

Reproving Friedlander’s inequality with the de Rham complex

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Abstract. Inequalities between Dirichlet and Neumann eigenvalues of the Laplacian and of other differential operators have been intensively studied in the past decades. The aim of this paper is to introduce differential forms and the de Rham complex in the study of such inequalities. We show how differential forms lie hidden at the heart of the work of Rohleder on inequalities between Dirichlet and Neumann eigenvalues for the Laplacian on planar domains. Moreover, we extend the ideas of Rohleder to a new proof of Friedlander’s inequality for any bounded Lipschitz domain.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be an open, connected, bounded domain with Lipschitz boundary. We write $\lambda_j(T)$ for the j -th eigenvalue, ordered increasingly, for a positive operator T with discrete spectrum. We denote by Δ_D and Δ_N the Dirichlet and Neumann realization of the Laplacian on Ω . The main goal of this paper is to introduce methods of differential forms and the de Rham complex as a tool for obtaining inequalities between eigenvalues of Δ_D and Δ_N .

The inequality $\lambda_2(\Delta_N) < \lambda_1(\Delta_D)$ appears already in the work of Pólya for $d = 2$, see [21], and it was extended by Payne to $\lambda_{j+2}(\Delta_N) < \lambda_j(\Delta_D)$ for all $j = 1, 2, \dots$ when Ω is a C^2 convex domain [20]. Later, Levine and Weinberger generalized the inequality $\lambda_{j+d}(\Delta_N) < \lambda_j(\Delta_D)$ considering a convex domain in $\mathbb{R}^d$ with C^2 boundary with Hölder continuous second derivatives [15]. As pointed out by Levine and Weinberger, the previous inequality can be extended by approximation to $\lambda_{j+d}(\Delta_N) \leq \lambda_j(\Delta_D)$ for all convex bounded domains. Moreover, they recovered the inequality $\lambda_{j+1}(\Delta_N) < \lambda_j(\Delta_D)$ for domains with C^2 boundary and non-negative mean curvature proven by Aviles [2].

In 1991, Friedlander [10] used properties of the Dirichlet-to-Neumann operator to prove

$$\lambda_{j+1}(\Delta_N) \leq \lambda_j(\Delta_D), \tag{1.1}$$

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for all $j = 1, 2, \dots$, and for C^1 domains and no curvature assumption. The smoothness assumption was removed and the inequality was proven to be strict by Filonov in [6] in a beautiful argument using Glazman's lemma, see more in the textbook [16]. We call equation (1.1) *Friedlander's inequality*. Recently, Rohleder [24] proved that for any simply connected, bounded, Lipschitz domain in $\mathbb{R}^2$, there is an inequality

$$\lambda_{j+2}(\Delta_N) \leq \lambda_j(\Delta_D), \quad (1.2)$$

for any $j \in \mathbb{N}$. We combine the ideas of [24] with the de Rham complex [4] into a common framework that in arbitrary dimension allows a proof of Friedlander's inequality (1.1) and a generalization of Rohleder's results on the curl curl operator [23], as well as a short proof of Rohleder's inequality (1.2) in dimension two.

The main novelty in this paper is found in the method we introduce. As mentioned, the method stems in the de Rham complex. Since our geometries have boundaries, we require an appropriate boundary condition. We use the so-called *absolute boundary condition* allowing us to rely on previous work [4] on this well studied Hilbert complex. What is promising with this method is that it allows us to give concise proofs of work of Rohleder [23, 24] and carries large dimensions of internal degrees of freedom that holds hope of pushing the estimates even further (cf. Remark 4.2 below).

It was conjectured that

$$\lambda_{j+d}(\Delta_N) < \lambda_j(\Delta_D) \quad (1.3)$$

holds for all domains $\Omega \subset \mathbb{R}^d$ with no convexity assumption [3]. If $d = 2, 3$, (1.3) is sharp because the unit ball is an edge case, namely $\lambda_{d+2}(\Delta_N) > \lambda_1(\Delta_D)$. For $d \geq 4$, it was observed in [5] that for the unit ball we have more than $d + 1$ Neumann eigenvalues strictly below the first Dirichlet eigenvalue. For example, if $d = 4$ we have $\binom{d}{d-1} + \binom{d+1}{d-1}$ Neumann eigenvalues strictly below the first Dirichlet eigenvalue. Recall that, for the unit ball, these binomial coefficients are connected with the dimension of the space of spherical harmonics of certain degrees (see [16, Section 1.2.3] for more details). In a recent work, Freitas [9] studied the gap between Dirichlet and Neumann eigenvalues with respect to the index j . It was conjectured that

$$\lambda_{j+\lfloor c(d,j) \rfloor}(\Delta_N) \leq \lambda_j(\Delta_D),$$

where $c(d, j) = \frac{dV_{d-1}}{2V_d^{1-2/d}} j^{1-1/d}$ and V_d is the volume of the d -dimensional unit ball. Moreover, for $d \geq 4$ and all $j \in \mathbb{N}$ we have the weaker inequality

$$\lambda_{j+\lfloor C_\Omega j^{1-3/d} \rfloor}(\Delta_N) \leq \lambda_j(\Delta_D),$$

where the constant C_Ω is not explicit (see [9, 25]). The de Rham complex introduces large binomial coefficients (cf. Remark 4.2 below) into the estimates that we hope can provide insight into the conjectural extra shift (1.3) for general domains.

The techniques used in proving inequalities for the Laplacian have been adapted to other differential operators. Frank, Helffer, and Laptev [7, 8] adapted such ideas to prove a similar inequality for the sub-Laplacian on an open set of a Carnot group, which in particular covers the Heisenberg group. Another example given by Mazzeo [18] is the adaptation of Friedlander's ideas to prove the same inequality for certain manifolds, e.g., for all symmetric spaces of non-compact type. However, for manifolds there are cases where the inequality (1.1) does not hold, for example any spherical cap larger than a hemisphere [18]. Recently, Lotoreichik explored these inequalities for the magnetic Laplacian with the homogeneous magnetic field in two and three dimensions [17]. Similar inequalities have also been proven for Schrödinger operators $-\Delta + V$ under convexity assumptions and further restrictions on the potential [22], and between the eigenvalues of a curl curl operator and the Dirichlet Laplacian [23].

The paper is organized as follows. In Section 2 we give a brief summary of the main tools we need in order to introduce the de Rham complex and prove the main results. We recall the de Rham complex on a manifold with boundary as well as the relevant technical results thereon in Section 3. In Section 4 we rephrase the results of [24] in general dimension using the de Rham complex, which lead us to our new proof of Friedlander's inequality (1.1). We compare our methods to Rohleder's work [23, 24] in dimension 2 and 3 in Section 5 where we provide a short proof for Rohleder's inequality (1.2).

There is a recent preprint [19] with results overlapping part of the results in this paper. In particular, [19, Theorem 1.5] can be obtained as a consequence of Lemma 4.3 for smooth domains in a similar way as we did for dimensions 2 and 3 in Section 5. In both cases, differential forms are used to obtain the results, but the techniques are different.

