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Essential norm resolvent estimates and essential numerical range

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Abstract. The main result of this paper are novel two-sided estimates of the essential resolvent norm for closed linear operators T. We prove that the growth of $\|(T-\lambda)^{-1}\|_{\mathrm{e}}$ is governed by the distance of a point $\lambda \in \rho(T) \setminus W_{\mathrm{e}}(T)$ to the essential numerical range $W_{\mathrm{e}}(T)$. We extend these bounds even to points $\lambda \in \mathbb{C} \setminus W_{\mathrm{e}}(T)$ outside the resolvent set $\rho(T)$ with $(T-\lambda)^{-1}$ replaced by the Moore–Penrose resolvent $(T-\lambda)^{\dagger}$. We use similar ideas to prove essential growth bounds in terms of the real part of the essential numerical range of generators of C_0-semigroups. Further, we study the essential approximate point spectrum $\sigma_{\mathrm{eap}}(T)$ and the essential minimum modulus $\gamma_{\mathrm{e}}(T)$, in particular, their relations to the various essential spectra and the essential norm of the Moore–Penrose inverse, respectively. An important consequence of our results are new perturbation results for the spectra and essential spectra (of type 2) for accretive and sectorial T. Applications e.g. to Schrödinger operators with purely imaginary rapidly oscillating potentials in $\mathbb{R}^d$ illustrate our results.

1. Introduction

Resolvent estimates are crucial tools not only in the theory of linear operators and semigroups, but also for perturbation theory and numerical approximations e.g. of eigenvalues or other types of spectra, see e.g. [2, 4, 16]. This is even more true for unbounded non-self-adjoint or, more generally, non-normal operators, for which small perturbations may cause large deviations of eigenvalues or spectra. The reason for this is that in the normal case the resolvent norm $\|(T-\lambda)^{-1}\| = (\operatorname{dist}(\lambda, \sigma(T)))^{-1}$ is controlled by the distance to the spectrum $\sigma(T)$, whereas in the non-normal case it is merely controlled by the distance to the numerical range $W(T) := \{(Tx, x) \in \mathbb{C} \mid x \in \mathcal{D}(T),\ \|x\| = 1\}$, more precisely, in the resolvent set $\rho(T) := \mathbb{C} \setminus \sigma(T)$

$$\frac{1}{\operatorname{dist}(\lambda, \sigma(T))} \leq \|(T-\lambda)^{-1}\| \leq \frac{1}{\operatorname{dist}(\lambda, W(T))}, \quad \lambda \in \rho(T) \setminus \overline{W(T)}. \tag{1.1}$$

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This classical resolvent estimate applies only outside of the closure of the numerical range, which may be much larger than the convex hull of the spectrum. Moreover, it does not distinguish between the very different parts of the spectrum such as isolated eigenvalues of finite algebraic multiplicity and essential spectrum.

The first aim, and the motivation, of this work is to establish an estimate of the so-called *essential norm* $\|(T-\lambda)^{-1}\|_{\mathrm{e}}$ of the resolvent which focuses on the non-discrete parts of the spectrum such as essential spectra of various types. Our main result shows that in this case, the role of the numerical range is taken over by the so-called *essential numerical range*, more precisely,

$$\frac{1}{\operatorname{dist}(\lambda, \sigma_{\mathrm{e}2}(T))} \leq \|(T-\lambda)^{-1}\|_{\mathrm{e}} \leq \frac{1}{\operatorname{dist}(\lambda, W_{\mathrm{e}}(T))}, \quad \lambda \in \rho(T) \setminus W_{\mathrm{e}}(T); \tag{1.2}$$

we also prove that this two-sided estimate extends to all points $\lambda \in \mathbb{C} \setminus W_{\mathrm{e}}(T)$ if we replace the resolvent $(T-\lambda)^{-1}$ by the so-called *Moore–Penrose resolvent* $(T-\lambda)^{\dagger}$.

In (1.2) the essential numerical range of an unbounded linear operator T is the set

$$\begin{aligned} W_{\mathrm{e}}(T) := \{\lambda \in \mathbb{C} \mid \exists\, (x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T),\ \|x_n\| = 1, \\ x_n \xrightarrow{w} 0,\ ((T-\lambda)x_n, x_n) \to 0\}, \end{aligned} \tag{1.3}$$

introduced and studied only recently in [5]; there it was also proved that $W_{\mathrm{e}}(T)$ is the smallest set capturing spectral pollution for any projection method. Further, the essential spectrum $\sigma_{\mathrm{e}2}(T)$ of type 2 is defined correspondingly, replacing the last condition in (1.3) by $\|(T-\lambda)x_n\| \to 0$, see (2.1), and the essential norm $\|\cdot\|_{\mathrm{e}}$ of a bounded linear operator B introduced by Calkin in [7] as the distance of B to the closed ideal of compact operators $\mathcal{K}(H)$, i.e.

$$\|B\|_{\mathrm{e}} := \operatorname{dist}(B, \mathcal{K}(H)) = \inf_{K \text{ compact}} \|B-K\|,$$

also plays a role in PDEs e.g. when studying double-layer potential operators, see [8].

As in the case of the essential numerical range itself, there are good reasons why it took so long to move forward from the essential resolvent norm estimates for bounded operators first derived by Stampfli and Williams in 1968 in the context of Banach algebras, see [36, Lemma 1 and Theorem 9]. While in the bounded case the essential numerical range can be lifted to a numerical range in the Calkin algebra, we show that this is no longer possible in the unbounded case and so we had to develop completely new techniques to derive (1.2).

The impact of our new essential resolvent estimates may be seen both from the results we obtain in establishing it and from the results we derive from it. The former include, firstly, a detailed study of the *essential approximate point spectrum*

$$\sigma_{\mathrm{eap}}(T) := \bigcap_{K \text{ compact}} \sigma_{\mathrm{ap}}(T+K),$$

first introduced by Rakocevic in [30] for bounded operators, see also [25]. Here the approximate point spectrum is defined as in (2.3) below. Secondly, we generalise and investigate the *essential minimum modulus*

$$\gamma_{\mathrm{e}}(T) := \inf \sigma_{\mathrm{e2}}(|T|) \in [0,\infty] \tag{1.4}$$

first introduced by Bouldin, see [6], to unbounded closed linear operators, in particular, we prove that, if T has closed range and $T^\dagger$ is its Moore–Penrose inverse,

$$\gamma_{\mathrm{e}}(T) = \|T^\dagger\|_{\mathrm{e}}^{-1}. \tag{1.5}$$

Thirdly, we show that, unlike the bounded case, the so-called *essential numerical range lifting problem* does *not* have a solution for unbounded T. This means that, in the alternative characterisation

$$W_{\mathrm{e}}(T) = \bigcap_{K \text{ compact}} \overline{W(T+K)}$$

of the essential numerical range, there need not exist a compact $K_0 \in \mathcal{K}(H)$ with $W_{\mathrm{e}}(T) = \overline{W(T+K_0)}$. Finally, our essential resolvent norm bounds inspired us to prove an estimate for the essential growth rate of a C_0-semigroup $(\tau(t))_{t\geq 0}$ with generator $-T$ which, if T is quasi-m-sectorial, takes the form

$$\|\tau(t)\|_{\mathrm{e}} \leq \mathrm{e}^{-t \inf \operatorname{Re} W_{\mathrm{e}}(T)}, \quad t \geq 0; \tag{1.6}$$

in the accretive case, an analogous result is derived.

The results we derive from the essential norm resolvent (1.2) split into two groups. The first group are perturbation results for the essential approximate point spectrum $\sigma_{\mathrm{eap}}(T+A)$ and the essential spectrum $\sigma_{\mathrm{e2}}(T+A)$ for accretive and sectorial T when the perturbation A is T-bounded with T-bound <1, e.g. we prove that

$$\begin{aligned}\sigma_{\mathrm{e2}}(T+A) &\subset \left\{\lambda \in \mathbb{C} \;\middle|\; \operatorname{Re}\lambda \geq -\frac{a}{1-b} - \frac{b}{1-b}|\operatorname{Im}\lambda|\right\} \\ &= -\frac{a}{1-b} + S_{\frac{\pi}{2}+\arctan\frac{b}{1-b}} \end{aligned}\tag{1.7}$$

if $\|Ax\| \leq a\|x\| + b\|Tx\|$, $x \in \mathcal{D}(T)$, with $b<1$ and an essential norm bound for the Moore–Penrose resolvent of $T+A$,

$$\|(T+A-\lambda)^\dagger\|_{\mathrm{e}} \leq \frac{1}{(1-b)|\operatorname{Re}\lambda| - (a+b|\operatorname{Im}\lambda|)}, \quad \lambda \notin -\frac{a}{1-b} + S_{\frac{\pi}{2}+\arctan\frac{b}{1-b}}.$$

Here $S_{\frac{\pi}{2}+\arctan\frac{b}{1-b}} := \left\{z \in \mathbb{C} \;\middle|\; |\arg z| \leq \frac{\pi}{2} + \arctan\frac{b}{1-b}\right\}$ denotes the sector with semi-angle $\frac{\pi}{2} + \arctan\frac{b}{1-b}$ and vertex 0.

$$\begin{tikzcd}
W_{\mathrm{e}5}(T) \arrow[r, dash, dashed, "\subset"] & W_{\mathrm{e}1}(T) \subset W_{\mathrm{e}}(T) \arrow[r, dash, dashed, "\subset"] & \overline{W(T)} & \\
\sigma_{\mathrm{e}2}(T) \arrow[u, dash, dashed, "\cup"] \arrow[r, dash, dashed, "\subset"] & \sigma_{\mathrm{eap}}(T) \arrow[u, dash, dashed, "\cup"] \arrow[r, dash, dashed, "\subset"] \arrow[dr, dash, dashed, "\subset"] & \sigma_{\mathrm{ap}}(T) \arrow[u, dash, dashed, "\cup"] \arrow[dr, dash, dashed, "\subset"] & \\
& & \sigma_{\mathrm{e}4}(T) \arrow[r, dash, dashed, "\subset"] & \sigma(T)
\end{tikzcd}$$

Figure 1. Connections of the essential approximate point spectrum with essential numerical ranges and essential spectra.

The second group of results we obtain employing the essential norm resolvent (1.2) are applications to concrete differential operators, including an advection-diffusion type operator, see e.g. [5], and Schrödinger operators with purely imaginary rapidly oscillating potentials in $\mathbb{R}^d$, see [29, 33, 38]. While for the latter the classical resolvent estimate (1.1) is not applicable/useful due to the lack of good enclosures of the numerical range, our new essential norm resolvent bound holds for all $\lambda \in \rho(T)$.

The paper is organised as follows. In Section 2, we study the essential approximate point spectrum of closed linear operators T and its relation to the various other, in general different types of essential spectra $\sigma_{\mathrm{e}i}(T)$ and essential numerical ranges $W_{\mathrm{e}i}(T)$, $i = 1, \dots, 5$, see Figure 1 for a schematic overview and Section 2 for all definitions. In particular, we show that

$$\sigma_{\mathrm{e}2}(T) \subset \sigma_{\mathrm{eap}}(T) \subset \sigma_{\mathrm{e}4}(T), \quad \sigma_{\mathrm{eap}}(T) \subset W_{\mathrm{e}}(T),$$

and that $\sigma_{\mathrm{eap}}(T)$ consists of $\sigma_{\mathrm{e}2}(T)$ plus possibly some 'holes' of $\sigma_{\mathrm{e}2}(T)$, which are the (bounded and unbounded) components of $\mathbb{C} \setminus \sigma_{\mathrm{e}2}(T)$, see Proposition 2.6. Further, in the chain of inclusions $W_{\mathrm{e}5}(T) \subset W_{\mathrm{e}1}(T) \subset W_{\mathrm{e}i}(T) = W_{\mathrm{e}}(T)$, $i = 1, \dots, 4$, see [5, 14, 15] we prove $\sigma_{\mathrm{eap}}(T) \subset W_{\mathrm{e}1}(T)$, but, in general, $\sigma_{\mathrm{eap}}(T) \not\subset W_{\mathrm{e}5}(T)$, see Theorem 2.13 and Example 2.14. Note that e.g. for singular non-symmetric perturbations of self-adjoint operators $W_{\mathrm{e}1}(T) = \mathbb{R}$, but $W_{\mathrm{e}}(T) = \mathbb{C}$ may occur, see [5, Example 3.5].

In Section 3 we introduce the essential minimum modulus $\gamma_{\mathrm{e}}(T)$ by means of the formula (1.4). We relate it to the essential spectrum of type 2 by showing that

$$\sigma_{\mathrm{e}2}(T) = \{\lambda \in \mathbb{C} \mid \gamma_{\mathrm{e}}(T - \lambda) = 0\}$$

and we prove two alternative characterisations of the essential minimum modulus, the first one in terms of compact operators under the assumption that $0 \notin \sigma_{\mathrm{eap}}(T)$,

$$\gamma_{\mathrm{e}}(T) = \sup_{K \text{compact}} \gamma(T + K),$$

and the second one by means of weakly null-sequences, see Proposition 3.3. The latter is our main tool to establish the identity (1.5) for the essential norm of the Moore–Penrose inverse.

In Section 4 we employ the results of Section 3 to establish the essential resolvent norm estimate (1.2), see Theorem 4.1 and Corollary 4.4. We illustrate this novel bound by applying it to Schrödinger operators $-\Delta + \mathrm{i}q$ in $L^2(\mathbb{R}^d)$ with rapidly oscillating purely imaginary potential $\mathrm{i}q$ such as $\mathrm{i}q(x) = \mathrm{i}(1+|x|^2)^{-1}\mathrm{e}^{|x|}\sin(\mathrm{e}^{|x|})$, $x \in \mathbb{R}^d$, see Example 4.8. For the corresponding m-sectorial operator T with $\sigma_{\mathrm{e}2}(T) = [0,\infty)$, see [29], we prove that

$$W_{\mathrm{e}}(T) = [0,\infty), \qquad \|(T-\lambda)^{-1}\|_{\mathrm{e}} = \begin{cases} \dfrac{1}{|\operatorname{Im}\lambda|}, & \operatorname{Re}\lambda \geq 0, \\ \dfrac{1}{|\lambda|}, & \operatorname{Re}\lambda < 0, \end{cases} \qquad \lambda \in \rho(T).$$

We also show that, unlike the bounded case, the essential numerical range of an unbounded closed linear operator is *not* the minimal closed convex set $W \subset \mathbb{C}$ so that the essential resolvent norm has at most linear growth $\|(T-\lambda)^{\dagger}\|_{\mathrm{e}} \leq (\operatorname{dist}(\lambda, W))^{-1}$, $\lambda \notin W$, see Example 4.10.

In Section 5 we prove a series of perturbation results for m-accretive and m-sectorial operators T with T-bounded perturbations A having T-bound < 1. The first group of results is for the perturbed approximate point spectrum $\sigma_{\mathrm{ap}}(T+A)$ and for the norm $\|(T+A)^{\dagger}\|$ of the perturbed Moore–Penrose inverse, see Theorem 5.1, the second group of results is for the perturbed essential spectrum $\sigma_{\mathrm{e}2}(T+A)$ of type 2 and for the essential norm $\|(T+A)^{\dagger}\|_{\mathrm{e}}$ of the perturbed Moore–Penrose inverse, see Theorem 5.11. All our enclosures for the perturbed spectra $\sigma_{\mathrm{ap}}(T+A)$, $\sigma_{\mathrm{e}2}(T+A)$ and the bounds for $\|(T+A)^{\dagger}\|$, $\|(T+A)^{\dagger}\|_{\mathrm{e}}$ are explicit in the T-boundedness constants a, b in $\|Ax\| \leq a\|x\| + b\|Tx\|$, $x \in \mathcal{D}(T)$, with $b < 1$ and in the sectoriality angle $\vartheta \in \left[0, \frac{\pi}{2}\right]$, with $\vartheta = \frac{\pi}{2}$ for accretive T, see (1.7) and (5.11). This dependence on the sectoriality angle ϑ is illustrated in Figure 2 below.

In Section 6 we disprove that the essential numerical range lifting problem has a solution for unbounded closed linear operators. This is in sharp contrast to the bounded case where the essential numerical range and the essential norm can even be lifted simultaneously, i.e. there exists a compact $K_0 \in \mathcal{K}(H)$ with $W_{\mathrm{e}}(T) = \overline{W(T+K_0)}$ and $\|T\|_{\mathrm{e}} = \|T+K_0\|$. This failure in the unbounded case is also the reason why it is *not* possible to reduce proofs for the essential numerical range and essential resolvent norms to results on the numerical range and the usual operator norm. Our counter-example shows, in fact, much more, since we even construct a normal m-sectorial operator T for which there exists an open neighbourhood U of $W_{\mathrm{e}}(T)$ such that $W(T+K) \not\subset U$ for all compact $K \in \mathcal{K}(H)$.

In the last Section 7 we prove bounds for the essential growth of C_0-semigroups $(\tau(t))_{t\geq 0}$ with quasi-m-accretive generator $-T$. While in the m-sectorial case, the growth bound may be estimated by the infimum $\beta_e(T) = \inf \operatorname{Re} W_e(T)$ of the real part of the essential numerical range, see (1.6), the upper bound for the growth bound in the m-accretive case is given by

$$\beta_e(T) := \inf_{(x_n)_{n\in\mathbb{N}} \in \mathcal{E}(T)} \liminf_{n\to\infty} \operatorname{Re}(Tx_n, x_n) \ \ (\leq \inf \operatorname{Re} W_e(T)),$$

see Theorem 7.1, and the inequality may be strict as one of our examples shows; here $\mathcal{E}(T)$ is the set of all normalised weakly null-sequences in $\mathcal{D}(T)$. Our result also proves the novel criterion $\beta_e(T) > 0$ for the semigroup $(\tau(t))_{t\geq 0}$ to be quasi-compact, or asymptotically compact according to [1, 12].

Throughout this paper, we use the following notation. By H we denote a (complex) infinite-dimensional separable Hilbert space with inner product $(\cdot,\cdot)$ and induced norm $\|\cdot\|$. A sequence $(x_n)_{n\in\mathbb{N}} \subset H$ is called *normalised* if $\|x_n\| = 1$, $n \in \mathbb{N}$, and a *weakly null-sequence* if $x_n \xrightarrow{w} 0$, i.e. $(x_n, x) \to 0$ for all $x \in H$. If $M \subset H$ is a (not necessarily closed) subspace, we denote by $M^\perp$ its orthogonal complement and, if in addition M is closed, we denote by $P_M \in L(H)$ the orthogonal projection in H onto M. For a closed linear operator T in H with domain $\mathcal{D}(T) \subset H$, we denote the spectrum, point spectrum, resolvent set, kernel and range of T by $\sigma(T), \sigma_p(T), \rho(T), \ker(T)$ and $\mathcal{R}(T)$, respectively. Further, T is called *accretive* if

$$W(T) \subset \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0\} =: \mathbb{H}_+$$

and *sectorial* if there exists $\theta \in \left[0, \frac{\pi}{2}\right)$ such that $W(T)$ is contained in the (closed) sector S_θ with vertex 0 and semi-angle θ, that is

$$W(T) \subset \{z \in \mathbb{C} \mid |\arg(z)| \leq \theta\} =: S_\theta \subset \mathbb{H}_+.$$

If T is accretive or sectorial and $\lambda \in \rho(T)$ for some (and hence for all) $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < 0$, we call T *m-accretive* or *m-sectorial*, respectively. Further, T is called *quasi-m-accretive/quasi-m-sectorial* if $T + \beta$ is m-accretive/m-sectorial for some $\beta \in \mathbb{R}$. We refer to [19, Section V.3.10] and [9] for more information on these and related concepts.

Finally, we use the common conventions $\sup \emptyset := -\infty$, $\inf \emptyset := \infty$ and $\frac{1}{\infty} := 0$, so that $\operatorname{dist}(\lambda, \emptyset) = \infty$ for every $\lambda \in \mathbb{C}$.

2. Essential approximate point spectrum and essential numerical range

For (bounded or unbounded) linear non-self-adjoint operators, there are various notions of essential spectra which are, in contrast to the self-adjoint case, no longer equivalent in general, see e.g. [11, Sections I.4 and IX.1]. Here we only need the essential spectra of type 2, see (2.1), and of type 4 given by

$$\sigma_{e2}(T) := \{\lambda \in \mathbb{C} \mid \exists\, (x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T),\ \|x_n\| = 1,\ x_n \xrightarrow{w} 0, \\ (T-\lambda)x_n \to 0\}, \tag{2.1}$$

$$\sigma_{e4}(T) := \bigcap_{K\in\mathcal{K}(H)} \sigma(T+K). \tag{2.2}$$

In this section we study the essential approximate point spectrum of closed linear operators T acting in a Hilbert space; this maybe not-so-well-known type of essential spectrum was first introduced by Rakocevic in [30] in the bounded case. The essential approximate point spectrum may be viewed as the essential version of the *approximate point spectrum* of T, which is defined as

$$\sigma_{ap}(T) := \{\lambda \in \mathbb{C} \mid (x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T),\ \|x_n\| = 1,\ \|(T-\lambda)x_n\| \to 0\} \subset \sigma(T). \tag{2.3}$$

Definition 2.1. Let T be a closed linear operator. We define the *essential approximate point spectrum* $\sigma_{eap}(T)$ of T as

$$\sigma_{eap}(T) := \bigcap_{K\in\mathcal{K}(H)} \sigma_{ap}(T+K).$$

Note that the notion of essential approximate point spectrum is used differently by different authors; e.g. in [26] it denotes the set $\sigma_{ap}(T)$.

The following properties of $\sigma_{eap}(T)$ are immediate from its definition and the closedness of the approximate point spectrum.

