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Existence and classification of maximal growth distributions

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Abstract. This article tackles the problem of the existence and classification of maximal growth distributions on smooth manifolds. We show that maximal growth distributions of rank > 2 abide by a full h-principle in all dimensions. We make use of M. Gromov's higher order convex integration and, on the way, we establish a new criterion for checking ampleness of a differential relation.

As a consequence, we answer in the positive, for $k > 2$, the long-standing open question posed by M. Kazarian and B. Shapiro more than 25 years ago about whether any parallelizable manifold admits a k-rank distribution of maximal growth. We also answer several related open questions.

For completeness, we show that the differential relation of maximal growth for rank-2 distributions is not ample in any ambient dimension. Non-ampleness of the Engel and the $(2, 3, 5)$-conditions follow as particular cases.

1. Introduction

This article tackles the long-standing open problem of existence and classification of maximal growth distributions on smooth manifolds. This question has garnered much interest in the mathematical community in recent years.

Every generic distribution germ has maximal growth [2]; i.e., *locally*, Lie brackets of vector fields tangent to generic distributions generate new directions as fast as possible. This condition implies, in particular, that such vector fields eventually generate the whole tangent bundle by Lie bracket operation.

This situation drastically changes when one passes to the *global* setting. Indeed, there are global obstructions to the existence of distributions of maximal growth. The existence of orientable Engel distributions (maximal growth rank-2 distributions on 4-manifolds) requires, for example, parallelizability of the ambient manifold [17]. A breakthrough in this direction was the paper [32] by T. Vogel, where he showed that parallelizability was also a sufficient condition. Later on, there has been further development regarding the classification problem of Engel structures (see [5, 6, 8, 9, 11]).

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Some other remarkable and well studied distributions are contact and even-contact structures. These are maximally non-integrable hyperplane fields, the former in odd-dimensional manifolds, whereas the latter in even-dimensional manifolds. D. McDuff has shown in [23] that even-contact structures abide by a full h-principle, providing thus a classification result from a homotopic viewpoint. This result follows from Gromov's convex integration and contrasts with the case of contact structures.

In general, contact structures do not abide by a complete h-principle, unless one restricts to particular subclasses called *overtwisted*. M. S. Borman, Y. Eliashberg and E. Murphy showed in [4] that there is a one-to-one correspondence between formal classes of contact structures and overtwisted classes with a fixed overtwisted disk. The existence of tight contact distributions (non-overtwisted) thus shows that an h-principle for contact structures cannot hold in general. See [3] for their existence in dimension 3. As for the higher dimensional case, see, for instance, K. Niederkrüger's work [24], which implies tightness of semi-positive symplectically fillable contact structures. See also the work [21] by P. Massot, K. Niederkrüger and C. Wendl for the existence and tightness of weakly fillable contact structures in higher dimensions.

The existence and classification problem for maximal growth distributions on open manifolds follows from Gromov's *h-principle for open manifolds*, see [14], p. 79. See also Theorems 8.3.3 and 8.3.5 in [7]. Nonetheless, not so much was known for the case of closed manifolds and several open questions have been posed in the literature throughout the years about this problem.

In [29], M. Kazarian and B. Shapiro described some topological obstructions for maximal growth distributions to exist in general. Regarding these results, and very much in the spirit of the h-principle philosophy, they comment:

> *"The basic problem related to the above topological obstructions is to what extent vanishing of these obstructions guarantees the existence of a maximal growth subbundle. The result of T. Vogel on the existence of Engel structures on parallelizable 4-manifolds brings a certain amount of optimism about this problem".*

Then they pose the following open question[1].

Question 1.1 (Kazarian and Shapiro (1996), [29]). *Does every closed parallelizable m-dimensional manifold admit a maximal growth distribution of rank $1 < n < m$?*

They state the following related problem as well.

Question 1.2 (Kazarian and Shapiro (1996), [29]). *When does a manifold admit a distribution whose associated flags have constant (and maximal-possible) ranks throughout the manifold?*

We will address both questions in this article. Our main theorem reads as follows.

Theorem 1.3. *Let M be a smooth manifold. The complete C^0-close h-principle holds for maximal growth k-distributions on M if $k > 2$.*

[1] The article [29] first appeared in 1996 as a preprint in arXiv, alg-geom/9611016v1, including Question 1.1.

Proposition 5.9 guarantees the existence of the formal underlying objects associated to maximal growth distributions on parallelizable manifolds (see Section 5 for further details). Therefore, combined with Theorem 1.3, we get the following *existence* result.

Theorem 1.4. *Every parallelizable n-dimensional smooth manifold M admits a k-distribution of maximal growth if $2 < k \leq n$.*

Theorem 1.4 gives a positive answer to Question 1.1 by M. Kazarian and B. Shapiro for all values $2 < n < m$, and Theorem 1.3 goes further by establishing a whole classification in terms of the formal data.

Theorem 1.3 also implies Corollary 1.5 below, which provides an answer for $k > 2$ to Question 1.2 in terms of an algebro-topological condition; i.e., in terms of existence of formal distributions.

Corollary 1.5. *A smooth manifold admits a rank > 2 maximal growth distribution if and only if it admits a formal rank > 2 distribution of maximal growth.*

The following question, which we also tackle, was posed in 2019 by Á. del Pino in [10] regarding classification results of distributions in terms of the underlying topological/algebraic data.

Question 1.6 (Del Pino, [10]). *Given a class of distributions whose nilpotentisation is fiberwise isomorphic to some generic graded Lie algebra $\mathfrak{g}$ (or a generic family of them), can we tell whether some classic h-principle technique (say, convex integration) applies to provide a complete classification result, purely in terms of $\mathfrak{g}$?*

In this paper, we deal with maximal growth distributions of rank-k on smooth n-dimensional manifolds M. We work within the framework of regular distributions (see Remark 3.5); the rank-k is assumed to be constant and independent of the choice of point. We also denote by r the step of $\mathcal{D}$ (see Definition 3.1). Specifically, Theorem 1.3 gives a positive answer to this question for $k > 2$ and formal nilpotentisations of maximal growth (see Definition 5.5). For the case $k = 2$, we provide a negative answer to the applicability of convex integration by showing that the associated differential relation, which we denote by $\mathcal{R}^{\text{step-}r} \subset J^r(\mathrm{Gr}_2(TM))$, fails to be ample, as the following theorem shows.

Theorem 1.7. *Let M be a smooth manifold of dimension n. The differential relation of maximal growth for rank-2 distributions $\mathcal{R}^{\text{step-}r} \subset J^r(\mathrm{Gr}_2(TM))$ is not ample in principal directions for any $n \geq 3$.*

Note that non-ampleness of the Engel and the $(2, 3, 5)$-conditions follow from Theorem 1.7 as particular cases. Thus, some other techniques may be necessary in order to establish flexibility results for rank-2 distributions. Flexibility for the Engel case was explored by Á. del Pino and T. Vogel in [11], where they showed that, analogously to the contact case, there exist overtwisted Engel classes abiding by a complete h-principle. See also [5].

Theorem 1.4 also answers, for $k > 2$, an open question raised during the workshop "Engel structures" (see [12]) held in April 2017 at AIM (American Institute of Mathematics, San Jose, California).

Question 1.8 (AIM Problem List (2017), Problem 6.2 in [1]). *Are there examples of pairs (n, k) with $n > k \geq 2$ such that for any parallelizable n-manifold, there exists a k-plane field $\mathcal{D} \subset TM$ with maximal growth vector?*

Additionally, the following generalisation of the previous question was posed as well.

Question 1.9 (AIM Problem List (2017), Remark to Problem 6.2 in [1]). *More generally, instead of parallelizability, one would like to assume only that one is dealing with manifolds admitting appropriate partial flags.*

Question 1.8 was further refined in [20], in line with the remark above, to ask whether any formal distribution can be homotoped inside the space of formal distributions to an actual maximal growth distribution.

Question 1.10 (Del Pino and Martínez-Aguinaga, [20]). *Does any formal distribution of maximal growth admit a holonomic representative up to homotopy?*

In [20], Á. del Pino and the author showed that the result holds for step $r = 2$, rank > 2, letting the general case $r, k > 2$ open. This article now answers in the positive, for $k > 2$, this question by Theorem 1.3 as well as Question 1.8, Question 1.9 and Question 1.10.

Note that this is the optimal range for the rank of the distribution $\mathcal{D}$ where one could expect a general h-principle statement to hold, since the contact $(2, 3)$-case is well known not to abide by an h-principle [3].

The approach in this article is essentially different from the one in [20]. There the problem was tackled from the point of view of differential forms, whereas here we deal with the description of distributions in terms of frames. This approach allows to apply the general higher-order version of convex integration and thus solve the general higher order case. Along the way, we establish a new criterion for checking ampleness of a differential relation (Section 6). This constitutes a result of independent interest within the general theory of convex integration.

It is worth noting that there has been some serious development of foundational nature regarding convex integration in recent years ([20, 22, 30, 31]). P. Massot and M. Theillière showed in [22] that convex integration implies the holonomic approximation theorem at the level of order-1 jets, providing further evidence of the broad scope of this technique. See also [13] for a recent application of convex integration to the holomorphic setting, where local h-principles are shown to hold for complex Engel and complex even-contact structures.

Let us fix some notations for the rest of the paper.

- We will use $\langle v_1, \ldots, v_n \rangle$ to denote the linear space spanned by vectors $v_1, \ldots, v_n$. Analogously, for a given set of vectors A, $\langle A \rangle$ will denote the linear space spanned by vectors in A.
- Following Section 1.3 of [7] and [14], we will use $\mathcal{O}p(p)$ to denote an arbitrarily small but not explicitly specified open neighborhood of a point p in a smooth manifold M.

2. Convex integration

The theory of convex integration will be key in the proof of the main theorem in this article. We will show that the maximal growth condition defined for distributions of rank greater than 2 yields a differential relation that falls within the scope of this theory. More specifically, we will show that this relation is ample. The goal of this section is to briefly recall some notation and notions needed to apply this theory.

2.1. Jet spaces: local description

For a smooth fiber bundle $X \to M$, we denote by $J^r(X)$ the corresponding space of r-jets. For $s < r$, we write $\rho^r_s : J^r(X) \to J^s(X)$ for the corresponding projections, as well as $\rho^r_{bs} : J^r(X) \to M$ for the projections onto the base. Given $F \in J^r(X)$, we often use the notation $j^i(F)$ for denoting $\rho^r_i(F) \in J^i(X)$ if $i \leq r$.

Working in a local chart $\mathcal{U}$ of X, we can locally identify M with $\mathbb{R}^n$ and X with $\mathbb{R}^n \times \mathbb{R}^m$, where the fibers of the fibration identify with the $\mathbb{R}^m$ factor. By doing so, we have the following local description of the corresponding jet spaces:

$$\begin{aligned} J^r(X) \supset J^r(\mathcal{U}) &\simeq J^r(\mathbb{R}^n \times \mathbb{R}^m) \\ &\simeq \mathbb{R}^n \times \mathbb{R}^m \times \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^m) \times \operatorname{Sym}^2(\mathbb{R}^n, \mathbb{R}^m) \times \cdots \times \operatorname{Sym}^r(\mathbb{R}^n, \mathbb{R}^m); \end{aligned}$$

here $\operatorname{Sym}^d(\mathbb{R}^n, \mathbb{R}^m)$ denotes the space of symmetric homogenous polynomials of degree d with entries in $\mathbb{R}^n$ and taking values in $\mathbb{R}^m$. Note that this identification works by identifying each jet $F \in J^r(\mathbb{R}^n \times \mathbb{R}^m)$ with its corresponding r-th order Taylor polynomial over the point $\rho^r_{bs}(F) \in \mathbb{R}^n$.

It turns out that ρ^1_0 is a vector bundle, whereas ρ^r_{r-1} is in general an affine fibration that can be understood as the map that assigns to each order-r Taylor polynomial its truncated order-$(r-1)$ part.

2.2. Principal subspaces

The following notion formalises the idea of two r-jets that agree except along a pure derivative of order r.

Definition 2.1. Given a hyperplane $\eta \subset T_pM$ and two sections $f, g : M \to X$ with $f(p) = g(p)$, we say that f and g have the same $\perp(\eta, r)$-jet at $p \in M$ if

$$D_p|_\eta j^{r-1} f = D_p|_\eta j^{r-1} g,$$

where $D_p|_\eta$ means taking the differential at p and restricting it to η.

When η is a hyperplane field, the $\perp(\eta, r)$-jets form a bundle, which we denote by $J^{\perp(\eta,r)}(X)$. There are affine fibrations

$$\rho^r_{\perp(\eta,r)} : J^r(X) \to J^{\perp(\eta,r)}(X) \quad \text{and} \quad \rho^{\perp(\eta,r)}_{r-1} : J^{\perp(\eta,r)}(X) \to J^{r-1}(X).$$

Given a section $f : M \to X$, we write

$$j^{\perp(\eta,r)} f : M \to J^{\perp(\eta,r)}(X)$$

for the corresponding section of $\perp(\eta, r)$-jets. A section of this form is called *holonomic*.

Definition 2.2. The fibers of $\rho^r_{\perp(\eta,r)}$ are said to be the *principal subspaces* associated to η (and r). They are all fiberwise affine subspaces parallel to one another. Given $F \in J^r(X)$, we write

$$\mathrm{Pr}^r_{\eta,F} := (\rho^r_{\perp(\eta,F)})^{-1}(\rho^r_{\perp(\eta,r)}(F))$$

for the principal subspace that contains it.

Remark 2.3. For the most common used flavors of convex integration (ampleness in principal coordinate directions, see 1.1.4 in [20], and ampleness in principal directions, see Definition 2.9), all the hyperplane fields $\eta \in TM$ we work with integrate to a codimension-1 foliation.

Then, when one passes to a local chart (and, thus, a choice of coordinates has been made), instead of talking about hyperplane fields, we will often identify the hyperplanes with the direction they define by duality (given by the Euclidean metric of $\mathbb{R}^n$ in the chart). We then write $\perp(\partial_i, r) := \perp(\ker(dx_i), r)$, where in local coordinates ∂_i is the dual vector field of dx_i.

Working in local coordinates, and assuming $M = \mathbb{R}^n$ and $\eta := \ker(dt)$, let $\mathbb{R}^n$ split as a product $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$ with coordinate t for $\mathbb{R}$ and coordinates $x_2, \dots, x_n$ for $\mathbb{R}^n$. The r-th order jet $j^r(f)$ of a section $f: \mathbb{R}^n \to X$ is determined by all the partial derivatives

$$\{\partial^\alpha_X \partial^\beta_t f\}, \quad \alpha = (\alpha_2, \dots, \alpha_n), \quad |\alpha| = \alpha_2 + \cdots + \alpha_n \leq r - \beta.$$

Here $\partial^\alpha_X f$ denotes the partial derivative of f with respect to $x_2, \dots, x_n$ with orders $\alpha = (\alpha_2, \dots, \alpha_n)$ for each direction, respectively. Analogously, $\partial^\beta_t f := \partial^\beta f / \partial t^\beta$.

Note that the space $J^{\perp(\eta,r)}$ corresponds to the set of equivalence classes of partial derivatives $\{\partial^\alpha_X \partial^\beta_t f\}$ for which $\beta \leq r - 1$ (abusing notation and referring here to the 0-jet part as an "order-0 derivative"). Then the splitting of $\mathbb{R}^n$ induces an splitting of $j^r(f)$ as follows:

$$j^r(f) = j^{\perp(\eta,r)} f \oplus \partial^r_t f,$$

which can be understood as a splitting in "all mixed partial derivatives, except the pure derivatives of order r in the direction of t" (including pure derivatives of order r or less in some other directions as well) and "pure derivatives of order r in the direction of t".

This same splitting is thus inherited in the space of formal solutions as well:

$$F = j^{\perp(\eta,r)}(F) \oplus j^r_t(F), \tag{2.1}$$

where the component $j^r_t(F)$ represents *the order-r pure formal derivative of F with respect to ∂_t*. Analogously, $j^{\perp(\eta,r)}(F)$ denotes the component corresponding to the rest of mixed formal partial derivatives.

Remark 2.4. This splitting depends on the splitting of $\mathbb{R}^n = \mathbb{R}_t \times \mathbb{R}^{n-1}$ but not on the coordinates chosen within $\mathbb{R}^{n-1}$.

Given a differential relation $\mathcal{R} \subset J^r(X)$ of order r, the projection of $\mathcal{R}$,

$$\rho^r_i(\mathcal{R}) \subset J^i(X),$$

is a differential relation of order $i < r$. If $\mathcal{R}$ is open, then $\rho^r_i(\mathcal{R})$ also is since the maps ρ^r_i are submersive and thus open.

Given $F \in J^r(X)$, we write

$$\mathcal{R}^r_{\eta,F} := \mathcal{R} \cap \Pr^r_{\eta,F}$$

for the restriction of $\mathcal{R}$ to the principal subspace $\Pr^r_{\eta,F}$ containing F.

Remark 2.5. Sometimes we are interested in considering the analogous objects for the projection of a given relation $\mathcal{R}$ and point F to the space of lower order jets; i.e., for $\rho^r_i(\mathcal{R})$ over $\rho^r_i(F)$, where $i < r$. So, for the ease of notation and whenever it is clear from the context, we will still denote $\mathcal{R}$ for $\rho^r_i(\mathcal{R})$ and we identify F with $\rho^r_i(F)$. This way, we will often use the following notation:

$$\Pr^i_{\eta,F} := \Pr^i_{\eta,\rho^r_i(F)}, \quad \mathcal{R}^i_{\eta,F} := \left(\rho^r_i(\mathcal{R})\right)^i_{\eta,\rho^r_i(F)}.$$

Similarly, using the notation $j^i(F)$ for denoting $\rho^r_i(F)$ (beginning of Subsection 2.1), there are well-defined decompositions

$$j^i(F) = j^{\perp(\eta,i)}(F) \oplus j^i_t(F). \tag{2.2}$$

2.2.1. Principal subspaces in coordinates. It may be useful to have a description of principal subspaces in coordinates. Recall that the principal subspaces $\Pr^s_{\partial_t,F}$ correspond to the space of all s-jets that differ with F only on the formal pure order-s partial derivatives $j^s_t(F) = \left(j^s_t(F_i)\right)^k_{i=1}$ with respect to ∂_t (where the usual identification of F with $\rho^r_s(F)$ has been made, recall Remark 2.5).

Then, a principal subspace $\Pr^s_{\partial_t,F}$ can be expressed as

$$\Pr^s_{\partial_t,F} = \left\{((j^{\perp(\partial_t,s)}(F_i),0) + (0,w_i))^k_{i=1} : w_i \in \mathbb{R}^n\right\}, \tag{2.3}$$

where the free $w_i \in \mathbb{R}^n$ parametrize all possible order-s pure formal derivatives of F_i with respect to ∂_t.

2.3. Ampleness in principal directions

Ampleness of a differential relation is a key notion within the theory of convex integration. We define ampleness for subsets of affine spaces first. We later adapt this notion to differential relations in jet spaces.

