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Algebraic Groups

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Abstract. The field of Linear Algebraic Groups is still a very active research area in contemporary mathematics. It has rich connections to algebraic geometry, representation theory, algebraic combinatorics, number theory, algebraic topology, and differential equations. The foundations of this theory were laid by A. Borel, C. Chevalley, J.-P. Serre, T. A. Springer, R. Steinberg, and J. Tits in the second half of the 20th century. The Oberwolfach workshops on algebraic groups, led by Springer and Tits, played an important role in this effort as a forum for researchers, meeting at regular intervals since 1971. The present workshop continued this tradition, covering a range of topics, with an emphasis on recent developments in the subject.

Mathematics Subject Classification (2020): 14Lxx, 17Bxx, 20Gxx, 14Mxx.

Introduction by the Organizers

For the past 70 years the theory of algebraic groups has been at the forefront of research in algebra and algebraic geometry. In particular, it plays an important role in the construction of moduli spaces in algebraic geometry, in the Langlands program, and in the classification of finite simple groups.

Many important discoveries in the field of algebraic groups were first announced at a series of Oberwolfach workshops, originated by Springer and Tits. These workshops met at (approximately) three year intervals since the 1971; the last meeting took place in April, 2021 (in hybrid form during the pandemic). This time, there were 52 participants (46 in Oberwolfach, 6 online) from 12 countries: Canada, China, France, Germany, Great Britain, India, Israel, Italy, the Netherlands, Russia, Switzerland, and the United States. The scientific program consisted of 21 lectures mainly on the following research topics:

(1) Stacks and Geometric Invariant Theory
(2) Branching rules, Schubert geometry and Geometric Langlands theory
(3) Representations of Frobenius kernels and small quantum groups.

Additionally, the two topics *Algebraic groups and automorphism groups* and *Coulomb branches* were considered.

More precisely:

(1) Stacks have been introduced about 50 years ago as a tool to study moduli problems. Meanwhile they became objects to be studied by themselves. In particular, there is a general theory available as witnessed by, e.g., the "stacks project." Now the focus turned towards a detailed structure theory for certain classed of stacks. Most stacks occurring in "nature" look locally like a quotient stack $[X/G]$; this means that locally these stacks can be studied using the powerful machinery of Geometric Invariant Theory (GIT). After Mumford's GIT techniques for the construction of moduli spaces have been gradually replaced by the stack machinery, GIT is now experiencing a powerful resurgence as a tool to study stacks. We can mention here the talks of Heinloth, Mayeux and Premet.

(2) The Horn problem is to provide the inequalities satisfied by the eigenvalues of a sum $A + B$ of two hermitian matrices in terms of the eigenvalues of A and B. This problem has been solved by Klyachko and Knutson-Tao. It involves certain triples of Schubert classes in the singular cohomology of Grassmannians. These are related to Littlewood-Richardson coefficients arising in the decomposition of tensor products of simple representations of $GL(n, \mathbb{C})$. More generally, one can pose this problem for arbitrary reductive group G. The talks of Francone and Kumar contributed to this area.

(3) Representations of algebraic reductive groups over an algebraically closed field in positive characteristic is a central topic in representation theory. It has deep connection to the geometry of affine flag varieties and representations of affine Lie algebras and those of quantum groups at roots of unity. There has been a lot of work on the tensor structure of representations of reductive algebraic groups. A major obstacle is that tensor products do not preserve blocks. Jonathan Gruber explained how to define a new tensor structure on the principal block by introducing a new notion called generic direct summands. Then he posed a conjecture relating the multiplicity of generic direct summands to coefficients of Schubert classes in the cohomology ring of affine Grassmannian.

The small quantum group is a characteristic zero analogue of a Frobenius kernel of an algebraic group G. Their representation theory captures important information on the representation of G. In the talk of Anna Lachowska, she explained recent progresses and new results on the study of Hochschild cohomology of the small quantum group, conjectural relationships to diagonal coinvariant rings, and relations to the cohomology of affine Springer fibers.

(4) Algebraic groups occur often as automorphism groups of algebraic structures and/or of algebraic varieties. For example the projective linear group PGL_n is the automorphism group of the matrix algebra $M_n(k)$ and also of the projective space of dimension $n-1$. An important question is the following. If G is a smooth connected algebraic group defined over a field k, can we realize it as of the automorphism group of a smooth projective variety? It is called the inverse Galois problem for connected algebraic groups. In the last decade, fundamental results have been established. There were interesting connections also with Cremona groups of birational morphisms of projective spaces. We can mention here talks by Blanc, Kraft and Schroer.

(5) Several years ago, Braverman-Finkelberg-Nakajima gave a mathematical definition of a Coulomb branch associated to a 3 dimensional N=4 supersymmetric gauge theory. The construction uses equivariant K-theory of loop spaces attached to a group G and a representation N, which admits natural quantizations if one imposes equivariance for the loop rotation torus action. These quantized algebras have been shown to be related to shifted quantum affine algebras of ADE type when G and N arise from a quiver of the same type. Vasserot explained recent progress in the study of Coulomb branches associated with a quiver with symmetrizer, which provides a geometric realization of shifted quantum affine algebras of nonsymmetric type, and applications of this realization in the study of category O for shifted quantum affine algebras.

Workshop: Algebraic Groups

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Abstracts

An analogue of Whittaker reduction for group-valued moment maps

Ana Bălibanu

Any semisimple complex Lie algebra $\mathfrak{g}$ has a canonical Kirillov–Kostant Poisson structure, whose symplectic leaves are the orbits of the adjoint action. Each nilpotent element f of $\mathfrak{g}$ determines a Slodowy slice $\mathcal{S}$, introduced by Kostant [7] and Slodowy [15], which is transverse to the adjoint orbits and strictly transverse to the nilpotent orbit containing f. Work of Gan and Ginzburg [5] shows that the Kirillov–Kostant symplectic form on any adjoint orbit $\mathcal{O}$ restricts to a symplectic form on the intersection $\mathcal{S} \cap \mathcal{O}$, and the resulting symplectic foliation induces on $\mathcal{S}$ a natural Poisson structure.

Let G be a semisimple complex group integrating $\mathfrak{g}$, let M be a complex Poisson variety on which G acts by Hamiltonian Poisson automorphisms, and let

$$\mu : M \longrightarrow \mathfrak{g}$$

be the corresponding moment map. Whittaker reduction is a type of Hamiltonian reduction, first defined by Kostant [8], that takes place along μ at the nilpotent element f. It can be realized either as a symplectic reduction of M with respect to the action of a unipotent subgroup N opposite to f, or as the preimage under μ of the Poisson transversal $\mathcal{S}$. These two constructions fit into the diagram

$$\begin{tikzcd} \mu^{-1}(\mathcal{S}) \arrow[r, hook] \arrow[dr, "\sim"'] & \mu_N^{-1}(f) \arrow[r, hook] \arrow[d, two heads] & M \\ & \mu_N^{-1}(f)/N, & \end{tikzcd}$$

where μ_N is the moment map of the N-action, f is viewed as an element of $\mathfrak{n}^*$ using the Killing form, and the diagonal map is an isomorphism. From this perspective, Whittaker reduction encodes the Poisson geometry of M in a direction transversal to the orbits of G.

Poisson manifolds frequently exhibit natural symmetries that fail to preserve the Poisson bracket, and it was observed by Semenov-Tian-Shansky [13] that this phenomenon is due to the fact that the group of symmetries itself carries an additional Poisson structure. In other words, the group G which acts on the Poisson variety M is a Poisson–Lie group in the sense of Drinfeld [3], and the action map is a Poisson map.

A Hamiltonian theory for such actions was developed by Lu [9], and the associated moment maps take values in the dual Poisson–Lie group G^*. In the case when G is a semisimple complex group equipped with the standard Poisson–Lie group structure, there is a local open embedding

$$G^* \cong B \times_T \overline{B} \longrightarrow G,$$

where B and $\overline{B}$ are a pair of opposite Borel subgroups and T is their common maximal torus. The image of this map is the maximal Bruhat cell, and Lu's definition can be extended to consider moment maps valued in the group G itself. This is a special case of the theory of D/G-valued moment maps introduced by Alekseev and Kosmann-Schwarzbach [1] and later studied by Bursztyn and Crainic [2]. We develop an analogue of Whittaker reduction along these G-valued moment maps.

The multiplicative counterpart of the Kostant slice was constructed by Steinberg [16] and is given by $U_w w$, where w is a minimal-length Coxeter element in the Weyl group of G and U_w is the subgroup generated by positive roots which are flipped by w^{-1}. This Steinberg slice consists entirely of regular elements and is strictly transverse to the regular orbits in G—in particular, it is strictly transverse to the regular unipotent orbit.

Steinberg's construction indicates that transversal slices to unipotent orbits in G are linked to conjugacy classes in the Weyl group. Slices associated to non-Coxeter conjugacy classes have been studied by He and Lusztig [6], by Sevostyanov [14], and most recently by Duan [4]. While there are subtle technical differences between their constructions, they share two fundamental features. First, they associate to an element w of the Weyl group a slice of the form

$$\Sigma := U_w Z w,$$

where U_w is defined as above and Z is the reductive subgroup of G generated by T^w and by the roots fixed by w. Second, each slice has the property that the conjugation map gives an isomorphism

$$U \times \Sigma \longrightarrow UZwU := \Omega,$$

where U is the unipotent subgroup generated by the positive roots not fixed by w. This isomorphism implies, in particular, that Σ is transverse to the conjugacy classes of G.

The work of Sevostyanov [14] shows that the slice Σ has a natural Poisson structure inherited from the Semenov–Tian–Shansky Poisson structure on G. More recently, results of Duan [4] make the connection to unipotent orbits precise by showing that, when the conjugacy class of w is "close to elliptic," the slice Σ is strictly transverse to the unipotent orbit associated to w under the Lusztig map [10, 11, 12]. The slice Σ therefore exhibits multiplicative analogues of the key geometric features of Slodowy slices.

We introduce an analogue of Whittaker reduction for Poisson actions of Poisson–Lie groups, which takes place along the slice Σ and its U-saturation Ω. Concretely, equip the semisimple complex group G with the standard Poisson–Lie group structure, and suppose that it has a Hamiltonian Poisson action on a complex Poisson variety M with corresponding moment map $\mu : M \longrightarrow G$. There is a commutative diagram

$$\begin{tikzcd} \mu^{-1}(\Sigma) \arrow[r, hook] \arrow[dr, "\sim"'] & \mu^{-1}(\Omega) \arrow[r, hook, "\jmath"] \arrow[d, two heads, "q"] & M \\ & Q := \mu^{-1}(\Omega)/U & \end{tikzcd}$$

in which the diagonal map is an isomorphism.

Theorem. *Let $\mathfrak{c}$ be the orthogonal complement of the fixed-point set $\mathfrak{t}^w$ in the maximal Cartan $\mathfrak{t}$ of the Lie algebra $\mathfrak{g}$. Then the quotient Q carries a natural Poisson bracket $\{\cdot,\cdot\}_Q$ which is uniquely characterized by the property that*

$$q^*\{f,g\}_Q = \jmath^*\{F,G\}$$

for all functions $f,g \in \mathcal{O}_Q$ and all $\mathfrak{c}$-invariant lifts $F,G \in \mathcal{O}_M$ that satisfy $q^*f = \jmath^*F$ and $q^*g = \jmath^*G$.

In the special case when M is the group G itself, the resulting Poisson structure on Σ agrees with the Poisson structure originally introduced by Sevostyanov.

Moreover, we show that, under the diagonal isomorphism in this diagram, the Poisson structure on Q can also be viewed as a pullback of the Poisson structure on M in the sense of Dirac geometry. This allows us to characterize the symplectic leaves of the reduction.

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Motivic factorization of KZ local systems and deformations of representation and fusion rings

Prakash Belkale
(joint work with Najmuddin Fakhruddin, Swarnava Mukhopadhyay)

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Let $\mathcal{C}_n$ be the configuration space of n-distinct points on the affine line. Let P^+ be the set of dominant integral weights of $\mathfrak{g}$ corresponding to a fixed Cartan decomposition. For $\lambda \in P^+$, let V_λ denote the corresponding irreducible finite dimensional irreducible representation of $\mathfrak{g}$.

Let $\lambda_1, \ldots, \lambda_n \in P^+$. The KZ connection is defined on the constant vector bundle on $\mathcal{C}_n$ with fibre $V(\vec{\lambda}) = V_{\lambda_1} \otimes V_{\lambda_2} \otimes \ldots \otimes V_{\lambda_n}$. Using variables $z_1, z_2, \ldots, z_n$ on $\mathcal{C}_n$, the connection equations are:

$$\kappa \frac{\partial}{\partial z_i} f = \Big(\sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} \Big) f.$$

Here f is any local section of $V(\vec{\lambda}) \otimes \mathcal{O}_{\mathcal{C}_n}$, Ω_{ij} is the normalised Casimir element acting on the i and j tensor factors, and $\kappa \in \mathbb{C}^\times$.

This connection, and the related WZW/Hitchin connections, especially in genus zero, and their q-analogs appear in many areas of mathematics, e.g., representation theory, enumerative geometry, algebraic geometry, number theory, and also mathematical physics.

The KZ connection is flat and commutes with the diagonal action of $\mathfrak{g}$, hence it induces a connection on the constant bundle of coinvariants,

$$\mathbb{A}(\vec{\lambda}, \nu^*) = (V(\vec{\lambda}) \otimes V_\nu^*)_{\mathfrak{g}} = \frac{V(\vec{\lambda}) \otimes V_\nu^*}{\mathfrak{g}(V(\vec{\lambda}) \otimes V_\nu^*)}, \text{ where } V(\vec{\lambda}) = V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n}.$$

We will think of these spaces as being attached to the representations $V_{\lambda_1}, \ldots, V_{\lambda_n}$ at $z_1, \ldots, z_n$ respectively, and the representation V_ν^* at $\infty \in \mathbb{P}^1_{\mathsf{k}}$.

Let $\kappa \in \mathbb{Q}^\times$. The mathematical theory of integral representations of solutions of KZ equations for generic κ [8], and the extension to arbitrary κ in [7, 2](also see [3]), allows one to construct motivic local systems $\mathcal{KZ}_\kappa(\vec{\lambda}, \nu^*)$ on $\mathcal{C}_n$ whose Betti realisations give the duals of classical KZ local systems over $\mathbb{C}$. When $\kappa = \ell + h^\vee$, one constructs motivic local systems of conformal blocks $\mathcal{CB}_\kappa(\vec{\lambda}, \nu^*)$ on $\mathcal{C}_n$ which are natural subsystems of $\mathcal{KZ}_\kappa(\vec{\lambda}, \nu^*)$. The fibres of these local systems over points in $\mathcal{C}_n(\mathsf{k})$ are Nori motives over a suitable finite extension of k with coefficients in a cyclotomic field (here $\mathsf{k} \subseteq \mathbb{C}$ is the field of definition of the Lie algebra $\mathfrak{g}$). The same constructions carried out in the derived category of mixed Hodge modules leads to variations of mixed Hodge structures. These variations were introduced by Looijenga [7]. The conformal block variations are pure, but the KZ variations are in general mixed.

In [4] we prove a basic factorisation for the nearby cycles of these motivic local systems as some of the n points coalesce. This leads to the construction of a family (parametrised by κ) of deformations over $\mathbb{Z}[t]$ of the representation ring of $\mathfrak{g}$—we call these enriched representation rings—which allows one to compute the ranks of the Hodge filtration of the associated variations of mixed Hodge structure; in turn, this has applications to both the local and global monodromy of the KZ connection. The key property is that the Hodge filtration on nearby cycles in Saito's mixed Hodge modules category is a suitable limit of the Hodge filtration along the degeneration, and this property implies the associativity property of these enriched rings.

In the case of $\mathfrak{sl}_n$ we give an explicit algorithm for computing all products in the enriched representation rings, which we use to prove that if $1/\kappa \in \mathbb{Z}$ then the global monodromy is finite and scalar.

We can make the enriched representation ring explicit in the case of $\mathfrak{sl}_2$: It has a $\mathbb{Z}$-basis given by dominant fundamental weights $[a]$ with $a \in \mathbb{Z}_{\geq 0}$, with $[0]$ the identity. The motivic Pieri formula is the following. Here $a \in \mathbb{Z}_{>0}$ is a dominant fundamental weight and $\varpi_1 = 1$ is the vector representation.

$$[a] \star [\varpi_1] = [a + \varpi_1] + [a : \kappa] \cdot [a - \varpi_1]$$

where $[a : \kappa]$ is 1 or t. For example when $1/\kappa$ is not an integer and $\kappa > 0$, then

(1) If a/κ and $(1+a)/\kappa$ are both in $\mathbb{Q} - \mathbb{Z}$, then $[a;\kappa]$ is determined from the integer $\langle -1/\kappa\rangle + \langle -a/\kappa\rangle + \langle (1+a)/\kappa\rangle$: If this integer is 2 then $[a;\kappa]$ is t, and $[a;\kappa]$ is 1 if this integer is 1. (Here $\langle \alpha\rangle = \alpha - \lfloor \alpha \rfloor$.)
(2) $a/\kappa \in \mathbb{Z}$ then $[a;\kappa]$ is 1, and if $(1+a)/\kappa \in \mathbb{Z}$ then $[a;\kappa]$ is t.

We also prove a similar factorisation result for motivic local systems associated to conformal blocks in genus 0; this leads to the construction of a family of deformations of the fusion rings.

If $\kappa = r/s$, with $r, s \in \mathbb{Z}$, then all the mixed sheaves $\mathcal{KZ}_\kappa(\vec{\lambda}, \nu^*)$ have coefficients in $\mathbb{Q}(\mu_N)$ for an explicit integer N depending on r. If $\sigma \in \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$, we have Galois twists $\mathcal{KZ}^\sigma_\kappa(\vec{\lambda}, \nu^*)$ (resp. $\mathcal{CB}^\sigma_\kappa(\vec{\lambda}, \nu^*)$) of the motivic sheaves $\mathcal{KZ}_\kappa(\vec{\lambda}, \nu^*)$ (resp. $\mathcal{CB}_\kappa(\vec{\lambda}, \nu^*)$ for $\kappa = \ell + h^\vee$) given by precomposing the $\mathbb{Q}(\mu_N)$-structure with σ^{-1}. If $\sigma(\zeta_N) = \zeta_N^a$ for some positive integer a with $(a, N) = 1$, then $\mathcal{KZ}^\sigma_\kappa(\vec{\lambda}, \nu^*) \simeq \mathcal{KZ}_{\kappa/a}(\vec{\lambda}, \nu^*)$.

The Hodge numbers of Galois twists corresponding to complex conjuagates of $\mathcal{KZ}_\kappa(\vec{\lambda}, \nu^*)$ are related to the weight filtrations of these motives, particularly when the ranks of these motives are one.

0.1. **The weight filtration of KZ motives and quantum groups.** Assume that $\mathfrak{g}$ is of type A in this section. We consider the category $\mathcal{C}_\kappa$ of finite dimensional representations (of Type I) of the associated (Lusztig's) quantum group over $\mathbb{C}$

at an integer level $\kappa > h$ (which is often denoted by ℓ in the quantum groups literature). This has the structure of a rigid braided monoidal category, in fact it is a ribbon category.

Denote the monodromy representation of the local system underlying $\mathcal{KZ}_\kappa(\vec{\lambda}, \nu^*)$ by $KZ(\vec{\lambda}; \nu^*)$. The Kazhdan-Lusztig equivalence [5, 6] gives an isomorphism of $KZ(\vec{\lambda}; \nu^*)$ (with the Hom below in the category of QG representations)

$$\mathrm{Hom}(\Delta(\vec{\lambda}), \nabla(\nu))^\vee \text{ where } \Delta(\vec{\lambda}) = \Delta(\lambda_1) \otimes \cdots \otimes \Delta(\lambda_n),$$

the latter becoming a pure braid group representation via the action in the category of the modules for the quantum group. Here $\Delta(\lambda)$ and $\nabla(\lambda)$) are the Weyl and dual Weyl modules respectively corresponding to λ.

For any tilting module Q and dominant weight μ, Andersen defines in [1, Section 1.3] a decreasing filtration $\mathrm{Hom}(Q, \nabla(\mu))^\bullet$ on $\mathrm{Hom}(Q, \nabla(\mu))$ which is functorial in Q. Note that Andersen's filtration is actually on invariants rather than coinvariants, so we are using the dual of his filtrations.

Andersen's filtration gives rise to a polynomial $f_\mu(Q) \in \mathbb{Z}[w]$ with non-negative coefficients which encodes the dimensions of the successive subquotients of the filtration [1, 1.5].

0.1.1. The conjecture below is a special case of a conjecture of N. Fakhruddin (in type A). Ongoing work of P. Belkale and N. Fakhruddin provides evidence for this conjecture.

Let $\vec{\lambda}$ be an n-tuple of dominant weights such that $\Delta(\lambda_i)$ is simple for all i and $\nu \in P^+$. Then each $\Delta(\lambda_i)$ is a tilting module, and hence $\Delta(\vec{\lambda})$ is a tilting module. Let M be the number of simple roots appearing in the expression of $\sum \lambda_i - \nu$ as a sum of simple roots. This is the "expected" weight of the motivic local system $\mathcal{KZ}_\kappa(\vec{\lambda}, \nu^*)$.

Conjecture 1. *(Fakhruddin) The weight filtration on $KZ(\vec{\lambda}; \nu^*)$ coincides (up to a shift by M) with the dual of the Andersen filtration on* $\mathrm{Hom}(\Delta(\vec{\lambda}), \nabla(\nu))$. *In particular, all weights of $KZ(\vec{\lambda}; \nu^*)$ are $\geq M$ and the weight polynomial is equal to $w^M f_\nu(\Delta(\vec{\lambda}))$.*

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Actions of algebraic groups: difference between positive and zero characteristic

Jérémy Blanc

Let us work with an algebraically closed field k.

In characteristic zero, Rentschler proved in 1968 that every action of the additive group $\mathbb{G}_a$ on the affine plane $\mathbb{A}^2$ is conjugate to

$$(x, y) \mapsto (x, y + tq(x))$$

for some $q \in \mathrm{k}[x]$ (see [11]). If the characteristic is $p > 0$, Miyanishi proved in 1971 that every $\mathbb{G}_a$ action on $\mathbb{A}^2$ is conjugate to

$$(x, y) \mapsto (x, y + tq_0(x) + t^p q_1(x) + \cdots + t^{p^r} q_r(x))$$

for some $r \geq 1$ and $q_0, \ldots, q_r \in \mathrm{k}[x]$ (see [10]).

