J. Spectr. Theory 15 (2025), 1409–1475
DOI 10.4171/JST/585

The Calderón problem revisited: Reconstruction with resonant perturbations

Ahcene Ghandriche and Mourad Sini

Abstract. The original Calderón problem consists in recovering the potential (or the conductivity) from the knowledge of the related Neumann to Dirichlet map (or Dirichlet to Neumann map). Here, we first perturb the medium by injecting small-scaled and highly heterogeneous particles. Such particles can be bubbles or droplets in acoustics or nanoparticles in electromagnetism. They are distributed, periodically for instance, in the whole domain where we want to do reconstruction. Under critical scales between the size and contrast, these particles resonate at specific frequencies that can be well computed. Using incident frequencies that are close to such resonances, we show that (1) the corresponding Neumann to Dirichlet map of the composite converges to the one of the homogenised medium. In addition, the equivalent coefficient, which consists in the sum of the original potential and the effective coefficient, is negative valued with a controllable amplitude; (2) as the equivalent coefficient is negative valued, then we can linearise the corresponding Neumann to Dirichlet map using the effective coefficient's amplitude; (3) from the linearised Neumann to Dirichlet map, we reconstruct the original potential using explicit complex geometrical optics solutions (CGOs).

Contents

Mathematics Subject Classification 2020: 35R30 (primary); 35C20 (secondary).
Keywords: inverse problems, Neumann-to-Dirichlet map, Newtonian operator, acoustic imaging, asymptotic expansions, spectral theory, Lippmann–Schwinger equation, droplets.

1. Introduction and statement of the results

1.1. General introduction

The original Calderón problem stated in the acoustic framework reads as follows. Let $n := c^{-1}$ be the index of refraction, where c stands for the speed of acoustic sound. In turn, this speed of sound is given by $c := \sqrt{\frac{k}{\rho}}$ where ρ is the mass density and k is the bulk modulus. In the time-harmonic regime, the propagation of the acoustic waves is modelled by

$$\begin{cases} (\Delta + \omega^2 n^2(\cdot)) p^f = 0 & \text{in } \Omega, \\ \partial_\nu p^f = f & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where p^f is the acoustic pressure generated by the applied source f. The Neumann to Dirichlet (NtD) operator Λ_c corresponds to any $f \in \mathbb{H}^{-1/2}(\partial\Omega)$, the trace on $\partial\Omega$ of the induced pressure p^f, i.e., $\Lambda_c(f) := p^f|_{\partial\Omega}$. The Calderón problem consists in recovering the sound speed c from the knowledge of the NtD map Λ_c. According to the model (1.1), the mass density ρ is assumed to be a constant, while the bulk modulus k is variable in a smooth domain Ω. We assume k to be a[1] $\mathbb{W}^{1,\infty}(\Omega)$ and positive function and Ω of class C^2. In addition, we assume that (1.1) has a unique solution, i.e., ω^2 is not an eigenvalue of $-n^{-2}\Delta$ with zero Neumann boundary condition on $\partial\Omega$.

The Calderón problem was the object of an intensive study since the early 1980s. The reader can see the following references for more information [17, 26, 38, 45]. A model of particular interest is the *electrical impedance tomography* (EIT) problem, also called *Calderón's problem,* which consists in identifying the conductivity γ using Cauchy data $(u|_{\partial\Omega}, \gamma\nabla u \cdot \nu|_{\partial\Omega})$ of the solution of equation $\nabla \cdot \gamma\nabla u = 0$, in $\Omega \subset \mathbb{R}^3$, where ν is the outward unit normal vector to $\partial\Omega$. The uniqueness question of this problem is reduced to the construction of the so-called *complex geometrical optics solutions* (in short CGOs), see [44], where γ is a positive C^2-smooth function. The regularity of γ is reduced to $C^{3/2+\varepsilon}, \varepsilon > 0$, in [8], then to $\mathbb{W}^{3/2,\infty}$ in [37] and to $\mathbb{W}^{3/2,p}$, $p > 6$ in [10]. Finally, in [14,24] this condition is reduced to $\mathbb{W}^{1,\infty}$ and then to $\mathbb{W}^{1,3}$ in [23]. The corresponding Calderón problem in the 2D-setting was solved in [35]. In [12] the author shows, for the Schrödinger equation given by $\Delta u + \mathfrak{q} u = 0$, in $\Omega \subset \mathbb{R}^2$, the uniqueness of a reconstruction of the potential $\mathfrak{q}(\cdot) \in \mathbb{L}^p(\Omega)$, $p > 2$, from the Cauchy data, i.e., $(u|_{\partial\Omega}; \partial_\nu u|_{\partial\Omega})$, see [12, Theorem 3.5]. In [35], we find a justification of the uniquely determination, from the knowledge of the Dirichlet-to-Neumann map, of the coefficient γ of the elliptic equation[2] $\nabla \cdot (\gamma\nabla u) = 0$ in

[1]This condition can be replace by an $\mathbb{L}^\infty$-regularity.

[2]The substitution $\tilde{u} = \sqrt{\gamma} u$ in $\nabla \cdot (\gamma\nabla u) = 0$ yields $\Delta\tilde{u} + \mathfrak{q}\tilde{u} = 0$, with $\mathfrak{q} = -\frac{1}{\sqrt{\gamma}}\Delta(\sqrt{\gamma})$.

a two-dimensional domain. In [16], it is proved that if for all cubes $\mathbf{Q} \subset \mathbb{R}^n$ the condition on the smallness of $\sup_{\mathbf{Q}} \|\mathbf{Q}\|^{(2p-n)/(np)} \|\mathfrak{q}\|_{\mathbb{L}^p(\mathbf{Q})}$, $p > \frac{(n-1)}{2}$, is satisfied, or $\mathfrak{q} \in \mathbb{L}^p$ with $p > \frac{n}{2}$, then the Dirichlet-to-Neumann map determines the potential $\mathfrak{q}$. Let us also cite [13, 30] regarding Dirac-type singular potentials. For more details, and without being exhaustive, we refer the readers to the following works [1,2,9,11,28,34,37,41,44] and the references therein. Let us mention, however, that apart from few works, like [34], where we find a reconstruction algorithm, most of these works are devoted to unique identifiability questions or stability estimates.

In this work, we propose a different approach for solving constructively this problem. The motivation of this approach comes from the engineering literature, see for instance [4, 6, 15, 29, 32, 36, 40] and many more, where it is suggested to inject small-scaled contrast agents into the region of interest to create the contrasts that are missing to generate clearer images. Such contrast agents could be injected in isolation, as single (or isolated), Dimers, or in general as designed Polymers. They can also be injected as a cluster "distributed" in the region of interest. Based on these ideas, we proposed in our recent works to use resonant contrast agents for solving inverse problems appearing in some imaging modalities, as ultrasound, optics or photo-acoustic imaging modalities, [19–22, 42, 43]. In those works, we use the measurements created after injecting single contrast agents (acoustic bubbles or nano-particles) as follows:

(1) in the time-harmonic regime, we recover the induced resonances (as the Minnaert or plasmonic ones) from which, we could recover the wave speeds (or related coefficients), see [19–21];

(2) in the time-domain regime, we recover the internal values of the travel time function. From the Eikonal equation, we extract the values of the speed, see [22, 42, 43].

In those works, we use contrast agents injected in isolation. This means, for each single injected agent we collect the generated measurements. However, it is of importance to emphasise that we measure only on one single point. In terms of dimensionality, this is advantageous.

In the current work, we inject all the contrast agents at once and then collect the measurements for multiple incident waves. In short, we collect the NtD mapping after injecting the collection of contrast agents all at once. With such measurements, we propose an approach to perform the reconstruction of the index of refraction $n^2(\cdot)$. This approach is divided into two steps.

(1) In the first step, we show that the NtD map generated by the coefficient $n^2(\cdot)$ and the collection of contrast agents converges to the one generated by a sum of $n^2(\cdot)$ and an effective coefficient. This effective coefficient is negative-valued and one can tune its amplitude. The negativity of the effective coefficient, which is key, is due to the resonant character of the injected

contrast agents. Therefore, we can tune these injected agents so that the sum of $n^2(\cdot)$ and the effective coefficients is negative valued with a controllable amplitude.

(2) From the effective NtD map, we reconstruct the coefficient $n^2(\cdot)$. To do so, we show that, due to the negativity of the effective coefficients, mentioned above, we can linearise the effective NtD map. Finally, from this linearised map, we derive an explicit formula to recover $n^2(\cdot)$ in terms of (explicit) CGO-solutions.

To go further into details, let us take as contrast agents droplets, which are bubbles filled in with water, having the following properties. They are modelled as D_j, $j = 1, \dots, M$, of the form $D_j = z_j + aB$ with B as a smooth domain containing the origin and maximum radius as unity, such that $D \equiv \bigcup_{j=1}^{M} D_j$. Their mass density ρ_j are equal and estimated as $\rho_j = \rho_0$, for $1 \leq j \leq M$, with ρ_0 is a constant independent on the parameter a, while their bulk modulus are very small and of order

$$k_j = k_0 a^2, \tag{1.2}$$

with k_0 being as fixed constant independent of a. The maximum radius a of this droplet is of order micrometer, therefore we take

$$a \ll 1.$$

We introduce the Newtonian operator $N_{D_j}(\cdot): \mathbb{L}^2(D_j) \to \mathbb{L}^2(D_j)$, with the image in $\mathbb{H}^2(D_j)$, given by the expression

$$N_{D_j}(f)(x) := \int_{D_j} \Phi_0(x;y) f(y)\, dy, x \in D_j, \tag{1.3}$$

where $\Phi_0(\cdot;\cdot)$ is the fundamental solution of the free-space Laplacian operator, i.e.,

$$\Delta_x \Phi_0(x;y) = -\boldsymbol{\delta}_y(x), \quad \text{with } x, y \in \mathbb{R}^3, \tag{1.4}$$

given by

$$\Phi_0(x;y) := \frac{1}{4\pi|x-y|}, \quad x \neq y. \tag{1.5}$$

This operator is self-adjoint and compact, therefore it enjoys a positive sequence of eigenvalues $\{\lambda_n^{D_j}\}_{n\in\mathbb{N}}$ and they scale as $\lambda_n^{D_j} = a^2 \lambda_n^B$, where $\{\lambda_n^B\}_{n\in\mathbb{N}}$ is the sequence of eigenvalues associated to the operator $N_B(\cdot)$, see [5, 39]. We fix any $n_0 \in \mathbb{N}$; $j \in \{1; \cdots; M\}$ and we consider the eigenvalue $\lambda_{n_0}^{D_j}$. The incident frequency ω that we use, in this acoustic model, is taken of the form

$$\frac{\omega^2}{\omega_0^2} = 1 - \frac{c_{n_0} a^h}{k_0}, \tag{1.6}$$

where the parameter h is positive, and $c_{n_0} \in \mathbb{R}$ satisfies $c_{n_0} < 0,$ is a parameter which is independent of a. The quantity ω_0 is defined as

$$\omega_0 := \sqrt{\frac{k_j}{\lambda_{n_0}^{D_j}}} = \sqrt{\frac{k_0}{\lambda_{n_0}^{B}}},$$

where the last equality is a consequence of the eigenvalues scales and (1.2).

Next, we make the following necessary assumption about the distribution of the D_m-s to derive the first main result of this work, i.e., Theorem 1.1.

Assumption 1. The droplets are distributed periodically inside Ω. More precisely, let Ω be a bounded domain of unit volume, containing the droplets D_j, with $j = 1, \dots, M$. We divide Ω as

$$\Omega \equiv \Omega_{\text{cube}} \cup \Omega_r \tag{1.7}$$

with

$$\Omega_{\text{cube}} \equiv \bigcup_{j=1}^{M} \Omega_j \quad \text{and} \quad \Omega_r \equiv \bigcup_{j=1}^{\aleph} \Omega_j^\star, \quad M = M(a),\ \aleph = \aleph(a) \in \mathbb{N},$$

where Ω_j-s are cubes located strictly within the interior of the domain Ω, i.e., they do not intersect with $\partial\Omega$ ($\Omega_{\text{cube}} \subsetneq \Omega$). Each subdomain Ω_j contains one D_j such that $z_j \in D_j \subset \Omega_j$ and $|\Omega_j| = a^{1-h}$, for $j = 1, \dots, M$ and $0 \leq h < 1$, while the $\Omega_j^\star$-s do not contain any, see Figure 1 for a schematic representation. Therefore, by denoting a reference subdomain as Ω_0, the distribution of Ω_j-s is constructed by appropriate translations of Ω_0. In a concise manner, the distribution of the droplets can be written as

$$D \equiv \Omega_{\text{cube}} \cap d\left(\mathbb{Z}^3 + \left(z + \frac{a}{d} B\right)\right), \tag{1.8}$$

where d is the minimal distance given by (1.12), z is a point contained in unit cell domain, and B is a Lipschitz domain in $\mathbb{R}^3$, such that $\operatorname{diam}(B) \sim 1$. Besides, we assume that Ω_{cube} is away from the boundary $\partial\Omega$, such that

$$\operatorname{dist}(\partial\Omega_{\text{cube}}; \partial\Omega) \sim \kappa(a) \sim a^{(1-h)/3}, \quad \text{with } a \ll 1 \text{ and } 0 < h < 1. \tag{1.9}$$

Hence, from (1.8) and (1.9), the droplets D are away from the boundary $\partial\Omega$, such that

$$\operatorname{dist}(D; \partial\Omega) \sim \kappa(a) \sim a^{(1-h)/3}, \quad \text{with } a \ll 1 \text{ and } 0 < h < 1. \tag{1.10}$$

We are concerned with the case where we have the number M of droplets of the order

$$M \sim a^{h-1}, a \ll 1 \quad \text{with } 0 \leq h < 1, \tag{1.11}$$

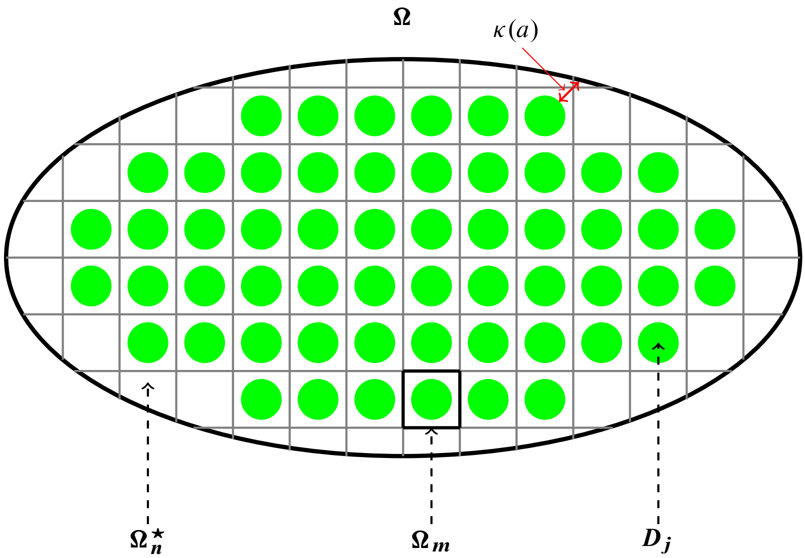

Figure 1. An illustration of how the droplets are distributed in Ω.

and, then, the minimum distance between the droplets is

$$d := \min_{\substack{i \neq j \\ 1 \leq i,j \leq M}} |z_i - z_j| \sim a^{(1-h)/3}, \quad a \ll 1, \quad \text{with } 0 \leq h < 1, \tag{1.12}$$

as $M \sim d^{-3}$. The choice in (1.11) is dictated by the behavior of scattering coefficient (or the polarisation tensor) in (A.32) which is of the order a^{1-h} and M should be inversely proportional to it. This behavior allows to generate a non-trivial effective medium in the homogenisation process.

With these notations at hand, let us state the perturbed problem as follows:

$$\begin{cases} \left(\Delta + \omega^2 n^2(\cdot)(1 - \chi_D) + \omega^2 \frac{\rho_1}{k_1} \chi_D\right) v^g = 0 & \text{in } \Omega, \\ \partial_\nu v^g = g & \text{on } \partial\Omega. \end{cases} \tag{1.13}$$

Under the condition that ω^2 is not an eigenvalue of $-n^{-2}\Delta$ with zero Neumann boundary condition on $\partial\Omega$ and that c_{n_0} and a are small enough, the problem (1.13) is well posed. Indeed, it is clear that the operator solution of the problem in (1.13) is a compact perturbation of the problem (1.1) which is well posed. By Lemma 4.1, we deduce the uniqueness of the solution of (1.13). Actually, by Lemma 4.1 we also derive the related estimate of the well-posedness for problem (1.13).

1.2. From the original NtD map Λ_D to the effective NtD map Λ_P: Theorem 1.1

Let $v^g(\cdot)$ be the solution of the problem (1.13). Multiplying (1.13) by $p^f(\cdot)$ (the solution of (1.1)) and integrating over Ω, we obtain

$$\begin{aligned}&\langle \Lambda_0(f); g\rangle_{\mathbb{H}^{1/2}(\partial\Omega)\times\mathbb{H}^{-1/2}(\partial\Omega)} \\ &\quad = \langle \nabla v^g; \nabla p^f\rangle_{\mathbb{L}^2(\Omega)} - \omega^2 \langle n^2 v^g; p^f\rangle_{\mathbb{L}^2(\Omega)} - \omega^2 \Big\langle \Big(\frac{\rho_1}{k_1} - n^2\Big) v^g; p^f \Big\rangle_{\mathbb{L}^2(D)},\end{aligned} \tag{1.14}$$

where $\Lambda_0(\cdot)$ is the NtD map defined from $\mathbb{H}^{-1/2}(\partial\Omega)$ to $\mathbb{H}^{1/2}(\partial\Omega)$ by

$$\langle \Lambda_0(f); g\rangle_{\mathbb{H}^{1/2}(\partial\Omega)\times\mathbb{H}^{-1/2}(\partial\Omega)} := \int\limits_{\partial\Omega} p^f(x) g(x) d\sigma(x)$$

where we use integrals to simplify notations. We set $\Lambda_D(\cdot)$ to be the NtD map of the background after injecting a cluster of droplets, i.e., the problem (1.13). Multiplying (1.1) by $v^g(\cdot)$ and integrating over Ω, using the self-adjointness of Λ_D and (1.14), we end up with the coming formula

$$\begin{aligned}&\langle \Lambda_D(f); g\rangle_{\mathbb{H}^{1/2}(\partial\Omega)\times\mathbb{H}^{-1/2}(\partial\Omega)} - \langle \Lambda_0(f); g\rangle_{\mathbb{H}^{1/2}(\partial\Omega)\times\mathbb{H}^{-1/2}(\partial\Omega)} \\ &\quad = \omega^2 \Big\langle \Big(\frac{\rho_1}{k_1} - n^2\Big) v^g; p^f \Big\rangle_{\mathbb{L}^2(D)},\end{aligned} \tag{1.15}$$

In a similar way, we define $u^g(\cdot)$ to be the solution of

$$\begin{cases} (\Delta + \omega^2 n^2(\cdot) - P^2) u^g = 0 & \text{in } \Omega, \\ \partial_\nu u^g = g & \text{on } \partial\Omega. \end{cases} \tag{1.16}$$

Here[3]

$$P^2 := \frac{-k_0(\langle 1; \bar{e}_{n_0}\rangle_{\mathbb{L}^2(B)})^2}{\lambda^B_{n_0} c_{n_0}}, \tag{1.17}$$

where $\bar{e}_{n_0}(\cdot)$ is the eigenfunction associated to the eigenvalue $\lambda^B_{n_0}$ related to the Newtonian operator, given by (1.3), defined in the domain B. We set $\Lambda_P(\cdot)$ to be the NtD map of the equivalent background, then we obtain

$$\begin{aligned}&\langle \Lambda_P(f); g\rangle_{\mathbb{H}^{1/2}(\partial\Omega)\times\mathbb{H}^{-1/2}(\partial\Omega)} - \langle \Lambda_0(f); g\rangle_{\mathbb{H}^{1/2}(\partial\Omega)\times\mathbb{H}^{-1/2}(\partial\Omega)} \\ &\quad = -P^2 \langle u^g; p^f\rangle_{\mathbb{L}^2(\Omega)},\end{aligned} \tag{1.18}$$

[3]The assumption that $\langle 1; \bar{e}_{n_0}\rangle_{\mathbb{L}^2(B)} \neq 0$ is reasonable. When B is a ball, we have an infinite sequence of eigenvalues $\lambda^B_{n_0}$ for which the corresponding eigenfunctions satisfy $\langle 1; \bar{e}_{n_0}\rangle_{\mathbb{L}^2(B)} \neq 0$, see [27] for instance.

From (1.15) and (1.18), we see that

$$\begin{aligned}&\langle(\Lambda_D-\Lambda_P)(f);g\rangle_{\mathbb{H}^{1/2}(\partial\Omega)\times\mathbb{H}^{-1/2}(\partial\Omega)}\\&\quad=\omega^2\Big\langle\Big(\frac{\rho_1}{k_1}-n^2\Big)v^g;p^f\Big\rangle_{\mathbb{L}^2(D)}+P^2\langle u^g;p^f\rangle_{\mathbb{L}^2(\Omega)}.\end{aligned}\tag{1.19}$$

In the sequel, we prove that when M is large, or a is small, the perturbed medium, after injecting a cluster of M droplets, behaves like the equivalent background. In other words, the map $\Lambda_D(\cdot)$ converges to $\Lambda_P(\cdot)$.

Theorem 1.1. *Let the domain Ω be C^2-regular. Supposed that the index of refraction $n^2(\cdot)\in\mathbb{W}^{1,\infty}(\Omega)$, the used frequency ω satisfying (1.6), the parameter h be such that $\frac{1}{3}<h<1$, and the droplets D_m-s are distributed as explained in Assumption 1. Then, we have the following convergence:*

$$\langle\Lambda_D(f);g\rangle_{\mathbb{H}^{1/2}(\partial\Omega)\times\mathbb{H}^{-1/2}(\partial\Omega)}\xrightarrow{a\to 0}\langle\Lambda_P(f);g\rangle_{\mathbb{H}^{1/2}(\partial\Omega)\times\mathbb{H}^{-1/2}(\partial\Omega)},$$

uniformly in terms of $(f,g)\in\mathbb{H}^{-1/2}(\partial\Omega)\times\mathbb{H}^{-1/2}(\partial\Omega)$. Precisely, we have the following rate:[4]

$$\|\Lambda_D-\Lambda_P\|_{\mathcal{L}(\mathbb{H}^{-1/2}(\partial\Omega),\mathbb{H}^{1/2}(\partial\Omega))}\lesssim a^{(1-h)(9-5\delta)/(18(3-\delta))}P^6,\quad a\ll 1,\tag{1.20}$$

where δ a sufficiently small but arbitrarily positive number.

Remark 1.2. Two comments are in order.

(1) Since $M\sim a^{h-1}$ and δ is very small, we can rewrite (1.20) as

$$\|\Lambda_D-\Lambda_P\|_{\mathcal{L}(\mathbb{H}^{-1/2}(\partial\Omega),\mathbb{H}^{1/2}(\partial\Omega))}\lesssim M^{(5\delta-9)/(18(3-\delta))}P^6,\quad M\gg 1.$$

We can choose M, i.e., a, such that

$$M^{(5\delta-9)/(18(3-\delta))}P^6\ll 1.\tag{1.21}$$

(2) The parameter δ in (1.20) is linked to the $\mathbb{L}^{3-\delta}(\Omega)$-integrability of the fundamental solution $\Phi_0(\cdot;\cdot)$, given by (1.5).

As we assume to know the NtD map $\Lambda_D(\cdot)$, for M large, the previous theorem suggests the following result.

Corollary 1.3. *Under the condition of Theorem 1.1, the NtD map $\Lambda_P(\cdot)$ is approximately known.*

[4]The $\mathbb{W}^{1,\infty}$-regularity of k, and hence n, is used to derive the rate in (1.20). The $\mathbb{L}^\infty(\Omega)$-regularity is enough to derive the convergence (without rates).

The proof of Theorem 1.1 is based on the point-interaction approximation, or the so-called *Foldy–Lax approximation.* We first approximate the left part in (1.20) by a linear combination of elements of a vector which is the solution of an algebraic system. This algebraic system captures the multiple scattering between the injected droplets through an interaction matrix where the interaction coefficients, that are also called scattering coefficients, are all positive due to the choice made in (1.6) of the sign of c_{n_0}. To prove the invertibility of this algebraic system, uniformly of the large number M of droplets, we first justify the invertibility of the related continuous integral equation and then, we show, with quite tedious computations, that the algebraic equation is "a discrete form" of this continuous integral equation.

Remark 1.4. Two remarks are in order.

(1) In (1.17), we take the constant $c_{n_0} < 0$ and the parameter P^2 such that

$$P^2 > \omega_0^2 \|n^2\|_{\mathbb{L}^\infty(\Omega)} := P_{\min},$$

where, we recall that, $\omega_0^2 = \frac{k_0}{\lambda_{n_0}^B}$. This is possible if we choose the parameter c_{n_0} to satisfy[5]

$$-\rho^{-1} \inf_{y\in\Omega} |k(y)| (\langle 1; \bar{e}_{n_0} \rangle_{\mathbb{L}^2(B)})^2 < c_{n_0} < 0 \quad \text{and} \quad c_{n_0} \to 0^-.$$

We recall that the parameter c_{n_0} appears in (1.6) and we have (1.17). The coefficient c_{n_0} is taken small, and hence P large, but satisfies (1.21).

(2) The parameter h appearing in (1.6) and (1.11) models how dilute, or dense, is the distribution of the injected droplets in Ω. If h is close to 0, we have a dense distribution and when h is close to 1 we have a light distribution.

1.3. The linearisation of the effective NtD map $\Lambda_P(\cdot)$: Theorem 1.5

Theorem 1.5. *We have the following linearisation of $\Lambda_P(\cdot)$, in the $\mathbb{H}^{1/2}(\partial\Omega)$ sense:*

$$\Lambda_P(f) - q^f = \omega^2 \gamma(\mathbf{W}^{q^f}) + \mathcal{O}\Big(\|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \frac{1}{P^4}\Big), \quad f(\cdot) \in \mathbb{H}^{-1/2}(\partial\Omega), \tag{1.22}$$

where $\gamma(\cdot)$ is the trace operator defined from $\mathbb{H}^s(\Omega)$ to $\mathbb{H}^{s-1/2}(\partial\Omega)$, $s \geq \frac{1}{2}$, $q^f(\cdot)$ is the solution of

$$\begin{cases} (\Delta - P^2) q^f = 0 & \text{in } \Omega, \\ \partial_\nu q^f = f & \text{on } \partial\Omega, \end{cases} \tag{1.23}$$

[5] We assume that we have an a priori information on $\inf_{y\in\Omega}|k(y)|$.

and $\mathbf{W}^{q^f}(\cdot)$ *satisfies*

$$\begin{cases} (\Delta - P^2 I)\mathbf{W}^{q^f} = -n^2 q^f & \text{in } \Omega, \\ \partial_\nu \mathbf{W}^{q^f} = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, knowing $\Lambda_P(f)$ allows us to construct $\mathbf{W}^{q^f}$, for $f \in \mathbb{H}^{-1/2}(\partial\Omega)$.