Definition 2.6. Let X be an affine space and $Y \subset X$ a subset. Given $y \in Y$, we write Y_y for the path-connected component containing it. We say that Y is *ample* if the convex hull $\mathrm{Conv}(Y,y) := \mathrm{Conv}(Y_y)$ of each $Y_y \subset Y$ is the whole of X. We say that ampleness holds trivially if Y_y equals the empty set or the total space.

Let us provide an example of an ample set which we will state as a lemma. Furthermore, this result will be key in the proof of Theorem 7.26. For completeness, we reproduce the proof from [20] with minor changes.

Lemma 2.7. *The space* $\mathrm{GL}(n)$ *of non-singular matrices of order* $n \times n$ *is ample inside the space* $\mathcal{M}_{n\times n}$ *of order* $n \times n$*-matrices if* $n \geq 2$ *and non-ample otherwise.*

Proof. The space $\mathrm{GL}(n)$ has two connected components: the space of positive determinant $n \times n$-matrices $\mathrm{GL}^+(n)$ and the one of negative determinant $\mathrm{GL}^-(n)$. In order to check that both components are ample inside $\mathcal{M}_{n\times n}$, we proceed as follows. Note that any $M \in \mathcal{M}_{n\times n}$ can be decomposed as a convex combination of non-singular matrices. Indeed, for $\mu \notin \mathrm{Spec}(M) \setminus \{0\}$, we have that $M = \frac{1}{2} \cdot (2(M - \mu \cdot \mathrm{Id})) + \frac{1}{2} \cdot (2\mu \cdot \mathrm{Id})$.

So, we just need to check that any matrix $M \in \mathrm{GL}^+(n)$ (alternatively, in $\mathrm{GL}^-(n)$) can be expressed as a convex combination of matrices in $\mathrm{GL}^-(n)$ (alternatively, in $\mathrm{GL}^+(n)$). Writing the matrix by columns $M = (v_1, \dots, v_n)$, we then define, for $\varepsilon > 0$,

$$M_1 = ((2+\varepsilon)\cdot v_1, -\varepsilon \cdot v_2, \dots, v_n) \quad \text{and} \quad M_2 = (-\varepsilon \cdot v_1, (2+\varepsilon)\cdot v_2, \dots, v_n).$$

Note that $M = \frac{1}{2} M_1 + \frac{1}{2} M_2$. Also, both M_1 and M_2 are non-singular and do not belong to the same connected component as M, thus yielding the claim. ■

Example 2.8. A classical example of a non-ample subset of a real affine space $\mathbb{A}$ is the complement of a hyperplane, $\mathbb{A} \setminus H$. Note that this set has two connected components, each of which is a convex set that coincides with its convex hull and, therefore, cannot be ample.

We now introduce the notion of ampleness along principal directions for differential relations. Note that there exist other more general notions of ampleness, see [20].

Definition 2.9. Take a bundle $X \to M$ and a differential relation $\mathcal{R} \subset J^r(X)$. Take a direction $\eta \in T_p M$. We say that $\mathcal{R}$ is *ample along the principal direction* (determined by) η if, for every $F \in$, $\mathcal{R}$ projecting to p, $\mathcal{R}_{\eta,F} \subset \mathrm{Pr}_{\eta,F}$ is ample.

More generally, if the relations $(\rho^r_{r'}(\mathcal{R}))_{r'=1,\dots,r}$ are ample along all non-zero directions η, then we say that $\mathcal{R}$ is *ample in principal directions*.

This notion of ampleness is the most common one and when it is satisfied we sometimes just say that $\mathcal{R}$ is *ample*.

Gromov's convex integration theorem was first proved for first order differential relations in Corollary 1.3.2 of [15], and later on for higher order differential relations in Section 2.4, p. 180, of [14]. Although it can be stated in more generality, we will state the following version which is enough for our purposes:

Theorem 2.10 (Convex integration). *The complete C^0-close h-principle holds for any open relation that is ample in all principal directions.*

The following lemma establishes a general situation where ampleness holds.

Lemma 2.11 (Corollary (E) on p. 173 of [14]). *Let $\Sigma \subset J^r(X)$ be a stratified subset of codimension ≥ 2 such that the intersection of Σ with every principal subspace has codimension ≥ 2 within the principal subspace. Then $J^r(X) \setminus \Sigma$ is an ample differential relation.*

Subsets Σ as in the previous lemma are called *thin singularities* or thin subsets. Instances of such subsets arise, for example, in the proofs of the h-principle for even-contact structures [23] or the h-principle for real and co-real immersions [14].

Remark 2.12. Note that the complement of $\mathrm{GL}(n)$ inside the space $\mathcal{M}_{n\times n}$ is the space of matrices with zero determinant (determined by the equation $\det(A) = 0$). Therefore, the associated singularity has codimension-1 and is, thus, not thin. Lemma 2.7 provides, thus, an example of an ample set not having a thin singularity.

3. Maximal growth distributions

A distribution $\mathcal{D}$ on a smooth manifold M is a subbundle of the tangent bundle TM. By subsequently applying the Lie bracket to sections $\Gamma(\mathcal{D})$ of $\mathcal{D}$, we get the following sequence of modules:

$$\Gamma^1(\mathcal{D}) \subset \Gamma^2(\mathcal{D}) \subset \Gamma^3(\mathcal{D}) \subset \cdots,$$
$$\text{where } \Gamma^1(\mathcal{D}) := \Gamma(\mathcal{D}), \quad \Gamma^{i+1}(\mathcal{D}) := [\Gamma^1(\mathcal{D}), \Gamma^i(\mathcal{D})].$$

It follows that $[\Gamma^i(\mathcal{D}), \Gamma^j(\mathcal{D})] \subset \Gamma^{i+j}(\mathcal{D})$. In the present article we will assume that all our distributions $\mathcal{D}$ are *regular*, i.e., there is a distribution $\mathcal{D}_i$ so that $\Gamma^i(\mathcal{D}) = \Gamma(\mathcal{D}_i)$.

A key observation is that under this assumption the flag

$$\mathcal{D}_1 = \mathcal{D} \subset \mathcal{D}_2 \subset \mathcal{D}_3 \subset \cdots$$

stabilises: i.e., there exists a natural number r such that $\mathcal{D}_i = \mathcal{D}_r$ for all $i \geq r$. By Frobenius' theorem, this is equivalent to $\Gamma^r(\mathcal{D})$ being involutive and therefore $\mathcal{D}_r$ being the tangent bundle of a foliation $\mathcal{F}$ on M. We call the *Lie flag* associated to/produced by $\mathcal{D}$ to the previous sequence.

Definition 3.1 (Bracket-generating distribution of step r). If $\mathcal{D}_r = \mathcal{F}$, we say that $\mathcal{D}$ *bracket-generates* $\mathcal{F}$ and if, moreover, $\mathcal{F} = TM$, then we say that $\mathcal{D}$ is *bracket-generating*. We call the first integer r satisfying $\mathcal{D}_r = TM$ the *step* of the distribution, and we denote it by $\operatorname{step}(\mathcal{D})$.

Definition 3.2 (Growth vector). Let $\mathfrak{n}_i = \dim(\mathcal{D}_i)$. The vector $\nu_{\mathcal{D}} = (\mathfrak{n}_1, \mathfrak{n}_2, \dots, \mathfrak{n}_i, \dots)$ is called the *growth vector* of $\mathcal{D}$.

Since we are working under the assumption that all our distributions are regular; this definition depends solely on $\mathcal{D}$ and not on any particular choice of point. Note that there exist distributions with non-constant growth vector. For instance, consider the Martinet distribution $(\mathbb{R}^3, \mathcal{D} := \ker(dy - z^2 dx))$, which has growth vector $(2, 3)$ everywhere except for the points in the hypersurface $\{z = 0\}$, which have growth vector $(2, 2, 3)$.

Let us discuss how we can establish a partial order in the set of growth vectors.

Definition 3.3 (Partial order on the set of growth vectors, [2]). We say that a distribution $\mathcal{D}_1$ with growth-vector $\nu_{\mathcal{D}_1} = (d_1, d_2, \dots)$ grows faster than a distribution $\mathcal{D}_2$ with growth vector $\nu_{\mathcal{D}_2} = (\tilde{d}_1, \tilde{d}_2, \dots)$ if $d_i \geq \tilde{d}_i$ for all $i \geq 1$ and, also, $d_j > \tilde{d}_j$ for some $j \geq 1$. This defines a partial order on the set of growth vectors.

Since we have defined a partial order we can talk about maximal elements in the space of growth vectors. There is, thus, a well-defined notion of *maximal growth vectors* for regular distributions on M. This gives rise to the following definition.

Definition 3.4 (Maximal growth distribution). We say that a distribution $\mathcal{D}$ on a smooth manifold M is a *maximal growth distribution* if its growth vector is maximal according to the partial order in Definition 3.3.

For each dimension $n \in \mathbb{N}$ and rank $k \in \mathbb{N}$, the entries of a maximal growth vector can be computed explicitly. This is explained in Subsection 3.2.

Remark 3.5. As stated in the Introduction, the maximal growth case is generic at the level of germs and, thus, so is being regular. Nonetheless, this is not true at the global level. Given a generic k-distribution $\mathcal{D}^k \subset TM$ it is expected that M contains a generally non-empty degeneracy locus Σ consisting of all points $p \in M$ for which the growth vector is not maximal. See [29] for a comprehensive discussion about the expected codimensions for such Σ. All our arguments work in the regular case but we do not tackle the case of non-regular distributions. This is left as an open question.

3.1. The nilpotentisation

Definition 3.6 (Nilpotentisation). We define the *nilpotentisation* $\mathcal{L}(\mathcal{D})$ of $\mathcal{D}$ as the graded vector bundle

$$\mathcal{D}_1 \oplus \mathcal{D}_2/\mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_i/\mathcal{D}_{i-1} \oplus \cdots \oplus \mathcal{D}_r/\mathcal{D}_{r-1}.$$

As mentioned earlier, we have that $[\Gamma^i(\mathcal{D}), \Gamma^j(\mathcal{D})] \subset \Gamma^{i+j}(\mathcal{D})$ and also note that the composition of taking the Lie bracket with the projection is C^∞-linear

$$\Gamma^j(\mathcal{D}) \times \Gamma^i(\mathcal{D}) \longrightarrow \Gamma^{i+j}(\mathcal{D}) \longrightarrow \Gamma^{i+j}(\mathcal{D})/\Gamma^{i+j-1}(\mathcal{D}).$$

Therefore, it descends to the bilinear map

$$\Omega_{i,j}(\mathcal{D}) : \mathcal{D}_j/\mathcal{D}_{j-1} \times \mathcal{D}_i/\mathcal{D}_{i-1} \longrightarrow \mathcal{D}_{i+j}/\mathcal{D}_{i+j-1}$$

that is called (i,j)-*curvature*.

All the curvatures together endow $\mathcal{L}(\mathcal{D})$ with a fiberwise Lie bracket compatible with the grading. $\mathcal{L}(\mathcal{D})$ is thus endowed with a step-r stratified Lie algebra structure (see Definition 5.1).

Let us introduce some more notation.

Definition 3.7. For a fixed integer k, define the set of ordered multi-indices $\mathbb{I}_\ell$ of order-ℓ as

$$\mathbb{I}_\ell := \{1, \dots, k\}^\ell.$$

Similarly, define $\mathbb{I}_0$ as the singleton containing the empty index $O = ()$. The set of ordered multi-indices of order $\leq i$ is defined as

$$\mathfrak{I}_i = \bigcup_{\ell=0}^{i} \mathbb{I}_\ell.$$

Consider a k-distribution $\mathcal{D}$ spanned by the (possibly local) vectors fields $X^1, \dots, X^n$: i.e., fix a (possibly local) frame $\mathfrak{Fr} = \{X^1, \dots, X^k\}$. For a given $I = (\ell_j, \dots, \ell_1) \in \mathbb{I}_j$, we denote

$$A_I := [X^{\ell_j}, \dots, [X^{\ell_3}, [X^{\ell_2}, X^{\ell_1}]] \dots].$$

We borrow the idea for such a compact notation from [19]. Note that if the length of I is 1, then the expression A_I denotes a single vector field.

Definition 3.8. For each $i > 1$, we define $\mathfrak{Br}^i$ as the set of brackets of vector fields $X^1, X^1, \dots, X^k$ (possibly with repetitions) in $\mathfrak{Fr}$ of length less than or equal to i:

$$\mathfrak{Br}^i := \{A_I : I \in \mathfrak{I}_i\}.$$

The frame $\mathfrak{Fr}$ itself coincides with the set $\mathfrak{Br}^1$ (where each element of the frame can be understood as a length-1 bracket).

The following proposition states some characterisations of the bracket-generating condition with a fixed growth vector:

Proposition 3.9. *A k-distribution $\mathcal{D}$ on a differentiable manifold M has growth vector $\nu = (\mathfrak{n}_i)_{i=1}^r$, if and only if any, and thus all, of the following equivalent conditions are satisfied*:

(i) *For $i = 1, \dots, r$, the i-th element $\mathcal{D}_i$ of the Lie flag has dimension* $\dim(\mathcal{D}_i) = \mathfrak{n}_i$.

(ii) *All local frames $\mathfrak{Fr}$ of $\mathcal{D}$ satisfy, for $i = 1, \dots, r$,* $\dim(\langle \mathfrak{Br}^i \rangle) = \mathfrak{n}_i$.

(iii) *There exists a local frame $\mathfrak{Fr}$ of $\mathcal{D}$ such that for $i = 1, \dots, r$,* $\dim(\langle \mathfrak{Br}^i \rangle) = \mathfrak{n}_i$.

Proof. Condition (i) is just a rephrasing of the Definition of growth vector. The equivalences of (i) with conditions (ii) and (iii) readily follow from the fact that the rank of a bilinear map does not depend on the choice of basis. ∎

3.2. Free Lie algebras and Hall bases

For fixed k, the step r of a maximal growth k-distribution depends on $n = \dim(M)$. But it is interesting to note that for a fixed step r, the first $r - 1$ entries of the growth vector $\nu_{\mathcal{D}} = (\mathfrak{n}_i)_{i=1}^r$ associated to a maximal growth k-rank distribution $\mathcal{D}$ only depend on the integer k and can be calculated explicitly. The last entry $\mathfrak{n}_r$ equals the dimension of M. Let us introduce some terminology prior to elaborating on this.

Definition 3.10 (Bracket expression, [10]). We say that a string x, depending on the variable x, is a length-1 *bracket expression*. Analogously, the string $[x_0, x_1]$, depending on the variables x_0 and x_1, is a *length-2 bracket expression*. Inductively, we define a *length-n bracket expression* to be a string of the form $[A(x_1, \dots, x_j), B(x_{j+1}, \dots, x_n)]$, where $1 < j < n$ and where A and B are bracket expressions of lengths j and $n - j$, respectively. We denote by $\ell(A)$ the length of A.

Given a set $X = \{x_1, \dots, x_m\}$, we denote by $M(X)$ the set of all possible bracket expressions of elements in X. Additionally, we denote by $M_i(X)$ the set of length-i bracket expressions in $M(X)$.

Remark 3.11. From an abstract algebra point of view, the set $M(X)$ can be understood as the free magma generated by the set X.

Let us introduce the notion of graded Lie algebra.

Definition 3.12. A Lie algebra $\mathfrak{g}$ is called *graded* if it is equipped with a grading compatible with the Lie bracket. It has a decomposition in vector spaces as follows:

$$\mathfrak{g} = \bigoplus_{i=1}^{r} \mathfrak{g}_i, \quad \text{where} \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}.$$

Let us fix some notation: the free Lie algebra with n generators is denoted by $\mathfrak{Lie}_n$, while we use the terminology $\mathfrak{Lie}_n^k$ for the linear subspace spanned by elements of length-k. We will regard the free Lie algebras as graded, where the grading is naturally given by the length.

We will now introduce the notion of a *Hall basis* associated to $\mathfrak{Lie}_n$. Bases for free Lie algebras appeared for the first time in the work of M. Hall [16] and, since, many articles appeared describing such bases. We will rather work with the definition from pp. 22–23 of the book [28] by J. P. Serre.

Definition 3.13 (Hall set associated to $\mathfrak{Lie}_n$ [28]). Consider an ordered set of n generators

$$X = \{X_1 < \cdots < X_n\}.$$

We say that a totally ordered subset $H \subset M(X)$ is a *Hall set* if the following conditions are satisfied:

(i) $X \subset H$,

(ii) if $\ell(a) < \ell(b)$ for $a, b \in H$, then $a < b$; i.e., brackets in H are ordered by length,

(iii) $[a, b] \in H$ if and only if the following two conditions are satisfied:

(iii.a) $a, b \in H$ and $a < b$,

(iii.b) either $b \in X$ or $b = [c, d]$ where $c, d \in H$ and $c \leq a$.

We denote by $\mathcal{V}_i$ the subset of length-i brackets in H; i.e., $\mathcal{V}_i := H \cap M_i(X)$.

We often refer to a Hall set produced by an ordered set X with n-generators as a *Hall basis* associated to X or, also, associated to $\mathfrak{Lie}_n$. This terminology is justified by the following well-known proposition.

Proposition 3.14 (Hall sets are graded bases [16, 26, 33]). *A Hall set H associated to $\mathfrak{Lie}_n$ constitutes a graded basis of $\mathfrak{Lie}_n$, whereas each subset $\mathcal{V}_k \subset H$ provides a basis for $\mathfrak{Lie}_n^k$. Moreover, the dimension $d_{n,k}$ of each $\mathfrak{Lie}_n^k$ is given by the expression*

$$d_{n,k} = \frac{1}{k} \sum_{p|(k)} \mu(p) n^{k/p},$$

where $\mu(\cdot)$ denotes the Möbius function from number theory.

Note that from a given local frame $\mathfrak{Fr}$ of a maximal growth distribution $\mathcal{D}$ we can produce a Hall basis associated to it as follows. Let

$$\mathfrak{Fr} = \{X_1, \ldots, X_k\} \quad \text{and} \quad \mathcal{D} = \langle X_1, \ldots, X_k \rangle.$$

Note that $\mathcal{D}_i = \langle \mathfrak{Br}^i \rangle$ for $1 \leq i \leq r$. It turns out that length-i brackets involving elements of the frame $\mathfrak{Fr}$ are generators of $\mathcal{D}_i / \mathcal{D}_{i-1}$ and, in the case that $i < r$, the only dependance relations among these brackets are the ones given by the antisymmetric property and Jacobi identity. Therefore, the first $r-1$ layers $\mathcal{V}_i$ in a Hall basis H associated to the frame $\mathfrak{Fr}$ constitute a graded basis of the first $r-1$ elements of the nilpotentisation. In other words, each $\mathcal{V}_i \subset H$ $(i \neq r)$ is a basis of $\mathcal{D}_i/\mathcal{D}_{i-1}$ and, so, $\dim(\mathcal{D}_i/\mathcal{D}_{i-1}) = \dim(\mathfrak{Lie}_k^i) = d_{k,i}$.