In particular, every additive action on the affine plane fixes a variable and is conjugate to a triangular automorphism.

The first example of a non-triangularisable additive action on $\mathbb{A}^3$ was found by Bass in 1984, see [1]. It works in any characteristic. In higher dimension, there are actually actions that do not fix any variable. This was first found in 1997 by Gene Freudenburg in [6], in characteristic zero. The first result in positive characteristic is much more recent, published in 2025 by Kuroda: [8].

If one asks for free actions on the affine space, every such action is conjugate to a translation in dimension 3, as proven by Kaliman in 2004, see [7]. In positive characteristic there are examples found by Kuroda of additive actions that do not fix any point and are not conjugate to translations, by Kuroda (see [9]). One can actually give a very simple as follows:

$$(x, y, z) \mapsto (x, y + tx, z + t^p).$$

The ring of invariants is $\mathrm{k}[x, y^p - xz]$ and as the morphism $\mathbb{A}^3 \to \mathbb{A}^2$ given by $(x, y) \mapsto (x, y^p - xz)$ has non-reduced fibres, the action is not conjugate to a translation. However, it is actually not really free, as the infinitesimal subgroup of $\mathbb{G}_a$ given by $t^p = 0$ fixes the plane $x = 0$.

The ring of invariants of an additive action on an affine space is not always finitely generated. There are examples in characteristic zero found in dimension 7 in 1990 (Roberts, [12]) and then 6 (Freudenburg, [5]) and 5 (Daigle and Freudenburg, [3]) in 1999-2000. In dimension 3 the ring of invariants is always finitely generated in any characteristic, by a classical result of 1954 of Zariski [13]. The situation in dimension 4 in characteristic zero and in any dimension ≥ 4 in positive characteristic are wide open. It is surprising that no known example is known in positive characteristic. Actually, Dufresne and Maurischat proved that the rings of invariants of the known counterexamples in dimension 5, 6 and 7 are finitely generated in positive characteristic [4].

In arbitrary characteristic, all additive actions on $\mathbb{A}^n$ are actually birationally conjugate for $n \leq 3$ by a result of Rosenlicht (see [2, Corollary 2.5.7]). The additive subgroups of $\mathrm{Bir}(\mathbb{A}^n)$ are in $1:1$ correspondence with varieties X such that $X \times \mathbb{A}^1$ is birational to $\mathbb{A}^n$, up to birational equivalence [2, Corollary 2.5.4].

However there are many other connected algebraic subgroups of $\mathrm{Bir}(\mathbb{P}^3)$. In characteristic zero, there is classification of all maximal connected algebraic subgroup made by Umemura in a serie of papers in the 80's, using analytic techniques. A version with birational geometry was then reproven in [2]. In many cases, the families obtained depend on discrete countable parameters (mostly degrees of polynomials). The only continuous family is given by automorphisms of quadric bundles $\mathcal{Q}_g \to \mathbb{P}^1$ that depend on polynomials g (see [2] for more details).

Together with Ronan Terpereau, we are dealing with the case of positive characteristic. We have found more continuous families, that are automorphisms of special kind of $\mathbb{P}^1$-bundles over Hirzebruch surfaces $\mathbb{F}_a$ with $a \geq 1$. Contrary to the case of characteristic zero the action on the base $\mathbb{F}_a$ is not always the whole group $\mathrm{Aut}(\mathbb{F}_a)$.

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Hecke operators over local non-archimedean fields, Shalika germs and symplectic duality

Alexander Braverman

We discuss recent developments about Hecke operators for moduli spaces of G-bundles on algebraic curves over a local non-archimedean field F. We discuss the notion of Hecke eigen-function and give some examples. In addition we discuss the question of how to describe the eigen-values and connect this question to the classical question about description of Shalika germs. We also briefly mention the connection to the subject of symplectic duality (in the form of a generalized Hikita conjecture).

This is a joint work in progress with David Kazhdan.

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Connected components of the moduli space of L-parameters

Sean Cotner

1. Introduction

Let p be a prime number, let F be a finite extension of $\mathbf{Q}_p$ with residue field $\mathbf{F}_q$, and let G be a quasi-split connected reductive group over F. Throughout, the letter ℓ will always denote a prime number different than p. The subject of this talk concerns the "Galois side" of the Langlands program, but the main theorem is easier to motivate on the "automorphic side", so we begin there.

Definition 1. If R is a commutative $\mathbf{Z}[1/p]$-algebra (e.g., $\mathbf{C}$, $\overline{\mathbf{F}}_\ell$, or $\mathbf{Z}[1/p]$ itself), then a *smooth R-representation* of $G(F)$ is an R-module M equipped with a $G(F)$-action such that every element of M has open stabilizer in $G(F)$.

A basic problem is to describe the *blocks* of $\mathrm{Rep}_R(G(F))$, i.e., to write

$$\mathrm{Rep}_R(G(F)) = \prod_{i\in I} \mathcal{C}_i$$

for some indecomposable abelian subcategories $\mathcal{C}_i \subset \mathrm{Rep}_R(G(F))$. When $R = \mathbf{C}$, this problem has been solved, as we now describe.

Given a parabolic subgroup $P \subset G$ with Levi L and a smooth R-representation (M,ρ) of $L(F)$, one can regard ρ as a representation of $P(F)$ via the quotient map $P \to L$ and form the parabolic induction $\mathrm{ind}_{P(F)}^{G(F)}(\rho)$.

If V is an irreducible smooth $\mathbf{C}$-representation of $G(F)$, then there is a minimal pair (L,ρ) such that V arises as a subquotient of $\mathrm{ind}_{P(F)}^{G(F)}(\rho)$ for some parabolic P with Levi factor L, and the pair (L,ρ) is called the *supercuspidal support* of V. For such a pair (L,ρ), let $\mathrm{Rep}_{\mathbf{C}}(G(F))_{(L,\rho)}$ be the subcategory of $\mathrm{Rep}_{\mathbf{C}}(G(F))$

consisting of representations all of whose Jordan–Hölder factors have supercuspidal support lying in the equivalence class of (L, ρ).

Theorem 2 (Bernstein–Deligne)**.** *There is a block decomposition*

$$\mathrm{Rep}_{\mathbf{C}}(G(F)) = \prod_{\{(L,\rho)\}/\sim} \mathrm{Rep}_{\mathbf{C}}(G(F))_{(L,\rho)},$$

where the product ranges over minimal pairs (L, ρ) considered up to a certain equivalence relation refining $G(F)$-conjugacy.

It is expected that similar block decompositions should exist over $\overline{\mathbf{F}}_\ell$, $\overline{\mathbf{Z}}_\ell$, and even $\overline{\mathbf{Z}}[1/p]$; see, e.g., [10], [7], [8], [9] for partial results centered around the case $G = \mathrm{GL}_n$. Even if one is only interested in $\mathbf{C}$-representations, it is profitable to consider rather general R: for instance, understanding the blocks of $\mathrm{Rep}_{\overline{\mathbf{Z}}_\ell}(G(F))$ amounts to understanding the ways in which blocks of $\mathrm{Rep}_{\mathbf{C}}(G(F))$ "collide" modulo ℓ. Similarly, the blocks of $\mathrm{Rep}_{\overline{\mathbf{Z}}[1/p]}(G(F))$ describe the ways in which blocks of $\mathrm{Rep}_{\mathbf{C}}(G(F))$ conspire to collide modulo all primes $\ell \neq p$. This idea was used by Sécherre–Stevens in [11] to deduce a certain compatibility between local Langlands and Jacquet–Langlands.

2. Dual groups

Let W_F be the Weil group of F, a certain dense subgroup of the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$. Let $\widehat{G}$ denote the Langlands dual group of G, a split reductive group scheme over $\mathbf{Z}$ equipped with a finite action of W_F. By definition, the root datum of $\widehat{G}$ is dual to the root datum of $G_{\overline{F}}$, and the W_F-action comes from the action of W_F on the absolute Dynkin diagram of G. We let ${}^LG = \widehat{G} \rtimes W_F$ denote the L-group of G.

Definition 3. If $\ell \neq p$ is prime, then an *L-parameter* is a continuous homomorphism ${}^L\varphi \colon W_F \to {}^LG(\overline{\mathbf{Q}}_\ell)$ which is a section to the natural projection ${}^LG(\overline{\mathbf{Q}}_\ell) \to W_F$. This is equivalent to a continuous 1-cocycle $\varphi \colon W_F \to \widehat{G}(\overline{\mathbf{Q}}_\ell)$.

The categorical local Langlands conjecture posits (see [5], [12], [6], [3] for more precise statements) that for a $\mathbf{Z}[1/p]$-algebra R, there should exist a fully faithful functor $D^+\mathrm{Rep}_R(G(F)) \to D^+_{\mathrm{QCoh}}(\mathscr{X}_{\widehat{G},R})$, where $\mathscr{X}_{\widehat{G},R}$ denotes the "moduli stack of L-parameters" defined in [5], [12], and [4]. In particular, if one could obtain a block decomposition of $D^+_{\mathrm{QCoh}}(\mathscr{X}_{\widehat{G},R})$, then one could deduce a product decomposition for $\mathrm{Rep}_R(G(F))$ (if not, perhaps, into blocks). As a step in this direction, we describe the set of connected components of $\mathscr{X}_{\widehat{G},R}$ when $R = \mathbf{Z}[1/p]$.

We will not give a precise definition of $\mathscr{X}_{\widehat{G}}$, referring the reader to [4] for the most "classical"-looking definition. However, we remark that $\mathscr{X}_{\widehat{G}}(\overline{\mathbf{Q}}_\ell)$ is the set of L-parameters $W_F \to {}^LG(\overline{\mathbf{Q}}_\ell)$ modulo $\widehat{G}(\overline{\mathbf{Q}}_\ell)$-conjugacy [4, Theorem 4.1 ii)]. Moreover, if the action of W_F on $\widehat{G}$ is tame, i.e., factors through the quotient W_F^{tame} which is topologically generated by Fr and s with relation $\mathrm{Fr}s\mathrm{Fr}^{-1} = s^q$, then $\mathscr{X}_{\widehat{G}}$ admits a closed substack $\mathscr{X}^{(1)}_{\widehat{G}}$ consisting essentially of 1-cocycles $W_F^{\mathrm{tame}} \to \widehat{G}$.

Theorem 4. [4, Section 3] *There exists an isomorphism*

$$\mathscr{X}_{\widehat{G},\overline{\mathbf{Z}}[1/p]} \cong \coprod_{i\in I} \mathscr{X}^{(1)}_{\widehat{H}_i,\overline{\mathbf{Z}}[1/p]}$$

for some reductive group schemes $\widehat{H}_i$ equipped with finite tame quasi-semisimple actions of W_F.

In particular, it is enough to understand $\pi_0(\mathscr{X}^{(1)}_{\widehat{G},\overline{\mathbf{Z}}[1/p]})$ for all $\widehat{G}$. Let $X^{(1)}_{\widehat{G}}$ denote the tautological $\widehat{G}$-torsor over $\mathscr{X}^{(1)}_{\widehat{G}}$, so $\pi_0(\mathscr{X}^{(1)}_{\widehat{G},\overline{\mathbf{Z}}[1/p]}) = \pi_0(X^{(1)}_{\widehat{G},\overline{\mathbf{Z}}[1/p]})$.

Example 5. Let $\widehat{G} = \mathrm{GL}_2$ equipped with trivial W_F-action, so

$$X^{(1)}_{\widehat{G}} = \{(\Phi,\Sigma) \in \mathrm{GL}_2 \times \mathrm{GL}_2 \colon \Phi\Sigma\Phi^{-1} = \Sigma^q\}.$$

Note that if (Φ,Σ) is a field-valued point of this scheme, then the characteristic polynomial of Σ is very constrained: if $\{\alpha,\beta\}$ is the set of eigenvalues of Σ, then $\{\alpha,\beta\} = \{\alpha^q,\beta^q\}$, so either $\alpha,\beta \in \mu_{q-1}$ or $\alpha \in \mu_{q^2-1}$ and $\beta = \alpha^q$.

Refining this argument somewhat, one shows that the map $\overline{\pi}\colon X^{(1)}_{\widehat{G}} \to \mathbf{A}^2$ sending (Φ,Σ) to the characteristic polynomial of Σ factors through the finite closed subscheme $(\mu^2_{q-1} \cup \mu_{q^2-1})/\sim$, where $\sim$ is the "flip" equivalence relation. One can show that $\overline{\pi}$ has connected fibers, and using this deduce

$$\pi_0(X^{(1)}_{\widehat{G}}) = \pi_0((\mu^2_{q-1} \cup \mu_{q^2-1})/\sim).$$

Over $\overline{\mathbf{Z}}[1/p]$, the latter set is a singleton, as follows from the fact that $q-1$ and q^2-1 are not divisible by p.

The following theorem generalizes this example and resolves a conjecture of Dat–Helm–Kurinczuk–Moss [4, Conjectures 4.3, 4.4].

Theorem 6. [2, Theorem 1.2] *If $\widehat{G}$ is a reductive $\overline{\mathbf{Z}}[1/p]$-group scheme equipped with a finite tame quasi-semisimple action of W_F, then $X^{(1)}_{\widehat{G},\overline{\mathbf{Z}}[1/p]}$ is connected.*

If $\widehat{G}$ has smooth center over $\overline{\mathbf{Z}}[1/p]$, then this theorem is [4, Theorem 4.5]. This covers many interesting cases, such as the case that G is simply connected or $G = \mathrm{GL}_n$, but it says little for $G = \mathrm{PSp}_{2n}$ and nothing for $G = \mathrm{PGL}_6$.

The proof of Theorem 6 is essentially a meditation on Example 5: one first reduces to the case that $\widehat{G}$ is semisimple and simply connected. Then one considers the map $\overline{\pi}\colon X^{(1)}_{\widehat{G}} \to \widehat{G}/\!/_s\widehat{G}$ given by $(\Phi,\Sigma) \mapsto [\Sigma]$, and shows that $\overline{\pi}$ has connected image and connected fibers, and deduces the result from a nontrivial topological argument, using other results of [4]. The most difficult part of the argument consists of proving connectedness of $\mathrm{im}(\overline{\pi})$. The proof is completely uniform if $\widehat{G}$ has no simple factors of type A, but the fact that groups of type A can have centers of order divisible by more than one prime causes serious difficulties.

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Branching problems through cluster algebras

Luca Francone

Given a complex reductive algebraic subgroup $\widehat{G}$ of the complex reductive group G, the branching problem of the pair $(G, \widehat{G})$ asks to understand how the irreducible representations of G decompose under the restricted $\widehat{G}$-action. A classical example arises when $G = \widehat{G} \times \widehat{G}$ and $\widehat{G}$ is diagonally embedded in G; in this case, the branching problem corresponds to decomposing the tensor product of irreducible representations of $\widehat{G}$. The solution to the branching problem of the pair $(G, \widehat{G})$ is encoded in a collection of integers known as *multiplicities*. These multiplicities can be realized as the dimensions of the homogeneous components of a graded algebra called the *branching algebra*, and denoted by $\mathrm{Br}(G, \widehat{G})$.

Recently, certain branching algebras have been proved to be *upper cluster algebras*. This has been done by Magee [5] and Gross-Hacking-Keel-Kontsevich [4] in the case of the tensor product decomposition of $SL(n)$, and by Fei [1] in the case of the tensor product decomposition of a simple, simply laced and simply connected algebraic group. In all these cases, the authors provide explicit polyhedral models for tensor product multiplicities through the theory of cluster algebras. One of the models recovered by Magee is the celebrated Knutson–Tao *hive model*, which played a central role in the proof of the saturation conjecture in type A. Previously, this model had only been constructed through an ad hoc approach. In addition, Fei's work provides an additive categorification of the branching algebras studied in [1] via the theory of *quivers with potentials*. These results suggest that cluster algebras can provide a formidable tool for the study of branching problems. Nevertheless, the techniques used in these works to identify branching algebras with upper cluster algebras do not readily generalize to broader families of branching problems. In this talk, we present a possible approach to addressing this gap, developed in [2] and [3].

We introduce the notion of *suitable for lifting schemes*. For any such scheme $\mathfrak{X}$, we present a homogenisation technique called *minimal monomial lifting* [2], which allows to construct an upper cluster subalgebra $\mathcal{U} \subseteq \mathcal{O}(\mathfrak{X})$, starting from a cluster structure on the ring of regular functions of a distinguished closed subscheme of $\mathfrak{X}$. The algebra $\mathcal{U}$ can be proved to be an optimal candidate for endowing $\mathcal{O}(\mathfrak{X})$ with the structure of an upper cluster algebra. In many interesting cases, it turns out that $\mathcal{U} = \mathcal{O}(\mathfrak{X})$. Since spectra of branching algebras are examples of suitable for lifting schemes, this procedure yields an upper cluster subalgebra of the branching algebra of many pairs $(G, \widehat{G})$ as above. This leads to the following result.

Theorem 1. *Assume that the subgroup $\widehat{G} \subseteq G$ belongs to the following list:*

(1) the group $\widehat{G}$ is a Levi subgroup of the semisimple, simply connected algebraic group G;

(2) the group $\widehat{G}$ is semisimple, simply connected, and diagonally embedded in $G = \widehat{G} \times \widehat{G}$.

Then, the branching algebra $\mathrm{Br}(G, \widehat{G})$ *is an upper cluster algebra.*

The previous result enables the use of the theory of cluster algebras to construct polyhedral models for tensor product multiplicities associated with any semisimple, simply laced, and simply connected algebraic group, as shown in [3]. Thanks to recent advancements, this construction can now be extended to the non-simply laced case as well.

The cluster-theoretic approach to branching problems described here extends beyond the specific pairs $(G, \widehat{G})$ considered in Theorem 1. In particular, it provides a framework for constructing cluster structures adapted to the study of branching problems involving *Kronecker* and *plethysm* coefficients, as well as those associated with *spherical subgroups of minimal rank*. When $\widehat{G}$ is a spherical subgroup of minimal rank of the semisimple, simply connected algebraic group G, it is shown in [3] that the branching algebra $\mathrm{Br}(G, \widehat{G})$ can be described as the coordinate ring of a blow-up of a cluster variety. We look forward to seeing new and interesting developments arising from this perspective.

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Generic direct summands of tensor products and multiplicity conjectures

Jonathan Gruber

Let G be a simply connected simple algebraic group over an algebraically closed field $\Bbbk$. This talk is concerned with the problem of understanding the structure of tensor products of finite-dimensional rational irreducible G-modules.

If $\mathrm{char}(\Bbbk) = 0$ then the category $\mathrm{Rep}(G)$ of finite-dimensional rational G-modules is semisimple, and so the problem of understanding tensor products is essentially a combinatorial one (e.g. via the Littlewood–Richardson rule). Here we instead focus on the case where char $\mathrm{char}(\Bbbk) = p > 0$, and for technical reasons, we assume that $p \geq h$, the Coxeter number of G. Then the category $\mathrm{Rep}(G)$ is non-semisimple, and it decomposes into blocks according to the so-called p-dilated dot action of the affine Weyl group of G. Writing $\mathrm{Rep}_0(G)$ for the principal block of G, we have the following paremetrizations of simple G-modules:

$$\begin{aligned} X^+ &\xrightarrow{1:1} \{ \text{ simple } G\text{-modules } \} / \cong \\ \lambda &\longmapsto L(\lambda) \\ W_{\mathrm{aff}}^+ \cong W_{\mathrm{fin}} \backslash W_{\mathrm{aff}} &\xrightarrow{1:1} \{ \text{ simple } G\text{-modules in } \mathrm{Rep}_0(G) \} / \cong \\ x &\longmapsto L_x \coloneqq L(x \cdot 0) \end{aligned}$$

Here X^+ denotes the set of dominant weights and W_{aff}^+ denotes the set of minimal coset representatives of the finite Weyl group W_{fin} in the affine Weyl group W_{aff}.

Many problems in the modular representation theory of algebaic groups can be reduced to questions about the principal block, and then resolved using combinatorial and categorical tools arising from the affine Weyl group (e.g. p-Kazhdan–Lusztig polynomials and Soergel bimodules). Here, we want to propose an approach to this program that may allow us to describe the structure of tensor products, the key difficulty being that the principal block is a priori not closed under tensor products. We overcome this problem by showing the existence of a certain "generic direct summand" $G(x, y)$ in the tensor product $L_x \otimes L_y$, for $x, y \in W_{\mathrm{aff}}^+$, which belongs to $\mathrm{Rep}_0(G)$ and captures some of the most important properties of the tensor product. We can further establish a reduction to the principal block, by which in order to understand tensor products of arbitrary simple G-modules, it essentially suffices to understand tensor products in $\mathrm{Rep}_0(G)$. Therefore, the next important challenge would be to understand the structure of the generic direct summands (in terms of combinatorics arising from W_{aff}). To that end, we propose the following approach: Let $\mathcal{G}r$ be the affine Grassmannian corresponding to the Langlands dual group over $\mathbb{C}$, with its stratification by Schubert cells

$\mathcal{G}r = \bigsqcup_{w \in W_{\text{aff}}^+} \Omega_w$. The fundamental classes of the Schubert varieties $X_w = \overline{\Omega_w}$ afford a basis of the homology ring

$$H_\bullet(\mathcal{G}r) = \operatorname{span}_{\mathbb{Z}}\{[X_w] \mid w \in W_{\text{aff}}^+\},$$

and as the latter is a Hopf algebra with convolution product, we can define structure constants via

$$[X_x] \cdot [X_y] = \sum_{z \in W_{\text{aff}}^+} a_{x,y}^z \cdot [X_z].$$

Conjecture. For $x, y, z \in W_{\text{aff}}^+$ such that $\ell(z) = \ell(x) + \ell(y)$, we have

$$\big[G(x,y) : L_z\big] = a_{x,y}^z.$$

Note that the structure constants $a_{x,y}^z$ also have a combinatorial interpretation if terms of the nil-affine Hecke and (in type A_{k-1}) via k-Schur functions, by work of Thomas Lam, in particular they can be computed explicitly. Some evidence for the conjecture comes from a forthcoming result of Sherman–Williamson et al., which loosely speaking relates tensor products of G-modules to tensor products of modules over the regular nilpotent centralizer. The latter can be seen equivalently as tensor products of $H^\bullet(\mathcal{G}r)$-modules, by an observation of Ginzburg and Yun–Zhu, and these tensor products are defined via the comultiplication on $H^\bullet(\mathcal{G}r)$. The latter is dual to the convolution product on $H_\bullet(\mathcal{G}r)$, which is in turn encoded in the structure constants $a_{x,y}^z$.