Remark 2.2. Let T be a closed linear operator. Then $\sigma_{eap}(T) = \sigma_{eap}(T+K)$ for every compact operator K, $\sigma_{eap}(T)$ is closed and

$$\sigma_{eap}(T) \subset \sigma_{e4}(T).$$

In order to relate the essential approximate point spectrum to $\sigma_{e2}(T)$ in (2.1) and $\sigma_{ap}(T)$, we establish an equivalent characterisation of $\sigma_{eap}(T)$ in terms of Fredholm properties. Recall that a closed linear operator T is called *upper semi-Fredholm* if $\mathcal{R}(T)$ is closed and $\dim\ker(T) < \infty$ and *lower semi-Fredholm* if $\mathcal{R}(T)$ is closed and $\dim \mathcal{R}(T)^\perp < \infty$, while T is called *Fredholm/semi-Fredholm* if T is upper and/or lower semi-Fredholm. If T is a semi-Fredholm operator, we define the *index* of T as

$$\mathrm{i}(T) := \dim\ker(T) - \dim \mathcal{R}(T)^\perp \in [-\infty, \infty],$$

see e.g. [11, Section I.3]. Then, since T is assumed to be closed, the *essential resolvent sets of type* 2 and 4, respectively, satisfy

$$\rho_{e2}(T) := \mathbb{C} \setminus \sigma_{e2}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda \text{ is upper semi-Fredholm}\},$$
$$\rho_{e4}(T) := \mathbb{C} \setminus \sigma_{e4}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda \text{ is Fredholm, } \mathrm{i}(T-\lambda) = 0\},$$

see e.g. [11, Theorems IX.1.3 (i) and IX.1.4]. Further, it is not difficult to see that *the set of regular points* of T can be described as

$$\rho_{\mathrm{ap}}(T) := \mathbb{C} \setminus \sigma_{\mathrm{ap}}(T) = \{\lambda \in \mathbb{C} \mid \mathcal{R}(T-\lambda) \text{ is closed, } T - \lambda \text{ is injective}\}$$

and hence $\rho_{\mathrm{ap}}(T) \subset \rho_{e2}(T)$ or, equivalently, $\sigma_{e2}(T) \subset \sigma_{\mathrm{ap}}(T)$.

Note that the assumption that T is densely defined is not needed in the proofs of the above equalities for $\rho_{e2}(T)$ and $\rho_{e4}(T)$ due to [11, Remark I.3.27] and by working with $\mathcal{R}(T-\lambda)^{\perp}$ instead of $\ker(T^* - \bar{\lambda})$, compare the proof of Lemma 2.3 below.

To characterise $\sigma_{\mathrm{eap}}(T)$ in terms of Fredholm properties, we generalise a result due to Yood in [39], see also [25, Theorem 8.7.1], to unbounded operators.

Lemma 2.3. *Let T be a closed linear operator with $\lambda \in \rho_{e2}(T)$. Then $\lambda \in \rho_{e2}(T)$ with* $\mathrm{i}(T-\lambda) \le 0$ *if and only if there exists $K \in \mathcal{K}(H)$ with*

(i) $\dim \mathcal{R}(K) = \dim\ker(T-\lambda) < \infty$*;*

(ii) $\lambda \in \rho_{\mathrm{ap}}(T+K)$.

Proof. The proof in [25, Theorem 8.7.1] for bounded operators in Banach spaces may be carried over to the case of closed linear operators in a Hilbert space. For the convenience of the reader, we briefly sketch it here, also because we do not assume that T is densely defined.

For the implication "$\Longrightarrow$", note that $\lambda \in \rho_{e2}(T)$ with $\mathrm{i}(T-\lambda) \le 0$ implies that $\dim\ker(T-\lambda) < \infty$. Let $x_1, \dots, x_n \in \ker(T-\lambda)$ be an orthonormal basis. Since $\mathrm{i}(T-\lambda) \le 0$, we have $\dim \mathcal{R}(T-\lambda)^{\perp} \ge \dim\ker(T-\lambda) = n$ and hence there exists an orthonormal system in $y_1, \dots, y_n \in \mathcal{R}(T-\lambda)^{\perp}$. Then the operator

$$K := \sum_{i=1}^{n} (\cdot, x_i) y_i$$

satisfies $K \in \mathcal{K}(H)$ with $\dim \mathcal{R}(K) = n = \dim\ker(T-\lambda)$ and $\mathcal{R}(T+K-\lambda)$ is closed due to the stability of $\rho_{e2}(T)$ under compact perturbations, see [11, Theorem I.3.21 and Remark I.3.27]. To show that $T+K-\lambda$ is injective, let $v_0 \in \ker(T+K-\lambda)$. Then, because $y_j \perp \mathcal{R}(T-\lambda)$, we obtain

$$\begin{aligned} 0 &= ((T+K-\lambda)v_0, y_j) \\ &= ((T-\lambda)v_0, y_j) + \sum_{i=1}^{n} (v_0, x_i)(y_i, y_j) = (v_0, x_j), \quad j = 1, \dots, n, \end{aligned}$$

which shows that $v_0 \in \ker(T-\lambda)^\perp$ and $Kv_0 = 0$. The latter implies that $(T-\lambda)v_0 = (T+K-\lambda)v_0 = 0$ and hence $v_0 \in \ker(T-\lambda) \cap \ker(T-\lambda)^\perp = \{0\}$.

For the implication "$\Longleftarrow$", we note that if $K \in \mathcal{K}(H)$ is such that (i) and (ii) hold, then $\lambda \in \rho_{\mathrm{ap}}(T+K) \subset \rho_{\mathrm{e2}}(T+K) = \rho_{\mathrm{e2}}(T)$ and the index satisfies $\mathrm{i}(T-\lambda) = \mathrm{i}(T+K-\lambda) = -\dim \mathcal{R}(T+K-\lambda)^\perp \le 0$ by [11, Theorem I.3.21 and Remark I.3.27]. ∎

Theorem 2.4. *Let T be a closed linear operator. Then*

$$\rho_{\mathrm{eap}}(T) = \{\lambda \in \mathbb{C} \mid \lambda \in \rho_{\mathrm{e2}}(T),\ \mathrm{i}(T-\lambda) \le 0\}.$$

Proof. First, let $\lambda \in \rho_{\mathrm{eap}}(T)$. Then, by Definition 2.1, there exists a $K \in \mathcal{K}(H)$ such that $\lambda \in \rho_{\mathrm{ap}}(T+K) \subset \rho_{\mathrm{e2}}(T+K)$ and $\mathrm{i}(T+K-\lambda) = -\dim \mathcal{R}(T+K-\lambda)^\perp \le 0$. Then the stability result for upper Fredholm operators and for the index, see [11, Theorem 1.3.21 and Remark I.3.27] implies that $\lambda \in \rho_{\mathrm{e2}}(T)$ and

$$\mathrm{i}(T-\lambda) = \mathrm{i}(T+K-\lambda) \le 0.$$

Vice versa, assume that $\lambda \in \rho_{\mathrm{e2}}(T)$ with $\mathrm{i}(T-\lambda) \le 0$. Then Lemma 2.3 immediately yields that there exists a $K \in \mathcal{K}(H)$ such that $\lambda \in \rho_{\mathrm{ap}}(T+K)$ and therefore $\lambda \in \rho_{\mathrm{eap}}(T)$ by Definition 2.1. ∎

Corollary 2.5. *Let T be a closed linear operator. Then*

$$\sigma_{\mathrm{e2}}(T) \subset \sigma_{\mathrm{eap}}(T) \subset \sigma_{\mathrm{ap}}(T) \subset \overline{W(T)}.$$

Proof. The first inclusion is immediate from Theorem 2.4; the last two inclusions are obvious from the respective definitions, see Definition 2.1 and (2.3). ∎

Proposition 2.6. *Let T be a closed linear operator. Then*

(i) $\sigma_{\mathrm{e2}}(T) \subset \sigma_{\mathrm{eap}}(T) \subset \sigma_{\mathrm{e4}}(T)$;

(ii) $\partial\sigma_{\mathrm{e4}}(T) \subset \partial\sigma_{\mathrm{eap}}(T) \subset \partial\sigma_{\mathrm{e2}}(T)$.

More precisely, $\sigma_{\mathrm{eap}}(T)$ consists of $\sigma_{\mathrm{e2}}(T)$ and possibly some components of $\rho_{\mathrm{e2}}(T)$, while $\sigma_{\mathrm{e4}}(T)$ consists of $\sigma_{\mathrm{eap}}(T)$ and possibly some components of $\rho_{\mathrm{eap}}(T)$.

Proof. (i) The claim was proved in Corollary 2.5 and Remark 2.2.

(ii) It is easy to see that, due to claim (i), it suffices to show that $\partial\sigma_{\mathrm{eap}}(T) \subset \sigma_{\mathrm{e2}}(T)$ and $\partial\sigma_{\mathrm{e4}}(T) \subset \sigma_{\mathrm{eap}}(T)$. Let $\lambda \in \partial\sigma_{\mathrm{eap}}(T)$. Then $\lambda \in \sigma_{\mathrm{eap}}(T)$ since the latter is closed. If $\lambda \notin \sigma_{\mathrm{e2}}(T)$, then $\mathrm{i}(T-\lambda) > 0$ by Theorem 2.4. Since $\rho_{\mathrm{e2}}(T)$ is open and the index is stable, see e.g. [19, Theorem IV.5.5.31], there exists $\varepsilon > 0$ such that all $\mu \in \mathbb{C}$ with $|\lambda - \mu| < \varepsilon$ satisfy $\mu \in \rho_{\mathrm{e2}}(T)$ and $\mathrm{i}(T-\mu) = \mathrm{i}(T-\lambda) > 0$, a contradiction to $\lambda \in \partial\sigma_{\mathrm{eap}}(T)$. The inclusion $\partial\sigma_{\mathrm{e4}}(T) \subset \sigma_{\mathrm{eap}}(T)$ is proved analogously.

To prove the last claims, note that if $\lambda \in \sigma_{\mathrm{eap}}(T) \setminus \sigma_{\mathrm{e2}}(T)$, then $T-\lambda$ is upper semi-Fredholm with $\mathrm{i}(T-\lambda) > 0$, see the proof of claim (ii). Denote by Δ the component of $\rho_{\mathrm{e2}}(T)$ with $\lambda \in \Delta$. Since the index is stable, see e.g. [19, Theorem IV.5.5.31], it is locally constant and hence $\mathrm{i}(T-\mu) = \mathrm{i}(T-\lambda) > 0$ for all $\mu \in \Delta$ and thus $\Delta \subset \sigma_{\mathrm{eap}}(T)$. The claim for $\sigma_{\mathrm{e4}}(T)$ is shown analogously. ∎

Remark 2.7. For bounded T, the bounded components of $\rho_{\mathrm{e2}}(T)$ and $\rho_{\mathrm{eap}}(T)$ are the holes of $\sigma_{\mathrm{e2}}(T)$ and $\sigma_{\mathrm{eap}}(T)$, respectively, see [25, Theorem 8.14.2]; for a comprehensive study of holes of various essential spectra of bounded operators see also [21]. For unbounded T also the unbounded components of $\rho_{\mathrm{e2}}(T)$ and $\rho_{\mathrm{eap}}(T)$ have to be taken into account which may be interpreted as 'holes at infinity'.

A simple example how large the difference between $\sigma_{\mathrm{eap}}(T)$ and $\sigma_{\mathrm{e4}}(T)$ is provided by maximal symmetric operators with non-equal defect numbers where an entire half-plane has to be added. The same may happen for the difference between $\sigma_{\mathrm{e2}}(T)$ and $\sigma_{\mathrm{eap}}(T)$, see Example 2.14 below.

Example 2.8. Let T be a closed symmetric operator with defect index $(0,k)$ with $k \in \mathbb{N}$, i.e. $\mathcal{R}(T-\mathrm{i}) = H$, $\mathcal{R}(T+\mathrm{i}) = \overline{\mathcal{R}(T+\mathrm{i})} \neq H$. Then

$$\sigma_{\mathrm{e2}}(T) = \sigma_{\mathrm{eap}}(T) = \sigma_{\mathrm{ap}}(T) = \mathbb{R}, \quad \sigma_{\mathrm{e4}}(T) = \sigma(T) = \{z \in \mathbb{C} \mid \operatorname{Im} z \leq 0\}.$$

Proof. Since T is symmetric, $W(T) \subset \mathbb{R}$, $\mathcal{R}(T-\lambda)$ is closed and $\ker(T-\lambda) = \{0\}$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$; because the defect index of T is $(0,k)$ with $k \neq 0$, we have

$$\mathrm{i}(T-\mathrm{i}) = -\dim \mathcal{R}(T-\mathrm{i})^{\perp} = 0, \quad \mathrm{i}(T+\mathrm{i}) = -\dim \mathcal{R}(T+\mathrm{i})^{\perp} = -k < 0, \tag{2.4}$$

and $\sigma(T) = \{z \in \mathbb{C} \mid \operatorname{Im} z \leq 0\}$. Then, by Corollary 2.5,

$$\sigma_{\mathrm{e2}}(T) \subset \sigma_{\mathrm{eap}}(T) \subset \sigma_{\mathrm{ap}}(T) \subset \mathbb{R}.$$

If $\sigma_{\mathrm{e2}}(T) \neq \mathbb{R}$, then $\rho_{\mathrm{e2}}(T)$ is connected and hence $\mathrm{i}(T-\mathrm{i}) = \mathrm{i}(T+\mathrm{i})$ since the index is locally constant, see e.g. [19, Theorem IV.5.31], a contradiction to (2.4). The local stability of the index and (2.4) also show that $\{z \in \mathbb{C} \mid \operatorname{Im} z > 0\} \subset \rho_{\mathrm{e4}}(T)$. On the other hand, $T-\lambda$ is Fredholm with $\mathrm{i}(T-\lambda) = -k < 0$ for $\lambda \in \mathbb{C}$, $\operatorname{Im}\lambda < 0$, and so $\{z \in \mathbb{C} \mid \operatorname{Im} z < 0\} \subset \sigma_{\mathrm{e4}}(T)$; thus $\sigma_{\mathrm{e4}}(T) = \{z \in \mathbb{C} \mid \operatorname{Im} z \leq 0\}$ since $\sigma_{\mathrm{e4}}(T)$ is closed. ∎

The following proposition shows that, like the essential spectra of type 1, 2, 3, 4, also the essential approximate point spectrum is not only invariant under compact perturbations by Definition 2.1, but also under relatively compact perturbations.

Recall that a linear operator A is called T*-compact* (or *relatively compact with respect to* T) if $\mathcal{D}(T) \subset \mathcal{D}(A)$ and $A|_{\mathcal{D}(T)}$ is compact as an operator from $\mathcal{D}(T)$

equipped with the graph norm $\|\cdot\|_T$ of T into H, i.e., whenever for every bounded sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T)$ with $(Tx_n)_{n\in\mathbb{N}}$ bounded, $(Ax_n)_{n\in\mathbb{N}}$ has a convergent subsequence, see e.g. [19, Section IV.1.3].

Proposition 2.9. *Let T be a closed densely defined operator. Then*

$$\sigma_{\mathrm{eap}}(T) = \bigcap_{K\ T\text{-compact}} \sigma_{\mathrm{ap}}(T+K);$$

in fact, $\sigma_{\mathrm{eap}}(T) = \sigma_{\mathrm{eap}}(T+K)$ for every T-compact K.

Proof. The inclusion "$\supseteq$" is immediate from Definition 2.1 since compact operators are T-compact. If K is an arbitrary T-compact operator, then Theorem 2.4 together with the stability theorem for $\sigma_{\mathrm{e}2}(T)$ and the index, see e.g. [19, Theorem IV.5.5.26], imply that $\sigma_{\mathrm{eap}}(T) = \sigma_{\mathrm{eap}}(T+K)$. Now, the inclusion "$\subseteq$" follows from this and Corollary 2.5 which yield

$$\sigma_{\mathrm{eap}}(T) = \sigma_{\mathrm{eap}}(T+K) \subset \sigma_{\mathrm{ap}}(T+K). \qquad \blacksquare$$

Theorem 2.4 also helps to prove that the essential approximate point spectrum satisfies a spectral mapping theorem like all other essential spectra, see [11, Theorem IX.2.3], and that it is stable when the resolvent difference of two operators is compact like the essential spectra of type 1, 2, 3, 4.

Theorem 2.10. *Let T be a closed linear operator and $\lambda \in \rho(T)$. Then, for every $\mu \in \mathbb{C}\setminus\{\lambda\}$,*

$$\mu \in \sigma_{\mathrm{eap}}(T) \iff (\mu-\lambda)^{-1} \in \sigma_{\mathrm{eap}}((T-\lambda)^{-1}).$$

Proof. If $\mu \in \sigma_{\mathrm{eap}}(T)$, then either $\mu \in \sigma_{\mathrm{e}2}(T)$ or $\mu \in \rho_{\mathrm{e}2}(T)$ but $\mathrm{i}(T-\mu) > 0$ by Theorem 2.4. In the former case, the spectral mapping theorem for $\sigma_{\mathrm{e}2}(T)$, see e.g. [11, Theorem IX.2.3 (iii)], yields that

$$(\mu-\lambda)^{-1} \in \sigma_{\mathrm{e}2}((T-\lambda)^{-1}) \subset \sigma_{\mathrm{eap}}((T-\lambda)^{-1}).$$

In the case $\mu \in \rho_{\mathrm{e}2}(T)$ but $\mathrm{i}(T-\mu) > 0$, then [11, Theorem IX.2.3 (iii)] yields that $(\mu-\lambda)^{-1} \in \rho_{\mathrm{e}2}((T-\lambda)^{-1})$. If $\mathrm{i}(T-\mu) = \infty$, i.e. μ is an eigenvalue of T with infinite geometric multiplicity, then, by [11, Theorem IX.2.3 (ii)], $(\mu-\lambda)^{-1}$ is an eigenvalue of $(T-\lambda)^{-1}$ with infinite geometric multiplicity, i.e.

$$\mathrm{i}((T-\lambda)^{-1} - (\mu-\lambda)^{-1}) = \infty.$$

Hence, Theorem 2.4 shows that $(\mu-\lambda)^{-1} \in \sigma_{\mathrm{eap}}((T-\lambda)^{-1})$. If, on the other hand, $0 < \mathrm{i}(T-\mu) < \infty$, then $T-\mu$ is Fredholm and, proceeding as in the proof of [11, Theorem IX.2.3 (iii)], we arrive at

$$\mathrm{i}((T-\lambda)^{-1} - (\mu-\lambda)^{-1}) = \mathrm{i}(T-\mu) > 0.$$

Thus $(\mu-\lambda)^{-1} \in \sigma_{\mathrm{eap}}((T-\lambda)^{-1})$ by Theorem 2.4, as required for "$\Longrightarrow$". The reverse implication "$\Longleftarrow$" follows analogously. ∎

The next corollary is an immediate consequence of Theorem 2.10 and the invariance of $\sigma_{\mathrm{eap}}(T)$ under compact perturbations, see Remark 2.2.

Corollary 2.11. *Let T and A be closed linear operators. Suppose that there exists $\lambda \in \rho(T) \cap \rho(A)$ so that $(T-\lambda)^{-1} - (A-\lambda)^{-1}$ is compact. Then $\sigma_{\mathrm{eap}}(T) = \sigma_{\mathrm{eap}}(A)$.*

Remark 2.12. So far, all results in this section also generalise to separable Banach spaces with essentially the same proofs, except for the proof of Lemma 2.3. There, for a closed (not necessarily densely defined) linear operator T in a separable Banach space, one follows the lines of the proof in the bounded case [25, Theorem 8.7.1], merely replacing every instance of $\ker(T^*)$ by the annihilator of $\mathcal{R}(T)$.

Next, we study the relation of the essential approximate point spectrum to the essential numerical range $W_{\mathrm{e}}(T)$, see (1.3), and its variants $W_{\mathrm{e}i}(T) \subset W_{\mathrm{e}}(T)$, $i = 1, 2, 3, 4, 5$. While all these closed and convex sets coincide for bounded T, see [13], this is no longer true in the unbounded case, see [5] for a comprehensive treatment of $W_{\mathrm{e}i}(T)$ for $i = 1, 2, 3, 4$ and [14] for $i = 5$. However, for a closed linear operator T, one still has $W_{\mathrm{e}}(T) = W_{\mathrm{e}3}(T) = W_{\mathrm{e}2}(T)$ or, spelled out,

$$W_{\mathrm{e}}(T) = \bigcap_{K \in \mathcal{K}(H)} \overline{W(T+K)} = \bigcap_{\substack{F \in \mathcal{K}(H) \\ \dim \mathcal{R}(F) < \infty}} \overline{W(T+F)}. \tag{2.5}$$

Since $\sigma_{\mathrm{ap}}(T+K) \subset \overline{W(T+K)}$, Definition 2.1 and (2.5) yield that $\sigma_{\mathrm{eap}}(T) \subset W_{\mathrm{e}}(T)$.

The other essential numerical ranges defined as

$$\begin{aligned} W_{\mathrm{e}1}(T) &:= \bigcap_{V \in \mathcal{V}_{<\infty}} \overline{W(T|_{V^\perp \cap \mathcal{D}(T)})}, \\ W_{\mathrm{e}4}(T) &:= \{\lambda \in \mathbb{C} \mid \exists\, (e_n) \subset \mathcal{D}(T), \text{ orthonormal with } (Tx_n, x_n) \to \lambda\}, \\ W_{\mathrm{e}5}(T) &:= \left\{ \lambda \in \mathbb{C} \;\middle|\; \begin{array}{l} \exists (x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T),\ \|x_n\| = 1,\ x_n \xrightarrow{w} 0 \\ \forall m \in \mathbb{N}\ (Tx_n, x_m) \to 0,\ (Tx_n, x_n) \to \lambda \end{array} \right\}, \end{aligned} \tag{2.6}$$

where $\mathcal{V}_{<\infty}$ is the set of all finite-dimensional subspaces of H, are in general no longer equal to $W_{\mathrm{e}}(T)$, they only satisfy

$$\sigma_{\mathrm{e}2}(T) \subset W_{\mathrm{e}5}(T) \subset W_{\mathrm{e}1}(T) \subset W_{\mathrm{e}i}(T) = W_{\mathrm{e}}(T), \quad i = 2, 3, 4, \tag{2.7}$$

see [5, Theorem 3.1], [14, Corollary 2.4 and Theorem 2.5], [15], and the two inclusions between the essential numerical ranges may be strict. The equality $W_{\mathrm{e}1}(T) = W_{\mathrm{e}}(T)$ holds if $\mathcal{D}(T) \cap \mathcal{D}(T^*)$ is dense or $W(T) \neq \mathbb{C}$, and $W_{\mathrm{e}5}(T) = W_{\mathrm{e}1}(T)$ holds if $\mathcal{D}(T) \cap \mathcal{D}(T^*)$ is a core for T.