The proof of Theorem 1.5 is based on the observation that the solution operator (i.e., the Lippmann–Schwinger operator) of the problem (1.16) can be seen as the one of the problem (1.23) plus a "small" perturbation. The smallness of this perturbation permits us to justify the related linearisation. The arguments of the analysis are based on the spectral and scaling properties of the Newtonian operator of the solution operator of (1.23) via Calderón–Zygmund-type estimates.

1.4. Construction of $n^2(\cdot)$ from the linearisation of $\Lambda_P(\cdot)$: Theorem 1.6

The next theorem describes a way how we can reconstruct the sound speed from the linearised part of $\Lambda_P(\cdot)$.

Theorem 1.6. *For every* $l := (l_1; l_2; l_3) \in \mathbb{Z}^3$, *we choose*

$$\xi = \frac{P^{2+\varsigma}|l|^{2+\varsigma}}{\sqrt{2}\sqrt{l_2^2 + l_3^2}} \begin{pmatrix} -i(l_2^2 + l_3^2) \\ -|l| l_3 + i l_1 l_2 \\ -|l| l_2 + i l_1 l_3 \end{pmatrix}, \quad \text{with } \varsigma \in \mathbb{R}^+. \tag{1.24}$$

Hence,

$$|\xi| = P^{2+\varsigma}|l|^{3+\varsigma}. \tag{1.25}$$

We set $q^f(\cdot) := q^{l,\xi}$ *the function defined by*

$$q^{l,\xi}(x) := e^{i\xi\cdot x}(e^{ix\cdot l} + r_1(x)), \quad x \in \mathbb{R}^3,$$

and $r_1(\cdot)$ *is such that*

$$(\Delta + 2i\xi\cdot\nabla - P^2) r_1(x) = (|l|^2 + P^2)e^{ix\cdot l}, \quad \text{in } \Omega. \tag{1.26}$$

In the same manner, we set $q^g(\cdot) := q^\xi(\cdot)$ *to be the function defined by*

$$q^\xi(x) := e^{-i\xi\cdot x}(1 + r_2(x)), \quad x \in \mathbb{R}^3,$$

where $r_2(\cdot)$ *is such that*

$$(-\Delta + 2i\xi\cdot\nabla + P^2) r_2(x) = -P^2, \quad \text{in } \Omega. \tag{1.27}$$

Then we have the following approximate reconstruction formula:

$$n^2(x) = (2\pi)^{-3} \sum_{\ell \in \mathbb{Z}^3} \langle \mathbf{W}^{q^{l,\xi}}; \partial_\nu q^\xi \rangle e^{i\ell\cdot x} + \mathcal{O}(P^{-\varsigma}), \tag{1.28}$$

in the $\mathbb{L}^2(\Omega)$ *sense.*

The justification of the existence and uniqueness of solutions corresponding to the problems (1.26) and (1.27) can be found in [41, Section 3.2]. More precisely, in [41, Theorem 3.7] the result is proved first for the free case equation, i.e., equation of the form $(\Delta + 2i\xi\cdot\nabla)r = f$, where $r(\cdot)$ is a correction term and f is the source data. Then, in [41, Theorem 3.8] the general case, i.e., equation of the form $(\Delta + 2i\xi\cdot\nabla + \mathbf{q})r = f$, where $\mathbf{q}$ is a potential, was proved under the conditions $\xi\cdot\xi = 0$ and $|\xi| \geq \max(C_0\|\mathbf{q}\|_{\mathbb{L}^\infty(\Omega)}; 1)$, where C_0 is a constant depending on the domain Ω and the space dimension. These nicely re-derived estimates were initially proved in the seminal work [44, Theorem 1.1, Proposition 2.1].

The key observation here is that these CGOs are solutions of fully explicit equations, see (1.26) and (1.27), which make the representation in (1.28) constructive.

Remark 1.7. In Theorem 1.6, we have shown how to construct $n^2(\cdot)$ using a discrete series expansion. Actually, we can also use the classical Calderón idea to construct the Fourier transform of $n^2(\cdot)$. Indeed, choosing any v which solves $(\Delta - P^2)v = 0$ and multiplying it with the PDE for $W^{q^f}(\cdot)$, we have

$$\int_\Omega n^2(x) q^f(x) v(x)\, dx = \int_{\partial\Omega} \partial_\nu v(x) W^{q^f}(x) ds(x). \tag{1.29}$$

Now, for $0 \neq \xi \in \mathbb{R}^3$, let us consider an orthonormal family $\{e_1 := \frac{\xi}{|\xi|}, e_2, e_3\}$ of $\mathbb{R}^3$. Using this basis, we take

$$\zeta_1 = \frac{|\xi|}{2} e_1 + i e_2 \sqrt{P^2 + \frac{|\xi|^2}{4}}, \quad \zeta_2 = \frac{|\xi|}{2} e_1 - i e_2 \sqrt{P^2 + \frac{|\xi|^2}{4}}$$

and then consider the two functions $q^f(x) := e^{i\zeta_1\cdot x}$ and $v(x) := e^{i\zeta_2\cdot x}$. We see that $(\Delta - P^2)q^f = 0$ and $(\Delta - P^2)v = 0$ since $\zeta_1\cdot\zeta_1 = \zeta_2\cdot\zeta_2 = -P^2$. We also see that $\zeta_1 + \zeta_2 = |\xi| e_1 = \xi$. With this choice, it is immediate that

$$\int_\Omega n^2(x) q^f(x) v(x)\, dx = \int_\Omega e^{i\xi\cdot x} n^2(x)\, dx, \quad \xi \in \mathbb{R}^3.$$

which allows to construct the Fourier transform of $n^2(\cdot)$ from (1.29).

The remaining parts of the paper are organised as follows. In Section 2, we discuss and justify the linearisation step and in Section 3 we deal with the reconstruction of

$n^2(\cdot)$ from the linearised NtD map. The justification for the effective NtD is stated in Section 4. This choice is taken because this step is technically the most involved part. Finally, we postpone several technical steps to be developed and justified in Appendix A.

2. Proof of Theorem 1.5

The goal of this section is to derive a linearisation, up to a first order term, of the NtD map of the equivalent background, i.e., $\Lambda_P(\cdot)$. Let $u^f(\cdot)$ be the solution of the following *Lippmann–Schwinger equation* (LSE in short):

$$u^f(x) - \omega^2 N^p(n^2 u^f)(x) = q^f(x), \quad x \in \Omega, \tag{2.1}$$

where $q^f(\cdot)$ is the solution of (1.23), and $N^p(\cdot)$ is the Newtonian operator defined, from $\mathbb{L}^2(\Omega)$ to $\mathbb{H}^2(\Omega)$, by

$$N^p(f)(x) := \int_\Omega G_p(x,y) f(y)\, dy, \quad x \in \Omega, \tag{2.2}$$

with $G_p(\cdot,\cdot)$ is the solution of

$$\begin{cases} (\Delta_x - P^2) G_p(x,y) = -\boldsymbol{\delta}_y(x) & \text{in } \Omega, \\ \partial_{\nu_x} G_p(x,y) = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

In effortless manner we can check that $u^f(\cdot)$ (the solution of (2.1)) is also the solution of (1.16). Moreover, by an induction process on the LSE, given by (2.1), we prove that

$$u^f(x) - q^f(x) = \omega^2 \gamma N^p(n^2 q^f)(x) + \sum_{j\geq 2} (K_j \overset{j}{\otimes} (n^2))(x), \quad x \in \partial\Omega, \tag{2.4}$$

where

$$(K_2 \overset{2}{\otimes} (n^2))(x) = (\omega^2)^2 \gamma N^p(n^2 N^p(n^2 q^f))(x)$$
$$(K_3 \overset{3}{\otimes} (n^2))(x) = (\omega^2)^3 \gamma N^p(n^2 N^p(n^2 N^p(n^2 q^f)))(x)$$
$$\vdots$$
$$(K_j \overset{j}{\otimes} (n^2)) = (\omega^2)^j \gamma \int_\Omega \cdots \int_\Omega G_p \cdots G_p n^2 \cdots n^2 q^f \, dy_1 \cdots dy_j,$$

where $N^p(\cdot)$ is the Newtonian operator defined by (2.2), and $\gamma(\cdot)$ is the trace operator defined from $\mathbb{H}^s(\Omega)$ to $\mathbb{H}^{s-1/2}(\partial\Omega)$, $s \geq \frac{1}{2}$, with Ω a smooth domain. The coming lemma is useful to study the convergence of the previous series with respect to the $\mathbb{H}^{1/2}(\partial\Omega)$-norm.

Lemma 2.1. *The Newtonian operator given by* (2.2) *admits the following estimations:*

$$\|N^p\|_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{L}^2(\Omega))} = \mathcal{O}\Big(\frac{1}{P^2}\Big), \tag{2.5}$$

and

$$\|\gamma N^p\|_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{H}^{1/2}(\partial\Omega))} = \mathcal{O}\Big(\frac{1}{P}\Big). \tag{2.6}$$

Proof. See Section A.2. ■

For the convergence of the series given into (2.4), we have

$$\Big\|\sum_{j\geq 2}(K_j \overset{j}{\otimes} (n^2))\Big\|_{\mathbb{H}^{1/2}(\partial\Omega)} \leq \sum_{j\geq 2}\|K_j \overset{j}{\otimes} (n^2)\|_{\mathbb{H}^{1/2}(\partial\Omega)}. \tag{2.7}$$

Now, we estimate the terms appearing in the previous series.

(1) For $j = 2$,

$$\begin{aligned}
&\|K_2 \overset{2}{\otimes} (n^2)\|_{\mathbb{H}^{1/2}(\partial\Omega)}\\
&\quad = (\omega^2)^2\|\gamma N^p(n^2 N^p(n^2 q^f))\|_{\mathbb{H}^{1/2}(\partial\Omega)}\\
&\quad \leq (\omega^2)^2\|\gamma N^p\|_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{H}^{1/2}(\partial\Omega))}\|n^2\|^2_{\mathbb{L}^\infty(\Omega)}\|N^p\|_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{L}^2(\Omega))}\|q^f\|_{\mathbb{L}^2(\Omega)}.
\end{aligned}$$

(2) For $j = 3$,

$$\begin{aligned}
&\|K_3 \overset{3}{\otimes} (n^2)\|_{\mathbb{H}^{1/2}(\partial\Omega)}\\
&\quad = (\omega^2)^3\|\gamma N^p(n^2 N^p(n^2 N^p(n^2 q^f)))\|_{\mathbb{H}^{1/2}(\partial\Omega)}\\
&\quad \leq (\omega^2)^3\|\gamma N^p\|_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{H}^{1/2}(\partial\Omega))}\|n^2\|^3_{\mathbb{L}^\infty(\Omega)}\|N^p\|^2_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{L}^2(\Omega))}\|q^f\|_{\mathbb{L}^2(\Omega)}.
\end{aligned}$$

(3) For an arbitrary j, by induction, we can prove that

$$\|K_j \overset{j}{\otimes} (n^2)\|_{\mathbb{H}^{1/2}(\partial\Omega)} \leq \Xi\,\|n^2\|_{\mathbb{L}^\infty(\Omega)}(\omega^2\|N^p\|_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{L}^2(\Omega))}\|n^2\|_{\mathbb{L}^\infty(\Omega)})^{j-1}, \tag{2.8}$$

where

$$\Xi := \omega^2\|q^f\|_{\mathbb{L}^2(\Omega)}\|\gamma N^p\|_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{H}^{1/2}(\partial\Omega))}. \tag{2.9}$$

Therefore, by going back to (2.7) and using the estimation (2.8), we obtain

$$\begin{aligned}
&\Big\|\sum_{j\geq 2}(K_j \overset{j}{\sum}\otimes(n^2))\Big\|_{\mathbb{H}^{1/2}(\partial\Omega)}\\
&\quad \leq \sum_{j\geq 2}\Xi\,\|n^2\|_{\mathbb{L}^\infty(\Omega)}(\omega^2\|N^p\|_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{L}^2(\Omega))}\|n^2\|_{\mathbb{L}^\infty(\Omega)})^{j-1}\\
&\quad = \Xi\,\|n^2\|^2_{\mathbb{L}^\infty(\Omega)}\omega^2\|N^p\|_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{L}^2(\Omega))_{j\geq 0}}\kappa^j,
\end{aligned} \tag{2.10}$$

where κ is the parameter given by $\kappa := \omega^2 \|n^2\|_{\mathbb{L}^\infty(\Omega)} \|N^p\|_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{L}^2(\Omega))}$. Under the condition

$$\kappa < 1, \tag{2.11}$$

the previous series converges. Now, because $\|N^p\|_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{L}^2(\Omega))} = \mathcal{O}(P^{-2})$, see (2.5), then with P large enough; knowing that $\omega^2 \|n^2\|_{\mathbb{L}^\infty(\Omega)}$ is a bounded term, we deduce that the condition (2.11) is satisfied. In addition, from (2.10), we have

$$\begin{aligned}
\Big\| \sum_{j\geq 2} (K_j \overset{j}{\otimes} (n^2)) \Big\|_{\mathbb{H}^{1/2}(\partial\Omega)} &= \mathcal{O}(\Xi \|N^p\|_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{L}^2(\Omega))}) \\
&\overset{(2.9)}{=} \mathcal{O}(\|N^p\|_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{L}^2(\Omega))} \|q^f\|_{\mathbb{L}^2(\Omega)} \|\gamma N^p\|_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{H}^{1/2}(\partial\Omega))}) \\
&\overset{\text{Lemma 2.1}}{=} \mathcal{O}\Big(\|q^f\|_{\mathbb{L}^2(\Omega)} \frac{1}{P^3}\Big).
\end{aligned} \tag{2.12}$$

The coming lemma is important to get an estimation of

$$\Big\| \sum_{j\geq 2} (K_j \overset{j}{\otimes} (n^2)) \Big\|_{\mathbb{H}^{1/2}(\partial\Omega)},$$

with respect to the data $f(\cdot)$ and the parameter P.

Lemma 2.2. *The function $q^f(\cdot)$ (the solution of (1.23)) satisfies*

$$\|q^f\|_{\mathbb{L}^2(\Omega)} = \mathcal{O}\Big(\|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \frac{1}{P}\Big). \tag{2.13}$$

Proof. The solution $q^f(\cdot)$ to the problem (1.23) can be represented as $q^f(x) = \mathrm{SL}^p(f)(x)$, for $x \in \Omega$, where $\mathrm{SL}^p(\cdot)$ is the single-layer operator defined, from $\mathbb{H}^{-1/2}(\partial\Omega)$ to $\mathbb{H}^1(\Omega)$, by

$$\mathrm{SL}^p(f)(x) := \int_{\partial\Omega} G_p(x,y) f(y) d\sigma(y), \quad x \in \Omega, \tag{2.14}$$

with $G_p(\cdot,\cdot)$ being the Green's kernel solution of (2.3). It is clear that $(\Delta - P^2) q^f = 0$ in Ω. In Section A.8, we show that $\partial_\nu \mathrm{SL}^p(f) = f$ on $\partial\Omega$. Multiplying the previous equation by $\overline{q^f}(\cdot)$ and integrating over Ω, gives us

$$\begin{aligned}
\|q^f\|^2_{\mathbb{L}^2(\Omega)} &= \langle q^f; \mathrm{SL}^p(f)\rangle_{\mathbb{L}^2(\Omega)} \\
&= \langle f; \gamma N^p(q^f)\rangle_{\mathbb{H}^{-1/2}(\partial\Omega)\times\mathbb{H}^{1/2}(\partial\Omega)} \\
&\leq \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \|\gamma N^p(q^f)\|_{\mathbb{H}^{1/2}(\partial\Omega)} \\
&\leq \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \|\gamma N^p\|_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{H}^{1/2}(\partial\Omega))} \|q^f\|_{\mathbb{L}^2(\Omega)}.
\end{aligned}$$

Then,

$$\|q^f\|_{\mathbb{L}^2(\Omega)} \leq \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \|\gamma N^P\|_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{H}^{1/2}(\partial\Omega))} \overset{(2.6)}{=} \mathcal{O}\Big(\|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \frac{1}{P}\Big).$$

This concludes the proof of Lemma 2.2. ■

Using (2.13), the estimation (2.12) becomes

$$\Big\| \sum_{j\geq 2} (K_j \overset{j}{\otimes} (n^2)) \Big\|_{\mathbb{H}^{1/2}(\partial\Omega)} = \mathcal{O}\Big(\|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \frac{1}{P^4}\Big).$$

Hence, from (2.4), we get

$$u^f(x) - q^f(x) = \omega^2 \gamma(\mathbf{W}^{q^f})(x) + \mathcal{O}\Big(\|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \frac{1}{P^4}\Big), \quad x \in \partial\Omega, \tag{2.15}$$

where $\mathbf{W}^{q^f} = N^P(n^2 q^f)$ is the function satisfying

$$\begin{cases} (\Delta - P^2 I)\mathbf{W}^{q^f} = -n^2 q^f & \text{in } \Omega, \\ \partial_\nu \mathbf{W}^{q^f} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.16}$$

Because on the boundary $\partial\Omega$, we have $u^f = \Lambda_P(\partial_\nu u^f) \overset{(1.16)}{=} \Lambda_P(f)$ and by plugging it into (2.15) we derive (1.22). This concludes the proof of Theorem 1.5.

3. Proof of Theorem 1.6

The purpose of this section is to explain how the linearised NtD map (measured on the boundary $\partial\Omega$) can be utilised with CGO solutions to reconstruct the Fourier coefficients associated with the unknown refraction index $n^2(\cdot)$. Hence, we reconstruct the refraction index of $n^2(\cdot)$ inside Ω as a discrete series expansion using the reconstructed Fourier coefficients. From the previous section, we deduce that measuring $u^f(\cdot) - q^f(\cdot)$ means measuring, approximately, $\mathbf{W}^{q^f}(\cdot)$, on the boundary $\partial\Omega$. We set $q^g(\cdot)$ to be the solution of

$$\begin{cases} (\Delta - P^2 I)q^g = 0 & \text{in } \Omega, \\ \partial_\nu q^g = g & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Multiplying the first equation of (2.16) with $q^g(\cdot)$ (the solution of (3.1)), and integrating over Ω, we get

$$\langle \nabla \mathbf{W}^{q^f}; \nabla q^g \rangle_{\mathbb{L}^2(\Omega)} + P^2 \langle \mathbf{W}^{q^f}; q^g \rangle_{\mathbb{L}^2(\Omega)} = \langle n^2 q^f; q^g \rangle_{\mathbb{L}^2(\Omega)}.$$

Moreover, by multiplying (3.1) with $\mathbf{W}^{q^f}$ (the solution of (2.16)), and integrating over Ω, we get

$$\langle \nabla \mathbf{W}^{q^f}; \nabla q^g \rangle_{\mathbb{L}^2(\Omega)} + P^2 \langle \mathbf{W}^{q^f}; q^g \rangle_{\mathbb{L}^2(\Omega)} = \langle \mathbf{W}^{q^f}; g \rangle_{\mathbb{H}^{1/2}(\partial\Omega)\times\mathbb{H}^{-1/2}(\partial\Omega)}.$$

Then, by subtracting the two previous equations we end up with

$$\langle \mathbf{W}^{q^f}; g \rangle_{\mathbb{H}^{1/2}(\partial\Omega)\times\mathbb{H}^{-1/2}(\partial\Omega)} = \langle n^2 q^f; q^g \rangle_{\mathbb{L}^2(\Omega)}, \tag{3.2}$$

for all $(f, g) \in \mathbb{H}^{-1/2}(\partial\Omega) \times \mathbb{H}^{-1/2}(\partial\Omega)$. Knowing that $\mathbf{W}^{q^f}$ can be measured, on the boundary $\partial\Omega$, and g is a data function, we deduce that the left-hand side is a known term. The goal is then to reconstruct $n^2(\cdot)$, inside Ω. To achieve this, we start by fixing $\eta \in \mathbb{R}^3$ and choosing $\xi \in \mathbb{C}^3$ such that

$$\xi \cdot \xi = 0. \tag{3.3}$$

We set $q^f(\cdot)$ the function defined by

$$q^f(x) := e^{i\xi\cdot x}(e^{ix\cdot\eta} + r_1(x)), \quad x \in \mathbb{R}^3, \tag{3.4}$$

where ξ is chosen such that[6]

$$\eta \cdot \xi = 0, \tag{3.5}$$

and $r_1(\cdot)$ is such that

$$(\Delta + 2i\xi \cdot \nabla - P^2) r_1(x) = (\|\eta\|^2 + P^2) e^{ix\cdot\eta} \quad \text{in } \Omega. \tag{3.6}$$

Observe that the right-hand side is depending on η and P, then $r_1(\cdot)$ will also depends on both η and P. Later, to mark this dependence, we note $r_{1,\eta,p}(\cdot)$ instead of $r_1(\cdot)$. Thanks to [41, Theorem 3.8], we know that under the condition

$$|\xi| \geq \max(C_0 P^2; 1) = C_0 P^2, \tag{3.7}$$

where the last equality is a consequence of the fact that $P \gg 1$, and C_0 is a constant depending on Ω, equation (3.6) has a solution $r_{1,\eta,p}(\cdot) \in \mathbb{H}^1(\Omega)$ satisfying

$$\|r_{1,\eta,p}\|_{\mathbb{L}^2(\Omega)} \leq \frac{C_0}{|\xi|}(|\eta|^2 + P^2)|\Omega|^{1/2} \tag{3.8a}$$

and

$$\|\nabla r_{1,\eta,p}\|_{\mathbb{L}^2(\Omega)} \leq C_0(|\eta|^2 + P^2)|\Omega|^{1/2}. \tag{3.8b}$$

[6] For every fixed $\eta \in \mathbb{R}^3$, we choose $\xi \in \mathbb{C}^3$ such that (3.3) and (3.5) will be fulfilled. Such ξ exists, see (1.24).

In the same manner, we set $q^g(\cdot)$ to be the function defined by

$$q^g(x) := e^{-i\xi\cdot x}(1 + r_2(x)), \quad x \in \mathbb{R}^3, \tag{3.9}$$

where $r_2(\cdot)$ is such that

$$(-\Delta + 2i\xi \cdot \nabla + P^2)r_2(x) = -P^2 \quad \text{in } \Omega. \tag{3.10}$$

Because the right-hand side is depending on P, a solution $r_2(\cdot)$ will also depends on P. Later, to mark this dependence, we note $r_{2,p}(\cdot)$ instead of $r_2(\cdot)$. Again, thanks to [41, Theorem 3.8], we know that under the condition (3.7), equation (3.10) has a solution $r_{2,p}(\cdot) \in \mathbb{H}^1(\Omega)$, satisfying

$$\|r_{2,p}\|_{\mathbb{L}^2(\Omega)} \leq \frac{C_0}{|\xi|} P^2 |\Omega|^{1/2} \quad \text{and} \quad \|\nabla r_{2,p}\|_{\mathbb{L}^2(\Omega)} \leq C_0 P^2 |\Omega|^{1/2}. \tag{3.11}$$

Now, we take unit vectors ω_1 and ω_2 in $\mathbb{R}^3$ such that $\{\omega_1; \omega_2; \eta\}$ is an orthogonal set. In addition, we choose $\xi = s(\omega_1 + i\omega_2)$, so that $|\xi| = s\sqrt{2}$ and $\xi \cdot \xi = 0$. Using the fact that $P \gg 1$ and taking the parameter s sufficiently large, such that (3.7) will be satisfied, we reduce the estimation of the $\mathbb{L}^2(\Omega)$−norm of $r_{1,\eta,p}(\cdot)$ and $r_{2,p}(\cdot)$ to

$$\|r_{1,\eta,p}\|_{\mathbb{L}^2(\Omega)} = \mathcal{O}\Big(\frac{P^2}{s}\Big) \quad \text{and} \quad \|r_{2,p}\|_{\mathbb{L}^2(\Omega)} = \mathcal{O}\Big(\frac{P^2}{s}\Big). \tag{3.12}$$

Next, by taking the product between $q^f(\cdot)$, given by (3.4), and $q^g(\cdot)$, given by (3.9), we obtain

$$(q^f \cdot q^g)(x) = e^{ix\cdot\eta} + r_{1,\eta,p}(x) + e^{ix\cdot\eta} r_{2,p}(x) + r_{1,\eta,p}(x) r_{2,p}(x), \tag{3.13}$$

and we would like to choose the solution in such a way that $(q^f \cdot q^g)(\cdot)$ is close to $e^{i\cdot\cdot\eta}$, since the functions $\{e^{i\cdot\cdot\eta}\}$ form a dense set, see [25, Theorem 1.1], in $\mathbb{L}^1(\Omega)$. By going back to (3.2), we have

$$\begin{aligned}
\langle \mathbf{W}^{q^f}; g\rangle_{\mathbb{H}^{1/2}(\partial\Omega)\times\mathbb{H}^{-1/2}(\partial\Omega)} &= \int_\Omega n^2(x) q^f(x) q^g(x)\, dx \\
&\overset{(3.13)}{=} \int_\Omega n^2(x) e^{ix\cdot\eta}\, dx + \mathrm{Error}(\eta, p),
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{Error}(\eta, p) := &\int_\Omega n^2(x) r_{1,\eta,p}(x)\, dx + \int_\Omega n^2(x) e^{ix\cdot\eta} r_{2,p}(x)\, dx \\
&+ \int_\Omega n^2(x) r_{1,\eta,p}(x) r_{2,p}(x)\, dx,
\end{aligned}$$

which can be estimated as

$$\begin{aligned}|\mathrm{Error}(\eta, p)| \leq \|n^2\|_{\mathbb{L}^\infty(\Omega)} \|r_{1,\eta,p}\|_{\mathbb{L}^2(\Omega)} |\Omega|^{1/2} + \|n^2\|_{\mathbb{L}^\infty(\Omega)} \|r_{2,p}\|_{\mathbb{L}^2(\Omega)} |\Omega|^{1/2} \\ + \|n^2\|_{\mathbb{L}^\infty(\Omega)} \|r_{1,\eta,p}\|_{\mathbb{L}^2(\Omega)} \|r_{2,p}\|_{\mathbb{L}^2(\Omega)},\end{aligned}$$

which, based on (3.8) and (3.11), can be reduced to

$$|\mathrm{Error}(\eta, p)| \lesssim \frac{(|\eta|^2 + P^2)}{|\xi|} \overset{(3.12)}{=} \mathcal{O}\Big(\frac{P^2}{s}\Big) = \mathcal{O}\Big(\frac{P^2}{|\xi|}\Big). \tag{3.14}$$