On the moment measure conjecture

Jochen Heinloth

(joint work with Xucheng Zhang)

To construct proper moduli spaces for a given moduli problem one often needs to introduce some stability condition, as there are usually so many degenerations that the space of all objects could not be Hausdorff. This is already appears for quotients by group actions $[X/G]$, where the orbit space will not be Hausdorff if the action has non-closed orbits. This issue always arises if G is a reductive group acting on a proper scheme. Unfortunately, we do not have many methods to find stability conditions. In the end these are often determined by the datum of a line bundle on the original moduli problem.

However, there are examples in which this does not cover all possibilities. As we also know geometric criteria that decide whether an open substack of a moduli problem admits a proper quotient, it seems natural to ask, whether there are other methods to find such open substacks.

The moment measure conjecture Bialynicki-Birula and Sommese from [1] gave a conjecturally complete combinatorial classification of such open subsets for quotients $[X/T]$ of torus actions on normal projective varieties, which in particular include quotients that are not projective and thus cannot be constructed through

GIT. We can now show [2] that this conjecture is true for quotients of the from $[X/T]$ where X is smooth, proper, carrying a line bundle that is positive on orbit closures, in particular this is satisfied if X is projective.

The conjecture was formulated in terms of a cell complex $\mathcal{C}(X)$ attached to $[X/T]$. This is defined as

$$\mathcal{C}(X) := \{c \subseteq \pi_0(X^T) \mid c = \mathrm{cell}(x) \text{ for some } x \in X\}$$

where $\pi_0(X^T)$ is the set of connected components of the fixed point set X^T and for any point cell(x) is the set of components that intersect the orbit closure $\overline{T}x$. A (geometric) moment-measure is then a particular choice of cells in $\mathcal{C}(X)$. Our main result is the following:

Theorem (Conjecture of Białynicki-Birula–Sommese, [1][2])**.** *Let X be a proper smooth variety over an algebraically closed field equipped with a torus action $T \times X \to X$ that admits a line bundle $\mathcal{L}$ that is ample on $\mathbb{G}_m$-orbit closures of closed points.*

Then a T-invariant open subscheme $U \subseteq X$ admits a proper (geometric) quotient if and only if it is defined by a (geometric) moment measure on the moment complex $\mathcal{C}(X)$.

The main idea of the argument is simple: In the case of geometric quotients, the rational top cohomology group of $[U/T]$ is one-dimensional, spanned by the cycle class of any closed point. For any closed point $x \in X$ the orbit closure $[\overline{Tx}/T]$ has an equivariant cycle class, or equivalently a cycle class in the cohomology of the quotient stack $[X/T]$, which restricts to a non-zero class in $H^*([U/T])$ if and only if $x \in U$. To prove the theorem we show that this class is uniquely determined by cell(x), by using that the cohomology of $[X/T]$ can be described by localization to the fixed point loci and an explicit computation of the localization of cycle classes of orbit closures.

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Automorphism Groups of Varieties – A Survey

Hanspeter Kraft

Some Basic Questions

Our base field $\Bbbk$ is algebraically closed. For a $\Bbbk$-variety X we denote by $\mathrm{Aut}(X)$ the group of *regular* automorphisms. Here are some basic questions:

- Topological structure? Algebraic structure?
- Which groups appear? (finite group, algebraic groups, simple groups? ...)

- Orbits of $\mathrm{Aut}(X)$ on X? Invariants? Transitive actions? ...
- Do we have a Lie algebra? Is it a subalgebra of the vector fields? Relation between the group and its Lie algebra?
- What implications for X can we retrieve from $\mathrm{Aut}(X)$? Is X determined by $\mathrm{Aut}(X)$?

Remark. If X is projective, then $\mathrm{Aut}(X)^\circ$ is an algebraic group scheme, and every connected algebraic group scheme appears (Brion-Schröer 2024, [BS24]). Thus we will mainly discuss the case where X is affine or quasi-affine.

Some History

Theorem 1 (Whittaker 1963, see [Whi63])**.** *If X and Y are "nice" topological spaces, then every isomorphism $\mathrm{Aut}_{cont}(X) \xrightarrow{\sim} \mathrm{Aut}_{cont}(Y)$ between the groups of continuous automorphisms is induced by a homeomorphism $X \xrightarrow{\sim} Y$. Thus X is completely determined by $\mathrm{Aut}_{cont}(X)$.*

(Here "nice" means a compact and locally euclidean manifold.)

Theorem 2 (Takens 1979, Filipkjewich 1982, see [Tak79], [Fil82])**.** *The same holds for differentiable manifolds X, Y where $\mathrm{Aut}_{diff}(X)$ is the group of diffeomorphisms.*

Corollary. Every automorphism of $\mathrm{Aut}_{cont}(X)$ and of $\mathrm{Aut}_{diff}(X)$ is inner.

We cannot expect such general results in the algebraic setting: There are many "nice" varieties with trivial automorphism group.

Ramanujam's Fundamental Result

Definition. A map $\varphi\colon A \to \mathrm{Aut}(X)$, A a variety, is a *morphism* if $A \times X \to X$, $(a, x) \mapsto \varphi(a)(x)$ is a morphism.

Products and inverses of morphisms are morphisms.

Theorem 3 (Ramanujam 1964, see [Ram64])**.** *Let $G \subseteq \mathrm{Aut}(X)$ be subgroup which is "connected" and "finite dimensional". Then G has a "universal structure" of a connected algebraic group.*

Definitions.
Connected: For every $g \in G$ there is a morphism $\varphi\colon A \to \mathrm{Aut}(X)$, A an irreducible variety, such that $\mathrm{id}_X, g \in \varphi(A) \subseteq G$.
Finite dimensional: There is an $N > 0$ such that, for any injective morphism $A \to \mathrm{Aut}(X)$ with image in G, one has $\dim A \le N$.
Universal structure: (a) The action map $G \times X \to X$ is a morphism; (b) If $A \to \mathrm{Aut}(X)$ is a morphism with image in G, then $A \to G$ is a morphism of varieties.

Remark. Let A be an irreducible variety and $\varphi\colon A \to \mathrm{Aut}(X)$ a morphism with $\mathrm{id}_X \in \varphi(A)$. Then the subgroup $G := \langle \varphi(A) \rangle$ is connected.

Definition. A subgroup $G \subseteq \mathrm{Aut}(X)$ generated by the image of a morphism $\varphi\colon A \to \mathrm{Aut}(X)$ where A is irreducible and $\mathrm{id}_X \in \varphi(A)$ is called *algebraically generated.*

Algebraically Generated Groups

Here are two very recent and far reaching generalizations of Ramanujam's Theorem.

Theorem 4 (S. Cantat, A. Regeta, J. Xie 2022, see [CRX23])**.** *Let X be an affine variety and $G \subseteq \mathrm{Aut}(X)$ an algebraically generated commutative subgroup. Then G is a connected commutative algebraic group.*

Theorem 5 (S. Cantat, H.K., A. Regeta, I. van Santen, 2025, to appear)**.** ($\mathrm{char}\,\Bbbk = 0$) *Let X be a quasi-affine variety and $G \subseteq \mathrm{Aut}(X)$ an algebraically generated solvable group. Then G is a connected solvable algebraic group.*

We will see later that these results have very strong consequences for the structure of $\mathrm{Aut}(X)$.

Ind-Varieties and ind-Groups

These objects have been introduced by Shafarevich in [Sha66] as "infinite-dimensional algebraic varieties"; see [FK18] and [Kum02] for a survey.

Basic example: A $\Bbbk$-vector space V^∞ of countable dimension.

- *Topology (Zariski-topology)*: $\mathcal{V} \subseteq V^\infty$ is closed $:\iff \mathcal{V} \cap V \subseteq V$ is closed for all finite dimensional subspaces $V \subset V^\infty$.
- *Morphisms:* $f\colon X \to V^\infty$ (X a variety) $:\iff$ (a) $f(X) \subseteq V \subset V^\infty$, V finite-dimensional subspace, and (b) $f\colon X \to V$ is a morphism.

Definition. An *(affine) ind-variety* is a set $\mathcal{V}$ with a filtration $\mathcal{V} = \bigcup_{k \geq 1} \mathcal{V}_k$ by (affine) algebraic varieties $\mathcal{V}_k$ such that $\mathcal{V}_k \subseteq \mathcal{V}_{k+1}$ is a closed subvariety.

Zariski-topology: A subset $U \subseteq \mathcal{V} = \bigcup_k \mathcal{V}_k$ is *closed* resp. *open* if the intersections $U \cap \mathcal{V}_k \subseteq \mathcal{V}_k$ are closed, resp. open.
Algebraic subset: A subset is *algebraic* if it is locally closed and contained in some $\mathcal{V}_k$.
Zariski tangent space: If $\mathcal{V} = \bigcup_k \mathcal{V}_k$ and $x \in \mathcal{V}$, then $T_x\mathcal{V} := \bigcup_k T_x\mathcal{V}_k$. This is a $\Bbbk$-vectors space of countable dimension.
Morphisms: Let $\mathcal{V} = \bigcup_k \mathcal{V}_k$, $\mathcal{W} = \bigcup_\ell \mathcal{W}_\ell$ be ind-varieties. A *morphism* $\varphi\colon \mathcal{V} \to \mathcal{W}$ is a map such that the following holds: For any k there is an ℓ such that
(i) $\varphi(\mathcal{V}_k) \subseteq \mathcal{W}_\ell$ and (ii) $\varphi\colon \mathcal{V}_k \to \mathcal{W}_\ell$ is a morphism of varieties.

Examples.
- Any variety X: $X_k := X$ for all k.
- A discrete countable set S; filtration by finite sets.
- $\mathrm{Mor}(\mathbb{A}^n, \mathbb{A}^n) \simeq \Bbbk[x_1, \ldots, x_n]^n$: filtration e.g. by degree.

Basic construction. Let R be a commutative $\Bbbk$-algebra of countable dimension (e.g. finitely generated), $\mathfrak{a} \subseteq R$ a subspace (e.g. an ideal), $P \subseteq \Bbbk[x_1, \dots, x_n]$ a set of polynomials. *Then the set*

$$\mathcal{V} := \{(a_1, \dots, a_n) \in R^n \mid f(a_1, \dots, a_n) \in \mathfrak{a} \textit{ for all } f \in P\} \subseteq R^n$$

is a closed subset of the ind-variety R^n, hence an affine ind-variety.

Example. Let $X \subseteq \Bbbk^n$ be a closed subset and R as above. Then the *R-rational points*

$$X(R) := \mathrm{Mor}(\operatorname{Spec} R, X) \subseteq \mathrm{Mor}(\operatorname{Spec} R, \Bbbk^n) = R^n$$

form a closed ind-subvariety.

Ind-Groups and Lie Algebras

Definitions (Ind-groups, tangent spaces, connected component)**.**
- An ind-variety $\mathcal{G}$ together with a group structure is an *ind-group* if the multiplication $\mu \colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and the inverse $\iota \colon \mathcal{G} \to \mathcal{G}$ are morphisms.
- $T_e\mathcal{G}$ has a natural structure of a *Lie algebra*, denoted by $\operatorname{Lie} \mathcal{G}$.
- The connected component $\mathcal{G}^\circ \subseteq \mathcal{G}$ is a closed ind-subgroup of countable index.

Examples.
- A linear algebraic group G.
- A countable group F; $\operatorname{Lie} F = \{0\}$.
- $\mathrm{GL}_n(R)$, R a commutative $\Bbbk$-algebra of countable dimension; then one gets $\operatorname{Lie} \mathrm{GL}_n(R) = \mathrm{M}_n(R)$.

Remarks. - Algebraic subgroups of $G \subseteq \mathcal{G}$ are closed, and $\operatorname{Lie} G \subseteq \operatorname{Lie} \mathcal{G}$.
- If $\operatorname{char} \Bbbk = 0$ and $G, H \subseteq \mathcal{G}$ are algebraic subgroups where H is connected and $\operatorname{Lie} H \subseteq \operatorname{Lie} G \subseteq \operatorname{Lie} \mathcal{G}$, then $H \subseteq G$. We will see later that this does not hold for closed ind-groups.

Automorphism Groups

Theorem 6 (Shafarevich 1966, see [Sha81])**.** *If X is affine, then* $\mathrm{Aut}(X)$ *is an affine ind-group.*

In fact, we have an embedding $\mathrm{Aut}(X) \hookrightarrow \mathrm{Mor}(X, X) \times \mathrm{Mor}(X, X)$, $g \mapsto (g, g^{-1})$, and the image is closed.

Theorem 7 (Furter-K. 2018, [FK18])**.** $\mathrm{Aut}(X) \underset{closed}{\subseteq} \mathrm{Dom}(X) \underset{open}{\subseteq} \mathrm{Mor}(X, X)$, *where* $\mathrm{Dom}(X)$ *is the set of dominant morphisms.*

Note: *A G-action on X is the same as a homomorphism $G \to$* $\mathrm{Aut}(X)$ *of ind-groups.*

Remark. We have a natural embedding $\operatorname{Lie} \mathrm{Aut}(X) \hookrightarrow \mathrm{Vec}(X)$ into the Lie algebra of algebraic vector fields on X, i.e., $\mathrm{Vec}(X) := \mathrm{Der}(\mathcal{O}(X))$.

What Kind of Groups Appear as $\mathrm{Aut}(X)$?

Proposition. ($\operatorname{char} \Bbbk = 0$)

(a) (Greenberg 1973) For any finite G there exists a compact Riemann surface S with $\mathrm{Aut}(S) \simeq G$.

(b) (Jelonek 2015), see [Jel15]) For any finite G there exists a smooth affine curve C with $\mathrm{Aut}(C) \simeq G$.

Examples.
- $\mathrm{Aut}(\mathbb{A}^1) = \mathrm{Aff}_1 = \Bbbk^* \ltimes \Bbbk^+$;
- $\mathrm{Aut}(\mathbb{A}^1 \setminus \{0\}) = \mathbb{Z}_2 \ltimes \Bbbk^*$;
- $\mathrm{Aut}(\mathbb{A}^1 \setminus \{a_1, \dots, a_k\})$ is trivial for $k \geq 3$ points in general position;
- $\dot{\mathbb{A}} := \mathbb{A}^1 \setminus \{0\}$, $\quad \mathrm{Aut}(\underbrace{\dot{\mathbb{A}} \times \dot{\mathbb{A}} \times \cdots \times \dot{\mathbb{A}}}_{m \text{ times}}) \simeq \mathrm{GL}_m(\mathbb{Z}) \ltimes (\Bbbk^*)^m$.

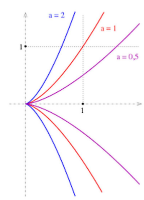
Proposition (Kraft 2017, see [Kra17]). If $\mathrm{Aut}(X)^\circ$ is an algebraic group, then either $X \simeq \mathbb{A}^1$ (and so $\mathrm{Aut}(X) = \Bbbk^* \ltimes \Bbbk^+$) or $\mathrm{Aut}(X)^\circ$ is a torus of dimension $\leq \dim X$.

Example ("Replica"). Consider an action of $\Bbbk^+$ on X given by $\rho\colon \Bbbk^+ \to \mathrm{Aut}(X)$. For $f \in J := \mathcal{O}(X)^{\Bbbk^+}$ we define a new action ("replica") setting $\rho_f(s)(x) := \rho(f(x)s)(x)$. It has generically the same orbits, if $f \neq 0$, hence the same invariants.

It follows that $f \mapsto \rho_f$ is a closed immersion $\Bbbk^+(J) = J^+ \hookrightarrow \mathrm{Aut}(X)$ of ind-groups. As a consequence, if X admits a non-trivial $\Bbbk^+$-action and $\dim X \geq 2$, then $\mathrm{Aut}(X)$ is infinite-dimensional.

Example (Furter-Kraft 2018, [FK18]). Let $C_1, \dots, C_m \subseteq \Bbbk^2$ be pairwise non-isomorphic cuspidal curves. *Then $\mathrm{Aut}(C_1 \times C_2 \times \cdots \times C_m)$ is an m-dimensional torus.*

Cuspidal curve: $C := \{x^r = y^s\} \subseteq \Bbbk^2$, $r \geq s \geq 2$ coprime:



Example ([FK18]). The smooth surface $S := \{x^2 + y^2 + z^2 = xyz\} \subseteq \Bbbk^3$ has a discrete (countable) automorphism group which contains the braid group B_3 with finite index.

Example. The automorphism group $\mathrm{Aut}(\mathbb{A}^2)$ contains Aff_2, the affine group, and Jonq_2, the de Jonquière subgroup:

$$\begin{aligned} \mathrm{Aff}_2 &:= \{(ax + by + r, cx + dy + s) \mid \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}_2(\Bbbk), r, s \in \Bbbk\} \\ &\simeq \mathrm{GL}_2(\Bbbk) \ltimes \underbrace{(\Bbbk^2)^+}_{\text{translations}} \qquad \text{(an algebraic group!)} \end{aligned}$$

$$\mathrm{Jonq}_2 := \{(ax + b, dy + p(x)) \mid a, d \in \Bbbk^*, p \in \Bbbk[x], b \in \Bbbk\} \quad (\infty\text{-dimensional!})$$

Proposition (Jung 1942, van den Kulk 1953, [Jun42], [vdK53]). The group $\mathrm{Aut}(\mathbb{A}^2)$ is the amalgamated product $\mathrm{Aff}_2 *_B \mathrm{Jonq}_2$ over $B := \mathrm{Aff}_2 \cap \mathrm{Jonq}_2$.

PROPERTIES OF X INDUCED BY $\mathrm{Aut}(X)$

Proposition (JELONEK 2015, see [Jel15]). (char $\Bbbk = 0$, X affine irreducible) If $\mathrm{Aut}(X)$ is infinite, then X is uniruled.

uniruled := $\exists$ a dominant rational map $\varphi \colon \mathbb{A}^1 \times Y \dashrightarrow X$, $\dim Y = \dim X - 1$.

For the next result we make the following definitions.

Definitions.
- Let $\mathcal{S}(X) \subseteq \mathrm{Aut}(X)$ be the subgroup generated by all $\Bbbk^+$-actions on X.
- X is *flexible* if $T_x(X)$ is spanned by the $\Bbbk^+$-orbits for all $x \in X_{\mathrm{reg}}$.

Proposition (I. ARZHANTSEV, H. FLENNER, S. KALIMAN, F. KUTZSCHEBAUCH, M. ZAIDENBERG 2013, see [AFK13]). (char $\Bbbk = 0$) X an affine variety of dimension ≥ 2. The following statements are equivalent:

(i) X is flexible;
(ii) $\mathcal{S}(X)$ is transitive on X_{reg};
(iii) $\mathcal{S}(X)$ is infinitely transitive on X_{reg}.

DOES $\mathrm{Aut}(X)$ DETERMINE X?

In her thesis JULIE DÉSERTI proved the following result related to the Theorems of WHITTAKER, TAKENS and FILIPKJEWICH, see Corollary to Theorem 1 and 2.

Proposition (J. DÉSERTI 2006, see [Dés06]). (char $\Bbbk = 0$) Every automorphism of $\mathrm{Aut}(\mathbb{A}^2)$ is inner, up to field automorphisms.

A partial generalization of this to the "tame" subgroup of $\mathrm{Aut}(\mathbb{A}^n)$ is obtained in [KS13].

Theorem 8 ([Kra17]). (char $\Bbbk = 0$) *Let X be connected. If* $\mathrm{Aut}(X) \simeq \mathrm{Aut}(\mathbb{A}^n)$ *as ind-groups, then $X \simeq \mathbb{A}^n$ as varieties.*

However, an isomorphism as ind-groups is a very strong assumption since it sends algebraic subgroups isomorphically onto algebraic subgroups!

Theorem 9 (S. CANTAT, A. REGETA, J. XIE 2022, see [CRX23]). (char $\Bbbk$ arbitrary) *Let X be a connected affine variety. If* $\mathrm{Aut}(X)$ *is isomorphic to* $\mathrm{Aut}(\mathbb{A}^n)$ *as an abstract group, then X is isomorphic to $\mathbb{A}^n$ as a variety.*

This is a consequence of Theorem 4: The image D of the subgroup of translations is (a) a *commutative algebraic group*, (b) is *unipotent*, (c) has *no invariants*, From that one concludes that D has a dense orbit, hence acts transitively and so X is an affine space.

From now one we assume char $\Bbbk = 0$.

Theorem 10 (LEUENBERGER-REGETA 2017, see [LR22]).

- *All generic* DANIELEWSKI-*surfaces D_p have abstractly isomorphic automorphism groups.*
  ($D_p := \{xy = p(z)\} \subseteq \Bbbk^3$), p a polynomial)

- *If* $\mathrm{Aut}(D_p)^\circ \simeq \mathrm{Aut}(D_q)^\circ$ *as ind-groups, or* $\mathrm{Lie}\,\mathrm{Aut}(D_p) \simeq \mathrm{Lie}\,\mathrm{Aut}(D_q)$*, or* $\mathrm{Vec}(D_p) \simeq \mathrm{Vec}(D_q)$ *as Lie-algebras, then* $D_p \simeq D_q$ *as varieties.*

This shows that there exist automorphism groups which are non-isomorphic as ind-groups, but isomorphic as abstract groups!

For surfaces we have the following result.

Theorem 11 (Liendo-Regeta-Urech 2018, see [LRU23])**.** *Let* S_1 *be an affine toric surface and* S_2 *a normal affine surface. If* $\mathrm{Aut}(S_1) \simeq \mathrm{Aut}(S_2)$*, then* $S_1 \simeq S_2$*.*

The main ingredient here is a *group-theoretic characterization* of the *algebraic elements* in $\mathrm{Aut}(S)$ for an affine surface S.

Definition. An automorphism $g \in \mathrm{Aut}(X)$ is *algebraic* if it belongs to an algebraic subgroup of $\mathrm{Aut}(X)$. It is *unipotent*, resp. *semisimple* if it belongs to a unipotent, resp. diagonalizable algebraic subgroup.

Lemma. Let S be an affine surface. Then an element $g \in \mathrm{Aut}(S)$ is algebraic if and only if a power g^k is divisible in $\mathrm{Aut}(S)$.

The proposition above was generalized to arbitrary affine toric varieties.