Our next theorem and Example 2.14 below reveal where the essential approximate point spectrum is located in the chain of inclusions (2.7), namely $\sigma_{\mathrm{eap}}(T) \subset W_{\mathrm{e}1}(T)$, but in general $\sigma_{\mathrm{eap}}(T) \not\subset W_{\mathrm{e}5}(T)$.

To prove this inclusion, we invoke a type of essential numerical range which was introduced in [5, Remark 4.2] merely to show that in (2.5) compact perturbations cannot be replaced by relatively compact ones; namely

$$\widetilde{W_{\mathrm{e}}}(T) := \bigcap_{K\ T\text{-compact}} \overline{W(T+K)} \subset W_{\mathrm{e}}(T).$$

Theorem 2.13. *Let T be a closed linear operator. Then*

$$\sigma_{\mathrm{eap}}(T) \subset \widetilde{W_{\mathrm{e}}}(T) \subset W_{\mathrm{e}1}(T) \subset W_{\mathrm{e}4}(T) \subset W_{\mathrm{e}i} \subset W_{\mathrm{e}}(T), \quad i = 2,3;$$

if $\overline{\mathcal{D}(T) \cap \mathcal{D}(T^*)}$ is a core for T, then $\sigma_{\mathrm{eap}}(T) \subset W_{\mathrm{e}i}(T) = W_{\mathrm{e}}(T)$, $i = 1,2,3,4,5$.

Proof. By Corollary 2.9, we have $\sigma_{\mathrm{eap}}(T) \subset \widetilde{W_{\mathrm{e}}}(T)$ and thus it suffices to show that $\widetilde{W_{\mathrm{e}}}(T) \subset W_{\mathrm{e}1}(T)$. Parts of our proof are similar to parts of the proof of [5, Theorem 3.1], but unlike there, we do not assume any density of domains here.

Suppose that $\widetilde{W_{\mathrm{e}}}(T) \subset W_{\mathrm{e}1}(T)$ is false, i.e. there exists $\lambda \in \widetilde{W_{\mathrm{e}}}(T)$ and $V \in \mathcal{V}_{<\infty}$ such that $\lambda \notin \overline{W(T|_{V^\perp \cap \mathcal{D}(T)})}$. Since the latter is a closed and convex set, the strong separation property, see e.g. [24, Theorem 3.6.9], shows that there exists a closed half-plane $\mathbb{H} \subset \mathbb{C}$ such that $\lambda \notin \mathbb{H}$ and $\overline{W(T|_{V^\perp \cap \mathcal{D}(T)})} \subset \mathbb{H}$. After possible shift and rotation, we may assume that $\mathbb{H} = \{\mu \in \mathbb{C} \mid \operatorname{Re}\mu \geq 0\}$ and so $\operatorname{Re}\lambda < 0$. As in the proof of [5, Theorem 3.1], we infer that there exists a (not necessarily orthogonal) finite-rank projection $P \in L(H)$ such that $\mathcal{R}(P) = V$ and $\mathcal{R}(P^*) \subset \mathcal{D}(T)$. As in the proof of [5, Theorem 3.1, p. 14], we set

$$K := -TP^* - PT + PTP^* + \mu PP^*, \quad \mathcal{D}(K) := \mathcal{D}(T),$$

which now no longer has a compact closure. Nonetheless, the properties of P imply that the operators PP^* and TP^* are bounded and of finite rank hence compact. If $(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T)$ is such that $x_n, Tx_n \to 0$ as $n \to \infty$, then, trivially, $PTx_n \to 0$ as $n \to \infty$. Hence, PT and thus also K, are T-bounded. Furthermore, $\mathcal{R}(K) \subset \operatorname{span}(\mathcal{R}(TP^*) \cup \mathcal{R}(P)) \in \mathcal{V}_{<\infty}$, and so K is T-degenerate and thus T-compact, see e.g. [19, Remark IV.1.1.13]. Further, we observe that

$$T + K = ((I-P)T(I-P^*) + \mu PP^*)|_{\mathcal{D}(T)} \tag{2.8}$$

and, for arbitrary $x \in \mathcal{D}(T)$ with $\|x\| = 1$,

$$(I-P^*)x \in \mathcal{R}(I-P^*) \cap \mathcal{D}(T) = \mathcal{R}(P)^\perp \cap \mathcal{D}(T) = V^\perp \cap \mathcal{D}(T). \tag{2.9}$$

Hence combining a simple computation with (2.8) and (2.9), we find that

$$((T+K)x,x) = (T(I-P^*)x,(I-P^*)x) + \mu\|P^*x\| \\ \in \{tz + s\mu \mid z \in W(T|_{V^\perp \cap \mathcal{D}(T)}),\ t,s \geq 0\}.$$

Since $\mu \in W(T|_{V^\perp \cap \mathcal{D}(T)}) \subset \mathbb{H}$, we deduce $W(T+K) \subset \mathbb{H}$. Hence, it follows that $\lambda \in \widetilde{W_\mathrm{e}}(T) \subset \overline{W(T+K)} \subset \mathbb{H}$, a contradiction to $\operatorname{Re}\lambda < 0$. ■

Finally, we give an example where $\sigma_{\mathrm{eap}}(T) \not\subset W_{\mathrm{e}5}(T)$ for which it was already proved in [14, Example 2.10] that $W_{\mathrm{e}5}(T) \subsetneq W_{\mathrm{e}1}(T)$.

Example 2.14. Let $H = \ell^2(\mathbb{N})$ and $(e_n)_{n\in\mathbb{N}} \subset \ell^2(\mathbb{N})$ be the standard basis of H. Consider the operator T in $\ell^2(\mathbb{N})$ given by

$$Te_1 := 0, \quad Te_n := \sum_{k=1}^{n-1} e_k, \quad n \geq 2, \quad \mathcal{D}(T) := \{x \in \ell^2(\mathbb{N}) : Tx \in \ell^2(\mathbb{N})\};$$

this means that T is a Toeplitz operator with (unbounded) symbol $p(z) = (z-1)^{-1}$, $z \in \mathbb{C} \setminus \{1\}$. In [14, Example 2.10] we showed that T has the following properties:

(i) T is quasi-m-accretive with $\sigma(T) = W(T) = \left\{\lambda \in \mathbb{C} \,\middle|\, \operatorname{Re}\lambda \geq -\frac{1}{2}\right\}$;

(ii) $\left\{\lambda \in \mathbb{C} \,\middle|\, \operatorname{Re}\lambda > -\frac{1}{2}\right\} \subset \sigma_p(T)$;

(iii) $\mathcal{D}(T) \cap \mathcal{D}(T^*)$ is dense in $\ell^2(\mathbb{N})$, but no core for T;

(iv) $W_{\mathrm{e}5}(T) = \left\{\lambda \in \mathbb{C} \,\middle|\, \operatorname{Re}\lambda = \frac{1}{2}\right\} \subsetneq W(T)$;

(v) $W_{\mathrm{e}i}(T) = W_\mathrm{e}(T) = W(T), i = 1, \ldots, 4$.

Here we prove that

$$\sigma_{\mathrm{eap}}(T) = W_{\mathrm{e}i}(T) = W_\mathrm{e}(T) \\ = \left\{\lambda \in \mathbb{C} \,\middle|\, \operatorname{Re}\lambda \geq -\frac{1}{2}\right\}, \quad i = 1,2,3,4, \tag{2.10}$$

$$\sigma_{\mathrm{e}2}(T) = \partial\sigma_{\mathrm{eap}}(T) = W_{\mathrm{e}5}(T) = \left\{\lambda \in \mathbb{C} \,\middle|\, \operatorname{Re}\lambda = -\frac{1}{2}\right\}. \tag{2.11}$$

This shows that $\sigma_{\mathrm{eap}}(T) \not\subseteq W_{\mathrm{e}5}(T)$ and illustrates that $\sigma_{\mathrm{eap}}(T)$ consists of $\sigma_{\mathrm{e}2}(T)$ and one of the two (unbounded) components of $\rho_{\mathrm{e}2}(T)$, namely the open right half-plane, see Proposition 2.6.

Proof. First we prove that $T - \lambda$ has dense range for any $\lambda \in \mathbb{C}$. Let $y = (y_n)_{n\in\mathbb{N}} \subset \mathcal{R}(T-\lambda)^\perp$. Then

$$0 = ((T-\lambda)e_1, y) = -\lambda y_1,$$

$$0 = ((T-\lambda)e_n, y) = \sum_{k=1}^{n-1}(e_k, y) - \lambda(e_n, y) = \sum_{k=1}^{n-1} y_k - \lambda y_n, \quad n = 2,3,\ldots.$$

Inductively, these equations imply that $y = 0$, which proves that $\mathcal{R}(T-\lambda)^{\perp} = \{0\}$ for any $\lambda \in \mathbb{C}$, as required. This and (ii) imply that, if $\operatorname{Re}\lambda > -\frac{1}{2}$, then

$$\mathrm{i}(T-\lambda) = \dim\ker(T-\lambda) - \dim \mathcal{R}(T-\lambda)^{\perp} = \dim\ker(T-\lambda) > 0.$$

Thus $\left\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > -\frac{1}{2}\right\} \subset \sigma_{\mathrm{eap}}(T)$ by Theorem 2.4. Since $\sigma_{\mathrm{eap}}(T) \subset \overline{W(T)}$, see Corollary 2.5, and $\sigma_{\mathrm{eap}}(T)$ is closed, see Remark 2.2, the claim in (2.10) follows.

Finally, because $\partial\sigma_{\mathrm{eap}}(T) \subseteq \partial\sigma_{\mathrm{e}2}(T) \subseteq \sigma_{\mathrm{e}2}(T) \subseteq W_{\mathrm{e}5}(T)$ by Proposition 2.6 (ii) and [14, Corollary 2.4] and since $\partial\sigma_{\mathrm{eap}}(T) = W_{\mathrm{e}5}(T)$ by (2.10), the claims in (2.11) follow. ∎

3. Essential minimum modulus and Moore–Penrose inverse

In this section we introduce the essential minimum modulus for closed linear operators. We relate it to the essential spectrum of type 2 and to the essential norm of the Moore–Penrose inverse. Further, we prove other characterisations of the essential minimum modulus which will be used to establish the essential resolvent norm bounds in the next section.

First, we recall the definition of the Moore–Penrose (generalised) inverse of a closed densely defined linear operator T with closed range. Under these assumptions, the operator T_1 between the Hilbert spaces $\ker(T)^{\perp}$ and $\mathcal{R}(T)$ given by

$$\mathcal{D}(T_1) := \mathcal{D}(T) \cap \ker(T)^{\perp}, \quad T_1 x = Tx, \quad x \in \mathcal{D}(T),$$

is closed and bijective, and hence has a bounded inverse T_1^{-1} by the closed graph theorem. Then the bounded operator

$$T^{\dagger} := T_1^{-1} P_{\mathcal{R}(T)} \in L(H)$$

is called *Moore–Penrose inverse* of T, see e.g. [22, 23] or [3, Chapter 9] (where $T^{\dagger}$ is called *maximal Tseng inverse*). It follows from the definition and [23, Proposition 3.2] that

$$TT^{\dagger} = P_{\mathcal{R}(T)}, \quad T^{\dagger}T \subset P_{\overline{\mathcal{R}(T^{\dagger})}}, \quad \mathcal{R}(T^{\dagger}) = \mathcal{D}(T) \cap \ker(T)^{\perp}, \tag{3.1}$$

i.e. $T^{\dagger}Tx = P_{\overline{\mathcal{R}(T^{\dagger})}}x$, $x \in \mathcal{D}(T)$, see also [3, Theorem 9.2]. Moreover, if

$$\gamma(T) := \inf_{x \in \mathcal{D}(T),\, \|x\|=1} \|Tx\| \in [0, \infty), \tag{3.2}$$

denotes the *minimum modulus* of T, see e.g. [11, Definition I.3.3], then [23, Proposition 3.9] implies that, whenever T is injective with closed range or, equivalently,

$0 \in \rho_{\mathrm{ap}}(T)$, we have that

$$\|T^{\dagger}\| = \|T^{-1}\| = \frac{1}{\gamma(T)}, \tag{3.3}$$

where $T^{-1} = T_1^{-1}$ is the inverse of T acting from the Hilbert space $\mathcal{R}(T)$ to H. For a closed densely defined linear operator T, it is obvious that

$$\sigma_{\mathrm{ap}}(T) = \{\lambda \in \mathbb{C} \mid \gamma(T-\lambda) = 0\}. \tag{3.4}$$

It is known that the norm of the Moore–Penrose resolvent $\|(T-\lambda)^{\dagger}\| = (\gamma(T-\lambda))^{-1}$, for $\lambda \in \rho_{\mathrm{ap}}(T)$ and $\lambda \notin \overline{W(T)}$, respectively, satisfies

$$\frac{1}{\operatorname{dist}(\lambda, \sigma_{\mathrm{ap}}(T))} \le \|(T-\lambda)^{\dagger}\| \le \frac{1}{\operatorname{dist}(\lambda, W(T))},$$

see [19, Theorem V.3.3.2].

To establish an 'essential version' of the minimum modulus, the characterisation

$$\gamma(T) = \inf \sigma(|T|) = \min \sigma(|T|)$$

with $|T| = (T^*T)^{\frac{1}{2}}$ is more useful, see [6, Theorem 1] for the case of bounded T. It follows easily from the spectral theorem for the self-adjoint operator $|T|$ since $\|Tx\| = \||T|x\|$, $x \in \mathcal{D}(T) = \mathcal{D}(|T|)$. Note also that $\gamma(T) = \gamma(|T|)$.

Definition 3.1. For a closed densely defined linear operator T, we define *the essential minimum modulus* $\gamma_{\mathrm{e}}(T)$ of T as

$$\gamma_{\mathrm{e}}(T) := \inf \sigma_{\mathrm{e}}(|T|) \in [0,\infty].$$

The following properties of the essential minimum modulus are immediate from its definition and since the essential spectrum is closed.

Remark 3.2. Clearly, $\gamma(T) \le \gamma_{\mathrm{e}}(T)$, $\gamma_{\mathrm{e}}(T) = \gamma_{\mathrm{e}}(|T|)$ and $\gamma_{\mathrm{e}}(T) = \min \sigma_{\mathrm{e}}(|T|)$ if $\sigma_{\mathrm{e}}(|T|) \neq \emptyset$.

Proposition 3.3. *Let T be a closed densely defined linear defined operator. Then*

(i) $\sigma_{\mathrm{e2}}(T) = \{\lambda \in \mathbb{C} \mid \gamma_{\mathrm{e}}(T-\lambda) = 0\}$*;*

(ii) *if we abbreviate, see* [11, Theorem IX.1.7]*,*

$$\mathcal{E}(T) := \{(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T) \mid (x_n)_{n\in\mathbb{N}} \text{ is a normalised weakly null-sequence}\},$$

then

$$\gamma_{\mathrm{e}}(T) = \inf_{(x_n)_{n\in\mathbb{N}} \in \mathcal{E}(T)} \liminf_{n\to\infty} \|Tx_n\| \le \operatorname{dist}(0, \sigma_{\mathrm{e2}}(T)); \tag{3.5}$$

in particular, $\gamma_{\mathrm{e}}(T) = \gamma_{\mathrm{e}}(T+K)$ *for all compact* $K \in \mathcal{K}(H)$*;*

(iii) *if* $0 \in \rho_{\mathrm{eap}}(T) (\subset \rho_{\mathrm{e2}}(T)$ *and hence* $\gamma_{\mathrm{e}}(T) > 0)$*, then*

$$\gamma_{\mathrm{e}}(T) = \sup_{K \in \mathcal{K}(H)} \gamma(T+K).$$

Proof of Proposition 3.3. (i) It follows from [13, Theorem (1.1)] that $\lambda \in \sigma_{\mathrm{e2}}(T)$ if and only if $\ker(T-\lambda)$ is infinite dimensional or 0 is an accumulation point of $\sigma(|T-\lambda|)$. Since $|T-\lambda|$ is self-adjoint, this is equivalent to $0 \in \sigma_{\mathrm{e}}(|T-\lambda|)$. This shows the inclusion "$\subset$" and, since the essential spectrum is closed, also "$\supset$".

(ii) Let $\lambda \in \sigma_{\mathrm{e}}(|T|)$. There exists a normalised weakly null-sequence $(y_n)_{n\in\mathbb{N}} \subset \mathcal{D}(|T|) = \mathcal{D}(T)$ with $\|(|T|-\lambda)y_n\| \to 0$ and hence $\||T|y_n\| \to \lambda$ as $n \to \infty$. Thus

$$\inf_{(x_n)_{n\in\mathbb{N}} \in \mathcal{E}(T)} \liminf_{n\to\infty} \|Tx_n\| \le \liminf_{n\to\infty} \|Ty_n\| = \lim_{n\to\infty} \||T|y_n\| = \lambda.$$

To prove the reverse inequality, let $\lambda \ge 0$ be such that there exists a normalised weakly null-sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T) = \mathcal{D}(|T|)$ with $\|Tx_n\| \to \lambda$ as $n \to \infty$. Then

$$0 \le (|T|x_n, x_n) \le \||T|x_n\| = \|Tx_n\| \to \lambda, \quad n \to \infty.$$

Thus we can assume, by passing to subsequences, that $(|T|x_n, x_n) \to \mu \le \lambda$ and so, by definition (1.3), it follows that $\mu \in W_{\mathrm{e}}(|T|)$. Since $|T|$ is self-adjoint and semi-bounded, [5, Theorem 3.8] implies that

$$\gamma_{\mathrm{e}}(T) = \inf \sigma_{\mathrm{e}}(|T|) = \inf W_{\mathrm{e}}(|T|) \le \mu \le \lambda.$$

The inequality in (3.5) follows from the equality therein, which we just proved, and from [11, Theorem IX.1.7].

Finally, from (3.5) and the definition of $\mathcal{E}(T)$, the last claim is evident.

(iii) Parts of the following proof of (iii) are similar to the proof for the bounded case, see [35]. By Remark 3.2 and the last claim in (ii), it follows that

$$\sup_{K\in\mathcal{K}(H)} \gamma(T+K) \le \sup_{K\in\mathcal{K}(H)} \gamma_{\mathrm{e}}(T+K) = \gamma_{\mathrm{e}}(T).$$

To prove the reverse inequality, we note that $\rho_{\mathrm{eap}}(T) \subset \rho_{\mathrm{e2}}(T)$ by Corollary 2.5 and hence $\gamma_{\mathrm{e}}(T) > 0$ by claim (i). Moreover, since $0 \in \rho_{\mathrm{eap}}(T)$, Definition 2.1 and equation (3.4) imply that there exists $K_0 \in \mathcal{K}(H)$ with $0 \in \rho_{\mathrm{ap}}(T+K_0)$ or, equivalently, $\gamma(|T+K_0|) = \gamma(T+K_0) > 0$. By (3.4), the latter implies that $0 \in \rho_{\mathrm{ap}}(|T+K_0|)$ and hence, since $|T+K_0|$ is self-adjoint, $0 \in \rho(|T+K_0|)$.

By Remark 3.2 and the last claim in (ii), we have

$$\gamma_{\mathrm{e}}(|T+K_0|) = \gamma_{\mathrm{e}}(T+K_0) = \gamma_{\mathrm{e}}(T) > 0.$$

Now, we choose an arbitrary sequence $(\lambda_n)_{n\in\mathbb{N}}$ such that $0 < \lambda_n < \gamma_{\mathrm{e}}(T)$, $n \in \mathbb{N}$, and $\lambda_n \nearrow \gamma_{\mathrm{e}}(T) = \gamma_{\mathrm{e}}(T + K_0)$ as $n \to \infty$. Let $E(\cdot)$ denote the spectral measure of $|T + K_0|$ and set

$$P_n := E([0, \lambda_n)), \quad K_n := \lambda_n P_n - |T + K_0|P_n, \quad n \in \mathbb{N}.$$

Then $\mathcal{R}(P_n) \subset \mathcal{D}(T)$ and $\dim(\mathcal{R}(P_n)) < \infty$ because $\lambda_n < \inf \lambda_{\mathrm{e}}(|T + K_0|)$, and hence the operators K_n are compact for all $n \in \mathbb{N}$; moreover,

$$|T + K_0| + K_n = \lambda_n P_n + |T + K_0|(I - P_n), \quad n \in \mathbb{N}.$$

Again by Remark 3.2 and the last claim in (ii), it now follows that

$$n\lambda_n \le \gamma(|T + K_0| + K_n) \le \gamma_{\mathrm{e}}(|T + K_0| + K_n) = \gamma_{\mathrm{e}}(T + K_0), \quad n \in \mathbb{N}.$$

Since $\lambda_n \nearrow \gamma_{\mathrm{e}}(T + K_0)$ as $n \to \infty$, we obtain that $\gamma(|T + K_0| + K_n) \nearrow \gamma_{\mathrm{e}}(T + K_0)$ as $n \to \infty$. By the polar decomposition, see e.g. [19, Section VI.2.7], there exists an isometry U from $\mathcal{R}(|T + K_0|) = H$ to $\mathcal{R}(T + K_0)$ such that $T + K_0 = U|T + K_0|$. Because K_0 and K_n are compact, so is $\widetilde{K}_n := K_0 + UK_n$ for $n \in \mathbb{N}$. If we use that $\|(T + \widetilde{K}_n)x\| = \|U(|T + K_0| + K_n)x\| = \|(|T + K_0| + K_n)x\|$ for $x \in \mathcal{D}(T) = \mathcal{D}(|T + K_0|)$ in the definition (3.2) of the minimum modulus, it follows that

$$\gamma(T + \widetilde{K}_n) = \gamma(|T + K_0| + K_n) \nearrow \gamma_{\mathrm{e}}(T + K_0) = \gamma_{\mathrm{e}}(T), \quad n \to \infty. \qquad \blacksquare$$

Corollary 3.4. *If T is self-adjoint, then, for $\lambda \in \mathbb{C}$,*

$$\gamma_{\mathrm{e}}(T - \lambda) = \operatorname{dist}(\lambda, \sigma_{\mathrm{e}i}(T)) = \operatorname{dist}(\lambda, \sigma_{\mathrm{eap}}(T)), \quad i = 1, \dots, 5,$$

where the essential spectra $\sigma_{\mathrm{e}i}(T)$, $i = 2, 4$, are as in (2.1), (2.2) *and $\sigma_{\mathrm{e}i}(T)$, $i = 1, 3, 5$, are defined as in* [11, Chapter IX].