This is encoded by the following well-known proposition.

Proposition 3.15 ([2]. Growth vector of a distribution of maximal growth). *The growth vector of a step-r, rank-k distribution $\mathcal{D}$ of maximal growth on a smooth manifold of dimension n is given by*

$$\nu_{\mathcal{D}} = (\mathfrak{n}_1, \mathfrak{n}_2, \ldots, \mathfrak{n}_{r-1}, \mathfrak{n}_r) = \Big(k = d_{k,1}, d_{k,1} + d_{k,2}, \ldots, \sum_{i=1}^{r-1} d_{k,i}, n\Big).$$

Remark 3.16. Since the entries of a maximal growth vector depend only on $k = \mathrm{rank}(\mathcal{D})$ and $n = \dim(M)$, we will often reserve the letters $\mathfrak{n}_i$ to denote the i-th entry of a maximal growth vector (k and n will be omitted from the notation whenever their values are clear from the context).

Definition 3.17. We say that a k-rank maximal growth distribution $\mathcal{D}$ is of *free type* if the last entry of the growth vector satisfies the following equality:

$$\mathfrak{n}_r = \sum_{i=1}^{r} d_{k,i}.$$

Remark 3.18. The condition in Definition 3.17 tantamounts to the r elements $\mathcal{D}_i$ $(1 \leq i \leq r)$ in the Lie flag of such a maximal growth distribution having the same dimensions as the first r subspaces $\mathfrak{Lie}_k^i$ in the free Lie algebra $\mathfrak{Lie}_k$.

Example 3.19. Some examples of maximal growth vectors of free type are $\nu_{\mathcal{D}} = (2,3,5)$, $\nu_{\mathcal{D}} = (2,3,5,8)$, $\nu_{\mathcal{D}} = (3,6,14)$ and $\nu_{\mathcal{D}} = (4,10,30)$; whereas $\nu_{\mathcal{D}} = (3,6,8)$ and $\nu_{\mathcal{D}} = (4,10,11)$ are examples of maximal growth vectors *not* of free type.

Let us elaborate on a concrete example.

Example 3.20. Consider a local frame $\mathfrak{Fr} = \{X_1, X_2, X_3\}$ and the maximal growth distribution it spans $\mathcal{D} = \langle \mathfrak{Fr} \rangle$. We can explicitly give the elements in the Hall basis associated to $\mathfrak{Fr}$ following Definition 3.13. Let us do this as an example for the three first elements in the flag:

$$\begin{aligned}
\mathcal{D}_1 &= \langle X_1, X_2, X_3 \rangle, \\
\mathcal{D}_2/\mathcal{D}_1 &= \langle [X_1,X_2], [X_1,X_3], [X_2,X_3] \rangle, \\
\mathcal{D}_3/\mathcal{D}_2 &= \langle [X_1,[X_1,X_2]], [X_1,[X_1,X_3]], [X_2,[X_1,X_2]], [X_2,[X_1,X_3]], \\
&\quad [X_2,[X_2,X_3]], [X_3,[X_1,X_2]], [X_3,[X_1,X_3]], [X_3,[X_2,X_3]] \rangle.
\end{aligned}$$

Note that there are length-3 brackets not appearing in the Hall basis:

$$[X_1,[X_2,X_3]], [X_3,[X_2,X_1]], \ldots$$

Indeed, they can be written as combinations of the ones in the Hall basis thanks to the Jacobi identity and the antisymmetric property. The growth vector for this distribution $\mathcal{D}$ reads as $\nu_{\mathcal{D}} = (3, 6, 14, \dots)$.

We will outline the following elementary fact as a lemma since it will be useful later.

Lemma 3.21. *Consider an ordered set* $\{X_1 < \cdots < X_k\}$ *and a Hall basis* $H \subset M(X)$ *associated to* X *as in Definition 3.13. Then,*

- *if* $j > i$*, the length-*ℓ *bracket* $[X_i, [X_i, [\dots, [X_i, [X_i, X_j]] \dots]]]$ *belongs to the Hall basis* H.
- *If* $j < i$*, then the bracket* $[X_i, [X_i, [\dots, [X_i, [X_j, X_i]] \dots]]]$ *belongs to the Hall basis* H.

Proof. First, note that any bracket formed by X_i, X_j, where the former appears $\ell - 1$ times and the latter exactly 1 time cannot be expressed as a combination of any other bracket with any other X_m involved, $m \neq i, j$, or X_i, X_j appearing a different number of times. Indeed, the number of times each element appears in each bracket is preserved both by Jacobi identity and the antisymmetric property. So, such a bracket formed by X_i, X_j must be part of any Hall basis associated to $\mathfrak{Fr}$. By the conditions in Definition 3.13, it follows that the only possibilities are the ones stated in this Lemma. ∎

Remark 3.22. This, in particular, implies that length-ℓ Lie brackets of the form

$$v_j := [X_i, [X_i, [\dots, [X_i, [X_i, X_j]] \dots]]]$$

are linearly independent in $\mathfrak{Lie}_k^\ell$. Also, by the same argument as that in the proof of Lemma 3.21 they cannot be expressed as linear combinations of any other bracket with any other X_m involved, $m \neq i, j$, or X_i, X_j appearing a different number of times.

Remark 3.23. Brackets $[X_i, [X_i, [\dots, [X_i, [X_i, X_j] \dots]$ as in Lemma 3.21 where X_i appears p times are often denoted by $\mathrm{ad}^p_{X_i}(X_j)$ in the literature (see, for example, Theorem 3 in [28]).

Example 3.24. For $\ell = 3$ in Example 3.20, the brackets $[X_1, [X_1, X_2]]$ and $[X_1, [X_1, X_3]]$ are of such type for $i = 1$, $[X_2, [X_1, X_2]]$ and $[X_2, [X_2, X_3]]$ for $i = 2$, and $[X_3, [X_1, X_3]]$ and $[X_3, [X_2, X_3]]$ for $i = 3$.

4. Formal bracket-generating distributions of step-r

4.1. Jet-coordinates

We will introduce jet-coordinates so that we can work in a comfortable and transparent way with expressions in jet spaces. We will focus our attention on $J^{r-1}\left(\bigoplus_k T\mathbb{R}^n\right)$ since this is the main jet space we will work with. We follow, adapting it to our case, the notation and elegant exposition from [25].

Definition 4.1. Define the set of *unordered multi-indices* $\mathbb{S}_\ell$ *of order-*ℓ as

$$\mathbb{S}_\ell := \mathbb{I}_\ell / \Sigma_\ell$$

(recall Definition 3.7). Similarly, we define $\mathbb{S}_0$ as the singleton containing the empty index $O = ()$. The set of *non-ordered multi-indices of length less than or equal to i* is defined as:

$$\mathcal{S}_i = \bigcup_{\ell=0}^{i} \mathbb{S}_\ell.$$

Taking the quotient by the symmetric group Σ_ℓ in Definition 4.1 just means that two multi-indices are equal if they coincide up to reordering their elements.

For $F = (F_1, \dots, F_k) \in J^{r-1}(\bigoplus_k T\mathbb{R}^n)$ over the point $x \in \mathbb{R}^n$, where $F_i \in J^{r-1}(T\mathbb{R}^n)$, we write $j^0(F_i) = (u_i^1, \dots, u_i^n)$. We think the u_i^j as smooth maps depending on $x = (x_1, \dots, x_n)$. This way, we write

$$F = (x, u_1^1, \dots, u_i^j, \dots, u_k^n, \dots, u_{i,I}^j, \dots), \tag{4.1}$$

for $x \in \mathbb{R}^n$, $i = 1, \dots, k$, $j = 1, \dots, n$ and $I \in \mathcal{S}_{r-1}$, where u_i^j denotes the j-th component of $j^0(F_i)$. Similarly, for $I \in \mathbb{S}_\ell$, the coordinate $u_{i,I}^j$ identifies with the partial derivative $\partial^I u_i^j / \partial x^I$, where we treat the jet variables u_i^j as smooth maps. We often use as well the following compact notation:

$$u_{i,I} := (u_{i,I}^1, \dots, u_{i,I}^n),$$

regarding it as a vector in $\mathbb{R}^n$.

Following the same terminology as [25], we introduce notion of differential polynomial. These will be fundamental for later defining the formal analogs of Lie brackets in the formal setting.

Definition 4.2. A *differential polynomial* is an n-dimensional expression

$$P(\cdot) = (P_1(\cdot), \dots, P_n(\cdot))$$

or, alternatively, $P(\cdot) = \sum_{s=1}^n P_s(\cdot)\,\partial_s$, where each $P_s(\cdot)$ is a polynomial function,

$$P_s(F) = P_s(u_1^1, \dots, u_i^j, \dots, u_n^n, \dots, u_{i,I}^j, \dots).$$

The *order* of P is defined as the maximum order I among all the individual jet-coordinates $u_{i,I}^j$ appearing in the expression. We denote by $\mathcal{P}_\ell$ the set of differential polynomials of order less than or equal to ℓ.

Remark 4.3. Since we are working under the framework of jets in $J^{r-1}(\bigoplus_k T\mathbb{R}^n)$, polynomials of order greater than $r-1$ will not be well defined.

Upon regarding the jet-coordinates as smooth maps depending on the variables x_i, a differential polynomial $P(\cdot)$ of order m can be thought as the symbol

$$P_{\mathsf{symb}} : J^{r-1}(\oplus_k T\mathbb{R}^n) \to J^0(\oplus_k T\mathbb{R}^n)$$

of some differential operator, where $m \leq r-1$.

For each coordinate direction $\partial_t \in \mathbb{R}^n$, we define the following operation within the space of differential polynomials. This is an operation of particular interest which will be fundamental for introducing further notions.

Definition 4.4. A *directional derivation* D_t is an operation defined in $\mathcal{P}_{r-2}$ which is linear, satisfies Leibniz rule and is defined as follows on individual coordinates:

$$D_t(u^j_{i,J}) = u^j_{i,(J,t)}, \quad J \in \mathcal{S}_{r-2},$$

where for $J = (i_1, \ldots, i_\ell)$ the notation (J,t) just denotes the multi-index concatenation $(J,t) := (i_1, \ldots, i_\ell, t)$. By regarding the variables $u^j_{i,J}$ as smooth maps, by linearity and Leibniz rule, the definition of D_t readily extends to the whole $\mathcal{P}_{r-2}$.

Note that D_t increases the order of a differential polynomial by one. Therefore, since we are working under the framework of $J^{r-1}(\bigoplus_k T\mathbb{R}^n)$, the assumption on the order of the differential polynomials (being less than $r-2$) is essential in order Definition 4.4 to make sense. Recall Remark 4.3.

Remark 4.5. Note that D_t just describes the symbol of the usual directional derivative with respect to the coordinate direction ∂_t in $\mathbb{R}^n$, $n \geq 1$.

Directional derivatives can be composed. For $J \in \mathbb{S}_s$, the composition of derivatives

$$D_{(m_1,\ldots,m_\ell)}(u^j_{i,J}) := D_{m_\ell} \circ \cdots \circ D_{m_1}(u^j_{i,J})$$

is well defined whenever $s + \ell \leq r-1$. We say that composed derivatives D_I have order k if $I \in \mathbb{S}_k$.

Note that we use unordered subindices $I \in \mathbb{S}_k$ to denote composed derivations. This reflects the commutativity of their composition.

Next lemma is an obvious remark that follows from the way jet-coordinates (4.1) were defined. It allows to dissect how

$$F_m = j^{\perp(\eta,r-1)}(F_m) \oplus j^{r-1}_t(F_m)$$

can be expressed in local jet-coordinates. We will make use of this description multiple times later in this work.

Lemma 4.6. *The splitting $F_m = j^{\perp(\eta,r-1)}(F_m) \oplus j^{r-1}_t(F_m)$ from equation* (2.1) *(or, equivalently, its analogous formula for lower order jets* (2.2)*) can be described as follows in jet-coordinates*:

(i) *$j^{r-1}_t(F_m)$ corresponds to the order-$(r-1)$ pure formal derivative of F with respect to ∂_t, and thus, in jet-coordinates, this component is described by*

$$u_{m,I} = \sum_{j=1}^{n} u^j_{m,I} \cdot \partial_j, \quad I = (t, \ldots, t) \in \mathbb{S}_{r-1}.$$

(ii) *Similarly, $j^{\perp(\eta,r)}(F_i)$ corresponds to the rest of mixed formal partial derivatives and the 0-jets; i.e., its components are of the form $u_{(m,J)}$ with $J \in \mathcal{S}_{r-1}$ and $J \neq (t, \ldots, t) \in \mathbb{S}_{r-1}$.*

Point (i) in Lemma 4.6 above motivates the following definition. It will introduce a differential operator that can be understood as a higher-order analog of the directional derivation in Definition 4.4.

Definition 4.7. We denote by $P_t^i(\cdot)$ the map that assigns to each jet $F = (F_1, \dots, F_k) \in J^{r-1}\left(\bigoplus_k T\mathbb{R}^n\right)$ its formal order-i pure derivative in the ∂_t-direction:

$$P_t^i(\cdot) : J^{r-1}(\oplus_k T\mathbb{R}^n) \to J^0(\oplus_k T\mathbb{R}^n),$$
$$F \mapsto (u_{1,I}^1, \dots, u_{m,I}^j, \dots, u_{k,I}^n), \quad I = (t, \dots, t) \in \mathbb{S}_i.$$

Note that $P_t^i(\cdot)$ in Definition 4.7 just describes, in line with Remark 4.5, the symbol of $\partial^i/\partial t^i$, the order-i derivative operator with respect to ∂_t.

4.2. Formal Lie brackets

The goal of this subsection is to introduce formal Lie brackets, a key notion in this work. Formal Lie brackets will be the analogs of Lie brackets in the formal setting.

Think of vector fields in $\mathbb{R}^n$ as sections $X_j : \mathbb{R}^n \to T\mathbb{R}^n$. Now, given k vector fields $X_1, \dots, X_k \in \mathrm{Sec}(T\mathbb{R}^n)$, the usual Lie bracket of X_i and X_j, which we denote by

$$A_{(i,j)}(X_1, \dots, X_k) = [X_i, X_j] \in \mathrm{Sec}(T\mathbb{R}^n),$$

is a new vector field produced out of these two.

In other words, each Lie bracket $A_{(i,j)}(X_1, \dots, X_k)$, where $(i, j) \in \mathbb{I}_2$, can be interpreted as a first order differential operator,

$$[\cdot,\cdot]_{(i,j)} : \mathrm{Sec}(\oplus_k T\mathbb{R}^n) \to \mathrm{Sec}(T\mathbb{R}^n).$$

We can thus describe its symbol as a fiberwise map

$$[\cdot,\cdot]_{(i,j)}^{\mathsf{symb}} : J^1(\oplus_k T\mathbb{R}^n) \to J^0(T\mathbb{R}^n).$$

More generally, a length-ℓ ($\ell \geq 1$) Lie bracket

$$A_{(a_\ell,\dots,a_1)}(X_1, \dots, X_k) := [X_{a_\ell}, [\dots, [X_{a_2}, X_{a_1}] \dots]], \quad (a_\ell, \dots, a_1) \in \mathbb{I}_\ell,$$

can be interpreted as a differential operator of order-$(\ell - 1)$,

$$A_{(a_\ell,\dots,a_1)}(\cdot, \dots, \cdot) : \mathrm{Sec}(\oplus_k T\mathbb{R}^n) \to \mathrm{Sec}(T\mathbb{R}^n),$$

which we will call *Lie bracket of length-ℓ with multi-index* $(a_1, \dots, a_\ell)$. Note that a length-1 Lie bracket is defined in the obvious manner; i.e., $A_{(a_i)}([X_1, \dots, X_k]) = X_{a_i}$, where $(i) \in \mathbb{I}_1$. Analogously, we consider the symbol of a length-ℓ Lie bracket as a map

$$A(\cdot, \dots, \cdot)_{(a_\ell,\dots,a_1)}^{\mathsf{symb}} : J^{\ell-1}(\oplus_k T\mathbb{R}^n) \to J^0(T\mathbb{R}^n),$$

which we will call *formal Lie bracket of length-ℓ with multi-index* $(a_1, \dots, a_\ell)$.

For practical reasons, and since there are well-defined projections $J^{r-1}(T\mathbb{R}^n) \to J^{\ell-1}(\oplus_k T\mathbb{R}^n)$ for $\ell - 1 < r - 1$, henceforth we will regard the symbols of all these operators as the obvious lifted fiberwise maps

$$A(\cdot, \dots, \cdot)_{(a_\ell,\dots,a_1)}^{\mathsf{symb}} : J^{r-1}(\oplus_k T\mathbb{R}^n) \to J^0(T\mathbb{R}^n);$$

i.e., we will consider $J^{r-1}(\oplus_k T\mathbb{R}^n)$ as the common domain for all of them.

From now on, for clarity and since we will mainly work with the symbols rather than with the operators themselves, we will drop the subindex symb from the notation. We will refer to the symbols of the various Lie bracket operators as *formal Lie brackets* but, again, we will often drop the word "formal" whenever it is clear from the context.

Next lemma describes the symbols of Lie brackets in jet-coordinates (we follow the same notation as in expression (4.1) for the coordinates of a generic jet $F = (F_1, \dots, F_k) \in J^{r-1}t(\bigoplus_k T\mathbb{R}^n)$. In other words, it provides an analytic description of formal Lie brackets in terms of jet-coordinates.

Lemma 4.8. *A formal Lie bracket of length-ℓ ($\ell \leq r$) with multi-index $(a_\ell, \dots, a_1) \in \mathbb{I}_\ell$ is a map*

$$A_{(a_\ell,\dots,a_1)} : J^{r-1}\big(\textstyle\bigoplus_k T\mathbb{R}^n\big) \longrightarrow J^0(T\mathbb{R}^n), \quad F = (F_1, \dots, F_k) \longmapsto [F_{a_\ell}, [\dots, [F_{a_2}, F_{a_1}] \dots]], \tag{4.2}$$

where, for any given $F = (F_1, \dots, F_k) \in J^{r-1}(\bigoplus_k T\mathbb{R}^n)$, the expression

$$[F_{a_\ell}, [\dots, [F_{a_2}, F_{a_1}] \dots]]$$

just depends on the components $F_{a_\ell}, \dots, F_{a_1}$ and can be described inductively on the length in terms of jet-coordinates as follows.

