $p > (d-1)q/(d-q)$. Then for all $\lambda > 0$ and for all $W_1, W_2 \in L_{\text{rad}}^{2q}(\mathbb{R}^d)$,*

$$\left\| W_1 \left(\frac{dE(\lambda)}{d\lambda} \right) W_2 \right\|_{\mathcal{S}^p(L^2(\mathbb{R}^d))} \leq C_{p,q,d} \lambda^{d/q-2} \|W_1\|_{L^{2q}} \|W_2\|_{L^{2q}}. \quad (2.2)$$

We will use the following two lemmas in the proof of Theorem 2.1. The first is a spherical harmonics decomposition of an operator that will appear in the proof. The second concerns bounds on integrals of Bessel functions.

In the following, $\mathcal{H}_k$ and $\mathcal{A}_k$ denote the k -th degree spherical harmonics and solid spherical harmonics, respectively, and J_ν denotes the Bessel function of the first kind of order ν .

Lemma 2.2. *For $k \in \mathbb{N}_0$, $s > 0$, let*

$$\varphi_k(s) := (2\pi)^{d/2} i^{-k} J_{(d+2k-2)/2}(s) s^{1-d/2}.$$

Then for all $H \in \mathcal{H}_k$ and $f_0 \in (L^1 \cap L^2)(\mathbb{R}_+, r^{d-1} dr)$, we have

$$(\mathcal{E}H)(\xi) = (-1)^k \varphi_k(|\xi|) H\left(\frac{\xi}{|\xi|}\right), \quad (2.3)$$

$$\mathcal{E}^*\left(x \mapsto f_0(|x|) H\left(\frac{x}{|x|}\right)\right)(\omega) = \langle \varphi_k, f_0 \rangle_{L^2(\mathbb{R}_+, r^{d-1} dr)} H(\omega). \quad (2.4)$$

Proof. Let $H \in \mathcal{H}_k$ and $f_0 \in (L^1 \cap L^2)(\mathbb{R}_+, r^{d-1} dr)$. We set

$$P(x) := |x|^k H\left(\frac{x}{|x|}\right), \quad f(x) := f_0(|x|) H\left(\frac{x}{|x|}\right), \quad \widetilde{f}_0(r) := r^{-k} f_0(r).$$

Then we can write $f(x) = \widetilde{f}_0(|x|) P(x)$. Since $P \in \mathcal{A}_k$ and $f \in (L^1 \cap L^2)(\mathbb{R}^d)$, it follows from [23, Theorem IV.3.10] that

$$\begin{aligned} \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} f(x) dx &= \widetilde{F}_0(|\xi|) P(\xi), \\ \widetilde{F}_0(r) &:= 2\pi i^{-k} r^{-(d+2k-2)/2} \int_0^\infty \widetilde{f}_0(s) J_{(d+2k-2)/2}(2\pi r s) s^{(d+2k)/2} ds. \end{aligned}$$

Replacing $f_0(r)$ by $f_0(2\pi r)$ and changing variables, we can also write this as

$$\begin{aligned} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f_0(|x|) H(x/|x|) dx &= F_0(r) H\left(\frac{x i}{|\xi|}\right), \\ F_0(r) &:= (2\pi)^{d/2} i^{-k} r^{-(d-2)/2} \int_0^\infty f_0(s) J_{(d+2k-2)/2}(rs) s^{d/2} ds. \end{aligned} \quad (2.5)$$

In particular,

$$\mathcal{E}^*\left(x \mapsto f_0(|x|)H\left(\frac{x}{|x|}\right)\right)(\omega) = F_0(1)H(\omega) = \langle \varphi_k, f_0 \rangle_{L^2(\mathbb{R}_+, r^{d-1}dr)} H(\omega).$$

This proves (2.4). Taking $f_0(r)$ in (2.5) to be an approximation of $\delta(r-1)$, a limiting argument yields

$$\int_{\mathbb{S}^{d-1}} e^{-i\omega \cdot \xi} H(\omega) dS(\omega) = \varphi_k(|\xi|) H\left(\frac{\xi}{|\xi|}\right).$$

Since $P(\xi) = |\xi|^k H(\xi/|\xi|)$ and P is a homogeneous polynomial of degree k , we have $H(-\omega) = (-1)^k H(\omega)$ and hence

$$\begin{aligned} (\mathcal{E}H)(\xi) &= \int_{\mathbb{S}^{d-1}} e^{i\omega \cdot \xi} H(\omega) dS(\omega) = (-1)^k \int_{\mathbb{S}^{d-1}} e^{-i\omega \cdot \xi} H(\omega) dS(\omega) \\ &= (-1)^k \varphi_k(|\xi|) H\left(\frac{\xi}{|\xi|}\right). \end{aligned}$$

This proves (2.3). ■

The following lemma is a special case of a result of Barcelo et al. [2]. The specific bound we require is stated below and can be inferred from [20, Lemma A.3].