Moreover, based on its construction, see (3.4), the function $q^f(\cdot)$ depends on η and this implies the dependency of $\mathbf{W}^{q^f}(\cdot)$ with respect to η. We mark explicitly this dependence and we write

$$\langle \mathbf{W}^{q^f}_\eta ; g \rangle_{\mathbb{H}^{1/2}(\partial\Omega)\times\mathbb{H}^{-1/2}(\partial\Omega)} - \mathrm{Error}(\eta, p) = \int_\Omega n^2(x) e^{i x\cdot\eta}\, dx, \tag{3.15}$$

which is valid in $\Lambda_\eta := \{\xi \in \mathbb{C}^3 \text{ such that } |\xi| \gg 1,\ \xi\cdot\xi = 0 \text{ and } \xi\cdot\eta = 0\}$, where η is fixed in $\mathbb{R}^3$. The set Λ_η is not empty, see (1.24). By restricting η to $\mathbb{Z}^3$, i.e., $\eta = -\ell$ with $\ell \in \mathbb{Z}^3$, we rewrite (3.15) as

$$\begin{aligned}\langle \mathbf{W}^{q^f}_{-\ell} ; g \rangle_{\mathbb{H}^{1/2}(\partial\Omega)\times\mathbb{H}^{-1/2}(\partial\Omega)} - \mathrm{Error}(-\ell, p) &= \int_\Omega n^2(x) e^{-i x\cdot\ell}\, dx \\ &= (2\pi)^3 \mathcal{F}(n^2 \chi_{\Omega}(\ell)),\end{aligned} \tag{3.16}$$

where $\mathcal{F}(\cdot)$ is the 3D-Fourier transform operator.[7] Now, thanks to [41, Theorem 2.3], we know that

$$n^2(x) = \sum_{\ell\in\mathbb{Z}^3} \mathcal{F}(n^2\chi_{\Omega})(\ell) e^{i\ell\cdot x}, \quad x \in \Omega,$$

with convergence in the $\mathbb{L}^2(\Omega)$-norm. Then, by gathering the previous expression and (3.16), we end up with

$$n^2(x) = (2\pi)^{-3} \sum_{\ell\in\mathbb{Z}^3} \int_{\partial\Omega} \mathbf{W}^{q^f}_{-\ell}(x) g(x) d\sigma(x) e^{i\ell\cdot x} + \mathbf{Error}(x, p), \tag{3.17}$$

[7] We recall that we have

$$\mathcal{F}(f)(\ell) := (2\pi)^{-3} \int_{\mathbb{R}^3} f(x) e^{-i\ell\cdot x}\, dx, \quad \ell \in \mathbb{Z}^3.$$

in the $\mathbb{L}^2(\Omega)$ sense, where $\mathbf{Error}(x,p)$ is a trigonometric series given by

$$\mathbf{Error}(x,p) := -(2\pi)^{-3} \sum_{\ell \in \mathbb{Z}^3} \mathrm{Error}(-\ell,p) e^{i\ell\cdot x}, \quad x \in \Omega.$$

Next, we estimate the $\mathbb{L}^2(\Omega)$ norm of $\mathbf{Error}(\cdot,p)$. We have

$$\|\mathbf{Error}(\cdot,p)\|_{\mathbb{L}^2(\Omega)} \lesssim \sum_{\ell \in \mathbb{Z}^3} |\mathrm{Error}(-\ell,p)| \overset{(3.14)}{\lesssim} \sum_{\ell \in \mathbb{Z}^3} \frac{(|\ell|^2 + P^2)}{|\xi|}.$$

At this stage, we recall that for every fixed $\ell \in \mathbb{Z}^3$, we choose $\xi \in \mathbb{C}^3$ such that

$$\xi \cdot \xi = 0, \ell \cdot \xi = 0 \text{ and } |\xi| \gg 1.$$

Such ξ exists, see (1.24). Without loss of generality, we take ξ satisfying (1.25), hence $|\xi| = P^{2+\varsigma}|\ell|^{3+\varsigma}$, with $\varsigma \in \mathbb{R}^+$. Then,

$$\|\mathbf{Error}(\cdot,p)\|_{\mathbb{L}^2(\Omega)} \lesssim \sum_{\ell \in \mathbb{Z}^3} \frac{(|\ell|^2 + P^2)}{P^{2+\varsigma}|\ell|^{3+\varsigma}} = P^{-2-\varsigma} \sum_{\ell \in \mathbb{Z}^3} \frac{1}{|\ell|^{1+\varsigma}} + P^{-\varsigma} \sum_{\ell \in \mathbb{Z}^3} \frac{1}{|\ell|^{3+\varsigma}}.$$

After that, we use the convergence of the two previous series to reduce the last estimation to

$$\|\mathbf{Error}(\cdot,p)\|_{\mathbb{L}^2(\Omega)} = \mathcal{O}(P^{-\varsigma}).$$

Hence, (3.17) becomes

$$n^2(x) = (2\pi)^{-3} \sum_{\ell \in \mathbb{Z}^3} \langle \mathbf{W}^{q^f}_{-\ell}; g \rangle_{\mathbb{H}^{1/2}(\partial\Omega)\times\mathbb{H}^{-1/2}(\partial\Omega)} e^{i\ell\cdot x} + \mathcal{O}(P^{-\varsigma}),$$

in the $\mathbb{L}^2(\Omega)$ sense. This ends the proof of Theorem 1.6.

4. Proof of Theorem 1.1

The purpose of this section is to prove Theorem 1.1. To ensure easy reading of the proof, we have divided this section into four subsections. The goal of the first subsection is to extract the dominant term of

$$\mathbf{I}_1 := \omega^2 \Big\langle \Big(\frac{\rho_1}{k_1} - n^2 \Big) v^g ; p^f \Big\rangle_{\mathbb{L}^2(D)},$$

where we prove that

$$\mathbf{I}_1 = \omega^2 \frac{\rho_1}{k_1} \sum_{j=1}^{M} p^f(z_j) \int_{D_j} v_j^g(x)\, dx + \mathrm{Error},$$

see (4.14). In the second subsection we derive and we justify the invertibility of the discrete algebraic system satisfied by the vector $\left(\int_{D_j} v_j^g(x)\,dx\right)_{j=1,\ldots,M}$, contained in $\mathbf{I}_1$, see (4.31) and Lemma 4.3. The third subsection consists in writing down the LSE, satisfied by $u^g(\cdot)$, where $u^g(\cdot)$ is the function appearing in

$$\mathbf{I}_2 := -P^2 \langle u^g; p^f \rangle_{\mathbb{L}^2(\Omega)},$$

see (4.34). Then, we prove that the discrete algebraic system can approximate the continuous LSE, see (4.57). The goal of the last subsection lies in the justification of the convergence of $\mathbf{I}_1$ to $\mathbf{I}_2$ for a large number of droplets, that is, $M \gg 1$.

To avoid making this section heavy and cumbersome, we have noted six lemmas without proofs. The proof of each lemma can be found in Section A.

4.1. Extraction of the dominant term of I_1

We set

$$\begin{aligned}
\mathbf{I}_1 &:= \omega^2 \frac{\rho_1}{k_1} \langle v^g; p^f \rangle_{\mathbb{L}^2(D)} - \omega^2 \langle n^2 v^g; p^f \rangle_{\mathbb{L}^2(D)} \\
&= \omega^2 \frac{\rho_1}{k_1} \sum_{j=1}^{M} \int_{D_j} v_j^g(x) p^f(x)\,dx - \omega^2 \langle n^2 v^g; p^f \rangle_{\mathbb{L}^2(D)},
\end{aligned}$$

where $v^g(\cdot)$ satisfies (1.13), $p^f(\cdot)$ is the solution of (1.1), and we have used the notation $v_j^g(\cdot) := v^g|_{D_j}(\cdot)$, for $j = 1, \ldots, M$. In addition, as the coefficients $n^2(\cdot)$ is $\mathbb{W}^{1,\infty}$-regular, then $p^f(\cdot)$, which is in $\mathbb{H}^1(\Omega)$, enjoys a $\mathbb{W}^{2,\infty}$-interior regularity. Based on this, we use Taylor expansion near the centres, z_j, to get

$$\mathbf{I}_1 = \omega^2 \frac{\rho_1}{k_1} \sum_{j=1}^{M} p^f(z_j) \int_{D_j} v_j^g(x)\,dx + J_1, \tag{4.1}$$

where

$$\begin{aligned}
J_1 := {} & \omega^2 \frac{\rho_1}{k_1} \sum_{j=1}^{M} \int_{D_j} v_j^g(x) \int_0^1 \nabla p^f(z_j + t(x - z_j)) \cdot (x - z_j)\,dt\,dx \\
& - \omega^2 \langle n^2 v^g; p^f \rangle_{\mathbb{L}^2(D)}.
\end{aligned}$$

We estimate the term J_1 as

$$\begin{aligned}
\|J_1\| \lesssim {} & a^{-2} \sum_{j=1}^{M} \|v_j^g\|_{\mathbb{L}^2(D_j)} \left\| \int_0^1 \nabla p^f(z_j + t(\cdot - z_j)) \cdot (\cdot - z_j)\,dt \right\|_{\mathbb{L}^2(D_j)} \\
& + \|v^g\|_{\mathbb{L}^2(D)} \|p^f\|_{\mathbb{L}^2(D)}
\end{aligned}$$

$$
\begin{aligned}
&\leq a^{-2}\|v^{g}\|_{\mathbb{L}^2(D)}\Big(\sum_{j=1}^{M}\Big\|\int_0^1 \nabla p^{f}(z_j+t(\cdot-z_j))\cdot(\cdot-z_j)\,dt\Big\|^2_{\mathbb{L}^2(D_j)}\Big)^{1/2}\\
&\quad+\|v^{g}\|_{\mathbb{L}^2(D)}\|p^{f}\|_{\mathbb{L}^2(D)}\\
&=\mathcal{O}(\|v^{g}\|_{\mathbb{L}^2(D)}[a^{-1}\|\nabla p^{f}\|_{\mathbb{L}^2(D)}+\|p^{f}\|_{\mathbb{L}^2(D)}]).
\end{aligned}
\tag{4.2}
$$

Moreover, based on (1.1) we deduce that $p^{f}(\cdot)$ can be represented as a single-layer with density $f(\cdot)$, i.e.,

$$
\begin{aligned}
p^{f}(x) &= \mathcal{S}(f)(x)\\
&:= \int_{\partial\Omega} G(x,y)f(y)d\sigma(y) = \langle G(x,\cdot); f\rangle_{\mathbb{H}^{1/2}(\partial\Omega)\times\mathbb{H}^{-1/2}(\partial\Omega)}, \quad x\in\Omega,
\end{aligned}
\tag{4.3}
$$

where $G(\cdot,\cdot)$ is Green's kernel defined by

$$
\begin{cases}
\Delta_x G(x,y)+\omega^2 n^2(x)G(x,y) = -\boldsymbol{\delta}_y(x) & \text{in } \Omega,\\
\partial_{\nu_x} G(x,y) = 0 & \text{on } \partial\Omega.
\end{cases}
\tag{4.4}
$$

The existence and the uniqueness of $G(\cdot,\cdot)$ and its singularity analysis, with point-wise estimates, can be found in [31]. Based on (4.3), we have

$$
\|p^{f}\|_{\mathbb{L}^2(D)} \leq \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)}\Big[\int_D \|G(x,\cdot)\|^2_{\mathbb{H}^{1/2}(\partial\Omega)}\,dx\Big]^{1/2}. \tag{4.5}
$$

In addition, we have

$$
\|G(x,\cdot)\|_{\mathbb{H}^{1/2}(\partial\Omega)} = \inf_{\substack{\mathsf{G}(x,\cdot)\in\mathbb{H}^1(\Omega)\\ \mathsf{G}(x,\cdot)|_{\partial\Omega}=G(x,\cdot)}} \|\mathsf{G}(x,\cdot)\|_{\mathbb{H}^1(\Omega)}.
$$

Let $\Omega^{\diamond}\equiv\Omega\setminus\overline{D}$, and let $\mathsf{G}(x,\cdot):=G(x,\cdot)\chi_{\Omega^{\diamond}}(\cdot)$. Thus, by its construction $\mathsf{G}(x,\cdot)\in\mathbb{H}^1(\Omega)$, for $x\in D$, and $\gamma(\mathsf{G}(x,\cdot))=G(x,\cdot)|_{\partial\Omega}$, on $\partial\Omega$, where $\gamma(\cdot)$ is the trace operator. This implies that

$$
\|G(x,\cdot)\|_{\mathbb{H}^{1/2}(\partial\Omega)} \leq \|G(x,\cdot)\chi_{\Omega^{\diamond}}(\cdot)\|_{\mathbb{H}^1(\Omega)} = \|G(x,\cdot)\|_{\mathbb{H}^1(\Omega^{\diamond})}. \tag{4.6}
$$

Then, by plugging (4.6) into (4.5), we deduce

$$
\|p^{f}\|_{\mathbb{L}^2(D)} \leq \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)}\Big[\int_D \|G(x,\cdot)\|^2_{\mathbb{H}^1(\Omega^{\diamond})}\,dx\Big]^{1/2}.
$$

Recalling that

$$\|G(x,\cdot)\|^2_{\mathbb{H}^1(\Omega^\diamond)} := \|\nabla G(x,\cdot)\|^2_{\mathbb{L}^2(\Omega^\diamond)} + \|G(x,\cdot)\|^2_{\mathbb{L}^2(\Omega^\diamond)},$$

we deduce, since $|\nabla G(x,y)| = \mathcal{O}(|x-y|^{-2})$, that

$$\begin{aligned}\|p^f\|_{\mathbb{L}^2(D)} &\lesssim \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)}\left[\int_D \|\nabla G(x,\cdot)\|^2_{\mathbb{L}^2(\Omega^\diamond)}\,dx\right]^{1/2} \\ &\lesssim \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)}\left[\int_D\int_{\Omega^\diamond} \frac{1}{|x-y|^4}\,dy\,dx\right]^{1/2}.\end{aligned} \tag{4.7}$$

We have, for $x \in D$ and $y \in \Omega^\diamond$, see Assumption 1, that

$$|x-y| \geq \operatorname{dist}(D,\partial\Omega^\diamond) = \operatorname{dist}(D,\partial\Omega) \geq \kappa(a), \tag{4.8}$$

which implies

$$\begin{aligned}\|p^f\|_{\mathbb{L}^2(D)} &\lesssim \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)}[|D|(\kappa(a))^{-4}]^{1/2} \\ &= \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)}[|D_{m_0}|M(\kappa(a))^{-4}]^{1/2} \\ &\overset{(1.10)}{=} \mathcal{O}(\|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)}a^{(2+7h)/6}).\end{aligned} \tag{4.9}$$

Similarly, using (4.3), the vector function $\nabla p^f(\cdot)$ can be expressed as

$$\begin{aligned}\nabla p^f(x) &= \int_x \nabla_{\partial\Omega} G(x,y) f(y)d\sigma(y) \\ &= \langle \nabla G(x,\cdot); f\rangle_{\mathbb{H}^{1/2}(\partial\Omega)\times\mathbb{H}^{-1/2}(\partial\Omega)}, \quad x\in\Omega,\end{aligned} \tag{4.10}$$

Then, by repeating the same computations as (4.5)–(4.7), we obtain

$$\begin{aligned}\|\nabla p^f\|_{\mathbb{L}^2(D)} &\lesssim \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)}\left[\int_D\int_{\Omega^\diamond} \frac{1}{|x-y|^6}\,dy\,dx\right]^{1/2} \\ &= \mathcal{O}(a^{3h/2}\|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)}).\end{aligned} \tag{4.11}$$

Hence, by returning to (4.2), using (4.9) and (4.11),

$$J_1 = \mathcal{O}(a^{3h/2-1}\|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)}\|v^g\|_{\mathbb{L}^2(D)}). \tag{4.12}$$

The following lemma gives us an a priori estimate satisfied by $v^g(\cdot)$.

Lemma 4.1. *We have the following a priori estimate:*

$$\|v^g\|_{\mathbb{L}^2(D)} \lesssim a^{(5-2h)/6}\|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}. \tag{4.13}$$

Proof. See Section A.4. ∎

Thanks to the previous lemma, the estimation of J_1 given by (4.12), we reduce the estimation of $\mathbf{I}_1$, given by (4.1), to

$$\mathbf{I}_1 = \omega^2 \frac{\rho_1}{k_1} \sum_{j=1}^{M} p^f(z_j) \int_{D_j} v_j^g(x)\, dx + \mathcal{O}\big(a^{(5h-2)/3} \| f \|_{\mathbb{H}^{-1/2}(\partial\Omega)} \| g \|_{\mathbb{H}^{-1/2}(\partial\Omega)}\big). \tag{4.14}$$

The goal of the following subsection is to derive the algebraic system satisfied by the vector $\big(\int_{D_j} v_j^g(x)\, dx\big)_{j=1,\dots,M}$ and justify its invertibility.

4.2. Algebraic system

We start with the following LSE, with $v^g(\cdot)$ as the solution of (1.13),

$$v^g(x) - \omega^2 \int_D G(x,y) v^g(y) \Big(\frac{\rho_1}{k_1} - n^2(y)\Big)\, dy = S(x), \quad x \in \Omega, \tag{4.15}$$

where $S(\cdot)$ is the solution of

$$\begin{cases} \Delta S + \omega^2 n^2(\cdot) S = 0 & \text{in } \Omega, \\ \partial_\nu S = g & \text{on } \partial\Omega. \end{cases} \tag{4.16}$$

and $G(\cdot,\cdot)$ is the Green's kernel solution of (4.4). Now, by restricting (4.15) into D, we obtain

$$v^g(x) - \omega^2 \int_D G(x,y) v^g(y) \Big(\frac{\rho_1}{k_1} - n^2(y)\Big)\, dy = S(x), \quad x \in D. \tag{4.17}$$

The coming lemma, on the decomposition of Green's kernel $G(\cdot;\cdot)$, is useful for the next step.

Lemma 4.2. *Green's kernel $G(\cdot;\cdot)$ (the solution of (4.4)) admits the following decomposition:*

$$G(x,y) = \Phi_0(x,y) + \mathcal{R}(x,y), \quad x \neq y, \tag{4.18}$$

where $\Phi_0(\cdot,\cdot)$ is given by (1.5), and the remainder term $\mathcal{R}(\cdot,\cdot)$ satisfies

$$\begin{cases} \Delta_x(\mathcal{R}(x,y)) + \omega^2 n^2(x) \mathcal{R}(x,y) = -\,\omega^2 n^2(x) \Phi_0(x,y) & \textit{in } \Omega, \\ \partial_{\nu_x}(\mathcal{R}(x,y)) = -\,\partial_{\nu_x}(\Phi_0(x,y)) & \textit{on } \partial\Omega. \end{cases} \tag{4.19}$$

For $x, y \in \Omega$, the term $\mathcal{R}(\cdot,\cdot)$ is such that[8]

$$|\mathcal{R}(x,y)| \lesssim \left(\frac{1}{\mathrm{dist}(x,\partial\Omega)}\right)^{1/3}\left(\frac{1}{\mathrm{dist}(y,\partial\Omega)}\right)^{2/3}. \tag{4.20}$$

Proof. See Section A.3. ■

For $x \in D_m$, we rewrite (4.17) as

$$\begin{aligned}\Big(I - \omega^2\frac{\rho_1}{k_1}N_{D_m}\Big)(v_m^g)(x) - \omega^2\frac{\rho_1}{k_1}\sum_{j\neq m}^{M}\int_{D_j} G(x,y)v_j^g(y)\,dy \\ = S_m(x) + \omega^2\frac{\rho_1}{k_1}\int_{D_m}\mathcal{R}(x,y)v_m^g(y)\,dy - \omega^2\int_D G(x,y)v^g(y)n^2(y)\,dy,\end{aligned} \tag{4.21}$$

where $S_m(\cdot) := S(\cdot)|_{D_m}$, $\mathcal{R}(\cdot,\cdot)$ is the solution of (4.19), and $N_{D_m}(\cdot)$ is the Newtonian operator defined, from $\mathbb{L}^2(D_m)$ to $\mathbb{L}^2(D_m)$, by (1.3). In both sides of (4.21), successively, we multiply by $\frac{k_1}{\omega^2\rho_1}$, we take the inverse operator of $\big(\frac{k_1}{\omega^2\rho_1}I - N_{D_m}\big)$ and integrate over D_m, the obtained equation, to get

$$\begin{aligned}\beta_m\int_{D_m} v_m^g(x)\,dx - \sum_{\substack{j=1\\ j\neq m}}^{M}\int_{D_m} W_m(x)\int_{D_j} G(x;y)v_j^g(y)\,dy\,dx \\ = \frac{k_1}{\omega^2\rho_1}\int_{D_m} W_m(x)S_m(x)\,dx \\ + \int_{D_m} W_m(x)\int_{D_m}\int_0^1 \nabla_y\mathcal{R}(x, z_m + t(y-z_m))\cdot(y-z_m)\,dt\,v_m^g(y)\,dy\,dx \\ - \frac{k_1}{\rho_1}\int_{D_m} W_m(x)\int_D G(x,y)v^g(y)n^2(y)\,dy\,dx,\end{aligned} \tag{4.22}$$

where $W_m(\cdot)$ is the solution of

$$\frac{k_1}{\omega^2\rho_1}W_m(x) - N_{D_m}(W_m)(x) = 1, \quad x \in D_m, \tag{4.23}$$

[8] In general, we can prove that

$$|\mathcal{R}(x,y)| \lesssim \left(\frac{1}{\mathrm{dist}(x,\partial\Omega)}\right)^{\mathfrak{q}}\left(\frac{1}{\mathrm{dist}(y,\partial\Omega)}\right)^{\mathfrak{p}}, \quad x, y \in \Omega,$$

where $\mathfrak{p}$ and $\mathfrak{q}$ are positive real numbers such that $\mathfrak{p} + \mathfrak{q} = 1$.

and $\beta_m \in \mathbb{C}$ is the constant given by

$$\beta_m := \left(1 - \int_{D_m} W_m(x)\mathcal{R}(x, z_m)\, dx\right). \tag{4.24}$$

We have

$$\begin{aligned}\left|\int_{D_m} W_m(x)\mathcal{R}(x, z_m)\, dx\right| &\leq \|W_m\|_{\mathbb{L}^2(D_m)} \|\mathcal{R}(\cdot, z_m)\|_{\mathbb{L}^2(D_m)} \\ &\underset{(4.78)}{\overset{(4.20)}{\lesssim}} \frac{a^{(1-h)}}{\operatorname{dist}(D_m; \partial\Omega)} \overset{(1.10)}{=} \mathcal{O}(a^{2(1-h)/3}).\end{aligned} \tag{4.25}$$

Hence,

$$\beta_m = 1 + \mathcal{O}(a^{2(1-h)/3}) \quad \text{for } m = 1, \dots, M. \tag{4.26}$$

Next, to derive the desired algebraic system, we expand in equation (4.22) Green's kernel $G(\cdot, \cdot)$ and the source term $S(\cdot)$, near the centres, to obtain

$$\beta_m \int_{D_m} v_m^g(x)\, dx - \alpha_m \sum_{\substack{j=1 \\ j \neq m}}^{M} G(z_m; z_j) \int_{D_j} v_j^g(x)\, dx = \frac{k_1}{\omega^2 \rho_1} \alpha_m S(z_m) + \mathrm{Rest}_m, \tag{4.27}$$

where α_m is the scattering coefficient given by

$$\alpha_m := \int_{D_m} W_m(x)\, dx, \tag{4.28}$$

and

$$\begin{aligned}\mathrm{Rest}_m :=& \sum_{\substack{j=1 \\ j \neq m}}^{M} \int_{D_m} W_m(x) \int_0^1 \nabla G(z_m + t(x - z_m); z_j) \cdot (x - z_m)\, dt\, dx \int_{D_j} v_j^g(y)\, dy \\ &+ \sum_{\substack{j=1 \\ j \neq m}}^{M} \int_{D_m} W_m(x) \int_{D_j} \int_0^1 \nabla G(x; z_j + t(y - z_j)) \cdot (y - z_j)\, dt\, v_j^g(y)\, dy\, dx \\ &+ \frac{k_1}{\omega^2 \rho_1} \int_{D_m} W_m(x) \int_0^1 \nabla S_m(z_m + t(x - z_m)) \cdot (x - z_m)\, dt\, dx \\ &+ \int_{D_m} W_m(x) \int_{D_m} \int_0^1 \nabla_y \mathcal{R}(x, z_m + t(y - z_m)) \cdot (y - z_m)\, dt\, v_m^g(y)\, dy\, dx \\ &- \frac{k_1}{\rho_1} \int_{D_m} W_m(x) \int_D G(x, y) v^g(y) n^2(y)\, dy\, dx.\end{aligned} \tag{4.29}$$

As D_j-s are translations and scales of the same domain B, i.e., $D_j = z_j + aB$ and $\rho_j = \rho_i$ with $k_j = k_i$ for $i, j = 1, \dots, M$,, we deduce that $\alpha_m = \alpha$, for $m = 1, \dots, M$. In addition, by multiplying its both sides by $\frac{\omega^2 \rho_1}{k_1 \alpha}$ and then, setting[9]

$$Y_m := \frac{\beta_m \omega^2 \rho_1}{\alpha k_1} \int_{D_m} v_m^g(x)\, dx \tag{4.30}$$

with $\alpha = -P^2 a^{1-h}$, we obtain

$$Y_m + \sum_{\substack{j=1 \\ j \neq m}}^{M} G(z_m; z_j) P^2 a^{1-h} \frac{1}{\beta_j} Y_j = S(z_m) + \frac{\omega^2 \rho_1}{k_1} \frac{\mathrm{Rest}_m}{\alpha}. \tag{4.31}$$

The next lemma ensures the invertibility of the previous algebraic system.

Lemma 4.3. *The algebraic system* (4.31) *is invertible from ℓ_2 to itself. In addition, the following estimation holds:*

$$\Big(\sum_{m=1}^{M} |Y_m|^2 \Big)^{1/2} \lesssim \Big(\sum_{m=1}^{M} |S(z_m)|^2 \Big)^{1/2} + a^{(h-3)} \Big(\sum_{m=1}^{M} |\mathrm{Rest}_m|^2 \Big)^{1/2}.$$

In particular,

$$\Big(\sum_{m=1}^{M} |Y_m|^2 \Big)^{1/2} \lesssim a^{2(h-1)/3} \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}. \tag{4.32}$$

Proof. See Section A.1. ∎

4.3. The LSE satisfied by $u^g(\cdot)$

We define $(\widetilde{Y}_1, \dots, \widetilde{Y}_M)$ as the solution of the unperturbed algebraic system related to (4.31). More precisely,

$$\widetilde{Y}_m + \sum_{\substack{j=1 \\ j \neq m}}^{M} G(z_m; z_j) P^2 a^{1-h} \widetilde{Y}_j \frac{1}{\beta_j} = S(z_m). \tag{4.33}$$

We set the following LSE:

$$Y(z) + P^2 \int_{\Omega} G(z; y) Y(y)\, dy = S(z), \quad z \in \Omega, \tag{4.34}$$

where $G(\cdot, \cdot)$ is the solution of (4.4) and $S(\cdot)$ is the solution of (4.16). We need the following lemma.