Proof. Since for a self-adjoint operator, all essential spectra and the essential approximate point spectrum coincide, see [11, Theorem IX.1.6] and Proposition 2.2 (ii), the claim is immediate from Proposition 3.3 (ii) and [11, Theorem IX.1.7]. $\blacksquare$

The following lemma shows, in particular, that in Proposition 3.3 (ii) the infimum in (3.5) is, in fact, a minimum (if we choose $M = \{0\}$ below).

Lemma 3.5. *Let T be a closed densely defined upper semi-Fredholm operator and let $M \subset \mathcal{D}(T)$ be a finite-dimensional subspace. Then there exists a normalised weakly null-sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T) \cap M^{\perp}$ such that*

$$\|Tx_n\| \to \gamma_{\mathrm{e}}(T), \quad n \to \infty.$$

Proof. Since T is upper semi-Fredholm, we have $0 \in \rho_{e2}(T)$ and so $\gamma_e(T) > 0$ by Proposition 3.3 (i). Further, Proposition 3.3 (ii) together with a standard diagonal sequence argument yields that there exists a normalised weakly null-sequence $(y_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T)$ such that $\|Ty_n\| \to \gamma_e(T)$ as $n \to \infty$. Since M is finite dimensional, P_M is compact and so we can assume, upon choosing a subsequence, that $P_M y_n \to 0$ and hence $\|(I - P_M)y_n\| \to 1$ as $n \to \infty$. Then

$$x_n := \frac{(I - P_M)y_n}{\|(I - P_M)y_n\|} \in \mathcal{D}(T) \cap M^\perp, \quad n \in \mathbb{N},$$

defines a normalised weakly null-sequence. Because $M \subset \mathcal{D}(T)$ is finite dimensional, it is easy to see that TP_M is everywhere defined and compact, so that we can assume, again upon choosing a subsequence, that $TP_M y_n \to 0$ and thus, altogether,

$$\|Tx_n\| = \frac{\|Ty_n - TP_M y_n\|}{\|(I - P_M)y_n\|} \to \gamma_e(T), \quad n \to \infty. \qquad \blacksquare$$

Remark 3.6. If we note that, in [17, pp. 257–258], it was shown that

$$\|B\|_e = \sup\{\limsup_{n\to\infty} \|Bx_n\| \mid (x_n)_{n\in\mathbb{N}} \text{ is a normalised weakly null-sequence}\}, \tag{3.6}$$

then this and a standard diagonal sequence argument imply that there exists a normalised weakly null-sequence $(x_n)_{n\in\mathbb{N}} \subset H$ such that $\|Bx_n\| \to \|B\|_e$ as $n \to \infty$.

Next we establish the ‘essential version’ of (3.3), i.e. we prove that the essential norm of the Moore–Penrose inverse coincides with the reciprocal of the essential minimum modulus. This result seems to be new even for bounded operators.

Theorem 3.7. *If T is a closed densely defined upper semi-Fredholm operator, then*

$$\|T^\dagger\|_e = \frac{1}{\gamma_e(T)}. \tag{3.7}$$

In particular, if $0 \in \rho(T)$, then T has compact resolvent or, equivalently, $|T|$ has compact resolvent if and only if $\gamma_e(T) = \infty$.

Proof. Since $0 \in \rho_{e2}(T)$ by assumption, $\ker(T)$ is finite dimensional, $\mathcal{R}(T)$ is closed and $\gamma_e(T) > 0$ by Proposition 3.3 (i).

First we prove the inequality “$\geq$”. By Lemma 3.5, there exists a normalised weakly null-sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T) \cap \ker(T)^\perp$ with $\|Tx_n\| \to \gamma_e(T) \in [0, \infty]$ as $n \to \infty$. In particular, $(Tx_n)_{n\in\mathbb{N}}$ is a bounded sequence and, upon passing to a subsequence, we may assume that it is weakly convergent. Since T is closed, its graph is also weakly closed, see e.g. [32, Theorem 3.12], which implies that $Tx_n \xrightarrow{w} 0$ as $n \to \infty$. Then the normalised sequence

$$y_n := \frac{Tx_n}{\|Tx_n\|} \in \mathcal{D}(T), \quad n \in \mathbb{N},$$

also satisfies $y_n \xrightarrow{w} 0$ as $n \to \infty$. Further, by (3.1) we have $x_n \in \mathcal{D}(T) \cap \ker(T)^\perp = \mathcal{R}(T^\dagger)$ and $\|T^\dagger T x_n\| = \|x_n\| = 1$, $n \in \mathbb{N}$. Altogether, it follows that

$$\frac{1}{\gamma_e(T)} = \lim_{n\to\infty} \frac{1}{\|Tx_n\|} = \lim_{n\to\infty} \frac{\|T^\dagger T x_n\|}{\|Tx_n\|} = \lim_{n\to\infty} \|T^\dagger y_n\| \le \|T^\dagger\|_e.$$

To prove the inequality "$\le$", we note that if $\|T^\dagger\|_e = 0$, there is nothing to show. Thus assume that $\|T^\dagger\|_e > 0$. Then, by (3.6), there exists a normalised weakly null-sequence $(y_n)_{n\in\mathbb{N}}$ such that $\|T^\dagger y_n\| \to \|T^\dagger\|_e$. Since $\mathcal{R}(T)$ is closed, we have $H = \mathcal{R}(T) \oplus \mathcal{R}(T)^\perp$. Hence there exist bounded sequences $(y_n^1)_{n\in\mathbb{N}} \subset \mathcal{R}(T)$, $(y_n^2)_{n\in\mathbb{N}} \subset \mathcal{R}(T)^\perp$ with $y_n = y_n^1 + y_n^2$ and $1 = \|y_n\|^2 = \|y_n^1\|^2 + \|y_n^2\|^2$ for $n \in \mathbb{N}$. By passing to subsequences, we may assume that $y_n^i \xrightarrow{w} 0$, $i = 1, 2$, and $\|y_n^1\| \to r \in [0, 1]$. Since $T^\dagger$ is bounded and $\ker(T^\dagger) = \mathcal{R}(T)^\perp$, see [23, Proposition 3.2], it follows that

$$0 < \|T^\dagger\|_e = \lim_{n\to\infty} \|T^\dagger y_n\| = \lim_{n\to\infty} \|T^\dagger y_n^1\| \le \|T^\dagger\| r, \tag{3.8}$$

which implies that $r > 0$. Therefore, $\big(\frac{y_n^1}{\|y_n^1\|}\big)_{n\in\mathbb{N}} \subset \mathcal{H}$ is a normalised weakly null-sequence. Then, by (3.6) and the equalities in (3.8), it follows that

$$\|T^\dagger\|_e \ge \limsup_{n\to\infty} \frac{\|T^\dagger y_n^1\|}{\|y_n^1\|} = \frac{\|T^\dagger\|_e}{r} \ge \|T^\dagger\|_e,$$

and so equality holds everywhere, which means that $r = 1$. Since $T^\dagger$ is bounded and $\|T^\dagger y_n^1\| \to \|T^\dagger\|_e > 0$ as $n \to \infty$,

$$x_n := \frac{T^\dagger y_n^1}{\|T^\dagger y_n^1\|} \in \mathcal{R}(T^\dagger) = \mathcal{D}(T) \cap \ker(T)^\perp, \quad n \in \mathbb{N},$$

defines a normalised weakly null-sequence $(x_n)_{n\in\mathbb{N}}$. Then we obtain that, because $(y_n^1)_{n\in\mathbb{N}} \subset \mathcal{R}(T)$ and by (3.1),

$$\frac{1}{\|Tx_n\|} = \frac{\|T^\dagger y_n^1\|}{\|TT^\dagger y_n^1\|} = \frac{\|T^\dagger y_n^1\|}{\|y_n^1\|} \to \|T^\dagger\|_e, \quad n \to \infty.$$

Since Proposition 3.3 (ii) implies that $\liminf_{n\to\infty} \|Tx_n\| \ge \gamma_e(T)$, we conclude that

$$\frac{1}{\gamma_e(T)} \ge \limsup_{n\to\infty} \frac{1}{\|Tx_n\|} = \|T^\dagger\|_e,$$

which completes the proof of (3.7).

Finally, if $0 \in \rho(T)$, then T has compact resolvent if and only if $T^\dagger = T^{-1}$ is compact or, equivalently, $\|T^\dagger\|_e = 0$. By (3.7) this is in turn equivalent to $\gamma_e(T) = \infty$. By definition, this holds if and only if $\sigma_{e2}(|T|) = \emptyset$. Since $|T|$ is self-adjoint, this is equivalent to the resolvent of $|T|$ being compact, see e.g. [34, Theorem V.5.12]. ∎

4. Resolvent estimates in the essential norm

In this section we establish two-sided estimates of the essential norm of the Moore–Penrose resolvent and the resolvent in terms of the essential numerical ranges and the essential spectrum of type 2. These bounds are new even in the case of bounded operators and they apply in regions inside of the numerical range where, up to now, no generally valid resolvent estimates have been available. Examples of differential operators show that e.g. for m-sectorial operators the essential norm of the resolvent may be estimated everywhere outside of $[0, \infty)$.

Theorem 4.1. *If T is a closed densely defined linear operator, then, for $\lambda \notin \sigma_{e2}(T)$,*

$$\frac{1}{\operatorname{dist}(\lambda, \sigma_{e2}(T))} \leq \|(T-\lambda)^\dagger\|_e \leq \frac{1}{\operatorname{dist}(\lambda, W_{ei}(T))}, \quad i = 1, \dots, 5. \tag{4.1}$$

Proof. First, we note that $\lambda \notin \sigma_{e2}(T)$ if and only if $\gamma_e(T-\lambda) > 0$ by Proposition 3.3, and the latter is equivalent to $\|(T-\lambda)^\dagger\|_e < \infty$ by Theorem 3.7.

By Theorem 3.7, the first inequality is equivalent to $\gamma_e(T-\lambda) \leq \operatorname{dist}(\lambda, \sigma_{e2}(T))$, which was proved in [11, Theorem IX.1.7].

To prove the second inequality, by (2.7) – see also [5, Theorem 3.1] and [14, Theorem 2.5] – it suffices to prove the claim for $i = 5$. If $\|(T-\lambda)^\dagger\|_e = 0$, there is nothing to prove. If $\|(T-\lambda)^\dagger\|_e > 0$, then $\gamma_e(T-\lambda) < \infty$ by Theorem 3.7. Due to Proposition 3.3 (ii) there exists a normalised weakly null-sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T)$ with

$$\|(T-\lambda)x_n\| \to \gamma_e(T-\lambda) = \frac{1}{\|(T-\lambda)^\dagger\|_e} < \infty, \quad n \to \infty; \tag{4.2}$$

in particular, $(Tx_n)_{n\in\mathbb{N}} \subset H$ is a bounded sequence. By passing to a subsequence if necessary, we may assume that $(Tx_n)_{n\in\mathbb{N}}$ is weakly convergent and $(Tx_n, x_n) \to \mu$ for some $\mu \in \mathbb{C}$. Since T is closed, its graph is also weakly closed, see e.g. [32, Theorem 3.12]. Hence, $x_n \xrightarrow{w} 0$ implies that $Tx_n \xrightarrow{w} 0$ as $n \to \infty$. Then it follows from the definition of $W_{e5}(T)$ in (2.6) that $\mu \in W_{e5}(T)$. Using (4.2), we conclude that

$$\|(T-\lambda)^\dagger\|_e = \lim_{n\to\infty} \frac{1}{\|(T-\lambda)x_n\|} \leq \lim_{n\to\infty} \frac{1}{|(Tx_n, x_n) - \lambda|} \leq \frac{1}{\operatorname{dist}(\lambda, W_{e5}(T))}. \quad ∎$$

Remark 4.2. If T is closed and densely defined, then, for $\lambda \notin \sigma_{\mathrm{eap}}(T)$,

$$\frac{1}{\operatorname{dist}(\lambda, \sigma_{\mathrm{eap}}(T))} = \frac{1}{\operatorname{dist}(\lambda, \sigma_{\mathrm{e2}}(T))} \leq \|(T-\lambda)^{\dagger}\|_{\mathrm{e}}. \tag{4.3}$$

Proof. The first equality follows from Proposition 2.6 and thus Theorem 4.1 implies the desired inequality; alternatively, we could also use the characterisation in Proposition 3.3 (iii) of the essential minimum modulus. ∎

The next corollary is obvious from Theorem 3.7, Corollary 3.4 and Theorem 3.7.

Corollary 4.3. *If T is self-adjoint, then, for $\lambda \in \rho(T)$,*

$$\|(T-\lambda)^{-1}\|_{\mathrm{e}} = \frac{1}{\operatorname{dist}(\lambda, \sigma_{\mathrm{e}i}(T))} = \frac{1}{\operatorname{dist}(\lambda, \sigma_{\mathrm{eap}}(T))}, \quad i = 1, \ldots, 5.$$

If we also have $\lambda \in \rho(T)$ rather than $\lambda \in \rho_{\mathrm{e2}}(T)$ in Theorem 4.1, then $(T-\lambda)^{\dagger} = (T-\lambda)^{-1}$ and the next result shows that then the lower bounds in (4.1) and (4.3) coincide.

To this end, we recall that if B is a bounded operator, then always $\sigma_{\mathrm{e}i}(B) \neq \emptyset$, $i = 1, \ldots, 5$, and so $\sigma_{\mathrm{eap}}(B) \neq \emptyset$ as well by Corollary 2.5. Moreover, the so-called *essential spectral radius* of B satisfies

$$r_{\mathrm{e}}(B) := \sup\{|\lambda| \mid \lambda \in \sigma_{\mathrm{e}i}(B)\} = \sup\{|\lambda| \mid \lambda \in \sigma_{\mathrm{eap}}(B)\} \leq \|B\|_{\mathrm{e}}, \quad i = 1, \ldots, 5,$$

no matter whether the various essential spectra of B coincide or not.

Proposition 4.4. *Let T be a closed densely defined linear operator and $i \in \{1, \ldots, 5\}$. Then, for $\lambda \in \rho(T)$,*

$$\begin{aligned} r_{\mathrm{e}}((T-\lambda)^{-1}) &= \frac{1}{\operatorname{dist}(\lambda, \sigma_{\mathrm{eap}}(T))} = \frac{1}{\operatorname{dist}(\lambda, \sigma_{\mathrm{e}i}(T))} \\ &\leq \|(T-\lambda)^{-1}\|_{\mathrm{e}} \leq \frac{1}{\operatorname{dist}(\lambda, W_{\mathrm{e}i}(T))}. \end{aligned}$$

Proof. Since $\lambda \in \rho(T)$, the two inequalities are immediate from Theorem 4.1 by the preceding remarks. To prove the first equality, we use that by the essential spectral mapping theorem, see [11, Theorem IX.2.3 (iii)], $\mu \in \sigma_{\mathrm{e}i}(T)$ if and only if we have $(\mu-\lambda)^{-1} \in \sigma_{\mathrm{e}i}((T-\lambda)^{-1}) \setminus \{0\}$ for $i = 1, \ldots, 5$. If $\sigma_{\mathrm{e2}}((T-\lambda)^{-1}) \setminus \{0\} \neq \emptyset$,this implies that

$$\frac{1}{\operatorname{dist}(\lambda, \sigma_{\mathrm{e}i}(T))} = \sup_{\mu \in \sigma_{\mathrm{e}i}(T)} |\mu-\lambda|^{-1} = \sup_{z \in \sigma_{\mathrm{e}i}((T-\lambda)^{-1}) \setminus \{0\}} |z| = r_{\mathrm{e}}((T-\lambda)^{-1})$$

for $i = 1, \ldots, 5$. If $\sigma_{\mathrm{e2}}((T-\lambda)^{-1}) \setminus \{0\} = \emptyset$, then it follows from the essential spectral mapping theorem that $\sigma_{\mathrm{e2}}(T) = \emptyset$ and hence $\operatorname{dist}(\lambda, \sigma_{\mathrm{e2}}(T)) = \infty$.

Since $\sigma_{e2}(T) \subset \sigma_{eap}(T) \subset \sigma_{e4}(T)$ by Proposition 2.6 (i), the second equality follows from the first one; alternatively, we could also use the essential spectral mapping theorem for $\sigma_{eap}(T)$, Theorem 2.10. ∎

Proposition 4.4 yields a useful information on operators with empty essential numerical range, which seems to be new. Note that $W_e(T) \neq \emptyset$ if T is bounded.

Corollary 4.5. *Let T be a densely defined linear operator with $W_e(T) = \emptyset$. Then either $\sigma(T) = \mathbb{C}$ or T has compact resolvent.*

Proof. If $\sigma(T) \neq \mathbb{C}$, then $\rho(T) \neq \emptyset$ and hence T is closed. Then $W_e(T) = \emptyset$ implies that, for all $\lambda \in \rho(T)$, we have $\operatorname{dist}(\lambda, W_e(T)) = \infty$ and thus $\|(T-\lambda)^{-1}\|_e = 0$ by Proposition 4.4, i.e. $(T-\lambda)^{-1}$ is compact. ∎

Remark 4.6. The converse of Corollary 4.5 is not true, not even in the self-adjoint case. In fact, if T is self-adjoint with compact resolvent, but not semi-bounded, then $W_e(T) = \mathbb{R}$, see [5, Theorem 3.8].

On the other hand, it is well known that a self-adjoint operator T has compact resolvent if and only if $\sigma_{e2}(T) = \emptyset$, see e.g. [34, Proposition V.5.12]; note that Corollary 4.3 yields another proof of this equivalence since $\sigma_{e2}(T) = \emptyset$ is equivalent to $\operatorname{dist}(\lambda, \sigma_{e2}(T)) = \infty$ and hence to $\|(T-\lambda)^{-1}\|_e = 0$, i.e. $(T-\lambda)^{-1}$ compact, for $\lambda \in \rho(T)$.

In the following we illustrate our results by applying them to two different examples from mathematical physics, first to an advection-diffusion type differential operator on $\mathbb{R}$ studied in [5, 10].

Example 4.7. For the advection-diffusion type differential operator

$$T := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + Q_1(x)\frac{\mathrm{d}}{\mathrm{d}x} + Q_0(x), \quad \mathcal{D}(T) := H^2(\mathbb{R}),$$

with complex-valued coefficients Q_0, $Q_1 \in L^\infty(\mathbb{R})$ such that $Q_1(x) \to -2$, and $Q_0(x) \to 0$ as $|x| \to \infty$, it was shown in [5, Example 7.3] that the essential spectrum of type 2 is a parabola and the essential numerical range is the convex hull of this parabola. Analogously, using Corollary 2.9 or Corollary 2.11, respectively, it is easy to see that $\sigma_{eap}(T) = \sigma_{e2}(T)$, so that altogether

$$\sigma_{eap}(T) = \sigma_{e2}(T) = \left\{\lambda \in \mathbb{C} \;\middle|\; \operatorname{Re}\lambda = \frac{(\operatorname{Im}\lambda)^2}{2}\right\},$$
$$W_e(T) = W_{ei}(T) = \operatorname{conv}(\sigma_{e2}(T)) = \left\{\lambda \in \mathbb{C} \;\middle|\; \operatorname{Re}\lambda \geq \frac{(\operatorname{Im}\lambda)^2}{2}\right\}, \quad i = 1, \dots, 5.$$

Then Corollary 4.4 implies that, ‘outside’ of the parabola,

$$\|(T-\lambda)^{-1}\|_e = \frac{1}{\operatorname{dist}(\lambda, W_e(T))} = \frac{1}{\operatorname{dist}(\lambda, \sigma_{e2}(T))}, \quad \lambda \in \rho(T) \setminus W_e(T).$$

In the particular case $Q_1 \equiv -2$, $Q_0 \equiv 0$ where it is also known that $\sigma(T) = \sigma_{e2}(T)$ and $\overline{W(T)} = W_e(T)$, see [10], Theorem 4.1 still yields that, also 'inside' of the parabola,

$$\|(T-\lambda)^{-1}\|_e \geq \frac{1}{\operatorname{dist}(\lambda, \sigma_{e2}(T))}, \quad \lambda \in \rho(T) \setminus \sigma_{e2}(T).$$

Our second example are Schrödinger operators with rapidly oscillating potentials on $\mathbb{R}^d$, $d \geq 3$, which were studied in [33, 38] in the case of real-valued potentials and in [29] for complex-valued potentials. For the purpose of illustrating our essential norm estimates of the resolvent, it is sufficient to focus on purely imaginary potentials.

Example 4.8 (Schrödinger operators with purely imaginary rapidly oscillating potentials). Let the operator T_0 in the Hilbert space $L^2(\mathbb{R}^d)$, $d \geq 3$, be given by

$$T_0 f(x) := -\Delta f(x) + \mathrm{i} q(x) f(x), \quad f \in \mathcal{D}(T_0) := C_0^\infty(\mathbb{R}^d),\ x \in \mathbb{R}^d,$$

where $q \in L^\infty_{loc}(\mathbb{R}^d)$ is a real-valued potential satisfying

$$\lim_{r_1, r_2 \to \infty} \sup_{\omega \in \partial B_1(0)} \left| \int_{r_1}^{r_2} q(r\omega)\, \mathrm{d}r \right| = 0.$$

This means that there exists a function $w\colon (0,\infty) \to [0,\infty)$ with $\lim_{R\to\infty} w(R) = 0$ such that for all $\omega \in \partial B_1(0)$ and $r_2 > r_1 > R$,

$$\left| \int_{r_1}^{r_2} q(r\omega)\, \mathrm{d}r \right| \leq w(R).$$

Typical examples of such potentials include $q(x) = |x|^3 \sin(|x|^5)$, $x \in \mathbb{R}^d$, and $q(x) = (1+|x|^2)^{-1} \mathrm{e}^{|x|} \sin(\mathrm{e}^{|x|})$, $x \in \mathbb{R}^d$.