Lemma 2.3. *For $\varrho \in \mathbb{R}$ and $p > \varrho + 1$, $2p/3 > \varrho + 1/3$, we have*

$$\int_1^\infty |J_\mu(2\pi s)|^{2p} s^\varrho ds \lesssim \max\{\mu^{-p+\varrho+1}, \mu^{-2p/3+\varrho+1/3}\}.$$

for all $\mu > 1/2$, $p \neq 2$. If $p = 2$ then the bound has an extra factor of $\log \mu$.

Proof of Theorem 2.1. By scaling, we may assume without loss of generality that $\lambda = 1$. By Hölder's inequality and a TT^* argument, we have

$$\begin{aligned} \|W_1 \mathcal{E} \mathcal{E}^* W_2\|_{\mathfrak{S}^p(L^2(\mathbb{R}^d))} &\leq \|W_1 \mathcal{E}\|_{\mathfrak{S}^{2p}(L^2(\mathbb{S}^{d-1}), L^2(\mathbb{R}^d))} \|\mathcal{E}^* W_2\|_{\mathfrak{S}^{2p}(L^2(\mathbb{R}^d), L^2(\mathbb{S}^{d-1}))} \\ &= \|W_1 \mathcal{E} \mathcal{E}^* \overline{W_1}\|_{\mathfrak{S}^p(L^2(\mathbb{R}^d))}^{1/2} \|W_2 \mathcal{E} \mathcal{E}^* \overline{W_2}\|_{\mathfrak{S}^p(L^2(\mathbb{R}^d))}^{1/2} \\ &= \|\mathcal{E}^* |W_1|^2 \mathcal{E}\|_{\mathfrak{S}^p(\mathbb{S}^{d-1})}^{1/2} \|\mathcal{E}^* |W_2|^2 \mathcal{E}\|_{\mathfrak{S}^p(\mathbb{S}^{d-1})}^{1/2}. \end{aligned}$$

Thus, by (2.1), it suffices to prove

$$\|\Sigma_V\|_{\mathfrak{S}^p(\mathbb{S}^{d-1})} \leq C_{p,q,d} \|V\|_{L^q} \quad \text{for all } V \in L^q_{\text{rad}}(\mathbb{R}^d),$$

where

$$\Sigma_V := \mathcal{E}^* V \mathcal{E}: L^2(\mathbb{S}^{d-1}) \rightarrow L^2(\mathbb{S}^{d-1}).$$

Since V is spherically symmetric, Lemma 2.2 implies that Σ_V is reduced by the decomposition $L^2(\mathbb{S}^{d-1}) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k$ and decomposes into an orthogonal sum

$$\Sigma_V = \bigoplus_{k=0}^{\infty} \Sigma_{V,k}, \quad \Sigma_{V,k} H = (-1)^k \lambda_{V,k} H, \quad \text{for all } H \in \mathcal{H}_k,$$

where

$$\lambda_{V,k} := (-1)^k \left(\int_0^{\infty} |\varphi_k(s)|^2 v(s) s^{d-1} ds \right), \quad v(|x|) := V(x).$$

Hence, $\Sigma_{V,k}$ is a multiple of the identity $\mathbf{1}_{\mathcal{H}_k}$, with eigenvalue $\lambda_{V,k}$ of multiplicity $\dim \mathcal{H}_k \approx k^{d-2}$. By Hölder,

$$\begin{aligned} |\lambda_{V,k}| &\leq (2\pi)^d \int_0^{\infty} |J_{(d+2k-2)/2}(s)|^2 |v(s)| s ds \\ &\leq (2\pi)^d \left(\int_0^{\infty} |J_{(d+2k-2)/2}(s)|^{2q'} s^{d-1+q'(2-d)} ds \right)^{1/q'} \left(\int_0^{\infty} |v(s)|^q s^{d-1} ds \right)^{1/q}. \end{aligned}$$

If we set $\varrho = d - 1 + q'(2 - d)$, then the conditions of Lemma 2.3 below are satisfied for $d \geq 2$ and

$$q' > \max \left\{ \frac{d}{d-1}, \frac{d - \frac{2}{3}}{d - \frac{4}{3}} \right\} = \frac{d}{d-1} \iff q < d.$$