[9]In the equation $\alpha = -P^2 a^{1-h}$, the term a^{1-h} comes from the estimation of α, see Lemma 4.7.

Lemma 4.4. *There exists one and only one solution $Y(\cdot)$ of the LSE (4.34), and it satisfies the estimate*

$$\|Y\|_{\mathbb{H}^1(\Omega)} \lesssim P^2 \|S\|_{\mathbb{H}^1(\Omega)}. \tag{4.35}$$

Proof. Equation (4.34) is invertible from $\mathbb{L}^2(\Omega)$ to $\mathbb{L}^2(\Omega)$ and this gives us the estimation

$$\|Y\|_{\mathbb{L}^2(\Omega)} \lesssim \|S\|_{\mathbb{L}^2(\Omega)} \leq \|S\|_{\mathbb{H}^1(\Omega)}. \tag{4.36}$$

Now, by taking the $\mathbb{H}^1(\Omega)$-norm in both sides of (4.34), we get

$$\|Y\|_{\mathbb{H}^1(\Omega)} \lesssim \|S\|_{\mathbb{H}^1(\Omega)} + P^2 \|\mathcal{N}(Y)\|_{\mathbb{H}^1(\Omega)},$$

where $\mathcal{N}(\cdot)$ is the Newtonian operator defined by

$$\mathcal{N}(f)(x) := \int_{\Omega} G(x,y) f(y)\, dy, \quad x \in \Omega. \tag{4.37}$$

Then, using the continuity of the Newtonian operator, from $\mathbb{L}^2(\Omega)$ to $\mathbb{H}^1(\Omega)$, we obtain

$$\|Y\|_{\mathbb{H}^1(\Omega)} \lesssim \|S\|_{\mathbb{H}^1(\Omega)} + P^2 \|Y\|_{\mathbb{L}^2(\Omega)} \overset{(4.36)}{\lesssim} P^2 \|S\|_{\mathbb{H}^1(\Omega)}.$$

This ends the proof of Lemma 4.4. ■

Remark 4.5. The function $S(\cdot)$ (the solution of (4.16)) can be represented as a single layer potential with density function given by $g(\cdot)$, i.e.,

$$S(x) = \mathcal{S}(g)(x) := \int_{\partial\Omega} G(x,y) g(y) d\sigma(y), \quad x \in \Omega, \tag{4.38}$$

with $G(\cdot,\cdot)$ is the Green's kernel solution of (4.4). Then, from (4.35), we obtain

$$\|Y\|_{\mathbb{H}^1(\Omega)} \lesssim \|\mathcal{S}(g)\|_{\mathbb{H}^1(\Omega)},$$

and using the continuity of the single layer operator, from $\mathbb{H}^{-1/2}(\partial\Omega)$ to $\mathbb{H}^1(\Omega)$, we end up with the following estimation:

$$\|Y\|_{\mathbb{H}^1(\Omega)} \lesssim \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}. \tag{4.39}$$

Based on the introduced notations in Assumption 1, in particular (1.7) and the fact that $|\Omega_j| = a^{1-h}$, for $1 \leq j \leq M$ and $0 \leq h < 1$,[10] we can rewrite (4.34) as

$$Y(z_m) + P^2 \sum_{\substack{j=1 \\ j \neq m}}^{M} G(z_m; z_j) a^{1-h} \frac{1}{\beta_j} Y(z_j) = S(z_m) - P^2 \widetilde{\mathbf{I}}(z_m), \tag{4.40}$$

[10]We have $|\Omega_j| = a^{1-h}$ and $|\Omega_j^\star| \sim a^{1-h}$ but, as these $\Omega_j^\star$-s intersect $\partial\Omega$, we cannot necessarily replace $\sim$ with $=$.

where

$$\widetilde{\mathbf{I}}(z_m) := \int_{\Omega} G(z_m; y) Y(y)\, dy - \sum_{\substack{j=1 \\ j \neq m}}^{M} G(z_m; z_j) Y(z_j) \frac{1}{\beta_j} |\Omega_j|. \tag{4.41}$$

To estimate $\widetilde{\mathbf{I}}(z_m)$, we first consider the term

$$\int_{\Omega} G(z_m; y) Y(y)\, dy := \int_{\Omega_{\text{cube}}} G(z_m; y) Y(y)\, dy + \int_{\Omega_r} G(z_m; y) Y(y)\, dy. \tag{4.42}$$

and show that the second term is negligible. In fact, for the domains $\Omega_j^\star$, which are not necessarily cubes, they have the property of non-empty intersection with $\partial\Omega$, i.e., $\partial\Omega_n^\star \cap \partial\Omega \neq \{\emptyset\}$, for $1 \leq n \leq \aleph$. Since each Ω_j has volume equal to a^{1-h}, and then its maximum radius is of the order $a^{(1-h)/3}$, then intersecting surfaces with $\partial\Omega$ have a volume of the order $a^{2(1-h)/3}$. As the volume of $\partial\Omega$ is of order one, we conclude that the number of such cubes will not exceed the order $a^{-\frac{2}{3}(1-h)}$. Hence, the volume of Ω_r will not exceed the order $a^{(1-h)/3} \xrightarrow{a \to 0} 0$, i.e.,

$$|\Omega_r| = \mathcal{O}(a^{(1-h)/3}). \tag{4.43}$$

Regarding the second term on the right-hand side of (4.42), we have[11]

$$\begin{aligned} \left| \int_{\Omega_r} G(z_m; y) Y(y)\, dy \right| &\leq \|Y\|_{\mathbb{L}^2(\Omega_r)} \|G(z_m; \cdot)\|_{\mathbb{L}^2(\Omega_r)} \\ &\leq |\Omega_r|^{1/3} \|Y\|_{\mathbb{L}^6(\Omega_r)} \|G(z_m; \cdot)\|_{\mathbb{L}^2(\Omega_r)} \\ &\leq |\Omega_r|^{1/3} \|Y\|_{\mathbb{L}^6(\Omega)} \|G(z_m; \cdot)\|_{\mathbb{L}^2(\Omega_r)}. \end{aligned}$$

Thanks to Lemma 4.4, we know that $Y(\cdot) \in \mathbb{H}^1(\Omega) \subset \mathbb{L}^6(\Omega)$. Then,

$$\begin{aligned} \left| \int_{\Omega_r} G(z_m; y) Y(y)\, dy \right| &\lesssim |\Omega_r|^{1/3} \|Y\|_{\mathbb{H}^1(\Omega)} \|G(z_m; \cdot)\|_{\mathbb{L}^2(\Omega_r)} \\ &\overset{(4.35)}{\lesssim} P^2 |\Omega_r|^{1/3} \|S\|_{\mathbb{H}^1(\Omega)} \|G(z_m; \cdot)\|_{\mathbb{L}^2(\Omega_r)}, \end{aligned}$$

which, by using equation (4.38) and the continuity of the single-layer operator from $\mathbb{H}^{-1/2}(\partial\Omega)$ to $\mathbb{H}^1(\Omega)$, can be reduced to

$$\begin{aligned} \left| \int_{\Omega_r} G(z_m; y) Y(y)\, dy \right| &\lesssim P^2 |\Omega_r|^{1/3} \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \|G(z_m; \cdot)\|_{\mathbb{L}^2(\Omega_r)} \\ &\lesssim P^2 |\Omega_r|^{(9-5\delta)/(6(3-\delta))} \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \|G(z_m; \cdot)\|_{\mathbb{L}^{3-\delta}(\Omega_r)}. \end{aligned}$$

[11] We recall from Lemma 4.2 that $G(z_m; \cdot) \in \mathbb{L}^{3-\delta}(\Omega)$, with z_m fixed, where δ is an arbitrarily and sufficiently small positive number.

Next, using the fact that $G(z_m;\cdot) \in \mathbb{L}^{3-\delta}(\Omega_r)$, hence $\|G(z_m;\cdot)\|_{\mathbb{L}^{3-\delta}(\Omega_r)} \sim 1$, we reduce the previous estimation to

$$\begin{aligned}\left| \int_{\Omega_r} G(z_m;y) Y(y)\,dy \right| &\lesssim P^2 |\Omega_r|^{(9-5\delta)/(6(3-\delta))} \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \\ &\overset{(4.43)}{=} \mathcal{O}\big(P^2 \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)} a^{(1-h)(9-5\delta)/(18(3-\delta))}\big). \end{aligned} \tag{4.44}$$

Therefore, by gathering (4.41), (4.42), and the estimation (4.44), we deduce

$$\widetilde{\mathbf{I}}(z_m) = \mathbf{I}(z_m) + \mathcal{O}\big(P^2 \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)} a^{(1-h)(9-5\delta)/(18(3-\delta))}\big) \quad \text{as } a \ll 1, \tag{4.45}$$

where

$$\mathbf{I}(z_m) := \sum_{\ell=1}^{M} \int_{\Omega_\ell} G(z_m;y) Y(y)\,dy - \sum_{\substack{j=1 \\ j\neq m}}^{M} G(z_m;z_j) Y(z_j) \frac{1}{\beta_j} |\Omega_j|.$$

Let us now estimate $\mathbf{I}(z_m)$. We write

$$\begin{aligned}\mathbf{I}(z_m) &= \sum_{\substack{\ell=1 \\ \ell\neq m}}^{M} \int_{\Omega_\ell} \Big[G(z_m;y) Y(y) - G(z_m;z_\ell) Y(z_\ell) \frac{1}{\beta_\ell} \Big] dy + \int_{\Omega_m} G(z_m;y) Y(y)\,dy \\ &\overset{(4.24)}{=} \sum_{\substack{\ell=1 \\ \ell\neq m}}^{M} \int_{\Omega_\ell} [G(z_m;y) Y(y) - G(z_m;z_\ell) Y(z_\ell)]\,dy + \int_{\Omega_m} G(z_m;y) Y(y)\,dy \\ &\quad - \sum_{\substack{\ell=1 \\ \ell\neq m}}^{M} G(z_m;z_\ell) Y(z_\ell) |\Omega_\ell| \frac{\int_{D_\ell} W_\ell(x) \mathcal{R}(x,z_\ell)\,dx}{1 - \int_{D_\ell} W_\ell(x) \mathcal{R}(x,z_\ell)\,dx}, \end{aligned}$$

and, by using Taylor expansion for the function $G(z_m,\cdot)Y(\cdot)$ near the point z_ℓ, we get

$$\begin{aligned}\mathbf{I}(z_m) &= \sum_{\substack{\ell=1 \\ \ell\neq m}}^{M} \int_{\Omega_\ell} \int_0^1 G(z_m; z_\ell + t(y-z_\ell)) \nabla Y(z_\ell + t(y-z_\ell)) \cdot (y-z_\ell)\,dt\,dy \\ &\quad + \sum_{\substack{\ell=1 \\ \ell\neq m}}^{M} \int_{\Omega_\ell} \int_0^1 Y(z_\ell + t(y-z_\ell)) \nabla G(z_m; z_\ell + t(y-z_\ell)) \cdot (y-z_\ell)\,dt\,dy \\ &\quad + \int_{\Omega_m} G(z_m;y) Y(y)\,dy \\ &\quad - \sum_{\substack{\ell=1 \\ \ell\neq m}}^{M} G(z_m;z_\ell) Y(z_\ell) |\Omega_\ell| \frac{\int_{D_\ell} W_\ell(x) \mathcal{R}(x,z_\ell)\,dx}{1 - \int_{D_\ell} W_\ell(x) \mathcal{R}(x,z_\ell)\,dx}. \end{aligned} \tag{4.46}$$

From Lemma 4.2, we know that $G(x,y) = \Phi_0(x,y) + \mathcal{R}(x,y)$, for $x \neq y$, where the dominant part $\Phi_0(\cdot,\cdot)$ is given by (1.5). In the sequel, we keep only the dominant part of $G(\cdot,\cdot)$. More precisely, we have

$$|G(x,y)| \lesssim \frac{1}{|x-y|} \quad \text{and} \quad |\nabla G(x,y)| \lesssim \frac{1}{|x-y|^2}, \quad \text{for } x \neq y. \tag{4.47}$$

By taking the modulus in both sides of (4.46) and using (4.47), we deduce

$$\begin{aligned} |\mathbf{I}(z_m)| &\lesssim a^{(1-h)/3} \sum_{\substack{\ell=1 \\ \ell \neq m}}^{M} \int_{\Omega_\ell} \int_0^1 \frac{1}{|z_m - z_\ell - t(y-z_\ell)|} |\nabla Y(z_\ell + t(y - z_\ell))| \, dt \, dy \\ &+ a^{(1-h)/3} \sum_{\substack{\ell=1 \\ \ell \neq m}}^{M} \int_{\Omega_\ell} \int_0^1 \frac{|Y(z_\ell + t(y-z_\ell))|}{|z_m - z_\ell - t(y-z_\ell)|^2} \, dt \, dy \\ &+ \int_{\Omega_m} |G(z_m; y)||Y(y)| \, dy + |\mathbf{L}(z_m)|, \end{aligned} \tag{4.48}$$

where $\mathbf{L}(z_m)$ is the term given by

$$\mathbf{L}(z_m) := \sum_{\substack{\ell=1 \\ \ell \neq m}}^{M} G(z_m; z_\ell) Y(z_\ell) |\Omega_\ell| \frac{\int_{D_\ell} W_\ell(x) \mathcal{R}(x, z_\ell) \, dx}{1 - \int_{D_\ell} W_\ell(x) \mathcal{R}(x, z_\ell) \, dx}. \tag{4.49}$$

Next, we delay the estimation of $\mathbf{I}(z_m)$ until we estimate first $\mathbf{L}(z_m)$. To do this, by taking the absolute value on both sides of (4.49), using the fact that $|\Omega_\ell| = a^{1-h}$, the estimation (4.47), with the estimation given by (4.25), gives us

$$\begin{aligned} |\mathbf{L}(z_m)| &\lesssim a^{5(1-h)/3} \sum_{\substack{\ell=1 \\ \ell \neq m}}^{M} \frac{1}{|z_m - z_\ell|} |Y(z_\ell)| \\ &\leq a^{5(1-h)/3} \Big(\sum_{\substack{\ell=1 \\ \ell \neq m}}^{M} \frac{1}{|z_m - z_\ell|^2} \Big)^{1/2} \Big(\sum_{\substack{\ell=1 \\ \ell \neq m}}^{M} |Y(z_\ell)|^2 \Big)^{1/2}. \end{aligned}$$

Besides, by applying a Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} &|\mathbf{L}(z_m)| \\ &\leq a^{5(1-h)/3} \Big(\sum_{\substack{\ell=1 \\ \ell \neq m}}^{M} \frac{1}{|z_m - z_\ell|^2} \Big)^{1/2} \Big[\Big(\sum_{\ell=1}^{M} |Y(z_\ell) - \widetilde{Y}_\ell|^2 \Big)^{1/2} + \Big(\sum_{\ell=1}^{M} |\widetilde{Y}_\ell|^2 \Big)^{1/2} \Big], \end{aligned}$$

where $(\widetilde{Y}_1; \cdots ; \widetilde{Y}_M)$ is the solution of (4.33). In addition, by using (4.32), it can be reduced to

$$|\mathbf{L}(z_m)| \lesssim \Big(\sum_{\substack{\ell=1 \\ \ell \neq m}}^{M} \frac{1}{|z_m - z_\ell|^2} \Big)^{1/2} \Big[a^{5(1-h)/3} \Big(\sum_{\ell=1}^{M} |Y(z_\ell) - \widetilde{Y}_\ell|^2 \Big)^{1/2} + a^{7(1-h)/3} \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \Big]. \tag{4.50}$$

Now, by making again use of the Cauchy–Schwarz inequality in (4.48) and using (4.50), we obtain

$$\begin{aligned} |\mathbf{I}(z_m)| &\lesssim a^{5(1-h)/6} \|\nabla Y\|_{\mathbb{L}^2(\Omega)} \Big(\sum_{\substack{\ell=1 \\ \ell \neq m}}^{M} \frac{1}{|z_m - z_\ell|^2} \Big)^{1/2} \\ &+ a^{5(1-h)/6} \|Y\|_{\mathbb{L}^2(\Omega)} \Big(\sum_{\substack{\ell=1 \\ \ell \neq m}}^{M} \frac{1}{|z_m - z_\ell|^4} \Big)^{1/2} \\ &+ \|Y\|_{\mathbb{L}^2(\Omega)} \Big(\int_{\Omega_m} \frac{1}{|z_m - y|^2} \, dy \Big)^{1/2} \\ &+ a^{5(1-h)/3} \Big(\sum_{\substack{\ell=1 \\ \ell \neq m}}^{M} \frac{1}{|z_m - z_\ell|^2} \Big)^{1/2} \Big(\sum_{\ell=1}^{M} |Y(z_\ell) - \widetilde{Y}_\ell|^2 \Big)^{1/2} \\ &+ a^{7(1-h)/3} \Big(\sum_{\substack{\ell=1 \\ \ell \neq m}}^{M} \frac{1}{|z_m - z_\ell|^2} \Big)^{1/2} \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}. \end{aligned}$$

The use of the lemma below allows for the estimation of the discrete sum of the inverse power-weighted distance between the centre of the droplets, and the discrete sum of the inverse power-weighted distance at the boundary.

Lemma 4.6. *Let $\{D_m = z_m + a B\}_{m=1}^{M} \subset \Omega$. Then, we have the following estimates.*

(1) *Inverse distance between the droplets centres,*

$$\sum_{\substack{j=1 \\ j \neq m}}^{M} |z_m - z_j|^{-k} = \begin{cases} \mathcal{O}(d^{-3}) & \text{for } k < 3, \\ \mathcal{O}(d^{-k}) & \text{for } k > 3. \end{cases} \tag{4.51}$$

(2) *Inverse distance to the boundary,*

$$\sum_{j=1}^{M} \frac{1}{\operatorname{dist}^k(D_j; \partial\Omega)} = \begin{cases} \mathcal{O}(d^{-3}) & \text{for } k < 3, \\ \mathcal{O}(d^{-k}) & \text{for } k > 3. \end{cases} \tag{4.52}$$

Proof. See Section A.7. ■

Thanks to (4.51) and the estimation (1.12), we have

$$\begin{aligned}
|\mathbf{I}(z_m)| \lesssim\ & a^{5(1-h)/6} \|\nabla Y\|_{\mathbb{L}^2(\Omega)} d^{-\frac{3}{2}} + a^{5(1-h)/6} \|Y\|_{\mathbb{L}^2(\Omega)} d^{-2} \\
& + \|Y\|_{\mathbb{L}^2(\Omega)} \Big(\int\limits_{\Omega_m} \frac{1}{|z_m - y|^2}\, dy \Big)^{1/2} \\
& + a^{7(1-h)/6} \Big(\sum_{\ell=1}^{M} |Y(z_\ell) - \widetilde{Y}_\ell|^2 \Big)^{1/2} + a^{11(1-h)/6} \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}. \qquad (4.53)
\end{aligned}$$

To achieve the estimation of $\mathbf{I}(z_m)$, we set and estimate the third term appearing on the right-hand side of the above equation

$$\mathbf{I}_3(z_m) := \int\limits_{B(z_m;r)} \frac{1}{|z_m - y|^2}\, dy + \int\limits_{\Omega_m \setminus B(z_m;r)} \frac{1}{|z_m - y|^2}\, dy,$$

where $B(z_m; r)$ is the ball of centre z_m and radius r, where r is such that $r \in \mathbf{I}_4 := \big[0; \frac{\sqrt{3}}{2} a^{(1-h)/3}\big]$. Then,

$$\begin{aligned}
\mathbf{I}_3(z_m) &\lesssim \int_0^r \int\limits_{\partial B(z_m;s)} \frac{1}{|z_m - y|^2} d\sigma(y) ds + \frac{1}{r^2} |\Omega_m \setminus B(z_m; r)| \\
&= \int_0^r \frac{1}{s^2} |\partial B(z_m; s)| ds + \frac{1}{r^2} |\Omega_m \setminus B(z_m; r)| \\
&= \frac{8\pi r}{3} + \frac{1}{r^2} a^{1-h} \leq \max_{r \in \mathbf{I}_4} \rho(r, a),
\end{aligned}$$

where

$$\rho(r, a) := \frac{8\pi}{3} r + \frac{1}{r^2} a^{1-h}.$$

We have that $\max_{r\in\mathbf{I}_4} \rho(r, a) = \rho(r_{\mathrm{sol}}, a)$, where r_{sol} is such that $\partial_r \rho(r_{\mathrm{sol}}, a) = 0$. Straightforward computations gives us $r_{\mathrm{sol}} = \big(\frac{3}{4\pi} a^{1-h}\big)^{1/3}$. Consequently,

$$\max_{r \in \mathbf{I}_4} \rho(r, a) = (48\pi^2)^{1/3} a^{(1-h)/3}.$$

Hence,

$$\mathbf{I}_3(z_m) = \mathcal{O}(a^{(1-h)/3}). \qquad (4.54)$$

Finally, by gathering (4.54) and (4.53) and using the fact that $d \sim a^{1-h/3}$, we end up with

$$\begin{aligned}|\mathbf{I}(z_m)| &\lesssim a^{(1-h)/3}\|\nabla Y\|_{\mathbb{L}^2(\Omega)} + a^{(1-h)/6}\|Y\|_{\mathbb{L}^2(\Omega)} \\ &\quad + a^{7(1-h)/6}\Big(\sum_{\ell=1}^{M}|Y(z_\ell) - \widetilde{Y}_\ell|^2\Big)^{1/2} + a^{11(1-h)/6}\|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \\ &\overset{(4.39)}{\lesssim} a^{(1-h)/6}\|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)} + a^{7(1-h)/6}\Big(\sum_{\ell=1}^{M}|Y(z_\ell) - \widetilde{Y}_\ell|^2\Big)^{1/2}.\end{aligned}$$

Hence, plugging the above estimation into (4.45), we get

$$\begin{aligned}\widetilde{\mathbf{I}}(z_m) = \mathcal{O}\Big(& P^2 a^{(9-5\delta)(1-h)/(18(3-\delta))}\|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \\ &+ a^{7(1-h)/6}\Big(\sum_{\ell=1}^{M}|Y(z_\ell) - \widetilde{Y}_\ell|^2\Big)^{1/2}\Big).\end{aligned} \tag{4.55}$$

Finally, by going back to (4.40) and making use of the estimation (4.55), we obtain

$$\begin{aligned}& Y(z_m) - \alpha \sum_{\substack{j=1 \\ j\neq m}}^{M} G(z_m; z_j)\frac{1}{\beta_j} Y(z_j) \\ &\quad = S(z_m) + \mathcal{O}(P^4 a^{(9-5\delta)(1-h)/18(3-\delta)}\|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}) \\ &\qquad + \mathcal{O}\Big(a^{7(1-h)/6} P^2\Big(\sum_{\ell=1}^{M}|Y(z_\ell) - \widetilde{Y}_\ell|^2\Big)^{1/2}\Big).\end{aligned} \tag{4.56}$$

Taking the difference between (4.33) and (4.56) gives us the following algebraic system:

$$\begin{aligned}& (\widetilde{Y}_m - Y(z_m)) + \sum_{\substack{j=1 \\ j\neq m}}^{M} G(z_m; z_j) P^2 a^{1-h}\frac{1}{\beta_j}(\widetilde{Y}_j - Y(z_j)) \\ &\quad = \mathcal{O}(P^4 a^{(9-5\delta)(1-h)/18(3-\delta)}\|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}) \\ &\qquad + \mathcal{O}\Big(a^{7(1-h)/6} P^2\Big(\sum_{\ell=1}^{M}\|Y(z_\ell) - \widetilde{Y}_\ell\|^2\Big)^{1/2}\Big).\end{aligned}$$

Consequently, using Lemma 4.3, we have

$$\begin{aligned}\Big(\sum_{m=1}^{M}|\widetilde{Y}_m - Y(z_m)|^2\Big)^{1/2} &\lesssim P^4 a^{-(1-h)(9-2\delta)/(9(3-\delta))}\|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \\ &\quad + a^{2(1-h)/3} P^2\Big(\sum_{\ell=1}^{M}|Y(z_\ell) - \widetilde{Y}_\ell|^2\Big)^{1/2},\end{aligned}$$

which, by knowing that $h < 1$, can be reduced to

$$\Big(\sum_{m=1}^{M}|\widetilde{Y}_m - Y(z_m)|^2\Big)^{1/2} = \mathcal{O}(P^4 a^{-(1-h)(9-2\delta)/(9(3-\delta))}\|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}). \tag{4.57}$$

The previous estimation confirm the convergence of the discrete algebraic system to the continuous LSE.