It was shown in [29, Theorems 6 and 10] that T_0 is sectorial and admits a Friedrichs extension $T = \overline{T_0}$ which is m-sectorial and satisfies $\sigma_{e2}(T) = [0,\infty)$. Here, we will show that, although the potential is purely imaginary, the essential numerical range is real and, by our previous results, we can determine the essential norm of the resolvent, more precisely,

$$W_e(T) = [0,\infty), \quad \|(T-\lambda)^{-1}\|_e = \begin{cases} \dfrac{1}{|\operatorname{Im}\lambda|}, & \operatorname{Re}\lambda \geq 0, \\ \dfrac{1}{|\lambda|}, & \operatorname{Re}\lambda < 0, \end{cases} \quad \lambda \in \rho(T);$$

note that it then follows directly that $\sigma_{e2}(T) = \sigma_{eap}(T) = W_e(T) = [0,\infty)$.

Proof. Due to [29, Theorem 10] and [5, Proposition 2.2] it suffices to show that $W_{\mathrm{e}}(T_0) \subset [0,\infty)$. If $\lambda \in W_{\mathrm{e}}(T_0)$, then there exists a normalised weakly null-sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T_0) = C_0^\infty(\mathbb{R}^d)$ such that $(T_0 f_n, f_n) \to \lambda$, i.e.

$$\begin{aligned}\operatorname{Re}(T_0 f_n, f_n) &= \|\nabla f_n\|^2_{L^2(\mathbb{R}^d)} \to \operatorname{Re}\lambda,\\ \operatorname{Im}(T_0 f_n, f_n) &= \int\limits_{\mathbb{R}^d} q(x)|f_n(x)|^2\,\mathrm{d}x \to \operatorname{Im}\lambda.\end{aligned} \tag{4.4}$$

To show that $\operatorname{Im}\lambda = 0$, let $\varepsilon > 0$ be arbitrary. Since $(f_n)_{n\in\mathbb{N}}$ is a normalised weakly null-sequence in $L_2(\mathbb{R}^d)$, we obtain that, for all $g \in C_0^\infty(\mathbb{R}^d)$,

$$\int\limits_{\mathbb{R}^d} \nabla f_n(x)\overline{\nabla g(x)}\,\mathrm{d}x = -\int\limits_{\mathbb{R}^d} f_n(x)\overline{\Delta g(x)}\,\mathrm{d}x \to 0, \quad n\to\infty.$$

Since $C_0^\infty(\mathbb{R}^d)$ is dense in $H^1(\mathbb{R}^d)$ and $(\nabla f_n)_{n\in\mathbb{N}}$ is bounded in $L_2(\mathbb{R}^d)$ by the first equation in (4.4), so that $(f_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^d)$, it follows that $(f_n)_{n\in\mathbb{N}}$ is a weak null-sequence in $H^1(\mathbb{R}^d)$.

Now set $C := \sup_{n\in\mathbb{N}} \|f_n\|_{H^1(\mathbb{R}^d)}$ and choose $R_0 \ge 1$ so that $w(R) \le \frac{1}{2}\frac{\varepsilon}{2C+d-1}$ for $R \ge R_0$. Due to the assumptions on q, we can apply [29, Lemma 7] which implies that, for all $n\in\mathbb{N}$ and $R\ge R_0$,

$$\begin{aligned}\Bigg|\int\limits_{|x|\ge R} q(x)|f_n(x)|^2\,\mathrm{d}x\Bigg| &\le w(R)(2\|f_n\|_{H^1(\mathbb{R}^d)} + d - 1)\\ &\le w(R)(2C + d - 1) \le \frac{\varepsilon}{2}.\end{aligned} \tag{4.5}$$

On the other hand, because $(f_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathbb{R}^d)$, the Rellich–Kondrachov theorem shows that there exists a subsequence such that

$$\|f_{n_k}\|_{L^1(B_R(0))} = \int\limits_{|x|<R} |f_{n_k}(x)|\,\mathrm{d}x \to 0, \quad k\to\infty. \tag{4.6}$$

Further, by the Gagliardo–Nirenberg inequality with $\theta = \frac{d}{d+2}$, there exists $M > 0$ such that

$$\begin{aligned}\Bigg|\int\limits_{|x|<R} q(x)|f_n(x)|^2\,\mathrm{d}x\Bigg| &\le \sup_{|x|\le R}|q(x)|\,\|f_n\|^2_{L^2(B_R(0))}\\ &\le M^2 \sup_{|x|\le R}|q(x)|\,\|f_n\|^{2\theta}_{H^1(\mathbb{R}^d)}\|f_n\|^{2-2\theta}_{L^1(B_R(0))}\\ &\le M^2C^{2\theta}\sup_{|x|\le R}|q(x)|\,\|f_n\|^{2-2\theta}_{L^1(B_R(0))} \le \frac{\varepsilon}{2}\end{aligned} \tag{4.7}$$

by (4.6) for $k \geq k_0$ with k_0 chosen sufficiently large. Using (4.5) and (4.7) in the second equation in (4.4), we conclude that $\operatorname{Im} |(T_0 f_{n_k}, f_{n_k})| \leq \varepsilon$ for $k \geq k_0$ and hence $\operatorname{Im} \lambda = 0$ as $\varepsilon > 0$ was arbitrary. ∎

It is well known that the closure $\overline{W(T)}$ of the numerical range is the minimal closed convex set $W \subset \mathbb{C}$ for which the resolvent satisfies the linear growth condition $\|(T-\lambda)^{-1}\| \leq (\operatorname{dist}(\lambda, W))^{-1}$ for $\lambda \notin W$. Note that, although in [28, 36] only bounded operators or elements of Banach algebras, respectively, were considered, the proofs therein also apply to closed densely defined operators with only minor adjustments. In particular, T is accretive if and only if $\|(T-\lambda)^{-1}\| \leq |\operatorname{Re} \lambda|^{-1}$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda < 0$, see [28, Lemma 2].

For bounded operators, a similar result for the essential numerical range was deduced in [36, Theorems 2 and 9], using that the essential numerical range $W_{\mathrm{e}}(T) = W_{\mathrm{e3}}(T) = \bigcap_{K \in \mathcal{K}(H)} \overline{W(T+K)}$ is a numerical range in the Calkin algebra.

Theorem 4.9 ([36]). *Let T be a bounded linear operator and $W \subset \mathbb{C}$ be any non-empty closed convex set. Then $W_{\mathrm{e}}(T) \subset W$ if and only if $\sigma_{\mathrm{e2}}(T) \subset W$ and*

$$\|(T-\lambda)^{\dagger}\|_{\mathrm{e}} \leq \frac{1}{\operatorname{dist}(\lambda, W)}, \quad \lambda \in \mathbb{C} \setminus W.$$

The next example shows that Theorem 4.9 does not generalise to unbounded operators for any type of essential numerical range, even if we add extra conditions on the operator T such as normality or m-accretivity or geometrical constraints on the convex set W such as being unbounded or similarly shaped as the numerical range.

Example 4.10. Let $H = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ and let A be the operator in $\ell^2(\mathbb{N})$ given by

$$\begin{aligned} \mathcal{D}(A) &:= \Big\{ (x_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}) \,\Big|\, \sum_{n=1}^{\infty} n^2 |x_n|^2 < \infty \Big\}, \\ A e_n &:= (1 + \mathrm{i}(-1)^n n) e_n, \end{aligned}$$

$n \in \mathbb{N}$, where $\{e_n \mid n \in \mathbb{N}\} \subset \ell^2(\mathbb{N})$ is the standard orthonormal basis. Then the operator

$$T := A \oplus 0 = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{D}(T) := \mathcal{D}(A) \oplus \ell^2(\mathbb{N}) \subset H,$$

is closed and densely defined, and has the following properties:

(i) T is a normal operator;

(ii) $\overline{W(T)} = [0,1] + \mathrm{i}\mathbb{R}$ and T is m-accretive;

(iii) $W_{\mathrm{e}i}(T) = W_{\mathrm{e}}(T) = \overline{W(T)} = [0,1] + \mathrm{i}\mathbb{R}$, $i = 1, \ldots, 5$;

(iv) $\sigma_{e2}(T) = \sigma_{eap}(T) = \{0\} \subset \mathrm{i}\mathbb{R}$;

(v) $\|(T-\lambda)^{\dagger}\|_e \leq |\lambda|^{-1} = (\mathrm{dist}(\lambda, \{0\}))^{-1}$ for all $\lambda \in \mathbb{C} \setminus \{0\}$ and hence, in particular, $\|(T-\lambda)^{\dagger}\|_e \leq |\operatorname{Re}\lambda|^{-1} = (\mathrm{dist}(\lambda, \mathrm{i}\mathbb{R}))^{-1}$ for all $\lambda \in \mathbb{C} \setminus \mathrm{i}\mathbb{R}$.

Proof. (i) Since $A = I + \mathrm{i}S$ where S is the self-adjoint operator given by $\mathcal{D}(S) = \mathcal{D}(A)$, $Se_n = (-1)^n e_n$, $n \in \mathbb{N}$, it is immediate that $A^* = I - \mathrm{i}S$ so that $\mathcal{D}(A^*) = \mathcal{D}(A)$ and $A^*A = AA^* = I + S^2$. Therefore, A, and hence T, is normal.

(ii) It is evident that $W(A) = 1 + \mathrm{i}\mathbb{R}$ and hence

$$\overline{W(T)} = \overline{\mathrm{conv}(W(A) \cup \{0\})} = [0,1] + \mathrm{i}\mathbb{R},$$

which shows that T is accretive. By claim (i), T is normal and so $\mathcal{D}(T) = \mathcal{D}(T^*)$. This implies that $W(T^*) = W(T)^* = \{\overline{(Tx,x)} \mid x \in \mathcal{D}(T)\}$ and hence T^* is accretive too. Thus T is m-accretive by [20, Theorem I.4.4].

(iii) Because $W(A)$ is a line and $\overline{W(T)}$ is a strip by claim (ii), [5, Corollary 2.5] yields that $W_e(A) = W(A)$ and that $W_e(T) \subset \overline{W(T)}$ is a strip or a line. Moreover, clearly, $W_e(A) = 1 + \mathrm{i}\mathbb{R} \subset W_e(T)$. This, $0 \in \sigma_{e2}(T) \subset W_e(T)$ and the fact that $W_e(T)$ is closed and convex yield that $W_e(T) = \overline{W(T)} = [0,1] + \mathrm{i}\mathbb{R}$. Since T is normal by claim (i), $\mathcal{D}(T) \cap \mathcal{D}(T^*) = \mathcal{D}(T)$ is a core of T and hence [14, Theorem 2.5] shows that $W_{ei}(T) = W_e(T)$ for $i = 1, \dots, 5$.

(iv) Since $\sigma_{e2}(T) \subset \sigma_{eap}(T)$ and $0 \in \sigma_{e2}(T)$, it suffices to show that $\sigma_{eap}(T) \subset \{0\}$. For this purpose, define the compact operators $\widetilde{K}_k$, $k \in \mathbb{N}$, in H by

$$\widetilde{K}_k := K_k \oplus 0, \quad K_k e_n := \begin{cases} -(1 + \mathrm{i}(-1)^k k)e_k, & n = k, \\ 0, & n \neq k, \end{cases} \quad n \in \mathbb{N}.$$

Since $T + \widetilde{K}_k$, $k \in \mathbb{N}$, are normal, [34, Proposition 3.26 iii) and Example 3.8] imply that

$$\sigma_{ap}(T + \widetilde{K}_k) = \sigma(T + \widetilde{K}_k) = \{0\} \cup \{1 + \mathrm{i}(-1)^n n \mid n \in \mathbb{N} \setminus \{k\}\}, \quad k \in \mathbb{N}.$$

Hence, by Definition 2.1

$$\sigma_{eap}(T) = \bigcap_{K \in \mathcal{K}(H)} \sigma_{ap}(T + K) \subset \bigcap_{k \in \mathbb{N}} \sigma_{ap}(T + \widetilde{K}_k) = \{0\}.$$

(v) Let $\lambda = a + ib \in \mathbb{C} \setminus \{0\}$. Note that the claim is equivalent to $\gamma_e(T - \lambda) \geq |\lambda|$ by Theorem 3.7. By Lemma 3.5 (applied with $M = \{0\}$) there exists a normalised weakly null-sequence $(v_n)_{n\in\mathbb{N}} = ((x_n, y_n)^t)_{n\in\mathbb{N}} \subset \mathcal{D}(T)$ such that $\gamma_e(T - \lambda) =$

$\lim_{n\to\infty}\|(T-\lambda)v_n\|^2$. Then

$$\begin{aligned}
\gamma_e(T-\lambda)^2 & K \in \mathcal{K}(H)\\
&= \lim_{n\to\infty}\|(T-\lambda)v_n\|^2 = \lim_{n\to\infty}(\|(A-\lambda)x_n\|^2 + \|\lambda y_n\|^2)\\
&= \lim_{n\to\infty}(|\lambda|^2 + \|Ax_n\|^2 - 2a\operatorname{Re}(Ax_n, x_n) + 2b\operatorname{Im}(Ax_n, x_n))\\
&= |\lambda|^2 + \lim_{n\to\infty}(\|Ax_n\|^2 - 2a\|x_n\|^2 - 2b\operatorname{Im}(Ax_n, x_n)).
\end{aligned}$$

Since $\|v_n\| = 1$, $n \in \mathbb{N}$, we may assume, by passing to a subsequence, that $\|x_n\| \to r$ with $r \in [0, 1]$ and hence

$$\gamma_e(T-\lambda)^2 = |\lambda|^2 - 2ar + \lim_{n\to\infty}(\|Ax_n\|^2 - 2b\operatorname{Im}(Ax_n, x_n)). \tag{4.8}$$

Next, we show that $(Ax_n)_{n\in\mathbb{N}}$ is bounded. Otherwise, there exists a subsequence with $\|Ax_{n_k}\| \to \infty$ as $k \to \infty$. Since $\|x_n\| \le \|v_n\| \le 1$, it follows that, for sufficiently large $k \in \mathbb{N}$,

$$\|Ax_{n_k}\|^2 - 2b\operatorname{Im}(Ax_{n_k}, x_{n_k}) \ge \|Ax_{n_k}\|\,\big|\|Ax_{n_k}\| - 2|b|\big| \to \infty, \quad k \to \infty.$$

Then (4.8) yields that $\gamma_e(T-\lambda) = \infty$. However, by Proposition 3.3 (ii), it follows that $\gamma_e(T-\lambda) \le \operatorname{dist}(0, \sigma_{e2}(T-\lambda)) = \operatorname{dist}(\lambda, \sigma_{e2}(T)) = |\lambda| < \infty$, a contradiction. Hence $(Ax_n)_{n\in\mathbb{N}} \subset H$ is bounded.

Next, we prove that $r = \lim_{n\to\infty}\|x_n\| = 0$. Otherwise, if $r > 0$, we may assume that $\|x_n\| > 0$, $n \in \mathbb{N}$. Then, with $x_n =: (x_{n,k})_{k\in\mathbb{N}}$, $n \in \mathbb{N}$, we have

$$\|Ax_n\|^2 = \sum_{k=1}^{\infty}(1+k^2)|x_{n,k}|^2 = \|x_n\|^2 + \||S|x_n\|^2, \quad n \in \mathbb{N}, \tag{4.9}$$

where $|S|$ is the non-negative self-adjoint operator with compact resolvent in $\ell^2(\mathbb{N})$ given by

$$|S|e_n = ne_n, \quad n \in \mathbb{N}, \quad \mathcal{D}(|S|) := \mathcal{D}(A).$$

By [5, Theorem 3.8], we know that $W_e(|S|) = \emptyset$. Since $\big(\frac{x_n}{\|x_n\|}\big)_{n\in\mathbb{N}} \subset \mathcal{D}(A) = \mathcal{D}(S)$ is a normalised weakly null-sequence, $\big(\frac{(|S|x_n, x_n)}{\|x_n\|^2}\big)_{n\in\mathbb{N}}$ cannot be bounded since in this case it would have a convergent subsequence which would converge to a point $\mu \in W_e(|S|)$. Hence there exists a subsequence such that

$$\infty = \lim_{k\to\infty}\frac{(|S|x_{n_k}, x_{n_k})}{\|x_{n_k}\|^2} \le \liminf_{k\to\infty}\frac{\||S|x_{n_k}\|}{r}.$$

Thus $\||S|x_{n_k}\| \to \infty$ as $k \to \infty$ and so, by (4.9), also $\|Ax_{n_k}\| \to \infty$ as $k \to \infty$, a contradiction to the boundedness of $(Ax_n)_{n\in\mathbb{N}}$.

Altogether, $\lim_{n\to\infty}\|x_n\| = r = 0$ and the boundedness of $(Ax_n)_{n\in\mathbb{N}}$ imply that $(Ax_n, x_n) \to 0$ as $n \to \infty$. Combining this with (4.8), we conclude that $\gamma_{\mathrm{e}}(T-\lambda) \geq |\lambda| \geq \operatorname{Re}\lambda$, as required. ■

5. Perturbation results

In this section we establish perturbation results for the approximate point spectrum, the essential approximate point spectrum and the essential spectrum of type 2 under relatively bounded perturbations. The latter two results are derived using the essential norm estimates for resolvents obtained in the previous section and are hence accompanied by essential norm estimates of the perturbed resolvents.

If T and A are linear operators in a Banach or Hilbert space H, then A is called *T-bounded* (or *relatively bounded with respect to T*) if $\mathcal{D}(T) \subset \mathcal{D}(A)$ and there exist constants $a, b \geq 0$ such that

$$\|Ax\| \leq a\|x\| + b\|Tx\|, \quad x \in \mathcal{D}(T); \tag{5.1}$$

the infimum of all $b \geq 0$ for which there exists an $a \geq 0$ such that (5.1) holds is called the *T-bound* or *relative bound with respect to T of A*, see e.g. [19, Section IV.1].

Our first theorem is a perturbation result for the approximate point spectrum of accretive and sectorial operators. To this end, for $\vartheta \in (-\pi, \pi]$ we define

$$S_\vartheta := \{z \in \mathbb{C} \mid |\arg(z)| \leq \vartheta\},$$

i.e. if $\vartheta = \frac{\pi}{2}$, then $S_\vartheta =: \mathbb{H}_+$ is the closed right half-plane and if $\vartheta < \frac{\pi}{2}$, then S_ϑ is the closed sector with vertex 0 and semi-angle ϑ.

Theorem 5.1. *Let T be a closed densely defined linear operator, let A be T-bounded with T-bound < 1 and $a, b \geq 0$, $b < 1$, as in (5.1), and let $\vartheta_b := \arctan\frac{b}{1-b} \in \left[0, \frac{\pi}{2}\right)$.*

(i) *If T is accretive, then*

$$\begin{aligned}\sigma_{\mathrm{ap}}(T+A) &\subset -\frac{a}{1-b} + S_{\frac{\pi}{2}+\vartheta_b}\\ &= \left\{\lambda \in \mathbb{C} \;\middle|\; \operatorname{Re}\lambda \geq -\frac{a}{1-b} - \frac{b}{1-b}|\operatorname{Im}\lambda|\right\}\end{aligned}$$

and

$$\|(T+A-\lambda)^\dagger\| \leq \frac{1}{(1-b)|\operatorname{Re}\lambda| - (a+b|\operatorname{Im}\lambda|)}$$

for $\lambda \notin -\frac{a}{1-b} + S_{\frac{\pi}{2}+\vartheta_b}$.

(ii) *If T is sectorial with semi-angle $\theta \in \left[0, \frac{\pi}{2}\right)$, then*

$$\begin{aligned}\sigma_{\mathrm{ap}}(T+A) &\subset \overline{B_{\frac{a}{1-b}}(0)} \cup \Big(-\frac{a}{b\cos\theta+(1-b)\sin\theta} + S_{\theta+\vartheta_b}\Big)\\ &= \overline{B_{\frac{a}{1-b}}(0)} \cup \Big\{\lambda \in \mathbb{C} \,\Big|\, \operatorname{Re}\lambda \geq -\frac{a}{b\cos\theta+(1-b)\sin\theta}\\ &\qquad -\frac{b\sin\theta-(1-b)\cos\theta}{b\cos\theta+(1-b)\sin\theta}|\operatorname{Im}\lambda|\Big\}\end{aligned}$$

and

$$\|(T+A-\lambda)^{\dagger}\| \leq \frac{1}{(1-b)|\lambda|-a}$$

for $\lambda \notin (\overline{B_{\frac{a}{1-b}}(0)} \cup S_{\frac{\pi}{2}+\theta})$, as well as

$$\|(T+A-\lambda)^{\dagger}\| \leq \frac{1}{d(\lambda; a, b, \theta)}$$

for $\lambda \in S_{\frac{\pi}{2}+\theta} \setminus \big(-\frac{a}{b\cos\theta+(1-b)\sin\theta} + S_{\theta+\vartheta_b}\big)$, where

$$\begin{aligned}d(\lambda; a, b, \theta) := (1-b)&|\cos\theta\operatorname{Im}\lambda - \sin\theta\operatorname{Re}\lambda|\\ &-(a+b|\cos\theta\operatorname{Re}\lambda + \sin\theta\operatorname{Im}\lambda|).\end{aligned}$$

Remark 5.2. The claims in (i) are identical with the claims in (ii) if we allow $\theta = \frac{\pi}{2}$ there; note that in this case $\cos\theta = 0$, $\sin\theta = 1$ and the disc $\overline{B_{\frac{a}{1-b}}(0)}$ is then contained in $-\frac{a}{b\cos\theta+(1-b)\sin\theta} + S_{\theta+\vartheta_b} = -\frac{a}{1-b} + S_{\frac{\pi}{2}+\vartheta_b}$.