Hence, for $q < d$, we obtain

$$\lambda_k \lesssim \max \{ k^{-q'+\varrho+1}, k^{-2q'/3+\varrho+1/3} \}^{1/q'} \|V\|_{L^q}.$$

By orthogonality, we then have

$$\|\Sigma\|_{\mathcal{B}^p}^p = \sum_{k=0}^{\infty} \|\Sigma_{V,k}\|_{\mathcal{B}^p}^p = \sum_{k=0}^{\infty} \dim \mathcal{H}_k \lambda_{V,k}^p \lesssim \sum_{k=0}^{\infty} k^{d-2} \lambda_{V,k}^p \lesssim \|V\|_{L^q}^p,$$

provided that $q < d$ and

$$d - 2 + \max \left(-q' + \varrho + 1, -\frac{2}{3}q' + \varrho + \frac{1}{3} \right) \frac{p}{q'} < -1 \iff p > \frac{(d-1)q}{d-q}.$$

This completes the proof of Theorem 2.1. ■

3. Resolvent estimates

3.1. Naive bound and simplifications

In the following, we use the abbreviation $R_0(z) := (-\Delta - z)^{-1}$, $z \in \mathbb{C} \setminus [0, \infty)$. To prove the resolvent estimate (1.7), we can assume $|z| = 1$ (by scaling). We only consider the case $\operatorname{Re} z > 0$, since the case $\operatorname{Re} z \leq 0$ is much easier. We split $R_0(z) = R_0(z)^{\text{high}} + R_0(z)^{\text{low}}$, where $R_0(z)^{\text{low}}$ has Fourier multiplier $(\xi^2 - z)^{-1} \chi(\xi/\sqrt{\operatorname{Re} z})$ and χ is a smooth, radial function on $\mathbb{R}^d$ that equals 1 on $B(0, 2)$ and is supported on $B(0, 4)$. Again, we only consider $R_0(z)^{\text{low}}$ since the estimate for $R_0(z)^{\text{high}}$ is much easier. Observing that

$$W_1 R_0(z)^{\text{low}} W_2 = \int_0^\infty \frac{\chi\left(\frac{\lambda}{\sqrt{\operatorname{Re} z}}\right)}{\lambda^2 - z} W_1 \frac{dE(\lambda)}{d\lambda} W_2 d\lambda \quad (3.1)$$

and using the spectral measure bound (2.2), we find

$$\|W_1 R_0(z)^{\text{low}} W_2\|_{\mathfrak{S}^p(L^2(\mathbb{R}^d))} \lesssim |\log |\operatorname{Im} z|| \|W_1\|_{L^{2q}} \|W_2\|_{L^{2q}}.$$

This is almost the desired bound, up to a logarithm. In the remainder of this section, we will remove the logarithm. We start with some simplifications regarding W_1, W_2 .

First simplification. The resolvent, unlike the spectral measure, is not a TT^* operator. However, since our proof only uses the spectral measure estimate, we are free to use TT^* arguments. Specifically, by the beginning of the proof of Theorem 2.1, we may assume without loss of generality that $W_2 = \overline{W_1}$. Nevertheless, for clarity and consistency, we continue to write W_1 and W_2 except where we revert to this simplification. As in the proof of Theorem 2.1, we will sometimes adopt the notation $V = |W_1|^2$ in this case. We warn the reader that this V is non-negative and thus differs from the one in the eigenvalue bounds (Theorem 1.1) by a modulus. However, no confusion should arise since we are solely concerned with resolvent estimates in this section.

Second simplification. Due to the spectral localization χ , we may also assume that W_1, W_2 are constant on unit cubes. In the remainder of Section 3.1, we detail the argument (essentially an instance of the uncertainty principle) leading to this simplification. Since this may be of independent interest, we first explain the idea without the radial symmetry assumption.

Consider a cover $\{Q\}$ of $\mathbb{R}^d$ by axis-parallel unit cubes, and define the family of weights

$$w_Q(x) := (1 + \operatorname{dist}(x, Q))^{-100d}, \quad x \in \mathbb{R}^d.$$

Given $V \in L^1(w_Q)$ (non-negative, but not necessarily radially symmetric), set

$$A(V) := \sum_Q \|V\|_{L^1(w_Q)} \mathbf{1}_Q.$$

By construction, $A(V)$ is constant on each cube Q . Adopting the first simplification, it suffices to show

$$\langle g, \mathcal{E}^* V \mathcal{E} g \rangle \lesssim \langle g, \mathcal{E}^* A(V) \mathcal{E} g \rangle, \quad \text{for all } g \in L^2(\mathbb{S}^{d-1}). \quad (3.2)$$