Theorem 12 (A. Regeta and I. van Santen 2022, see [RvS25])**.** *Let* X, Y *be irreducible normal affine varieties such that* $\mathrm{Aut}(X) \simeq \mathrm{Aut}(Y)$ *as abstract groups. If* X *is toric and not isomorphic to a torus, then* Y *is isomorphic to* X*. The same holds if* X *is smooth and spherical.*

The proof is based on the following lemma about unipotent elements.

Lemma. Let X, Y be irreducible affine varieties, and let $\theta\colon \mathrm{Aut}(X) \xrightarrow{\sim} \mathrm{Aut}(Y)$ be an isomorphism of (abstract) groups. Then θ maps unipotent elements to unipotent elements.

For the proof they show that *a maximal commutative subgroup consisting of unipotent elements is maximal among all commutative subgroups!*

Is Aut(X) Simple?

Let $\mathrm{SAut}(\mathbb{A}^n) \subseteq \mathrm{Aut}(\mathbb{A}^n)$ denote the kernel of the Jacobian determinant

$$\mathrm{jac}\colon \mathrm{Aut}(\mathbb{A}^n) \to \Bbbk^*,\ (f_1, \ldots, f_n) \mapsto \det(\frac{\partial f_i}{\partial x_j}).$$

It was shown in 1974 by Danilov that the group $\mathrm{SAut}(\mathbb{A}^2)$ is not simple as an abstract group [Dan74], cf. Furter-Lamy 2010, [FL10].

In his paper from 1966 Shafarevitch claims that $\mathrm{SAut}(\mathbb{A}^n)$ is simple as a topological group, i.e., there is no non-trivial closed normal subgroup:

- It is easy to see that $\mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n)$ is a simple Lie algebra.

- Then he “proves” that for a closed ind-subgroup $\mathcal{H} \subseteq \mathcal{G}$ of a connected ind-group $\mathcal{G}$ we have $\mathcal{H} = \mathcal{G}$ in case $\mathrm{Lie}\,\mathcal{H} = \mathrm{Lie}\,\mathcal{G}$, and he is done!

- But we gave a counterexample to this last statement in [FK18]!

Theorem 13 (J. Blanc 2024, see [Bla24])**.** (Any base field $\Bbbk$.) *A closed normal subgroup of* $\mathrm{SAut}(\mathbb{A}^n)$ *contains the tame subgroup, i.e., the subgroup generated by* SAff_n *and* SJonq_n*. In particular,* $\mathrm{SAut}(\mathbb{A}^2)$ *is simple as a topological group.*

Since one knows that $\langle \mathrm{Aff}_3, \mathrm{Jonq}_3 \rangle \subsetneqq \mathrm{Aut}(\mathbb{A}^3)$ (Shestakov-Umirbaev 2002, [SU03]), and so $\langle \mathrm{SAff}_3, \mathrm{SJonq}_3 \rangle \subsetneqq \mathrm{SAut}(\mathbb{A}^3)$, the problem is not yet solved!

Nested ind-Groups

Definition. An ind-group $\mathcal{G}$ is *nested* if it admits a filtration by algebraic groups.

We cannot expect that $\mathrm{Aut}(X)$ is nested, but large subgroups might be!

Examples.
- $(V^\infty)^+ = \bigcup_k V_k^+$, with V_k finite dimensional;
- $\mathrm{GL}_\infty(\Bbbk) = \bigcup_k \mathrm{GL}_k(\Bbbk)$, with obvious embeddings $\mathrm{GL}_k(\Bbbk) \hookrightarrow \mathrm{GL}_{k+1}(\Bbbk)$;
- The de Jonquière subgroup of $\mathrm{Aut}(\mathbb{A}^n)$:

$$\mathrm{Jonq}_n := \{(f_1, \ldots, f_n) \in \mathrm{Aut}(\mathbb{A}^n) \mid f_i \in \Bbbk[x_1, \ldots, x_i]\}.$$

Let R be a finitely generated integral $\Bbbk$-domain.

- $U_n(R)$ where $U_n \subseteq \mathrm{GL}_n$ are the upper triangular unipotent matrices;
- For a torus T we have $T(R)^\circ = T$, because $\Bbbk^*(R) = R^*$, the subgroup of invertible elements, and one knows that $R^*/\Bbbk^*$ is a finitely generated free abelian group.
- $B_n(R)^\circ$ where $B_n \subseteq \mathrm{GL}_n$ are the upper triangular matrices.

Groups Consisting of Algebraic Elements

As a consequence of the Theorems 4 and 5 by Cantat et al we get the following results.

Corollary.
(A) (char $\Bbbk$ arbitrary, X affine) A closed connected and commutative subgroup of $\mathrm{Aut}(X)$ is nested.
(B) (char $\Bbbk = 0$, X quasi-affine) A closed connected and solvable subgroup of $\mathrm{Aut}(X)$ is nested.

Theorem 14 (A. Perepechko 2024, [Per23], cf. [KPZ17])**.** *Let* $\mathcal{G} \subseteq \mathrm{Aut}(X)$ *be a connected nested subgroup. Then* $\mathcal{G}$ *is closed and of the form* $\mathcal{G} = G \ltimes R_u(\mathcal{G})$ *where* G *is a reductive algebraic subgroup and* $R_u(\mathcal{G})$ *a nested unipotent subgroup.* (Levi decomposition!)

In a nested subgroup of $\mathrm{Aut}(X)$ every element is algebraic. In the same paper Perepechko proves a partial converse of this.

Theorem 15 ([Per23])**.** *Let* $U \subseteq \mathrm{Aut}(X)$ *be a subgroup consisting of unipotent elements. Then the closure* $\bar{U} \subseteq \mathrm{Aut}(X)$ *is a nested unipotent subgroup.*

Relation between $\mathcal{G}$ and $\operatorname{Lie}\mathcal{G}$

The relation between a subgroup $\mathcal{G} \subseteq \operatorname{Aut}(X)$ and its Lie algebra $\operatorname{Lie}\mathcal{G} \subseteq \operatorname{Vec}(X)$ is still unclear. A brief overview of a few selected chapters can be found in [KZ24b]; it includes some open questions. Here is an interesting result in this direction.

Theorem 16 (Kraft-Zaidenberg 2014, see [KZ24a])**.** *Let $\mathcal{G} \subseteq \operatorname{Aut}(X)$ be a subgroup generated by a family $\{G_j\}_{j\in J}$ of algebraic groups. Then $\mathcal{G}$ is an algebraic group if and only if the Lie algebras $\operatorname{Lie} G_j \subseteq \operatorname{Vec}(X)$ generate a finite dimensional Lie algebra.*

We have already mentioned an example from [FK18] of a closed connected subgroup $\mathcal{G} \subseteq \operatorname{Aut}(\mathbb{A}^3)$ which contains a strict closed subgroup $\mathcal{H} \subsetneqq \mathcal{G}$ with $\operatorname{Lie}\mathcal{H} = \operatorname{Lie}\mathcal{G}$. Clearly, this cannot happen if $\mathcal{G}$ is nested.

Some Open Problems

(1) Let $\mathcal{G} \subseteq \operatorname{Aut}(X)$ be a connected closed subgroup and $\mathcal{N} \subsetneqq \mathcal{G}$ a proper normal and closed subgroup. *Does it follow that $\operatorname{Lie}\mathcal{N} \subsetneqq \operatorname{Lie}\mathcal{G}$?*

(2) Let $\mathcal{H}, \mathcal{G} \subseteq \operatorname{Aut}(X)$ be closed subgroups. Assume that $\mathcal{G}$ is connected, $\operatorname{Lie}\mathcal{G} \subseteq \operatorname{Lie}\mathcal{H}$, and that every element of $\mathcal{G}$ is algebraic. *Does it follow that $\mathcal{G} \subseteq \mathcal{H}$?*

(3) Let $\mathcal{G} \subseteq \operatorname{Aut}(X)$ be a closed subgroup consisting of algebraic elements. *Does this imply that $\mathcal{G}$ is nested?* More generally, let $G \subseteq \operatorname{Aut}(X)$ be an arbitrary subgroup consisting of algebraic elements. *Is the closure $\overline{G}$ nested?* (Cf. [PR23], [PR24])

(4) If $\operatorname{Aut}(X)$ is not discrete, i.e., $\operatorname{Aut}(X)^\circ$ is non-trivial, *does it follow that it contains a non-finite algebraic group, i.e., a copy of $\Bbbk^*$ or of $\Bbbk^+$?*

(5) *Is the closure of the normalizer of the tame subgroup $\langle \operatorname{SAff}_n, \operatorname{SJonq}_n\rangle \subseteq \operatorname{SAut}(\mathbb{A}^n)$ the whole group?*
This would imply that $\operatorname{SAut}(\mathbb{A}^n)$ is simple as a topological group.

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Alternating sign behavior for the product structure in T-equivariant quantum K-theory of flag varieties

Shrawan Kumar

Consider the T-equivariant (small) quantum K-theory $QK_T(X_P)$ defined by Kim and Givental, where $X_P = G/P$ for a semisimple group G and any parabolic subgroup P, and $T \subset P$ is a maximal torus. Let $\{[\mathscr{O}_w]\}_{w \in W/W_P}$ be the structure sheaf basis corresponding to the opposite Schubert variety $\overline{B^- wP/P} \subset X_P$ over the representation ring $R(T)$ of T, where W (resp. W_P) is the Weyl group of G (resp. P) and B^- is the opposite Borel subgroup. Then, it is conjectured that the structure constants of the product in $QK_T(X_P)$ in the structure sheaf basis exhibit an alternating sign behavior. Special cases of this conjecture have been obtained by Buch-Chaput-Mihalcea-Perrin, Weihong Xu, Lenart-Naito-Sagaki.

We recall the following result due to Syu Kato, where Gr_G is the infinite Grassmannian $G((t))/G[[t]]$: There exists a 'natural' isomorphism of algebras:

$$K_T(\mathrm{Gr}_G)_{\mathrm{loc}} \simeq QK_T(X_B)_{\mathrm{loc}},$$

which sends the Schubert basis of $K_T(\mathrm{Gr}_G)$ to the Schubert basis of $QK_T(X_B)$ (up to a Novikov monomial), where $K_T(\mathrm{Gr}_G)_{\mathrm{loc}}$ (resp. $QK_T(X_B)_{\mathrm{loc}}$) is a certain localization of $K_T(\mathrm{Gr}_G)$ (resp. $QK_T(X_B)$) and $K_T(\mathrm{Gr}_G)$ acquires its product via Pontryagin product. Moreover, for any standard parabolic subgroup P, there exists a surjective morphism of commutative algebras $QK_T(X_B) \to QK_T(X_P)$, which takes the Schubert basis of $QK_T(X_B)$ to the Schubert basis of $QK_T(X_P)$. Thus, the original alternating sign behavior in $QK_T(X_P)$ reduces to that of the corresponding alternating sign behavior for the Pontryagin product in the commutative ring $K_T(\mathrm{Gr})$ with respect to the Schubert basis. The aim of the talk is to report a precise conjecture on the alternating sign behavior in the Pontryagin product in $K_T(\mathrm{Gr})$ with respect to the Schubert basis and on the progress towards this conjecture.

Center of the small quantum group: towards a combinatorial model

Anna Lachowska

(joint work with Qi You, Nicolas Hemelsoet, Oskar Kivinen)

The small quantum group $\mathsf{u}_q(\mathfrak{g})$ is a finite-dimensional Hopf algebra associated to a semisimple Lie algebra $\mathfrak{g}$ and an (odd) lth root of unity q. It occurs as a subquotient of the big quantum group, as defined by Lusztig, generated by the divided powers of the usual Chevalley generators.

The problem of determining the structure of the centers of small quantum groups at roots of unity has a long history. Even before the small quantum group $\mathsf{u}_q(\mathfrak{g})$ was defined by Lusztig [8] for a semisimple Lie algebra $\mathfrak{g}$ and an l-th root of unity q, a similar problem was considered for the restricted enveloping algebra of a reductive algebraic group over a field of positive characteristic. In both quantum and modular cases the objects under consideration are finite-dimensional Hopf algebras whose structure are determined by a finite root system and an integer or a prime parameter. By the work of Andersen, Jantzen and Soergel [1], the principal blocks of both algebras are Morita equivalent to the same algebra (up to a base field change, with some restrictions on l), meaning that an answer for the structure of the center for one of them translates to the other.

Let

$$\mathsf{u}_q(\mathfrak{g}) = \oplus_{\lambda \in P/\tilde{W}_l} \mathsf{u}_\lambda$$

be the block decomposition of the Hopf algebra $\mathsf{u}_q(\mathfrak{g})$, where P is the root lattice and λ runs over the set of the orbits of the action of the extended affine Weyl group $\tilde{(W)}_l = W \ltimes lP$ in P. The same decomposition holds for the center of $\mathsf{u}_q(\mathfrak{g})$. A major breakthrough in the study of the center of $\mathsf{u}_q(\mathfrak{g})$ is given by the following result first established for the principal block containing the trivial representation [2] and then extended to the singular blocks in [6] [7] (with some restrictions on l):

Theorem 1. *Let $P \subset G$, $\mathfrak{g} = \mathrm{Lie}(G)$, be the parabolic subgroup whose Weyl group $W_P \subset W$ stabilizes λ. Let $\widetilde{\mathcal{N}}_P \simeq T^*(G/P)$ be the parabolic Springer resolution. Then there is an equivalence of triangulated categories*

$$D^b(\mathrm{Rep}(\mathsf{u})^\lambda) \cong D^b(\mathrm{Coh}_{\mathbb{C}^*}(\widetilde{\mathcal{N}}_P)).$$

Corollary 2. *Let $\lambda \in P$ be an integral weight of G, and P the parabolic subgroup whose Weyl group $W_P \subset W$ stabilizes λ. Then the Hochschild cohomology of the singular block $\mathrm{Rep}(\mathsf{u})^\lambda$ is isomorphic to the $\mathbb{C}^*$-equivariant Hochschild cohomology ring of the variety $\widetilde{\mathcal{N}}_P$, where $\mathbb{C}^*$ acts by dilations on the fibers:*

$$\mathrm{HH}^\star_{\mathbb{C}^*}(\widetilde{\mathcal{N}}_P) \cong \bigoplus_{i+j+k=\star} \mathrm{H}^i(\widetilde{\mathcal{N}}_P, \wedge^j T\widetilde{\mathcal{N}}_P)^k.$$

In particular, the center of the singular block $\mathsf{z}_\lambda := \mathsf{z}(\mathsf{u}_\lambda)$ is isomorphic as a commutative algebra

$$\mathsf{z}_\lambda \cong \mathrm{HH}^0_{\mathbb{C}^*}(\widetilde{\mathcal{N}}_P) \cong \bigoplus_{i+j+k=0} \mathrm{H}^i(\widetilde{\mathcal{N}}_P, \wedge^j T\widetilde{\mathcal{N}}_P)^k.$$

In general the right-hand side of the isomorphism in 2 is hard to compute. A careful study of the $\mathbb{C}^*$-equivariant cohomologies of the variety $\widetilde{\mathcal{N}}_P$ in several low rank cases allowed to formulate the combinatorial conjecture on the structure of the blocks of the center [6, 7] (with some restrictions on l):

Conjecture 3. *Let $W_P \subset W$ be the stabilizer subgroup of the weight λ, and let $\mathrm{DR}_W^{W_P}$ denote the canonical quotient of the diagonal coinvariant algebra $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{W_P}/\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W_+$, that is a module over the Cherednik algebra [4]. As a bigraded vector space the G-invariant part of the block of the center is isomorphic to $\mathrm{DR}_W^{W_P}$:*

$$\mathsf{z}_\lambda^G \cong \mathrm{DR}_W^{W_P}.$$

In particular, the dimension of z_λ^G equals to $(h+1)^{\mathrm{rk}(\mathfrak{g})}$, where h is the Coxeter number of $\mathfrak{g}$.

The next major advance is due to the work [3] where an isomorphism of algebras between the center $\mathsf{z}(\mathsf{u}_q(\mathfrak{g}))$ and certain cohomology of an affine Springer fiber was conjectured and later proved. The paper [5] in combination [3] uses this new geometric model to prove the following result, which partially confirms the conjecture 3 (with some restrictions on l):

Theorem 4. *Let $\lambda \in P$ be an integral weight of G, and P the parabolic subgroup whose Weyl group $W_P \subset W$ stabilizes λ. Then*

$$\dim \mathsf{z}_\lambda^G = \dim \mathrm{DR}_W^{W_P}.$$

Summing up for all blocks of $\mathsf{z}^G = \oplus_{\lambda \in P/\widetilde{W}_l} \mathsf{z}_\lambda^G$, we obtain the following result for the dimension of the G-invariant part of the center of the small quantum group [5]:

Theorem 5. *Let l be very good for $\mathfrak{g}$. Then*

$$\dim \mathsf{z}(\mathsf{u}_q(\mathfrak{g}))^G = \mathrm{Cat}_W((h+1)l - h, h),$$

where we denote by Cat_W the generalized rational Coxeter-Catalan numbers corresponding to the Weyl group W.
In particular, let $G = SL_n$ and suppose that n and $n+1$ are coprime to l. Then

$$\dim \mathsf{z}(\mathsf{u}_q(\mathfrak{g}))^G = \frac{1}{(n+1)l}\binom{(n+1)l}{n},$$

the rational $((n+1)l - n, n)$-Catalan number.

Several important questions remain open. In particular, the following conjecture was formulated in [7]:

Conjecture 6. *In type A we have $\mathsf{z}(\mathsf{u}_q(\mathfrak{g}))^G \cong \mathsf{z}(\mathsf{u}_q(\mathfrak{g}))$.*

It is known that this conjecture fails for types B_2, B_3 and G_2, but holds for A_n for $n \leq 4$.

Further, more precise combinatorial information on the structure of the center of the small quantum group is still missing. We would like to understand the bigraded structure of the center that is evident in its coherent geometric model 2, as well as its multiplicative structure. We would like to define a natural W-action on $\mathsf{z}(\mathsf{u}_q(\mathfrak{g}))$ that would turn the isomorphism 3 into an isomorphism of W-modules, up to tensoring with the sign representation. These are the directions of our subsequent research.

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Wonderful Embedding for group schemes in Bruhat–Tits theory

Shang Li

Let G be a connected quasi-split semisimple group of adjoint type over a strictly Henselian discretely valued field k with the ring of integers $\mathfrak{o}$, the maximal ideal $\mathfrak{m}$ and the perfect residue field κ. Let $\overline{G}$ be the wonderful compactification of G after De Concini and Procesi. This geometric object $\overline{G}$ plays an important role in Lie theory, enumerative geometry, representation theory, and arithmetic geometry. As a generalization of wonderful compactification, for a general reductive group (not necessarily adjoint), we have the theory of equivariant toroidal embeddings which are classified by combinatorial data.

In order to facilitate applying various useful results of wonderful compactification (and also of toroidal embedding) into Bruhat–Tits theory and arithmetic geometry, in this report, we propose a theory of wonderful embedding for concave function group schemes in the Bruhat–Tits theory.

To state our main results, we need to fix some notations. Let S be a maximal k-split torus which is contained in a Borel k-subgroup $B \subset G$. Let T be the centralizer of S in G. Then T is a maximal k-torus contained in B. Let B^- be the opposite Borel such that $B \bigcap B^- = T$, and let U and U^- be the unipotent radicals of B and B^-. Let $\Phi := \Phi(G, S)$ be the relative root system, and let $\hat{\Phi} = \Phi \bigcup \{0\}$. The choice of B gives a set of simple roots $\Delta \subset \Phi$.

Let $\mathcal{B}(G)$ be the Bruhat–Tits building of $G(k)$, and let $\mathcal{A}(S)$ be the apartment corresponding to S. To a point $x \in \mathcal{A}(S)$ and a concave function $f : \hat{\Phi} \to \mathbb{R}$, we can associate with an open bounded subgroup $G(k)_{x,f} \subset G(k)$ which is a central object of the Bruhat–Tits theory. These groups include parahoric subgroups, Moy–Prasad groups and Schneider–Stuhler groups as special cases. A fundamental algebro-geometric result of [3] and [6] is that $G(k)_{x,f}$ admits (necessarily uniquely) an integral model $\mathcal{G}_{x,f}$ over $\mathfrak{o}$. The group scheme $\mathcal{G}_{x,f}$ is rarely reductive, i.e., the unipotent radical $\mathscr{R}_u((\mathcal{G}_{x,f})_\kappa)$ of the special fiber $(\mathcal{G}_{x,f})_\kappa$ is rarely trivial. We denote by $\mathsf{G}_{x,f}$ the maximal reductive quotient of the special fiber $(\mathcal{G}_{x,f})_\kappa$.

Let $\mathscr{S}$ be the schematic closure of S in $\mathcal{G}_{x,f}$, and let S be the isomorphic image of the special fiber $\mathscr{S}_\kappa$ in the maximal reductive quotient $\mathsf{G}_{x,f}$. We will adopt the natural identifications of the character lattices and of cocharacter lattices:

$$X^*(S) \cong X^*(\mathscr{S}) \cong X^*(\mathrm{S});\ X_*(S) \cong X_*(\mathscr{S}) \cong X_*(\mathrm{S}). \tag{1}$$

Let $\Phi_{x,f} := \{a \in \Phi | f(a) + f(-a) = 0 \text{ and } f(a) \in \Gamma'_a\}$ which is identified with the root system of $\mathsf{G}_{x,f}$ with respect to the maximal torus S under the above identifications, where Γ'_a is the set of values of the root subgroup $U_a(k)$ with respect to x. Let $\Delta_{x,f} := \Delta \bigcap \Phi_{x,f} \subset \Phi_{x,f}$ which is a set of simple roots. Moreover, the negative Weyl chamber $\mathfrak{C} \subset X_*(S)_{\mathbb{R}}$ defined by the simple roots Δ is a cone inside the negative Weyl chamber $\mathfrak{C}_{x,f} \subset X_*(\mathrm{S})_{\mathbb{R}}$ defined by $\Delta_{x,f}$.

We establish the following result. For simplicity, in this report, we only state it under the assumption that G splits over k. The theorem is valid for quasi-split groups after replacing $\mathbb{A}_{1,\mathfrak{o}}$ with certain Weil restriction of it.