Proof of Theorem 5.1. (i) Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < 0$ and $x \in \mathcal{D}(T)$. Since T is accretive, we can estimate

$$\begin{aligned}\|(T-\lambda)x\|^2 &= \|(T-\mathrm{i}\operatorname{Im}\lambda)x\|^2 + |\operatorname{Re}\lambda|^2\|x\|^2 - 2\operatorname{Re}\lambda\operatorname{Re}((T-\mathrm{i}\operatorname{Im}\lambda)x, x)\\ &= \|(T-\mathrm{i}\operatorname{Im}\lambda)x\|^2 + |\operatorname{Re}\lambda|^2\|x\|^2 - 2\operatorname{Re}\lambda\operatorname{Re}((Tx, x))\\ &\geq \|(T-\mathrm{i}\operatorname{Im}\lambda)x\|^2\end{aligned}$$

and hence

$$\begin{aligned}\|Ax\| &\leq a\|x\| + b\|Tx\|\\ &\leq (a+b|\operatorname{Im}\lambda|)\|x\| + b\|(T-\mathrm{i}\operatorname{Im}\lambda)x\|\\ &\leq (a+b|\operatorname{Im}\lambda|)\|x\| + b\|(T-\lambda)x\|,\end{aligned}$$

which implies that

$$\begin{aligned}\|(T+A-\lambda)x\| &\geq \|(T-\lambda)x\| - \|Ax\|\\ &\geq (1-b)\|(T-\lambda)x\| - (a+b|\operatorname{Im}\lambda|)\|x\|.\end{aligned}$$

Taking the infimum over all $x \in \mathcal{D}(T)$, $\|x\| = 1$, by (3.2) we obtain that, for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < 0$,

$$\begin{aligned} \gamma(T + A - \lambda) &\geq (1-b)\gamma(T-\lambda) - (a + b|\operatorname{Im}\lambda|) \\ &\geq (1-b)|\operatorname{Re}\lambda| - (a + b|\operatorname{Im}\lambda|). \end{aligned} \tag{5.2}$$

Thus it follows that $\lambda \in \rho_{\mathrm{ap}}(T + A)$ or, equivalently, $\gamma(T + A - \lambda) > 0$ provided that $(1-b)|\operatorname{Re}\lambda| - (a + b|\operatorname{Im}\lambda|) > 0$; since $\operatorname{Re}\lambda < 0$ the latter is equivalent to the condition $\operatorname{Re}\lambda < -\frac{a+b|\operatorname{Im}\lambda|}{1-b}$, which proves the first claim. The norm estimate for the Moore–Penrose inverse for these λ follows from (5.2) and from $\|(T + A - \lambda)^\dagger\| = \frac{1}{\gamma(T+A-\lambda)}$, see (3.3).

(ii) The claims in the sectorial case all follow by applying (i) for the accretive case in two different situations in the upper and lower half-plane, depending on the position of $\lambda \notin S_\theta$. If $\lambda \in S_{\frac{\pi}{2}+\theta} \setminus S_\theta$ and $\operatorname{Im}\lambda > 0$ then the operator $\mathrm{e}^{\mathrm{i}(\frac{\pi}{2}-\theta)}T$ is accretive and we apply the claims in (i) to $\mathrm{e}^{\mathrm{i}(\frac{\pi}{2}-\theta)}T$ perturbed by $\mathrm{e}^{\mathrm{i}(\frac{\pi}{2}-\theta)}A$ at the point $\mathrm{e}^{\mathrm{i}(\frac{\pi}{2}-\theta)}\lambda$; analogously, we treat the case that $\operatorname{Im}\lambda < 0$ replacing the rotating factor $\mathrm{e}^{\mathrm{i}(\frac{\pi}{2}-\theta)}$ by $\mathrm{e}^{-\mathrm{i}(\frac{\pi}{2}-\theta)}$. If $\lambda \notin S_{\frac{\pi}{2}+\theta}$ and $\operatorname{Im}\lambda > 0$, then $\arg(\lambda) \in \big(\theta + \frac{\pi}{2}, \pi\big)$, the operator $\mathrm{e}^{\mathrm{i}(\pi-\arg(\lambda))}T$ is accretive and we apply the claims in (i) to $\mathrm{e}^{\mathrm{i}(\pi-\arg(\lambda))}T$ perturbed by $\mathrm{e}^{\mathrm{i}(\pi-\arg(\lambda))}A$ at the point $\mathrm{e}^{\mathrm{i}(\pi-\arg(\lambda))}\lambda$; analogously, we treat the case that $\operatorname{Im}\lambda < 0$ replacing the rotating factor $\mathrm{e}^{\mathrm{i}(\pi-\arg(\lambda))}$ by $\mathrm{e}^{-\mathrm{i}(\pi-\arg(\lambda))}$. We leave the remaining simple details of deriving the claimed formulas in (ii) to the reader. ∎

If more is known about the shape of the numerical range of T than being contained in a half-plane or in a sector, then the following local perturbation result may be applied to all rays perpendicular to supporting half-planes in boundary points of the numerical range. Here, for $\phi \in (-\pi, \pi]$ and $r \geq 0$, we denote the open ray emanating from the point $r\mathrm{e}^{\mathrm{i}\phi}$ with angle ϕ by

$$R_{r,\phi} := \{t\mathrm{e}^{\mathrm{i}\phi} \in \mathbb{C} \mid t > r\}.$$

Lemma 5.3. *Let T be a closed densely defined linear operator, $\mu \in \mathbb{C}$ and $\phi \in (-\pi, \pi]$. Suppose A is T-bounded with T-bound < 1 and constants $a, b \geq 0$, $b < 1$, as in (5.1). If $(\mu + R_{0,\phi}) \cap \overline{W(T)} = \emptyset$ and*

$$\operatorname{dist}(\mu + t\mathrm{e}^{\mathrm{i}\phi}, W(T)) \geq t, \quad t \geq 0, \tag{5.3}$$

i.e. the ray $\overline{\mu + R_{0,\phi}}$ is perpendicular to some line separating $\overline{W(T)}$ and $\mu + R_{0,\phi}$. Then

$$\mu + R_{\frac{a+b|\mu|}{1-b},\phi} \subset \rho_{\mathrm{ap}}(T + A)$$

and

$$\|(T + A - \lambda)^\dagger\| \leq \frac{1}{(1-b)\operatorname{dist}(\lambda, W(T)) - (a + b|\mu|)}, \quad \lambda \in R_{\frac{a+b|\mu|}{1-b},\phi}.$$

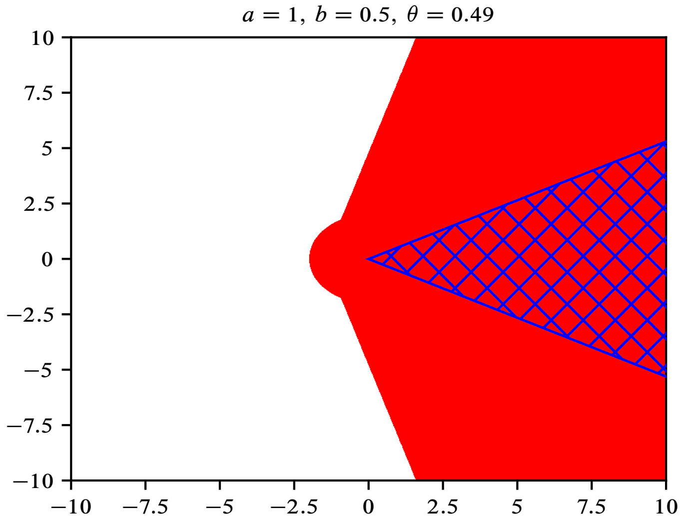

Figure 2. The blue hatched area shows S_θ for the indicated $\theta \in \left[0, \frac{\pi}{2}\right]$ and the red area shows the enclosure obtained for $\sigma_{\mathrm{ap}}(T + A)$ via Theorem 5.1 in this situation for $a = 1$ and $b = \frac{1}{2}$.

Proof. The proof is analogous to the proof of Theorem 5.1 (i) if we show that the operator $-\mathrm{e}^{-\mathrm{i}\phi}(T - \mu)$ is accretive. Suppose the contrary, i.e. there is a $z \in W(T)$ with

$$-\operatorname{Re}(\mathrm{e}^{-\mathrm{i}\phi}(z - \mu)) < 0.$$

This implies that

$$|\mu + t\mathrm{e}^{\mathrm{i}\phi} - z|^2 - t^2 = |\mu - z|^2 - 2t \operatorname{Re}(\mathrm{e}^{-\mathrm{i}\phi}(z - \mu)) \to -\infty, \quad t \to \infty.$$

Thus there exists $t_0 > 0$ such that

$$\operatorname{dist}(\mu + t\mathrm{e}^{\mathrm{i}\phi}, W(T)) \leq |\mu + t\mathrm{e}^{\mathrm{i}\phi} - z| < t, \quad t \geq t_0,$$

a contradiction to (5.3). ∎

In order to establish analogous enclosures for essential spectra and corresponding essential norm estimates of Moore–Penrose resolvents, we need two geometric lemmas relating $W_{\mathrm{e}}(T)$ and $\overline{W(T)}$. The first one is a generalisation of [5, Proposition 2.4].

Lemma 5.4. *Let T be an arbitrary linear operator. If $\lambda_0 \in W_{\mathrm{e}}(T)$ and there exists $\theta_0 \in (-\pi, \pi]$ with $R := \lambda_0 + \mathrm{e}^{\mathrm{i}\theta_0}[0, \infty) \subset \overline{W(T)}$, then $R \subset W_{\mathrm{e}}(T)$.*

Remark 5.5. In [5, Proposition 2.4] it was assumed that $R \subset W(T)$ rather than $R \subset \overline{W(T)}$. Note that parts of the proof of [5, Corollary 2.5] would have already required the stronger result Lemma 5.4; thus, although its proof is similar to the one of [5, Proposition 2.4], we sketch it here.

Proof of Lemma 5.4. After possible shift and rotation, we may assume, without loss of generality, that $\lambda_0 = \theta_0 = 0$, i.e. $R = [0, \infty)$. Let $\lambda > 0$ be arbitrary. By assumption $0 \in W|_{\mathrm{e}}(T)$ and hence there exists a normalised weakly null-sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T)$ such that $(Tx_n, x_n) \to 0$ as $n \to \infty$. Since $R \subset \overline{W(T)}$ there exists a normalised sequence $(y_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T)$ such that

$$(Ty_n, y_n) = n + \varepsilon_n, \quad n \in \mathbb{N},$$

where $(\varepsilon_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ is a bounded sequence. Since $x_n \xrightarrow{w} 0$ as $n \to \infty$, there exists a strictly increasing sequence $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ with

$$|(y_k, x_{n_k})| < \frac{1}{k}, \quad |(Ty_k, x_{n_k})| < \frac{1}{k}, \quad k \in \mathbb{N}.$$

Further, we choose $r_k \geq 0$ and $\theta_k \in (-\pi, \pi]$ so that

$$r_k^2 k + r_k |(Tx_{n_k}, y_k)| = \lambda, \quad \mathrm{e}^{-\mathrm{i}\theta_k}(Tx_{n_k}, y_k) \geq 0, \quad k \in \mathbb{N};$$

note that then $r_k \to 0$ as $k \to \infty$. Hence, if we set

$$u_k := x_{n_k} + r_k \mathrm{e}^{\mathrm{i}\theta_k} y_k, \quad v_k := \frac{u_k}{\|u_k\|}, \quad k \in \mathbb{N},$$

then $\|u_k\| \to 1$ as $k \to \infty$, $(v_k)_{k\in\mathbb{N}} \subset \mathcal{D}(T)$ is a normalised weakly null-sequence and, noting that $r_k^2 \varepsilon_k \to 0$ as $k \to \infty$ since $(\varepsilon_k)_{k\in\mathbb{N}}$ is bounded, it is not difficult to check that $(Tv_k, v_k) \to \lambda$ as $k \to \infty$ and hence $\lambda \in W_{\mathrm{e}}(T)$. ■

Recall that if T is bounded, then $W_{\mathrm{e}}(T)$ is a non-empty compact set. Examples show, see e.g. [5, Examples 2.6 and 3.5], that if T is unbounded, then $W_{\mathrm{e}}(T)$ may be empty or unbounded. Lemma 5.4 yields that either of these cases prevails.

Corollary 5.6. *Let T be an arbitrary unbounded linear operator. Then*

(i) *$W(T)$ is unbounded;*

(ii) *$W_{\mathrm{e}}(T)$ is either empty or unbounded.*

Remark 5.7. For densely defined operators, claim (i) was shown in [37, Proposition 2.51], but we are not aware of a reference for the case without dense domain.

Proof of Corollary 5.6. (i) Set $M := \overline{\mathcal{D}(T)}$ and consider the compression $T_M := P_M T|_M$ with $\mathcal{D}(T_M) = \mathcal{D}(T)$. Then T_M is densely defined in the Hilbert space M and therefore $W(T) = W(T_M)$ is unbounded by [37, Proposition 2.51].

(ii) Since $\overline{W(T)}$ is closed, convex and unbounded by (i), it contains some ray, see e.g. [31, Theorem 8.4]. If $W_e(T) \neq \emptyset$, then $W_e(T)$ therefore also contains some ray by Lemma 5.4 and is thus unbounded as well. ■

Lemma 5.8. *Let T be a linear operator with $W_e(T) \neq \emptyset$ and*

$$W_e(T) \subset \mathbb{H}_+ = \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0\}.$$

Then there exist $\phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\mu \in \mathbb{C}$, $\operatorname{Re} \mu \leq 0$, such that

$$\overline{W(T)} \subset \mu + e^{i\phi}\mathbb{H}_+.$$

Proof. In this proof we will frequently use the fact that if $C \subset \mathbb{C}$ is an unbounded closed convex set, then for every $z \in C$ there exists a ray R_z emanating from z such that $R_z \subset C$; this follows e.g. from [31, Theorem 8.4].

Let $\lambda_0 \in W_e(T)$. Since $W_e(T) \neq \mathbb{C}$ by assumption, it follows that $W(T) \neq \mathbb{C}$ by [5, Corollary 2.5]. If $\inf \operatorname{Re} W(T) > -\infty$, then the claim follows with $\phi := 0$ and $\mu := \min\{\inf \operatorname{Re} W(T), 0\}$. So it remains to consider the case that $W(T) \neq \mathbb{C}$ and $\inf \operatorname{Re} W(T) = -\infty$.

Then there exists a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset W(T)$ such that $\operatorname{Re} \lambda_n \to -\infty$ as $n \to \infty$. If $s := \sup_{n\in\mathbb{N}_0} |\operatorname{Im} \lambda_n| < \infty$, then $|\lambda_0| \leq s$ and

$$C_0 := (\mathbb{R} + i[-s, s]) \cap \{z \in \mathbb{C} \mid \operatorname{Re} z \leq \operatorname{Re} \lambda_0\} \cap \overline{W(T)}$$

is an unbounded closed convex set with $\lambda_0 \in C_0$ and $(\lambda_n)_{n\in\mathbb{N}} \subset C_0$. Hence, by [31, Theorem 8.4], there exists a ray R emanating from $\lambda_0 \in C_0$ with $R \subset C_0$. By definition of C_0, this ray must be of the form $R = \lambda_0 + (-\infty, 0]$. Now Lemma 5.4 implies that $R \subset W_e(T)$, a contradiction to $W_e(T) \subset \mathbb{H}_+$. Therefore $s = \sup_{n\in\mathbb{N}_0} |\operatorname{Im} \lambda_n| = \infty$ and hence at least one of $\sup_{n\in\mathbb{N}_0} \operatorname{Im} \lambda_n = \infty$ or $\inf_{n\in\mathbb{N}_0} \operatorname{Im} \lambda_n = -\infty$ prevails. Assume that the first case holds. Then

$$C_+ := \{z \in \mathbb{C} \mid \operatorname{Im} z \geq \operatorname{Im} \lambda_0\} \cap \{z \in \mathbb{C} \mid \operatorname{Re} z \leq \operatorname{Re} \lambda_0\} \cap \overline{W(T)}$$

is an unbounded closed convex set with $\lambda_0 \in C_+$ and $(\lambda_n)_{n\in\mathbb{N}} \subset C_+$. Hence, by [31, Theorem 8.4], there exists a ray R emanating from $\lambda_0 \in C_+$ with $R \subset C_+$. By definition of C_+, this ray must be of the form $R = \lambda_0 + e^{i\theta}[0, \infty)$ for some $\theta \in \left[\frac{\pi}{2}, \pi\right]$. Now Lemma 5.4 yields that $R \subset W_e(T)$. If $\theta > \frac{\pi}{2}$, this contradicts $W_e(T) \subset \mathbb{H}_+$ and therefore $\theta = \frac{\pi}{2}$, i.e.

$$R = \lambda_0 + i[0, \infty) \subset W_e(T) \subset \overline{W(T)}. \tag{5.4}$$

Moreover, [31, Theorem 8.3] yields that $R_n := (\lambda_n - \lambda_0) + R = \lambda_n + i[0, \infty) \subset \overline{W(T)}$ for $n \in \mathbb{N}_0$. Let $L_n \subset \mathbb{C}$, $n \in \mathbb{N}_0$, be the vertical lines with $R_n \subset L_n$, $n \in \mathbb{N}$. Suppose that there exists $n_0 \in \mathbb{N}$ such that $L_{n_0} \subset \overline{W(T)}$. Because $\lambda_n \in R_n \subset \overline{W(T)}$,

$n \in \mathbb{N}$, [31, Theorem 8.4] yields that $L_n \subset \overline{W(T)}$ for all $n \in \mathbb{N}_0$. Since $\operatorname{Re}\lambda_n \to -\infty$ as $n \to \infty$, this implies that

$$\{z \in \mathbb{C} \mid \operatorname{Re} z \le \operatorname{Re}\lambda_0\} \subset \overline{W(T)}.$$

Therefore and because $\operatorname{Re}\lambda_0 \ge 0$, $\overline{W(T)}$ must be a left half-plane of the form $\{z \in \mathbb{C} \mid \operatorname{Re} z \le z_0\}$ for some $z_0 \ge 0$. Then, by [5, Corollary 2.5], $W_e(T)$ is a half-plane as well and $W_e(T) \subset \overline{W(T)}$, a contradiction to $W_e(T) \subset \mathbb{H}_+$. Hence $L_n \not\subset \overline{W(T)}$ for all $n \in \mathbb{N}_0$. This and $\lambda_n \in L_n$ for $n \in \mathbb{N}_0$ imply that

$$L_n \cap \partial W(T) \neq \emptyset, \quad n \in \mathbb{N}_0.$$

Let $\mu_n \in L_n \cap \partial W(T)$, $n \in \mathbb{N}_0$. Then

$$\operatorname{Re}\mu_n = \operatorname{Re}\lambda_n, \quad n \in \mathbb{N}_0,$$

and hence $\operatorname{Re}\mu_n \to -\infty$ as $n \to \infty$. Next we show that $\sup_{n\in\mathbb{N}} \operatorname{Im}\mu_n = \infty$. Suppose the contrary, i.e. that $\tilde{s} := \sup_{n\in\mathbb{N}_0} \operatorname{Im}\mu_n < \infty$. Then

$$C_- := \{z \in \mathbb{C} \mid \operatorname{Im} z \le \tilde{s}\} \cap \{z \in \mathbb{C} \mid \operatorname{Re} z \le \operatorname{Re}\lambda_0\} \cap \overline{W(T)}$$

is an unbounded closed convex set. Since $\lambda_0 \in C_-$, [31, Theorem 8.4] yields that there exists a ray $\widetilde{R} \subset C_-$ emanating from $\lambda_0 \in C_-$. By definition of C_-, this ray must be of the form $\widetilde{R} = \lambda_0 + e^{i\tilde{\theta}}[0,\infty)$ for some $\tilde{\theta} \in \left[-\pi, -\frac{\pi}{2}\right]$. Now Lemma 5.4 yields that $R \subset W_e(T)$. If $\theta < -\frac{\pi}{2}$, this contradicts $W_e(T) \subset \mathbb{H}_+$ and therefore $\theta = -\frac{\pi}{2}$. But then we have $L_0 = R \cup \widetilde{R} \subset \overline{W(T)}$, a contradiction to $L_n \not\subset \overline{W(T)}$ for all $n \in \mathbb{N}_0$. Hence $\sup_{n\in\mathbb{N}_0} \operatorname{Im}\mu_n = \infty$. Together with $\operatorname{Re}\mu_n \to -\infty$ as $n \to \infty$, it follows that there exists $N \in \mathbb{N}$ with

$$\operatorname{Im}\mu_N > \operatorname{Im}\lambda_0, \quad \operatorname{Re}\mu_N < 0 \le \operatorname{Re}\lambda_0. \tag{5.5}$$

Since $\overline{W(T)}$ is closed and convex with $\mu_N \in \partial W(T)$, there exists a supporting hyperplane at μ_N by [24, Theorem 3.7.4], i.e. there is a $\phi \in (-\pi, \pi]$ such that

$$\overline{W(T)} \subset \mu_N + e^{i\phi}\mathbb{H}_+. \tag{5.6}$$

Since $\lambda_0 \in W_e(T) \subset \overline{W(T)}$ and, by [31, Theorem 8.4] and (5.4), $\mu_N + i[0,\infty) \in \overline{W(T)}$. This yields that $\phi \in [0, \pi]$. It is not difficult to check that the inequalities (5.5) together with $\mu_N \in \partial W(T)$ imply $\phi \in \left[0, \frac{\pi}{2}\right)$. This completes the proof in the case $\sup_{n\in\mathbb{N}_0} \operatorname{Im}\lambda_n = \infty$.

In the case $\inf_{n\in\mathbb{N}_0} \operatorname{Im}\lambda_n = -\infty$, the proof is analogous and here (5.6) holds with $\phi \in \left(-\frac{\pi}{2}, 0\right]$. ∎

Our next stability result generalises the general stability theorem for semi-Fredholm operators, see [18, Section 4, Theorem 1a] or [19, Theorem IV.5.22]. The latter shows that if T is injective with closed range, i.e. $0 \in \rho_{\mathrm{ap}}(T)$ and hence $\gamma(T) > 0$, A is T-bounded with T-bound < 1 and $a, b \geq 0$ in (5.1) with $b < 1$ satisfy

$$a < (1-b)\gamma(T),$$

then $T + A$ is injective with closed range, i.e., $0 \in \rho_{\mathrm{ap}}(T + A)$. In analogy to the proof of [19, Theorem IV.1.16], one can also show that in this case $\gamma(T + A) \geq (1-b)\gamma(T) - a > 0$ or, equivalently, see (3.3),

$$\|(T + A)^{\dagger}\| \leq \frac{\|T^{\dagger}\|}{(1-b) - a\|T^{\dagger}\|}.$$

Proposition 5.9. *Let T be a closed densely defined linear operator and let A be T-bounded with T-bound < 1. Suppose that T is upper semi-Fredholm, i.e., $0 \in \rho_{\mathrm{e2}}(T)$ (so that $\gamma_{\mathrm{e}}(T) > 0$), and a_{e}, $b_{\mathrm{e}} \geq 0$, $b_{\mathrm{e}} < 1$, are such that, for all normalised weakly null-sequences $(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T)$,*

$$\limsup_{n\to\infty} \|Ax_n\| \leq a_{\mathrm{e}} + b_{\mathrm{e}} \limsup_{n\to\infty} \|Tx_n\|. \tag{5.7}$$

If

$$a_{\mathrm{e}} < (1-b_{\mathrm{e}})\gamma_{\mathrm{e}}(T), \tag{5.8}$$

then $T + A$ is upper-semi-Fredholm, i.e. $0 \in \rho_{\mathrm{e2}}(T + A)$, with essential minimum modulus $\gamma_{\mathrm{e}}(T + A) \geq (1-b_{\mathrm{e}})\gamma_{\mathrm{e}}(T) - a_{\mathrm{e}} > 0$ or, equivalently,

$$\|(T + A)^{\dagger}\|_{\mathrm{e}} \leq \frac{\|T^{\dagger}\|_{\mathrm{e}}}{(1-b_{\mathrm{e}}) - a_{\mathrm{e}}\|T^{\dagger}\|_{\mathrm{e}}}. \tag{5.9}$$

Remark 5.10. The constants a_{e} and b_{e} in (5.7) can always be chosen equal to the constants a, b in (5.1), but, since (5.7) is a weaker condition, they may also be chosen differently to optimise the constants in the estimate (5.9).