Indeed, by the variational principle, (3.2) implies

$$\|\mathcal{E}^* V \mathcal{E} g\|_{\mathcal{E}^p(L^2(\mathbb{S}^{d-1}))} \lesssim \|\mathcal{E}^* A(V) \mathcal{E} g\|_{\mathcal{E}^p(L^2(\mathbb{S}^{d-1}))}.$$

Proof of (3.2). Since $\mathcal{E}^* V \mathcal{E}$ is an integral operator with kernel $\widehat{V}(\xi - \eta)$, for $\xi, \eta \in \mathbb{S}^{d-1}$, we have $\mathcal{E}^* V \mathcal{E} = \mathcal{E}^*(\varphi * V)\mathcal{E}$, whenever $\varphi \in \mathcal{S}$ is such that $\widehat{\varphi} = 1$ on $B(0, 2)$. Thus, we may assume from the beginning that $\widehat{V}$ is supported on $B(0, 2)$. It is then easy to show that V is locally constant on unit cubes, in the sense that

$$\|V\|_{L^\infty(Q)} \lesssim \|V\|_{L^1(w_Q)},$$

see, e.g., [10, Lemma 15]. This implies that, for $g \in L^2(\mathbb{S}^{d-1})$,

$$\begin{aligned} \langle g, \mathcal{E}^* V \mathcal{E} g \rangle &= \int_{\mathbb{R}^d} V(x) |\mathcal{E} g(x)|^2 dx \\ &= \sum_Q \int_Q V(x) |\mathcal{E} g(x)|^2 dx \\ &\lesssim \sum_Q \|V\|_{L^1(w_Q)} \int_Q |\mathcal{E} g(x)|^2 dx \\ &= \int_{\mathbb{R}^d} \sum_Q \|V\|_{L^1(w_Q)} \mathbf{1}_Q(x) |\mathcal{E} g(x)|^2 dx \\ &= \langle g, \mathcal{E}^* A(V) \mathcal{E} g \rangle, \end{aligned}$$

where we used Fubini's theorem in the third equality. ■

The modification for spherically symmetric $V(x) = v(|x|)$ is straightforward. Instead of unit cubes in $\mathbb{R}^d$, one uses unit intervals in $\mathbb{R}_+$ and applies the same argument to $v(r)$.

3.2. Smoothing

The first step is to trade the logarithmic loss in $|\operatorname{Im} z|$ into a logarithmic loss involving the size of the supports of W_1, W_2 . Hence, we will temporarily assume $\operatorname{supp}(W_1), \operatorname{supp}(W_2) \subset B_R$, where B_R is a ball in $\mathbb{R}^d$ with radius $R \gg 1$ (the center of the ball will not be important). The spatial localization to B_R smooths off the integrand in (3.1) on the $1/R$ scale, i.e., when using the triangle inequality, $|\lambda^2 - z|$ may be replaced by $|\lambda^2 - z| + 1/R$. To make this rigorous, observe that, by the convolution theorem,

$$\mathbf{1}_{B_R} m(D) \mathbf{1}_{B_R} = \mathbf{1}_{B_R} m_R(D) \mathbf{1}_{B_R}$$

whenever $m(D)$ is a Fourier multiplier, where $m_R := \varphi_R * m$, $\varphi_R(\xi) := R^d \varphi(R\xi)$ and φ is a spherically symmetric Schwartz function such that $\widehat{\varphi} = 1$ on $B(0, 2)$. The argument of the previous subsection then yields

$$\|W_1 R_0(z)^{\operatorname{low}} W_2\|_{\mathfrak{S}^p(L^2(\mathbb{R}^d))} \lesssim (\log R) \|W_1\|_{L^{2q}} \|W_2\|_{L^{2q}}. \quad (3.3)$$

We refer to [10, Section 6.2] for full details.

3.3. Removal of the logarithm

To remove the $\log R$ factor, we partly follow the strategy of the first-named author and Merz [10, Section 7.2]. The idea is reminiscent of an “epsilon removal lemma” of Tao [25] in the context of Fourier restriction theory. Compared to [10], the strategy is easier to explain in our setting since the potential is not random. Thus, we will only need to deal with the bilinear operator $(W_1, W_2) \mapsto W_1 R_0(z)^{\operatorname{low}} W_2$, as opposed to multilinear operators with more than one factor of $R_0(z)^{\operatorname{low}}$, which had to be considered in [10]. On the other hand, we work with Schatten norms, whereas [10] deals with operator norms.