2. Preliminaries

Let T be a positive, self-adjoint operator with discrete spectrum (i.e., the spectrum consists of isolated eigenvalues of finite multiplicity). We denote by $\lambda_j(T)$ the j -th eigenvalue of T ordered increasingly counting multiplicity. We write the counting function of T as

$$N(T, \lambda) := \#\{j : \lambda_j(T) \leq \lambda\}.$$

We let

$$m(T, \lambda) := \dim \ker(T - \lambda),$$

denote the multiplicity of an eigenvalue (or 0 if λ is not an eigenvalue). Note that

$$j \leq N(T, \lambda_j(T)) \leq j + m(T, \lambda_j(T)) - 1. \quad (2.1)$$

2.1. Variational principle

The results in [6, 24] rely on a variational principle, which will also be useful in our approach using the de Rham complex. Because of this, we recall Glazman's lemma describing the counting function by means of finite-dimensional subspace of the form domain.

Lemma 2.1 (Glazman's lemma). *Let T be a positive self-adjoint operator with discrete spectrum acting on a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ and q_T the quadratic form associated with T . Then*

$$N(T, \lambda) = \max_{\substack{V \subseteq \text{Dom}(q_T) \\ R[u] \leq \lambda \quad \forall u \in V \setminus \{0\}}} \dim V,$$

where $R[u] = \frac{q_T(u)}{\langle u, u \rangle}$ is the so-called Rayleigh quotient.

See [26, Proposition 9.5] for a proof of the Glazman's lemma. Hence, if we have a finite-dimensional subspace $V \subseteq \text{Dom}(q_T)$ such that

$$q_T(u) \leq \lambda \|u\|^2,$$

for $u \in V$, then using Glazman's lemma we obtain

$$N(T, \lambda) \geq \dim V.$$

We will also use the notation $N(q_T, \lambda) := N(T, \lambda)$. We use the notation $T: \mathcal{H} \dashrightarrow \mathcal{H}'$ for a densely defined operator between two Hilbert spaces. A fact we use is that if there is a closed densely defined operator $t: \mathcal{H} \dashrightarrow \mathcal{H}'$ for some Hilbert space $\mathcal{H}'$ such that $T = t^*t$, then $\text{Dom}(q_T) = \text{Dom}(t)$ and $q_T(u, v) = \langle tu, tv \rangle$.

2.2. Hilbert complexes

A key aspect in our study of inequalities between Dirichlet and Neumann Laplacian eigenvalues will be the usage of the de Rham complex on a domain in $\mathbb{R}^n$ with appropriate boundary conditions. It is helpful to set this in an abstract framework, so we first recall the notion of a Hilbert complex. We follow the presentation of [4] and refer the reader there for further details. Below in Section 3 we specialize to de Rham complexes.

Definition 2.2. A Hilbert complex written as $(\mathcal{H}_\bullet, T_\bullet)$ or

$$0 \rightarrow \mathcal{H}_0 \xrightarrow{T_0} \mathcal{H}_1 \xrightarrow{T_1} \cdots \xrightarrow{T_{d-2}} \mathcal{H}_{d-1} \xrightarrow{T_{d-1}} \mathcal{H}_d \rightarrow 0,$$

consists of Hilbert spaces $\mathcal{H}_0, \mathcal{H}_1, \dots, \mathcal{H}_d$ and closed, densely defined maps

$$T_k: \mathcal{H}_k \dashrightarrow \mathcal{H}_{k+1}$$

with the property that

$$\text{ran}(T_{k-1}) \subseteq \ker(T_k).$$

In other words, $\text{ran}(T_{k-1}) \subseteq \text{Dom}(T_k)$ and $T_k T_{k-1} = 0$.

We say that $(\mathcal{H}_\bullet, T_\bullet)$ is *Fredholm* if the cohomology groups

$$H^k(\mathcal{H}_\bullet, T_\bullet) := \ker(T_k) / \text{ran}(T_{k-1}),$$

are finite-dimensional. The Euler characteristic of a Fredholm Hilbert complex $(\mathcal{H}_\bullet, T_\bullet)$ is defined as

$$\chi(\mathcal{H}_\bullet, T_\bullet) := \sum_{k=0}^d (-1)^k \dim H^k(\mathcal{H}_\bullet, T_\bullet). \quad (2.2)$$

We say that $(\mathcal{H}_\bullet, T_\bullet)$ has *discrete spectrum* if for any k the densely defined, self-adjoint Laplacians

$$\Delta_{k,T_\bullet} := T_k^* T_k + T_{k-1} T_{k-1}^* : \mathcal{H}_k \dashrightarrow \mathcal{H}_k,$$

have discrete spectrum.

We note it is a stronger assumption to have discrete spectrum than being Fredholm. Moreover, each operator T_k has closed range if the Hilbert complex $(\mathcal{H}_\bullet, T_\bullet)$ is Fredholm. Note also that $\Delta_{k,T_\bullet} = (T_k + T_{k-1}^*)^* (T_k + T_{k-1}^*)$ and hence $\text{Dom } \Delta_{k,T_\bullet} = \text{Dom } T_k \cap \text{Dom } T_{k-1}^*$.

We will utilize Hilbert complexes $(\mathcal{H}_\bullet, T_\bullet)$ in order to compare the spectrum of the bottom Laplacian $\Delta_{0,T_\bullet} = T_0^* T_0$ and the top Laplacian $\Delta_{d,T_\bullet} := T_d T_d^*$. We shall see below that for the de Rham complex with an appropriate boundary condition, $\Delta_{0,T_\bullet}$ is the Neumann realization of the Laplacian and $\Delta_{d,T_\bullet}$ is up to the Hodge star the Dirichlet realization of the Laplacian. Assuming that $(\mathcal{H}_\bullet, T_\bullet)$ is Fredholm, we have the Hodge decomposition

$$\mathcal{H}_k = \ker(\Delta_{k,T_\bullet}) \oplus \text{ran}(T_k^*) \oplus \text{ran}(T_{k-1}). \quad (2.3)$$

In particular, if $(\mathcal{H}_\bullet, T_\bullet)$ in fact has discrete spectrum we can deduce that

$$N(\Delta_{k,T_\bullet}, \lambda) = \dim \ker(\Delta_{k,T_\bullet}) + N(T_k^* T_k; (0, \lambda]) + N(T_{k-1} T_{k-1}^*; (0, \lambda]). \quad (2.4)$$

Here we use the notation $N(T; (0, \lambda])$ for the number of eigenvalues of a self-adjoint operator T in the interval $(0, \lambda]$. Combining such terms in an alternating sum, and using that $T_k^* T_k$ and $T_k T_k^*$ has the same non-zero spectrum including multiplicities, we conclude the following lemma.