Proof of Proposition 5.9. Since A has T-bound < 1, there exist $a, b \geq 0$, $b < 1$, satisfying (5.1) and $T + A$ is closed by [19, Theorem IV.1.1]. If $(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T)$ is an arbitrary normalised weakly null-sequence, then

$$(1-b)\|Tx_n\| - a \leq \|(T + A)x_n\| \leq (1+b)\|Tx_n\| + a, \quad n \in \mathbb{N}. \tag{5.10}$$

Hence $\liminf_{n\to\infty} \|Tx_n\| = \infty$ if and only if $\liminf_{n\to\infty} \|(T + A)x_n\| = \infty$; in particular, $\gamma_{\mathrm{e}}(T) = \infty$ if and only if $\gamma_{\mathrm{e}}(T + A) = \infty$. Now let $\gamma_{\mathrm{e}}(T) < \infty$ and $(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T)$ be an arbitrary normalised weakly null-sequence such that

$\liminf_{n\to\infty} \|Tx_n\| < \infty$. Then there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ with $\liminf_{n\to\infty} \|Tx_n\| = \lim_{k\to\infty} \|Tx_{n_k}\|$. Since A is T-bounded, $(\|Ax_{n_k}\|)_{k\in\mathbb{N}} \subset \mathbb{C}$ is a bounded sequence and hence we may assume, upon choosing another subsequence, without loss of generality that $(\|Ax_{n_k}\|)_{k\in\mathbb{N}}$ converges in $\mathbb{C}$. Then, by (5.7), (5.8), and Proposition 3.3 (ii),

$$\begin{aligned}\liminf_{n\to\infty} \|(T+A)x_n\| &\geq \liminf_{n\to\infty} \|Tx_n\| - \limsup_{n\to\infty} \|Ax_n\| \\ &= \lim_{k\to\infty} \|Tx_{n_k}\| - \lim_{k\to\infty} \|Ax_{n_k}\| \\ &\geq (1-b_{\mathrm{e}}) \lim_{k\to\infty} \|Tx_{n_k}\| - a_{\mathrm{e}} \\ &\geq (1-b_{\mathrm{e}})\gamma_{\mathrm{e}}(T) - a_{\mathrm{e}} > 0.\end{aligned}$$

By Proposition 3.3 (ii) and (i), this yields $\gamma_{\mathrm{e}}(T+A) > 0$ and hence $0 \in \rho_{\mathrm{e}2}(T+A)$ as well as the estimate for $\gamma_{\mathrm{e}}(T+A)$ which, by Theorem 3.7, is equivalent to (5.9). ∎

Now we are ready to prove our perturbation result for the essential spectrum of type 2 and the essential approximate point spectrum, accompanied by essential norm estimates for the Moore–Penrose resolvents of the perturbed operators.

Theorem 5.11. *Let T be a closed densely defined linear operator, let A be T-bounded with T-bound < 1 and $a, b \geq 0$, $b < 1$, as in (5.1), and let $\vartheta_b := \arctan \frac{b}{1-b} \in \left[0, \frac{\pi}{2}\right)$.*

(i) *If $W_{\mathrm{e}5}(T) \subset \mathbb{H}_+ = S_{\frac{\pi}{2}}$, then*

$$\begin{aligned}\sigma_{\mathrm{e}2}(T+A) &\subset -\frac{a}{1-b} + S_{\frac{\pi}{2}+\vartheta_b} \\ &= \left\{\lambda \in \mathbb{C} \;\middle|\; \operatorname{Re}\lambda \geq -\frac{a}{1-b} - \frac{b}{1-b}|\operatorname{Im}\lambda|\right\}\end{aligned}$$

and

$$\|(T+A-\lambda)^{\dagger}\|_{\mathrm{e}} \leq \frac{1}{(1-b)|\operatorname{Re}\lambda| - (a+b|\operatorname{Im}\lambda|)}$$

for $\lambda \notin -\frac{a}{1-b} + S_{\frac{\pi}{2}+\vartheta_b}$.

(ii) *If $W_{\mathrm{e}5}(T) \subset S_\theta$ with $\theta \in \left[0, \frac{\pi}{2}\right)$, then*

$$\begin{aligned}\sigma_{\mathrm{e}2}(T+A) &\subset \overline{B_{\frac{a}{1-b}}(0)} \cup \left(-\frac{a}{b\cos\theta + (1-b)\sin\theta} + S_{\theta+\vartheta_b}\right) \\ &= \overline{B_{\frac{a}{1-b}}(0)} \cup \left\{\lambda \in \mathbb{C} \;\middle|\; \operatorname{Re}\lambda \geq -\frac{a}{b\cos\theta + (1-b)\sin\theta} \right. \\ &\qquad \left. - \frac{b\sin\theta - (1-b)\cos\theta}{b\cos\theta + (1-b)\sin\theta}|\operatorname{Im}\lambda|\right\}\end{aligned} \tag{5.11}$$

and

$$\|(T+A-\lambda)^{\dagger}\|_{\mathrm{e}} \leq \frac{1}{(1-b)|\lambda|-a}$$

for $\lambda \notin (\overline{B_{\frac{a}{1-b}}(0)} \cup S_{\frac{\pi}{2}+\theta})$*, as well as*

$$\|(T+A-\lambda)^{\dagger}\|_{\mathrm{e}} \leq \frac{1}{d(\lambda;a,b,\theta)},$$

for $\lambda \in S_{\frac{\pi}{2}+\theta} \setminus \big(-\frac{a}{b\cos\theta+(1-b)\sin\theta} + S_{\theta+\vartheta_b}\big)$*, where*

$$\begin{aligned} d(\lambda;a,b,\theta) := (1-b)|\cos\theta \operatorname{Im}\lambda - \sin\theta \operatorname{Re}\lambda| \\ - (a + b|\cos\theta \operatorname{Re}\lambda + \sin\theta \operatorname{Im}\lambda|). \end{aligned}$$

If $\mathcal{D}(T) \cap \mathcal{D}(T^*)$ *is a core of* T *or the assumptions in* (i) *or* (ii) *hold with* $W_{\mathrm{e}5}(T)$ *replaced by (the possibly larger set)* $W_{\mathrm{e}}(T)$*, then the respective claims in* (i) *or* (ii) *hold with* $\sigma_{\mathrm{e}2}(T+A)$ *replaced by (the possibly larger set)* $\sigma_{\mathrm{eap}}(T+A)$*.*

Remark 5.12. The claims in (i) are identical with the claims in (ii) if we allow $\theta = \frac{\pi}{2}$ there, compare Remark 5.2.

Proof of Theorem 5.11. (i) Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < 0$ and let $(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T)$ be any normalised weakly null-sequence. Then

$$\begin{aligned} \|(T-\lambda)x_n\|^2 &= \|(T-\mathrm{i}\operatorname{Im}\lambda)x_n\|^2 + |\operatorname{Re}\lambda|^2\|x_n\|^2 \\ &\quad - 2\operatorname{Re}\lambda \operatorname{Re}((T-\mathrm{i}\operatorname{Im}\lambda)x_n, x_n) \\ &= \|(T-\mathrm{i}\operatorname{Im}\lambda)x_n\|^2 + |\operatorname{Re}\lambda|^2 - 2\operatorname{Re}\lambda\operatorname{Re}((Tx_n,x_n)). \end{aligned} \tag{5.12}$$

Suppose that $\sup_{n\in\mathbb{N}}\|Tx_n\| < \infty$. Since A is T-bounded, $(\|Ax_{n_k}\|)_{k\in\mathbb{N}} \subset \mathbb{C}$ is a bounded sequence and hence we may assume, upon choosing another subsequence, without loss of generality that there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}} \subset (x_n)_{n\in\mathbb{N}}$ such that $(Tx_{n_k})_{k\in\mathbb{N}}$ is weakly convergent, $(\|(T-\mathrm{i}\operatorname{Im}\lambda)x_{n_k}\|)_{k\in\mathbb{N}}$ converges in $\mathbb{C}$ and

$$\begin{aligned} &\lim_{k\to\infty}\|Ax_{n_k}\| = \limsup_{n\to\infty}\|Ax_n\| < \infty, \\ &\lim_{k\to\infty}(Tx_{n_k}, x_{n_k}) = z \in \mathbb{C}; \end{aligned} \tag{5.13}$$

note that then, by (5.12), $(\|(T-\lambda)x_{n_k}\|)_{k\in\mathbb{N}}$ converges as well. Since T is closed, its graph is also weakly closed, see e.g. [32, Theorem 3.12], and so $Tx_{n_k} \xrightarrow{w} 0$ as $k\to\infty$ as $(x_{n_k})_{k\in\mathbb{N}}$ is a weakly null-sequence. Hence [14, Theorem 2.3] implies that $z \in W_{e5}(T)$. By assumption, it then follows that $\operatorname{Re} z > 0$ and thus, by (5.12) and (5.13),

$$\begin{aligned}\lim_{k\to\infty} \|(T - \mathrm{i}\operatorname{Im}\lambda)x_{n_k}\|^2 &\le \lim_{k\to\infty} \|(T - \mathrm{i}\operatorname{Im}\lambda)x_{n_k}\|^2 + |\operatorname{Re}\lambda|^2 - 2\operatorname{Re}\lambda\operatorname{Re} z \\ &= \lim_{k\to\infty} \|(T-\lambda)x_{n_k}\|^2 \le \limsup_{n\to\infty} \|(T-\lambda)x_n\|^2.\end{aligned}$$

imply that

$$\begin{aligned}\limsup_{n\to\infty} \|Ax_n\| &= \lim_{k\to\infty} \|Ax_{n_k}\| \\ &\le a + b \limsup_{k\to\infty} \|Tx_{n_k}\| \\ &\le (a + b|\operatorname{Im}\lambda|) + b \lim_{k\to\infty} \|(T - \mathrm{i}\operatorname{Im}\lambda)x_{n_k}\| \\ &\le (a + b|\operatorname{Im}\lambda|) + b \limsup_{n\to\infty} \|(T-\lambda)x_n\|.\end{aligned}$$

Now suppose that $\sup_{n\in\mathbb{N}} \|Tx_n\| = \infty$. Since, by the two-sided estimate (5.10), this is equivalent to $\sup_{n\in\mathbb{N}} \|Ax_n\| = \infty$, we have

$$\limsup_{n\in\mathbb{N}} \|Tx_n\| = \infty \iff \limsup_{n\in\mathbb{N}} \|Ax_n\| = \infty.$$

Hence, in all cases, condition (5.7) of Proposition 5.9 holds for the operators $T - \lambda$ and A with constants $a_e = a + b|\operatorname{Im}\lambda|$ and $b_e = b$ therein. Now all claims in (i) follow from Proposition 5.9 and from $\|(T + A - \lambda)^\dagger\|_e = (\gamma_e(T + A - \lambda))^{-1}$, see Theorem 3.7.

(ii) The proof of the case where $W_{e5}(T)$ is contained in a sector follows by applying the claims in the case where $W_{e5}(T)$ is contained in the closed right half-plane in the very same way as the sectorial case (ii) in Theorem 5.1 was derived from the accretive case (i) therein.

If $\mathcal{D}(T) \cap \mathcal{D}(T^*)$ is a core of T, then $W_{e5}(T) = W_e(T)$. Hence to prove the last claims, in both cases we can assume that even $W_e(T) \supset W_{e5}(T)$ instead of $W_{e5}(T)$ satisfies the enclosures in (i) and (ii). So assume that $W_e(T) \subset S_\theta$ for some $\theta \in \left[0, \frac{\pi}{2}\right]$. Then, since $W_{e5}(T) \subset W_e(T) \subset S_\theta$, we infer from (i) or (ii), respectively, that

$$\Delta_\theta := \mathbb{C} \setminus \left(\overline{B_{\frac{a}{1-b}}(0)} \cup \left(-\frac{a}{b\cos\theta + (1-b)\sin\theta} + S_{\theta+\vartheta_b}\right)\right) \subset \rho_{e2}(T + A). \tag{5.14}$$

Since the set Δ_θ is connected and the index is locally constant, See e.g. [19, Theorem IV.5.31], it suffices, in view of Theorem 2.4, to find a connected set $\Delta \subset \rho_{e2}(T + A)$ with

$$\Delta \cap \Delta_\theta \neq \emptyset, \quad \Delta \cap \rho_{ap}(T + A) \neq \emptyset; \tag{5.15}$$

note that, since $W_{\mathrm{e}}(T) \subset S_\theta$ does not imply that $W(T)$ lies in some right half-plane, compare Lemma 5.8, we cannot simply choose $\Delta = \Delta_\theta$ and use Theorem 4.1.

Assume first that $W_{\mathrm{e}}(T) = \emptyset$. Then Theorem 4.1 implies that $\gamma_{\mathrm{e}}(T-\lambda) = \infty$ for all $\lambda \in \mathbb{C}$ and so $\rho_{\mathrm{e}2}(T+A) = \mathbb{C}$ by Proposition 5.9. Since $W_{\mathrm{e}}(T) \neq \mathbb{C}$, [5, Corollary 2.5] yields $W(T) \neq \mathbb{C}$. Thus $\overline{W(T)}$ is contained in some closed half-plane $\mathbb{H}$. After some shift and rotation (which only change the constant a but leave the relative bound $b < 1$ unchanged) we can assume without loss of generality that $\mathbb{H} = \mathbb{H}_+$. Applying Theorem 5.1 (i), we obtain that $\rho_{\mathrm{ap}}(T+A) \neq \emptyset$ and hence (5.15) holds for $\Delta := \mathbb{C}$.

Assume now that $W_{\mathrm{e}}(T) \neq \emptyset$. Then Lemma 5.8 applies and shows that there exist $\phi_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu \leq 0$ such that

$$W(T) \subset \mu + \mathrm{e}^{\mathrm{i}\phi_0}\mathbb{H}_+. \tag{5.16}$$

Because $\mathrm{e}^{-\mathrm{i}\phi_0}A$ is relatively bounded with respect to $\mathrm{e}^{-\mathrm{i}\phi_0}(T-\mu)$ with constants $a + b|\mu|$ and $b < 1$, we can apply Theorem 5.1 to the accretive operator $\mathrm{e}^{-\mathrm{i}\phi_0}(T-\mu)$ to conclude that

$$\mathrm{e}^{\mathrm{i}\phi_0}\left(-\infty, -\frac{a+b|\mu|}{1-b}\right) + \mu \subset \rho_{\mathrm{ap}}(T+A) \tag{5.17}$$

Suppose that $\phi_0 \geq 0$ and let $\phi \in [0, \phi_0]$ be arbitrary; the case $\phi_0 \leq 0$ is analogous. Then, due to

$$W_{\mathrm{e}5}(T) \subset W_{\mathrm{e}}(T) \subset S_\theta \subset \mathbb{H}_+$$

and $W_{\mathrm{e}}(T) \subset \overline{W(T)}$, by (5.16) we obtain that

$$W_{\mathrm{e}5}(T-\mu) \subset (\mathbb{H}_+ - \mu) \cap \mathrm{e}^{\mathrm{i}\phi_0}\mathbb{H}_+ \subset \mathbb{H}_+ \cap \mathrm{e}^{\mathrm{i}\phi_0}\mathbb{H}_+ \subset \mathrm{e}^{\mathrm{i}\phi}\mathbb{H}_+. \tag{5.18}$$

By (5.18), we can apply claim (i) to the operator $\mathrm{e}^{-\mathrm{i}\phi}(T-\mu)$ ($\mathrm{e}^{-\mathrm{i}\phi}A$ is relatively bounded with respect to $\mathrm{e}^{-\mathrm{i}\phi}(T-\mu)$ with constants $a+b|\mu|$ and $b<1$) which yields

$$R_\phi := \mathrm{e}^{\mathrm{i}\phi}\left(-\infty, -\frac{a+b|\mu|}{1-b}\right) + \mu \subset \rho_{\mathrm{e}2}(T+A).$$

Set $\Delta := \bigcup_{\phi\in[0,\phi_0]} R_\phi \subset \rho_{\mathrm{e}2}(T+A)$. Then Δ is clearly connected and satisfies

$$\emptyset \neq R_0 \cap \Delta_\theta \subset \Delta \cap \Delta_\theta, \quad \emptyset \neq R_{\phi_0} \subset \Delta \cap \rho_{\mathrm{ap}}(T+A),$$

by (5.14) and (5.17), as required in (5.15). ∎

Remark 5.13. There exists an essential version of Lemma 5.3 which may be used to study perturbations of operators for which more is known on the shape of the essential numerical range than just lying in a half-plane or a sector; in this case either $W(T)$ is replaced by $W_{\mathrm{e}5}(T)$ and, consequently, $\rho_{\mathrm{ap}}(T)$ is replaced by $\rho_{\mathrm{e}2}(T)$ or $W(T)$ is replaced by $W_e(T)$ and, consequently, $\rho_{ap}(T)$ is replaced by $\rho_{eap}(T)$. In either of those cases, the norm of the Moore–Penrose resolvent is replaced by the essential norm of the Moore–Penrose resolvent.

6. Numerical range lifting problem for unbounded operators

In this section we show that the numerical range lifting problem does not have a solution for unbounded operators. This means that, unlike for bounded operators B where there always exists a compact operator $K \in \mathcal{K}(H)$ with $W_e(B) = \overline{W(B+K)}$, this is no longer true for unbounded operators.

In fact, we show that this property may fail even for normal m-sectorial operators. A fortiori, we give an example of a normal m-sectorial operator T such that there exists an open neighbourhood U of $W_e(T)$ such that $W(T+K) \not\subset U$ for all compact operators $K \in \mathcal{K}(H)$.

Example 6.1. Let I_H denote the identity operator and let S be a self-adjoint semi-bounded operator in a Hilbert space H with

$$\overline{W(S)} = [1,\infty), \quad W_e(S) = \emptyset;$$

e.g. we can choose $H = \ell_2(\mathbb{N})$ and $Se_n = ne_n$, $n \in \mathbb{N}$, with maximal domain $\mathcal{D}(S) = \{x \in \ell_2(\mathbb{N}) : Sx \in \ell_2(\mathbb{N})\}$. Then the operator $T := I_H \oplus (S + \mathrm{i}I_H)$ in $H^2 = H \oplus H$, i.e.

$$T = \begin{pmatrix} I_H & 0 \\ 0 & S + \mathrm{i}I_H \end{pmatrix}, \quad \mathcal{D}(T) := H \oplus \mathcal{D}(S) \subset H^2,$$

is densely defined and closed. In the following we prove:

(i) T is normal;

(ii) $\overline{W(T)} = \mathrm{i}[0,1] + [1,\infty)$ and T is m-sectorial with semi-angle $\frac{\pi}{4}$;

(iii) $W_e(T) = [1,\infty)$;

(iv) there exists $\varepsilon > 0$ with $W(T+K) \not\subset U_\varepsilon := (1-\varepsilon,\infty) + \mathrm{i}(-\varepsilon,\varepsilon)$ for all $K \in \mathcal{K}(H)$.

Proof. (i) It is easy to check that T is normal since S is self-adjoint and hence $T^* = I_H \oplus (S - \mathrm{i}I_H)$.

(ii) It is straightforward to check that $W(S + \mathrm{i}I_H) = \mathrm{i} + [1,\infty)$. Thus

$$\overline{W(T)} = \overline{\mathrm{conv}(W(I) \cup W(S+\mathrm{i}I))} = \overline{\mathrm{conv}(\{1\} \cup (\mathrm{i} + [1,\infty)))} = [1,\infty) + \mathrm{i}[0,1].$$

Hence T is sectorial with semi-angle $\frac{\pi}{4}$. Moreover, it is obvious that $0 \in \rho(T)$ and hence $\sigma(T) \subset \overline{W(T)}$, which shows that T is m-sectorial.

(iii) Suppose $\lambda = a + ib \in W_e(T)$ with $b \neq 0$. Then $a \in [1, \infty)$, $b \in (0, 1]$ since $W_e(T) \subset \overline{W(T)} = [1, \infty) + \mathrm{i}[0, 1]$ by (ii) and there exists a normalised weakly null-sequence $(v_n)_{n\in\mathbb{N}} = ((x_n, y_n)^t)_{n\in\mathbb{N}} \subset \mathcal{D}(T)$ such that

$$(Tv_n, v_n) = \|x_n\|^2 + \mathrm{i}\|y_n\|^2 + (Sy_n, y_n) \to \lambda = a + \mathrm{i}b, \quad n \to \infty. \tag{6.1}$$

Hence $\|y_n\|^2 \to b > 0$ and so $\left(\frac{y_n}{\|y_n\|}\right)_{n\in\mathbb{N}} \subset \mathcal{D}(S)$ is a normalised weakly null-sequence. Since S is self-adjoint and semi-bounded with $W_e(S) = \emptyset$, it follows that $(Sy_n, y_n) \to \infty$, a contradiction to (6.1). Hence $W_e(T) \subset [1, \infty)$. Since, obviously, $\{1\} = W_e(I_H) \subset W_e(T)$ and $[1, \infty) \subset \overline{W(T)}$ by (ii), [5, Proposition 2.4] or also Lemma 5.4, respectively, yield that $W_e(T) = [1, \infty)$.