To illustrate the method, we will first prove a version without the radial symmetry assumption.

Proposition 3.1. *Let $d \geq 2$, $1 \leq p_0, q_0 < \infty$, and assume that the estimate*

$$\|W \mathfrak{E} \mathfrak{E}^* \overline{W}\|_{\mathfrak{S}^{p_0}(L^2(\mathbb{R}^d))} \leq C(\log R)^s \|W\|_{L^{2q_0}}^2 \quad (3.4)$$

holds for all $W \in L^{2q_0}(\mathbb{R}^d)$ supported in a ball of radius R and constant on unit cubes, for some $s > 0$. Then, for $1 \leq q < q_0$ and $1 \leq p < p_0$ satisfying

$$\frac{\frac{1}{p} - \frac{1}{p_0}}{\frac{1}{q} - \frac{1}{q_0}} < \frac{1 - \frac{1}{p_0}}{1 - \frac{1}{q_0}}, \quad (3.5)$$

there exists a constant $C_{p,q,s,d}$ such that

$$\|W_1 R_0(z)^{\text{low}} W_2\|_{\mathfrak{S}^p(L^2(\mathbb{R}^d))} \leq C_{p,q,s,d} \|W_1\|_{L^{2q}} \|W_2\|_{L^{2q}} \quad (3.6)$$

for all $W_1, W_2 \in L^{2q}(\mathbb{R}^d)$ and for all $z \in \mathbb{C} \setminus [0, \infty)$ with $|z| = 1$.

Remark 3. By [19, Theorem 2], the assumed bound (3.4) is satisfied if and only if $q_0 \leq (d+1)/2$ and $p_0 \leq (d-1)q/(d-q)$. The resolvent bound (3.6), including the endpoint $q = q_0$, follows from [19, Theorem 12]. The Schatten exponent in both theorems there is $p = (d-1)q/(d-q)$. Using only [19, Theorem 2], Proposition 3.1 asserts that (3.6) holds for all $q < (d+1)/2$, with a slightly worse Schatten exponent p strictly greater than, but arbitrarily close to, $(d-1)q/(d-q)$ (note that the latter is an increasing function of q). Thus, in practice, Proposition 3.1 does not give anything new. What is important is the technique, which will be used to prove Theorem 1.2.

To free up notation, we will use W, W' instead of W_1, W_2 in the proof of Proposition 3.1. We will perform several decompositions of W, W' , and we start explaining these for a non-negative simple function W , supported on a finite union of c -cubes contained in a ball B_R .

- Horizontal dyadic decomposition: we write $W = \sum_{i \in \mathbb{N}} W_i$, where

$$W_i := W \mathbf{1}_{H_i \geq W \geq H_{i+1}}, \quad H_i := \inf\{t > 0 : |\{W > t\}| \leq 2^{i-1}\},$$

then

$$\|H_i 2^{i/q}\|_{\ell^r_i(\mathbb{Z}_+)} \asymp \|W\|_{L^{q,r}},$$

where $L^{q,r}$ denotes a Lorentz space (see e.g., [26, Theorem 6.6]). Also note that $L^{q,q} = L^q$.

- Sparse decomposition: we write

$$W_i = \sum_{j=1}^{K_i} \sum_{k=1}^{N_i} W_{ijk}, \quad (3.7)$$

where, for fixed i, j , the functions W_{ijk} are supported on a “sparse collection” of balls $\{B(x_k, R_i)\}_{k=1}^{N_i}$. By this, we mean that the support of W_{ijk} is contained in $B(x_k, R_i)$ and that the following definition is satisfied (cf. [25, Definition 3.1]) for some sufficiently large γ (to be fixed later).

Definition 1. A collection $\{B(x_k, R)\}_{k=1}^N$ is γ -sparse if the centers x_k are $(RN)^\gamma$ separated.

For fixed $\gamma > 0$ and $K \geq 1$, [25, Lemma 3.3] asserts that (3.7) holds with

$$K_i = \mathcal{O}(K2^{i/K}), \quad N_i = \mathcal{O}(2^i), \quad R_i = \mathcal{O}(2^{i\gamma^K}). \quad (3.8)$$

To apply the result of [25], we need the assumption that W is constant on unit cubes (it is clear that W_i are also constant on the same cubes).

We will also need the following estimate.