Lemma 2.3. *Assume that $(\mathcal{H}_\bullet, T_\bullet)$ is a Hilbert complex with discrete spectrum. Then for any $\lambda \geq 0$, we have an equality*

$$\sum_{k=0}^d (-1)^k N(\Delta_{k, T_\bullet}, \lambda) = \chi(\mathcal{H}_\bullet, T_\bullet).$$

3. The de Rham complex

We now turn to studying the de Rham complex with boundary conditions. The material in this section is well known and can be found in the literature [4, 11]. The reader uninitiated in differential forms and the de Rham complex can consult [28] for more details, or the more elementary text [12]. We take a smooth, oriented, compact manifold with Lipschitz boundary M , or in other words, M is a precompact domain with Lipschitz boundary in a smooth, oriented manifold. We write d for the dimension of M . To carry out spectral geometry, we need to choose a Riemannian metric g on M . We denote the Riemannian volume form by dV and the Riemannian volume density by dx . The volume density will only be used as a measure in integrals so its difference to differentials such as dx_j will be clear. We write $T^*M \rightarrow M$ for the cotangent bundle on M . Abusing the notation, we write $\bigwedge^k T^*M \rightarrow M$ for the complexified bundle of degree k -forms on M and $\bigwedge^* T^*M := \bigoplus_{k=0}^d \bigwedge^k T^*M$ for the bundle of all complexified forms on M .

For our application to the Friedlander's inequality (1.1), we consider Lipschitz domains in $\mathbb{R}^d$ with the Euclidean metric. *For notational clarity, we reserve the letter Ω for domains in $\mathbb{R}^d$ and M for general manifolds.* If Ω is a domain in $\mathbb{R}^d$, the basis vectors of $\mathbb{R}^d$ defines a frame and trivializations $\bigwedge^k T^*\Omega \cong \Omega \times \bigwedge^k \mathbb{C}^d$.

The exterior differential between differential forms is a well studied differential operator. We write the exterior differential on forms of degree k as

$$d_k: C^\infty(M, \bigwedge^k T^*M) \rightarrow C^\infty(M, \bigwedge^{k+1} T^*M) \quad (3.1)$$

and the exterior differential on all forms as

$$d: C^\infty(M, \bigwedge^* T^*M) \rightarrow C^\infty(M, \bigwedge^* T^*M).$$

We also write $\delta: C^\infty(M, \bigwedge^* T^*M) \rightarrow C^\infty(M, \bigwedge^* T^*M)$ for the formal adjoint of d , decomposing over the form degrees into

$$\delta_k: C^\infty(M, \bigwedge^{k+1} T^*M) \rightarrow C^\infty(M, \bigwedge^k T^*M). \quad (3.2)$$

The operator δ_k takes the form

$$\delta_k = (-1)^{k+1} \star^{-1} d_{d-k-1} \star,$$

where $\star$ denotes the Hodge star. The Hodge star $\star\omega$ of a k -form ω is the $n - k$ -form with the property that for any real k -form ω' ,

$$\omega' \wedge \star\omega = \langle \omega', \omega \rangle_{\wedge^k T^*M} dV,$$

where $\langle \cdot, \cdot \rangle_{\wedge^k T^*M}$ denotes the inner product on k -forms and dV denotes the Riemannian volume form. The operator $\not{D} := d + \delta$ is an elliptic first order differential operator, called the *Hodge–Dirac* operator, and $\not{D}^2$ coincides with the Hodge Laplacian on forms, see [11].

So far, the discussion has only been concerned with differential expressions. Now, we turn to realizations of these operators on L^2 -spaces. We consider the Hilbert spaces

$$\mathcal{H}_k := L^2(M; \wedge^k T^*M), \quad k = 0, 1, 2, \dots, d.$$

There are several ways to realize the exterior differential and its adjoint on $\mathcal{H}_k$ as closed operators, notably via the *ideal boundary conditions* defined as those boundary conditions ensuring that we obtain a Hilbert complex. Such boundary conditions are discussed in detail in [4]. We will only use the so-called *maximal realization* but for completeness we also discuss the minimal realization.

- We shall write $d_{k,\max}$ for the maximal realization of d_k , i.e., so $u \in \text{Dom}(d_{k,\max})$ if and only if $u \in L^2(M; \wedge^k T^*M)$ satisfies that $du \in L^2(M; \wedge^{k+1} T^*M)$ where the exterior differential is applied in a distributional sense. The reader should beware that, for $k > 0$ the domain of $d_{k,\max}$ is substantially larger than $H^1(M; \wedge^k T^*M)$. It follows from [4, Section 4] that $H^1(M; \wedge^k T^*M)$ is a core for $d_{k,\max}$.
- The minimal realization $d_{k,\min}$ of d_k , i.e., the graph closure of d_k acting on $C_c^\infty(M^\circ; \wedge^k T^*M)$. Also in this case, the reader should be aware that for $k < d$ the domain of $d_{k,\min}$ is larger than $H_0^1(M; \wedge^k T^*M)$ even if it follows from [4, Section 4] that $H_0^1(M; \wedge^k T^*M)$ forms a core for the operator.

Unless otherwise state, we use the maximal realization. We use the notation

$$\Delta_{k,a} := d_{k,\max}^* d_{k,\max} + d_{k-1,\max} d_{k-1,\max}^*.$$

The index a refers to its defining boundary condition which is called the *absolute boundary condition*, it is called so for reasons that will become apparent in Theorem 3.3 and Remark 3.4 below. Note that $d_{k-1,\max}^*$ is the minimal realization of δ_{k-1} . We make the following observations from quadratic form considerations. We have that

$$\Delta_{0,a} = d_{0,\max}^* d_{0,\max} = \Delta_N$$

is defined from the quadratic form with domain $H^1(M)$, so it is the Neumann realization of the Hodge Laplacian on 0-forms. We have that

$$\Delta_{d,a} = d_{d,\max} d_{d,\max}^* = \Delta_D$$

is defined from the quadratic form with domain $H_0^1(M, \bigwedge^d T^*M)$, so it is the Dirichlet realization of the Hodge Laplacian on d -forms. Indeed, the Hodge star $L^2(M) \rightarrow L^2(M; \bigwedge^d T^*M)$, $f \mapsto f \, dV$ implements a canonical identification of $\Delta_{d,a}$ with the Dirichlet realization of the Laplacian on 0-forms. Below in Lemma 3.2 we will see that the domain of $\Delta_{k,a}$ is contained in $H^1(M; \bigwedge^k T^*M)$, so by the Rellich theorem we obtain that the Hilbert complex $(L^2(M; \bigwedge^\bullet T^*M), d_\bullet)$ has discrete spectrum as soon as M is a compact manifold with Lipschitz boundary.