For length-1 formal brackets, $[F_{a_1}]$ just corresponds to the components u_{a_1}:

$$[F_{a_1}] := \sum_{i=1}^{n} u^i_{a_1} \partial_i .$$

We can write length $\ell - 1$ brackets as $[F_{a_{\ell-1}}, [\dots, [F_{a_2}, F_{a_1}] \dots]] = \sum_{i=1}^n p^i \partial_i$, where $p = (p^1, \dots, p^n)$ is a differential polynomial in the jet-coordinates of F. Length-ℓ brackets then read as

$$[F_{a_\ell}, [F_{a_{\ell-1}}, [\dots, [F_{a_2}, F_{a_1}] \dots]]] := \sum_{i,j=1}^{n} \big(u^j_{a_\ell} D_j(p^i) - p^j D_j(u^i_{a_\ell})\big) \partial_i . \tag{4.3}$$

Remark 4.9. Note that, potentially, F_{a_i} could denote the same component of F as F_{a_j} in (4.2) for some $i \neq j$. That would be the case if $a_i = a_j$ and, thus, the corresponding component appeared twice in the entries of $[F_{a_\ell}, [\dots, [F_{a_2}, F_{a_1}] \dots]]$.

Remark 4.10. We shall refer to $F_{a_\ell}, \dots, F_{a_1}$ as the entries of the bracket expression $[F_{a_\ell}, [\dots, [F_{a_2}, F_{a_1}] \dots]]$ and we will interchangeably write either $[F_{a_\ell}, [\dots, [F_{a_2}, F_{a_1}] \dots]]$ or $A_{(a_\ell,\dots,a_1)}(F)$ at our convenience.

Remark 4.11. Note that Lemma 4.8 is just a rephrasing, in terms of jet-coordinates, of the well-known formulas for Lie brackets. As such, we can recover usual properties of the Lie bracket, as Remark 4.12 and Lemma 4.13 illustrate. Equivalently, both readily follow from Lemma 4.8.

Remark 4.12. Each bracket expression $[F_{a_\ell}, [\dots, [F_{a_2}, F_{a_1}] \dots]]$ is multilinear in its entries.

Lemma 4.13. *For any multi-index $I = (a_\ell, \dots, a_2, a_1)$, $\ell \geq 2$, consider the multi-index $\tilde{I} = (a_\ell, \dots, a_1, a_2)$ defined by reversing the last two entries of I. Then the following equality holds*:

$$A_I(F) = -A_{\tilde{I}}(F). \tag{4.4}$$

Proof. One could argue that this is true for usual brackets and, being a formal bracket simply its symbol, this condition readily follows. Equivalently, it follows from Lemma 4.8 as well. Note that condition (4.4) for length-2 brackets is equivalent to antisymmetry, which readily follows from Lemma 4.8.

For higher order length-brackets, (4.4) follows in the same manner: just note that the iterative process described by equation (4.3) in Lemma 4.8 implies that Lie bracket expressions are antisymmetric in their last two entries. Thus exchanging the positions of a_2 and a_1 produces a change of sign in the whole expression. ■

We will now introduce in Definition 4.14 what we call the simplified form of a given formal bracket. This will provide a simple way of expressing formal brackets in terms of their 0-th order information and derivations which will be useful later in this work.

Definition 4.14. We say that a length-ℓ formal bracket $A_I(F)$, $(I \in \mathbb{I}_\ell)$, is expressed in *simplified form* if it is written as an expression involving solely non-zero order-0 jet-coordinates u_i^j and order $\leq \ell - 1$ derivations of those, $D_J(u_i^j)$, $J \in \mathcal{S}_{\ell-1}$.

Let us examine a simple example of formal brackets in simplified form.

Example 4.15. Length-2 formal brackets $[F_{a_2}, F_{a_1}]$ in simplified form read as

$$[F_{a_2}, F_{a_1}] := \sum_{i,j=1}^{n} \big(u_{a_2}^j D_j(u_{a_1}^i) - u_{a_1}^j D_j(u_{a_2}^i)\big)\,\partial_i. \tag{4.5}$$

Indeed, Expression (4.5) only involves order-0 jet-coordinates $u_{a_1}^j$ and $u_{a_2}^j$ and order-1 derivations of those. An example of the same bracket not expressed in simplified form would just consist on replacing each $D_j(u_{a_\alpha}^i)$ by order-1 jet-coordinates $u_{a_\alpha,(j)}^i$.

It is natural to ask whether every formal bracket admits a simplified form. This is answered by the following result.

Lemma 4.16. *Each length-ℓ bracket $A_{(a_1,\dots,a_\ell)}(F)$ admits a simplified form for $\ell \geq 1$.*

Proof. It follows from Lemma 4.8, since the lemma itself describes an iterative process that allows to decompose any length-ℓ bracket as a polynomial expression involving only order-0 jet-coordinates u_i^j of F together with successive derivations $D_{j_m} \circ \cdots \circ D_{j_1}(u_i)$ up to order $\ell - 1$. ■

Finally, we conclude this subsection with Lemma 4.17, which shows that each length-ℓ formal bracket depends solely on the $(\ell - 1)$-order information. This will be key in the proof of the main theorem in this work.

Lemma 4.17. *The expression $A_{(a_1,\dots,a_\ell)}(F)$ depends solely on the $(\ell-1)$-order information of the input F; i.e., $A_{(a_1,\dots,a_\ell)}(\cdot)$ factors through some map h as follows*:

$$A_{(a_1,\dots,a_\ell)}(\cdot) = h \circ j^{\ell-1}(\cdot).$$

Proof. Since $A_\ell(F_{a_1},\dots,F_{a_\ell})$ admits a simplified form (Lemma 4.16), it can be defined by the order-0 jet-coordinates u_i^j of F and derivations of those up to order $\ell-1$, thus yielding the claim. ∎

4.3. Formal Lie flags

Note that since we have well-defined formal brackets of jets, we can now define the analogs of the sets $\mathfrak{Br}^i$ (Definition 3.8) in the formal setting.

Definition 4.18. Let $F = (F_i)_{i=1}^k \in J^{r-1}(\oplus_k T\mathbb{R}^n)$. For $i \geq 1$, we define $\mathfrak{Br}_F^i$ as the set of brackets of F of length less than or equal to i $(i \leq r)$:

$$\mathfrak{Br}_F^i := \{A_I(F) : I \in \mathfrak{I}_i\}.$$

Similarly, we can now define the analog notion of Lie flags in the formal setting as well:

Definition 4.19. Let $F = (F_i)_{i=1}^k \in J^{r-1}(\oplus_k T\mathbb{R}^n)$. For each $i = 1,\dots,r$, we define the planes

$$\mathcal{D}_i(F) := \langle \mathfrak{Br}_F^i \rangle.$$

We call the *formal Lie flag* associated to/produced by F to the following flag of inclusions:

$$\mathcal{D}_1(F) \subset \mathcal{D}_2(F) \subset \cdots \subset \mathcal{D}_r(F).$$

Similarly, we say that $\mathcal{D}_1(F) = \mathcal{D}_F$ has (formal) *growth vector*

$$(n_i)_{i=1}^r := (\dim(\mathcal{D}_i(F))_{i=1}^r.$$

Take $p \in \mathbb{R}^n$ and a distribution $\mathcal{D} \subset T\mathbb{R}^n$. Take a local frame $\mathfrak{Fr} = (X_1,\dots,X_k)$ of $\mathcal{D}$ over $\mathcal{O}p(p)$ and consider its associated sets $\mathfrak{Br}^i$. Similarly, consider the jet $F = (F_i)_{i=1}^k \in J^{r-1}(\oplus_k T\mathbb{R}^n)$ associated to the frame; i.e., where $F_i = j^{r-1}(X_i)$ $(i = 1,\dots,k)$. Note that the sets $\mathfrak{Br}^i$ associated to $\mathfrak{Fr}$ coincide with its formal analogs $\mathfrak{Br}_F^i$ over each point in $\mathcal{O}p(p)$.

Consequently, the Lie flag associated to $\mathcal{D}$ coincides with the formal Lie flag associated to F; i.e., for each $i = 1,\dots,r$, $\mathcal{D}_i = \mathcal{D}_i(F)$ over each point in $\mathcal{O}p(p)$.

Remark 4.20. For the ease of notation, and whenever it is clear from the context, we will drop the letter F from the notation of the just defined notions; i.e., we will just write $\mathfrak{Br}^i$ for $\mathfrak{Br}_F^i$ and $\mathcal{D}_i$ for $\mathcal{D}_i(F)$. Similarly, we often write $\mathcal{D}_F$, or plainly $\mathcal{D}$, for $\mathcal{D}_1(F)$.

Recall that usual Lie bracket operators behave well with respect to changes of coordinates. Since formal brackets are just the symbols of the usual bracket operators, this readily translates to the formal setting. We phrase this fact as a lemma.

Lemma 4.21. *Let* $F = (F_i)_{i=1}^k \in J^{r-1}(\oplus_k T\mathbb{R}^n)$ *and let* $f \in \mathrm{Diff}(\mathbb{R}^n, \mathbb{R}^n)$. *Consider then the pushforward* $f_* F = (f_* F_i)_{i=1}^k$. *Then we have*

$$f_* A_{(a_1,\dots,a_\ell)}(F) = A_{(a_1,\dots,a_\ell)}(f_* F)$$

Lemma 4.21 implies that the definition of the elements $\mathcal{D}_i(F)$ does not depend on the choice of coordinates for $\mathbb{R}^n$. Therefore, we can extend the definition of the Lie flag (Definition 4.19) to arbitrary smooth manifolds.

Indeed, we can just define the Lie flag in local charts $U \subset M$ and, by the invariance under changes of coordinates (see Lemma 4.21), this definition does not depend on the choice of chart. In other words, these objects are intrinsically well defined for any element $F \in J^{r-1}(\oplus_k TM)$.

4.4. Formal distributions

Definition 4.22. We define the space of *k-frames over M* as the subspace $\mathrm{Fr}_k(M)$ of linearly independent k-tuples inside the space of k-tuples $\bigoplus_k TM$. The projection $\mathrm{Fr}_k(M) \to M$ is a fiber bundle.

Note that there is a natural bundle projection that maps a k-tuple of pointwise linearly independent vector fields $(X_1, \dots, X_k)$ (also called a k-frame) to the linear distribution they span:

$$\pi : \mathrm{Fr}_k(M) \longrightarrow \mathrm{Gr}_k(TM),$$

which induces a map at the level of r-jets:

$$j^r\pi : J^r(\mathrm{Fr}_k(M)) \longrightarrow J^r(\mathrm{Gr}_k(TM)).$$

We will often denote the map $j^r\pi$ by π^r.

Definition 4.23. Given an r-tuple of integers of the form $\nu = (\mathfrak{n}_1 = k, \mathfrak{n}_2, \dots, \mathfrak{n}_r = \dim M)$, the differential relation

$$\mathcal{S}^\nu \subset J^{r-1}(\mathrm{Fr}_k(M))$$

is defined by the set of elements $F \in J^{r-1}(\mathrm{Fr}_k(M))$ for which $\dim(\mathcal{D}_i(F)) = \mathfrak{n}_i$ for every $1 \le i \le r$. We say that $\mathcal{S}^\nu$ is the differential relation of *formal bracket-generating frames* with growth vector ν.

Definition 4.24. In the particular case of $\nu = (\mathfrak{n}_1, \dots, \mathfrak{n}_{r-1}, \mathfrak{n}_r = \dim M)$ being the maximal growth vector corresponding to the ambient manifold M (where $\mathfrak{n}_{r-1} < \dim M$), sections to the differential relation $\mathcal{S}^\nu$ are called formal frames of *maximal growth* of step-r, and we denote this differential relation by $\mathcal{S}^{\text{step-}r}$.

Definition 4.25. A jet $F \in J^{r-1}(\mathrm{Gr}_k TM)$ is called *formally bracket-generating of step-r* if F is the projection of some element $\tilde{F} \in \mathcal{S}^{\text{step-}r}$; i.e., if $\pi^{r-1}(\tilde{F}) = F$. The subset of those elements F will be denoted by $\mathcal{R}^{\text{step-}r} \subset J^{r-1}(\mathrm{Gr}_k TM)$.

Definition 4.26. We say that a *formal maximal growth distribution* of step-r on M is a smooth section $s: M \to \mathcal{R}^{\text{step-}r} \subset J^{r-1}(\mathrm{Gr}_k TM)$.

$$
\begin{array}{ccccc}
\mathcal{S}^{\text{step-}r} \subset & J^{r-1}(\mathrm{Fr}_k(M)) & \xrightarrow{\pi^{r-1}} & J^{r-1}(\mathrm{Gr}_k(TM)) & \supset \mathcal{R}^{\text{step-}r} \\
& \downarrow & & \downarrow & \\
& \mathrm{Fr}_k(M) & \xrightarrow{\pi} & \mathrm{Gr}_k(TM) &
\end{array}
$$

Figure 1. The differential relation $\mathcal{S}^{\text{step-}r}$ gets mapped to $\mathcal{R}^{\text{step-}r}$ via π^{r-1}.

We often refer to sections as in Definition 4.26 as rank-k formal maximal growth distributions or, plainly, formal maximal growth distributions.

Remark 4.27. Note that the set of (germs of) holonomic sections to $\mathcal{R}^{\text{step-}r}$ coincides with the set of (germs of) maximal growth distributions on M (recall Definition 3.4).

The terminology $\mathcal{R}^{\text{step-}2}$ was introduced in [20] to denote the differential relation defined by distributions of step-2, which are always of maximal growth. We then use the notation $\mathcal{R}^{\text{step-}r}$ to denote the differential relation defined by step-r maximal growth distributions, naturally extending the terminology from [20].

Lemma 4.28. *Consider jets $G, \tilde{G} \in \mathcal{S}^{\text{step-}r}$ over a point $p \in M$ so that $\pi^{r-1}(G) = \pi^{r-1}(\tilde{G})$. Then the Lie flag associated to G coincides with the one associated to $\tilde{G}$ at the point p.*

Proof. Any jet $F \in J^{r-1}(\mathrm{Fr}_k(M))$ over p can be regarded as the equivalence class of sections which coincide up to the $(r-1)$-order information over the point p. Take two sections $X = \{X_1, \dots, X_k\}\colon M \to TM$ and $\tilde{X} = \{\tilde{X}_1, \dots, \tilde{X}_k\}\colon M \to TM$ realizing, over p, the jets $G = (G_i)_{i=1}^k$ and $\tilde{G} = (\tilde{G})_{i=1}^k$, respectively; i.e., so that

$$j^{r-1}(X) = G \quad \text{and} \quad j^{r-1}(\tilde{X}) = \tilde{G} \quad \text{over } p.$$

It is well known that the Lie flag of a smooth distribution does not depend on the choice of frame. Therefore, since X and $\tilde{X}$ coincide up to order-$(r-1)$ at p (because $G = \tilde{G}$), then they have the same Lie flag over p. But since $j^{r-1}(X) = G$, the Lie flag produced by $j^{r-1}(G)$ over p coincides with the one produced by G (and so is the case for $j^{r-1}(\tilde{X})$ and $\tilde{G}$, respectively).

Putting everything together, we conclude that the Lie flag produced by G coincides with the one produced by $\tilde{G}$, yielding the claim. ∎

Lemma 4.28 implies that any given $F \in \mathcal{R}^{\text{step-}r}$ has a well-defined associated Lie flag. Indeed, any two lifts $G, \tilde{G} \in \mathcal{S}^{\text{step-}r}$ with $\pi^{r-1}(G) = \pi^{r-1}(\tilde{G}) = F$ define the same formal Lie flag by Lemma 4.28. This gives rise to the following definition.

Definition 4.29. We define the (formal) *Lie flag* associated to $F \in \mathcal{R}^{\text{step-}r}$ as the Lie flag associated to any of its lifts; that is, as the Lie flag associated to any $\tilde{F} \in \mathcal{S}^{\text{step-}r}$ with $\pi^{r-1}(\tilde{F}) = F$.

Let us finish this subsection with the following lemma, which is an immediate consequence of Lemma 4.21. It shows that the differential relation $\mathcal{S}^{\text{step-}r}$ does not depend on the choice of coordinates.

Lemma 4.30. *The differential relation $\mathcal{S}^{\text{step-}r}$ is* $\mathrm{Diff}(M)$ *invariant.*

5. Existence of maximal growth formal distributions

We aim to make this exposition as self-contained as possible, so let us review some elementary notions from Lie group theory that will be relevant in the upcoming discussion.

Definition 5.1 ([18]). Consider a graded Lie algebra $\mathfrak{g} = \bigoplus_{i=1}^r \mathfrak{g}_i$, where $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ and $\mathfrak{g}_r \neq 0$. If $\mathfrak{g}_1$ generates $\mathfrak{g}$ as an algebra, then we say that $\mathfrak{g}$ is a *stratified algebra*; i.e., we have that

$$\mathfrak{g} = \bigoplus_{i=1}^{r} \mathfrak{g}_i, \quad \text{where} \quad [\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}$$

and where $\mathfrak{g}_{r+1} = 0$ (which implies $[\mathfrak{g}_1, \mathfrak{g}_r] = 0$). The integer r is called the *step* of the algebra $\mathfrak{g}$.

Definition 5.2. Similarly, we say that a Lie group G is a *stratified group* of step-r if it is simply connected with a step-r stratified Lie algebra $\mathfrak{g} = \bigoplus_{i=1}^r \mathfrak{g}_i$. We denote by $e \in G$ its identity element and we identify T_eG with $\mathfrak{g}$.

Stratified Lie algebras are nilpotent and, thus, so are stratified Lie groups.

Remark 5.3. Note that, on stratified groups, the degree-one layer $(\mathfrak{g}_1)_e \subset T_eG$ defines, by left translation, a left-invariant distribution $\mathcal{D} \subset TG$. Also, by definition, $\mathfrak{g}_1$ generates the whole Lie algebra.

Remark 5.4. The Baker–Campbell–Hausdorff formula is a well-known result in the theory of Lie groups (see, for instance, Section 1.3 of [27]). Consider a Lie algebra $\mathfrak{g}$ with corresponding connected Lie group G and exponential map $\exp\colon \mathfrak{g} \to G$. Given two elements $X, Y \in \mathfrak{g}$, this formula is a formal series (not necessarily convergent):

$$Z(X,Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] - \frac{1}{24}[Y,[X,[X,Y]]] + \cdots$$

When this series is convergent, it produces the element $Z \in \mathfrak{g}$ that solves the equation $\exp(X)\exp(Y) = \exp(Z)$.

Note that in the case of nilpotent Lie algebras and groups, the Baker–Campbell–Hausdorff formula only contains a finite number of non-zero terms, and thus becomes a closed finite expression. This readily allows to identify $\mathfrak{g}$ with G, via this formula, and $\mathfrak{g}$ is thus endowed with a Lie group structure by inheriting the group structure of G.

The following notion is the analog of the nilpotentisation (recall Definition 3.6) in the formal setting.

Definition 5.5. Let M be a smooth n-manifold. Consider a flag of subbundles of TM,

$$\mathcal{D}_1 \subset \cdots \subset \mathcal{D}_{r-1} \subset \mathcal{D}_r \subseteq TM,$$

together with a fiberwise stratified Lie algebra structure of step-r (for some fiberwise Lie bracket) on

$$\mathcal{D}_1 \oplus \mathcal{D}_2/\mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_i/\mathcal{D}_{i-1} \oplus \cdots \oplus \mathcal{D}_r/\mathcal{D}_{r-1}.$$

For $\dim(\mathcal{D}_1) = k$, such a fiberwise Lie algebra structure is called a rank-k *formal nilpotentisation* of step-r. If, additionally, each dimension $\dim(\mathcal{D}_i/\mathcal{D}_{i-1}))$ coincides with the i-th entry of a maximal growth vector for ambient dimension n (in particular implying $\mathcal{D}_r = TM$), we then say that the formal nilpotentisation is of *maximal growth*.