Lemma 3.2. *There exists a constant such that for all $z \in \mathbb{C} \setminus [0, \infty)$, $|z| = 1$, for all $W, W' \in L^q(\mathbb{R}^d)$ supported in balls of radius R with $\text{dist}(\text{supp}(W), \text{supp}(W')) > 0$, and for all $\delta \in [0, 1]$, we have*

$$\begin{aligned} & \|WR_0(z)^{\text{low}}W'\|_{\mathfrak{S}^{1+\delta}} \\ & \leq C(\log R) \text{dist}(\text{supp}(W), \text{supp}(W'))^{-(d-1)\delta/(1+\delta)} \|W\|_{L^2} \|W'\|_{L^2}. \end{aligned} \quad (3.9)$$

Proof. The result will follow from interpolation between the two estimates

$$\|WR_0(z)^{\text{low}}W'\|_{\mathfrak{S}^2} \lesssim \text{dist}(\text{supp}(W), \text{supp}(W'))^{-(d-1)/2} \|W\|_{L^2} \|W'\|_{L^2} \quad (3.10)$$

and

$$\|WR_0(z)^{\text{low}}W'\|_{\mathfrak{S}^1} \lesssim (\log R) \|W\|_{L^2} \|W'\|_{L^2}. \quad (3.11)$$

The first estimate follows from the kernel bound

$$|R_0(z)^{\text{low}}(x - y)| \lesssim (1 + |x - y|)^{-(d-1)/2},$$

which in turn is a consequence of a standard stationary phase argument (see e.g., [7, 24]). The second estimate follows from

$$\|W\mathcal{E}\mathcal{E}^*W'\|_{\mathfrak{S}^1} \lesssim \|W\|_{L^2} \|W'\|_{L^2} \quad (3.12)$$

together with the smoothing argument leading to (3.3). Since $\mathbb{S}^{d-1}$ is compact, we have the obvious Hilbert–Schmidt bounds

$$\|W\mathcal{E}\|_{\mathfrak{S}^2} \lesssim \|W\|_{L^2}, \quad \|\mathcal{E}^*W'\|_{\mathfrak{S}^2} \lesssim \|W'\|_{L^2},$$

which immediately imply (3.12). Interpolating (3.10) and (3.11) yields, for any $p \in [1, 2]$,

$$\begin{aligned} & \|WR_0(z)^{\text{low}}W'\|_{\mathfrak{S}^p} \\ & \lesssim (\log R)^{2/p-1} \text{dist}(\text{supp}(W), \text{supp}(W'))^{-(d-1)(1-1/p)} \|W\|_{L^2} \|W'\|_{L^2}. \end{aligned}$$

Setting $p = 1 + \delta$ and recalling that $R \gg 1$ yields the claim. ■

Proof of Proposition 3.1. We write W, W' instead of W_1, W_2 and apply the above decomposition:

$$W = \sum_{\alpha} W_{\alpha}, \quad W' = \sum_{\alpha'} W'_{\alpha'},$$

where $\alpha = (i, j, k)$, $\alpha' = (i', j', k')$ and $i, i' \in \mathbb{Z}_+$, $1 \leq j \leq K_i$, $1 \leq j' \leq K_{i'}$, $1 \leq k \leq N_i$ and $1 \leq k' \leq N_{i'}$. Set

$$\begin{aligned} L_{\alpha, \alpha'} &:= 1 + \text{dist}(\text{supp}(W_{\alpha}), \text{supp}(W'_{\alpha'})), \\ \varrho_{\alpha, \alpha'} &:= 2(1 + L_{\alpha, \alpha'} + R_i + R_{i'}). \end{aligned}$$

In the following, we simply write R_0 instead of $R_0(z)^{\text{low}}$. By (3.3), and using that $\mathfrak{S}_{p_0} \subset \mathfrak{S}_{p_0}^w$,

$$s_n(W_{\alpha} R_0 W'_{\alpha'}) \lesssim n^{-1/p_0} (\log \varrho_{\alpha, \alpha'})^{s+1} \|W_{\alpha}\|_{L^{2q_0}} \|W'_{\alpha'}\|_{L^{2q_0}},$$

whereas (3.9) implies

$$s_n(W_{\alpha} R_0 W'_{\alpha'}) \lesssim n^{-1/(1+\delta)} \log(\varrho_{\alpha, \alpha'}) L_{\alpha, \alpha'}^{-(d-1)\delta/(1+\delta)} \|W_{\alpha}\|_{L^2} \|W'_{\alpha'}\|_{L^2}. \quad (3.13)$$