Theorem 1. *There is a unique smooth quasi-projective integral model $\overline{\mathcal{G}_{x,f}}$ over $\mathfrak{o}$ of $\overline{G}$ satisfying*

- (i) *the $(G \times_k G)$-equivariant open immersion $G \hookrightarrow \overline{G}$ extends to a $(\mathcal{G}_{x,f} \times_{\mathfrak{o}} \mathcal{G}_{x,f})$-equivariant open immersion $\mathcal{G}_{x,f} \hookrightarrow \overline{\mathcal{G}_{x,f}}$;*
- (ii) *the canonical k-open immersion $\overline{\Omega} := U^- \times_k \prod_\Delta \mathbb{A}_{1,k} \times_k U^+ \hookrightarrow \overline{G}$ extends to an $\mathfrak{o}$-open immersion*

$$\overline{\Omega_{x,f}} := \mathcal{U}^- \times_{\mathfrak{o}} \prod_{\Delta} \mathbb{A}_{1,\mathfrak{o}}^{(f(0))} \times_{\mathfrak{o}} \mathcal{U}^+ \hookrightarrow \overline{\mathcal{G}_{x,f}},$$

  where $\mathcal{U}^-$ and $\mathcal{U}^+$ are the schematic closures of U^- and U^+ in $\mathcal{G}_{x,f}$ and $\mathbb{A}_{1,\mathfrak{o}}^{(f(0))}$ is the unique integral model of $\mathbb{A}_{1,k}$ over $\mathfrak{o}$ such that $\mathbb{A}_{1,\mathfrak{o}}^{(f(0))}(\mathfrak{o}) = 1 + \mathfrak{m}^{f(0)}$;
- (iii) *$(\mathcal{G}_{x,f} \times_{\mathfrak{o}} \mathcal{G}_{x,f}) \cdot \overline{\Omega_{x,f}} = \overline{\mathcal{G}_{x,f}}$, i.e., the group action morphism is surjective.*

Moreover we have

1. *if $f(0) = 0$, the quotient sheaf $(\overline{\mathcal{G}_{x,f}})_\kappa / (\mathscr{R}_u((\mathcal{G}_{x,f})_\kappa) \times_\kappa \mathscr{R}_u((\mathcal{G}_{x,f})_\kappa))$ is represented by a scheme and the natural morphism*

$$\mathsf{G}_{x,f} \longrightarrow (\overline{\mathcal{G}_{x,f}})_\kappa / (\mathscr{R}_u((\mathcal{G}_{x,f})_\kappa) \times_\kappa \mathscr{R}_u((\mathcal{G}_{x,f})_\kappa))$$

   induced by the open immersion $\mathcal{G}_{x,f} \hookrightarrow \overline{\mathcal{G}_{x,f}}$ is an equivariant toroidal embedding *of the split reductive group $\mathsf{G}_{x,f}$ determined by the cone $\mathfrak{C}$ in the negative Weyl chamber $\mathfrak{C}_{x,f}$ in the sense of* [1, Definition 6.2.2]*;*
2. *if $f(0) > 0$, the special fiber of $\overline{\mathcal{G}_{x,f}}$ degenerates to the special fiber $(\mathcal{G}_{x,f})_\kappa$;*
3. *$\overline{\mathcal{G}_{x,f}}$ is* projective *over $\mathfrak{o}$* if and only if *$\mathcal{G}_{x,f}$ is* a reductive group scheme *over $\mathfrak{o}$. This is the case, for instance, if $f = 0$ and $x \in \mathcal{B}(G)$ is a hyperspecial point;*
4. *the boundary $\overline{\mathcal{G}_{x,f}} \backslash \mathcal{G}_{x,f}$ is covered by $(\mathcal{G}_{x,f} \times_{\mathfrak{o}} \mathcal{G}_{x,f})$-stable smooth $\mathfrak{o}$-relative effective Cartier divisor S_α for $\alpha \in \Delta$ with $\mathfrak{o}$-relative normal crossings.*

We first remark that the existence part of Theorem 1 is only nontrivial when $f(0) = 0$ because the (2) says that, when $f(0) > 0$, the $\overline{\mathcal{G}_{x,f}}$ is simply a gluing of $\overline{G}$ and $\mathcal{G}_{x,f}$ along the open subscheme G. This is similar to the structure of the group scheme $\mathcal{G}_{x,f}$ when $f(0) > 0$. Moreover, our wonderful embedding $\overline{\mathcal{G}_{x,f}}$ is compatible with dilatation operation on the group scheme $\mathcal{G}_{x,f}$.

We study the Picard group of $\overline{\mathcal{G}_{x,f}}$ and its generators, which are similar to those of the classical wonderful compactification $\overline{G}$.

Theorem 2. *The boundary $\overline{\mathcal{G}_{x,f}}\backslash\overline{\Omega_{x,f}}$ is covered by prime effective Cartier divisors*

$$\mathbf{D}_\alpha := \overline{B\cdot \dot{s}_\alpha \cdot B^-}, \alpha\in\Delta$$

where the bar indicates taking the schematic closure in $\overline{\mathcal{G}_{x,f}}$ and $\dot{s}_\alpha \in N_G(S)(k)$ is a representative of the simple reflection along α in the relative Weyl group of G with respect to S. Moreover the Picard group Pic$(\overline{\mathcal{G}_{x,f}})$ is freely generated by $\{\mathbf{D}_\alpha|\alpha\in\Delta\}$.

Beyond the quasi-split case, we construct our wonderful embedding by *étale descent.* We now consider an adjoint reductive (not necessarily quasi-split) group G over a Henselian discretely valued field k with perfect residue field and ring of integers $\mathfrak{o}$. Let K be a maximal unramified extension of k, and let $\mathcal{O}\subset K$ be the ring of integers. By a theorem of Steinberg, G_K is quasi-split over K. We choose a maximal k-split torus S with $\Phi(S) := \Phi(S,G)$ and $\hat{\Phi}(S) := \Phi(S)\bigcup\{0\}$. Let T be a special k-torus containing S. For a concave function $\widetilde{f} : \hat{\Phi}(T_K)\to\mathbb{R}$ the concave function obtained by compositing a concave function $f : \hat{\Phi}(S) \longrightarrow \mathbb{R}$ with the restriction map $X^*(T_K)\to X^*(S_K) = X^*(S)$ and a point x in the apartment $\mathcal{A}(S)$ of the building $\mathcal{B}(G)$ corresponding to S, by Bruhat–Tits theory, we have the $\mathfrak{o}$-group scheme $\mathcal{G}_{x,f}$ obtained by étale descent from the $\mathcal{O}$-group scheme $\mathcal{G}_{x,\widetilde{f}}$. Let $\overline{\mathcal{G}_{x,\widetilde{f}}}$ be the integral model over $\mathcal{O}$ by applying Theorem 1 to $\mathcal{G}_{x,\widetilde{f}}$.

Theorem 3. *The $(\mathcal{G}_{x,\widetilde{f}}\times_{\mathcal{O}}\mathcal{G}_{x,\widetilde{f}})$-scheme $\overline{\mathcal{G}_{x,\widetilde{f}}}$ over $\mathcal{O}$ descends to a smooth quasi-projective $(\mathcal{G}_{x,f}\times_{\mathfrak{o}}\mathcal{G}_{x,f})$-scheme $\overline{\mathcal{G}_{x,f}}$ over $\mathfrak{o}$ which equivariantly contains $\mathcal{G}_{x,f}$ as an open dense subscheme, and the boundary $\overline{\mathcal{G}_{x,f}}\backslash\mathcal{G}_{x,f}$ is an $\mathfrak{o}$-relative Cartier divisor with $\mathfrak{o}$-relative normal crossings. Moreover, $\overline{\mathcal{G}_{x,f}}$ is $\mathfrak{o}$-projective if and only if $\mathcal{G}_{x,f}$ is a reductive group scheme over $\mathfrak{o}$.*

Since the scheme $\overline{\mathcal{G}_{x,f}}$ behaves in many aspects similarly to the wonderful compactification $\overline{G}$ (when $f=0$), Theorem 1 and Theorem 3 can serve as a first step towards the Bruhat–Tits theory for wonderful compactification. It is also expected to expend our results to more general wonderful varieties in the sense of [4].

Remark 4. Recall that, when G splits over k, the set of the $(G\times_k G)$-orbits of the wonderful compactification $\overline{G}$ are bijective to the set $\Delta\times\{0,1\}$. This has the following affine analogue. If $f(0)=0$, the $(\mathcal{G}_{x,f}(\mathfrak{o})\times\mathcal{G}_{x,f}(\mathfrak{o}))$-orbits of $\overline{\mathcal{G}_{x,f}}(\mathfrak{o})$ are bijective to the set $\prod_{\alpha\in\Delta}(\Gamma_\alpha\bigcap\mathbb{R}_{\geq 0})$. Given such a parallel between $\overline{G}$ and $\overline{\mathcal{G}_{x,f}}$, we ask if there is a classification of $(G(k)_{\mathcal{C},0}\times G(k)_{\mathcal{C},0})$-orbits in $\overline{\mathcal{G}_{x,f}}(\mathfrak{o})$, where $\mathcal{C}\subset\mathcal{B}(G)$ is a chamber containing x and $G(k)_{\mathcal{C},0}$ is thus an Iwahoric subgroup, as an analogue of the classification of $(B\times_k B)$-orbits in $\overline{G}$ [2, § 2.1]; if we have a description of $\mathcal{G}_{x,f}(\mathfrak{o})_{\mathrm{Diag}}$-stable pieces in $\overline{\mathcal{G}_{x,f}}(\mathfrak{o})$ as an analogue of [5].

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Modular ramified geometric Satake equivalence

João Lourenço

(joint work with Pramod Achar, Timo Richarz, Simon Riche)

Recall that the geometric Satake equivalence of Mirković–Vilonen [MV07] consists of an equivalence of monoidal categories between perverse sheaves on the Hecke stack Hk_G of a reductive group G and representations of the Langlands dual $G^\vee$. Nowadays, many variants for different flavors of sheaf theories and geometric objects have been established.

In our talk, we presented work [ALRR22, ALRR24] on a ramified version of the geometric Satake equivalence with integral and modular coefficients. This builds on a theme initiated by Zhu–Richarz [Zhu15, Ric16] for ℓ-adic coefficients with $\ell \neq p$, which allows for ramification in the reductive group under consideration. To be more precise, let us introduce some notation: we let k denote an algebraic closure of $\mathbb{F}_p$, $O = k[\![t]\!]$ its power series ring, and $F = k(\!(t)\!)$ its Laurent series field; G is a reductive F-group and $\mathcal{G}$ a special parahoric O-model of G. Note that we no longer assume G is a split reductive k-group, but rather *possibly nonsplit* reductive F-group. We now extend the previous work of Zhu–Richarz to coefficients in an algebraic extension of $\mathbb{F}_\ell$ or an integral closure of $\mathbb{Z}_\ell$ in an algebraic extension of $\mathbb{Q}_\ell$ with $\ell \neq p$:

Theorem 1. *There is a symmetric monoidal equivalence* $\mathrm{Perv}(\mathrm{Hk}_{\mathcal{G}}) \simeq \mathrm{Rep}((G^\vee_\Lambda)^I)$, *intertwinning nearby cycles* $\mathrm{Perv}(\mathrm{Hk}_G) \to \mathrm{Perv}(\mathrm{Hk}_{\mathcal{G}})$ *with the restriction map* $\mathrm{Rep}(G^\vee_\Lambda) \to \mathrm{Rep}((G^\vee_\Lambda)^I)$.

In the next sections, we explain the various concepts appearing in the statement and sketch the main ideas in the proofs.

1. Pinned fixed points as group schemes

In Zhu–Richarz, the dual group is given by the scheme-theoretic inertia fixed points $(G^\vee_{\mathbb{Q}_\ell})^I$ over $\mathbb{Q}_\ell$, where the natural I-action preserves the canonical pinning of $G^\vee$. This $\mathbb{Q}_\ell$-group is possibly disconnected, but always reductive. (It also only depends on G, and not on the choice of special facet, which is surprising, given that odd unitary groups have 2 conjugacy classes of special parahorics with quite distinct reductive quotients.) In the case of modular or integral coefficients Λ, it is thus natural to expect the dual group to equal the *scheme-theoretic* fixed points $(G^\vee_\Lambda)^I$ of fixed points of the action as above on the dual split reductive group scheme $G^\vee_\Lambda$ over the given ring of coefficients Λ. Little seemed to be known about the $\mathbb{Z}_\ell$-scheme, so we proved the following result in [ALRR22]:

Theorem 2. *The $\mathbb{Z}_\ell$-group scheme $G^\vee_{\mathbb{Z}_\ell}$ is flat with reduced fibers being isogenous reductive group schemes. It is smooth if and only if there is no pair of roots $\{\alpha, \ell\alpha\} \subset \Phi_G$ and $X_*(T)_I$ is ℓ-torsion free. It is connected if and only if $X_*(T)_I$ has no ℓ-primary torsion.*

The proof proceeds by a big cell argument and then a case-by-case analysis of the various fixed points, exploiting crucially the stability of the pinning under the I-action. The conditions on $X_*(T)_I$ refer to toral behavior, and the condition of $\{\alpha, \ell\alpha\}$ reflects the non-smoothness of $\mathrm{GL}^{\mathbb{Z}/2\mathbb{Z}}_{2n+1,\mathbb{Z}_2}$ arising from $G = \mathrm{U}_{2n+1}$ and only when $\ell = 2$. This recovers only one of the many examples of non-reductive quasi-reductive groups in the sense of Prasad–Yu [PY06].

2. Ramified affine Grassmannians

Let $L^+\mathcal{G}$ be the positive loop group of our fixed special parahoric model and LG be the loop group of our reductive group. We define $\mathrm{Gr}_\mathcal{G}$ as the étale quotient $LG/L^+\mathcal{G}$, and this is an ind-projective ind-scheme. The assumption that $\mathcal{G}$ fixes a special facet is necessary to ensure a certain parity condition on $L^+\mathcal{G}$-orbit dimensions in $\mathrm{Gr}_\mathcal{G}$, without which a geometric Satake equivalence cannot hold. Note that the reduction of $\mathrm{Gr}_\mathcal{G}$ is the colimit of all its $L^+\mathcal{G}$-orbit closures $\mathrm{Gr}_{\mathcal{G},\leq\bar{\nu}}$, where $\bar{\nu} \in X_*(T)^+_I$. As usual, we also study semi-infinite orbits in $\mathrm{Gr}_\mathcal{G}$ and their relationship to attractors, filling some gaps on their geometry in the paper of Haines–Richarz [HR21]. Intersecting these orbits with Schubert varieties, we get the corresponding Mirković–Vilonen cycles, and the topological results required for proving semi-smallness of convolution in the ramified case stem from [AGLR22].

There is another important geometric input: the Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{\mathcal{G},\mathcal{O}}$ arising as the base change of a construction in [BD91]. Its special fiber is precisely the ind-scheme $\mathrm{Gr}_\mathcal{G}$ considered in the previous paragraph and its generic fiber identifies with the constant affine Grassmannian Gr_G (now with residue field F instead of k). We can take nearby cycles along this geometric space, and this allows us to link the unramified with the ramified Hecke category. This is how the intertwinning of unramified and ramified geometric Satake arises in our main theorem, and plays a major role in computing the Tannakian dual group.

3. The fiber functor

With characteristic 0 coefficients, the ramified statement can be deduced almost immediately from the unramified one thanks to semisimplicity of $\mathrm{Rep}((G^\vee_{\mathbb{Q}_\ell})^I)$ and to a result of Bezrukavnikov regarding central functors from representations of algebraic groups. While nearby cycles still work integrally, the remaining facts become inaccessible. We therefore focus on following [MV07].

The first step lies in studying weight functors arising from constant terms CT_P (also known as Braden's hyperbolic localization for the corresponding attractor correspondence), so that we can prove monoidality of the obvious fiber functor H^* of total cohomology (and of all the CT_P themselves). In [MV07] the fusion product induced by the Beilinson–Drinfeld deformation is used to endow the fiber functor with a monoidal structure: unfortunately, this deformation takes us from ramified to unramified groups generically, so it is not of great help for us. The fusion product is even more crucial when it comes to proving symmetry of the monoidal structure, but we handle this differently in the next section.

We bypass the difficulty regarding the lack of a fusion product in a different way, constructing this monoidal structure by exploiting parity properties of orbit dimensions. Indeed, it is relatively easy to write down a filtration on total cohomology relating to closures of semi-infinite orbits, and still easy to identify the successive subquotients. It is much harder to show that the filtration is naturally split, and that is where parity enters.

We warn the reader that there is an extremely pervasive error in the literature going back to [MV07], claiming that one can construct a complementary filtration via natural geometric operations: every single paper claiming this contains a sign mistake, so that the two complementary filtrations are one and the same.

4. Tannakian reconstruction

The next goal is to identify the monoidal category $\mathrm{Perv}(\mathrm{Hk}_{\mathcal{G}}, \Lambda)$ with the category of finitely generated $\mathrm{B}_{\mathcal{G}}(\Lambda)$-comodules for some flat Λ-bialgebra $\mathrm{B}_{\mathcal{G}}(\Lambda)$. We cannot directly apply the Tannakian formalism as Λ is not a field, but nonetheless an easy variant of Barr–Beck yields the existence of the desired Λ-bialgebra. The formation of $\mathrm{B}_{\mathcal{G}}$ is compatible with extension of scalars in Λ. The missing symmetry constraint can be deduced now by checking commutativity of the bialgebra as follows: the $\mathbb{Q}_\ell$-bialgebra is already known by Zhu–Richarz to be commutative; the $\mathbb{Z}_\ell$-bialgebra embeds in the previous one by flatness; and the $\mathbb{F}_\ell$-bialgebra is obtained as a quotient. It is not too hard to check now that its spectrum is a Λ-group scheme $\mathcal{G}^\vee_\Lambda$.

Next, we exploit the nearby cycles functor and the Tannakian formalism to get a map of Λ-groups $\mathcal{G}^\vee_\Lambda \to G^\vee_\Lambda$. This is easily seen to factor through the inertia fixed points $(G^\vee_\Lambda)^I$ and our goal is to show the isomorphy of the resulting map. To prove this, we first treat the case of relative rank 1, and exploit once again the possibility of transferring information between the different coefficient rings. The SL_2 case is quite straightforward and one simply follows the argument of Fargues–Scholze [FS21]. In the SU_3, there is some extra subtlety involved only when $\ell = 2$, and we rely on the proof in [ALRR22] that $(G^\vee_{\mathbb{Z}_2})^I$ cannot be dilated into becoming a group scheme with reduced fiber (it is crucial that no square root of 2 is allowed in the coefficients! Afterwards, one can calmly base change to $\bar{\mathbb{Z}}_2$). During the talk, Sean Cotner asked if we were simply proving normality of this $\mathbb{Z}_2$-group, but this does not hold true: in fact, the normalization turns out not to be a group scheme in this particular case.

Once again, we would like to emphasize the weirdness of these group schemes: it is quite stunning that they naturally arise as the Tannakian group for a category of perverse sheaves. Finally, we should mention that the result also holds for the Witt Grassmannians of Zhu, and even with motivic $\mathbb{Z}[1/p]$-coefficients by work of van den Hove [vdH24].

5. Underlying motivation and future directions

The main motivation for the constructions in [Zhu15, Ric16] was the application to some properties of Shimura varieties. At this point it does not seem that our integral and modular versions lead to any specific new application in this direction; in fact our desire to establish them came from representation theory. Namely, in many cases the group $(G^\vee_{\mathbb{F}_\ell})^I$ is still a reductive group. It was conjectured by Brundan [Bru98], and proved by him in most cases, that the restriction along $(G^\vee_{\mathbb{F}_\ell})^I \to G^\vee_{\mathbb{F}_\ell}$ preserves tilting modules. The remaining cases were later treated by van der Kallen [vdK01] on a case-by-case basis. Our hope is that the geometric description of the restriction functor in terms of nearby cycles will shed some light behind the actual reason behind this property. Recently, Fargues–Scholze [FS21] give a uniform proof of this result, assuming the I-action factors through a p-prime quotient, but not assuming it respects a pinning, see [FS21, Theorem VIII.5.15]. It would be interesting to see whether the coprimality assumption can be removed in our setting.

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Parabolic and horospherical subgroup schemes of reductive groups in small characteristics

Matilde Maccan

(joint work with Ronan Terpereau)

This report summarizes the results presented in [8] and [9], along with the strategy of the ongoing work in [10] and [11]. It is structured in three parts. The first section presents a complete classification of non-reduced parabolic subgroup schemes of reductive groups across all characteristics. The second discusses the connected component of the automorphism group of the associated homogeneous spaces, extending classical results of Demazure. The final section outlines the first steps towards a classification of horospherical subgroups, which naturally generalize parabolics.

We work over an algebraically closed field of prime characteristic $p > 0$. Let G be a simple adjoint group with a fixed Borel subgroup B and maximal torus $T \subset B$. For a simple root α (with respect to the chosen pinning), the associated maximal reduced parabolic subgroup is denoted P^α; it is characterized by having trivial intersection with the negative root subgroup associated to $-\alpha$. A key object is the kernel ${}_mG$ of the m-th iterated relative Frobenius homomorphism of G, which is by construction an infinitesimal subgroup. All group schemes considered are of finite type over the base field and need not be smooth.

1. Parabolic subgroup schemes

Flag varieties are among the few classes of algebraic varieties that allow for concrete computations. In positive characteristic, there exist twisted counterparts, namely projective homogeneous spaces with possibly non-reduced parabolic stabilizers. Their geometry appears to differ significantly from that of classical flag varieties, and their classification–originally due to Haboush, Lauritzen, and Wenzel for characteristics at least five–has now been completed in full generality.

In small characteristics (two and three), some additional exotic objects arise. More precisely, if the Dynkin diagram of G has an edge of multiplicity p–which occurs for types B_n, C_n, F_4 when $p = 2$, and type G_2 when $p = 3$–one can define the *very special isogeny* of such a group. This is a finite map with source G whose kernel

K is minimal among normal noncentral subgroups killed by Frobenius; see [3] and the original work of [2] for more details. The subgroup K is uniquely determined by its Lie algebra, which is generated by the subspaces corresponding to *short* roots.

When G is an exceptional group of type G_2 and $p = 2$, there are two additional parabolic subgroups Q_1 and Q_2 whose reduced part is P^{α_1}, where α_1 denotes the short simple root and α_2 the long one. Their respective Lie subalgebras are:

$$\mathrm{Lie} Q_1 = \mathrm{Lie} P^{\alpha_1} \oplus \mathfrak{g}_{-2\alpha_1-\alpha_2}, \quad \mathrm{Lie} Q_2 = \mathrm{Lie} P^{\alpha_1} \oplus \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{-\alpha_1-\alpha_2}.$$

We now have all the ingredients to state the main classification results.