We will show that every formal distribution has an associated formal nilpotentisation. Let us introduce some notions first.

Consider $\tilde{F} = (\tilde{F}_i)_{i=1}^k \in J^{r-1}(\oplus_k T\mathbb{R}^n)$ with associated formal Lie flag

$$\mathcal{D}_1(\tilde{F}) \subset \mathcal{D}_2(\tilde{F}) \subset \cdots \subset \mathcal{D}_r(\tilde{F}).$$

Recall that the elements $\mathcal{D}_i(\tilde{F})$ in this flag are defined by using formal Lie brackets (Definition 4.19). Furthermore, since formal Lie brackets are just the symbol of usual Lie brackets, it follows that the natural stratified Lie algebra associated to a distribution (Definition 3.1) translates to the formal setting as well; i.e., we have that

$$\mathcal{D}_1(\tilde{F}) \oplus \mathcal{D}_2(\tilde{F})/\mathcal{D}_1(\tilde{F}) \oplus \cdots \oplus \mathcal{D}_i(\tilde{F})/\mathcal{D}_{i-1}(\tilde{F}) \oplus \cdots \oplus \mathcal{D}_r(\tilde{F})/\mathcal{D}_{r-1}(\tilde{F}) \tag{5.1}$$

with formal Lie brackets carries a stratified Lie algebra structure. We call it the *stratified Lie algebra associated to* $\tilde{F} \in J^{r-1}(\oplus_k T\mathbb{R}^n)$.

On the other hand, recall that for any given $F \in \mathcal{R}^{\text{step-}r}$ there is an associated formal Lie flag (Definition 4.29) defined as the formal Lie flag (5.1) associated to any of its lifts $\tilde{F} \in \mathcal{S}^{\text{step-}r}$. Since the nilpotentisation of a distribution does not depend on the choice of frame, we also have a well-defined notion of stratified Lie algebra associated to an element $F \in \mathcal{R}^{\text{step-}r}$ as the following definition shows.

Definition 5.6. Let $F \in \mathcal{R}^{\text{step-}r}$. We define its *associated stratified Lie algebra* as the stratified Lie algebra (5.1) associated to any of its lifts $\tilde{F} \in \mathcal{S}^{\text{step-}r}$.

Therefore, every formal distribution of maximal growth on a smooth manifold M, i.e., a section $s\colon M \to \mathcal{R}^{\text{step-}r}$, yields a fiberwise stratified Lie algebra structure of step-r. In other words, it has a natural *associated formal nilpotentisation of maximal growth*. We may simply refer to it as its associated formal nilpotentisation.

Remark 5.7. Note that when a formal distribution of maximal growth is holonomic; i.e., it is an actual maximal growth distribution, then its associated formal nilpotentisation clearly coincides with its usual nilpotentisation (Definition 3.1).

The next lemma shows that the same phenomenon occurs in the other direction. This result and its proof are due to Álvaro del Pino.

Lemma 5.8. *If M admits a rank-k formal nilpotentisation of maximal growth, then it admits a rank-k formal maximal growth distribution whose associated formal nilpotentisation is precisely the given rank-k formal nilpotentisation.*

Proof. Assume, by hypothesis, that we have a flag of subbundles

$$\mathcal{D}_1 \subset \cdots \subset \mathcal{D}_r = TM \tag{5.2}$$

whose corresponding graded vector space

$$\mathcal{D}_1 \oplus \mathcal{D}_2/\mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_i/\mathcal{D}_{i-1} \oplus \cdots \oplus \mathcal{D}_r/\mathcal{D}_{r-1} \tag{5.3}$$

identifies fiberwise with a stratified Lie algebra $\mathfrak{g} = \bigoplus_{i=1}^r \mathfrak{g}_i$.

Take a point $p \in M$ and $T_pM \simeq \mathfrak{g}$. The Baker–Campbell–Hausdorff formula (see Remark 5.4) identifies $\mathfrak{g}$ with the corresponding nilpotent Lie group G which is stratified of step-r (because $\mathfrak{g}$ was). We can thus consider the left-invariant distribution $\mathcal{D} \subset TG$ associated to the first layer of the grading $\mathfrak{g}_1$ (recall Remark 5.3). By construction, the nilpotentisation associated to $\mathcal{D}$ (recall Subsection 3.1) over any point of G is isomorphic to the given one (5.3).

Fix a metric g on M and use the exponential map to identify $T_pM \simeq \mathfrak{g}$ with a small open neighborhood $\mathcal{O}p(p) \subset M$. We thus identify $\mathcal{O}p(p) \subset M$ with a neighborhood of identity in G. The $(r-1)$-jet $j^{r-1}(\mathcal{D})$ defines a formal distribution over $\mathcal{O}p(p)$ with associated formal Lie flag (5.2).

This construction clearly works with parameters. Therefore we can choose a fine enough covering of M and carry out this process over every open set in the covering simultaneously. We thus end up constructing a global formal distribution on M with formal Lie flag (5.2) and associated nilpotentisation (5.3). ∎

By means of the Lemma 5.8, we can finally prove the main result in this subsection.

Proposition 5.9 (Parallelizability implies existence of a formal structure). *Let M be an n-dimensional parallelizable manifold and fix $1 < k < n$. Then M admits a k-rank formal maximal growth distribution.*

Proof. By Lemma 5.8, it suffices to show that M admits a formal nilpotentisation of maximal growth. Denote by $(\mathfrak{n}_i)_{i=1}^r$ a maximal growth vector with $\mathfrak{n}_1 = k, \mathfrak{n}_r = n$.

Choose any stratified Lie algebra $\mathfrak{g} = \bigoplus_{i=1}^r \mathfrak{g}_i$ so that $\dim(\mathfrak{g}_i) = \mathfrak{n}_i$. By the parallelizability of M, trivialise TM and thus consider a fiberwise identification of TM with $\mathfrak{g}$; i.e., we have fiberwise linear isomorphisms $i_p : \mathfrak{g} \to T_pM$, varying smoothly with $p \in M$. We can just translate the stratification of $\mathfrak{g}$ onto TM. In other words, we can consider the pushforward $(\mathcal{D}_i)_p := (i_p)_*(\mathfrak{g}_i)$ of each $\mathfrak{g}_i$ and readily translate the stratified Lie Algebra structure of $\mathfrak{g}$ onto TM.

We thus produce a flag of subbundles $\mathcal{D}_1 \subset \cdots \subset \mathcal{D}_{r-1} \subset TM$ whose fiberwise associated graded vector bundle

$$\mathcal{D}_1 \oplus \mathcal{D}_2/\mathcal{D}_1 \oplus \cdots \oplus \mathcal{D}_i/\mathcal{D}_{i-1} \oplus \cdots \oplus \mathcal{D}_r/\mathcal{D}_{r-1}$$

inherits the stratified Lie algebra structure of $\mathfrak{g}$ where each $\mathcal{D}_i$ identifies with $\mathfrak{g}_i$. Thus, M admits a rank-k formal nilpotentisation of maximal growth and, therefore, by Lemma 5.8, a rank-k formal maximal growth distribution. ∎

Remark 5.10. Note that the converse is not true; i.e., the existence of a formal maximal growth distribution does not imply parallelizability. Indeed, there are infinitely many non-parallelizable manifolds M admitting a formal maximal growth distribution. For instance, the odd-dimensional spheres $(n \geq 1)$

$$\mathbb{S}^{2n+1} = \{z \in \mathbb{C}^{n+1} : |z| = 1\}$$

admit a contact structure defined by

$$\xi := T\mathbb{S}^{2n+1} \cap i(T\mathbb{S}^{2n+1})$$

and, therefore, an underlying formal maximal growth distribution. However, $\mathbb{S}^{2n+1}, n \geq 1$, is not parallelizable for $n \neq 1, 3$.

6. A fibered criterion for ampleness

The goal of this section is to present a new criterion, within the theory of convex integration, that allows to check ampleness of certain differential relations.

There are several criteria to check ampleness of differential relations in certain contexts: thinness of its complement (Lemma 2.11; see also Subsection 26.4.2 of [7]), ampleness criterion for A-directed immersions (Subsection 27.1.1 of [7]), elliptic operators $\mathcal{D}$ having rank ≥ 2 for relations defined by linear independence of systems of $\mathcal{D}$-sections (pp. 181–182 in [14], Subsection 28.4.1 of [7]), etc. Once any of these conditions is checked, one can directly invoke the theorem of convex integration (Theorem 2.10) and the complete h-principle follows.

We will thus present a new criterion of such type (that to our knowledge has not been explained elsewhere) that shows that ampleness of certain differential relations reduces, in a precise sense, to ampleness of some other differential relations fibering over them. As we will show later in Subsection 7.1 via a concrete example, checking ampleness of these auxiliary relations can in some cases be easier, thus showing that the criterion we present is not void.

The philosophical idea behind this criterion is as follows: in order to prove ampleness of a differential relation in $J^r(X)$, one may find easier to work with an auxiliary space Y (which may be the space of frames/coordinates/additional structure of the space X). In particular, this allows to check (in a precise sense) ampleness of a certain Diff(M) invariant differential relation by just checking it for certain choice of local coordinates.

Let us fix the general framework. Consider $\rho_Y : Y \to M$, $\rho_X : X \to M$ two smooth fiber bundles over the same base manifold and $\pi : Y \to X$ a surjective and submersive bundle map (that can often be thought as a projection or a quotient map). We thus have the induced map $\pi^r : J^r(Y) \to J^r(X)$. Consider an open differential relation $\mathcal{S} \subset J^r(Y)$ and an open differential relation $\mathcal{R} \subset J^r(X)$ so that the former is the pullback of the latter; i.e., $\mathcal{S} := (\pi^r)^*(\mathcal{R})$. In other words, $F \in \mathcal{S}$ if and only if $\pi^r(F) \in \mathcal{R}$. We will henceforth work under this framework. In particular, we have that $\pi^r(\mathcal{S}) = \mathcal{R}$, where fibers are mapped to fibers. This is summed up by the commutative diagram in Figure 2.

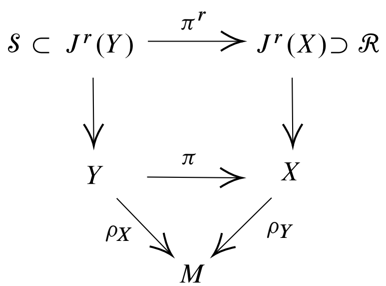

Figure 2. Submersive bundle map $\pi : Y \to X$ between smooth fiber bundles $Y \to M$ and $X \to M$ so that the differential relation $\mathcal{S}$ is the pullback of $\mathcal{R}$ by the associated r-jet extension π^r. The map π can often be thought as a quotient map.

Lemma 6.1. *Every $F \in J^r(X)$ has a lift $\tilde{F} \in J^r(Y)$ so that $\pi^r(\tilde{F}) = F$.*

Proof. This follows by the surjectivity and the submersive condition of π. Indeed, choose a local model where π is regarded, locally, as a projection map and thus the statement readily follows. ∎

The following lemma shows that $\pi^r\colon J^r(Y) \to J^r(X)$ is locally an affine bundle that maps principal subspaces to principal subspaces.

Lemma 6.2. *For any $q \in Y$, there exists an open neighborhood $\mathcal{U} \subset Y$ of q such that*:

(i) $\pi^r\colon J^r(\mathcal{U}) \to J^r(\mathcal{V})$ *is an affine bundle, where $\mathcal{V} := \pi(\mathcal{U})$.*

(ii) *The images by π^r of the principal subspaces of the fibration $J^r(\mathcal{U}) \to \rho_X(\mathcal{U})$ are the principal subspaces of $J^r(\mathcal{V}) \to \rho_Y(\mathcal{V})$*:

$$\pi^r(\mathrm{Pr}^r_{\eta,F}) = \mathrm{Pr}^r_{\eta,\pi^r(F)}.$$

Proof. For any point $q \in Y$ and $x := \pi(q) \in X$, choose local charts $q \in \mathcal{U} \subset Y$ and $x \in \mathcal{V} = \pi(\mathcal{U}) \subset X$, so that both fibrations are trivial; i.e., $\mathcal{U} \simeq \mathbb{R}^n \times \mathbb{R}^m$, $\mathcal{V} \simeq \mathbb{R}^n \times \mathbb{R}^{\tilde{m}}$, where $n = \dim(M)$ and the fiber $\mathbb{R}^m$ factor submerses onto the fiber $\mathbb{R}^{\tilde{m}}$ factor via π.

We can thus extend these coordinates (recall Section 2.1) to the level of r-jets as follows:

$$\begin{aligned}(6.1)\quad J^r(Y) \supset J^r(\mathcal{U}) &\simeq J^r(\mathbb{R}^n \times \mathbb{R}^m)\\ &\simeq \mathbb{R}^n \times \mathbb{R}^m \times \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \times \mathrm{Sym}^2(\mathbb{R}^n, \mathbb{R}^m) \times \cdots \times \mathrm{Sym}^r(\mathbb{R}^n, \mathbb{R}^m),\end{aligned}$$

$$\begin{aligned}(6.2)\quad J^r(X) \supset J^r(\mathcal{V}) &\simeq J^r(\mathbb{R}^n \times \mathbb{R}^{\tilde{m}})\\ &\simeq \mathbb{R}^n \times \mathbb{R}^{\tilde{m}} \times \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^{\tilde{m}}) \times \mathrm{Sym}^2(\mathbb{R}^n, \mathbb{R}^{\tilde{m}}) \times \cdots \times \mathrm{Sym}^r(\mathbb{R}^n, \mathbb{R}^{\tilde{m}}).\end{aligned}$$

Therefore, the induced map $\pi^r\colon J^r(\mathcal{U}) \to J^r(\mathcal{V})$ maps each of the factors in the fibers accordingly; i.e., each factor in (6.1) is mapped to its homologous factor in (6.2). We have that π (which acts on the j^0-part of the fibers) is a trivial affine bundle (and note that π is just the truncated part of π^r acting on the first two factors $\mathbb{R}^n \times \mathbb{R}^m$ of the product, i.e., $\pi\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^{\tilde{m}}$). The rest of the factors of the fibers are affine spaces which are clearly mapped affinely by π^r, thus yielding (i).

As for (ii), we have that, locally,

$$J^r(\mathcal{U}) = J^{\perp(\eta,r)}(\mathcal{U}) \times \mathcal{W} \quad \text{and} \quad J^r(\mathcal{V}) = J^{\perp(\eta,r)}(\mathcal{V}) \times \tilde{\mathcal{W}},$$

where $\mathcal{W}$ and $\tilde{\mathcal{W}}$ just denote the fibers of the respective bundles of $\perp(\eta, r)$-jets. Assume again that $\mathcal{U} \simeq \mathbb{R}^n \times \mathbb{R}^m$ and $\mathcal{V} \simeq \mathbb{R}^n \times \mathbb{R}^{\tilde{m}}$, where $n = \dim(M)$, and that the fiber $\mathbb{R}^m$ factor submerses onto the fiber $\mathbb{R}^{\tilde{m}}$ factor via π. We can also assume, without loss of generality, that $\eta = \ker(dt)$ where $\mathbb{R}^n = \mathbb{R}_t \times \mathbb{R}^{n-1}$ (recall Subsection 2.2). Extending now these coordinates to the level of r-jets as before, we have that $\pi^r\colon J^r(\mathcal{U}) \overset{\pi^r}{\to} J^r(\mathcal{V})$ maps $J^{\perp(\eta,r)}(\mathcal{U})$ to $J^{\perp(\eta,r)}(\mathcal{V})$ and $\mathcal{W}$ to $\tilde{\mathcal{W}}$.

We will show that for an arbitrary element $F \in J^r(\mathcal{U})$, the following equality holds:

$$\pi^r(\mathrm{Pr}^r_{\eta,F}) = \mathrm{Pr}^r_{\eta,\pi^r(F)}.$$

Take $G \in \mathrm{Pr}^r_{\eta,F}$; i.e., so that $j^{\perp(\eta,r)}(G) = j^{\perp(\eta,r)}(F)$. Since π^r maps $J^{\perp(\eta,r)}(\mathcal{U})$ to $J^{\perp(\eta,r)}(\mathcal{V})$ and $\mathcal{W}$ to $\tilde{\mathcal{W}}$, this implies that π^r maps F and G to the same principal subspace downstairs; i.e., $j^{\perp(\eta,r)}(\pi^r(F)) = j^{\perp(\eta,r)}(\pi^r(G))$. This shows that

$$\pi^r(\mathrm{Pr}^r_{\eta,F}) \subseteq \mathrm{Pr}^r_{\eta,\pi^r(F)}.$$

In order to conclude, let us check that the reverse inclusion $\pi^r(\mathrm{Pr}^r_{\eta,F}) \supseteq \mathrm{Pr}^r_{\eta,\pi^r(F)}$ also holds.

By (i), π^r becomes, upon choosing small enough neighborhoods, a trivial affine fibration $\pi^r\colon J^r(\mathcal{V}) \times \mathcal{L} \to J^r(\mathcal{V})$, where $\mathcal{L}$ denotes the fiber. We can thus write

$$F = (\pi^r(F), \ell) \in J^r(\mathcal{U}) = J^r(\mathcal{V}) \times \mathcal{L}, \quad \text{for some } \ell \in \mathcal{L}.$$

Take now an arbitrary element $\widehat{H} \in \mathrm{Pr}^r_{\eta,\pi^r(F)}$ and we will show that $\widehat{H} \in \pi^r(\mathrm{Pr}^r_{\eta,F})$. This will yield the required inclusion. To do so, define the element

$$H := (\widehat{H}, \ell) \in J^r(\mathcal{U}).$$

By construction, we have that

$$\widehat{H} = \pi^r(H) \quad \text{and} \quad j^{\perp(\eta,r)}(\widehat{H}) = j^{\perp(\eta,r)}(\pi^r(F)).$$

On the other hand, we have that $j^{\perp(\eta,r)}(H) = j^{\perp(\eta,r)}((\widehat{H}, \ell))$ clearly coincides with $j^{\perp(\eta,r)}(F) = j^{\perp(\eta,r)}((\pi^r(F), \ell))$. Indeed, this is the case since $F = (\pi^r(F), \ell)$ and $H = (\widehat{H}, \ell)$ only differ on their projections $\pi^r(F)$ and $\widehat{H}$, respectively, and precisely we have that $j^{\perp(\eta,r)}(\pi^r(F)) = j^{\perp(\eta,r)}(\widehat{H})$. This shows that $H \in \mathrm{Pr}^r_{\eta,F}$ and therefore $\widehat{H} \in \pi^r(\mathrm{Pr}^r_{\eta,F})$, yielding the inclusion $\pi^r(\mathrm{Pr}^r_{\eta,F}) \supseteq \mathrm{Pr}^r_{\eta,\pi^r(F)}$ and thus completing the proof. ■

Remark 6.3. Note that if we restrict the whole construction to lower jet spaces $J^i(Y) \subset J^r(Y)$, $J^i(X) \subset J^r(Y)$, $i < r$, the same result follows for i-jets. Indeed, we have the projected relations $\pi^r_i(\mathcal{S})$ and $\pi^r_i(\mathcal{R})$, and the map π^i is just the truncated part of π^r acting on i-jets. Therefore, the same proof applies verbatim in this context.