4.4. Finishing the proof of Theorem 1.1

We define $\mathbf{J}$ as

$$\begin{aligned}\mathbf{J} &:= \omega^2\frac{\rho_1}{k_1}\langle v^g; p^f\rangle_{\mathbb{L}^2(D)} + P^2\langle u^g; p^f\rangle_{\mathbb{L}^2(\Omega)} - \omega^2\langle n^2 v^g; p^f\rangle_{\mathbb{L}^2(D)} \\ &= \sum_{j=1}^{M}\Big[\omega^2\frac{\rho_1}{k_1}\langle v_j^g; p^f\rangle_{\mathbb{L}^2(D_j)} + P^2\langle u_j^g; p^f\rangle_{\mathbb{L}^2(\Omega_j)}\Big] - \omega^2\langle n^2 v^g; p^f\rangle_{\mathbb{L}^2(D)}.\end{aligned} \tag{4.58}$$

Then, by using the Taylor expansion for the function $p^f(\cdot)$ near the centres, we get

$$\mathbf{J} = \sum_{j=1}^{M} p^f(z_j)\Big[\omega^2\frac{\rho_1}{k_1}\int_{D_j} v_j^g(x)\,dx + P^2\int_{\Omega_j} u_j^g(x)\,dx\Big] + \mathbf{Err}_j, \tag{4.59}$$

where

$$\begin{aligned}\mathbf{Err}_j &:= \omega^2\frac{\rho_1}{k_1}\sum_{j=1}^{M}\int_{D_j} v_j^g(x)\int_0^1 \nabla p^f(z_j + t(x-z_j))\cdot(x-z_j)\,dt\,dx \\ &\quad + P^2\sum_{j=1}^{M}\int_{\Omega_j} u_j^g(x)\int_0^1 \nabla p^f(z_j + t(x-z_j))\cdot(x-z_j)\,dt\,dx \\ &\quad - \omega^2\langle n^2 v^g; p^f\rangle_{\mathbb{L}^2(D)},\end{aligned}$$

which can be estimated, by recalling that $\rho_1 \sim 1; k_1 \sim a^2$ and $|\Omega_j| \sim a^{1-h}$, as

$$\begin{aligned}\|\mathbf{Err}_j\| &\lesssim a^{-1}\sum_{j=1}^{M}\|v_j^g\|_{\mathbb{L}^2(D_j)}\|\nabla p^f\|_{\mathbb{L}^2(D_j)} \\ &\quad + a^{(1-h)/3}P^2\sum_{j=1}^{M}\|u_j^g\|_{\mathbb{L}^2(\Omega_j)}\|\nabla p^f\|_{\mathbb{L}^2(\Omega_j)} \\ &\quad + \|v^g\|_{\mathbb{L}^2(D)}\|p^f\|_{\mathbb{L}^2(D)}\end{aligned}$$

$$
\begin{aligned}
&\lesssim a^{-1}\|v^g\|_{\mathbb{L}^2(D)}\|\nabla p^f\|_{\mathbb{L}^2(D)} + a^{(1-h)/3}P^2\|u^g\|_{\mathbb{L}^2(\Omega)}\|\nabla p^f\|_{\mathbb{L}^2(\Omega)} \\
&\quad + \|v^g\|_{\mathbb{L}^2(D)}\|p^f\|_{\mathbb{L}^2(D)} \\
&\underset{(4.9)}{\overset{(4.11)}{\lesssim}} a^{(3h-2)/2}\|v^g\|_{\mathbb{L}^2(D)}\|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \\
&\quad + a^{(1-h)/3}P^2\|u^g\|_{\mathbb{L}^2(\Omega)}\|\nabla p^f\|_{\mathbb{L}^2(\Omega)}. \qquad (4.60)
\end{aligned}
$$

Next, we estimate $\|\nabla p^f\|_{\mathbb{L}^2(\Omega)}$. To do this, we recall from (4.10) that $\nabla p^f = \nabla \mathcal{S}(f)$, in Ω, where $\mathcal{S}(\cdot)$ is the single-layer operator defined by (4.38). Hence, as

$$
\mathcal{S}(\cdot)\colon \mathbb{H}^{-1/2}(\partial\Omega) \to \mathbb{H}^1(\Omega),
$$

we have

$$
\|\nabla p^f\|_{\mathbb{L}^2(\Omega)} = \mathcal{O}(\|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)}). \qquad (4.61)
$$

Then, by using (4.13) and (4.61) into (4.60), we have

$$
|\mathbf{Err}_j| \lesssim \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)}[\|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}a^{(7h-1)/6} + a^{(1-h)/3}P^2\|u^g\|_{\mathbb{L}^2(\Omega)}]. \qquad (4.62)
$$

Let us now estimate $u^g(\cdot)$ in terms of $g(\cdot)$. As $u^g(\cdot)$ is the solution of (1.16), then it satisfies the following integral equation:

$$
u^g(\cdot) + P^2\mathcal{N}(u^g)(\cdot) = \mathcal{S}(g)(\cdot) \quad \text{in } \Omega, \qquad (4.63)
$$

where $\mathcal{N}(\cdot)$ is the Newtonian operator defined by (4.37). Hence, by gathering (4.34), (4.63), and (4.39), we deduce that

$$
\|u^g\|_{\mathbb{L}^2(\Omega)} \lesssim \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}. \qquad (4.64)
$$

Then, by plugging (4.64) into (4.62), and using the fact that $h > \frac{1}{3}$, we deduce that

$$
|\mathbf{Err}_j| \lesssim a^{(1-h)/3}P^2\|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}\|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)}.
$$

Going back to (4.59), we obtain

$$
\begin{aligned}
\mathbf{J} &= \sum_{j=1}^{M} p^f(z_j)\Big[\omega^2\frac{\rho_1}{k_1}\int_{D_j} v_j^g(x)\,dx + P^2\int_{\Omega_j} u_j^g(x)\,dx\Big] \\
&\quad + \mathcal{O}(a^{(1-h)/3}P^2\|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)}\|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}).
\end{aligned}
$$

Remark that $u^g(\cdot)$ (the solution of (1.16)), is the solution of the LSE given by (4.34), i.e., $u^g(\cdot) = Y(\cdot)$, in Ω. Using this, we obtain

$$
\int_{\Omega_j} u_j^g(x)\,dx = \int_{\Omega_j} Y(x)\,dx = Y(z_j)|\Omega_j| + \int_{\Omega_j}\int_0^1 (x-z_j)\cdot\nabla Y(z_j + t(x-z_j))\,dt\,dx.
$$

Then,

$$\mathbf{J} = \sum_{j=1}^{M} p^f(z_j)\Big[\omega^2 \frac{\rho_1}{k_1} \int_{D_j} v_j^f(x)\,dx + P^2 Y(z_j)|\Omega_j|\Big] + P^2 \sum_{j=1}^{M} p^f(z_j) \int_{\Omega_j} \int_0^1 (x-z_j)\cdot\nabla Y(z_j + t(x-z_j))\,dt\,dx + \mathcal{O}(a^{(1-h)/3} P^2 \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}). \tag{4.65}$$

We estimate the second term on the right-hand side as

$$T_2 := P^2 \sum_{j=1}^{M} p^f(z_j) \int_{\Omega_j} \int_0^1 (x-z_j)\cdot\nabla Y(z_j + t(x-z_j))\,dt\,dx$$

$$\begin{aligned}|T_2| &\lesssim P^2 \sum_{j=1}^{M} |p^f(z_j)| \Big| \int_{\Omega_j} \int_0^1 (x-z_j)\cdot\nabla Y(z_j + t(x-z_j))\,dt\,dx \Big| \\ &\leq P^2 \Big(\sum_{j=1}^{M} |p^f(z_j)|^2\Big)^{1/2} \Big(\sum_{j=1}^{M} \Big|\int_{\Omega_j} \int_0^1 (x-z_j)\cdot\nabla Y(z_j + t(x-z_j))\,dt\,dx\Big|^2\Big)^{1/2} \\ &= \mathcal{O}\Big(P^2 \Big(\sum_{j=1}^{M} |p^f(z_j)|^2\Big)^{1/2} a^{5(1-h)/6} \|\nabla Y\|_{\mathbb{L}^2(\Omega)}\Big).\end{aligned} \tag{4.66}$$

At this stage, we need first to estimate $\sum_{j=1}^{M} |p^f(z_j)|^2$. To achieve this, we recall that $p^f(\cdot)$ is the solution of (1.1) and we introduce $\tilde{p}^f(\cdot)$ as the solution of

$$\begin{cases} (\Delta + \omega^2)\tilde{p}^f = 0 & \text{in } \Omega, \\ \partial_\nu \tilde{p}^f = f & \text{on } \partial\Omega. \end{cases} \tag{4.67}$$

Now, by subtracting (1.1) from (4.67), we get

$$\begin{cases} (\Delta + \omega^2 n^2)(p^f - \tilde{p}^f) = \omega^2(-n^2+1)\tilde{p}^f & \text{in } \Omega, \\ \partial_\nu (p^f - \tilde{p}^f) = 0 & \text{on } \partial\Omega. \end{cases}$$

Its solution takes the following form:

$$(p^f - \tilde{p}^f)(z) = \omega^2 \int_\Omega G(z,y)(-n^2+1)(y)\tilde{p}^f(y)\,dy, \quad z \in \Omega,$$

where $G(\cdot,\cdot)$ is the Green's kernel solution of (4.4). By taking the modulus, we get

$$|(p^f - \tilde{p}^f)(z)| \leq \|\omega^2(-n^2+1)\|_{\mathbb{L}^\infty(\Omega)} \|G(z,\cdot)\|_{\mathbb{L}^2(\Omega)} \|\tilde{p}^f\|_{\mathbb{L}^2(\Omega)} \lesssim \|\tilde{p}^f\|_{\mathbb{L}^2(\Omega)}, \tag{4.68}$$

where the last estimation is a consequence of the $\mathbb{L}^2(\Omega)$-integrability of Green's kernel $G(z,\cdot)$, uniformly on $z \in \Omega$, and the boundedness of $\|n^2\|_{\mathbb{L}^\infty(\Omega)}$. In addition, we use the fact that (4.67) is a well-posed problem to derive

$$\|\tilde{p}^f\|_{\mathbb{L}^2(\Omega)} \lesssim \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)}. \tag{4.69}$$

Then, by gathering (4.68) and (4.69), we obtain

$$|(p^f - \tilde{p}^f)(z)| \lesssim \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)}. \tag{4.70}$$

As $\tilde{p}^f(\cdot)$ satisfies a Helmholtz equation in Ω, see (4.67), then we have the following mean value integral formula:

$$\frac{\sin(\omega r')}{\omega r'}\tilde{p}^f(z_j) = \frac{1}{|\partial\mathcal{B}_j(r')|}\int\limits_{\partial\mathcal{B}_j(r')} \tilde{p}^f(x)d\sigma(x), \tag{4.71}$$

where $\mathcal{B}_j(r')$ is the ball, centreed at z_j with radius r', contained in the cube Ω_j. See [18, formula (36), p. 288]. Now, by integrating both sides of (4.71) with respect to the variable r', from 0 to r, where r is such that $\mathcal{B}_j := \mathcal{B}_j(r)$ is the largest ball, centreed at z_j with radius r', contained in the cube Ω_j, we obtain

$$\tilde{p}^f(z_j) = \frac{\omega^3}{4\pi(\sin(\omega r) - \omega r\cos(\omega r))}\int\limits_{\mathcal{B}_j} \tilde{p}^f(x)\,dx. \tag{4.72}$$

In addition, because r is small, the following approximation holds:

$$4\pi(\sin(\omega r) - \omega r\cos(\omega r)) = \frac{4\pi}{3}\omega^3 r^3 + \mathcal{O}(r^5) = \omega^3|\mathcal{B}_j| + \mathcal{O}(r^5). \tag{4.73}$$

Then, by plugging (4.73) into (4.72), we obtain

$$\tilde{p}^f(z_j) = \frac{1}{|\mathcal{B}_j| + \mathcal{O}(r^5)}\int\limits_{\mathcal{B}_j} \tilde{p}^f(x)\,dx.$$

We observe that, for $1 \leq j \leq M$, we have $|\mathcal{B}_j| = |\mathcal{B}_{j_0}| \sim a^{1-h} \sim M^{-1}$. Then, using the Cauchy–Schwarz inequality, in the above formula, we deduce that

$$|\tilde{p}^f(z_j)| \lesssim |\mathcal{B}_j|^{-1/2}\|\tilde{p}^f\|_{\mathbb{L}^2(\mathcal{B}_j)}. \tag{4.74}$$

Therefore,

$$\sum_{j=1}^{M}|p^f(z_j)|^2 = \sum_{j=1}^{M}|\tilde{p}^f(z_j) + (p^f(z_j) - \tilde{p}^f(z_j))|^2$$

$$\lesssim \sum_{j=1}^{M} |\tilde{p}^f(z_j)|^2 + \sum_{j=1}^{M} |p^f(z_j) - \tilde{p}^f(z_j)|^2.$$

By making use of (4.74) and (4.70), we obtain

$$\begin{aligned}\sum_{j=1}^{M} |p^f(z_j)|^2 &\lesssim \sum_{j=1}^{M} |\mathcal{B}_j|^{-1} \|\tilde{p}^f\|^2_{\mathbb{L}^2(\mathcal{B}_j)} + M\|f\|^2_{\mathbb{H}^{-1/2}(\partial\Omega)} \\ &\lesssim |\mathcal{B}_{j_0}|^{-1} \|\tilde{p}^f\|^2_{\mathbb{L}^2(\bigcup_{j=1}^{M}\mathcal{B}_j)} + M\|f\|^2_{\mathbb{H}^{-1/2}(\partial\Omega)}.\end{aligned}$$

As $|\mathcal{B}_{j_0}|^{-1} \sim M$ and $\bigcup_{j=1}^{M} \mathcal{B}_j \subset \Omega$, we obtain

$$\sum_{j=1}^{M} |p^f(z_j)|^2 \lesssim M(\|\tilde{p}^f\|^2_{\mathbb{L}^2(\Omega)} + \|f\|^2_{\mathbb{H}^{-1/2}(\partial\Omega)}) \overset{(4.69)}{\lesssim} M\|f\|^2_{\mathbb{H}^{-1/2}(\partial\Omega)}. \tag{4.75}$$

We continue with our estimation of (4.66) by using (4.75) to get

$$\begin{aligned}\|T_2\| &\lesssim P^2 M^{1/2} \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)} a^{5(1-h)/6} \|\nabla Y\|_{\mathbb{L}^2(\Omega)} \\ &\overset{(4.39)}{=} \mathcal{O}(P^2 a^{(1-h)/3} \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}).\end{aligned}$$

Then, using the above estimation of the term $\mathbf{J}$, given by (4.65), becomes

$$\begin{aligned}\mathbf{J} = &\sum_{j=1}^{M} p^f(z_j) \left[\omega^2 \frac{\rho_1}{k_1} \int_{D_j} v_j^g(x)\,dx + P^2 Y(z_j)|\Omega_j|\right] \\ &+ \mathcal{O}(a^{(1-h)/3} P^2 \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}).\end{aligned} \tag{4.76}$$

To see how the parameter P^2 is related to the scattering coefficient α, we set the following lemma.

Lemma 4.7. *The scattering coefficient α, given by (4.28), admits the following estimation*

$$\alpha = -P^2 a^{1-h} + \mathcal{O}(a), \tag{4.77}$$

where $0 < h < 1$, and

$$P^2 := \frac{-k_0(\langle 1; \bar{e}_{n_0}\rangle_{\mathbb{L}^2(B)})^2}{\lambda^B_{n_0} c_{n_0}}.$$

In addition, the following estimation holds:

$$\|W_m\|_{\mathbb{L}^2(D_m)} \lesssim a^{-(2+h)} \|1\|_{\mathbb{L}^2(D_m)}, \tag{4.78}$$

where $W_m(\cdot)$ is the solution of (4.23).

Proof. See Section A.5. ∎

Knowing that $|\Omega_j| = a^{1-h}$ and using (4.77), we deduce that $P^2 Y(z_j)|\Omega_j| = -\alpha Y(z_j)$. In addition, from (4.30), we have

$$\frac{\omega^2 \rho_1 \beta_j}{k_1} \int\limits_{D_j} v_j^g(x)\, dx = \alpha Y_j.$$

Hence, equation (4.76) becomes

$$\mathbf{J} = \sum_{j=1}^{M} p^f(z_j)\alpha\Big[\frac{1}{\beta_j} Y_j - Y(z_j)\Big] + \mathcal{O}(a^{(1-h)/3} P^2 \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)})$$

$$\mathbf{J} = \sum_{j=1}^{M} p^f(z_j)\alpha[\widetilde{Y}_j - Y(z_j)] + \sum_{j=1}^{M} p^f(z_j)\alpha\Big[\frac{1}{\beta_j} Y_j - \widetilde{Y}_j\Big] + \mathcal{O}(a^{(1-h)/3} P^2 \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}). \tag{4.79}$$

Next, we set and estimate the second term on the right-hand side. To do this, we have

$$\mathbf{Q} := \sum_{j=1}^{M} p^f(z_j)\alpha\Big[\frac{1}{\beta_j} Y_j - \widetilde{Y}_j\Big] \overset{(4.24)}{=} \sum_{j=1}^{M} p^f(z_j)\alpha[Y_j - \widetilde{Y}_j] + \sum_{j=1}^{M} p^f(z_j)\alpha \frac{\int_{D_j} W_j(x)\mathcal{R}(x, z_j)\, dx}{\beta_j} Y_j.$$

Then, using (4.77), (4.25), and (4.26), we obtain

$$\begin{aligned} |\mathbf{Q}| &\lesssim P^2 a^{1-h}\Big(\sum_{j=1}^{M} |p^f(z_j)|^2\Big)^{1/2}\Big[\Big(\sum_{j=1}^{M} |Y_j - \widetilde{Y}_j|^2\Big)^{1/2} + a^{2(1-h)/3}\Big(\sum_{j=1}^{M} |Y_j|^2\Big)^{1/2}\Big] \\ &\overset{(4.75)}{\lesssim} P^2 a^{1-h} M^{1/2} \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)}\Big[\Big(\sum_{j=1}^{M} |Y_j - \widetilde{Y}_j|^2\Big)^{1/2} + a^{2(1-h)/3}\Big(\sum_{j=1}^{M} |Y_j|^2\Big)^{1/2}\Big] \\ &\overset{(4.32)}{\lesssim} P^2 a^{1-h} M^{1/2} \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)}\Big[\Big(\sum_{j=1}^{M} |Y_j - \widetilde{Y}_j|^2\Big)^{1/2} + a^{4(1-h)/3} \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}\Big]. \end{aligned} \tag{4.80}$$

In addition, by subtracting (4.31) from (4.33), we derive the following algebraic system

$$(Y_m - \widetilde{Y}_m) + \sum_{\substack{j=1 \\ j \neq m}}^{M} G(z_m; z_j) P^2 a^{1-h} \frac{1}{\beta_j} (Y_j - \widetilde{Y}_j) = \frac{\omega^2 \rho_1}{k_1} \frac{\mathrm{Rest}_m}{\alpha}.$$

Then, thanks to Lemma 4.3, the fact that $k_1 \sim a^2$ and $\alpha \sim a^{1-h}$, the following estimation holds:

$$\Big(\sum_{j=1}^{M} |Y_j - \widetilde{Y}_j|^2\Big)^{1/2} \lesssim a^{h-3}\Big(\sum_{j=1}^{M} |\mathrm{Rest}_j|^2\Big)^{1/2} \overset{(A.43)}{\lesssim} a^{3+h}\|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}.$$

Hence, by plugging the above estimation into (4.80) and using the fact that $M \sim a^{h-1}$, we obtain

$$|\mathbf{Q}| \lesssim P^2 a^{11(1-h)/6} \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}.$$

Taking the modulus in both sides of (4.79), using the above estimation, we get

$$\begin{aligned}
|\mathbf{J}| &\lesssim |\alpha| \Big(\sum_{j=1}^{M} |\widetilde{Y}_j - Y(z_j)|^2\Big)^{1/2} \Big(\sum_{j=1}^{M} |p^f(z_j)|^2\Big)^{1/2} \\
&\quad + a^{(1-h)/3} P^2 \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \\
&\overset{(4.75)}{\lesssim} |\alpha| M^{1/2} \Big(\sum_{j=1}^{M} |\widetilde{Y}_j - Y(z_j)|^2\Big)^{1/2} \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \\
&\quad + a^{(1-h)/3} P^2 \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}.
\end{aligned}$$

Noticing that $M \sim a^{h-1}$, using the fact that $\alpha \sim P^2 a^{1-h}$, see (4.77), and taking into account the estimation derived in (4.57), we obtain

$$\begin{aligned}
|\mathbf{J}| &\lesssim a^{(1-h)(9-5\delta)/(18(3-\delta))} P^6 \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)} && (4.81)\\
&\quad + a^{(1-h)/3} P^2 \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \\
&= \mathcal{O}\big(a^{(1-h)(9-5\delta)/(18(3-\delta))} P^6 \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}\big). && (4.82)
\end{aligned}$$

Now, by gathering (1.19), (4.58), and the estimation (4.81), we obtain

$$\begin{aligned}
&|\langle (\Lambda_D - \Lambda_P)(f); g\rangle_{\mathbb{H}^{1/2}(\partial\Omega)\times\mathbb{H}^{-1/2}(\partial\Omega)}| \\
&\quad = |\mathbf{J}| \overset{(4.81)}{\lesssim} a^{(1-h)(9-5\delta)/(18(3-\delta))} P^6 \|f\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}.
\end{aligned}$$

This suggest,

$$\|\Lambda_D - \Lambda_P\|_{\mathcal{L}(\mathbb{H}^{-1/2}(\partial\Omega);\mathbb{H}^{1/2}(\partial\Omega))} \lesssim a^{(1-h)(9-5\delta)/(18(3-\delta))} P^6.$$

This proves (1.20) and ends the proof of Theorem 1.1.

A. Appendix – Proofs of auxiliary results

This section is organised as follows. We start by proving Lemma 4.3 related to the invertibility of the algebraic system (4.31). Then, we prove Lemma 2.1 related to the smallness of the Newtonian operator $N^{P}(\cdot)$ with respect to the parameter P. Next, it is important to first examine the proof of Lemma 4.2, on the analysis of the Green's kernel decomposition $G(\cdot;\cdot) = \Phi_0(\cdot;\cdot) + \mathcal{R}(\cdot;\cdot)$, before moving on to the proof of Lemma 4.1, giving us an a priori estimation satisfied by the acoustic field $v^{g}(\cdot)$. Later, we examine the proof of Lemma 4.7, which gives us an estimation of the scattering coefficient α. Finally, we conclude this section by proving Lemma A.1.

A.1. Proof of Lemma 4.3

The goal of this subsection is to prove the invertibility of the algebraic system (4.31). To accomplish this, we link it to a continuous integral equation, for which we demonstrate its invertibility through variational formulation techniques. As a result, the algebraic system (4.31) can be inverted. From (4.31), we have

$$Y_m + P^2 \sum_{\substack{j=1 \\ j \neq m}}^{M} G(z_m; z_j) a^{1-h} \frac{1}{\beta_j} Y_j = S(z_m) + \frac{\omega^2 \rho_1}{k_1 \alpha} \operatorname{Rest}_m .$$

where Y_m is defined by (4.30) and Rest_m is given by (4.29). Then, by using the fact that $|\Omega_j| = a^{1-h}$, for $1 \leq j \leq M$, we rewrite the previous equation as

$$Y_m + P^2 \sum_{\substack{j=1 \\ j \neq m}}^{M} G(z_m; z_j) |\Omega_j| \frac{1}{\beta_j} Y_j = S(z_m) + \frac{\omega^2 \rho_1}{k_1 \alpha} \operatorname{Rest}_m$$

$$Y_m + P^2 \sum_{\substack{j=1 \\ j \neq m}}^{M} \int_{\Omega} G(z_m; z_j) \chi_{\Omega_j}(x) \frac{1}{\beta_j} Y_j \, dx = S(z_m) + \frac{\omega^2 \rho_1}{k_1 \alpha} \operatorname{Rest}_m .$$

Multiplying the two sides of the previous equation with $\chi_{\Omega_m}(\cdot)$ and summing up with respect to the index m, we get

$$\sum_{m=1}^{M} \chi_{\Omega_m}(\cdot) Y_m + P^2 \sum_{m=1}^{M} \chi_{\Omega_m}(\cdot) \sum_{\substack{j=1 \\ j \neq m}}^{M} \int_{\Omega} G(z_m; z_j) \chi_{\Omega_j}(x) \frac{1}{\beta_j} Y_j \, dx$$
$$= \sum_{m=1}^{M} \chi_{\Omega_m}(\cdot) S(z_m) + \frac{\omega^2 \rho_1}{k_1 \alpha} \sum_{m=1}^{M} \chi_{\Omega_m}(\cdot) \operatorname{Rest}_m .$$

which can be rewritten using the notations

$$\mathbf{Y}(\cdot) := \sum_{m=1}^{M} \chi_{\Omega_m}(\cdot) Y_m; \mathbf{S}(\cdot) := \sum_{m=1}^{M} \chi_{\Omega_m}(\cdot) S(z_m) \quad \text{and} \quad \mathbf{R}(\cdot) := \sum_{m=1}^{M} \chi_{\Omega_m}(\cdot)\, \mathrm{Rest}_m, \tag{A.1}$$

as

$$\mathbf{Y}(\cdot) + P^2 \sum_{m=1}^{M} \chi_{\Omega_m}(\cdot) \sum_{\substack{j=1 \\ j \neq m}}^{M} \int_{\Omega} G(z_m; z_j) \chi_{\Omega_j}(x) \frac{1}{\beta_j} Y_j \, dx = \mathbf{S}(\cdot) + \frac{\omega^2 \rho_1}{k_1 \alpha} \mathbf{R}(\cdot). \tag{A.2}$$

The goal of the next lemma is to prove that the second term on the left-hand side converges, in $\mathbb{L}^2(\Omega)$, to a function which belongs to the range of the Newtonian operator $\mathcal{N}(\cdot)$, see (4.37) for its definition.