The operator $\Delta_{k,a}$ is a realization of the Hodge Laplacian on k -forms, and the realization is described by a boundary condition. Let us clarify the boundary condition defining $\Delta_{k,a}$ for $0 < k < d$ and describe their form domains. We do so using the Hodge–Dirac operator $\mathbb{D} = d + \delta$ and the results from [4]. The results in [4] are described for smooth manifolds with boundary, but using [13, 27] the results extend ad verbatim to Lipschitz manifolds with boundary. To state the results, we need further notation. Write x_n for the inwards pointing normal coordinate near the boundary. For a k -form ω , we can near the boundary write

$$\omega = \omega_1 + dx_n \wedge \omega_2, \quad (3.3)$$

where ω_1 and ω_2 are defined near the boundary and take values in $\bigwedge^k T^*\partial M$ and $\bigwedge^{k-1} T^*\partial M$, respectively. In other words, (3.3) uniquely decomposes ω into components ω_1 and ω_2 not containing dx_n . Following [11, Section 2.7.1], we can define the relative boundary condition B_r and the absolute boundary condition B_a by

$$B_r\omega := \omega_1|_{\partial M} = 0 \quad \text{and} \quad B_a\omega := \omega_2|_{\partial M} = 0,$$

Theorem 3.1. *Let M be a compact Lipschitz manifold with boundary. The operator $D_a := d_{\max} + d_{\max}^*$ is a self-adjoint realization of the Hodge–Dirac operator $\mathbb{D} = d + \delta$ with domain contained in the Sobolev space $H^1(M; \bigwedge^* T^*M)$. In fact,*

$$\text{Dom}(D_a) := \{u \in H^1(M; \bigwedge^* T^*M) : B_a u = 0\},$$

and in the special case that M is a smooth manifold with smooth boundary then D_a is a Shapiro–Lopatinski elliptic boundary value problem.

The reader can find more details about Shapiro–Lopatinski elliptic boundary value problems in [1]. We refer the reader to [4, Theorem 4.1.1] for a proof of Theorem 3.1. But to give the reader a feeling for the argument, we recall its salient features. The main idea is to go to the doubled Lipschitz manifold $\tilde{M} := 2M$ and let $\alpha: \tilde{M} \rightarrow \tilde{M}$

denote the flip map which is an involutive diffeomorphism; in [4] they remain within the smooth category. We equip $\tilde{M}$ with the Riemannian structure making α isometric. Now, as in [4], the Hilbert complex $(\tilde{H}_\bullet, \tilde{d}_\bullet)$ defined from the de Rham complex on $\tilde{M}$ has only one ideal boundary condition (the minimal and maximal realization coincides). We decompose into the ± 1 -eigenspaces for α^* as

$$(\tilde{H}_\bullet, \tilde{d}_\bullet) = \underbrace{(\tilde{H}_\bullet^a, \tilde{d}_\bullet^a)}_{=\ker(\alpha^*-1)} \oplus \underbrace{(\tilde{H}_\bullet^r, \tilde{d}_\bullet^r)}_{=\ker(\alpha^*+1)}.$$

As in [4], one proves that

$$(\tilde{H}_\bullet^a, \tilde{d}_\bullet^a)|_M = (H_\bullet, d_{\bullet, \max}) \quad \text{and} \quad (\tilde{H}_\bullet^r, \tilde{d}_\bullet^r)|_M = (H_\bullet, d_{\bullet, \min}).$$

From here, the proof proceeds as in [4] to show that $d_{\max} + d_{\max}^*$ is the realization defined from the boundary condition B_a . We note here that the relative boundary condition B_r arises in the same way but from the minimal realization $d_{\min}$.

By construction, we have that

$$D_a^2 = \bigoplus_{k=0}^d \Delta_{k,a},$$

so we can describe the form domain of $\Delta_{k,a}$ rather easily using Theorem 3.1.

Lemma 3.2. *The quadratic form $q_{k,a}$ associated with $\Delta_{k,a}$ takes the form*

$$q_{k,a}(\omega) := \int_M (|d_k \omega|^2 + |\delta_{k-1} \omega|^2) dx,$$

and its domain is given by

$$\begin{aligned} \text{Dom}(q_{k,a}) &:= \text{Dom}(d_{k,\max}) \cap \text{Dom}(d_{k-1,\max}^*) \\ &\equiv \{u \in H^1(M; \bigwedge^k T^*M) : B_a u = 0\}. \end{aligned}$$

When M and its boundary are smooth, we can even describe the domain of $\Delta_{k,a}$. We define the boundary conditions $\mathcal{B}_a$ for the Hodge Laplacian on k -forms Δ_k as

$$\mathcal{B}_a \omega := (B_a \omega, B_a (d + \delta) \omega).$$

The identity $D_a^2 = \bigoplus_{k=0}^d \Delta_{k,a}$, Theorem 3.1 and elliptic regularity for Shapiro–Lopatinski elliptic boundary value problems implies the following.

Theorem 3.3. *Let M be a smooth compact manifold with smooth boundary. The operator*

$$\Delta_{k,a} := d_{k,\max}^* d_{k,\max} + d_{k-1,\max} d_{k-1,\max}^*,$$

is a self-adjoint realization of the Hodge Laplacian on k -forms with domain contained in the Sobolev space $H^2(M; \bigwedge^k T^*M)$. In fact, $\Delta_{k,a}$ is a Shapiro–Lopatinski elliptic boundary value problem and

$$\text{Dom}(\Delta_{k,a}) := \{u \in H^2(M; \bigwedge^k T^*M) : \mathcal{B}_a u = 0\}.$$

We here impose the assumption that M is smooth to ensure that the Sobolev space $H^2(M; \bigwedge^k T^*M)$ is well defined and to be able to employ elliptic regularity. For instance, Theorem 3.3 covers Euclidean domains with smooth boundary. For our considerations, we only need the quadratic form domain (as described in Lemma 3.2) in the proofs of eigenvalue inequalities. We note also that [11, Lemma 2.7.2] ensures that $(\Delta_k, \mathcal{B}_a)$ is self-adjoint from first principles, and not only from that it coincides with our operator $\Delta_{k,a}$.

Let us verify again that $\Delta_{0,a}$ coincides with the Neumann realization of the Hodge Laplacian. In degree zero, $\mathcal{B}_a(d + \delta)\omega = \mathcal{B}_a d_0 \omega = \partial_{x_n} \omega|_{\partial M}$. So by definition, for $\omega \in C^\infty(M) = C^\infty(M, \bigwedge^0 T^*M)$,

$$\mathcal{B}_a \omega = (0, \partial_{x_n} \omega|_{\partial M}).$$

In particular, in degree zero,

$$\mathcal{B}_a \omega = 0 \iff \partial_{x_n} \omega|_{\partial M} = 0,$$

which gives the desired Neumann boundary condition. If one wants to check that $\Delta_{d,a}$ coincides with the Dirichlet realization, one computes that for $\omega = \omega_0 dV \in C^\infty(M, \bigwedge^d T^*M)$, with $\omega_0 \in C^\infty(M)$ and $dV = \star(1)$ the Riemannian volume form, that

$$\mathcal{B}_a \omega = ((dx_n \lrcorner \omega)|_{\partial M}, 0), \quad \text{so} \quad \mathcal{B}_a \omega = 0 \iff \omega_0|_{\partial M} = 0.$$

Here $dx_n \lrcorner \omega$ denotes the contraction of ω along the normal covector dx_n . Moreover, by [11, Lemma 2.7.1] the Hodge star $\star$ implements an identification of $\bigoplus_{k=0}^d \Delta_{k,a}$ with the corresponding relative/minimal realization $\bigoplus_{k=0}^d \Delta_{k,r}$ at the cost of flipping degree k forms to degree $d - k$ -forms. Indeed, $\mathcal{B}_a \omega = 0 \iff \mathcal{B}_r \star \omega = 0$.