As a consequence of Lemma 6.2, we can establish the following lemmas.

Lemma 6.4. *Consider $F \in J^r(X)$ and a lift $\tilde{F} \in J^r(Y)$; i.e., so that $\pi^r(\tilde{F}) = F$. Take a principal codirection η. Then, for any $i \leq r$, ampleness of $\mathcal{S}^i_{\eta,\tilde{F}} \subset \mathrm{Pr}^i_{\eta,\tilde{F}}$ is equivalent to ampleness of $\mathcal{R}^i_{\eta,F} \subset \mathrm{Pr}^i_{\eta,F}$.*

Proof. By Lemma 6.2, π^r becomes an affine fibration when restricted to small enough open sets of $J^r(Y)$ that also maps principal subspaces upstairs $\mathrm{Pr}^r_{\eta,\tilde{F}}$ to principal subspaces downstairs $\mathrm{Pr}^r_{\eta,F}$ (and also $\mathcal{S}^r_{\eta,\tilde{F}}$ upstairs to $\mathcal{R}^r_{\eta,F}$ downstairs). Take now an arbitrary element $G \in \mathrm{Pr}^r_{\eta,F}$ and a lift $\tilde{G} \in \mathrm{Pr}^r_{\eta,\tilde{F}}$; i.e., so that $\pi^r(\tilde{G}) = G$. We then have that convex combinations within $\mathcal{S}^r_{\eta,\tilde{F}}$ upstairs averaging[2] $\tilde{G}$ translate, via π^r, to the corresponding convex combinations in $\mathcal{R}^r_{\eta,F}$ downstairs averaging G. Therefore ampleness of $\mathcal{S}^r_{\eta,\tilde{F}} \subset \mathrm{Pr}^r_{\eta,\tilde{F}}$ readily implies ampleness of $\mathcal{R}^r_{\eta,F} \subset \mathrm{Pr}^r_{\eta,F}$.

[2]We say that a convex combination $\sum_i \lambda_i p_i$ (where $\sum_i \lambda_i = 1$) averages F if $\sum_i \lambda_i p_i = F$.

Conversely, take now an arbitrary $\tilde{Z} \in \mathrm{Pr}^r_{\eta,\tilde{F}}$ and consider

$$Z := \pi^r(\tilde{Z}) \in \mathrm{Pr}^r_{\eta,F}.$$

By ampleness of $\mathcal{R}^r_{\eta,F} \subset \mathrm{Pr}^r_{\eta,F}$, we can find a convex combination of points in $\mathcal{R}^r_{\eta,F}$ averaging Z; i.e., $Z = \sum_j \lambda_j p_j$ where $\sum_j \lambda_j = 1$ and $p_j \in \mathcal{R}^r_{\eta,F}$. If we show that these $p_j \in \mathcal{R}^r_{\eta,F}$ can be lifted to $\mathcal{S}^r_{\eta,\tilde{F}}$ so that their lifts average $\tilde{Z}$ we will be done since this would yield ampleness of $\mathcal{S}^r_{\eta,\tilde{F}} \subset \mathrm{Pr}^r_{\eta,\tilde{F}}$.

By Lemma 6.2, π^r locally becomes, upon choosing small enough neighborhoods, a trivial affine fibration $\pi^r: J^r(\mathcal{V}) \times \mathcal{L} \to J^r(\mathcal{V})$, where $\mathcal{L}$ denotes the affine fiber. Let $J^r(\mathcal{V}) \times \{\ell\}$ be the leaf containing $\tilde{F}$; i.e., $\tilde{F} = (F, \ell)$. It is then clear that the convex combination $\sum_j \lambda_j \tilde{p}_j$ does the job, where $\tilde{p}_j := (p_j, \ell)$. Indeed, $\pi^r(\tilde{p}_j) = p_j$ and, since $\mathcal{S} \subset J^r(Y)$ is the pullback of $\mathcal{R} \subset J^r(X)$ (recall the framework we fixed at the beginning of Section 6), then $\tilde{p}_j \in \mathcal{S}$. The fact that they average $\tilde{Z}$ follows by construction.

Finally, we will also show that $\tilde{p}_j \in \mathrm{Pr}^r_{\eta,\tilde{F}}$ and thus $\tilde{p}_j \in \mathcal{S}^r_{\eta,\tilde{F}}$. Indeed, we can choose coordinates exactly as in the proof of Lemma 6.2 where, locally,

$$J^r(Y) = J^r(\mathbb{R}^n, \mathbb{R}^m), \quad J^r(X) = J^r(\mathbb{R}^n \times \mathbb{R}^{\tilde{m}}) \quad \text{and} \quad \eta = \ker(dt).$$

It is then clear that $j^{\perp(\eta,r)}(\tilde{p}_j) = j^{\perp(\eta,r)}(\tilde{F})$. This is the case because $\tilde{p}_j$ and $\tilde{F}$ only differ on their projections p_j and F, respectively, and, precisely, $j^{\perp(\eta,r)}(p_j) = j^{\perp(\eta,r)}(F)$ by construction.

This yields ampleness of $\mathcal{S}^r_{\eta,\tilde{F}} \subset \mathrm{Pr}^r_{\eta,\tilde{F}}$.

Arguing in the same fashion for $\mathcal{R}^i_{\eta,F}$ and $\mathrm{Pr}^i_{\eta,F}$ (recall the notation from Remark 2.5), for $i \leq r$, the claim follows as well (Remark 6.3). ∎

Let us state an obvious consequence of Lemma 6.4.

Lemma 6.5. *Let $F, G \in J^r(Y)$ so that $\pi^r(F) = \pi^r(G)$. Then, for $1 \leq i \leq r$, ampleness of $\mathcal{S}^i_{\eta,F} \subset \mathrm{Pr}^i_{\eta,F}$ is equivalent to ampleness of $\mathcal{S}^i_{\eta,G} \subset \mathrm{Pr}^i_{\eta,G}$ for any principal codirection η.*

We now state the ampleness-criterion.

Proposition 6.6 (Fibered ampleness-criterion). *Ampleness of $\mathcal{S} \subset J^r(Y)$ is equivalent to ampleness of $\mathcal{R} \subset J^r(X)$.*

Proof. If $\mathcal{S}$ is ample, then for any codirection η, any $i \leq r$ and any $F \in J^r(X)$, there exists a lift $\tilde{F} \in J^r(Y)$ (Lemma 6.1) so that $\pi^r(\tilde{F}) = F$ and $\mathcal{S}^i_{\eta,\tilde{F}} \subset \mathrm{Pr}^i_{\eta,\tilde{F}}$ is ample. Therefore, it follows by Lemma 6.4 that, for any codirection η, any $i \leq r$ and any $F \in J^r(X)$, $\mathcal{R}^i_{\eta,F} \subset \mathrm{Pr}^i_{\eta,F}$ is ample, yielding ampleness of $\mathcal{R}$.

Conversely, if $\mathcal{R}$ is ample, then for any codirection η, any $i \leq r$ and any $G \in J^r(Y)$, we have that $\mathcal{R}^i_{\eta,\pi^r(G)} \subset \mathrm{Pr}^i_{\eta,\pi^r(G)}$ is ample. Then, by Lemma 6.4, we have that for any codirection η, any $i \leq r$ and any $G \in J^r(Y)$ it follows that $\mathcal{R}^i_{\eta,G} \subset \mathrm{Pr}^i_{\eta,G}$ is ample. This yields ampleness of $\mathcal{S}$. ∎

We will apply the previous criterion for the case of a differential relation in the context of distributions on manifolds.

7. h-principle for distributions of maximal growth

The goal of this last section is to prove the following Theorem 7.1. Note that by Theorem 2.10, this will imply the main Theorem 1.3 in this article.

Theorem 7.1. *Let M be a smooth manifold of dimension n. The differential relation $\mathcal{R}^{\text{step-}r} \subset J^{r-1}(\mathrm{Gr}_k(TM))$ is ample for $k > 2$.*

Recall that $\mathcal{R}^{\text{step-}r}$ is defined in Definition 4.25 and $\mathcal{S}^{\text{step-}r}$ in Definition 4.24.

In Subsection 7.1, we apply the fibered criterion of ampleness to the context of formal maximal growth distributions and formally bracket-generating frames of step-r. This will constitute the first reduction step since it will allow us to reduce the problem of checking ampleness of $\mathcal{R}^{\text{step-}r}$ to checking ampleness of $\mathcal{S}^{\text{step-}r}$. On a second reduction step (Subsection 7.2), we show that we can reduce the study of ampleness to checking ampleness just along some particular principal directions called *non-normal*. In Subsections 7.3 and 7.4, we introduce and elaborate on some particular models where checking ampleness becomes somehow tractable. Finally, we carry out the argument for the proof of ampleness of $\mathcal{S}^{\text{step-}r}$ in Section 7.5.

7.1. Application of the fibered criterion to distributions

We will apply the fibered criterion in order to reduce ampleness of $\mathcal{R}^{\text{step-}r}$ to some easier to check condition on the bundle of k-frames. Recall (Section 4) that we have the surjective and submersive bundle map $\pi\colon \mathrm{Fr}_k(M) \to \mathrm{Gr}_k(TM)$ which maps each k-frame to the k-plane it spans and thus induces a map at the level of $(r-1)$-jets.

Following the notation from the beginning of Section 6 and Proposition 6.6, we make the choices $Y = \mathrm{Fr}_k(M)$ and $X = \mathrm{Gr}_k(TM)$, and the corresponding differential relations

$$\mathcal{S}^{\text{step-}r} \subset J^{r-1}(\mathrm{Fr}_k(M)) \quad \text{and} \quad \mathcal{R}^{\text{step-}r} \subset J^{r-1}(\mathrm{Gr}_k(TM)).$$

Note also that Lemma 4.28 together with Definition 4.24 and Definition 4.25 imply that $\mathcal{S}^{\text{step-}r}$ is the pullback by π^{r-1} of $\mathcal{R}^{\text{step-}r}$; i.e.,

$$\mathcal{S}^{\text{step-}r} = (\pi^{r-1})^*(\mathcal{R}^{\text{step-}r}).$$

By Proposition 6.6, ampleness of $\mathcal{R}^{\text{step-}r}$ is equivalent to ampleness of $\mathcal{S}^{\text{step-}r}$. We will prove that $\mathcal{S}^{\text{step-}r}$ is ample. This will, thus, yield ampleness of $\mathcal{R}^{\text{step-}r}$ as a consequence.

7.2. Dealing with normal principal directions

The goal of this step is to check that ampleness of $\mathcal{S}^{\text{step-}r}$ with respect to certain directions called "normal" follows trivially. This will imply that, after this step, we can reduce the study of ampleness of $\mathcal{S}^{\text{step-}r}$ to non-normal principal directions. This will be encoded by Lemma 7.5.

Fix a formal solution $F = (F_i)_{i=1}^k \in \mathcal{S}^{\text{step-}r} \subset J^{r-1}(\mathrm{Fr}_k(M))$ and write

$$\mathcal{D}_F = \langle j^0(F_1), \ldots, j^0(F_k) \rangle$$

(we often drop the subindex and just write $\mathcal{D}$ if it is clear from the context).

Remark 7.2. Since convex integration works locally, i.e., it is implemented chart by chart, we can assume for the rest of the section that $M = \mathbb{R}^n$. Up to changes of coordinates, when checking ampleness with respect to a certain direction $X_i(p) \in T_p\mathbb{R}^n$ over a point $p \in \mathbb{R}^n$, we can assume without loss of generality (recall Lemma 4.30) that this direction coincides with the first coordinate direction; i.e., $X_i(p) = \partial_t$.

Similarly, we will consider the splitting $\mathbb{R}^n = \mathbb{R}_t \times \mathbb{R}^{n-1}$ which induces the splitting of F in equation (2.1). So, if it is not explicitly stated otherwise, whenever we write ∂_t we are referring to ∂_1. Nonetheless, we will consistently use the letter t for didactical reasons along the exposition.

Whenever we talk about metric properties such as orthogonality, we will be referring to the Euclidean metric of $\mathbb{R}^n$. For the ease of notation, we will denote $\mathcal{S}$ for $\mathcal{S}^{\text{step-}r}$ along the rest of the subsection.

Definition 7.3. A locally defined non-vanishing direction/vector field $X_i \colon \mathcal{U} \subset \mathbb{R}^n \to T\mathbb{R}^n$ is said to be *normal* at a point $p \in \mathbb{R}^n$ with respect to F if it is normal to the distribution $\mathcal{D}_F$ with respect to the Euclidean metric of $\mathbb{R}^n$ over p; i.e., $X_i(p) \perp \mathcal{D}_F(p)$. Otherwise, we say that it is *non-normal*.

7.2.1. Normal directions and trivial ampleness. Before proving the aforementioned Lemma 7.5, we first state a more general result (Lemma 7.4) which will not only imply Lemma 7.5 but which will also be useful later on in this article.

Consider the first coordinate direction $\partial_t \in T_p\mathbb{R}^n$ over $p \in \mathbb{R}^n$ and a bracket expression $A_{(a_1,\dots,a_\mathrm{r})}(\cdot)$ of length $\mathrm{r} \leq r$. Next lemma states that if the expression $A_{(a_1,\dots,a_\mathrm{r})}(F)$ involves two components of F with j^0-components orthogonal to ∂_t, then the whole expression does not depend on $j_t^{\mathrm{r}-1}(F)$.

Lemma 7.4. *Consider $F = (F_i)_{i=1}^k \in J^{r-1}(\mathrm{Fr}_k(\mathbb{R}^n))$ over $p \in \mathbb{R}^n$. If a formal length-r bracket $A_{(a_1,\dots,a_\mathrm{r})}(F)$ involves at least two components F_{a_i} and F_{a_j} $(i, j \leq \mathrm{r})$ from F satisfying*

$$j^0(F_{a_i}), j^0(F_{a_j}) \perp \partial_t, \tag{7.1}$$

then it does not depend on $j_t^{\mathrm{r}-1}(F)$. More precisely, there exists a map h such that for all $F \in J^{r-1}(\mathrm{Fr}_k(\mathbb{R}^n))$ whose $j^0(F_{a_i})$ and $j^0(F_{a_j})$ components satisfy condition (7.1), the expression $A_{(a_1,\dots,a_\mathrm{r})}$ factors through h as follows:

$$A_{(a_1,\dots,a_\mathrm{r})}(F) = h \circ j^{\perp(\partial_t,\mathrm{r}-1)}(F).$$

Proof. Since $\partial_t \perp j^0(F_{a_i}), j^0(F_{a_j})$, then these two vectors do not have ∂_t-component when written in coordinates:

$$j^0(F_{a_i}) = 0 \cdot \partial_t + u_2^{a_i} \cdot \partial_2 + \dots + u_n^{a_i} \cdot \partial_n,$$
$$j^0(F_{a_j}) = 0 \cdot \partial_t + u_2^{a_j} \cdot \partial_2 + \dots + u_n^{a_j} \cdot \partial_n.$$

Express the bracket $A_{(a_1,\dots,a_r)}(F)$ in simplified form (Lemma 4.16). The maximal number m of iterated derivations of the form $D_t \circ \dots \circ D_t(u_{\ell_i}^q)$ that could potentially appear in the expression is at most $\mathrm{r} - 2$ (one per each contribution of the ∂_t-component of each $j^0(F_{a_m})$, $a_m \neq a_i, a_j$). This means that such expression can be expressed purely in terms of jet-coordinates $u_{(i,J)}$ with $J \in \mathcal{S}_{\mathrm{r}-1}$ and $J \neq (t,\dots,t) \in \mathbb{S}_{\mathrm{r}-1}$.

We thus conclude (Lemma 4.6) that $A_{(a_1,\dots,a_\mathrm{r})}(\tilde{F})$ does not depend on $j_t^{\mathrm{r}-1}(F)$ or, equivalently, there is a factorization through some map h as follows: $A_{(a_1,\dots,a_\mathrm{r})}(\cdot) = h \circ j^{\perp(\partial_t,\mathrm{r}-1)}(\cdot)$ for such jets satisfying condition (7.1). ∎

The following lemma shows that any change to the pure order-(r − 1) information with respect to a direction ∂_t perpendicular to $\mathcal{D}$ does not contribute at all to changing the length-r brackets of elements in $\mathcal{D}$ (and, so, neither the bracket-generating condition defined based on them).

Lemma 7.5. *For jets $F = (F_i)_{i=1}^k \in J^{r-1}(\mathrm{Fr}_k(\mathbb{R}^n))$ over $p \in \mathbb{R}^n$ for which $\partial_t \perp \mathcal{D}_F$, length-r brackets $A_{(a_1,\dots,a_\mathrm{r})}(F)$ do not depend on $j_t^{\mathrm{r}-1}(F)$. Equivalently, there exists a map h so that for such jets the bracket factors as*

$$A_{(a_1,\dots,a_\mathrm{r})}(F) = h \circ j^{\perp(\partial_t,\mathrm{r}-1)}(F).$$

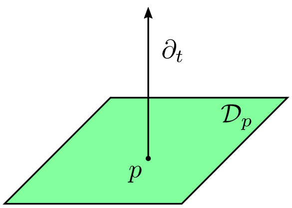

Figure 3. When $\partial_t \perp \mathcal{D}_F$ at a point $p \in \mathbb{R}^n$, any formal bracket of length r involving components of F does not depend on the pure (r − 1)-order information in the direction of ∂_t (Lemma 7.5). Therefore, ampleness of $\mathcal{S}^i_{\partial_t,F}$ within $\mathrm{Pr}^i_{\partial_t,F}$ holds trivially for all $1 \le i \le r-1$ in this case (Corollary 7.7).

Proof. Since $\partial_t \perp \mathcal{D}_p$, no $j^0(F_a)$ has ∂_t-component; i.e., for any $a \in \{1,\dots,k\}$ we can write

$$j^0(F_a) = \sum_{i=1}^n a_i \partial_i = 0 \cdot \partial_t + a_2 \cdot \partial_2 + \cdots + a_n \cdot \partial_n.$$

The result then readily follows by Lemma 7.4. ∎

Remark 7.6. Note that we did not use the fact that $k > 2$ in the previous lemmas; i.e., they apply to the rank-2 case as well.