Theorem 1. *Let $P \subset G$ be a parabolic subgroup with reduced part P^α for some simple root α. Up to taking the quotient by some Frobenius kernel ${}_G$, the parabolic P is one of*

$$P^\alpha, \quad KP^\alpha, \quad Q_1, \quad Q_2,$$

where K denotes the kernel of the very special isogeny, and Q_i are the two exotic parabolics.

Theorem 2. *Let $P \subset G$ be a parabolic subgroup. Suppose P_{red} is the intersection of $P^{\beta_1}, \ldots, P^{\beta_r}$ for simple roots β_i. Then*

$$P = \bigcap_{i=1}^{r} Q^i,$$

where each Q^i is the smallest subgroup containing both P and P^{β_i}. In particular, any parabolic subgroup is an intersection of parabolics with maximal reduced part.

2. The connected automorphism group of a flag variety

Very little is known about the geometry of homogeneous varieties $X = G/P$, where G is simple adjoint and P is a non-reduced parabolic. A few works on the subject include [5], [7], and [12]. A natural question is to determine exactly when

$$\iota \colon G \hookrightarrow \underline{\mathrm{Aut}}^0_X$$

is an isomorphism. Moreover, when ι is a strict inclusion, one can ask about the structure of the connected automorphism group, viewed as a (possibly non-reduced) group scheme. In the case of a reduced parabolic, Demazure showed that ι is always an isomorphism, except for the following three Picard rank one cases:

- $G = \mathrm{PSp}_{2n}, \alpha = \alpha_1$, whose associated homogeneous space is the projective space of dimension $2n - 1$;
- $G = \mathrm{SO}_{2n+1}, \alpha = \alpha_n$, whose space parametrizes totally isotropic subspaces of k^{2n+1} of dimension n;
- $G = G_2, \alpha = \alpha_1$, whose space is a smooth quadric in $\mathbb{P}^6$.

We refer to such P^α as *exceptional*.

In characteristic $p > 0$, the only known example to the author is [1, Proposition 4.3.4], which proves non-reducedness via Lie algebra computations. The following generalizes this example to all types and characteristics, and is the main result of the ongoing work in [10].

Theorem 3. *Let $X = G/P$ of Picard rank at least two, where the stabilizer is of the form*

$$P = P_J \cap (\ker \xi) P'.$$

Assume ξ is a non-central isogeny and is minimal with respect to inclusion.

(1) If $P_J \neq P^\alpha$ for an exceptional root α, then $\underline{\mathrm{Aut}}^0_X = G$.
(2) If $P_J = P^\alpha$ is exceptional, there is a unique $m \geq 0$ such that

$$\underline{\mathrm{Aut}}^0_X = {}_m\hat{G} \cdot G \subset \hat{G},$$

where $\hat{G}$ denotes the connected automorphism group of G/P^α.

A very short sketch of the proof is to combine Demazure's theorem with results from [8] on contractions of Schubert curves on X, Blanchard's Lemma, and explicit computations on the Lie algebra of $\hat{G}$ as a G-module.

3. Horospherical subgroup schemes

A natural generalization of parabolic subgroups is the notion of *horospherical subgroups.* These are, up to conjugation, subgroups of G containing the unipotent radical U of the Borel subgroup B. In characteristic zero, they are well studied because the associated homogeneous spaces–and their equivariant compactifications–offer a good compromise between spherical and flag varieties, with a rich combinatorial structure. This final section is joint work in progress with Ronan Terpereau.

A subgroup $H \subset G$ is *horospherical* if it contains a maximal unipotent subgroup. We say that it is *strongly horospherical* if its normalizer $N_G(H)$ is a parabolic subgroup scheme. In characteristic zero, all horospherical subgroups are strongly horospherical. This fails in positive characteristic, as shown by the following example, due to [6].

Example 4. Let $p = 2$ and define

$$H = \left\{ \begin{pmatrix} a & b \\ a + a^3 & a^3 + (1 + a^2)b \end{pmatrix} : a^4 = 1 \right\} \subset \mathrm{SL}_2.$$

Then $H_{\mathrm{red}} = U$, but $N_{\mathrm{SL}_2}(H) = H$, so H is horospherical but not strongly horospherical.

Conjecture 5. *Let $H \subset G$ be a horospherical subgroup over a field of characteristic $p \geq 3$. Then H is strongly horospherical.*

Currently, the authors can only show that if $p \geq 3$, then Lie$H$ is normalized by T. In characteristic two, we are working towards a classification of strongly horospherical subgroups using root systems and the known classification of parabolics.

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Quotients of LLG

Andrea Maffei

(joint work with Valerio Melani, Gabriele Vezzosi)

This is a report on our joint work contained in [3]. Given a smooth affine algebraic group G over an algebraically closed field k, the aim of our work is to study certain quotients of $G(k(\!(x)\!)(\!(y)\!))$, for example by the group $G(k(\!(x)\!)[\![y]\!])$ or by the group $G(k[\![x]\!](\!(y)\!))$. We want to thank Philippe Gille for explaining to us, during the conference in Oberwolfach, how to extend a result of our paper to a more general class of groups (see Remark 4).

Let us recall some definitions. If $\mathcal{F}$ is a functor from k-algebras to sets, the loop space $L\mathcal{F}$ of $\mathcal{F}$ and the jet space $J\mathcal{F}$ of $\mathcal{F}$, on a k-algebra R, are defined as

$$L\mathcal{F}(R) = \mathcal{F}\big(R((t))\big) \qquad J\mathcal{F}(R) = \mathcal{F}\big(R[[t]]\big).$$

If G is an affine algebraic group over k, its affine Grassmannian is the presheaf $R \mapsto LG(R)/JG(R)$. By a result of Česnavičius, this presheaf is actually a sheaf for the fppf topology (see [1] and [3], Corollary 1.3). It can be shown that the affine Grassmannian is an ind-scheme, and moreover, we can interpret the affine Grassmannian as a space of G-torsors on the formal disk of a point on a smooth curve, with a trivialization on the punctured disk. In our work, we focused on studying analogous properties for quotients of the group LLG.

Ind-representability. The quotients of LLG that we considered in our study are the following.

The affine Grassmannian of the loop group is the presheaf quotient LLG/JLG. We denote it as

$$\mathcal{G}r^L_G : R \longmapsto \frac{LLG(R)}{JLG(R)} = \frac{G\big(R(\!(t)\!)(\!(s)\!)\big)}{G\big(R[\![t]\!](\!(s)\!)\big)}.$$

This is an analogue of the affine Grassmannian for the loop group LG.

The loop space of the affine Grassmannian is the presheaf quotient LLG/LJG. We denote it as

$${}^L\mathcal{G}r_G : R \longmapsto \frac{LLG(R)}{LJG(R)} = \frac{G\big(R(\!(t)\!)(\!(s)\!)\big)}{G\big(R(\!(t)\!)[\![s]\!]\big)}.$$

This is the loop presheaf of the usual affine Grassmannian of G.

The big Grassmannian is the presheaf quotient LLG/JJG. We denote it as follows:

$$\mathcal{G}r^{big}_G : R \longmapsto \frac{LLG(R)}{JJG(R)} = \frac{G\big(R(\!(t)\!)(\!(s)\!)\big)}{G\big(R[\![t]\!][\![s]\!]\big)}.$$

The 2-dimensional local field affine Grassmannian is the presheaf quotient $LLG/G^{(2)}$, where for a given a k-algebra R we put $\mathcal{O}''(R) = R[\![t]\!] + \sum_{i>0} R(\!(t)\!)s^i \subset R(\!(t)\!)[\![s]\!]$, and consider the subgroup of $LLG(R)$ given by $G^{(2)}(R) = G(\mathcal{O}''(R))$. The 2-dimensional local field affine Grassmannian is denoted as

$$\mathcal{G}r^{(2)}_G : R \longmapsto \frac{LLG(R)}{G^{(2)}(R)} = \frac{LLG(R)}{G(\mathcal{O}''(R))}.$$

Theorem 1 (Theorem 1.7 of [3])**.** *If G is solvable then the above Grasmmannians are represented by ind-schemes.*

We notice that these indschemes are over filtered sets, and the transition maps are closed immersion, however they are over uncountable sets. In particular if G is solvable the presheaves defining these Grassmannians are already sheaves. For a general G we denote with $\mathcal{G}r^{L,\sharp}_G$, ${}^L\mathcal{G}r^{\sharp}_G$, etc. the sheaves associated to the presheaves defined above.

Geometric interpretation. Let X be a smooth surface over k, D an effective Cartier divisor in X, and $Z \subset X$ a closed subscheme of dimension zero. We define the following spaces of G torsors corresponding to the flag $Z \subset D \subset X$. For simplicity, in this exposition, we will assume $X = Spec\, A$ is affine.

The geometric Grassmannian of the loop group is defined as

$$\mathbf{Gr}^L_{D,Z} := \mathbf{Bun}^G_{(\widehat{Z})^{\mathrm{aff}} \smallsetminus D} \times_{\mathbf{Bun}^G_{(\widehat{Z}^{\widehat{D}} \smallsetminus Z)^{\mathrm{aff}} \smallsetminus D}} \{*\}.$$

The geometric loop Grassmannian is defined as

$${}^L\mathbf{Gr}_{D,Z} := \mathbf{Bun}^G_{(\widehat{Z}^{\widehat{D}} \smallsetminus Z)^{\mathrm{aff}}} \times_{\mathbf{Bun}^G_{(\widehat{Z}^{\widehat{D}} \smallsetminus Z)^{\mathrm{aff}} \smallsetminus D}} \{*\}.$$

The geometric big Grassmannian is defined as

$$\mathbf{Gr}^{\mathrm{big}}_{D,Z} := \mathbf{Bun}^G_{\widehat{Z}} \times_{\mathbf{Bun}^G_{(\widehat{Z}^{\widehat{D}} \smallsetminus Z)^{\mathrm{aff}} \smallsetminus D}} \{*\}.$$

The geometric 2-dimensional local field Grassmannian is defined as

$$\mathbf{Gr}^{(2)}_{D,Z} := (\,\mathbf{Bun}^G_{(\widehat{Z}^{\widehat{D}} \smallsetminus Z)^{\mathrm{aff}}} \times_{\mathbf{Bun}^G_{\widehat{Z}^D \smallsetminus Z}} \mathbf{Bun}^G_{\widehat{Z}^D}\,) \times_{\mathbf{Bun}^G_{(\widehat{Z}^{\widehat{D}} \smallsetminus Z)^{\mathrm{aff}} \smallsetminus D}} \{*\}.$$

The definition has some formal subtleties that we now explain in the case of the geometric grassmannian of the loop group (for the other cases see [3]). First we explain what kind of objects are $\widehat{Z} \smallsetminus D$ and $(\widehat{Z}^{\widehat{D}} \smallsetminus Z) \smallsetminus D$. To every k-algebra R over X (that is with a given map $\varphi : Spec\, R \longrightarrow X$) they associate a scheme or an indscheme (functors from the category of k algebras over X to the category of schemes or indscheme or more generally stacks were introduced in [2] and they called them fiber functors). We denote by D_R and Z_R the pull back of D and Z via φ. Assume that D_R is defined by the ideal I_R and Z_R by the ideal J_R, then

- $\hat{Z}(R)$ is the formal neighborhood of Z_R in X_R, this is the colimit of the schemes $Spec\, R/J^n_R$.
- $(\hat{Z})^{\mathrm{aff}}(R) = Spec\, \hat{R}_Z$ where $\hat{R}_Z$ is the limit of R/J^n_R.
- D_R determines a closed subset in $(\hat{Z})^{\mathrm{aff}}(R)$. Define $((\hat{Z})^{\mathrm{aff}} \smallsetminus D)(R) = (\hat{Z})^{\mathrm{aff}}(R) \smallsetminus D_R$.
- We now define $(\hat{Z}^{\hat{D}} \smallsetminus Z)(R)$. To define this object we first consider the formal neighborhood $(\hat{Z})^{D_n}$ of Z in D_n, where $D_n = Spec\, R/I^n_R$. So that $(\hat{Z})^{D_n}$ is the colimit over m of $Spec\, R/I^n_R + J^m_R$. Its affinization is then $Spec\, \hat{R}_n$ where R_n is the limit over m of $R/I^n_R + J^m_R$. We remove Z_R from this affinization and we obtain a scheme $(\hat{Z}^{D_n})^{\mathrm{aff}}(R) \smallsetminus Z_R$. The indscheme $(\hat{Z}^{\hat{D}} \smallsetminus Z)(R)$ is the colimit of these schemes.
- Finally $\big((\widehat{Z}^{\widehat{D}} \smallsetminus Z)^{\mathrm{aff}} \smallsetminus D\big)(R)$ is defined by removing the closed subschemes D_R by the affinization of $(\hat{Z}^{\hat{D}} \smallsetminus Z)(R)$.

On a ring over R the grassmannian $\mathbf{Gr}^L_{D,Z}(R)$ is then defined as the grupoid of G-torsors on $\big((\widehat{Z})^{\mathrm{aff}} \smallsetminus D\big)(R)$ with a trivialization on $\big((\widehat{Z}^{\widehat{D}} \smallsetminus Z)^{\mathrm{aff}} \smallsetminus D\big)(R)$.

To compare these grassmannian with the quotients of LLG, for a k-algebra R (not over X) we define $\underline{\mathbf{Gr}}^L_{D,Z}(R) = \mathbf{Gr}^L_{D,Z}(R \otimes A)$, where recall $X = Spec\, A$.

For example if $X = \mathbb{A}^2$ with coordinates t and s, D is the divisor $s = 0$ and Z is the origin then $\underline{\mathbf{Gr}}^L_{D,Z}(R)$ is the grupoid of G-torsors on $Spec\, R[[t]]((s))$ with a trivilization on $Spec\, R((t))((s))$.

Finally we define $\underline{\mathbf{Gr}}^{L,\sharp}_{D,Z}$ as the stackification of $\underline{\mathbf{Gr}}^L_{D,Z}$.

The following result gives the geometric interpretation of the quotients of LLG.

Theorem 2 (Theorem 2.14 and Corollary 3.11 of [3]). *Assume D is smooth and Z is a simple point. Then*

$${}^L\underline{\mathbf{Gr}}^{\sharp}_{D,Z} \simeq {}^L\mathcal{G}r^{\sharp}_G \quad \underline{\mathbf{Gr}}^{(2),\sharp}_{D,Z} \simeq \mathcal{G}r^{(2),\sharp}_G, \quad \underline{\mathbf{Gr}}^{\mathrm{big},\sharp}_{D,Z} \simeq \mathcal{G}r^{\mathrm{big},\sharp}_G.$$

In the case of the affine grassmannian of the loop group, our result is weaker.

Theorem 3 (Proposition 2.16 and Corollary 3.11 of [3]). *Assume G is connected and solvable or G is semisimple connected and simply connected. Then*

$$\underline{\mathbf{Gr}}^{L}_{D,Z}(k) \simeq \mathcal{G}r^{L}_{G}(k).$$

The last statement immediately generalizes to connected algebraic groups which have a filtration $\{e\} = G_0 \subset G_1 \subset \cdots \subset G_n = G$ such that G_i is normal in G_{i+1} and the quotients G_i/G_{i-1} are connected and solvable or connected semisimple and simply connected. For example it holds

Remark 4. *The last theorem is stated in* [3] *only for G solvable and G semisimple and special. We thank Philippe Gille for explaining us, how to extend this result to any G simply connected.*

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The theory of edifices

Benjamin Martin

(joint work with Michael Bate, Gerhard Roehrle)

1. Spherical buildings and optimality

In this talk I describe how to generalise the notions of vector and spherical buildings from connected reductive groups to arbitrary connected linear algebraic groups. For now, let G be a connected reductive algebraic group over a field k. The *spherical building* Δ_G of G is the simplicial complex formed by the set of parabolic (i.e., k-parabolic) subgroups of G, ordered by reverse inclusion. Given a maximal split torus T of G, the *apartment* Δ_T is the subcomplex of Δ_G consisting of the parabolic subgroups that contain T. We denote the geometric realisations by $|\Delta_G|$ and $|\Delta_T|$; the latter is an $(r-1)$-sphere, where r is the semisimple rank of G. Since any two parabolic subgroups of G contain a common maximal split torus, it follows that any two simplices of Δ_G are contained in a common apartment (we say that G satisfies the *common apartment property*).

There is a beautiful construction of $|\Delta_G|$ using the set Y_G of cocharacters of G. Our motivation is the following application to geometric invariant theory. Suppose $k = \overline{k}$. Let X be an affine G-variety and let $x \in X$ such that $G \cdot x$ is not closed. We can associate to x a so-called "optimal destabilising cocharacter" λ_{opt} with corresponding parabolic subgroup P_{opt}; these have the property that $x' := \lim_{a\to 0} \lambda(a) \cdot x$ exists, $G \cdot x'$ is closed and P_{opt} contains the stabiliser G_x.

Now let k be arbitrary again. We would like to extend these constructions. We can associate to X and x a closed convex subset Σ of $|\Delta_G|$; roughly speaking, Σ is the set of cocharacters λ such that $\lim_{a\to 0}\lambda(a)\cdot x$ exists. If we knew that Σ was a subcomplex of Δ_G then we could use Tits's Centre Conjecture — which is now a theorem — to deduce the existence of an appropriate analogue of P_{opt}. Unfortunately Σ is not a subcomplex in general.

We do not know how to construct P_{opt} for arbitrary k. While investigating this, however, we have looked further into the construction of $|\Delta_G|$ in terms of cocharacters. It turns out that this makes sense for arbitrary connected groups, producing objects we call the *spherical edifice* and *vector edifice* of G. We describe these now.

From here on we let G be an arbitrary connected linear algebraic group over k. If λ is a cocharacter of G then we define subgroups P_λ and L_λ of G by

$$P_\lambda(\overline{k}) = \{g\in G(\overline{k})\mid \lim_{a\to 0}\lambda(a)\cdot x \text{ exists}\}$$

and

$$L_\lambda(\overline{k}) = \{g\in G(\overline{k})\mid \lim_{a\to 0}\lambda(a)\cdot x = x\}.$$

We call P_λ a *Richardson parabolic* (R-parabolic) subgroup of G, and L_λ a *Richardson Levi* (R-Levi) subgroup of G. If G is reductive then P_λ is a parabolic subgroup of G, L_λ is a Levi subgroup of G, and all parabolic and Levi subgroups of G arise in this way.

Set $\mathbb{K}=\mathbb{Q}$ or $\mathbb{R}$. Given a maximal split torus T of G, we define $Y_T\subseteq Y_G$ to be the set of cocharacters of T. Then Y_T is an abelian group, so we it makes sense to define $Y_T(\mathbb{K}) := Y_T\otimes_{\mathbb{Z}}\mathbb{K}$. We can associate an R-parabolic subgroup P_λ and an R-Levi subgroup L_λ to any $\lambda\in Y_T(\mathbb{K})$. The group $G(k)$ acts on $\mathcal{Y} := \bigcup_T Y_T(\mathbb{K})$ in a natural way, and we have $P_{g\cdot\lambda} = gP_\lambda g^{-1}$ for any λ and any g. Given $\lambda\in Y_T(\mathbb{K})$ and $\lambda'\in Y_{T'}(\mathbb{K})$, we define $\lambda'\sim\lambda$ if there exists $g\in P_\lambda(k)$ such that $\lambda' = g\cdot\lambda$. This gives an equivalence relation on $\mathcal{Y}$.

Definition 1. We define $V_G(\mathbb{K}) = \mathcal{Y}/{\sim}$, and we call this the *vector edifice* of G. We write ϕ_G for the canonical projection from $\mathcal{Y}$ to $V_G(\mathbb{K})$. If $\zeta\in V_G(\mathbb{K})$ then we define P_ζ to be P_λ for any $\lambda\in\mathcal{Y}$ such that $\phi_G(\lambda)=\zeta$ (this does not depend on the choice of λ).

There is a natural way to endow $V_G(\mathbb{K})$ with a metric.

Theorem 2 ([1, Prop. 6.8])**.** *$V_G(\mathbb{K})$ is a complete metric space.*

We define Δ_G to be the poset of R-parabolic subgroups of G ordered by reverse inclusion; note that Δ_G need not be a simplicial complex. We call Δ_G the *combinatorial edifice* of G. Given a maximal split torus T of G, we define Δ_T to be the set of R-parabolic subgroups of G that contain T, and we call this an *apartment* of Δ_G. (There is a notion of apartment for the vector building as well: we define $V_T(\mathbb{K}) = \phi_G(Y_T(\mathbb{K}).)$

The point: If G is reductive then Δ_G is precisely the spherical building of G from before, and $V_G(\mathbb{R})$ is the vector building of G. If G is semisimple then we may identify $|\Delta_G|$ with the unit sphere S in $V_G(\mathbb{R})$: given a parabolic subgroup P of G, we can associate to P the topological simplex $\{\zeta \in S \mid P_\zeta \supseteq P\}$.

Even if one is interested only in reductive G, this more general construction can be useful: for instance, one can describe the projection map (in the general sense of buildings) from Δ_G onto a Levi sphere Δ_L in terms of the vector edifices $V_P(\mathbb{K})$ and $V_L(\mathbb{K})$, where P is a parabolic subgroup of G such that L is a Levi subgroup of P.

It is known that if G is pseudo-reductive then the spherical edifice Δ_G is a spherical building. Our next result identifies the class of groups for which the analogous result holds. Let $G_{t,s}$ be the subgroup generated by all the split tori of G and let $R_{u,s}(G)$ be the largest split connected normal unipotent subgroup of G.

Theorem 3 (BMR 2024)**.** *The following are equivalent.*

(a) Δ_G *is a spherical building.*
(b) G *has the common apartment property.*
(c) $R_{u,s}(G)$ *commutes with* $G_{t,s}$*.*

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Algebraic Magnetism

Arnaud Mayeux

Introduction

The talk was an introduction to a new invariant in the setting of an arbitrary algebraic action a of a diagonalizable group scheme $D(M)_S$. I also computed this invariant in the case where a is the adjoint action of a maximal split torus on a split reductive group and explained that it encodes classical invariants in this very special case (root system, standard parabolic groups, root groups).