Remark 3.4. The content of [4, Theorem 4.1.2] is precisely that

$$\ker(\Delta_{k,a}) \cong H^k(M; \mathbb{C}).$$

This is the motivation for using the term absolute boundary conditions, since the associated space of harmonic forms realizes the absolute cohomology groups $H^*(M; \mathbb{C})$. In particular, the definition of Euler characteristic (see equation (2.2)) and the Hodge decomposition (2.3) implies that

$$\chi(L^2(M; \bigwedge^\bullet T^*M), d_{\bullet, \max}) = \chi(M).$$

So, for any $\lambda \geq 0$, Lemma 2.3 implies

$$\sum_{k=0}^d (-1)^k N(\Delta_{k,a}, \lambda) = \chi(M).$$

If we use the minimal/relative realization, we instead have the equalities

$$\ker(\Delta_{k,r}) \cong H^k(M, \partial M; \mathbb{C}) \quad \text{and} \quad \chi(L^2(M; \bigwedge^\bullet T^*M), d_{\bullet, \min}) = \chi(M, \partial M),$$

where $H^k(M, \partial M; \mathbb{C})$ denotes the k -th relative cohomology with respect to the boundary inclusion $\partial M \hookrightarrow M$ and $\chi(M, \partial M) := \sum_{k=0}^d (-1)^k \dim_{\mathbb{C}} H^k(M, \partial M; \mathbb{C})$. This is the motivation for using the term relative boundary conditions, since the associated space of harmonic forms realizes the relative cohomology groups $H^*(M, \partial M; \mathbb{C})$.

3.1. Some computations on domains in $\mathbb{R}^d$

We are primarily interested in $\Omega \subseteq \mathbb{R}^d$ being a domain. In this case, we use the standard basis for the exterior algebra. That is, we construct an ON-basis dx_I for the k -forms labeled by ordered sets $I = \{i_1 < i_2 < \cdots < i_k\}$, where $i_j \in \{1, \dots, d\}$, as

$$dx_I := dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}.$$

In particular, we see that

$$\text{rk} \bigwedge^k T^*\Omega = \dim \bigwedge^k \mathbb{C}^d = \binom{d}{k}.$$

For instance, $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_d$ is the basis element of choice for $\bigwedge^d T^*\Omega$. In these bases, for a function $f \in C^\infty(\Omega)$ we have that

$$\mathcal{D}(f dx_I) = \sum_{j \notin I} \partial_{x_j} f dx_j \wedge dx_I - \sum_{j \in I} \text{sign}(j, I) \partial_{x_j} f dx_{I \setminus \{j\}},$$

where $\text{sign}(j, I) \in \{-1, 1\}$ is determined by $dx_j \wedge dx_{I \setminus \{j\}} = \text{sign}(j, I) dx_I$. We then have

$$\mathcal{D}^2(f dx_I) = (\Delta f) dx_I.$$

In particular, $\Delta_{k,a}$ is a realization of the scalar Laplacian on each of the basis vectors of $\bigwedge^k \mathbb{C}^d$.

4. Spectral properties of the de Rham complex

In this section we use the notions and results introduced in the previous sections to obtain estimates using the ideas from [24] applied to the de Rham complex. We combine them to prove Friedlander's inequality (1.1) and in the next section they are used to prove Rohleder's inequality (1.2). We will henceforth only consider a domain $\Omega \subseteq \mathbb{R}^d$ which is bounded, connected and has Lipschitz boundary. We provide a series of rough estimates leading up to a new proof of Friedlander's inequality (1.1). In particular, Lemma 4.3 provide a higher-dimensional analog to estimates appearing in [23, 24]. We believe they are of significant interest for future considerations improving estimates between eigenvalues of Dirichlet and Neumann Laplacians, for instance the conjectural bound (1.3).

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^d$ be a connected and bounded domain with Lipschitz boundary for $d \geq 2$ and $(\Delta_{k,a})_{k=0,\dots,d}$ be the corresponding Hodge Laplacians on k -forms with absolute boundary conditions. For $\lambda \geq 0$ and $0 \leq k \leq d$ we have that*

$$\binom{d}{k} N(\Delta_{d,a}, \lambda) \leq N(\Delta_{k,a}, \lambda) \leq \binom{d}{k} N(\Delta_{0,a}, \lambda).$$

Proof. Consider the quadratic form

$$\tilde{q}_k(u) := \sum_{l=1}^d \int_{\Omega} |\partial_l u|^2 dx,$$

with domain $\text{Dom}(\tilde{q}_k) = H^1(\Omega, \bigwedge^k T^*\Omega)$. We also write $\tilde{q}_{k,(0)}$ for the restriction of $\tilde{q}_k$ to $H_0^1(\Omega, \bigwedge^k T^*\Omega)$. We now explain how $\tilde{q}_k$ and $\tilde{q}_{k,(0)}$ identifies with $\binom{d}{k}$ copies of $q_{0,a}$ and $q_{d,a}$ respectively, and that there is a chain of extensions

$$\tilde{q}_{k,(0)} \subseteq q_{k,a} \subseteq \tilde{q}_k,$$

of quadratic forms. The lemma then follows from Glazman's lemma (see Lemma 2.1 above).

Firstly, we compare $\tilde{q}_k$ and $\tilde{q}_{k,(0)}$ to $q_{0,a}$ and $q_{d,a}$, respectively. Let e_l for $l = 1, \dots, \binom{d}{k}$ denote the standard ON-basis for $\bigwedge^k \mathbb{C}^d$ inducing a frame for $\bigwedge^k T^*\Omega$, that is, elements of the form $dx_{l_1} \wedge \dots \wedge dx_{l_k}$ with $\{l_1, \dots, l_k\} \subseteq \{1, \dots, d\}$. From this ON-basis, we obtain a unitary mapping

$$L^2(\Omega, \mathbb{C}^{\binom{d}{k}}) \rightarrow L^2(\Omega, \bigwedge^k T^*\Omega),$$

which maps

$$H^1(\Omega, \mathbb{C}^{\binom{d}{k}}) \rightarrow H^1(\Omega, \bigwedge^k T^*\Omega) \quad \text{and} \quad H_0^1(\Omega, \mathbb{C}^{\binom{d}{k}}) \rightarrow H_0^1(\Omega, \bigwedge^k T^*\Omega),$$

and hence identifies $\tilde{q}_k$ and $\tilde{q}_{k,(0)}$ with $\binom{d}{k}$ copies of $q_{0,a}$ and $q_{d,a}$ respectively.