Corollary 7.7. *If $\partial_t \perp \mathcal{D}_F$, then $\mathcal{S}^i_{\partial_t,F}$ is trivially ample in $\mathrm{Pr}^i_{\partial_t,F}$ for $1 \le i \le r-1$.*

Proof. If $F \in \mathcal{S}^{\text{step-}r}$, then $\dim(\langle \mathfrak{Br}^j \rangle) = \mathfrak{n}_j$ for all $1 \le j \le r$, where $(\mathfrak{n}_j)_{j=1}^r$ is a maximal growth vector (by Definition 4.24). Take any $i = 1,\dots,r-1$. Any other $\tilde{F} \in \mathrm{Pr}^i_{\partial_t,F}$ only differs from F in the pure order-i formal partial derivatives with respect to ∂_t (equation 2.3) and, therefore, by Lemma 7.5, it has the same associated sets $\mathfrak{Br}^j$ for $1 \le j \le r$

(thus satisfying the formally maximal growth condition as well). We conclude thus that any $\tilde{F} \in \mathrm{Pr}^i_{\partial_t,F}$ is a formal solution of $\mathcal{S}^{\text{step-}r}$, yielding trivial ampleness of $\mathcal{S}^i_{\partial_t,F}$ within $\mathrm{Pr}^i_{\partial_t,F}$ for $1 \leq i \leq r-1$. ■

7.3. Adapted frames with respect to non-normal directions

The goal of this subsection is to introduce some local frames associated to a given distribution and a non-normal direction where ampleness of $\mathcal{S}^{\text{step-}r}$ can be easily checked.

Definition 7.8. Let $\mathcal{D}$ be a distribution on $\mathbb{R}^n$. We say that a frame $\mathfrak{Fr} = \{X_1, \dots, X_k\}$ of $\mathcal{D}$ is *adapted to* the direction ∂_t if the following conditions hold pointwise:

(i) X_1 is the orthogonal projection of ∂_t onto the distribution $\mathcal{D}$.

(ii) For $i \geq 2$, the vector fields X_i are orthogonal to X_1; i.e., they lie in $\langle X_1 \rangle^\perp \cap \mathcal{D}$.

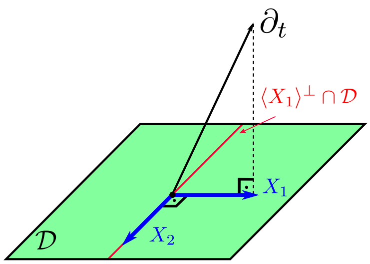

Figure 4. Example of a rank-2 distribution $\mathcal{D}$ and a frame $\mathfrak{Fr} = \{X_1, X_2\}$ (in blue) adapted to the non-normal direction ∂_t: note that X_1 is the normal projection of ∂_t onto $\mathcal{D}$ by the Euclidean metric of $\mathbb{R}^3$ and X_2 lies in $\langle X_1 \rangle^\perp \cap \mathcal{D}$ (in red).

Remark 7.9. Conditions (i) and (ii) imply, in particular, that $X_i \perp \partial_t$ for $i \geq 2$. See Figure 4.

The definition above naturally extends to the context of $(r-1)$-jets.

Definition 7.10. Take $F = (F_i)_{i=1}^k \in J^{r-1}(\mathrm{Fr}_k(\mathbb{R}^n))$. We say that F at $p \in \mathbb{R}^n$ is *adapted to* ∂_t if the frame $\{j^0(F_1), \dots, j^0(F_k)\}$ of $\mathcal{D} := \langle j^0(F_1), \dots, j^0(F_k) \rangle$ is adapted to ∂_t at p.

Since we are working in coordinates where ∂_t denotes the first coordinate direction of $\mathbb{R}^n$ (recall Remark 7.2), the definition above implies the elements $j^0(F_j)$ to admit expressions of the form:

$$j^0(F_1) = 1 \cdot \partial_t + \sum_{j=2}^n u_1^j \partial_j, \quad u_j^1 \in \mathbb{R}, \tag{7.2}$$

$$j^0(F_i) = 0 \cdot \partial_t + \sum_{j=2}^n u_i^j \partial_j, \quad \text{for } i \geq 2,\ u_i^j \in \mathbb{R}. \tag{7.3}$$

Lemma 7.11 (Existence of ∂_t-adapted frames). *Consider a distribution $\mathcal{D}$ and a point $p \in \mathbb{R}^n$. If the integral field $X_t \subset T\mathbb{R}^n$ of the coordinate direction ∂_t is non-normal with respect to $\mathcal{D}$, then there exists a local frame of $\mathcal{D}$ which is adapted to X_t in a small neighborhood $\mathcal{O}p(p)$.*

Proof. Since non-normality is an open condition, there exists, on a small neighborhood $\mathcal{O}p(p)$ of p, a non-zero smooth projection X_1 of X_t onto $\mathcal{D}$. We can always locally complete X_1 to a frame $\mathfrak{Fr} = \{X_1, \dots, X_k\}$ of $\mathcal{D}$ adapted to ∂_t by considering a frame of $\langle X_1 \rangle^\perp$ inside $\mathcal{D}$. This proves the local existence of adapted frames. ■

Lemma 7.11 readily implies, in terms of jets, the following lemma.

Lemma 7.12 (Existence of ∂_t-adapted jets). *Consider $\tilde{F} \in J^{r-1}(\mathrm{Gr}_k(T\mathbb{R}^n))$ over a point $p \in \mathbb{R}^n$ such that the direction $\partial_t \in T_p\mathbb{R}^n$ is not normal to $\mathcal{D}_{\tilde{F}} = j^0(\tilde{F})$. Then there always exists an element $F = (F_i)_{i=1}^k \in J^{r-1}(\mathrm{Fr}_k(\mathbb{R}^n))$ projecting to $\tilde{F}$ by π^r (recall Section 4) which is adapted to ∂_t.*

7.4. Principal subspaces and adapted frames

We will give in this subsection a description of principal subspaces and their intersection with $\mathcal{S}^{\text{step-}r}$ in terms of adapted frames. We will introduce some notation first.

Consider some jet $\tilde{F} \in J^{r-1}(\mathrm{Gr}_k(T\mathbb{R}^n))$. Take $F = (F_1, \dots, F_k) \in J^{r-1}(\mathrm{Fr}_k(\mathbb{R}^n))$ adapted to ∂_t and so that $\pi^{r-1}(F) = \tilde{F}$ (which exists by Lemma 7.12).

Consider
$$\mathfrak{Br}^1 := \{j^0(F_1), j^0(F_2), \dots, j^0(F_k)\},$$
so that $\mathcal{D}_F = \langle j^0(F_1), j^0(F_2), \dots, j^0(F_k)\rangle$.

Let us split the set of indices
$$\mathfrak{I}_i = \{(\ell_1, \dots, \ell_j) : j \leq i \text{ and } \ell_1, \dots, \ell_j \in \{1, \dots, k\}\} \cup \{O\}$$
(recall Definition 3.7) into
$$\mathfrak{I}_i = \mathfrak{I}_i^t \sqcup \tilde{\mathfrak{I}}_i, \tag{7.4}$$
where $\tilde{\mathfrak{I}}_i$ denotes the set of multi-indices of length less than or equal to i where the index 1 does not appear $i-1$ times. More precisely, denote by Σ_i the symmetric group of degree i. Then
$$\mathfrak{I}_i^t := \left\{(\ell_1, \dots, \ell_i) : \exists \sigma \in \Sigma_i, \exists m \leq k,\ \sigma(\ell_1, \dots, \ell_i) = (\underbrace{1, \dots, 1}_{i-1}, m)\right\} \quad \text{and} \quad \tilde{\mathfrak{I}}_i := \mathfrak{I}_i \setminus \mathfrak{I}_i^t.$$

Remark 7.13. The case $j = 0$ in the definition of the set $\tilde{\mathfrak{I}}_i$ corresponds to the empty subindex $O = ()$ which also belongs to the set.

Splitting (7.4) induces a splitting in $\mathfrak{Br}^i$ (recall Definition 4.18 and Remark 4.20) as
$$\mathfrak{Br}^i = \mathfrak{Br}^i_{\perp,t} \sqcup \mathfrak{Br}^i_t,$$

where

$$\mathfrak{Br}^i_{\perp,t} = \{A_J(F) : J \in \tilde{\mathfrak{I}}_i\} \quad \text{and} \quad \mathfrak{Br}^i_t = \{A_J(F) : J \in \mathfrak{I}^t_i\}. \tag{7.5}$$

We then (pointwise) define the vector spaces $\mathcal{D}^i_\perp := \langle \mathfrak{Br}^i_{\perp,t} \rangle$ and $\mathcal{D}^i_t := \langle \mathfrak{Br}^i_t \rangle$, where

$$\mathcal{D}_i = \mathcal{D}^i_\perp + \mathcal{D}^i_t. \tag{7.6}$$

Remark 7.14. The notation chosen for denoting the planes $\mathcal{D}^i_\perp$ and $\mathcal{D}^i_t$ will be justified in the context of adapted $(r-1)$-jets by Remark 7.22. We may write $\mathcal{D}^i_\perp(F)$ and $\mathcal{D}^i_t(F)$ whenever we want to make it explicit that these objects are associated to a specific jet $F = (F_1, \dots, F_k) \in J^{r-1}(\mathrm{Fr}_k(\mathbb{R}^n))$.

Remark 7.15. Note that the sum in equation (7.6) is not necessarily a direct sum in general. Lemma 7.17 will show that it actually is under additional hypothesis.

Denote by $\mathfrak{h}^i_m$ and $\mathfrak{p}^i_m$ the length-i multi-indices of the form $\mathfrak{h}^i_m := (t, \dots, t, m) \in \mathcal{I}^t_i$ and $\mathfrak{p}^i_m = (t, \dots, t, m, t) \in \mathcal{I}^t_i$, respectively.

Remark 7.16. Note that, by Lemma 4.13, the only length-i brackets in $\mathfrak{Br}^i_t(F)$ which are potentially non-zero are the ones of the form $A_{\mathfrak{h}^i_m}(F)$ and $A_{\mathfrak{p}^i_m}(F)$, $m \neq t$. Moreover, Lemma 4.13 also implies that the following equality holds:

$$A_{\mathfrak{h}^i_m}(F) = -A_{\mathfrak{p}^i_m}(F).$$

Thus, brackets $A_{\mathfrak{h}^i_m}(F)$, $m \geq 2$, are generators of $\mathcal{D}^i_t(F)$.

Lemma 7.17. *If $F \in \mathcal{S}^{\text{step-}r}$ and $i < r$, then*

(i) $\mathcal{D}_i(F) = \mathcal{D}^i_\perp(F) \oplus \mathcal{D}^i_t(F)$ *and, moreover,*

(ii) $\dim(\mathcal{D}^i_t(F)) = k - 1$.

Proof. First note that, for $i < r$, the elements $\mathcal{D}_i(F)/\mathcal{D}_{i-1}(F)$ have the same dimensions as the subspaces $\mathfrak{Lie}^i_n$ in the free Lie algebra $\mathfrak{Lie}_n$ (by Definition 4.24 together with Lemma 3.15). This means that the only relations between different length-i brackets are Jacobi identity and the antisymmetric property. By Lemma 3.21 and Remark 3.22, the $k-1$ elements $A_{\mathfrak{h}^i_m}(F)$ $(m = 2, \dots, k)$ cannot be expressed as combinations of elements from $\mathfrak{Br}^i_\perp$, yielding (i). Additionally, they conform a basis of $\mathfrak{Br}^i_t$, since they are linearly independent (by Lemma 3.21 and Remark 3.22) and they generate $\mathfrak{Br}^i_t$ (by Remark 7.16). This yields (ii). ■

The following result is just a rephrasing of the notion of (formal) growth vector from Definition 4.19 in terms of equation (7.6). We will phrase it as a lemma.

Lemma 7.18. *The condition for $\mathcal{D}$ having growth vector $\nu_{\mathcal{D}} = (\mathfrak{n}_i)^r_{i=1}$ is equivalent to:*

$$\text{for each } i \geq 1, \quad \dim(\mathcal{D}^i_\perp + \mathcal{D}^i_t) = \mathfrak{n}_i, \tag{7.7}$$

or, equivalently,

$$\text{for each } i \geq 1, \quad \dim(\mathcal{D}^i_t / \mathcal{D}^i_\perp) = \mathfrak{n}_i - \dim(\mathcal{D}^i_\perp). \tag{7.8}$$

Lemma 7.19. *Let $F \in \mathcal{S}^{\text{step-}r}$ and $m_i := \dim(\mathcal{D}^i_\perp(F))$. The following statements hold:*

(i) *if $i = r$, then $\mathfrak{n}_i = n$,*

(ii) *if $i < r$, then $m_i + k - 1 = \mathfrak{n}_i < n$.*

Proof. The statement (i) just follows from F being formally bracket generating of step-r. The first equality in (ii) follows from (7.7) together with Lemma 7.17, whereas the inequality $\mathfrak{n}_i < n$ follows from the fact that we are not in the last step; i.e., $\mathcal{D}_i \neq \mathcal{D}_r = T\mathbb{R}^n$. ■

7.4.1. Principal subspaces and adapted frames. The following two lemmas state that the only brackets in $\mathfrak{Br}^{\mathfrak{m}}$ that depend on $j_t^{\mathfrak{m}-1}(F)$ are the ones in $\mathfrak{Br}_t^{\mathfrak{m}}$.

Lemma 7.20. *Take a ∂_t-adapted jet $F = (F_i)_{i=1}^k \in J^{r-1}(\mathrm{Fr}_k(\mathbb{R}^n))$. Consider a length $\mathfrak{m} \geq 2$ formal bracket of the form $A_J(F)$, $J \in \tilde{\mathfrak{I}}_{\mathfrak{m}}$ (recall equation (7.5)). Then $A_J(F)$ does not depend on $j_t^{\mathfrak{m}-1}(F)$; i.e., the bracket expression factors, for adapted $(r-1)$-jets, through some map h as follows:*

$$A_J(F) = h \circ j^{\perp(\partial_t, \mathfrak{m}-1)}(F).$$

Proof. Let $J = (\ell_{\mathfrak{m}}, \dots, \ell_1)$. Since $J \in \tilde{\mathfrak{I}}_{\mathfrak{m}}$, then the expression

$$A_J(F) = [F_{\ell_{\mathfrak{m}}}, [\dots[F_{\ell_3}, [F_{\ell_2}, F_{\ell_1}]], \dots]$$

belongs to $\mathfrak{Br}^{\mathfrak{m}}_{\perp,t}$ and it involves at least two components F_{ℓ_a}, F_{ℓ_b} different from F_1. Now, since F is ∂_t-adapted, then $j^0(F_{\ell_a})$, $j^0(F_{\ell_b}) \perp \partial_t$, and the claim follows from Lemma 7.4. ■

By Remark 7.16, brackets of the form $A_I(F)$ with $I = (t, \dots, t, j) \in \mathfrak{I}_{\mathfrak{i}}^t$ constitute a set of generators of the whole $\mathcal{D}_t^i(F)$. We thus focus our study on those jets in the following lemma.

Lemma 7.21. *Let $I = (t, \dots, t, j) \in \mathfrak{I}_{\mathfrak{m}}^t$, where $\mathfrak{m} \geq 2$, $j \neq t$. Then the bracket $A_I(\cdot)$ admits the following decomposition for ∂_t-adapted jets $F = (F_i)_{i=1}^k \in J^{r-1}(\mathrm{Fr}_k(\mathbb{R}^n))$:*

$$A_I(F) = P_t^{\mathfrak{m}-1}(F_j) + \nu_j(F_j)$$

(recall Definition 4.7), where $\nu_j(F_j)$ is an expression not depending on $j_t^{\mathfrak{m}-1}(F)$ or, more precisely: there is a map h_j such that for ∂_t-adapted jets F,

$$\nu_j(F_j) = h_j(j^{\perp(\partial_t, \mathfrak{m}-1)}(F_j)).$$

Proof. Since F is ∂_t-adapted, recall from (7.2) and (7.3) that we can write, in jet-coordinates (introduced in Subsection 4.1),

$$F_t = (1, u_t^2, \dots, u_t^n, \dots, u_{t,J}^\ell, \dots), \quad 1 \leq \ell \leq n, \ J \in \mathbb{S}_{r-1}, \tag{7.9}$$

$$F_j = (0, u_j^2, \dots, u_j^n, \dots, u_{j,J}^\ell, \dots), \quad 1 \leq \ell \leq n, \ J \in \mathbb{S}_{r-1}, \tag{7.10}$$

where $F_t = F_1$ (recall Remark 7.2). We expand the bracket $A_I(F)$ by writing it in simplified form (Definition 4.14) following Lemma 4.8 and we isolate the terms involving order-$(\mathfrak{m}-1)$ derivatives with respect to ∂_t from the rest of the expression as follows:

$$\begin{aligned} A_I(F) &= [F_t, [F_t, \dots, [F_t, F_j] \dots]] \\ &= \sum_{i=1}^{n} (u_t^t)^{\mathfrak{m}-1} \cdot \underbrace{D_t \circ \cdots \circ D_t}_{\mathfrak{m}-1}(u_j^i) \cdot \partial_i + \sum_{i=1}^{n} \nu_j^i \cdot \partial_i . \end{aligned} \tag{7.11}$$

Note that each term $\left(u_t^t\right)^{\mathfrak{m}-1} \cdot D_{(t,\dots,t)}(u_j^i)$ on the right-hand side comes from the contribution of the $\mathfrak{m}-1$ u_t^1-entries of each F_t together with the entry u_j^i from F_j. Additionally, note from (7.9) that $u_t^1 = 1$.

Since F_j does not have u_j^t-component ($u_j^t = 0$ by (7.10)), there are no more order-$(\mathfrak{m}-1)$ derivations with respect to ∂_t in the expression. So, $\sum_{i=1}^{n} \nu_j^i$ is just a polynomial expression not containing a single term involving order-$(\mathfrak{m}-1)$ derivatives $D_t \circ \cdots \circ D_t$.

By noting that

$$\sum_{i=1}^{n} D_t \circ \cdots \circ D_t(u_j^i) \cdot \partial_i = \sum_{i=1}^{n} u^j_{i,(t,\dots,t)}$$

and by denoting $\nu_j := \sum_{i=1}^{n} \nu_j^i$, we conclude that $A_I(F) = P_t^{\mathfrak{m}-1}(F_j) + \nu_j$, where ν_j does not depend on $j_t^{\mathfrak{m}-1}(F)$ or, equivalently, is of the form $\nu_j = h_j(j^{\perp(\partial_t,\mathfrak{m}-1)}(F_j))$ for some map h_j. ∎

Remark 7.22. Lemma 7.20 implies that $\mathcal{D}_\perp^i(F)$ only depends on $j^{\perp(\partial_t,i-1)}(F_j)$ whereas, by Lemma 7.21 together with Remark 7.16, $\mathcal{D}_t^i$ can be expressed as

$$\mathcal{D}_t^i(F) = \langle P_t^{i-1}(F_2) + \nu_2, \dots, P_t^{i-1}(F_k) + \nu_k \rangle, \tag{7.12}$$

where $\nu_j = h_j(j^{\perp(\partial_t,i-1)}(F_j))$ for some map h_j.