(iv) Suppose that (iv) is false. Then for all $\varepsilon > 0$ there exists $K_\varepsilon \in \mathcal{K}(H)$ with

$$W(T + K_\varepsilon) \subset U_\varepsilon = (1 - \varepsilon, \infty) + \mathrm{i}(-\varepsilon, \varepsilon).$$

Let $\varepsilon > 0$ and $v = (P_1 v, P_2 v)^t \in \mathcal{D}(T)$ with $\|v\| = 1$ be arbitrary, where P_1 and P_2 denote the orthogonal projections in H^2 onto $H \oplus \{0\}$ and $\{0\} \oplus H$, respectively. Since K_ε is compact, the self-adjoint operators $\operatorname{Re} K_\varepsilon = \frac{1}{2}(K_\varepsilon + K_\varepsilon^*)$ and $\operatorname{Im} K_\varepsilon = \frac{1}{2\mathrm{i}}(K_\varepsilon - K_\varepsilon^*)$ are compact as well. Then, for all $\varepsilon > 0$,

$$\begin{aligned} |\operatorname{Im}(((T + K_\varepsilon)v, v))| &= |\|P_2 v\|^2 + (\operatorname{Im} K_\varepsilon v, v)| \\ &= |((P_2 + \operatorname{Im} K_\varepsilon)v, v)| < \varepsilon. \end{aligned} \tag{6.2}$$

Since $\operatorname{Im} K_\varepsilon$ and P_2 (as an orthogonal projection) are self-adjoint, $P_2 + \operatorname{Im} K_\varepsilon$ is self-adjoint as well. Hence, (6.2) and the fact that $\mathcal{D}(T)$ is dense in H imply that

$$\|P_2 + \operatorname{Im} K_\varepsilon\| = \sup_{v\in\mathcal{D}(T),\, \|v\|=1} |((P_2 + \operatorname{Im} K_\varepsilon)v, v)| \leq \varepsilon. \tag{6.3}$$

Inequality (6.3) shows that $\operatorname{Im} K_\varepsilon \to P_2$ uniformly as $\varepsilon \searrow 0$. Since $\operatorname{Im} K_\varepsilon$ are compact for all $\varepsilon > 0$, it follows that P_2 is compact as well, a contradiction because $P_2 = 0 \oplus I_H$ is a non-zero infinite-rank orthogonal projection. ■

Remark 6.2. It may not be obvious why the unboundedness of T in Example 6.1 is necessary. However, it is essential that $W_e(S) = \emptyset$ which is impossible if S is bounded. Indeed, it is easy to check that if we replace S in Example 6.1 with any (bounded or unbounded) self-adjoint operator such that $W(S) \subset [1, \infty)$ and $\lambda_0 \in W_e(S)$, then claim (iv) continues to hold, but it is no longer a contradiction to the numerical range lifting problem because $\mathrm{i} + \lambda_0 \in W_e(T)$ and hence $W_e(T) \not\subset U_\varepsilon$ for $\varepsilon < 1$. E.g. if $\emptyset \neq W_e(S) = [\lambda_0, \lambda_1] \subset \overline{W(S)} \subset [1, \infty)$, then $W_e(T)$ is the triangle with vertices 1, $\lambda_0 + \mathrm{i}$ and $\lambda_1 + \mathrm{i}$.

7. Essential norm of C_0-semigroups and essential growth bounds

In this last section we derive estimates of the essential norm of C_0-semigroups with generator $-T$ for which T is quasi-m-accretive and we obtain a criterion for a C_0-semigroup to be quasi-compact.

Recall that if T is quasi-m-accretive, i.e. its numerical range lies in some left half-plane and T is m-accretive, then $-T$ generates a C_0-semigroup $(\tau(t))_{t\geq 0}$. A simple application of the Lumer–Philipps theorem, see [12, Theorem II.3.15], shows that

$$\|\tau(t)\| \leq \mathrm{e}^{-t\beta(T)}, \quad t > 0, \tag{7.1}$$

where $\beta(T)$ is given by

$$\beta(T) := \inf \operatorname{Re} W(T).$$

Furthermore, a C_0-semigroup $(\tau(t))_{t\geq 0}$ is called *quasi-compact*, see e.g. [12, Definition V.3.4], if

$$\lim_{t\to\infty} \|\tau(t)\|_{\mathrm{e}} = 0. \tag{7.2}$$

In order to estimate the essential norm of $(\tau(t))_{t\geq 0}$, we introduce the essential analogue of the quantity $\beta(T)$ by setting, for an arbitrary linear operator T,

$$\beta_{\mathrm{e}}(T) := \inf_{(x_n)_{n\in\mathbb{N}}\in\mathcal{E}(T)} \liminf_{n\to\infty} \operatorname{Re}(Tx_n, x_n), \tag{7.3}$$

where $\mathcal{E}(T)$ is the set of all normalised weakly null-sequences, see Proposition 3.3.

Clearly,

$$\beta(T) \leq \beta_{\mathrm{e}}(T) \leq \inf \operatorname{Re} W_{\mathrm{e}}(T) \tag{7.4}$$

and both inequalities may be strict. Especially for the second inequality in (7.4), we give an example for strict inequality, see Example 7.4 below, and we establish criteria for equality, see Proposition 7.3.

Theorem 7.1. *Let T be a quasi-m-accretive linear operator in H and let $(\tau(t))_{t\geq 0}$ be the associated C_0-semigroup generated by $-T$. Then*

$$\|\tau(t)\|_{\mathrm{e}} \leq \mathrm{e}^{-t\beta_{\mathrm{e}}(T)}, \quad t > 0; \tag{7.5}$$

in particular, if $\beta_{\mathrm{e}}(T) > 0$, then $(\tau(t))_{t\geq 0}$ is quasi-compact.

Proof. Let $t_0 > 0$ be arbitrary. If $\|\tau(t_0)\|_{\mathrm{e}} = 0$, then there is nothing to show. If $\|\tau(t_0)\|_{\mathrm{e}} > 0$, then, because $\mathcal{D}(T)$ is dense and thus a core for the bounded operator $\tau(t_0)$, it is easy to see that, by (3.6) and a standard diagonal sequence argument, there exists a normalised weakly null-sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T)$ with $\|\tau(t_0)x_n\| \to \|\tau(t_0)\|_{\mathrm{e}}$ as $n \to \infty$. Since $-T$ is the generator of $(\tau(t))_{t\geq 0}$, the function $y_n(t) := \tau(t)x_n$ is the unique classical solution of the abstract Cauchy problem

$$\begin{cases} y_n'(t) = -Ty_n(t), & t > 0, \\ y_n(0) = x_n, \end{cases} \tag{7.6}$$

see e.g. [12, Proposition II.6.2], i.e. the function y_n is differentiable on $(0,\infty)$ and satisfies (7.6). Next we show that $\liminf_{n\to\infty} \|y_n(t)\| > 0$, $t \in [0, t_0]$; otherwise, if we had $\liminf_{n\to\infty} \|y_n(t)\| = 0$ for some $t \in (0, t_0)$, then also

$$\begin{aligned} \|\tau(t_0)\|_{\mathrm{e}} = \liminf_{n\to\infty} \|\tau(t_0)x_n\| &= \liminf_{n\to\infty} \|\tau(t_0 - t)\tau(t)x_n\| \\ &= \liminf_{n\to\infty} \|\tau(t_0 - t)y_n(t)\| = 0, \end{aligned}$$

a contradiction to $\|\tau(t_0)\|_{\mathrm{e}} > 0$. Thus, upon choosing subsequences, the sequences

$$(v_n(t))_{n\in\mathbb{N}} := \left(\frac{y_n(t)}{\|y_n(t)\|}\right)_{n\in\mathbb{N}} \subset \mathcal{D}(T), \quad t \in [0, t_0],$$

are normalised weakly null-sequences. Moreover, by (7.6) we have, for $t \in [0, t_0]$ and $n \in \mathbb{N}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|y_n(t)\|^2 = 2\,\mathrm{Re}(y_n'(t), y_n(t)) = -\frac{2\,\mathrm{Re}(Ty_n(t), y_n(t))}{\|y_n(t)\|^2}\|y_n(t)\|^2,$$

and, therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln\|y_n(t)\| = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\|y_n(t)\|}{\|y_n(t)\|}\frac{\|y_n(t)\|}{\|y_n(t)\|} = \frac{\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|y_n(t)\|^2}{\|y_n(t)\|^2} = -\frac{\mathrm{Re}(Ty_n(t), y_n(t))}{\|y_n(t)\|^2}.$$

Hence, noting that $\|x_n\| = \|y_n(0)\| = 1$, we can write

$$\|\tau(t_0)x_n\| = \|y_n(t_0)\| = \exp\left(-\int_0^{t_0} \frac{\mathrm{Re}(Ty_n(t), y_n(t))}{\|y_n(t)\|^2}\mathrm{d}t\right), \quad n \in \mathbb{N}. \tag{7.7}$$

Now we set $f(t) := \liminf_{n\to\infty} f_n(t) \in [\beta_{\mathrm{e}}(T), \infty]$, $t \in [0, t_0]$, where

$$f_n(t) := \frac{\mathrm{Re}(Ty_n(t), y_n(t))}{\|y_n(t)\|^2} \in [\beta(T), \infty), \quad t \in [0, t_0].$$

Then, by (7.7) and Fatou's lemma,

$$\begin{aligned} \|\tau(t_0)\|_{\mathrm{e}} &= \lim_{n\to\infty} \|\tau(t_0)x_n\| = \limsup_{n\to\infty} \|\tau(t_0)x_n\| \\ &= \exp\left(-\liminf_{n\to\infty}\int_0^{t_0} f_n(t)\mathrm{d}t\right) \leq \exp\left(-\int_0^{t_0} f(t)\mathrm{d}t\right) \leq \exp(-t_0\beta_{\mathrm{e}}(T)), \end{aligned}$$

which proves (7.5) since $t_0 > 0$ was arbitrary. If $\beta_e(T) > 0$, then (7.5) immediately yields that $\lim_{t\to\infty} \|\tau(t)\|_e = 0$, i.e. $(\tau(t))_{t\geq 0}$ is quasi-compact, see (7.2). ∎

Theorem 7.1 gives some information on the so-called *growth bound* $\omega(-T)$ and *essential growth bound* $\omega_e(-T)$ of the C_0-semigroup $(\tau(t))_{t\geq 0}$ with generator $-T$ which may be defined as

$$\omega(-T) := \inf_{t>0} \frac{1}{t} \log \|\tau(t)\|, \quad \omega_e(-T) := \inf_{t>0} \frac{1}{t} \log \|\tau(t)\|_e,$$

see [12, Definition I.5.6, Proposition IV.2.2, and Definition IV.2.9]; in fact, $\omega(-T)$ is the infimum of all $\omega \in \mathbb{R}$ so that there exists $M_\omega \geq 1$ with $\|\tau(t)\| \leq M_\omega e^{t\omega}$, $t \geq 0$. It is well known that $\omega(-T)$ is related to the so-called *spectral bound* $s(-T)$ of $-T$ defined as

$$s(-T) := \sup_{\lambda\in\sigma(-T)} \operatorname{Re}\lambda = -\inf_{\lambda\in\sigma(T)} \operatorname{Re}(\lambda),$$

see [12, Definition IV.2.1], via the formula

$$\omega(-T) = \max\left\{s(-T), \omega_e(-T)\right\},$$

see [12, Corollary IV.2.11] or [27, Section 2.3]. The next corollary yields a new sufficient condition for the so-called *spectral growth bound condition* $\omega(-T) = s(-T)$.

Corollary 7.2. *Let T be quasi-m-accretive. Then*

$$\omega_e(-T) \leq -\beta_e(T), \quad \omega(-T) \leq \max\{s(-T), -\beta_e(T)\};$$

in particular, if $\beta_e(T) = \infty$, the spectral growth bound condition $\omega(-T) = s(-T)$ holds.

Finally, we give a criterion for equality in $\beta_e(T) \leq \inf \operatorname{Re} W_e(T)$, see (7.4).

Proposition 7.3. *Let T be an arbitrary linear operator with $W_e(T) \neq \emptyset$. Assume that $\overline{W(T)}$ or, equivalently, $W_e(T)$ does not contain any vertical ray. Then*

$$\beta_e(T) = \inf \operatorname{Re} W_e(T); \tag{7.8}$$

in particular, (7.8) holds if T is quasi-sectorial.

Proof. Since $W_e(T) \neq \emptyset$, Lemma 5.4 implies that $\overline{W(T)}$ contains a vertical ray if and only if $W_e(T)$ does. If $\inf \operatorname{Re} W_e(T) = -\infty$, then the claim follows from the inequality $\beta_e(T) \leq \inf \operatorname{Re} W_e(T)$. If T is bounded, the claim follows trivially. So suppose that T is unbounded. If $\inf \operatorname{Re} W_e(T) > -\infty$, then

$$W_e(T) \subset \{z \in \mathbb{C} \mid \operatorname{Re} z \geq \inf \operatorname{Re} W_e(T)\} \neq \mathbb{C}, \tag{7.9}$$

and hence $W_e(T) \neq \mathbb{C}$. By [5, Corollary 2.5] it follows that $W(T) \neq \mathbb{C}$. If $\overline{W(T)}$ is a half-plane, then so is $W_e(T)$ by [5, Corollary 2.5]. Then, due to (7.9) and since $\overline{W(T)} \supset W_e(T)$, we obtain

$$\overline{W(T)} = \{z \in \mathbb{C} \mid \operatorname{Re} z \geq \beta(T)\},$$

a contradiction to the assumption that $\overline{W(T)}$ does not contain any vertical ray. If $\overline{W(T)}$ is a strip or a line, then so is $W_e(T)$ by [5, Corollary 2.5]. Thus (7.9) and $\overline{W(T)} \supset W_e(T)$ yield that

$$\overline{W(T)} = \{z \in \mathbb{C} \mid \beta(T) \leq \operatorname{Re} z \leq \sup \operatorname{Re} W(T)\},$$

again a contradiction to the assumption that $\overline{W(T)}$ does not contain any vertical ray. Altogether, $\overline{W(T)}$ is neither $\mathbb{C}$, nor a half-plane, nor a strip or a line. Because $\overline{W(T)}$ is convex, and hence the intersection of all its supporting half-planes, it must be contained in a sector. Moreover, since $\overline{W(T)}$ does not contain any vertical rays, this sector can be chosen to be of the form

$$\overline{W(T)} \subset \{z \in \mathbb{C} \mid |\arg(z-\beta)| \leq \theta\} \quad \text{or} \quad \overline{W(T)} \subset \{z \in \mathbb{C} \mid |\arg(-z+\beta)| \leq \theta\}$$

for some $\beta \in \mathbb{R}$ and $\theta \in \left[0, \frac{\pi}{2}\right)$. In the second case, (7.9) yields that $W_e(T)$ is bounded. However, since $W_e(T) \neq \emptyset$ by assumption and T is unbounded, Corollary 5.6 yields that $W_e(T)$ must be unbounded, a contradiction. Hence we have $\overline{W(T)} \subset \{z \in \mathbb{C} \mid |\arg(z-\beta)| \leq \theta\}$.

By definition of $\beta_e(T)$ in (7.3) and a standard diagonal sequence argument, there is a normalised weakly null-sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T)$ such that $\operatorname{Re}((Tx_n, x_n)) \searrow \beta_e(T)$, $n \to \infty$. Since $\overline{W(T)} \subset \{z \in \mathbb{C} \mid |\arg(z-\beta)| \leq \theta\}$, it follows that the sequence $(|\operatorname{Im}(Tx_n, x_n)|)_{n\in\mathbb{N}}$ is bounded. Hence, by passing to a subsequence $(x_{n_k})_{k\in\mathbb{N}}$, we have $\lim_{k\to\infty}(Tx_{n_k}, x_{nk}) \in W_e(T)$ and hence $\beta_e(T) = \inf \operatorname{Re} W_e(T)$. ■

The next example shows that Theorem 7.1 is sharp. In fact, if $W(T)$ or, equivalently, $W_e(T)$ contains vertical rays, then the strict inequality $\beta_e(T) < \inf \operatorname{Re} W_e(T)$ may hold and the semigroup decay (7.5) need not hold with $\beta_e(T)$ replaced by the quantity $\inf \operatorname{Re} W_e(T)$.

Example 7.4. Let I_H denote the identity operator and let S be a self-adjoint semibounded operator in a Hilbert space H with

$$\overline{W(S)} = [1, \infty), \quad W_e(S) = \emptyset.$$

e.g., we can choose $H = \ell_2(\mathbb{N})$ and $Se_n = ne_n$, $n \in \mathbb{N}$,

$$\mathcal{D}(S) = \{x \in \ell_2(\mathbb{N}) \mid Sx \in \ell_2(\mathbb{N})\}.$$

Then the operator $T := I_H \oplus \mathrm{i}(S - I_H)$ in $H^2 = H \oplus H$, i.e.

$$T = \begin{pmatrix} I_H & 0 \\ 0 & \mathrm{i}(S - I_H) \end{pmatrix}, \quad \mathcal{D}(T) := H \oplus \mathcal{D}(S) \subset H^2,$$

is densely defined and closed. In the following, we prove:

(i) T is normal;

(ii) $\overline{W(T)} = [0, 1] + \mathrm{i}[0, \infty]$ and T is m-accretive;

(iii) $W_{\mathrm{e}}(T) = 1 + \mathrm{i}[0, \infty)$ and hence $\inf \operatorname{Re} W_{\mathrm{e}}(T) = 1$;

(iv) $\beta(T) = \beta_{\mathrm{e}}(T) = 0$;

(v) the C_0-semigroup $(\tau(t))_{t\geq 0}$ generated by $-T$ is not quasi-compact, more precisely, $\|\tau(t)\|_{\mathrm{e}} = \|\tau(t)\| = 1 = \mathrm{e}^{-t\beta_{\mathrm{e}}(T)}$, $t \geq 0$, i.e. the decay in (7.5) is sharp.

Proof. (i) and (ii) follow similarly to claims (i) and (ii) in Example 6.1, respectively.

(iii) Clearly, $1 \in W_{\mathrm{e}}(I_H) \subset W_{\mathrm{e}}(T)$. Hence, by claim (ii) and Lemma 5.4, it follows that $1 + \mathrm{i}[0, \infty) \subset W_{\mathrm{e}}(T)$. To prove the reverse inclusion, assume that $\lambda = a + ib \in W_{\mathrm{e}}(T) (\subset \overline{W(T)})$. Then $a \in [0, 1]$ by (ii) and there exists a normalised weakly null-sequence $(v_n)_{n\in\mathbb{N}} = ((x_n, y_n)^t)_{n\in\mathbb{N}} \subset \mathcal{D}(T)$ with

$$(Tv_n, v_n) = \|x_n\|^2 + \mathrm{i}(Sy_n, y_n) - \mathrm{i}\|y_n\|^2 \to \lambda = a + \mathrm{i}b, \quad n \to \infty. \tag{7.10}$$

If $\liminf_{n\to\infty} \|y_n\| > 0$, then $\left(\frac{y_n}{\|y_n\|}\right)_{n\in\mathbb{N}} \subset \mathcal{D}(S)$ is a normalised weakly null-sequence. Since $W_{\mathrm{e}}(S) = \emptyset$ and S is semi-bounded, we obtain that $(Sy_n, y_n) \to \infty$ as $n \to \infty$, a contradiction to (7.10) because $\|y_n\| \leq 1$, $n \in \mathbb{N}$. Hence, by passing to a subsequence, we can assume without loss of generality that $y_n \to 0$ as $n \to \infty$. Since $(v_n)_{n\in\mathbb{N}}$ is normalised, this implies that $\|x_n\| \to 1$ as $n \to \infty$. Inserting this into (7.10) yields $a = 1$, which proves the claim.

(iv) By claim (ii), we obtain $0 = \beta(T) \leq \beta_{\mathrm{e}}(T)$. Let $(y_n)_{n\in\mathbb{N}} \subset \mathcal{D}(S)$ be a normalised weakly null-sequence and define $v_n := (0, y_n)^t$, $n \in \mathbb{N}$. Then $(v_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T)$ is a normalised weakly null-sequence as well and $\operatorname{Re}(Tv_n, v_n) = 0$, $n \in \mathbb{N}$. Now, by the definition of $\beta_{\mathrm{e}}(T)$ in (7.3) and since $\beta_{\mathrm{e}}(T) \geq 0$, it follows that $\beta_{\mathrm{e}}(T) = 0$.

(v) It is easy to see that the C_0-semigroup generated by $-T$ is given by

$$\tau(t)\begin{pmatrix} x \\ y \end{pmatrix} = \left(\begin{pmatrix} (\mathrm{e}^{-t} x_n)_{n\in\mathbb{N}} \\ (U(t)y_n)_{n\in\mathbb{N}} \end{pmatrix}\right) \quad x = (x_n)_{n\in\mathbb{N}}, y = (y_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}),\ t \geq 0,$$

where $U(\cdot)$ is the C_0-semigroup in the Hilbert space H generated by $-\mathrm{i}(S - I_H)$. Since $S - I_H$ is self-adjoint, $U(\cdot)$ extends to a C_0-group of unitary operators by Stone's theorem, see e.g. [12, Theorem 3.24]. If $(e_n)_{n\in\mathbb{N}} \subset H$ is any orthonormal system, then the sequence $(y_n)_{n\in\mathbb{N}} := (U(-t)e_n)_{n\in\mathbb{N}}$, is an orthonormal system as well because $U(-t)$ is unitary, thus, in particular, it is a normalised weakly null-sequence. Applying (3.6), we readily see that

$$\|\tau(t)\|_{\mathrm{e}} \geq \limsup_{n\to\infty} \left\|\tau(t)\begin{pmatrix}0\\ y_n\end{pmatrix}\right\| = \limsup_{n\to\infty} \left\|\begin{pmatrix}0\\ U(t)U(-t)e_n\end{pmatrix}\right\| = 1, \quad t \geq 0. \quad (7.11)$$

Since $\beta(T) = 0$, we have $\|\tau(t)\| \leq \mathrm{e}^{-t\beta(T)} = 1$ by (7.1) and hence, together with (7.11), $\|\tau(t)\| = \|\tau(t)\|_{\mathrm{e}} = 1$ for $t \geq 0$. ∎

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