Lemma A.1. *We set*

$$T_1(\cdot) := \mathcal{N}(\mathbf{Y})(\cdot) - \sum_{m=1}^{M} \chi_{\Omega_m}(\cdot) \sum_{\substack{j=1 \\ j \neq m}}^{M} \int_{\Omega} G(z_m; z_j) \chi_{\Omega_j}(x) \frac{1}{\beta_j} Y_j \, dx \quad \text{in } \Omega, \tag{A.3}$$

where $\mathcal{N}(\cdot)$ is the Newtonian operator defined by (4.37). Then, we have the following estimation:

$$\| T_1 \|_{\mathbb{L}^2(\Omega)} \lesssim a^{(1-h)/6} \| \mathbf{Y} \|_{\mathbb{L}^2(\Omega)}. \tag{A.4}$$

Proof. See Section A.6. ∎

Thanks to the previous lemma, we rewrite (A.2) as

$$(I + P^2 \mathcal{N})(\mathbf{Y})(\cdot) = \mathbf{S}(\cdot) + \mathbf{r}(\cdot) \quad \text{in } \Omega, \tag{A.5}$$

where $\mathbf{S}(\cdot)$ is the function given by (A.1), and $\mathbf{r}(\cdot)$ is the function defined by

$$\mathbf{r}(\cdot) := \frac{\omega^2 \rho_1}{k_1 \alpha} \mathbf{R}(\cdot) + P^2 T_1(\cdot), \tag{A.6}$$

with $\mathbf{R}(\cdot)$ as the function given by (A.1), and $T_1(\cdot)$ is the function defined by (A.3). Then, in the distributional sense, we have from (A.5)

$$(\Delta + \omega^2 n^2 - P^2)(\mathbf{Y}) = (\Delta + \omega^2 n^2)(\mathbf{S} + \mathbf{r}) =: \mathfrak{f} \quad \text{in } \Omega. \tag{A.7}$$

As by construction, see (A.1), we have $\mathbf{Y}(\cdot) = 0$ near $\partial\Omega$, then equation (A.7) can be stated in $\mathbb{R}^3$ by extending $\mathbf{Y}(\cdot)$ and $\mathfrak{f}(\cdot)$ by zero in $\mathbb{R}^3 \setminus \Omega$. Keeping the same notations for $\mathbf{Y}(\cdot)$ and $\mathfrak{f}(\cdot)$ with their extensions to $\mathbb{R}^3$, we have

$$\Delta(\mathbf{Y}) = (-\omega^2 n^2 + P^2)(\mathbf{Y}) + \mathfrak{f} \quad \text{in } \mathbb{R}^3,$$

with $\mathfrak{f} \in \mathbb{H}^{-2}_{comp}(\mathbb{R}^3)$. Therefore,

$$\mathbf{Y} + N_{\mathbb{R}^3}((P^2 - \omega^2 n^2)(\mathbf{Y})) = - N_{\mathbb{R}^3}(\mathfrak{f}) \quad \text{in } \mathbb{L}^2(\mathbb{R}^3), \tag{A.8}$$

with $N_{\mathbb{R}^3}(\cdot)$ is the Newtonian operator defined by (1.3). To study the existence and uniqueness of the solution corresponding to (A.8), we start by multiplying its both sides by the function $(P^2 - \omega^2 n^2) > 0$, for $P \gg 1$, to get

$$(P^2 - \omega^2 n^2)\mathbf{Y} + (P^2 - \omega^2 n^2) N_{\mathbb{R}^3}((P^2 - \omega^2 n^2)(\mathbf{Y})) = -(P^2 - \omega^2 n^2) N_{\mathbb{R}^3}(\mathfrak{f}), \tag{A.9}$$

in $\mathbb{L}^2(\mathbb{R}^3)$. Next, by taking the $\mathbb{L}^2(\mathbb{R}^3)$-inner product on both sides of (A.9), we get

$$\beth_1(\mathbf{Y}; \mathbf{Z}) = \beth_2(\mathbf{Z}), \tag{A.10}$$

where the bilinear form $\beth_1(\cdot;\cdot)$ is given by

$$\begin{aligned}\beth_1(\mathbf{Y}; \mathbf{Z}) := {} & \langle \sqrt{P^2 - \omega^2 n^2}\mathbf{Y}; \sqrt{P^2 - \omega^2 n^2}\mathbf{Z} \rangle_{\mathbb{L}^2(\mathbb{R}^3)} \\ & + \langle N_{\mathbb{R}^3}((P^2 - \omega^2 n^2)\mathbf{Y}); (P^2 - \omega^2 n^2)\mathbf{Z} \rangle_{\mathbb{L}^2(\mathbb{R}^3)},\end{aligned}$$

and the bilinear form $\beth_2(\cdot)$ is given by

$$\beth_2(\mathbf{Z}) := - \langle N_{\mathbb{R}^3}(\mathfrak{f}); (P^2 - \omega^2 n^2)\mathbf{Z} \rangle_{\mathbb{L}^2(\mathbb{R}^3)}.$$

We see that $\beth_1(\cdot,\cdot)$ is continuous and admits the following estimation:

$$\begin{aligned}|\beth_1(\mathbf{Y}, \mathbf{Z})| \leq {} & \|\mathbf{Y}\|_{\mathbb{L}^2(\mathbb{R}^3)} \|\mathbf{Z}\|_{\mathbb{L}^2(\mathbb{R}^3)} \|P^2 - \omega^2 n^2\|_{\mathbb{L}^\infty(\mathbb{R}^3)} \\ & + \|\mathbf{Y}\|_{\mathbb{L}^2(\mathbb{R}^3)} \|\mathbf{Z}\|_{\mathbb{L}^2(\mathbb{R}^3)} \|P^2 - \omega^2 n^2\|^2_{\mathbb{L}^\infty(\mathbb{R}^3)} \|N_{\mathbb{R}^3}\|_{\mathcal{L}}.\end{aligned}$$

Besides, $\beth_1(\cdot,\cdot)$ is coercive satisfying

$$\begin{aligned}\beth_1(\mathbf{Y}, \mathbf{Y}) = {} & \|\sqrt{P^2 - \omega^2 n^2}\mathbf{Y}\|^2_{\mathbb{L}^2(\mathbb{R}^3)} \\ & + \langle N_{\mathbb{R}^3}((P^2 - \omega^2 n^2)\mathbf{Y}); (P^2 - \omega^2 n^2)\mathbf{Z} \rangle_{\mathbb{L}^2(\mathbb{R}^3)} \\ \geq {} & \|\sqrt{P^2 - \omega^2 n^2}\mathbf{Y}\|^2_{\mathbb{L}^2(\mathbb{R}^3)} \\ \geq {} & \inf_{\mathbb{R}^3}(P^2 - \omega^2 n^2) \|\mathbf{Y}\|^2_{\mathbb{L}^2(\mathbb{R}^3)},\end{aligned} \tag{A.11}$$

where the before last estimation is due to the positivity of the Newtonian operator. In addition, the linear form $\beth_2(\cdot)$ is continuous and satisfy the following estimation:

$$|\beth_2(\mathbf{Z})| \leq \|N_{\mathbb{R}^3}(\mathfrak{f})\|_{\mathbb{L}^2(\mathbb{R}^3)} \|P^2 - \omega^2 n^2\|_{\mathbb{L}^\infty(\mathbb{R}^3)} \|\mathbf{Z}\|_{\mathbb{L}^2(\mathbb{R}^3)}. \tag{A.12}$$

Hence, thanks to Lax–Milgram theorem, we deduce the existence and uniqueness of the solution corresponding to (A.8). Furthermore, by gathering (A.10), (A.11), and (A.12), we derive the following estimation:

$$\inf_{\mathbb{R}^3}(P^2 - \omega^2 n^2) \|\mathbf{Y}\|_{\mathbb{L}^2(\mathbb{R}^3)} \leq \|N_{\mathbb{R}^3}(\mathfrak{f})\|_{\mathbb{L}^2(\mathbb{R}^3)} \|P^2 - \omega^2 n^2\|_{\mathbb{L}^\infty(\mathbb{R}^3)}. \tag{A.13}$$

Besides, as $P^2 \gg 1$, we have

$$\inf_{\mathbb{R}^3}(P^2 - \omega^2 n^2) \sim P^2 \quad \text{and} \quad \|P^2 - \omega^2 n^2\|_{\mathbb{L}^\infty(\mathbb{R}^3)} \sim P^2,$$

which, by plugging it into (A.13), gives us

$$\|\mathbf{Y}\|_{\mathbb{L}^2(\mathbb{R}^3)} \leq \|N_{\mathbb{R}^3}(\mathfrak{f})\|_{\mathbb{L}^2(\mathbb{R}^3)} \lesssim \|\mathfrak{f}\|_{\mathbb{H}^{-2}(\mathbb{R}^3)},$$

where we have used the continuity of the Newtonian operator to derive the last estimation. Knowing that $\mathbf{Y}(\cdot) = 0$, in $\mathbb{R}^3 \setminus \overline{\Omega}$, and $\mathfrak{f}(\cdot) = 0$, in $\mathbb{R}^3 \setminus \overline{\Omega}$, we deduce

$$\|\mathbf{Y}\|_{\mathbb{L}^2(\Omega)} \lesssim \|\mathfrak{f}\|_{\mathbb{H}^{-2}(\Omega)} \overset{(A.7)}{:=} \|(\Delta + \omega^2 n^2)(\mathbf{S} + \mathbf{r})\|_{\mathbb{H}^{-2}(\Omega)},$$

which, by keeping the dominant part on the right-hand side, can be reduced to

$$\|\mathbf{Y}\|_{\mathbb{L}^2(\Omega)} \lesssim \|\Delta(\mathbf{S} + \mathbf{r})\|_{\mathbb{H}^{-2}(\Omega)} \leq \|\Delta\|_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{H}^{-2}(\Omega))}\|\mathbf{S} + \mathbf{r}\|_{\mathbb{L}^2(\Omega)}.$$

Thus, the following inequalities hold:

$$\begin{aligned}
\|\mathbf{Y}\|_{\mathbb{L}^2(\Omega)} &\lesssim \|\mathbf{S}\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{r}\|_{\mathbb{L}^2(\Omega)} \\
&\overset{(A.6)}{\lesssim} \|\mathbf{S}\|_{\mathbb{L}^2(\Omega)} + \frac{1}{\|k_1\alpha\|}\|\mathbf{R}\|_{\mathbb{L}^2(\Omega)} + P^2\|T_1\|_{\mathbb{L}^2(\Omega)} \\
&\overset{(A.4)}{\lesssim} \|\mathbf{S}\|_{\mathbb{L}^2(\Omega)} + \frac{1}{\|k_1\alpha\|}\|\mathbf{R}\|_{\mathbb{L}^2(\Omega)} + P^2 a^{(1-h)/6}\|\mathbf{Y}\|_{\mathbb{L}^2(\Omega)},
\end{aligned}$$

which, under the fact that $P^2 a^{(1-h)/6} \ll 1$, as $a \ll 1$, which is satisfied because of (1.20) (or (1.21)), can be reduced to

$$\|\mathbf{Y}\|_{\mathbb{L}^2(\Omega)} \lesssim \|\mathbf{S}\|_{\mathbb{L}^2(\Omega)} + \frac{1}{|k_1\alpha|}\|\mathbf{R}\|_{\mathbb{L}^2(\Omega)}.$$

Now, using the fact that $k_1 \sim a^2$, see (1.2), and the estimation of $\alpha \sim a^{1-h}$, see (4.77), we deduce

$$\|\mathbf{Y}\|_{\mathbb{L}^2(\Omega)} \lesssim \|\mathbf{S}\|_{\mathbb{L}^2(\Omega)} + a^{(h-3)}\|\mathbf{R}\|_{\mathbb{L}^2(\Omega)} \overset{(A.44)}{\lesssim} \|\mathbf{S}\|_{\mathbb{L}^2(\Omega)} + a^{(1+h)/2}\|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)},$$

which, by using (A.1) and (4.38), can be reduced to

$$\|\mathbf{Y}\|_{\mathbb{L}^2(\Omega)} \lesssim a^{(1-h)/2}\sqrt{\sum_{m=1}^{M}|\mathbb{S}(g)(z_m)|^2} + a^{(1+h)/2}\|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}. \tag{A.14}$$

Let us estimate

$$\sum_{m=1}^{M}|\mathbb{S}(g)(z_m)|^2.$$

To do this, we have

$$
\begin{aligned}
\sum_{m=1}^{M} |\mathcal{S}(g)(z_m)|^2 &\leq \|g\|^2_{\mathbb{H}^{-1/2}(\partial\Omega)} \sum_{m=1}^{M} \|G(z_m,\cdot)\|^2_{\mathbb{H}^{1/2}(\partial\Omega)} \\
&\overset{(4.6)}{\leq} \|g\|^2_{\mathbb{H}^{-1/2}(\partial\Omega)} \sum_{m=1}^{M} \|G(z_m,\cdot)\|^2_{\mathbb{H}^{1}(\Omega^\diamond)} \\
&\lesssim \|g\|^2_{\mathbb{H}^{-1/2}(\partial\Omega)} \sum_{m=1}^{M} \frac{1}{\operatorname{dist}^4(D_m;\partial\Omega)} \\
&\overset{(4.52)}{\lesssim} \|g\|^2_{\mathbb{H}^{-1/2}(\partial\Omega)} d^{-4} \\
&\overset{(1.12)}{=} \mathcal{O}(\|g\|^2_{\mathbb{H}^{-1/2}(\partial\Omega)} a^{-4(1-h)/3}).
\end{aligned}
$$

Then, plugging the above estimation into (A.14),

$$
\|\mathbf{Y}\|_{\mathbb{L}^2(\Omega)} \lesssim a^{(h-1)/6} \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}.
$$

In addition, as by construction,

$$
\|\mathbf{Y}\|^2_{\mathbb{L}^2(\Omega)} = \sum_{m=1}^{M} |Y_m|^2 |\Omega_m| = |\Omega_{m_0}| \sum_{m=1}^{M} |Y_m|^2, \tag{A.15}
$$

we deduce the estimate

$$
\Big(\sum_{m=1}^{M} |Y_m|^2\Big)^{1/2} \lesssim a^{2(h-1)/3} \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}. \tag{A.16}
$$

This implies the injectivity of (A.5). In addition, it is known that any injective linear map between two finite-dimensional vector spaces of the same dimension is surjective. This proves the surjectivity and, consequently, the bijectivity of (A.5). Hence, we have also the invertibility of the algebraic system (4.31). This concludes the proof of Lemma 4.3.

A.2. Proof of Lemma 2.1

From the spectral theory, we have

$$
\|N^p\|_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{L}^2(\Omega))} = \|\mathcal{R}(P^2;\Delta)\|_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{L}^2(\Omega))} \leq \frac{1}{\operatorname{dist}(P^2;\sigma(\Delta))},
$$

where $\sigma(\Delta)$ stands for the spectrum of the Neumann Laplacian operator in $\mathbb{L}^2(\Omega)$. It is known that $\sigma(\Delta) := \{\mu_n\}_{n\geq 1}$ such that $0 = \mu_1 > \mu_2 > \mu_3 > \cdots \to -\infty$. Hence, we get $\mathrm{dist}(P^2;\sigma(\Delta)) = P^2$. Consequently,

$$\|N^p\|_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{L}^2(\Omega))} \leq \frac{1}{P^2}.$$

This proves (2.5). To prove (2.6), we start by remarking that for an arbitrary function $f \in \mathbb{L}^2(\Omega)$, the function $N^p(f)$ satisfies the problem

$$\begin{cases} (\Delta - P^2 I)N^p(f) = -f & \text{in } \Omega, \\ \partial_\nu N^p(f) = 0 & \text{on } \partial\Omega. \end{cases}$$

Multiplying both sides of the first equation by $N^p(f)$ and integrating in Ω, we get

$$\begin{aligned} \|\nabla N^p(f)\|^2_{\mathbb{L}^2(\Omega)} &\leq P^2\|N^p(f)\|^2_{\mathbb{L}^2(\Omega)} + \|f\|_{\mathbb{L}^2(\Omega)}\|N^p(f)\|_{\mathbb{L}^2(\Omega)} \\ &\leq P^2\|N^p\|^2_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{L}^2(\Omega))}\|f\|^2_{\mathbb{L}^2(\Omega)} \\ &\quad + \|f\|^2_{\mathbb{L}^2(\Omega)}\|N^p\|_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{L}^2(\Omega))}. \end{aligned}$$

Hence,

$$\begin{aligned} \|\nabla N^p\|^2_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{L}^2(\Omega))} &\leq P^2\|N^p\|^2_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{L}^2(\Omega))} + \|N^p\|_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{L}^2(\Omega))} \\ &\overset{(2.5)}{=} \mathcal{O}\left(\frac{1}{P^2}\right). \end{aligned} \tag{A.17}$$

Then,

$$\|N^p\|_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{H}^1(\Omega))} := [\|N^p\|^2_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{L}^2(\Omega))} + \|\nabla N^p\|^2_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{L}^2(\Omega))}]^{1/2},$$

which, using (2.5) and (A.17), becomes $\|N^p\|_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{H}^1(\Omega))} = \mathcal{O}\left(\frac{1}{P}\right)$ and, by taking the trace operator, we end up with the following estimation:

$$\|\gamma N^p\|_{\mathcal{L}(\mathbb{L}^2(\Omega);\mathbb{H}^{1/2}(\partial\Omega))} = \mathcal{O}\left(\frac{1}{P}\right).$$

This proves (2.6) and ends the proof of Lemma 2.1.

A.3. Proof of Lemma 4.2

Multiplying both sides of (4.19) by $\Phi_0(\cdot,\cdot)$ (the solution of (1.4)), integrating by parts over the domain Ω, and using the fact that

$$\partial_{\nu_x}(\mathcal{R}(x,y)) = -\partial_{\nu_x}(\Phi_0(x,y)), \quad x \in \partial\Omega \text{ and } y \in \Omega,$$

we obtain

$$\begin{aligned} \mathcal{R}(x,y) = &- \mathrm{DL}_{\partial\Omega}(\mathcal{R}(\cdot,y))(x) + \omega^2 N_\Omega(n^2(\cdot)\mathcal{R}(\cdot,y))(x) \\ &+ \omega^2 N_\Omega(n^2(\cdot)\Phi_0(\cdot,y))(x) - \mathrm{SL}_{\partial\Omega}(\partial_\nu(\Phi_0(\cdot,y)))(x), \quad x \in \Omega, \end{aligned} \tag{A.18}$$

where $y \in \Omega$ is taken as a parameter, $N_\Omega(\cdot)$ is the Newtonian operator defined by (1.3), $\mathrm{SL}_{\partial\Omega}(\cdot)$ is the single-layer operator defined by

$$\mathrm{SL}_{\partial\Omega}(f)(x) := \int_{\partial\Omega} \Phi_0(x,y) f(y) d\sigma(y), \quad x \in \Omega,$$

and $\mathrm{DL}_{\partial\Omega}(\cdot)$ is the double-layer operator defined by

$$\mathrm{DL}_{\partial\Omega}(f)(x) := \int_{\partial\Omega} \frac{\partial \Phi_0(x,y)}{\partial \nu(y)} f(y) d\sigma(y), \quad x \in \Omega,$$

Besides, thanks to [33, Proposition 4.3], we have the following singularity analysis:

$$|G(x,y)| \lesssim \frac{1}{|x-y|}, \quad x \neq y,$$

which by plugging it into the right-hand side of (A.18), and up to an additive uniformly bounded part, gives us

$$\mathcal{R}(x,y) \simeq -\mathrm{SL}_{\partial\Omega}(\partial_\nu(\Phi_0(\cdot,y)))(x), \quad x \in \Omega.$$

As near the boundary $\partial\Omega$, i.e., $\mathrm{dist}(y,\partial\Omega) \simeq \kappa(a)$ and $\mathrm{dist}(x,\partial\Omega) \simeq \kappa(a)$, we have

$$|\mathcal{R}(x,y)| \lesssim \int_{\partial\Omega} \frac{1}{|t-x|} \frac{1}{|t-y|^2} d\sigma(t),$$

which, by using the Holder inequality, gives us

$$\begin{aligned} |\mathcal{R}(x,y)| &\lesssim \left(\int_{\partial\Omega} \frac{1}{|t-x|^3} d\sigma(t) \right)^{1/3} \left(\int_{\partial\Omega} \frac{1}{|t-y|^3} d\sigma(t) \right)^{2/3} \\ &\lesssim \left(\frac{1}{\mathrm{dist}(x,\partial\Omega)} \right)^{1/3} \left(\frac{1}{\mathrm{dist}(y,\partial\Omega)} \right)^{2/3}, \end{aligned} \tag{A.19}$$

see [46, Lemma 4.6]. This concludes the proof of Lemma 4.2.

A.4. Proof of Lemma 4.1

We start by recalling, from (4.17), that $v^g(\cdot)$ is the solution of

$$v^g(x) - \omega^2 \int_D G(x,y) v^g(y) \Big(\frac{\rho_1}{k_1} - n^2(y) \Big)\, dy = S(x), \quad x \in D. \tag{A.20}$$

In the sequel, we divide the proof into two steps.

(1) *The case of one droplet.* Using the decomposition (4.18), of Green's kernel $G(\cdot,\cdot)$, we rewrite (A.20) as

$$
\begin{aligned}
v^g(x) &- \omega^2 \frac{\rho_1}{k_1} \int\limits_D \Phi_0(x,y) v^g(y)\, dy \\
&= S(x) + \omega^2 \frac{\rho_1}{k_1} \int\limits_D \mathcal{R}(x,y) v^g(y)\, dy - \omega^2 \int\limits_D G(x,y) v^g(y) n^2(y)\, dy.
\end{aligned}
$$

Next, we denote by $(\lambda_n^D; e_n)_{n\in\mathbb{N}}$ the eigensystem associated to the Newtonian operator $N_D(\cdot)$ in $\mathbb{L}^2(D)$. Then, after taking the inner product with respect to $e_n(\cdot)$ and the square modulus in both sides of the previous equation, we get

$$
\begin{aligned}
|\langle v^g; e_n\rangle_{\mathbb{L}^2(D)}|^2 \lesssim \frac{|k_1|^2}{|k_1 - \omega^2 \rho_1 \lambda_n^D|^2} \Bigg[&|\langle S; e_n\rangle_{\mathbb{L}^2(D)}|^2 \\
&+ \left| \left\langle \int\limits_D G(\cdot,y) n^2(y) v^g(y)\, dy; e_n \right\rangle_{\mathbb{L}^2(D)} \right|^2 \\
&+ |k_1|^{-2} \left| \left\langle \int\limits_D \mathcal{R}(\cdot,y) v^g(y)\, dy; e_n \right\rangle_{\mathbb{L}^2(D)} \right|^2 \Bigg].
\end{aligned}
$$

Then, by summing up with respect to the index n and taking into account the relations (A.31) and (1.2) we obtain

$$
\begin{aligned}
\|v^g\|^2_{\mathbb{L}^2(D)} \lesssim a^{-2h} \Bigg[&\|S\|^2_{\mathbb{L}^2(D)} + a^{-4} \left\| \int\limits_D \mathcal{R}(\cdot,y) v^g(y)\, dy \right\|^2_{\mathbb{L}^2(D)} \\
&+ \left\| \int\limits_D G(\cdot,y) n^2(y) v^g(y)\, dy \right\|^2_{\mathbb{L}^2(D)} \Bigg].
\end{aligned} \tag{A.21}
$$

Next, we estimate the second term and the third term on the right-hand-side as

$$
\mathbf{R}_1 := \left\| \int\limits_D \mathcal{R}(\cdot,y) v^g(y)\, dy \right\|^2_{\mathbb{L}^2(D)} \leq \int\limits_D \int\limits_D |\mathcal{R}(x,y)|^2\, dy\, dx \|v^g\|^2_{\mathbb{L}^2(D)}.
$$

Thanks to (A.19), we have

$$
\int\limits_D \int\limits_D |\mathcal{R}(x,y)|^2\, dy\, dx \lesssim \int\limits_D \frac{1}{\operatorname{dist}^{2/3}(x,\partial\Omega)}\, dx \int\limits_D \frac{1}{\operatorname{dist}^{4/3}(y,\partial\Omega)}\, dy,
$$

which, by using the fact that $\mathrm{dist}(x, \partial\Omega) \geq \kappa(a)$ and $\mathrm{dist}(y, \partial\Omega) \geq \kappa(a)$, gives us[12]

$$\int_D \int_D |\mathcal{R}(x,y)|^2 \, dy \, dx \lesssim \left(\frac{|D|}{\kappa(a)} \right)^2 \overset{(1.10)}{=} \mathcal{O}(a^{2(8+h)/3}). \tag{A.22}$$

Hence,

$$\mathbf{R}_1 := \left\| \int_D \mathcal{R}(\cdot, y) v^g(y) \, dy \right\|^2_{\mathbb{L}^2(D)} \lesssim a^{2(8+h)/3} \| v^g \|^2_{\mathbb{L}^2(D)}. \tag{A.23}$$

Furthermore,

$$\begin{aligned}
\mathbf{R}_2 &:= \left\| \int_D G(\cdot, y) n^2(y) v^g(y) \, dy \right\|^2_{\mathbb{L}^2(D)} \\
&\overset{(4.18)}{\lesssim} \| N_D(n^2 v^g) \|^2_{\mathbb{L}^2(D)} + \left\| \int_D \mathcal{R}(\cdot, y) n^2(y) v^g(y) \, dy \right\|^2_{\mathbb{L}^2(D)} \\
&\leq [\| N_D \|^2_{\mathcal{L}(\mathbb{L}^2(D);\mathbb{L}^2(D))} + \int_D \int_D |\mathcal{R}(x,y)|^2 \, dy \, dx] \| n^2 \|^2_{\mathbb{L}^\infty(D)} \| v^g \|^2_{\mathbb{L}^2(D)},
\end{aligned}$$

which, by using the fact that $\| n^2 \|_{\mathbb{L}^\infty(D)} = \mathcal{O}(1)$, $\| N_D \|_{\mathcal{L}(\mathbb{L}^2(D);\mathbb{L}^2(D))} = \mathcal{O}(a^2)$, and the estimation (A.22), can be reduced to

$$\mathbf{R}_2 \lesssim a^4 \| v^g \|^2_{\mathbb{L}^2(D)}. \tag{A.24}$$

Then, by plugging (A.23) and (A.24) into (A.21), we obtain

$$\begin{aligned}
\| v^g \|^2_{\mathbb{L}^2(D)} &\lesssim a^{-2h} \| S \|^2_{\mathbb{L}^2(D)} + a^{4(1-h)/3} \| v^g \|^2_{\mathbb{L}^2(D)} \\
&\lesssim a^{-2h} \| S \|^2_{\mathbb{L}^2(D)},
\end{aligned} \tag{A.25}$$

as $0 < h < 1$ and $a \ll 1$. Besides, thanks to (4.38) we know that $S = \mathcal{S}(g)$. Hence,

$$\| v^g \|^2_{\mathbb{L}^2(D)} \lesssim a^{-2h} \int_D |\mathcal{S}(g)(x)|^2 \, dx. \tag{A.26}$$

[12] If the droplet D is away from the boundary $\partial\Omega$, the estimation (A.22), will be reduced to

$$\int_D \int_D |\mathcal{R}(x,y)|^2 \, dy \, dx = \mathcal{O}(a^6).$$

Thus, the estimation (A.22) corresponds to the worst case.