Secondly, we make some observations concerning the Hodge–Dirac operator $\not{D} = d + \delta$ using Stokes' theorem. For $u, v \in H^1(\Omega, \bigwedge^* T^* \Omega)$ we have that

$$\begin{aligned} & \langle \not{D}u, v \rangle_{L^2(\Omega, \bigwedge^* T^* \Omega)} - \langle u, \not{D}v \rangle_{L^2(\Omega, \bigwedge^* T^* \Omega)} \\ &= \langle B_a u, B_r v \rangle_{L^2(\partial\Omega, \bigwedge^* T^* \partial\Omega)} - \langle B_r u, B_a v \rangle_{L^2(\partial\Omega, \bigwedge^* T^* \partial\Omega)}, \end{aligned} \quad (4.1)$$

which can be found in [11, equation (2.7.12)]. When u and v are smooth k -forms, equation (4.1) and the identity $\Delta = \not{D}^2$ imply

$$\begin{aligned} \tilde{q}_k(u, v) &= \langle \Delta_k u, v \rangle_{L^2(\Omega, \bigwedge^k T^* \Omega)} - \langle \partial_{x_n} u, v \rangle_{L^2(\partial\Omega, \bigwedge^k T^* \Omega)} \\ &= \langle \not{D}u, \not{D}v \rangle_{L^2(\Omega, \bigwedge^* T^* \Omega)} - \langle \partial_{x_n} u, v \rangle_{L^2(\partial\Omega, \bigwedge^* T^* \Omega)} \\ &\quad + \langle B_a \not{D}u, B_r v \rangle_{L^2(\partial\Omega, \bigwedge^* T^* \partial\Omega)} - \langle B_r \not{D}u, B_a v \rangle_{L^2(\partial\Omega, \bigwedge^* T^* \partial\Omega)} \\ &= \langle \not{D}u, \not{D}v \rangle_{L^2(\Omega, \bigwedge^* T^* \Omega)} + \langle \not{D} \partial B_a u, B_r v \rangle_{L^2(\partial\Omega, \bigwedge^* T^* \partial\Omega)} \\ &\quad - \langle \not{D} \partial B_r u, B_a v \rangle_{L^2(\partial\Omega, \bigwedge^* T^* \partial\Omega)}, \end{aligned}$$

where $\not{D} \partial$ denotes the Hodge–Dirac operator on the boundary $\partial\Omega$. The last step uses the facts that

$$B_a \not{D} = -\not{D} \partial B_a + B_r \partial_{x_n} \quad \text{and} \quad B_r \not{D} = \not{D} \partial B_r - B_a \partial_{x_n}. \quad (4.2)$$

See for example [11, equation (2.7.6)] for how one deduces (4.2). By an approximation argument, we see that

$$\begin{aligned} \tilde{q}_k(u, v) &= \langle \not{D}u, \not{D}v \rangle_{L^2(\Omega, \bigwedge^* T^* \Omega)} + \langle \not{D} \partial B_a u, B_r v \rangle_{L^2(\partial\Omega, \bigwedge^* T^* \partial\Omega)} \\ &\quad - \langle \not{D} \partial B_r u, B_a v \rangle_{L^2(\partial\Omega, \bigwedge^* T^* \partial\Omega)}, \end{aligned}$$

for $u, v \in H^1(\Omega, \bigwedge^k T^* \Omega)$. The fact that we have an extension

$$\tilde{q}_{k,(0)} \subseteq q_{k,a},$$

is now immediate. Moreover, we can conclude that we have an extension

$$q_{k,a} \subseteq \tilde{q}_k,$$

from Theorem 3.1 and the fact that

$$q_{k,a}(u) = \|\not{D}u\|_{L^2(\Omega, \bigwedge^* T^* \Omega)}^2$$

for $u \in \text{Dom}(q_{k,a}) = \text{Dom}(D_a) \cap L^2(T^* \Omega, \bigwedge^k T^* \Omega)$. ■

Remark 4.2. The naive estimate in Lemma 4.1 gives a first hint towards the appearance of binomial coefficients in counting function estimates. This is interesting since as the discussion in the introduction indicates, the optimal shift $c = c(d)$ for which $\lambda_{j+c(d)}(\Delta_N) \leq \lambda_j(\Delta_D)$ for all j is conjecturally a binomial coefficient, or a sum of binomial coefficients, in d . Indeed, a small improvement in the choice of the subspace V in the proof of Lemma 4.1 could lead to an improvement of the right size with respect to the conjecture.

Next, we give a natural generalization to higher dimensions of the main idea in [24]. In Section 5, we will give further details on how this lemma translates into the results in [14, 23, 24].

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^d$ be a connected and bounded domain with Lipschitz boundary with $d \geq 2$ and $(\Delta_{k,a})_{k=0,\dots,d}$ be the corresponding Hodge Laplacians on k -forms with absolute boundary conditions. For $\lambda \geq 0$,*

$$d N(\Delta_{d,a}, \lambda) + m(\Delta_{d,a}, \lambda) \leq N(\Delta_{d-1,a}, \lambda).$$

Proof. Similar to the proof of Lemma 4.1, we let $N := N(\Delta_{d,a}, \lambda)$ and select N orthonormal eigenfunctions $f_1 dV, \dots, f_N dV$ of $\Delta_{d,a}$ with eigenvalues less or equal to λ . Consider the dN -dimensional space

$$V_1 := \text{Span}\{f_j \widehat{dx}_l : j = 1, \dots, N, l = 1, \dots, d\} \subseteq H_0^1(\Omega; \bigwedge^{d-1} T^* \Omega)$$

where $\widehat{dx}_l := dx_1 \wedge \dots \wedge dx_{l-1} \wedge dx_{l+1} \wedge \dots \wedge dx_d$ denotes the standard ON-basis for $\bigwedge^{d-1} \mathbb{C}^d$. We can estimate $q_{d-1,a}(u) \leq \lambda \|u\|_{L^2(\Omega; \bigwedge^{d-1} T^* \Omega)}^2$ for $u \in V_1$.

Next, we will consider the space

$$V_2 := \delta_{d-1} \ker(\Delta_{d,a} - \lambda)$$

for which $\dim V_2 = \dim \delta_{d-1}(\ker(\Delta_{d,a} - \lambda)) = \dim(\ker(\Delta_{d,a} - \lambda)) = m(\Delta_{d,a}, \lambda)$. We will show that $V_1 \cap V_2 = 0$, that is, if $\delta_{d-1}g \in V_1$ for some $g \in \ker(\Delta_{d,a} - \lambda)$, then g is identically zero (this proof is similar to [16, Lemma 3.2.36]). We denote by γ_0 the trace operator. Since $g \in \ker(\Delta_{d,a} - \lambda)$, we have that $g \in H_0^1(\Omega; \bigwedge^d T^* \Omega)$, which implies that $\gamma_0 g = 0$. Moreover, since $\delta_{d-1}g \in V_1 \subseteq H_0^1(\Omega; \bigwedge^{d-1} T^* \Omega)$, we have that $\gamma_0 \delta_{d-1}g = 0$. This means that we can extend g by zero to $\tilde{g} \in H^1(\mathbb{R}^d; \bigwedge^d \mathbb{R}^d)$. For any $v \in C_c^\infty(\mathbb{R}^d; \bigwedge^d \mathbb{R}^d)$,