Setting

Let S be a base scheme, M an abelian group, X a scheme over S, and let a be an S-action of the diagonalizable group scheme $D(M)_S$ (cf. [2] for diagonalizable group schemes) on X. For simplicity of exposition, we assume that M is finitely generated and that the morphism $X \to S$ is separated and locally of finite presentation. For statements and details in more general settings, see [4, 5]. We refer to [7] for monoids.

Attractors and Magnets

We proceed with the above setting. If N is a submonoid of M, then the diagonalizable group scheme $D(M)_S$ acts canonically on the diagonalizable monoid scheme $A(N)_S := \mathrm{Spec}(\mathbb{Z}[N]) \times_{\mathrm{Spec}(\mathbb{Z})} S$ over S. We now introduce a contravariant functor X^N from the category of schemes over S to the category of sets.

Definition 1. $X^N : (T \to S) \mapsto \underset{(D(M)_T\text{-equivariant morphisms})}{\mathrm{Hom}_T^{D(M)_T}(A(N)_T, X_T)}$.

The functor X^N is called the attractor associated to the monoid N under the action a. This terminology comes from dynamic, e.g. $\mathbb{G}_m$-actions. If $M = \mathbb{Z}$ and $N = \mathbb{N}$, then $D(M)_S \cong \mathbb{G}_{m,S}$, $A(N)_S \cong \mathbb{A}^1_S$ and X^N is denoted X^+, cf. the numerous references in [4, 5]. However, the references on $\mathbb{G}_m$-actions cited in [4, 5] (including [1, 8, 3]) do not make use of the language of monoid schemes associated with commutative monoids [7], which is central to our theory. Indeed, if k is a commutative ring, $\mathrm{Spec}(k[t])$ is primarily regarded as the affine line $\mathbb{A}^1_k$ in those works.

Theorem 2.

(1) *The functor X^N is representable by a scheme.*
(2) *If $X \to S$ is affine, then X^N is a closed subscheme of X. Explicitely, if $\mathcal{A}$ denotes the quasi-coherent algebra of X so that $X = \mathrm{Spec}_S(\mathcal{A})$, then $X^N \cong \mathrm{Spec}_S(\mathcal{A}/\mathcal{J}_N)$ where $\mathcal{J}_N = \langle \mathcal{A}_m | m \in M \setminus N\rangle$ (an action of $D(M)_S$ on an S-affine scheme corresponds to an M-grading on the associated quasi-coherent algebra, so that we have in particular a decomposition $\mathcal{A} = \bigoplus_{m\in M} \mathcal{A}_m$).*
(3) *If $N = Z$ is a group, then $X^Z = X^{D(M/Z)_S}$, i.e. X^Z identifies with the fixed points scheme of X under the action of $D(M/Z)_S$ on X ($D(M/Z)_S$ acts on X via the morphism $D(M)_S \to D(M/Z)_S$ associated to the projection $M \to M/Z$).*
(4) *In the extremal cases, we have $X^M = X^{D(M/M)_S} = X$ and $X^0 = X^{D(M)_S}$.*
(5) *If $N \subset L$ are submonoids of M, then we have a canonical monomorphism $X^N \to X^L$ (we use Definition 1 and the equivariant morphism $A(L)_S \to A(N)_S$ associated to the canonical morphism of M-graded rings $\mathbb{Z}[N] \to \mathbb{Z}[L]$).*
(6) *We obtain a canonical monomorphism $X^N \to X^M = X$ (in general $X^N \to X$ is not a closed immersion). We think about N as a magnet which attracts X^N.*
(7) *There are canonical actions of $A(N)_S$ and $D(M)_S$ on X^N.*
(8) *If L is another submonoid of M, then $(X^N)^L = X^{N\cap L}$ (we use (7)).*
(9) *If $F \subset N$ is a face of N, then we have a canonical morphism $X^N \to X^F$ called a face morphism (to obtain it, observe that the M-graded projection $\mathbb{Z}[N] \to \mathbb{Z}[F]$ is a morphism of rings so that we have an equivariant morphism $A(F)_S \to A(N)_S$). The composition $X^F \overset{(5)}{\to} X^N \to X^F$ is the identity. Moreover if X/S is smooth, then $X^N \to X^F$ is smooth.*

(10) *Let $f : X \to Y$ be a $D(M)_S$-equivariant morphism, then we obtain a morphism $f^N : X^N \to Y^N$ on attractors. Moreover if f is étale/smooth/a closed or open immersion, so is f^N (this is wrong for flatness in general).*
(11) *The face morphism $X^N \to X^{N^*}$ induces a bijection on connected components (here $N^* = \{x \in N | \exists y \in N \text{ such that } x + y = 0\}$ denotes the face of invertible elements). If X/S is smooth and a is Zariski locally linearizable, then $X^N \to X^{N^*}$ is an affine bundle.*
(12) *If $X \to S$ is a monoid/group scheme, so is $X^N \to S$.*
(13) *If $f : M \to Z$ is a morphism of abelian groups and Y is a submonoid of Z such that $f^{-1}(Y) = N$, then $X^{Y \subset Z} = X^{N \subset M}$ ($D(Z)_S$ acts on X via $D(Z)_S \to D(M)_S$ dual to $M \to Z$).*

Recall that a denotes the action of $D(M)_S$ on X.

Definition 3. A magnet of a is a submonoid of M. The set of magnets is denoted $m(a)$, it only depends on M.

Proposition 4. *$X^N = X^{E(N)}$ where $E(N) = \bigcap_{\substack{L \in m(a) \\ X^N = X^L}} L$.*

Definition 5. A pure magnet of a is a magnet of a of the form $E(N)$. Equivalently, $N \in m(a)$ is a pure magnet if and only if for all $L \in m(a)$ such that $X^N = X^L$, we have $N \subset L$. The set of pure magnets of a is denoted $\mho(a)$. We have a decomposition

$$m(a) = \bigsqcup_{N \in \mho(a)} m^N(a)$$

where $m^N(a) = \{L \in m(a) | X^L = X^N\}$.

We obtain a bijection between (classes of) attractors and pure magnets

$$\{X^N | N \in m(a)\} \leftrightarrow \mho(a).$$

The set $\mho(a)$ is a poset relatively to the relation of inclusions of submonoids of M. This poset together with associated attractors form the new invariant attached to a, as announced in the introduction. We are often also interested in the cardinalities of minimal generating sets of pure magnets (as monoids). Assume now, for simplicity, that a is Zariski locally linearizable. We then have the following finiteness result.

Proposition 6. *Assume that $X \to S$ is finitely presented, then $\mho(a)$ is finite.*

Definition 7. Let $Z \to X^{N^*}$ be a monomorphism, we put $X^N_Z = X^N \times_{X^{N^*}} Z$.

In a work in progress, Algebraic Magnetism is used and studied for algebraic stacks [6].

Examples

Let G be a Chevalley group scheme over $\mathbb{Z}$. Let $T = D(M)$ be a maximal split torus of G and let $\Phi = \Phi(G, T) \subset M$ be the associated root system (we omit $S = \mathrm{Spec}(\mathbb{Z})$). Let a and $\mathfrak{a}$ be the adjoint actions of T on G and $\mathrm{Lie}(G)$.

Proposition 8. *We have* $\mho(a) \cong \mho(\mathfrak{a}) \overset{\text{bijection}}{\longleftrightarrow} \{$*additively stable subsets of* $\Phi\}$. *The bijection sends a magnet* N *to the additively stable subset* $\Phi \cap N$. *Conversely, a stable subset* Σ *is sent to the submonoid* $[\Sigma\rangle$ *of* M *generated by* Σ *(a subset* $\Sigma \subset \Phi$ *is called additively stable if* $[\Sigma\rangle \cap \Phi = \Sigma$*).*

So Φ identifies with the mono-generated pure magnets. Let B be a Borel subgroup of G containing T and $\mathcal{B}$ be the associated base of Φ. Recall the bijection b between the set of subsets of $\mathcal{B}$ and the set of parabolic subgroups of G containing B.

Proposition 9. *The bijection* b *is given by* $\Gamma \mapsto G^{[\mathcal{B}\cup-\Gamma\rangle}$ *(*$[\mathcal{B}\cup -\Gamma\rangle$ *denotes the monoid generated by* $\mathcal{B}\cup -\Gamma$*).*

Proposition 10. *Let* $\alpha \in \Phi$ *be a root and* e_G *be the neutral subgroup scheme. Then* $G_{e_G}^{[\alpha\rangle}$ *is the usual root group* U_α.

Assume $G = SL_3$, so Φ is the root system of type A_2. Let $\alpha_1, \dots, \alpha_6$ be the roots in Φ, indexed $\mathbb{Z}/6\mathbb{Z}$-cyclically. Then the additively stable subsets of Φ are (ordered by cardinality): Cardinality 0: $\{\emptyset\}$, Cardinality 1:$\{\{\alpha_i\}|1 \leq i \leq 6\}$, Cardinality 2: $\{\{\alpha_i, \alpha_{i+1}\}|1 \leq i \leq 6\}, \{\{\alpha_i, \alpha_{i+3}\}|1 \leq i \leq 3\}$, (note that $\{\alpha_i, \alpha_{i+2}\}$ is not stable because $\alpha_{i+1} = \alpha_i + \alpha_{i+2}$), Cardinality 3: $\{\{\alpha_i, \alpha_{i+1}, \alpha_{i+2}\}|1 \leq i \leq 6\}$, Cardinality 4: $\{\{\alpha_i, \alpha_{i+1}, \alpha_{i+2}, \alpha_{i+3}\}|1 \leq i \leq 6\}$, Cardinality 5: There is no stable subset of Φ of cardinality 5, Cardinality 6: $\{\Phi\}$. So $\#\mho(a) = 1 + 6 + 9 + 6 + 6 + 0 + 1 = 29$.

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Hesselink strata and Lusztig–Xue pieces

Alexander Premet

Let G be a connected reductive algebraic group of rank ℓ over an algebraically closed field $\mathbf{k}$ and T a maximal torus of G. Let Σ be the root system of G with respect to T and Π a basis of simple roots of Σ. Write $X(T)$ (resp. $X_*(T)$) for the lattice of rational characters (resp. cocharacters) of T and $X_*^+(T)$ for the intersection of $X_*(T)$ with the dual Weyl chamber of $X_*(T)_\mathbb{R} := X_*(T) \otimes_\mathbb{Z} \mathbb{R}$

associated with Π. We fix an inner product $(\cdot,\cdot)$ on $X_*(T)_{\mathbb{R}}$ invariant under the action of the Weyl group $W(\Sigma)$. The set $X_*(G) = \{gX_*(T)g^{-1} \mid g \in G\}$ of all rational cocharacters of G admits an $(\operatorname{Ad} G)$-invariant norm $\|\cdot\|$ such that $\|\lambda\| = \sqrt{(\lambda,\lambda)}$ for all $\lambda \in X_*(T)$.

Each nonzero $\lambda \in X_*(G)$ gives rise to a $\mathbb{Z}$-grading

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(\lambda,i), \qquad \mathfrak{g}(\lambda,i) = \{x \in \mathfrak{g} \mid (\operatorname{Ad}\lambda(t))\,x = t^i x \ \text{ for all } \ t \in \mathbf{k}^\times\}$$

of the Lie algebra $\mathfrak{g} = \operatorname{Lie}(G)$. For $d \in \mathbb{Z}$, we put $\mathfrak{g}(\lambda, \geq d) := \bigoplus_{i\geq d} \mathfrak{g}(\lambda,i)$ and denote by $P(\lambda) = L(\lambda)R_u(\lambda)$ the parabolic subgroup of G associated with λ. Here $L(\lambda) = Z_G(\lambda)$ is a Levi subgroup of G. Recall that $\operatorname{Lie}(P(\lambda)) = \mathfrak{g}(\lambda,\geq 0)$ and $\operatorname{Lie}(L(\lambda)) = \mathfrak{g}(\lambda,0)$. Following [1] we denote by $L^\perp(\lambda)$ the normal subgroup of $L(\lambda)$ generated by $(L(\lambda),L(\lambda))$ and the subtorus $T^\lambda := \langle \mu(\mathbf{k}^\times) \mid (\mu,\lambda) = 0\rangle$ of T. Note that $L^\perp(\lambda)$ is a connected reductive group of rank is $\ell - 1$ and it has codimension 1 in $L(\lambda)$.

We write $\mathcal{N}(\mathfrak{g})$ for the nilpotent cone of $\mathfrak{g}$, the variety of all $(\operatorname{Ad} G)$-unstable vectors of $\mathfrak{g}$, and denote by $\mathfrak{D}_G$ the set of all Bala–Carter labels attached to the nilpotent orbits of a complex Lie algebra with root system Σ. As explained in [1], the Hesselink strata $\mathcal{H}(\Delta)$ of $\mathcal{N}(\mathfrak{g})$ are parameterised by the set of cocharacters $\tau_\Delta \in X_*^+(T)$ with $\Delta \in \mathfrak{D}_G$, and they form a partition of $\mathcal{N}(\mathfrak{g})$, so that

$$\mathcal{N}(\mathfrak{g}) = \bigsqcup_{\Delta\in\mathfrak{D}_G} \mathcal{H}(\Delta). \tag{1}$$

The cocharacter τ_Δ can be read off the weighted Dynkin diagram $(a_1,\ldots,a_\ell)$ associated with Δ as follows: if x is a root vector of $\mathfrak{g}$ associated with $\alpha_i \in \Pi$ then $(\operatorname{Ad}\tau_\Delta(t)))(x) = t^{a_i}x$ for all $t \in \mathbf{k}^\times$. The Hesselink stratum attached to Δ has the form

$$\mathcal{H}(\Delta) = (\operatorname{Ad} G)\left(\mathcal{V}(\tau_\Delta,2)_{ss} + \mathfrak{g}(\tau_\Delta,\geq 3)\right)$$

where $\mathcal{V}(\tau_\Delta,2)_{ss}$ is the set of all $(\operatorname{Ad} L^\perp(\tau_\Delta))$-semistable vectors of $\mathfrak{g}(\tau_\Delta,2)$; see [1] for more detail.

Given $\Delta \in \mathfrak{D}_G$ we write $\mathfrak{g}_2^{\Delta,!}$ for the set of all $x \in \mathfrak{g}(\tau_\Delta,2)$ such that $G_x \subset P(\tau_\Delta)$ where $G_x = Z_G(x)$ is the stabiliser of x in G. As explained in [1] each set $\mathfrak{g}_2^{\Delta,!}$ contains $\mathcal{V}(\tau_\Delta,2)_{ss}$, a *nonempty* Zariski open subset of $\mathfrak{g}(\tau_\Delta,2)$. The set

$$\operatorname{LX}(\Delta) := (\operatorname{Ad} G)\big(\mathfrak{g}_2^{\Delta,!} + \mathfrak{g}(\tau_\Delta,\geq 3)\big)$$

containing $\mathcal{H}(\Delta)$ will be referred to as the *Lusztig-Xue piece* of $\mathcal{N}(\mathfrak{g})$ associated with Δ. The pieces $\operatorname{LX}(\Delta)$ and their analogues for $\mathcal{N}(\mathfrak{g}^*)$ and for the unipotent variety of G were introduced by Lusztig. Viability of these pieces has to do with the fact that $\mathfrak{g}_2^{\Delta,!}$ is defined in a more transparent fashion than its elusive subset $\mathcal{V}(\tau_\Delta,2)_{ss}$.

In [4, Appendix A], Lusztig and Xue proved that the pieces $\operatorname{LX}(\Delta)$ form a partition of $\mathcal{N}(\mathfrak{g})$ in the case where G is a classical group. Very recently, the same property was established by Voggesberger [8] for groups of type G_2, F_4 and E_6 with the help of Magma. These results imply that $\operatorname{LX}(\Delta) = \mathcal{H}(\Delta)$ for all $\Delta \in \mathfrak{D}_G$ provided that G is not of type E_7 or E_8.

The partition property of the coadjoint analogues of $\operatorname{LX}(\Delta)$ was established by Lusztig [5] and Xue [9] in all cases where G is a simple algebraic group and

$p = \mathrm{char}(\mathbf{k})$ equals the ratio of the squared lengths of long and short roots in Σ. In all other cases there is a G-equivariant bijection between $\mathcal{N}(\mathfrak{g})$ and $\mathcal{N}(\mathfrak{g}^*)$ which enables one to identify the nilpotent coadjoint orbits and pieces of $\mathfrak{g}^*$ with those of $\mathfrak{g}$; see [7, Section 5.6].

In [6], we prove the following:

Theorem 1. *Let G be a connected reductive group over an algebraically closed field $\mathbf{k}$ of characteristic $p \geq 0$. Then $\mathcal{H}(\Delta) = \mathrm{LX}(\Delta)$ for all $\Delta \in \mathfrak{D}_G$ and hence*

$$\mathcal{N}(\mathfrak{g}) = \textstyle\bigsqcup_{\Delta \in \mathfrak{D}_G} \mathrm{LX}(\Delta).$$

Our proof is computer-free, but relies heavily on some results of Liebeck–Seitz obtained in [3]. In view of (1) this theorem confirms Voggesberger's conjecture and earlier expectations of Lusztig (pertaining to the nilpotent and unipotent case).

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Complete reducibility and Semisimplification

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(joint work with M. Bate, S. Böhm, B. Martin and L. Voggesberger)

This is a report on the joint papers [1] and [5].

1. Cocharacter-closed orbits

Following [8], [7], and [3], we regard an affine variety over a field k as a variety X over the algebraic closure $\overline{k}$ together with a choice of k-structure. We write $X(k)$ for the set of k-points of X and $X(\overline{k})$ (or just X) for the set of $\overline{k}$-points of X. By a subvariety of X we mean a closed $\overline{k}$-subvariety of X; a k-subvariety is a subvariety that is defined over k.

Below G denotes a possibly non-connected reductive linear algebraic group over k. By a subgroup of G we mean a closed $\overline{k}$-subgroup and by a k-subgroup we mean a subgroup that is defined over k. We define $Y_k(G)$ to be the set of k-defined cocharacters of G and $Y(G) := Y_{\overline{k}}(G)$ to be the set of all cocharacters of G. Let H be a subgroup of G.

Next we recall some basic notation concerning parabolic subgroups in (non-connected) reductive groups G from [4, §6] and [7]. Given $\lambda \in Y(G)$, we define

$$P_\lambda = \{g \in G \mid \lim_{a\to 0} \lambda(a)g\lambda(a)^{-1} \text{ exists}\}$$

and $L_\lambda = C_G(\mathrm{Im}(\lambda))$. We call P_λ an *R-parabolic subgroup* of G and L_λ an *R-Levi subgroup* of P_λ. We have $P_\lambda = L_\lambda = G$ if $\mathrm{Im}\,(\lambda)$ belongs to the centre of G.

We denote the canonical projection from P to L by c_L; this is k-defined if P and L are. If we are given $\lambda \in Y(G)$ such that $P = P_\lambda$ and $L = L_\lambda$ then we often write c_λ instead of c_L. We have $c_\lambda(g) = \lim_{a\to 0} \lambda(a)g\lambda(a)^{-1}$ for $g \in P_\lambda$; the kernel of c_λ is the unipotent radical $R_u(P_\lambda)$ and the set of fixed points of c_λ is L_λ.

Let $m \in \mathbb{N}$. Below we consider the action of G on G^m by simultaneous conjugation: $g \cdot (g_1, \ldots, g_m) = (gg_1g^{-1}, \ldots, gg_mg^{-1})$. Given $\lambda \in Y(G)$, we have a map $P_\lambda^m \to L_\lambda^m$ given by $\mathbf{g} \mapsto \lim_{a\to 0} \lambda(a)\cdot \mathbf{g}$; we abuse notation slightly and also call this map c_λ. For any $\mathbf{g} \in P_\lambda^m$, there exists an R-Levi k-subgroup L of P_λ with $\mathbf{g} \in L^n$ if and only if $c_\lambda(\mathbf{g}) = u \cdot \mathbf{g}$ for some $u \in R_u(P_\lambda)(k)$.

Our main tool from GIT is the notion of cocharacter-closure from [7] and [3].

Definition 1. Let X be an affine G-variety and let $x \in X$ (we do not require x to be a k-point). We say that the orbit $G(k)\cdot x$ is *cocharacter-closed over* k if for all $\lambda \in Y_k(G)$ such that $x' := \lim_{a\to 0} \lambda(a)\cdot x$ exists, x' belongs to $G(k)\cdot x$. If $k = \overline{k}$ then it follows from the Hilbert-Mumford Theorem that $G(k)\cdot x$ is cocharacter-closed over k if and only if $G(k)\cdot x$ is closed [9, Thm. 1.4]. If $\mathcal{O}$ is a $G(k)$-orbit in X then we say that $\mathcal{O}$ is *accessible from* x *over* k if there exists $\lambda \in Y_k(G)$ such that $x' := \lim_{a\to 0} \lambda(a)\cdot x$ belongs to $\mathcal{O}$.

Theorem 2 (Rational Hilbert-Mumford Theorem ([3, Thm. 1.3]))**.** *Let G, X, x be as above. Then there is a unique $G(k)$-orbit $\mathcal{O}$ such that $\mathcal{O}$ is cocharacter-closed over k and accessible from x over k.*

2. G-complete reducibility

Definition 3. Let H be a subgroup of G. We say that H is *G-completely reducible over k* (G-cr over k) if for any R-parabolic k-subgroup P of G such that P contains H, there is an R-Levi k-subgroup L of P such that L contains H. We say that H is *G-irreducible over k* (G-ir over k) if H is not contained in any proper R-parabolic k-subgroup of G at all. We say that H is G-cr if H is G-cr over $\overline{k}$.

For more on G-complete reducibility, see [11] and [4]. Note that the definition make sense even if H is not k-defined. We have $P_{g\cdot\lambda} = gP_\lambda g^{-1}$ and $L_{g\cdot\lambda} = gL_\lambda g^{-1}$ for any $\lambda \in Y(G)$ and any $g \in G$ (cf. [4, §6]). It follows that if H is G-cr over k (resp., G-ir over k) then so is any $G(k)$-conjugate of H.

Fix a k-embedding $G \to \mathrm{GL}_n$ for some $n \in \mathbb{N}$. Let H be a subgroup of G. Let $m \in \mathbb{N}$ and let $\mathbf{h} = (h_1, \ldots, h_m) \in H^m$. We call $\mathbf{h}$ a *generic tuple for* H if $h_1, \ldots, h_m$ generates the subalgebra of M_n generated by H [7, Def. 5.4], where

M_n denotes the associative algebra of $n \times n$ matrices over k. Note that we don't insist that $\mathbf{h}$ is a k-point.