Interpolating the last two displays yields, for any $\theta \in (0, 1)$,

$$\begin{aligned} s_n(W_{\alpha} R_0 W'_{\alpha'}) &\lesssim_{\theta} n^{-\theta/(1+\delta) - (1-\theta)/p_0} L_{\alpha, \alpha'}^{-\theta(d-1)\delta/(2(1+\delta))} \\ &\quad \times \log^{s+1}(2 + R_i + R_{i'}) H_i H'_{i'} 2^{(i+i')(\theta/2 + (1-\theta)/2q_0)}, \end{aligned}$$

where we used the fact that

$$L_{\alpha, \alpha'}^{-\theta(d-1)\delta/(1+\delta)} (\log^{s+1} \varrho_{\alpha, \alpha'})^{1-\theta} \lesssim_{\theta} L_{\alpha, \alpha'}^{-(d-1)\delta/(2(1+\delta))} \log^{s+1}(2 + R_i + R_{i'}).$$

Summing (3.13) first over k , then k' ,

$$\begin{aligned} &\sum_{k, k'} s_n(W_{\alpha} R_0 W'_{\alpha'}) \\ &\lesssim_{\theta} n^{-\theta/(1+\delta) - (1-\theta)/p_0} \log^{s+1}(2 + R_i + R_{i'}) H_i H'_{i'} 2^{(i+i')(\theta/2 + (1-\theta)/2q_0)}. \end{aligned} \quad (3.14)$$

Here we have used the fact that, for $\alpha = (i, j, k)$, $\alpha' = (i', j', k')$, and i, j, i', j', k' fixed, the sum over k is bounded,

$$\begin{aligned} &\sum_{k \leq N_i} (1 + \text{dist}(B(x_k, R_i), B(x_{k'}, R_{i'})))^{-\theta(d-1)\delta/(2(1+\delta))} \\ &\leq 1 + 2N_i (N_i R_i)^{-\gamma\theta(d-1)\delta/(2(1+\delta))} \lesssim 1 \end{aligned}$$

uniformly in i, j, i', j', k' , provided $\gamma\theta(d-1)\delta/(2(1+\delta)) \geq 1$. We will momentarily fix θ, δ , and then fix γ such that $\gamma\theta(d-1)\delta/(2(1+\delta)) = 1$. Note that, even though the balls in the last display may belong to different sparse families, we have

$$\text{dist}(B(x_k, R_i), B(x_{k'}, R_{i'})) \geq \frac{1}{2}(N_i R_i)^\gamma$$

for all but at most one k (and fixed $x_{k'}, R_i, R_{i'}$). Indeed, suppose for contradiction that there exist two distinct values k_1, k_2 such that

$$\text{dist}(B(x_{k_n}, R_i), B(x_{k'}, R_{i'})) < \frac{1}{2}(N_i R_i)^\gamma, \quad n = 1, 2.$$

Then, by the triangle inequality,

$$\text{dist}(B(x_{k_1}, R_i), B(x_{k_2}, R_i)) \leq \sum_{n=1}^2 \text{dist}(B(x_{k_n}, R_i), B(x_{k'}, R_{i'})) < (N_i R_i)^\gamma,$$

which contradicts the sparsity of the collection $\{B(x_k, R_i)\}_{k=1}^{N_i}$.

Summing (3.14) over j then j' (recall $j \leq K_i, j' \leq K_{i'}$),

$$\begin{aligned} & \sum_{j, j'} \sum_{k, k'} s_n(W_\alpha R_0 W_{\alpha'}) \\ & \lesssim_\theta K_i K_{i'} n^{-\theta/(1+\delta)-(1-\theta)/p_0} \log^{s+1}(2 + R_i + R_{i'}) H_i H_{i'} 2^{(i+i')(\theta/2+(1-\theta)/2q_0)}. \end{aligned}$$

Finally, summing over i, i' and using (3.8),

$$\begin{aligned} & \sum_{\alpha, \alpha'} s_n(W_\alpha R_0 W_{\alpha'}) \lesssim_\theta n^{-\theta/(1+\delta)-(1-\theta)/p_0} K^2 \sum_{i, i'} \log^{s+1}(2 + R_i + R_{i'}) \\ & \quad \times H_i H_{i'} 2^{(i+i')(\theta/2+(1-\theta)/2q_0+1/K)} \\ & \lesssim_{\theta, K, s} n^{-\theta/(1+\delta)-(1-\theta)/p_0} \sum_i H_i 2^{i(\theta/2+(1-\theta)/(d+1)+2/K)} \\ & \quad \times \sum_{i'} H_{i'} 2^{i'(\theta/2+(1-\theta)/2q_0+2/K)}, \end{aligned}$$

where we estimated $\log^{s+1} R_i \lesssim_{K, s} 2^{i/K}$ in the second line. We now fix $\theta, \delta > 0$ and $K \geq 1$ in such a way that

$$\begin{aligned} & \frac{\theta}{2} + \frac{1-\theta}{2q_0} + \frac{2}{K} = \frac{1}{2q}, \\ & \frac{\theta}{1+\delta} + \frac{1-\theta}{p_0} < \frac{1}{p}. \end{aligned}$$