$$\begin{aligned} \langle \delta_{d-1} \tilde{g}, \delta_{d-1} v \rangle_{L^2(\mathbb{R}^d; \bigwedge^{d-1} \mathbb{R}^d)} &= \langle \delta_{d-1} g, \delta_{d-1} v \rangle_{L^2(\Omega; \bigwedge^{d-1} T^* \Omega)} \\ &= \langle \Delta_d g, v \rangle_{L^2(\Omega; \bigwedge^d T^* \Omega)} \\ &= \lambda \langle g, v \rangle_{L^2(\Omega; \bigwedge^d T^* \Omega)} \\ &= \lambda \langle g, v \rangle_{L^2(\mathbb{R}^d; \bigwedge^d \mathbb{R}^d)}, \end{aligned}$$

where the boundary term is zero because $\gamma_0 \delta_{d-1} g = 0$. Therefore, $\tilde{g} \in H^1(\mathbb{R}^d; \bigwedge^d \mathbb{C}^d)$ is a solution of $\Delta_d \tilde{g} = \lambda \tilde{g}$ in the weak sense on $\mathbb{R}^d$. By elliptic regularity, we get that $\tilde{g}$ is real-analytic. Since $\tilde{g}|_{\mathbb{R}^d \setminus \Omega} \equiv 0$, then unique continuation results imply that $\tilde{g} \equiv 0$. This ensures that the space $V := V_1 + V_2$ has

$$\dim(V) = dN + m(\Delta_{d,a}, \lambda).$$

Lastly, for $u \in V$ of the form $u = v_1 + v_2$ with $v_1 \in V_1 \subseteq H_0^1(\Omega, \bigwedge^{d-1} T^* \Omega)$ and $v_2 \in V_2 \subseteq \ker(\Delta_{d-1,a} - \lambda)$, we see that

$$\begin{aligned} q_{d-1,a}(u) &= \|d_{d-1} v_1 + d_{d-1} v_2\|_{L^2(\Omega; \bigwedge^d T^* \Omega)}^2 + \|\delta_{d-2} v_1\|_{L^2(\Omega; \bigwedge^{d-2} T^* \Omega)}^2 \\ &= q_{d-1,a}(v_1) + \|d_{d-1} v_2\|_{L^2(\Omega; \bigwedge^d T^* \Omega)}^2 \\ &\quad + 2 \operatorname{Re} \langle d_{d-1} v_1, d_{d-1} v_2 \rangle_{L^2(\Omega; \bigwedge^d T^* \Omega)} \\ &= q_{d-1,a}(v_1) + \lambda \|v_2\|_{L^2(\Omega; \bigwedge^{d-1} T^* \Omega)}^2 + 2\lambda \operatorname{Re} \langle v_1, v_2 \rangle_{L^2(\Omega; \bigwedge^d T^* \Omega)} \\ &\leq \lambda \|u\|_{L^2(\Omega; \bigwedge^{d-1} T^* \Omega)}^2 \end{aligned}$$

The result follows from Glazman's lemma using the linear subspace V . ■

Proof of Friedlander's inequality (1.1). Note that Lemmas 4.1 and 4.3 imply that

$$dN(\Delta_D, \lambda) + m(\Delta_D, \lambda) \leq dN(\Delta_N, \lambda).$$

From this, inequality we conclude Friedlander's inequality (1.1),

$$\lambda_{j+1}(\Delta_N) \leq \lambda_j(\Delta_D).$$

Indeed, take $\lambda = \lambda_j(\Delta_D)$ and use equation (2.1) in combination with the fact that $N(T, \lambda) \geq x$ if and only if $\lambda_{[x]}(T) \leq \lambda$. ■

5. Comparing forms to vectors in two and three dimensions

The aim of this section is to rewrite the results of Section 4 in vector operators in two and three dimensions to see how our results compare to those in [14, 23, 24].

5.1. Inequalities in dimension 2

Let $\Omega \subset \mathbb{R}^2$ be a connected and bounded domain with Lipschitz boundary. In two dimensions, the exterior differential on 1-forms introduced in (3.1) acts as

$$d_1(u_1 dx_1 + u_2 dx_2) = (-\partial_{x_2} u_1 + \partial_{x_1} u_2) dx_1 \wedge dx_2,$$

and the formal adjoint of the exterior differential on 0-forms introduced in (3.2) as

$$\delta_0(u_1 dx_1 + u_2 dx_2) = \partial_{x_1} u_1 + \partial_{x_2} u_2,$$

which can be identified with the differential expressions $\omega(u) := \partial_{x_1} u_2 - \partial_{x_2} u_1$ and $\operatorname{div} u$ introduced in [24]. In other words, the form $q_{1,a}$ associated with $\Delta_{1,a}$ with domain $\operatorname{Dom}(q_{1,a}) = \operatorname{Dom}(d_{1,\max}) \cap \operatorname{Dom}(\delta_{0,\max})$ is exactly the same as

$$\alpha[u, v] = \int_{\Omega} (\operatorname{div} u \overline{\operatorname{div} v} + \omega(u) \overline{\omega(v)}) dx,$$

with

$$\operatorname{Dom} \alpha = \{u \in L^2(\Omega)^2 : \operatorname{div} u, \omega(u) \in L^2(\Omega), \langle u|_{\partial\Omega}, \nu \rangle = 0\},$$

where ν the unit normal vector, introduced in [24, Section 3]. This means that the operator A introduced in [24, Proposition 3.1] coincides with $\Delta_{1,a}$. Next, we present [24, Theorem 4.1] and give an analogous proof using Lemma 4.3.

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^2$ be a connected and bounded domain with Lipschitz boundary. Let $j = 1, 2, \dots$; then*

$$\lambda_{j+\chi(\Omega)+m(\Delta_D, \lambda_j(\Delta_D))}(\Delta_N) \leq \lambda_j(\Delta_D). \quad (5.1)$$

Proof. We let $j \in \mathbb{N}$, and fix $\lambda := \lambda_j(\Delta_{2,a})$ as well as $m := m(\Delta_{2,a}, \lambda_j(\Delta_{2,a}))$. By Lemma 4.3, we know that

$$2N(\Delta_{2,a}, \lambda) + m \leq N(\Delta_{1,a}, \lambda), \quad (5.2)$$

and by Lemma 2.3, we have

$$N(\Delta_{0,a}, \lambda) - N(\Delta_{1,a}, \lambda) + N(\Delta_{2,a}, \lambda) = \chi(\Omega). \quad (5.3)$$

Combining (5.2) with (5.3),

$$N(\Delta_{0,a}, \lambda) \geq \chi(\Omega) + N(\Delta_{2,a}, \lambda) + m$$

which gives (5.1) since $\Delta_N = \Delta_{0,a}$ and $\Delta_D = \Delta_{2,a}$ (see Section 3). ■

Corollary 5.2 ([24, Theorem 4.1]). *Let $\Omega \subset \mathbb{R}^2$ be a simply connected and bounded domain with Lipschitz boundary. Then*

$$\lambda_{j+2}(\Delta_N) \leq \lambda_j(\Delta_D)$$

for all $j = 1, 2, 3, \dots$

Proof. If $\Omega \subset \mathbb{R}^2$ is simply connected, then $\chi(\Omega) = 1$, so the result follows from Proposition 5.1 and $m(\Delta_D, \lambda_j(\Delta_D)) \geq 1$. ■

Remark 5.3. If $\Omega \subseteq \mathbb{R}^2$ is non-simply connected, Proposition 5.1 contains no new information beyond Friedlander's inequality (1.1). In fact, if Ω has $g = 1$ holes we retrieve Friedlander's inequality (1.1), for $g = 2$ holes then $\chi(\Omega) = -1$ and Proposition 5.1 gives the same bound as simply applying Glazman's lemma to the fact that the form domain of Δ_D is contained in the form domain of Δ_N . If Ω has $g > 2$ holes, Proposition 5.1 gives a worse bound than variational principles.