Now, by using the continuity of the single-layer operator from $\mathbb{H}^{-1/2}(\partial\Omega)$ to $\mathbb{H}^{1}(\Omega)$, the continuous embedding of $\mathbb{H}^{1}(\Omega)$ into $\mathbb{L}^{6}(\Omega)$, see [7, Corollary 9.14], and the Hölder inequality, we deduce that

$$\begin{aligned}
\|v^{g}\|^{2}_{\mathbb{L}^{2}(D)} &\lesssim a^{-2h}\Big(\int_{D} |\mathcal{S}(g)(x)|^{6}\,dx\Big)^{1/3} |D|^{2/3} \\
&= a^{2-2h}\Big(\int_{D} \Big\| \int_{\partial\Omega} G(x,y)g(y)d\sigma(y)\Big\|^{6}\,dx\Big)^{1/3} \\
&= a^{2-2h}\Big(\int_{D} \|\langle G(x,\cdot);g\rangle_{\mathbb{H}^{1/2}(\partial\Omega)\times\mathbb{H}^{-1/2}(\partial\Omega)}\|^{6}\,dx\Big)^{1/3} \\
&\leq a^{2-2h}\|g\|^{2}_{\mathbb{H}^{-1/2}(\partial\Omega)}\Big(\int_{D} \|G(x,\cdot)\|^{6}_{\mathbb{H}^{1/2}(\partial\Omega)}\,dx\Big)^{1/3}. \qquad \text{(A.27)}
\end{aligned}$$

Repeating the same computations done in (4.5)–(4.7), we derive the following estimation:

$$\begin{aligned}
\|v^{g}\|^{2}_{\mathbb{L}^{2}(D)} &\lesssim a^{2-2h}\|g\|^{2}_{\mathbb{H}^{-1/2}(\partial\Omega)}\Big(\int_{D}\int_{\Omega^{\diamond}} \frac{1}{|x-y|^{12}}\,dy\,dx\Big)^{1/3} \\
&\overset{(4.8)}{\lesssim} a^{2-2h}\|g\|^{2}_{\mathbb{H}^{-1/2}(\partial\Omega)}(\kappa(a)^{-12}|D|)^{1/3} \\
&\overset{(1.10)}{=} \mathcal{O}(a^{(5-2h)/3}\|g\|^{2}_{\mathbb{H}^{-1/2}(\partial\Omega)}). \qquad \text{(A.28)}
\end{aligned}$$

Finally,

$$\|v^{g}\|_{\mathbb{L}^{2}(D)} \lesssim a^{(5-2h)/6}\|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}.$$

(2) *The case of multiple droplets.* From (A.20), by taking $x \in D_m$, we get

$$\begin{aligned}
\Big(I - \frac{\omega^{2}\rho_{1}}{k_{1}}N_{D_m}\Big)(v^{g}_{m})(x) = S_{m}(x) &+ \frac{\omega^{2}\rho_{1}}{k_{1}}\sum_{\substack{j=1\\ j\neq m}}^{M}\int_{D_j} G(x,y)v^{g}_{j}(y)\,dy \\
&+ \frac{\omega^{2}\rho_{1}}{k_{1}}\int_{D_m} \mathcal{R}(x,y)v^{g}_{m}(y)\,dy \\
&- \omega^{2}\int_{D_m} G(x,y)n^{2}(y)v^{g}_{m}(y)\,dy \\
&- \omega^{2}\sum_{\substack{j=1\\ j\neq m}}^{M}\int_{D_j} G(x,y)n^{2}(y)v^{g}_{j}(y)\,dy
\end{aligned}$$

where $N_{D_m}(\cdot)$ is the Newtonian operator given by (1.3). Besides, taking a Taylor expansion for the functions $G(x,\cdot)$ and $G(x,\cdot)n^2(\cdot)$, and inverting the operator $\big(I - \frac{\omega^2\rho_1}{k_1} N_{D_m}\big)$, we derive

$$
\begin{aligned}
v_m^g = {} & \Big(I - \frac{\omega^2\rho_1}{k_1} N_{D_m}\Big)^{-1} \\
& \times \Bigg[S_m + \frac{\omega^2\rho_1}{k_1} \sum_{\substack{j=1\\ j\neq m}}^{M} G(\cdot, z_j) \int_{D_j} v_j^g(y)\, dy \\
& \quad + \frac{\omega^2\rho_1}{k_1} \sum_{\substack{j=1\\ j\neq m}}^{M} \int_{D_j} \int_0^1 \nabla_y G(\cdot, z_j + t(y-z_j)) \cdot (y - z_j)\, dt\, v_j^g(y)\, dy \\
& \quad + \frac{\omega^2\rho_1}{k_1} \int_{D_m} \mathcal{R}(\cdot, y) v_m^g(y)\, dy - \omega^2 \int_{D_m} G(\cdot, y) n^2(y) v_m^g(y)\, dy \\
& \quad - \omega^2 \sum_{\substack{j=1\\ j\neq m}}^{M} G(\cdot, z_j) n^2(z_j) \int_{D_j} v_j^g(y)\, dy \\
& \quad - \omega^2 \sum_{\substack{j=1\\ j\neq m}}^{M} \int_{D_j} \int_0^1 \nabla(G(\cdot,\cdot)n^2(\cdot))(z_j + t(y-z_j)) \cdot (y - z_j)\, dt\, v_j^g(y)\, dy \Bigg],
\end{aligned}
$$

in D_m. Introducing the notation (4.30), the above equation can be rewritten as

$$
\begin{aligned}
v_m^g = {} & \Big(I - \frac{\omega^2\rho_1}{k_1} N_{D_m}\Big)^{-1} \\
& \times \Bigg[S_m + \alpha \sum_{\substack{j=1\\ j\neq m}}^{M} G(\cdot, z_j) Y_j \\
& \quad + \frac{\omega^2\rho_1}{k_1} \sum_{\substack{j=1\\ j\neq m}}^{M} \int_{D_j} \int_0^1 \nabla_y G(\cdot, z_j + t(y-z_j)) \cdot (y - z_j)\, dt\, v_j^g(y)\, dy \\
& \quad + \frac{\omega^2\rho_1}{k_1} \int_{D_m} \mathcal{R}(\cdot, y) v_m^g(y)\, dy - \omega^2 \int_{D_m} G(\cdot, z_m) n^2(y) v_m^g(y)\, dy \\
& \quad - \frac{\alpha k_1}{\rho_1} \sum_{\substack{j=1\\ j\neq m}}^{M} G(\cdot, z_j) n^2(z_j) Y_j \\
& \quad - \omega^2 \sum_{\substack{j=1\\ j\neq m}}^{M} \int_{D_j} \int_0^1 \nabla(G(\cdot,\cdot)n^2(\cdot))(z_j + t(y-z_j)) \cdot (y - z_j)\, dt\, v_j^g(y)\, dy \Bigg],
\end{aligned}
$$

in D_m. Taking the $\mathbb{L}^2(D_m)$-norm, using (A.23), as well as $\rho_1 \sim 1, k_1 \sim a^2$, and

$$\left\| \left(I - \frac{\omega^2 \rho_1}{k_1} N_{D_m}\right)^{-1} \right\|_{\mathcal{L}(\mathbb{L}^2(D_m);\mathbb{L}^2(D_m))} \lesssim a^{-h},$$

proved for the case of one droplet, see (A.25), we get

$$\begin{aligned}
&\|v_m^g\|_{\mathbb{L}^2(D_m)} \\
&\quad \lesssim a^{-h}\|S_m\|_{\mathbb{L}^2(D_m)} + a^{-h}|\alpha| \sum_{\substack{j=1\\ j\neq m}}^{M} \|G(\cdot, z_j)\|_{\mathbb{L}^2(D_m)} |Y_j| \\
&\qquad + a^{-2-h} \sum_{\substack{j=1\\ j\neq m}}^{M} \left\| \int_{D_j} \int_0^1 \nabla_y G(\cdot, z_j + t(y - z_j)) \cdot (y - z_j)\, dt\, v_j^g(y)\, dy \right\|_{\mathbb{L}^2(D_m)} \\
&\qquad + a^{2(1-h)/3}\|v_m^g\|_{\mathbb{L}^2(D_m)} + a^{-h}\left\| \int_{D_m} G(\cdot, y) n^2(y) v_m^g(y)\, dy \right\|_{\mathbb{L}^2(D_m)} \\
&\qquad + a^{2-h}|\alpha| \sum_{\substack{j=1\\ j\neq m}}^{M} \|G(\cdot, z_j)\|_{\mathbb{L}^2(D_m)} |Y_j| \\
&\qquad + a^{-h} \sum_{\substack{j=1\\ j\neq m}}^{M} \left\| \int_{D_j} \int_0^1 \nabla(G(\cdot,\cdot) n^2(\cdot))(z_j + t(y - z_j)) \cdot (y - z_j)\, dt\, v_j^g(y)\, dy \right\|_{\mathbb{L}^2(D_m)}.
\end{aligned}$$

Now, by estimating the terms containing Green's kernel $G(\cdot,\cdot)$ appearing on the right-hand side of the above inequality, and using the fact that $h < 1$, the previous inequality can be reduced to

$$\begin{aligned}
&\|v_m^g\|_{\mathbb{L}^2(D_m)} \\
&\quad \overset{(4.47)}{\lesssim} a^{-h}\|S_m\|_{\mathbb{L}^2(D_m)} + a^{3/2-h}|\alpha| \Bigg(\sum_{\substack{j=1\\ j\neq m}}^{M} \frac{1}{|z_m - z_j|^2} \Bigg)^{1/2} \Bigg(\sum_{\substack{j=1\\ j\neq m}}^{M} |Y_j|^2 \Bigg)^{1/2} \\
&\qquad + a^{2-h} \Bigg(\sum_{\substack{j=1\\ j\neq m}}^{M} \frac{1}{|z_m - z_j|^4} \Bigg)^{1/2} \Bigg(\sum_{\substack{j=1\\ j\neq m}}^{M} \|v_j^g\|^2_{\mathbb{L}^2(D_j)} \Bigg)^{1/2} \\
&\qquad + a^{2-h}\|v_m^g\|_{\mathbb{L}^2(D_m)} + a^{7/2-h}|\alpha| \Bigg(\sum_{\substack{j=1\\ j\neq m}}^{M} \frac{1}{|z_m - z_j|^2} \Bigg)^{1/2} \Bigg(\sum_{\substack{j=1\\ j\neq m}}^{M} |Y_j|^2 \Bigg)^{1/2} \\
&\qquad + a^{4-h} \Bigg(\sum_{\substack{j=1\\ j\neq m}}^{M} \frac{1}{|z_m - z_j|^4} \Bigg)^{1/2} \Bigg(\sum_{\substack{j=1\\ j\neq m}}^{M} \|v_j^g\|^2_{\mathbb{L}^2(D_j)} \Bigg)^{1/2},
\end{aligned}$$

which, by using (4.51), can be reduced to

$$\|v_m^g\|_{\mathbb{L}^2(D_m)} \lesssim a^{-h}\|S_m\|_{\mathbb{L}^2(D_m)} + a^{1/2-h}|\alpha| d^{-\frac{3}{2}}\Big(\sum_{j=1}^{M}|Y_j|^2\Big)^{1/2} + a^{2-h} d^{-2}\Big(\sum_{\substack{j=1 \\ j\neq m}}^{M}\|v_j^g\|^2_{\mathbb{L}^2(D_j)}\Big)^{1/2}.$$

Besides, taking the square in both sides of the above equation, using the fact that $d \sim a^{(1-h)/3}$, see (1.12), and the estimation $\alpha \sim a^{1-h}$, see (4.77), we deduce

$$\|v_m^g\|^2_{\mathbb{L}^2(D_m)} \lesssim a^{-2h}\|S_m\|^2_{\mathbb{L}^2(D_m)} + a^{4-3h}\sum_{j=1}^{M}|Y_j|^2 + a^{2(4-h)/3}\sum_{j=1}^{M}\|v_j^g\|^2_{\mathbb{L}^2(D_j)},$$

which, by summing up with respect to the index m gives us

$$\sum_{m=1}^{M}\|v_m^g\|^2_{\mathbb{L}^2(D_m)} \lesssim a^{-2h}\sum_{m=1}^{M}\|S_m\|^2_{\mathbb{L}^2(D_m)} + a^{4-3h} M\sum_{j=1}^{M}|Y_j|^2 + a^{2(4-h)/3} M\sum_{j=1}^{M}\|v_j^g\|^2_{\mathbb{L}^2(D_j)}.$$

Since $M \sim a^{h-1}$, see (1.11), we obtain

$$\sum_{m=1}^{M}\|v_m^g\|^2_{\mathbb{L}^2(D_m)} \lesssim a^{-2h}\sum_{m=1}^{M}\|S_m\|^2_{\mathbb{L}^2(D_m)} + a^{3-2h}\sum_{j=1}^{M}|Y_j|^2 + a^{5(1-h)/3}\sum_{j=1}^{M}\|v_j^g\|^2_{\mathbb{L}^2(D_j)},$$

which, by knowing that $h < 1$, can be reduced to

$$\sum_{m=1}^{M}\|v_m^g\|^2_{\mathbb{L}^2(D_m)} \lesssim a^{-2h}\sum_{m=1}^{M}\|S_m\|^2_{\mathbb{L}^2(D_m)} + a^{3-2h}\sum_{j=1}^{M}|Y_j|^2,$$

$$\|v^g\|^2_{\mathbb{L}^2(D)} \underset{(4.38)}{\overset{(A.16)}{\lesssim}} a^{-2h}\|\mathcal{S}(g)\|^2_{\mathbb{L}^2(D)} + a^{(5-2h)/3}\|g\|^2_{\mathbb{H}^{-1/2}(\partial\Omega)}. \tag{A.29}$$

Let us estimate $\|\mathcal{S}(g)\|^2_{\mathbb{L}^2(D)}$. To do this, as done for the case of a single droplet, see (A.26)–(A.28), by using the Holder inequality, we can derive the following inequalities:

$$
\begin{aligned}
\|\mathcal{S}(g)\|^2_{\mathbb{L}^2(D)} &\leq \sum_{m=1}^{M} \|\mathcal{S}(g)\|^2_{\mathbb{L}^6(D_m)} \|1\|^2_{\mathbb{L}^3(D_m)} \\
&\lesssim a^2 \sum_{m=1}^{M} \left[\int_{D_m} \left\| \int_{\partial\Omega} G(x,y) g(y)\, dy \right\|^6 dx \right]^{1/3} \\
&\leq a^2 \|g\|^2_{\mathbb{H}^{-1/2}(\partial\Omega)} \sum_{m=1}^{M} \left[\int_{D_m} \|G(x,\cdot)\|^6_{\mathbb{H}^{1/2}(\partial\Omega)}\, dx \right]^{1/3} \\
&\underset{(A.28)}{\overset{(A.27)}{\leq}} a^2 \|g\|^2_{\mathbb{H}^{-1/2}(\partial\Omega)} \sum_{m=1}^{M} \left(\int_{D_m} \int_{\Omega^{\diamond}} \frac{1}{|x-y|^{12}}\, dy\, dx \right)^{1/3} \\
&\lesssim a^3 \|g\|^2_{\mathbb{H}^{-1/2}(\partial\Omega)} \sum_{m=1}^{M} \frac{1}{\operatorname{dist}^4(D_m;\partial\Omega)} \\
&\overset{(4.52)}{\lesssim} a^3 \|g\|^2_{\mathbb{H}^{-1/2}(\partial\Omega)} d^{-4} \\
&= (a^{(5+4h)/3} \|g\|^2_{\mathbb{H}^{-1/2}(\partial\Omega)}).
\end{aligned}
$$

Thus, by plugging the above estimation into (A.29), we obtain

$$
\|v^g\|_{\mathbb{L}^2(D)} \lesssim a^{(5-2h)/6} \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}. \tag{A.30}
$$

This ends the proof of Lemma 4.1.

A.5. Proof of Lemma 4.7

We know that, for m fixed,

$$
\alpha := \int_{D_m} W_m(x)\, dx = \int_{D_m} \left(\frac{k_1}{\omega^2 \rho_1} I - N_{D_m} \right)^{-1} (1)(x)\, dx.
$$

By expanding the constant function 1 over the basis of the Newtonian operator $N_{D_m}(\cdot)$, we obtain

$$
\begin{aligned}
\alpha &= \sum_n \langle 1; e_n \rangle_{\mathbb{L}^2(D_m)} \int_{D_m} \left(\frac{k_1}{\omega^2 \rho_1} I - N_{D_m} \right)^{-1} (e_n)(x)\, dx \\
&= \sum_n (\langle 1; e_n \rangle_{\mathbb{L}^2(D_m)})^2 \frac{\omega^2 \rho_1}{(k_1 - \omega^2 \rho_1 \lambda_n^{D_m})}.
\end{aligned}
$$

We choose ω the solution of following coming dispersion equation:

$$k_1 - \omega^2 \rho_1 \lambda_{n_0}^{D_m} = c_{n_0} a^{2+h}, \quad c_{n_0} \in \mathbb{R}.$$

By solving the previous quadratic equation, we obtain

$$\omega^2 = \frac{k_1 - c_{n_0} a^{2+h}}{\rho_1 \lambda_{n_0}^{D_m}}.$$

Hence,

$$|k_1 - \omega^2 \rho_1 \lambda_n^{D_m}| = \begin{cases} a^{2+h} & \text{if } n = n_0, \\ a^2 & \text{otherwise.} \end{cases} \tag{A.31}$$

Then,

$$\alpha = (\langle 1; e_{n_0} \rangle_{\mathbb{L}^2(D_m)})^2 \frac{\omega^2 \rho_1}{(k_1 - \omega^2 \rho_1 \lambda_{n_0}^{D_m})} + \sum_{n \neq n_0} (\langle 1; e_n \rangle_{\mathbb{L}^2(D_m)})^2 \frac{\omega^2 \rho_1}{(k_1 - \omega^2 \rho_1 \lambda_n^{D_m})}.$$

We estimate the second term on the right-hand side as

$$\|T_{\text{RHS}}\| \lesssim \sum_{n \neq n_0} \frac{|\langle 1; e_n \rangle_{\mathbb{L}^2(D_m)}|^2}{|k_1 - \omega^2 \rho_1 \lambda_n^{D_m}|} \overset{(A.31)}{\lesssim} a^{-2} \|1\|^2_{\mathbb{L}^2(D_m)} = \mathcal{O}(a),$$

then

$$\alpha = (\langle 1; e_{n_0} \rangle_{\mathbb{L}^2(D_m)})^2 \frac{\omega^2 \rho_1}{(k_1 - \omega^2 \rho_1 \lambda_{n_0}^{D_m})} + \mathcal{O}(a).$$

Knowing that $\langle 1; e_{n_0} \rangle_{\mathbb{L}^2(D_m)} = a^{3/2} \langle 1; \bar{e}_{n_0} \rangle_{\mathbb{L}^2(B)}$, and using the fact that $k_1 = a^2 k_0$, $\lambda_{n_0}^{D_m} = a^2 \lambda_{n_0}^B$, we rewrite the previous equation like

$$\begin{aligned} \alpha &= \frac{(k_0 - c_{n_0} a^h)}{\lambda_{n_0}^B c_{n_0}} (\langle 1; \bar{e}_{n_0} \rangle_{\mathbb{L}^2(B)})^2 a^{1-h} + \mathcal{O}(a) \\ &= \frac{k_0}{\lambda_{n_0}^B c_{n_0}} (\langle 1; \bar{e}_{n_0} \rangle_{\mathbb{L}^2(B)})^2 a^{1-h} + \mathcal{O}(a). \end{aligned} \tag{A.32}$$

We define P^2 as be the scaled dominant part of α, i.e.,

$$P^2 := \frac{-k_0 (\langle 1; \bar{e}_{n_0} \rangle_{\mathbb{L}^2(B)})^2}{\lambda_{n_0}^B c_{n_0}},$$

and we end up with the following formula:

$$\alpha = -P^2 a^{1-h} + \mathcal{O}(a).$$

To estimate $\|W_m\|_{\mathbb{L}^2(D_m)}$, we use the same above arguments to derive

$$\|W_m\|^2_{\mathbb{L}^2(D_m)} = \sum_n \frac{|\omega^2 \rho_1|^2}{|k_1 - \omega^2 \rho_1 \lambda_n^{D_m}|^2} |\langle 1; e_n \rangle_{\mathbb{L}^2(D_m)}|^2 \overset{(A.31)}{\lesssim} a^{-2(2+h)} \|1\|^2_{\mathbb{L}^2(D_m)}.$$

Hence,

$$\|W_m\|_{\mathbb{L}^2(D_m)} \lesssim a^{-(2+h)}\|1\|_{\mathbb{L}^2(D_m)} = \mathcal{O}(a^{-(\frac{1}{2}+h)}).$$

This concludes the proof of Lemma 4.7.

A.6. Proof of Lemma A.1

We compute the $\mathbb{L}^2(\Omega)$-norm of the term $T_1(\cdot)$ defined by (A.3):

$$\|T_1\|^2_{\mathbb{L}^2(\Omega)} := \int_\Omega \Bigg| \int_\Omega G(y,x)\mathbf{Y}(x)\,dx - \sum_{m=1}^M \chi_{\Omega_m}(y) \sum_{\substack{j=1 \\ j\neq m}}^M \int_\Omega G(z_m;z_j)\chi_{\Omega_j}(x)\frac{1}{\beta_j}Y_j\,dx\Bigg|^2 dy.$$

In contrast to Section 4.3, where the cutting of Ω onto $\bigcup_{j=1}^M \Omega_j$ and $\bigcup_{j=1}^{\aleph} \Omega_j^\star$ was critically important to derive the exact dominant term related to $\int_{\Omega_j} u^g(x)\,dx$, for $1 \leq j \leq M$, here we need only to estimate functions (not to extract dominant term) defined in Ω, thus involving both $\bigcup_{j=1}^M \Omega_j$ and $\bigcup_{j=1}^{\aleph} \Omega_j^\star$. Because, for every $1 \leq j \leq M$ and $1 \leq k \leq \aleph$, we have $|\Omega_j| \sim a^{1-h} \sim |\Omega_k^\star|$, we do not need to specify, in our comping computations, if we are dealing with $\{\Omega_j\}_{j=1}^M$ or $\{\Omega_j^\star\}_{j=1}^{\aleph}$. Moreover, to write short, we use the notation Ω_j for the domains $\Omega_j^\star$. Then,

$$\begin{aligned}
\|T_1\|^2_{\mathbb{L}^2(\Omega)} &= \int_\Omega \Bigg| \sum_{m=1}^M \Bigg(\int_\Omega G(y,x)\mathbf{Y}(x)\,dx - \sum_{\substack{j=1 \\ j\neq m}}^M \int_\Omega G(z_m;z_j)\chi_{\Omega_j}(x)\frac{1}{\beta_j}Y_j\,dx\Bigg)\chi_{\Omega_m}(y)\Bigg|^2 dy \\
&= \sum_{m=1}^M \int_{\Omega_m} \Bigg| \int_\Omega G(y,x)\mathbf{Y}(x)\,dx - \sum_{\substack{j=1 \\ j\neq m}}^M \int_\Omega G(z_m;z_j)\chi_{\Omega_j}(x)\frac{1}{\beta_j}Y_j\,dx\Bigg|^2 dy.
\end{aligned}$$

Using the definition of $\mathbf{Y}(\cdot)$, see (A.1), and the triangular inequality, we rewrite the previous equation as

$$\begin{aligned}
\|T_1\|^2_{\mathbb{L}^2(\Omega)} \lesssim & \sum_{m=1}^M |Y_m|^2|\Omega_m| \int_{\Omega_m}\int_{\Omega_m} |G(y,x)|^2\,dx\,dy \\
&+ \sum_{m=1}^M \sum_{\substack{j=1 \\ j\neq m}}^M |Y_j|^2 \sum_{\substack{j=1 \\ j\neq m}}^M |\Omega_j| \int_{\Omega_m}\int_{\Omega_j} \Bigg| G(y,x) - G(z_m;z_j)\frac{1}{\beta_j}\Bigg|^2 dx\,dy.
\end{aligned} \tag{A.33}$$

Furthermore, by using (4.26), we have

$$|G(y,x) - G(z_m;z_j)\frac{1}{\beta_j}|^2 \lesssim |G(y,x) - G(z_m;z_j)|^2 + a^{4(1-h)/3}|G(z_m;z_j)|^2.$$

Hence, by plugging the above estimation into (A.33), we obtain

$$\begin{aligned}\|T_1\|^2_{\mathbb{L}^2(\Omega)} \leq \sum_{m=1}^{M}|Y_m|^2\Bigg[&\max_{1\leq m\leq M}\left(|\Omega_m|\int_{\Omega_m}\int_{\Omega_m}|G(y,x)|^2\,dx\,dy\right)\\ &+\sum_{m=1}^{M}\sum_{\substack{j=1\\ j\neq m}}^{M}|\Omega_j|\int_{\Omega_m}\int_{\Omega_j}|G(y,x)-G(z_m;z_j)|^2\,dx\,dy\Bigg]\\ &+a^{10(1-h)/3}\sum_{j=1}^{M}|Y_j|^2\sum_{\substack{j=1\\ j\neq m}}^{M}|G(z_m;z_j)|^2.\end{aligned}$$

Using Taylor expansion for the function $G(\cdot;\cdot)$, near the centres, we get

$$\begin{aligned}G(y,x) - G(z_m;z_j) = &\int_0^1 \nabla G(z_m;z_j + t(x-z_j))\cdot(x-z_j)\,dt\\ &+\int_0^1 \nabla G(z_m + t(y-z_m);x)\cdot(y-z_m)\,dt.\end{aligned}$$

We plug the previous expansion into the previous estimation and we use (4.47) to reduce the previous estimation to

$$\begin{aligned}\|T_1\|^2_{\mathbb{L}^2(\Omega)} \lesssim \sum_{m=1}^{M}|Y_m|^2\Bigg[&\max_{1\leq m\leq M}\left(|\Omega_m|\int_{\Omega_m}\int_{\Omega_m}\frac{1}{|y-x|^2}\,dx\,dy\right)\\ &+\sum_{m=1}^{M}\sum_{\substack{j=1\\ j\neq m}}^{M}|\Omega_j|\int_{\Omega_m}\int_{\Omega_j}\frac{|x-z_j|^2}{|y-z_j|^4}\,dx\,dy\Bigg]\\ &+a^{10(1-h)/3}\sum_{j=1}^{M}|Y_j|^2\sum_{\substack{j=1\\ j\neq m}}^{M}\frac{1}{|z_m-z_j|^2}.\end{aligned}$$

Besides, by knowing that $|\Omega_m| = a^{1-h}$, with $1\leq m\leq M$, we deduce the following estimation:

$$\max_{1\leq m\leq M}\left(|\Omega_m|\int_{\Omega_m}\int_{\Omega_m}\frac{1}{|y-x|^2}\,dx\,dy\right)\lesssim a^{7(1-h)/3}.$$

Then,

$$
\begin{aligned}
&\|T_1\|^2_{\mathbb{L}^2(\Omega)} \\
&\lesssim \sum_{m=1}^{M} |Y_m|^2 \left[a^{7(1-h)/3} + \max_{1\leq j\leq M}(|\Omega_j|) \sum_{m=1}^{M} \sum_{\substack{j=1\\ j\neq m}}^{M} \int_{\Omega_m} \frac{1}{|y-z_j|^4}\, dy \int_{\Omega_j} |x-z_j|^2\, dx \right] \\
&\quad + a^{10(1-h)/3} \sum_{j=1}^{M} |Y_j|^2 \sum_{\substack{j=1\\ j\neq m}}^{M} \frac{1}{|z_m-z_j|^2}.
\end{aligned}
$$

In addition, by Taylor expansion, we have

$$
\begin{aligned}
&\int_{\Omega_m} \frac{1}{|y-z_j|^4}\, dy \\
&\quad = \frac{1}{|z_m-z_j|^4} |\Omega_m| + \int_{\Omega_m} \int_0^1 \nabla(|\cdot - z_j|^{-4})(z_m + t(y-z_m)) \cdot (y-z_m)\, dt\, dy,
\end{aligned}
$$

hence

$$
\begin{aligned}
&\|T_1\|^2_{\mathbb{L}^2(\Omega)} \\
&\quad \lesssim \sum_{m=1}^{M} |Y_m|^2 \left[a^{7(1-h)/3} + a^{(1-h)} \max_{1\leq j\leq M} \left(\int_{\Omega_j} |x-z_j|^2\, dx \right) \right. \\
&\qquad\qquad \left. \times \sum_{m=1}^{M} |\Omega_m| \sum_{\substack{j=1\\ j\neq m}}^{M} \frac{1}{|z_m-z_j|^4} \right] \\
&\qquad + a^{10(1-h)/3} \sum_{j=1}^{M} |Y_j|^2 \sum_{\substack{j=1\\ j\neq m}}^{M} \frac{1}{|z_m-z_j|^2}.
\end{aligned}
$$

The following estimations hold:

$$
\begin{aligned}
\max_{1\leq j\leq M} \left(\int_{\Omega_j} |x-z_j|^2\, dx \right) &= \mathcal{O}(a^{5(1-h)/3}) \\
\sum_{m=1}^{M} |\Omega_m| \sum_{\substack{j=1\\ j\neq m}}^{M} \frac{1}{|z_m-z_j|^4} &\overset{(4.51)}{\lesssim} \sum_{m=1}^{M} d^{-4} |\Omega_m| = \mathcal{O}(d^{-4}) \\
\sum_{\substack{j=1\\ j\neq m}}^{M} \frac{1}{|z_m-z_j|^2} &\overset{(4.51)}{=} \mathcal{O}(d^{-3});
\end{aligned}
$$

then, by recalling that $d \sim a^{(1-h)/3}$ and using the fact that $|\Omega_{m_0}| = a^{1-h}$, we obtain

$$\begin{aligned}\|T_1\|^2_{\mathbb{L}^2(\Omega)} &\lesssim \sum_{m=1}^{M} |Y_m|^2 [a^{7(1-h)/3} + a^{4(1-h)/3}] \\ &= \mathcal{O}\Big(a^{4(1-h)/3} \sum_{m=1}^{M} |Y_m|^2\Big) \overset{(A.15)}{=} \mathcal{O}(a^{(1-h)/3} \|\mathbf{Y}\|^2_{\mathbb{L}^2(\Omega)}).\end{aligned}$$

Finally,

$$\|T_1\|_{\mathbb{L}^2(\Omega)} = \mathcal{O}(a^{(1-h)/6} \|\mathbf{Y}\|_{\mathbb{L}^2(\Omega)}).$$

This concludes the proof of Lemma A.1.