This is always possible since $q < q_0$ and p satisfies (3.5). Set

$$\varepsilon := \frac{1}{p} - \frac{\theta}{1+\delta} - \frac{1-\theta}{p_0} > 0.$$

Then

$$\sum_{\alpha, \alpha'} s_n(W_\alpha R_0 W'_{\alpha'}) \lesssim_q n^{-1/p-\varepsilon} \|W\|_{L^{2q,1}} \|W'\|_{L^{2q,1}},$$

and for any $N \in \mathbb{N}$, by Fubini,

$$\sum_{\alpha, \alpha'} \sum_{n=1}^N s_n(W_\alpha R_0 W'_{\alpha'}) \lesssim_q N^{1-1/p-\varepsilon} \|W\|_{L^{2q,1}} \|W'\|_{L^{2q,1}}. \quad (3.15)$$

Set

$$\frac{1}{r} := \frac{1}{p} + \varepsilon.$$

Then, by the triangle inequality and (3.15),

$$\|WR_0 W'\|_{\mathfrak{S}_w^r} \leq \sup_{N \in \mathbb{N}} N^{-1+1/r} \sum_{\alpha, \alpha'} \sum_{n=1}^N s_n(W_\alpha R_0 W'_{\alpha'}) \lesssim_q \|W\|_{L^{2q,1}} \|W'\|_{L^{2q,1}}.$$

In particular, since $r < p$,

$$\|WR_0 W'\|_{\mathfrak{S}^p} \lesssim_q \|W\|_{L^{2q}} \|W'\|_{L^{2q}},$$

where we used that $\mathfrak{S}_w^r \subset \mathfrak{S}^p$ and $L^{2q,1} \subset L^{2q}$. ■

3.4. Proof of Theorem 1.2

The proof is similar to that of Proposition 3.1. The difference is that we use (2.2), which only holds for radial functions, instead of (3.4) as input. Again, we write W, W' instead of W_1, W_2 and decompose

$$w = \sum_{\alpha} w_{\alpha}, \quad w' = \sum_{\alpha'} w'_{\alpha'},$$

where $W(x) = w(|x|)$ and $W'(x) = w'(|x|)$. The balls in the sparse collections (in the proof of Proposition 3.1) are now intervals, but (3.8) still holds (this is just the one-dimensional case in [25, Lemma 3.3]). The remainder of the proof is the same as before.

4. Eigenvalue sums

4.1. Proof of Theorem 1.1

A special case ($\varrho = 0$) of [16, Theorem 3.1] states that if $p \geq 1$, $\sigma > 0$ and if $K(z)$, $z \in \mathbb{C} \setminus [0, \infty)$, is an analytic family of operators satisfying

$$\|K(z)\|_{\mathfrak{S}^p} \leq M|z|^{-\sigma} \quad \text{for all } z \in \mathbb{C} \setminus [0, \infty), \quad (4.1)$$

then the finite-type eigenvalues z_j of $1 + K$ satisfy

$$\sum_j \delta(z_j) |z_j|^{-\frac{1}{2} + \frac{1}{2}(2p\sigma - 1 + \varepsilon)_+} \leq C_{p,\sigma,\varepsilon} M^{\frac{1}{2\sigma}(1 + (2p\sigma - 1 + \varepsilon)_+)} \quad (4.2)$$

for any $\varepsilon > 0$. We use this with $K(z) = \sqrt{|V|} R_0(z) \sqrt{V}$, in which case the z_j are the eigenvalues of $H = -\Delta + V$ (see [16] for details). By Theorem 1.2, since V is assumed to be radial, (4.1) holds with $M = C_{p,q,d} \|V\|_{L^q}$, $\sigma = 1 - d/(2q)$ and any $p > (d-1)q/(d-q)$. Since this implies that $p > q/(2q-d)$, we have $2p\sigma - 1 > 0$, and hence we can omit the notation for the positive part, $(\dots)_+$, in (4.2). Since we are imposing an open condition for p , we can absorb ε into p and omit it from the inequality. The result is (1.6), so the proof of Theorem 1.1 is complete.

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