Remark 5.4. In the previous corollary we used the fact that $m(\Delta_D, \lambda_j(\Delta_D)) \geq 1$. Note that keeping the multiplicity term in (5.1) will give

$$\lambda_{j+1+m(\Delta_D, \lambda_j(\Delta_D))}(\Delta_N) \leq \lambda_j(\Delta_D).$$

This inequality can be observed in the case of the unit disc where $m(\Delta_D, \lambda_2(\Delta_D)) = 2$, i.e., $\lambda_2(\Delta_D) = \lambda_3(\Delta_D)$. For the disc, we know that $\lambda_5(\Delta_N) < \lambda_3(\Delta_D)$, but $\lambda_6(\Delta_N) = \lambda_3(\Delta_D)$, where this equality comes from the fact that the zeros $j_{m,n}$ of the m -th Bessel function $J_m(r)$ and the positive zeros $j'_{m,n}$ of the derivative $J'_m(r)$ fulfill $j_{1,n} = j'_{0,n+1}$ for $n \in \mathbb{N}$.

Remark 5.5. Rohleder was able to obtain strict inequality in (5.1) for simply connected domains if $\lambda_k(\Delta_D)$ is a simple eigenvalue or $\partial\Omega$ contains a straight line segment. We refer to [24, Theorem 4.1] for the proof.

5.2. Rohleder's bound on eigenvalues for the curl curl operator

Let $\Omega \subset \mathbb{R}^d$ be a connected and bounded domain with Lipschitz boundary. We can define a positive, self-adjoint operator

$$\mathbb{C} := d_{d-2, \max} d_{d-2, \max}^*,$$

which is densely defined on the Hilbert subspace

$$\ker(d_{d-1, \max}) \subseteq L^2(\Omega, \bigwedge^{d-1} T^*\Omega).$$

The form associated with the operator $\mathbb{C}$ takes the form

$$q_{\mathbb{C}}(u) := \int_{\Omega} |\delta_{d-2} u|^2 dx,$$

that by Lemma 3.2 has the domain

$$\begin{aligned} \text{Dom}(q_{\mathbb{C}}) &\equiv \ker(d_{d-1, \max}) \cap \text{Dom}(d_{d-2, \max}^*) \\ &= \{u \in \ker(d_{d-1, \max}) \cap H^1(\Omega; \bigwedge^{d-1} T^*\Omega) : B_a u = 0\}. \end{aligned}$$

For a $(d-1)$ -form u , near the boundary we can write $u = u_0 dV_\partial + dx_n \wedge u_2$ where u_0 is a scalar function, dV_∂ the volume form on $\partial\Omega$ induced from the Euclidean metric, and u_2 is a section to $\bigwedge^{d-2} T^*\partial\Omega$. In particular, if $u \in \ker(d_{d-1,\max}) \cap H^1(\Omega; \bigwedge^{d-1} T^*\Omega)$, then $B_a u = 0$ if and only if $u_2|_{\partial\Omega} = 0$. In analogy with [23], we call $\mathfrak{C}$ the *curl curl operator*.

Proposition 5.6. *Let $\Omega \subset \mathbb{R}^d$ be a connected and bounded domain with Lipschitz boundary and write $\mathfrak{C}$ for its curl curl operator. Let $j = 1, 2, \dots$; then*

$$\lambda_{(d-1)j+m(\Delta_D, \lambda_j(\Delta_D))}(\mathfrak{C}) \leq \lambda_j(\Delta_D). \quad (5.4)$$

Proof. By the definition of $\mathfrak{C}$ and the Hodge decomposition, we have for $\lambda \geq 0$ that

$$N(\mathfrak{C}, \lambda) = \dim(\ker(\Delta_{d-1,a})) + N(d_{d-2,\max}^* d_{d-2,\max}; (0, \lambda)).$$

In particular, using equation (2.4)

$$\begin{aligned} N(\mathfrak{C}, \lambda) &= N(\Delta_{d-1,a}, \lambda) - N(d_{d-1,\max}^* d_{d-1,\max}; (0, \lambda)) \\ &= N(\Delta_{d-1,a}, \lambda) - N(\Delta_D, \lambda). \end{aligned}$$

From Lemma 4.3, we see that

$$N(\mathfrak{C}, \lambda) \geq (d-1) N(\Delta_D, \lambda) + m(\Delta_D, \lambda),$$

and the proof is complete. ■

In three dimensions, the exterior codifferential δ_1 on 2-forms acts as

$$\begin{aligned} \delta_1(u_1 dx_2 \wedge dx_3 + u_2 dx_3 \wedge dx_1 + u_3 dx_1 \wedge dx_2) \\ = (\partial_2 u_3 - \partial_3 u_2) dx_1 + (\partial_1 u_3 - \partial_3 u_1) dx_2 + (\partial_2 u_1 - \partial_1 u_2) dx_3, \end{aligned}$$

so up to the Hodge star we can identify δ_1 with the curl operator in three dimensions. A similar computation shows that d_2 can be identified with the divergence of vector fields. We note the discussion above shows that a 2-form u belongs to $\text{Dom}(q_{\mathfrak{C}})$ if and only if $d_2 u = 0$ in distributional sense and $\star u$ restricts to the zero form on $\partial\Omega$. If we identify 2-forms with vector fields via the Hodge star, this means that u belongs to $\text{Dom}(q_{\mathfrak{C}})$ if and only if $\text{div}(u) = 0$ in distributional sense and $u \times \nu = 0$ on $\partial\Omega$. We see that in dimension 3, $\mathfrak{C}$ coincides with the curl curl operator defined in [23] and in the notation of [23], $\lambda_j(\mathfrak{C}) = \alpha_j$. In particular, Proposition 5.6 extends [23, Theorem 1.1] from dimension three to arbitrary dimension.

Remark 5.7. In [23, Theorem 1.1] strict inequality in (5.4) is attained when Ω is a polyhedron or $\lambda_k(\Delta_D)$ is a simple eigenvalue. An analogous proof could be carried on to obtain strict inequality between $\mathfrak{C}$ and Δ_D for $d \geq 2$.

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