Recall (Lemma 4.6 and Lemma 4.7) that $P_t^i(\cdot)$ just encodes the formal order-i pure derivative in the ∂_t-direction. We can thus notice that a principal subspace $\mathrm{Pr}^i_{\partial_t,F}$ satisfies the following condition in terms of $\mathcal{D}_\perp^{i+1}$ and $\mathcal{D}_t^{i+1}$:

$$\mathcal{D}_{i+1}(G) = \overbrace{\mathcal{D}_\perp^{i+1}(F)}^{\mathcal{D}_\perp^{i+1}(G)} + \overbrace{\langle w_1 + \nu_2, \dots, w_{k-1} + \nu_k \rangle}^{\mathcal{D}_t^{i+1}(G)}, \quad w_i \in \mathbb{R}^n, \tag{7.13}$$

where $\nu_j = h_j(j^{\perp(\partial_t,i-1)}(F_j))$ for some map h_j. Blue has been used to denote the variables w_i that parametrize the space and black is reserved for the fixed variables $\mathcal{D}_\perp^{i+1}(F)$ and ν_j determined by F.

7.5. Ampleness of $\mathcal{S}^{\text{step-}r}$

This subsection is devoted to the proof of ampleness of $\mathcal{S}^{\text{step-}r}$ for distributions of rank > 2 on manifolds of dimension $n \geq 3$. By the fibered ampleness criterion (Proposition 6.6), this will imply Theorem 7.1 and, ultimately, Theorem 1.3.

We will first state some lemmas about ampleness of subspaces of matrices that will be useful later in the proof of the theorem. We isolate them here for clarity and because they are interesting on their own.

Let us fix some notation. Consider the space of $(\ell \times q)$-matrices with first k columns fixed for some choice of $k < q$ vectors in $\mathbb{R}^\ell$ and denote it by $\mathcal{A}^k_{\ell\times q}$. Denote by

$$\overline{\mathcal{A}^k_{\ell\times q}} \subset \mathcal{A}^k_{\ell\times q}$$

the subset of maximal rank matrices in $\mathcal{A}^k_{\ell\times q}$. We will study ampleness of such subspace depending on the values of k, ℓ and q.

Let us begin studying the case $\ell = q$:

Lemma 7.23. *If $q - k \geq 2$, then the space of maximal rank matrices with first k columns fixed $\overline{\mathcal{A}^k_{q\times q}}$ is ample within the space of all matrices with identical k columns fixed $\mathcal{A}^k_{q\times q}$ for any choice of fixed columns.*

Proof. Regard the columns of a matrix $A \in \mathcal{A}^k_{q\times q}$ as vectors in $\mathbb{R}^q$. If the first fixed k columns are not k linearly independent vectors, then the space $\overline{\mathcal{A}^k_{q\times q}}$ is empty and trivially ample.

Otherwise, we can assume, up to a linear isomorphism, that the first k columns correspond, respectively, to the first k vectors in the canonical basis of $\mathbb{R}^q$. There is no loss of generality in this assumption since convex combinations of vectors are preserved by linear isomorphisms.

Consider now the projection map $\tilde{\pi}\colon \mathcal{M}_{q\times q} \to \mathcal{M}_{q-k\times q-k}$ that projects any matrix $A_{q\times q}$ to its submatrix $M_{q-k\times q-k}$ resulting from removing its first k columns and rows. This map can be thought as a quotient by the subspace spanned by the first k column vectors. Ampleness now follows by ampleness of $\mathrm{GL}(q-k)$ (Lemma 2.7). ■

We now study separately the case $\ell \neq q$. We will also assume in the statement that the first k columns are linearly independent since, otherwise, ampleness follows trivially.

Lemma 7.24. *Let $k < \ell < q$. Then the space $\overline{\mathcal{A}^k_{\ell\times q}}$ of maximal rank matrices with first k columns linearly independent and fixed is the complement of a thin singularity within the space of matrices with identical k columns fixed $\mathcal{A}^k_{q\times q}$ and is, thus, ample.*

Proof. The result follows from the fact that the space of non-maximal rank $(\ell \times q)$-matrices has codimension greater than 1 (as a stratified subset) within the space of $(\ell \times q)$-matrices (see, for example, Subsection 2.2.1 of [7]). Therefore, the claim follows by Lemma 2.11. ■

Note that if the first fixed k columns constitute a maximal rank submatrix themselves, then ampleness follows trivially. This is encoded by the following lemma, whose proof is trivial.

Lemma 7.25. *Let $k = \ell < q$. The space $\overline{\mathcal{A}^k_{\ell\times q}}$ of maximal rank matrices with first k columns linearly independent and fixed coincides with the whole space of matrices with identical k columns fixed $\mathcal{A}^k_{q\times q}$ and is, thus, trivially ample.*

The case $\ell = q$ and $k = q - 1$ is left for the last section of the paper (see Lemma 8.1).

Let us finally prove ampleness of $\mathcal{S}^{\text{step-}r} \subset J^{r-1}(\mathrm{Fr}_k(M))$ for $k > 2$.

Theorem 7.26. $\mathcal{S}^{\text{step-}r} \subset J^{r-1}(\mathrm{Fr}_k(M))$ *is ample if* $k > 2$.

Proof. Since convex integration works locally and is implemented chart by chart, we assume $M = \mathbb{R}^n$. Consider a point $p \in \mathbb{R}^n$ and take $Y = (Y_1, \dots, Y_k) \in J^{r-1}(\mathrm{Fr}_k(\mathbb{R}^n))$ over p. Take a principal codirection $\eta \in T_p^*\mathbb{R}^n$ which, by the duality given by the Euclidean metric, defines a principal direction ∂_t (recall Remark 2.3). We will check ampleness with respect to ∂_t.

By Subsection 7.2, in order to check ampleness, we can assume that ∂_t is non-normal to $\mathcal{D} = \langle j^0(Y_1), \dots, j^0(Y_k)\rangle$. Choose another $F = (F_1, \dots, F_k) \in J^{r-1}(\mathrm{Fr}_k(M))$ such that $\pi^{r-1}(F) = \pi^{r-1}(Y)$ and so that it is an adapted $(r-1)$-jet with respect to the direction ∂_t (Lemma 7.12); that is,

(i) $\pi^{r-1}(F) = \pi^{r-1}(Y)$; i.e., they define the same element in $J^{r-1}(\mathrm{Gr}_k(TM))$, and
(ii) $j^0(F_1)$ is the orthogonal projection of ∂_t to $\mathcal{D}$ with respect to the Euclidean metric of $\mathbb{R}^n$.

For the ease of notation, we still denote by $\mathcal{S}^{\text{step-}r}$ the projected relation $\rho^{r-1}_{i-1}(\mathcal{S}^{\text{step-}r})$. Recall that, by Lemma 6.5, $\mathcal{S}^{\text{step-}r}_{\partial_t,Y} \subset \mathrm{Pr}^{i-1}_{\partial_t,Y}$ is ample if and only if $\mathcal{S}^{\text{step-}r}_{\partial_t,F} \subset \mathrm{Pr}^{i-1}_{\partial_t,F}$ is ample. So, we will check ampleness of the latter.

Consider the subspaces $\mathcal{D}^i_\perp(F)$ and $\mathcal{D}^i_t(F)$ associated to F. By equation (7.13), we can express the principal subspace as

(7.14)
$$\mathrm{Pr}^{i-1}_{\partial_t,F} = \left\{ G \in J^{i-1}(\mathrm{Fr}_k(\mathbb{R}^n)) \;\middle|\; \begin{array}{l} j^{\perp(\partial_t,i-1)}(G) = j^{\perp(\partial_t,i-1)}(F), \\ \mathcal{D}_i(G) = \overbrace{\mathcal{D}^i_\perp(F)}^{\mathcal{D}^i_\perp(G)} + \overbrace{\langle w_1 + \nu_2, \dots, w_{k-1} + \nu_k\rangle}^{\mathcal{D}^i_t(G)}, \\ w_i \in \mathbb{R}^n \end{array} \right\}.$$

Combining (7.14) with (7.8), the intersection $\mathcal{S}^{i-1}_{\partial_t,F} := \mathrm{Pr}^{i-1}_{\partial_t,F} \cap \mathcal{S}^{\text{step-}r}$ of the principal subspace with $\mathcal{S}^{\text{step-}r}$ reads as

$$\mathcal{S}^{i-1}_{\partial_t,F} = \left\{ G \in J^{i-1}(\mathrm{Fr}_k(\mathbb{R}^n)) \;\middle|\; \begin{array}{l} j^{\perp(\partial_t,i-1)}(G) = j^{\perp(\partial_t,i-1)}(F), \\ \dim\Big(\overbrace{\langle w_1 + \nu_2, \dots, w_{k-1} + \nu_k\rangle}^{w_i \text{ are free, } \nu_i \text{ are frozen.}} / \overbrace{\mathcal{D}^i_\perp(F)}^{\text{frozen data}}\Big) \\ \quad = \overbrace{\mathfrak{n}_i - \dim(\mathcal{D}^i_\perp(F))}^{\text{frozen data}}, \quad w_i \in \mathbb{R}^n \end{array} \right\}. \tag{7.15}$$

Note that all data are fixed, except for the $k-1$ vectors $w_i \in \mathbb{R}^n$ corresponding to the pure formal partial derivatives of order $i-1$ with respect to the principal direction ∂_t. Denote $m_i := \dim(\mathcal{D}^i_\perp(F))$.

Note that in order to check ampleness of $\mathcal{S}^{i-1}_{\partial_t,F}$ within $\mathrm{Pr}^{i-1}_{\partial_t,F}$, the constants $\nu_2, \dots, \nu_k$ are irrelevant since they are fixed and can thus be subsummed within the free variables $w_1, \dots, w_{k-1}$. We can, thus, assume without loss of generality that $\nu_2 = \dots = \nu_k = 0$.

Take a generating set $\{v_1, \dots, v_{m_i}\}$ of $\mathcal{D}^i_\perp(F)$, i.e., so that $\langle v_1, \dots, v_{m_i}\rangle = \mathcal{D}^i_\perp(F)$. Define $\mathcal{B}_{n\times(m_i+k-1)} \subset \mathcal{M}_{n\times(m_i+k-1)}$ as the subset of matrices $B \in \mathcal{M}_{n\times(m_i+k-1)}$ whose first m_i columns are formed by the vectors $v_1, \ldots, v_{m_i}$. Additionally, denote by

$$\overline{\mathcal{B}}_{n\times(m_i+k-1)} \subset \mathcal{B}_{n\times(m_i+k-1)}$$

the subset of matrices in $\mathcal{B}_{n\times(m_i+k-1)}$ of $\text{rank}(B) = \mathfrak{n}_i$, where $\mathfrak{n}_i$ denotes the i-th entry of a maximal growth vector (Remark 3.16).

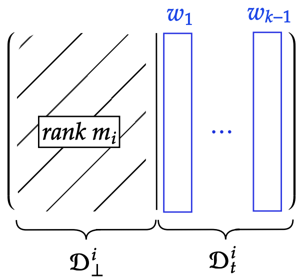

Figure 5. The subspace $\mathcal{B}_{n\times(m_i+k-1)}$ is formed by those matrices with first m_i columns fixed (striped area) and generating the m_i-dimensional subspace $\mathcal{D}^i_\perp(F)$. The other $k-1$ columns (in blue) are not fixed. Checking ampleness of $\mathcal{S}^{i-1}_{\partial_t,F} \subset \text{Pr}^{i-1}_{\partial_t,F}$ tantamounts to checking ampleness of the space of $\mathfrak{n}_i$-rank matrices $\overline{\mathcal{B}}_{n\times(m_i+k-1)} \subset \mathcal{B}_{n\times(m_i+k-1)}$.

Note that showing ampleness of $\mathcal{S}^{i-1}_{\partial_t,F} \subset \text{Pr}^{i-1}_{\partial_t,F}$ tantamounts to showing ampleness of the space $\overline{\mathcal{B}}_{n\times(m_i+k-1)} \subset \mathcal{B}_{n\times(m_i+k-1)}$. We distinguish several cases in order to conclude:

(i) If $i \neq r$, then $m_i + k - 1 = \mathfrak{n}_i < n$ (Lemma 7.19) and ampleness of $\overline{\mathcal{B}}_{n\times(m_i+k-1)} \subset \mathcal{B}_{n\times(m_i+k-1)}$ tantamounts to ampleness of the subset of rank $= m_i + k - 1 < n$ matrices (maximal rank) with m_i columns fixed inside the space of all matrices with the same columns fixed. This subset has a thin complement by Lemma 7.24 and, thus, ampleness follows.

(ii) If $i = r$, then $\mathfrak{n}_i = n$ (Lemma 7.19) and ampleness of $\overline{\mathcal{B}}_{n\times(m_r+k-1)} \subset \mathcal{B}_{n\times(m_r+k-1)}$ tantamounts to ampleness of the subset of rank $= n$ matrices (maximal rank) with m_r columns fixed inside the space of all matrices with the same columns fixed. Note that the case $n > m_r + k - 1$ cannot take place since this would contradict the fact that F was a formal solution in the first place. We have thus two possible cases:

  (ii.a) If $n < m_r + k - 1$, then there are two possibilities. If $m_r = n$, then ampleness holds trivially by Lemma 7.25. Otherwise, the space of maximal rank matrices with m_r columns fixed has a thin complement in $\mathcal{B}_{n\times(m_r+k-1)}$ by Lemma 7.24, and ampleness follows as well.

  (ii.b) If $n = m_r + k - 1$, ampleness follows since the space of maximal rank matrices with m_r columns fixed in $\mathcal{B}_{n\times(m_r+k-1)}$ is ample if $k - 1 \geq 2$ by Lemma 7.23 (although it does not have a thin complement, recall Remark 2.12).

This completes the proof. ■

The following result, which we isolate here since it is of independent interest, follows from the proof of Theorem 7.26 (precisely, from case (i) in the case distinction).

Lemma 7.27. *Fix $i < r$. Let $p \in \mathbb{R}^n$ and let $\partial_t \in T_p\mathbb{R}^n$ be a non-normal direction. Then for a ∂_t-adapted jet F over p, $\mathcal{S}^{i-1}_{\partial_t,F}$ has a thin complement within $\mathrm{Pr}^{i-1}_{\partial_t,F}$.*

Remark 7.28 (Associated singularities are not thin in the last step of the free case). Note that, for $i = r$ in the free case (Definition 3.17), the equality $m_r + k - 1 = n$ always holds (Lemma 7.17) and thus case $ii.a$) in the case distinction above can never take place. This means that $\mathcal{S}^{r-1}_{\partial_t,F} \subset \mathrm{Pr}^{r-1}_{\partial_t,F}$ is either trivially ample or its ampleness follows by ampleness of $\mathrm{GL}(m_r + k - 1)$. Therefore, either ampleness of $\mathcal{S}^{r-1}_{\partial_t,F} \subset \mathrm{Pr}^{r-1}_{\partial_t,F}$ holds trivially or it holds without yielding a thin singularity (Remark 2.12).

8. The rank-2 case

In this last section, we treat separately the rank-2 case. The section is devoted to prove Theorem 1.7; i.e., we will show that the maximal growth condition for rank-2 distributions does not give rise to ample differential relations. By the arguments carried out until this point (see the application of the fibered criterion in Subsection 7.1), ampleness of $\mathcal{R}^{\text{step-}r}$ is equivalent to ampleness of $\mathcal{S}^{\text{step-}r}$. We will show that $\mathcal{S}^{\text{step-}r}$ is not ample for rank-2 distributions, thus showing that neither is $\mathcal{R}^{\text{step-}r}$.

Lemma 8.1. *The space $\overline{\mathcal{A}^{q-1}_{q\times q}}$ of maximal rank matrices with the first $q-1$ columns fixed (and linearly independent) is the complement of a hyperplane within the space $\mathcal{A}^{q-1}_{q\times q}$ of all matrices with identical fixed columns. Therefore, $\overline{\mathcal{A}^{q-1}_{q\times q}} \subset \mathcal{A}^{q-1}_{q\times q}$ is not ample.*

Proof. The space $\overline{\mathcal{A}^{q-1}_{q\times q}}$ has two connected components, the one corresponding to matrices with positive determinant and the one with negative determinant. The complement of such space in $\mathcal{A}^{q-1}_{q\times q}$ corresponds to matrices with zero determinant. These matrices are determined by a linear equation (given by $det(A) = 0$, which is linear in the entries of the non-fixed column) and thus represent a hyperplane within $\mathcal{A}^{q-1}_{q\times q}$. Non-ampleness follows (recall Example 2.8). ■

Theorem 8.2. *Let M be a differentiable manifold of dimension $n \geq 3$ and consider the differential relation $\mathcal{S}^{\text{step-}r} \subset J^{r-1}(\mathrm{Fr}_2(M))$. Along any principal subspace, $\mathrm{Pr}^{r-1}_{\partial_t,F} \cap \mathcal{S}^{\text{step-}r}$ is either not ample or trivially ample.*

Proof. Take $F = (F_1, \ldots, F_k) \in J^{r-1}(\mathrm{Fr}_2(M))$ over a point $p \in M$ and a principal direction $\partial_t \in T_pM$. If $\partial_t \perp \mathcal{D}(F)$, then ampleness follows trivially (Corollary 7.7). In other case, we can reproduce verbatim the same argument from the proof of Theorem 7.26. In that case, denoting by $m_r = \mathrm{rank}\left(\mathcal{D}^r_\perp\right)$, we have that ampleness of $\mathrm{Pr}^{r-1}_{\partial_t,F} \cap \mathcal{S}^{\text{step-}r}$ is equivalent to ampleness of the space of rank-n matrices with m_r columns fixed (i.e., all but one) $\overline{\mathcal{A}^{m_r}_{n\times(m_r+1)}}$ within the space of all matrices with the same fixed columns $\mathcal{A}^{m_r}_{n\times(m_r+1)}$.

Considering the same case distinction as in the proof of Theorem 7.26, and following the same notation for its subcases: (ii.a) and (ii.b), we encounter two possibilities:

(ii.a) If $n < m_r + 1$, this means that $m_r = n$, since m_r cannot be greater than n for dimensional reasons. Therefore, ampleness follows trivially by Lemma 7.25.

(ii.b) If $n = m_r + 1$, then non-ampleness follows since the space $\overline{\mathcal{A}}^{m_r}_{n\times n}$ of maximal rank matrices with $m_r = n - 1$ columns fixed within the space of all matrices with the same columns fixed $\mathcal{A}^{m_r}_{n\times n}$ is non-ample (with its singularity being a hyperplane) by Lemma 8.1.

This completes the proof. ∎

The fibered-criterion of ampleness readily implies Theorem 1.7.

Remark 8.3. Non-ampleness of the Engel and the $(2,3,5)$-conditions follow as particular cases.

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