Here is one of the pivotal results from [3].

Theorem 4 ([3, Thm. 9.3])**.** *Let H be a subgroup of G and let $\mathbf{h} \in H^m$ be a generic tuple for H. Then H is G-completely reducible over k if and only if $G(k) \cdot \mathbf{h}$ is cocharacter-closed over k.*

Using this result one can derive many results on G-complete reducibility: for instance, see [4] for the algebraically closed case and [7], [3] for arbitrary k. For more on G-complete reducibility for subgroups of G, see [11], [4].

Next we recall the definition of G-complete reducibility for Lie subalgebras of $\mathrm{Lie}(G)$ and also the link between this concept and GIT using generating tuples, due to Richardson.

Definition 5. A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is *G-completely reducible over k* (G-cr over k) if for any parabolic k-subgroup P of G such that $\mathfrak{h} \subseteq \mathrm{Lie}(P)$, there is a Levi k-subgroup L of P such that $\mathfrak{h} \subseteq \mathrm{Lie}(L)$ (see [2, Def. 5.3]).

As in the subgroup case, we say that $\mathfrak{h}$ is *G-completely reducible*, if it is G-completely reducible over $\overline{k}$.

For $k = \overline{k}$, this notion is due to McNinch, see [10] and also [7, §5.3].

We now recall the link to GIT. For this, we need the following definition.

Definition 6. Let $\mathfrak{h}$ be a Lie algebra. Call $\mathbf{x} = (x_1, \ldots, x_m) \in \mathfrak{h}^m$ (some $m \in \mathbb{N}$) a *generating tuple for* $\mathfrak{h}$ if $x_1, \ldots, x_m$ is a generating set for $\mathfrak{h}$ as a Lie algebra.

The next theorem is [2, Thm. 5.4]; see also [10, Thm. 1(i)] for the case $k = \overline{k}$.

Theorem 7 ([2, Thm. 5.4])**.** *Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\mathbf{x} \in \mathfrak{g}^m$ be a generating tuple for $\mathfrak{h}$, and let G act on $\mathfrak{g}^m$ by simultaneous conjugation (via* Ad*). Then $\mathfrak{h}$ is G-completely reducible over k if and only if the $G(k)$-orbit of $\mathbf{x}$ is cocharacter-closed in $\mathfrak{g}^m$ over k.*

3. k-semisimplification

Definition 8. Let H be a subgroup of G. We say that a subgroup H' of G is a *k-semisimplification of H* if there exist an R-parabolic k-subgroup P of G and an R-Levi k-subgroup L of P such that $H \subseteq P$, $H' = c_L(H)$ and H' is G-completely reducible (or equivalently, L-completely reducible, by [5, Prop. 3.6(b)]) over k. We say *the pair (P, L) yields H'*.

Remarks 9. (a). Let H be a subgroup of G. If H is G-cr over k then clearly H is a k-semisimplification of itself, yielded by the pair (G, G).

(b). Suppose (P, L) yields a k-semisimplification H' of H. Let L_1 be another R-Levi k-subgroup of P. Then $L_1 = uLu^{-1}$ for some $u \in R_u(P)(k)$, so $c_{L_1}(H) = uc_L(H)u^{-1}$. Hence (P, L_1) also yields a k-semisimplification of H. We say that P *yields a k-semisimplification of H*.

(c). For G connected and H a subgroup of $G(k)$, Definition 8 generalizes Serre's "G-analogue" of a semisimplification from [11, §3.2.4].

Remark 10. Let $\mathbf{h} = (h_1, \ldots, h_m) \in H^m$ be a generic tuple for H. It is easy to see that $c_\lambda(\mathbf{h}) = (c_\lambda(h_1), \ldots, c_\lambda(h_m))$ is a generic tuple for $c_\lambda(H)$. Hence by Theorem 4, $c_\lambda(H)$ is a k-semisimplification of H if and only if $G(k) \cdot c_\lambda(\mathbf{h})$ is cocharacter-closed over k. Owing to Theorem 2, H admits at least one k-semisimplification: choose $\lambda \in Y_k(G)$ such that $G(k) \cdot c_\lambda(\mathbf{h})$ is cocharacter-closed over k, so $c_\lambda(H)$ is a k-semisimplification of H, yielded by (P_λ, L_λ).

Here is the main result from [5], which was proved in the special case $k = \overline{k}$ in [7, Prop. 5.14(i)], cf. [11, Prop. 3.3(b)]. The uniqueness statement is akin to the theorem of Jordan–Hölder.

Theorem 11 ([5, Thm. 4.5])**.** *Let H be a subgroup of G. Then any two k-semisimplifications of H are $G(k)$-conjugate.*

Remark 12. Given a reductive k-group G and a subgroup H of G, we may regard G as a $\overline{k}$-group by forgetting the k-structure, so it makes sense to consider the $\overline{k}$-semisimplification of H. It can happen that H is G-cr over k but not G-cr, or vice versa: see [4, Ex. 5.11] and [6, Ex. 7.22]. So there is no direct relation between the notions of k-semisimplification and $\overline{k}$-semisimplification in general.

We now come to the analogue of Definition 8 for subalgebras of $\mathfrak{g}$.

Definition 13 ([1, Def. 5.4])**.** Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. A Lie subalgebra $\mathfrak{h}'$ of $\mathfrak{g}$ is a *k-semisimplification of* $\mathfrak{h}$ *(for G)* if there exist a parabolic k-subgroup P of G and a Levi k-subgroup L of P such that $\mathfrak{h} \subseteq \mathrm{Lie}(P)$, $\mathfrak{h}' = c_{\mathrm{Lie}(L)}(\mathfrak{h})$ and $\mathfrak{h}'$ is G-completely reducible over k. We say *the pair (P, L) yields* $\mathfrak{h}'$.

Remarks 14. (i). Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. If $\mathfrak{h}$ is already G-cr over k then clearly $\mathfrak{h}$ is a k-semisimplification of itself, yielded by the pair (G, G).

(ii). Suppose (P, L) yields a k-semisimplification $\mathfrak{h}'$ of $\mathfrak{h}$. Let L_1 be another Levi k-subgroup of P. Then $L_1 = uLu^{-1}$ for some $u \in R_u(P)(k)$ by [1, Lem. 2.3(iii)], so consequently $c_{\mathrm{Lie}(L_1)}(\mathfrak{h}) = u \cdot c_{\mathrm{Lie}(L)}(\mathfrak{h})$. Hence (P, L_1) also yields a k-semisimplification of $\mathfrak{h}$. Because of this, when the choice of L doesn't matter we simply say that P *yields a k-semisimplification of* $\mathfrak{h}$.

As in the group case (Remark 9(ii)) a k-semisimplification of an arbitrary subalgebra of $\mathfrak{g}$ always exists, due to the rational Hilbert-Mumford Theorem 2:

Remark 15. Suppose $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$. Let $\mathbf{h} = (h_1, \ldots, h_m) \in \mathfrak{h}^m$ be a generating tuple for $\mathfrak{h}$. Then $c_\lambda(\mathbf{h}) = (c_\lambda(h_1), \ldots, c_\lambda(h_m))$ is a generating tuple for $c_\lambda(\mathfrak{h})$, for any $\lambda \in Y_k(G)$, and hence $c_\lambda(\mathfrak{h})$ is a k-semisimplification of $\mathfrak{h}$ if and only if $G(k) \cdot c_\lambda(\mathbf{h})$ is cocharacter-closed over k, by Theorem 7. It follows from Theorem 2 that $\mathfrak{h}$ admits at least one k-semisimplification: for we can choose $\lambda \in Y_k(G)$ such that $G(k) \cdot c_\lambda(\mathbf{h})$ is cocharacter-closed over k, so $c_\lambda(\mathfrak{h})$ is a k-semisimplification of $\mathfrak{h}$, yielded by (P_λ, L_λ).

Here is the analogue of Theorem 11 in the Lie algebra setting, which can again be viewed as a kind of Jordan–Hölder theorem. Since the adjoint action is k-linear, the proof is easier than the one for Theorem 11 in [5], where a descending chain argument is needed.

Theorem 16 ([1, Thm. 5.8]). *Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. Then any two k-semisimplifications of $\mathfrak{h}$ are $\mathrm{Ad}(G(k))$-conjugate.*

It turns out that the notions of k-semisimplifications for subgroups and subalgebras are compatible in the following natural fashion.

Theorem 17 ([1, Thm. 5.9]). *Let H be a subgroup of G and let H' be a k-semisimplification of H. Then $\mathrm{Lie}(H')$ is a k-semisimplification of $\mathrm{Lie}(H)$.*

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Realization of algebraic groups as automorphism group schemes

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(joint work with Michel Brion)

Let k be a ground field of characteristic $p \geq 0$, and X a proper scheme. According to a result of Matsumura and Oort [2], the automorphism group scheme $\mathrm{Aut}_{X/k}$ exists and the connected component of the identity $\mathrm{Aut}^0_{X/k}$ is an *algebraic group*, that is, a group scheme of finite type. Note that the underlying scheme may be singular for $p > 0$.

It is natural to wonder if each connected algebraic group arises in this way. We show that this is indeed the case ([1], Theorem 2.1): Given such G there is an integral projective scheme X where $\mathrm{Aut}^0_{X/k}$ is isomorphic to G. For smooth G, one may choose $\dim(X) = 2\dim(G)$. For singular G, however, there is little control on dimensions. By the work of Lombardo and Maffei [4] and Blanc and Brion [3], some connectedness assumptions are inevitable.

The idea of the proof for smooth G is as follows: Choose some equivariant compactification $G \subset V$, formed with respect to the left-right action of $G \times G$. Then interpret $G = G \times \{e\}$ as the scheme of fixed points inside $\mathrm{Aut}^0_{V/k}$ with respect to some finite étale subscheme $\{e\} \times F$ in $G \times G$. On the product $Y = V \times V$, this gives

$$G = \mathrm{Aut}^0_{(Y,Z)/k} \subset \mathrm{Aut}^0_{Y/k},$$

for the scheme of graphs $Z \subset Y$, defined as the schematic image of $F \times V \to V \times V = Y$. With the Blanchard Lemma, one then deduce that the blowing-up $X = \mathrm{Bl}_Z(V)$ has the desired properties.

The argument for singular G in characteristic $p > 0$ is more involved. First choose some

$$G \subset H \quad \text{and} \quad H = \mathrm{Aut}^0_{Y/k}$$

for some smooth connected algebraic group H, and some projective Y. One may assume that Y is geometrically integral, also normal, and with free H-action on some H-stable dense open U that is smooth. Now choose some finite étale $F \subset U/G$, and take $Z \subset Y$ as the closure of $U \times_{U/G} F$. Again the blowing-up $X = \mathrm{Bl}_Z(Y)$ has the desired properties. This relies on a careful computation of Fitting ideals for complete intersections, the local structure of the ring $\mathscr{O}^\wedge_{G,e}$ after passing to perfect closure, and Blanchard's Lemma.

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Representations of shifted affine quantum groups and Coulomb branches

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(joint work with Michela Varagnolo)

Given an arbitrary Cartan matrix $\mathbf{c}$, a mathematical definition of the Coulomb branch of a 3D, N=4 quiver gauge theory associated with two I-graded vector spaces V and W was given by Nakajima and Weekes in [6]. It was proved in [6] that the quantization of the Coulomb branch is a truncated shifted Yangian, and the fixed point set of some $\mathbb{C}^\times$-action on the space of triples (i.e., the BFN space) associated with the Coulomb branch was computed. Some consequences for the module category of the truncated shifted Yangian were also discussed. A related construction in physics in the context of 4D, N=2 quiver gauge theory was considered by Kimura and Pestun in [4].

In this paper we consider the Coulomb branch with symmetrizers of the 3D, N=4 quiver gauge theory associated with the Cartan matrix $\mathbf{c}$. We relate it to a truncated shifted quantum loop group of type $\mathbf{c}$, generalizing the work of Finkelberg-Tsymbaliuk [2] in the symmetric case. The BFN space is infinite dimensional. We use the formalism of Cautis-Williams [1] to represent it by an ind-geometric derived ∞-stack. We then prove a version of the Segal-Thomason localization theorem which relates the Coulomb branch to the K-theory and Borel-Moore homology of the fixed point subset of the $\mathbb{C}^\times$-action on the BFN space.

This result yields an equivalence from the integral category $\mathcal{O}$ of the truncated shifted quantum loop group to the category of nilpotent modules of a new version of quiver Hecke algebras, which we call an integral $\mathbb{Z}$-quiver Hecke algebra. This quiver Hecke algebra is attached to the symmetric Cartan matrix $\underline{C}$ obtained by unfolding $\mathbf{c}$, and depends also on a grading given by $\mathbf{c}$ (and not by $\underline{C}$ only). While the presence of this unfolding was already observed in [6], the role of the integral $\mathbb{Z}$-quiver Hecke algebra is new and important. For symmetric $\mathbf{c}$ the integral $\mathbb{Z}$-quiver Hecke algebra coincides with the parity quiver Hecke algebra of type $\mathbf{c}$ considered in [3]. For non symmetric $\mathbf{c}$, the definition of the integral $\mathbb{Z}$-quiver Hecke algebra differs from the definition of the parity quiver Hecke algebra of $\underline{C}$.

This new quiver Hecke algebra allows us to decategorify the integral category $\mathcal{O}$ of the truncated shifted quantum loop group in term of a finite dimensional module over the simple Lie algebra whose Cartan matrix is $\underline{C}$. This finite-dimensional module is not generally known. We provide a few conditions it satisfies and compute it in type B_2. We also give a (partly conjectural) combinatorial rule to compute this representation. This rule uses a crystal which generalizes Nakajima's monomial crystal.

Notably, the integral $\mathbb{Z}$-quiver Hecke algebra admits a cohomological grading, as it is a convolution algebra in Borel-Moore homology. Consequently, our equivalence of categories yields a grading on the integral category $\mathcal{O}$. Furthermore, although we focus on finite types, many of our results extend naturally to the case of symmetrizable generalized Cartan matrices. We will return to this elsewhere.

Another motivation for this work comes from [8], where we provide a geometrization of (shifted) quantum loop groups of arbitrary types via the critical K-theory of quiver varieties, generalizing Nakajima's work on symmetric types in [5]. Quiver varieties are the 3D mirror duals of Coulomb branches. We aim to better understand the relationships between these two constructions.

First, we describe quiver Hecke algebras modeled over spaces of $\mathbb{Z}$-flags, i.e., spaces of sequencew of finite-dimensional vector spaces labeled by $\mathbb{Z}$. We compare them with the tensor product algebras introduced by Webster. Next, we fix a non-symmetric Cartan matrix $\mathbf{c}$ whose Dynkin diagram a folded Dynkin diagram of a symmetric Cartan matrix $\underline{C}$. We introduce the integral $\mathbb{Z}$-quiver Hecke algebras of type $\mathbf{c}$, which are modeled on $\mathbb{Z}$-quiver Hecke algebras of type $\underline{C}$, with an additional integrality condition that generalizes the parity quiver Hecke algebras from [3] in the symmetric case. We then prove that the module categories of integral $\mathbb{Z}$-quiver Hecke algebras are quotients of the module categories of tensor product algebras of type $\underline{C}$. Let $\mathfrak{g}$ be complex simple Lie algebra of $\mathbf{c}$, and $\underline{\mathfrak{g}}$ the complex simple Lie algebra of $\underline{C}$. Subsequently, we decategorify the integral $\mathbb{Z}$-quiver Hecke algebras ${}^0\mathcal{T}^{\rho}_{\boldsymbol{\mu}}$ by weight subspaces in some $\underline{\mathfrak{g}}$-modules. While we do not explicitly compute these modules, we discuss their connections to [3] in the symmetric case and we compute them in certain specific scenarios, such as the generic case.

Next, we introduce the BFN space with symmetrizers $\mathcal{R}$, following [6]. To facilitate the application of K-theory later, we employ a variation of the formalism from [1], which uses ind-tamely presented ∞-stacks of ind-geometric type. We then describe the fixed point locus of certain automorphisms. Finally, we introduce the Coulomb branches of 4D, N=2 quiver gauge theories with symmetrizers $\mathcal{A}^{\lambda}_{\mu,R}$.

Finally, we introduce shifted quantum groups $\mathbf{U}_{\boldsymbol{\mu},R}$ and their integral category $\mathcal{O}$. Next, we introduce truncated shifted quantum groups and their module category ${}^0\mathcal{O}^{\rho} = \bigoplus_{\boldsymbol{\mu}} {}^0\mathcal{O}^{\rho}_{\boldsymbol{\mu}}$, along with the surjective algebra homomorphism $\Phi : \mathbf{U}_{\boldsymbol{\mu},R} \otimes R_{T_W} \to \mathcal{A}^{\lambda}_{\mu,R}$ that maps to Coulomb branches with symmetrizers. We then prove a localization theorem for Coulomb branches, employing techniques similar to those in [7], which identify the Coulomb branch $\mathcal{A}^{\lambda}_{\mu}$ with the integral $\mathbb{Z}$-quiver Hecke algebras ${}^0\widetilde{\mathcal{T}}^{\rho}_{\boldsymbol{\mu}}$ after suitable completions. Leveraging the localization theorem, we establish a connection between the truncated shifted category $\mathcal{O}$ and integral $\mathbb{Z}$-quiver Hecke algebras, and we discuss a few implications at the decategorified level. The main result is the following.

Theorem.

(1) ${}^0\mathcal{O}^{\rho}_{\boldsymbol{\mu}}$ is equivalent to a category of nilpotent modules over the integral $\mathbb{Z}$-quiver Hecke algebra ${}^0\mathcal{T}^{\rho}_{\boldsymbol{\mu}}$.

(2) There is a representation of $\underline{\mathfrak{g}}$ in $K({}^0\mathcal{O}^{\rho})$ and an embedding of $K({}^0\mathcal{O}^{\rho})$ into a tensor product of fundamental modules of $\underline{\mathfrak{g}}$ which takes the simple modules into the dual canonical basis.

The representation of $\underline{\mathfrak{g}}$ in the Grothendieck group $K({}^0\mathcal{O}^{\rho})$ is not known in general. We define a crystal of type $\underline{C}$ which is, conjecturally, isomorphic to the crystal of the $\underline{\mathfrak{g}}$-module $K({}^0\mathcal{O}^{\rho})$. This yields a combinatorial rule to compute the ℓ-highest weight of all simple modules in ${}^0\mathcal{O}^{\rho}$ which holds true in type B_2. We will come back to this elsewhere.

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Complementary spherical subalgebras and compatible Poisson brackets

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(joint work with Dmitri Panyushev)

Let $\mathfrak{g}$ be a complex reductive Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra. Then $\mathfrak{g}$ can be contracted to $\mathfrak{g}_{(0)} := \mathfrak{h} \ltimes (\mathfrak{g}/\mathfrak{h})^{\rm ab}$, where $(\mathfrak{g}/\mathfrak{h})^{\rm ab}$ is an Abelian ideal. The index of a Lie algebra is the codimension of a generic coadjoint orbit and under contraction it can only increase. Our first observation is that $\operatorname{ind}\mathfrak{g}_{(0)} = \operatorname{rk}\mathfrak{g}$ if and only if $\mathfrak{h}$ is a spherical subalgebra [2, Sect. 2].

If there is a complementary subalgebra $\mathfrak{r} \subset \mathfrak{g}$, i.e., $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r}$ as a vectror space, then the Lie–Poisson brackets $\{\ ,\ \}$ and $\{\ ,\ \}_0$ of $\mathfrak{g}$ and $\mathfrak{g}_{(0)}$ are *compatible*. By definition this means that $\{\ ,\ \}_{a,b} := a\{\ ,\ \} + b\{\ ,\ \}_0$ is a Poisson bracket on the symmetric algebra $\mathcal{S}(\mathfrak{g})$ for any $a, b \in \mathbb{C}$. Suppose that $\mathfrak{h}$ and $\mathfrak{r}$ are spherical. Then each non-zero bracket $\{\ ,\ \}_{a,b}$ has the same rank as $\{\ ,\ \}$, which is $\dim\mathfrak{g} - \operatorname{rk}\mathfrak{g}$. This is important for constructions of integrable systems.

If $Z_{a,b} \subset \mathcal{S}(\mathfrak{g})$ denotes the Poisson centre of $(\mathcal{S}(\mathfrak{g}), \{\ ,\ \}_{a,b})$, then the subalgebra $\mathcal{Z} = \mathcal{Z}(\mathfrak{h},\mathfrak{r}) \subset \mathcal{S}(\mathfrak{g})$ generated by all $Z_{a,b}$ is Poisson-commutative w.r.t. $\{\ ,\ \}$ and $\{\ ,\ \}_0$. Furthermore, $\operatorname{tr.deg}\mathcal{Z}$ takes the maximal possible value $\boldsymbol{b}(\mathfrak{g}) := \frac{1}{2}(\dim\mathfrak{g} + \operatorname{rk}\mathfrak{g})$ and $\mathcal{Z}$ is complete on generic coadjoint orbits. In many interesting cases, $\mathcal{Z}$ can be described explicitly and it turns out to be a polynomial ring. The most striking instance is related to a triangular decomposition $\mathfrak{g} = \mathfrak{u}_- \oplus \mathfrak{t} \oplus \mathfrak{u}$. Set $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{u}$. Then $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{u}_-$.

The algebra $\mathcal{Z} = \mathcal{Z}(\mathfrak{b},\mathfrak{u}_-)$ is described in [2]. For any generating set $\{F_i \mid 1 \le i \le \operatorname{rk}\mathfrak{g}\} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ consisting of homogeneous elements, the bi-homogeneous components

$$(F_i)_{j,d_i-j} \in \mathcal{S}^j(\mathfrak{b})\mathcal{S}^{d_i-j}(\mathfrak{u}_-) \quad \text{with} \quad 1 \le j < d_i := \deg F_i \ \text{ and } 1 \le i \le \operatorname{rk}\mathfrak{g}$$

together with a basis of $\mathfrak{t}$ freely generate $\mathcal{Z} = \mathcal{Z}(\mathfrak{b},\mathfrak{u}_-)$. Furthermore, $\mathcal{Z}$ is complete on each regular coadjoint orbit and it is a maximal Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})$ [2, Thm. 4.4 & Thm. 5.5]. Recently, $\mathcal{Z}$ was lifted to a commutative subalgebra of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ [1].

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