A.7. Proof of Lemma 4.6

To prove (4.51) we refer the readers to [3, Section 3.3]. To justify (4.52), we define $\mathbf{\Omega}_n$ as

$$\mathbf{\Omega}_n := \{x \in \Omega, (n-1)d \leq \operatorname{dist}(x, \partial\Omega) \leq nd\}, \quad \text{for } n = 1, \dots, [d^{-1}],$$

where d is the minimum distance given by (1.12). Then, $\Omega \subset \bigcup_{n=1}^{[d^{-1}]} \mathbf{\Omega}_n$, and

$$|\mathbf{\Omega}_n| \lesssim (nd)^2 d = \mathcal{O}(n^2 d^3).$$

Then, the number of droplets in $\mathbf{\Omega}_n$ is of order n^2. Hence,

$$\begin{aligned}\sum_{j=1}^{M} \frac{1}{\operatorname{dist}^k(D_j;\partial\Omega)} &= \sum_{n=1}^{[d^{-1}]} \sum_{D_j \subset \Omega_n} \frac{1}{\operatorname{dist}^k(D_j;\partial\Omega)} \\ &\lesssim \sum_{n=1}^{[d^{-1}]} n^2 \frac{1}{((n-1)d)^k} \simeq \frac{1}{d^k} \sum_{n=1}^{[d^{-1}]} \frac{1}{n^{k-2}}.\end{aligned}$$

Therefore,

$$\sum_{j=1}^{M} \frac{1}{\operatorname{dist}^k(D_j;\partial\Omega)} = \begin{cases} \mathcal{O}(d^{-3}) & \text{for } k < 3, \\ \mathcal{O}(d^{-k}) & \text{for } k > 3. \end{cases}$$

This ends the proof of Lemma 4.6.

A.8. Normal derivative of $\mathrm{SL}^p(\cdot)$.

We recall, from (2.14), the following definition of the single-layer operator $\mathrm{SL}^p(\cdot)$:

$$\mathrm{SL}^p(f)(x) := \int_{\partial\Omega} G_p(x,y) f(y) d\sigma(y), \quad x \in \Omega, \tag{A.34}$$

where $G_p(\cdot,\cdot)$ is the solution of (2.3). The goal of this subsection is to compute the jumping coefficient of the normal derivative related to (A.34). To do this, we define $\Phi_{ip}(\cdot,\cdot)$ as the fundamental solution of $(\Delta - P^2)\Phi_{ip}(\cdot,\cdot) = -\delta_{\cdot}(\cdot)$, in $\mathbb{R}^3$. Multiplying (2.3) by $\Phi_{ip}(\cdot,\cdot)$ and integrating over Ω allows us to deduce that

$$G_p(\cdot,y) + D^{ip}(G_p(\cdot,y))(\cdot) = \Phi_{ip}(\cdot,y) \quad \text{in } \Omega, \tag{A.35}$$

where $y \in \Omega$ is taken as a parameter, and $D^{ip}(\cdot)$ is the double-layer operator associated to $\Phi_{ip}(\cdot,\cdot)$. Besides, from (A.35), we deduce that

$$G_p(\cdot,y) = \Big(\frac{1}{2}I + K^{ip}\Big)^{-1}(\Phi_{ip}(\cdot,y)) \quad \text{on } \partial\Omega, \tag{A.36}$$

where $y \in \Omega$ is taken as a parameter, and $K^{ip}(\cdot)$ is the Neumann–Poincaré operator associated to $\Phi_{ip}(\cdot,\cdot)$. Furthermore, by denoting $S^{ip}(\cdot)$ the single-layer operator associated to $\Phi_{ip}(\cdot,\cdot)$ and using (A.36), we deduce that, for $x \in \Omega$,

$$\begin{aligned}
S^{ip}(f)(x) &:= \int_{\partial\Omega} \Phi_{ip}(x,y) f(y) d\sigma(y) \\
&\overset{(A.36)}{=} \int_{\partial\Omega} \Big(\frac{1}{2}I + K^{ip}\Big)(G_p(\cdot,y))(x) f(y) d\sigma(y) \\
&= \int_{\partial\Omega} G_p(x,y) \Big(\frac{1}{2}I + K^{ip}\Big)^{\star}(f)(y) d\sigma(y) \\
&= \int_{\partial\Omega} G_p(x,y) \Big(\frac{1}{2}I + K^{-ip}\Big)(f)(y) d\sigma(y) \\
&\overset{(A.34)}{=} \mathrm{SL}^p\Big(\Big(\frac{1}{2}I + K^{-ip}\Big)(f)\Big)(x).
\end{aligned}$$

As $f(\cdot)$ is an arbitrary function, we deduce the resulting relation

$$\mathrm{SL}^p(f) = S^{ip}\Big(\Big(\frac{1}{2}I + K^{-ip}\Big)^{-1}(f)\Big) \quad \text{in } \Omega;$$

hence by taking the normal derivative on both sides of the above equation, we obtain

$$\frac{\partial}{\partial\nu}[\mathrm{SL}^p(f)] = \frac{\partial}{\partial\nu}\Big[S^{ip}\Big(\Big(\frac{1}{2}I + K^{-ip}\Big)^{-1}(f)\Big)\Big];$$

by using on the right-hand side the jumping properties for the single-layer operator $S^{ip}(\cdot)$, we deduce

$$\begin{aligned}
\frac{\partial}{\partial\nu}[\mathrm{SL}^p(f)] &= \Big(\frac{1}{2} + K^{ip}\Big)^{\star}\Big(\Big(\frac{1}{2}I + K^{-ip}\Big)^{-1}(f)\Big) \\
&= \Big(\frac{1}{2} + K^{-ip}\Big)\Big(\Big(\frac{1}{2}I + K^{-ip}\Big)^{-1}(f)\Big) = f, \quad \text{on } \partial\Omega.
\end{aligned}$$

A.9. Estimating $\|\mathbf{R}\|_{\mathbb{L}^2(\Omega)}$

We recall from (A.1) that

$$\mathbf{R}(\cdot) := \sum_{m=1}^{M} \chi_{\Omega_m}(\cdot)\,\mathrm{Rest}_m,$$

with Rest_m is given by (4.29). Then, using the fact that Ω_m-s are disjoint sets, we obtain

$$\|\mathbf{R}\|^2_{\mathbb{L}^2(\Omega)} = \sum_{m=1}^{M} |\Omega_m||\mathrm{Rest}_m|^2 = |\Omega_{m_0}| \sum_{m=1}^{M} |\mathrm{Rest}_m|^2 = a^{(1-h)} \sum_{m=1}^{M} |\mathrm{Rest}_m|^2. \tag{A.37}$$

Besides, by taking the absolute value in both sides of (4.29), we obtain

$$\begin{aligned}
&|\mathrm{Rest}_m| \\
&\lesssim \sum_{\substack{j=1\\ j\neq m}}^{M} \left| \int_{D_m} W_m(x) \int_0^1 \nabla G(z_m + t(x-z_m); z_j) \cdot (x - z_m)\, dt\, dx \right| \left| \int_{D_j} v_j^g(y)\, dy \right| \\
&\quad + \sum_{\substack{j=1\\ j\neq m}}^{M} \left| \int_{D_m} W_m(x) \int_{D_j} \int_0^1 \nabla G(x; z_j + t(y - z_j)) \cdot (y - z_j)\, dt\, v_j^g(y)\, dy\, dx \right| \\
&\quad + a^2 \left| \int_{D_m} W_m(x) \int_0^1 \nabla S_m(z_m + t(x - z_m)) \cdot (x - z_m)\, dt\, dx \right| \\
&\quad + \left| \int_{D_m} W_m(x) \int_{D_m} \int_0^1 \nabla_y \mathcal{R}(x, z_m + t(y - z_m)) \cdot (y - z_m)\, dt\, v_m^g(y)\, dy\, dx \right| \\
&\quad + a^2 \left| \int_{D_m} W_m(x) \int_{D} G(x, y) v^g(y) n^2(y)\, dy\, dx \right|.
\end{aligned} \tag{A.38}$$

In addition, for the third term on the right-hand side, we have

$$\begin{aligned}
\zeta_3 &:= a^2 \left| \int_{D_m} W_m(x) \int_0^1 \nabla S_m(z_m + t(x - z_m)) \cdot (x - z_m)\, dt\, dx \right| \\
&\leq a^2 \|W_m\|_{\mathbb{L}^2(D_m)} \left\| \int_0^1 \nabla S_m(z_m + t(\cdot - z_m)) \cdot (\cdot - z_m)\, dt \right\|_{\mathbb{L}^2(D_m)} \\
&\leq a^2 \|W_m\|_{\mathbb{L}^2(D_m)} \left[\int_0^1 \frac{1}{t} \int_{B(z_m, ta)} |\nabla S_m(y)|^2 |y - z_m|^2\, dy\, dt \right]^{1/2},
\end{aligned}$$

where $B(z_m, ta)$ is a ball of centre z_m and radius ta. Then,

$$\zeta_3 \leq a^3 \|W_m\|_{\mathbb{L}^2(D_m)} \Big[\int_0^1 \int_{B(z_m,ta)} |\nabla S_m(y)|^2 \, dy \, dt \Big]^{1/2}$$
$$\leq a^3 \|W_m\|_{\mathbb{L}^2(D_m)} \Big[\int_0^1 \int_{D_m} |\nabla S_m(y)|^2 \, dy \, dt \Big]^{1/2},$$

as $B(z_m, ta) \subset D_m$. Then,

$$\zeta_3 = \mathcal{O}(a^3 \|W_m\|_{\mathbb{L}^2(D_m)} \|\nabla S_m\|_{\mathbb{L}^2(D_m)}). \tag{A.39}$$

Now, by using (4.38), (4.47), and (A.39), we derive from the inequality (A.38)

$$|\mathrm{Rest}_m| \lesssim \|W_m\|_{\mathbb{L}^2(D_m)} \Big[a^4 \sum_{\substack{j=1 \\ j \neq m}}^{M} \frac{1}{|z_m - z_j|^2} \|v_j^g\|_{\mathbb{L}^2(D_j)} + a^3 \|\nabla \mathcal{S}(g)\|_{\mathbb{L}^2(D_m)}$$
$$+ a \Big[\int_{D_m} \int_{D_m} |\nabla \mathcal{R}(x,y)|^2 \, dy \, dx \Big]^{1/2} \|v_m^g\|_{\mathbb{L}^2(D_m)}$$
$$+ a^4 \|v_m^g\|_{\mathbb{L}^2(D_m)} \Big]$$
$$+ a^{(9/2-h)} \sum_{\substack{j=1 \\ j \neq m}}^{M} \frac{1}{|z_m - z_j|} \|v_j^g\|_{\mathbb{L}^2(D_j)}.$$

Using the Cauchy–Schwarz inequality and the estimation given by (4.51), we deduce

$$|\mathrm{Rest}_m| \lesssim \|W_m\|_{\mathbb{L}^2(D_m)} [a^{(10+2h)/3} \|v^g\|_{\mathbb{L}^2(D)} + a^3 \|\nabla \mathcal{S}(g)\|_{\mathbb{L}^2(D_m)}]$$
$$+ \|W_m\|_{\mathbb{L}^2(D_m)} \Big[a \Big[\int_{D_m} \int_{D_m} |\nabla \mathcal{R}(x,y)|^2 \, dy \, dx \Big]^{1/2} + a^4 \Big] \|v_m^g\|_{\mathbb{L}^2(D_m)}$$
$$+ a^{(8-h)/2} \|v^g\|_{\mathbb{L}^2(D)}. \tag{A.40}$$

The following estimation holds:

$$\|\nabla \mathcal{S}(g)\|_{\mathbb{L}^2(D_m)} \leq \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \Big[\int_{D_m} \|\nabla G(x,\cdot)\|^2_{\mathbb{H}^{1/2}(\partial\Omega)} \, dx \Big]^{1/2}$$
$$\overset{(4.6)}{\leq} \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \Big[\int_{D_m} \|\nabla G(x,\cdot)\|^2_{\mathbb{H}^{1}(\Omega^\diamond)} \, dx \Big]^{1/2}$$

$$\lesssim \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \left[\int_{D_m} \frac{1}{\operatorname{dist}^6(x,\partial\Omega)}\, dx \right]^{1/2}$$

$$\overset{(1.10)}{\lesssim} \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \frac{a^{3/2}}{\operatorname{dist}^3(D_m;\partial\Omega)}. \tag{A.41}$$

Besides, similarly to (4.20), for the $\nabla\mathcal{R}(\cdot,\cdot)$, we can prove that

$$|\nabla_y \mathcal{R}(x,y)| \lesssim \left(\frac{1}{\operatorname{dist}(x,\partial\Omega)}\right)^{2/3} \left(\frac{1}{\operatorname{dist}(y,\partial\Omega)}\right)^{4/3} \quad \text{for } x \neq y.$$

Then,

$$\int_{D_m}\int_{D_m} |\nabla\mathcal{R}(x,y)|^2\, dy\, dx \lesssim \int_{D_m}\int_{D_m} \frac{1}{\operatorname{dist}^{4/3}(x,\partial\Omega)} \frac{1}{\operatorname{dist}^{8/3}(y,\partial\Omega)}\, dy\, dx$$

$$\lesssim \frac{|D_m|^2}{\operatorname{dist}^4(D_m,\partial\Omega)}. \tag{A.42}$$

Hence, by returning to (A.40), and using (A.41) and (A.42), we derive the following estimation:

$$\begin{aligned}
|\mathrm{Rest}_m| &\lesssim \|W_m\|_{\mathbb{L}^2(D_m)} \left[a^{(10+2h)/3} \|v^g\|_{\mathbb{L}^2(D)} + \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \frac{a^{9/2}}{\operatorname{dist}^3(D_m;\partial\Omega)} \right] \\
&\quad + \|W_m\|_{\mathbb{L}^2(D_m)} a^4 \frac{1}{\operatorname{dist}^2(D_m;\partial\Omega)} \|v_m^g\|_{\mathbb{L}^2(D_m)} + a^{(8-h)/2} \|v^g\|_{\mathbb{L}^2(D)} \\
&\overset{(4.78)}{\lesssim} \|1\|_{\mathbb{L}^2(D_m)} \left[a^{(4-h)/3} \|v^g\|_{\mathbb{L}^2(D)} + \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)} \frac{a^{(5-2h)/2}}{\operatorname{dist}^3(D_m;\partial\Omega)} \right] \\
&\quad + \|1\|_{\mathbb{L}^2(D_m)} \frac{a^{(2-h)}}{\operatorname{dist}^2(D_m;\partial\Omega)} \|v_m^g\|_{\mathbb{L}^2(D_m)} + a^{(8-h)/2} \|v^g\|_{\mathbb{L}^2(D)}.
\end{aligned}$$

Thus, by using (4.52), we obtain

$$\sum_{m=1}^{M} |\mathrm{Rest}_m|^2 \lesssim a^{(14+h)/3} \|v^g\|^2_{\mathbb{L}^2(D)} + a^6 \|g\|^2_{\mathbb{H}^{-1/2}(\partial\Omega)} \overset{(A.30)}{\lesssim} a^6 \|g\|^2_{\mathbb{H}^{-1/2}(\partial\Omega)}. \tag{A.43}$$

Then, by plugging the above estimation into (A.37), we obtain

$$\|\mathbf{R}\|_{\mathbb{L}^2(\Omega)} = \mathcal{O}(a^{(7-h)/2} \|g\|_{\mathbb{H}^{-1/2}(\partial\Omega)}). \tag{A.44}$$

Acknowledgements. The authors are grateful to the anonymous referees for their reading and helpful comments.

Funding. Mourad Sini was partially supported by the Austrian Science Fund (FWF) : P30756-NBL.

References

[1] G. Alessandrini, Stable determination of conductivity by boundary measurements. *Appl. Anal.* **27** (1988), no. 1-3, 153–172 Zbl 0616.35082 MR 0922775

[2] G. Alessandrini, Singular solutions of elliptic equations and the determination of conductivity by boundary measurements. *J. Differential Equations* **84** (1990), no. 2, 252–272 Zbl 0778.35109 MR 1047569

[3] H. Ammari, D. P. Challa, A. P. Choudhury, and M. Sini, The point-interaction approximation for the fields generated by contrasted bubbles at arbitrary fixed frequencies. *J. Differential Equations* **267** (2019), no. 4, 2104–2191 Zbl 1447.35371 MR 3950478

[4] C. R. Anderson, X. Hu, H. Zhang, J. Tlaxca, A-E. Declèves, R. Houghtaling, K. Sharma, M. Lawrence, K. W. Ferrara, and J. J. Rychak, Ultrasound molecular imaging of tumor angiogenesis with an integrin targeted microdroplet contrast agent. *Invest. Radiol.* **46** (2011), no. 4, 215–224

[5] J. M. Anderson, D. Khavinson, and V. Lomonosov, Spectral properties of some integral operators arising in potential theory. *Quart. J. Math. Oxford Ser. (2)* **43** (1992), no. 172, 387–407 Zbl 0764.31001 MR 1188382

[6] G. Bellizzi, O. M. Bucci, and I. Catapano, Microwave cancer imaging exploiting magnetic nanoparticles as contrast agent. *IEEE Transactions on Biomedical Engineering* **58** (2011), no. 9, 2528–2536

[7] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*. Universitext, Springer, New York, 2011 Zbl 1218.46002 MR 2759829

[8] R. M. Brown, Global uniqueness in the impedance-imaging problem for less regular conductivities. *SIAM J. Math. Anal.* **27** (1996), no. 4, 1049–1056 Zbl 0867.35111 MR 1393424

[9] R. M. Brown, Global uniqueness in the impedance-imaging problem for less regular conductivities. *SIAM J. Math. Anal.* **27** (1996), no. 4, 1049–1056 Zbl 0867.35111 MR 1393424

[10] R. M. Brown and R. H. Torres, Uniqueness in the inverse conductivity problem for conductivities with $3/2$ derivatives in $L^p,\ p > 2n$. *J. Fourier Anal. Appl.* **9** (2003), no. 6, 563–574 Zbl 1051.35105 MR 2026763

[11] R. M. Brown and G. A. Uhlmann, Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions. *Comm. Partial Differential Equations* **22** (1997), no. 5-6, 1009–1027 Zbl 0884.35167 MR 1452176

[12] A. L. Bukhgeim, Recovering a potential from Cauchy data in the two-dimensional case. *J. Inverse Ill-Posed Probl.* **16** (2008), no. 1, 19-–33 Zbl 1142.30018 MR 2387648

[13] P. Caro and A, Garcia, Scattering with critically-singular and δ-shell potentials. *Comm. Math. Phys* **379** (2020), no. 2, 543–587 Zbl 1450.35204 MR 4156217

[14] P. Caro and K. M. Rogers, Global uniqueness for the Calderón problem with Lipschitz conductivities. *Forum Math. Pi* **4** (2016), article no. e2 Zbl 1330.35525 MR 3456182

[15] C. F. Caskey, K. W. Ferrara, and S. Qin, Ultrasound contrast microdroplets in imaging and therapy: physical principles and engineering. *Phys Med Biol* **54** (2009), no. 6, article no. R27

[16] S. Chanillo, A problem in electrical prospection and an n-dimensional Borg–Levinson theorem. *Proc. Amer. Math. Soc.* **108** (1990), no. 3, 761–767 Zbl 0702.35035 MR 0998731

[17] D. Colton and R. Kress, *Inverse acoustic and electromagnetic scattering theory*. Appl. Math. Sci. 93, Springer, Cham, 2019 Zbl 1425.35001 MR 3971246

[18] R. Courant and D. Hilbert, *Methods of mathematical physics.* Vol. II: Partial differential equations. Interscience, New York and London, 1962 Zbl 0729.35001 MR 0140802

[19] A. Dabrowski, A. Ghandriche, and M. Sini, Mathematical analysis of the acoustic imaging modality using bubbles as contrast agents at nearly resonating frequencies. *Inverse Probl. Imaging* **15** (2021), no. 3, 555–597 Zbl 1467.35345 MR 4235670

[20] A. Ghandriche and M. Sini, Mathematical analysis of the photo-acoustic imaging modality using resonating dielectric nano-particles: the $2D$ TM-model. *J. Math. Anal. Appl.* **506** (2022), no. 2, article no. 125658 Zbl 1478.92101 MR 4316026

[21] A. Ghandriche and M. Sini, Photo-acoustic inversion using plasmonic contrast agents: The full Maxwell model. *Journal of Differential Equations* **341** (2022), 1–78 Zbl 1504.35524 MR 4486150

[22] A. Ghandriche and M. Sini, Simultaneous reconstruction of optical and acoustical properties in photoacoustic imaging using plasmonics. *SIAM J. Appl. Math.* **83** (2023), no. 4, 1738–1765 Zbl 1522.35581 MR 4631437

[23] B. Haberman, Uniqueness in Calderón's problem for conductivities with unbounded gradient. *Comm. Math. Phys.* **340** (2015), no. 2, 639–659 Zbl 1456.35230 MR 3397029

[24] B. Haberman and D. Tataru, Uniqueness in Calderón's problem with Lipschitz conductivities. *Duke Math. J.* **162** (2013), no. 3, 496–516 Zbl 1260.35251 MR 3024091

[25] V. Isakov, Completeness of products of solutions and some inverse problems for PDE. *J. Differential Equations* **92** (1991), no. 2, 305–316 Zbl 0728.35141 MR 1120907

[26] V. Isakov, *Inverse problems for partial differential equations*. Second edn., Appl. Math. Sci. 127, Springer, New York, 2006 Zbl 1092.35001 MR 2193218

[27] T. S. Kalmenov and D. Suragan, A boundary condition and spectral problems for the Newton potential. In *Modern aspects of the theory of partial differential equations*, pp. 187–210, Oper. Theory Adv. Appl. 216, Birkhäuser/Springer, Basel, 2011 MR 2848241

[28] R. Kohn and M. Vogelius, Determining conductivity by boundary measurements. *Comm. Pure Appl. Math.* **37** (1984), no. 3, 289–298 Zbl 0586.35089 MR 0739921

[29] W. Li and X. Chen, Gold nanoparticles for photoacoustic imaging. *Nanomedicine* **10** (2015), no. 2, 299–320

[30] A. Mantile, A. Posilicano, and M. Sini, Uniqueness in inverse acoustic scattering with unbounded gradient across Lipschitz surfaces. *J. Differential Equations* **265** (2018), no. 9, 4101–4132 Zbl 1396.81211 MR 3843295

[31] W. McLean, *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge, 2000 Zbl 0948.35001 MR 1742312

[32] E. McLeod and A. Ozcan, *https://doi.org/10.1016/j.nantod.2014.08.005*Nano-imaging enabled via self-assembly. *Nanotoday* **9** (2014), no. 5, 560–573

[33] M. Mitrea and M. Taylor, Potential theory on Lipschitz domains in Riemannian manifolds: L^P Hardy, and Hölder space results. *Comm. Anal. Geom.* **9** (2001), no. 2, 369–421 Zbl 0994.31001 MR 1846208

[34] A. I. Nachman, Reconstructions from boundary measurements. *Ann. of Math. (2)* **128** (1988), no. 3, 531–576 Zbl 0675.35084 MR 0970610

[35] A. I. Nachman, Global uniqueness for a two-dimensional inverse boundary value problem. *Ann. of Math. (2)* **143** (1996), no. 1, 71–96 Zbl 0857.35135 MR 1370758

[36] V. Ntziachristos, Going deeper than microscopy: the optical imaging frontier in biology. *Nature Methods* **7** (2010), no. 8, 603–614

[37] L. Päivärinta, A. Panchenko, and G. Uhlmann, Complex geometrical optics solutions for Lipschitz conductivities. *Rev. Mat. Iberoamericana* **19** (2003), no. 1, 57–72 Zbl 1055.35144 MR 1993415

[38] A. G. Ramm, *Inverse problems*. Mathematical and analytical techniques with applications to Ef.ngineering, Springer, New York, 2005 Zbl 1083.35002 MR 2838778

[39] G. Rozenblum, M. Ruzhansky, and D. Suragan, Isoperimetric inequalities for Schatten norms of Riesz potentials. *J. Funct. Anal.* **271** (2016), no. 1, 224–239 Zbl 1381.35111 MR 3494249

[40] S. Qin, C. F. Caskey, and K. W. Ferrara, Ultrasound contrast microbubbles in imaging and therapy: physical principles and engineering. *Phys. Med. Biol.* **54** (2009) **54**, no. 6, article no. R27

[41] M. Salo, Calderón problem, Lecture Notes, Citeseer, 2008 https://users.jyu.fi/~salomi/lecturenotes/calderon_lectures.pdf visited on 28 September 2025

[42] S. Senapati, M. Sini, and H. Wang, Recovering both the wave speed and the source function in a time-domain wave equation by injecting contrasting droplets. *Discrete Contin. Dyn. Syst.* **44** (2024), no. 5, 1446–1474 Zbl 1534.35450 MR 4714546

[43] M. Sini and H. Wang, The inverse source problem for the wave equation revisited: a new approach. *SIAM J. Math. Anal.* **54** (2022), no. 5, 5160–5181 Zbl 1497.35524 MR 4476945

[44] J. Sylvester and G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem. *Ann. of Math. (2)* **125** (1987), no. 1, 153–169 Zbl 0625.35078 MR 0873380

[45] G. Uhlmann, Inverse problems: seeing the unseen. *Bull. Math. Sci.* **4** (2014), no. 2, 209–279 Zbl 1325.35286 MR 3228575

[46] N. Valdivia, Uniqueness in inverse obstacle scattering with conductive boundary conditions. *Appl. Anal.* **83** (2004), no. 8, 825—851 Zbl 1076.35135 MR 2072311 Zbl 1076.35135

Received 12 August 2023; revised 22 May 2025.

Ahcene Ghandriche
Nanjing Center for Applied Mathematics, 211135 Nanjing, P. R. China;
gh.hsen.math@gmail.com

Mourad Sini
RICAM, Austrian Academy of Sciences, Altenbergerstrasse 69, 4040 Linz, Austria;
mourad.sini@oeaw.ac.at