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Enveloping Algebras and Geometric Representation Theory

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ABSTRACT. The workshop brought together experts investigating algebraic Lie theory from the categorical, combinatorial, geometric and noncommutative viewpoints. In general, the talks during the workshop focused on structural results and connections to other areas of mathematics.

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Introduction by the Organizers

This workshop continues a series of conferences on enveloping algebras, as the first part of the title suggests, but the focus of these meetings and also the organisers have changed over the years to reflect the newest developments in the field of algebraic Lie theory. This year the main focus was on a broad mix of categorical, combinatorial, geometric, and noncommutative methods, with an eye to structural results and connections to other areas of mathematics.

The workshop was almost fully in person, with four colleagues taking the option of participating online. In total the workshop had 51 participants and featured 24 talks (19 long talks, 5 short talks), of which one was online. Wednesday afternoon was reserved for a walk to Saint Roman for Schwarzwälder Kirschtorte.

Quantum groups and their generalizations, including vertex algebras, Coulomb branches, and cohomological Hall algebras, were prominently featured this year in the talks by Arakawa, Chari, Creutzig, Davison, Hernandez, Padurariu, Reineke,

Shan, and Vasserot, each offering distinct perspectives. Shan's talk focused on the connection between Coulomb branches and skein algebras, while the talks by Davison and Padurariu primarily addressed cohomological Hall algebras. Chari, Creutzig, Hernandez, Reineke, and Vasserot, the latter two with a geometric approach, discussed the representation theory of quantum groups, and Creutzig specifically highlighted a connection between representations of certain super quantum groups and vertex algebras. Arakawa, for his part, showed how certain vertex algebras can be viewed as chiral quantizations of symplectic singularities.

Some Grothendieck rings of quantum affine groups are endowed with a cluster algebra structure. Leclerc discussed a geometric model for these structures, and Fan Qin's talk explored some remarkable bases of cluster algebras.

Many talks focused on representations in positive characteristic. Etingof discussed theorems on the support of holonomic $\mathcal{D}$ -modules in positive characteristic and their discrete counterparts at roots of unity. Achar reported on using Smith–Treumann localization towards a conjecture for the character of the Steinberg module in positive characteristic, while Losev described the Harish-Chandra center for affine Kac–Moody algebras.

Faergeman, Nehme, Serganova, and Yun focused on tensor categories and their module categories. The geometry of flag varieties, toric varieties, and symplectic resolutions was addressed in the talks of Braverman, Dumanski, Feigin, Costa Cesari, and Iezzi. Williams discussed an approach to the spectral data of a dimer model via new objects, tropical Lagrangian coamoebae.

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Workshop: Enveloping Algebras and Geometric Representation Theory

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Abstracts

Smith–Treumann localization and the Steinberg module

PRAMOD N. ACHAR

(joint work with Paul Sobaje)

This talk is a report on work-in-progress that is motivated by three questions stated in the paragraphs below. At the end of the abstract, I will state some results related to these questions.

The first question involves the representation theory of a semisimple, simply connected algebraic group $\mathbf{G}$ over an algebraically closed field $\mathbb{k}$ of characteristic $p > 0$. Let $\mathbf{X}$, resp. $\mathbf{X}^+$, denote the weight lattice, resp. set of dominant weights, for $\mathbf{G}$, and let W be the Weyl group. For any finite-dimensional $\mathbf{G}$ -module M , one can consider its *character* $\text{ch } M$, an element of the ring $\mathbb{Z}[\mathbf{X}]^W$.

For $\lambda \in \mathbf{X}^+$, let $\mathbf{T}(\lambda)$ denote the indecomposable tilting $\mathbf{G}$ -module of highest weight λ . A particularly important example is the *Steinberg module*, denoted by

$$\text{St} = \mathbf{T}((p-1)\rho).$$

In previous work [8, 9], Sobaje observed that for all $\lambda \in \mathbf{X}^+$, the character of $\mathbf{T}((p-1)\rho + \lambda)$ is divisible in the ring $\mathbb{Z}[\mathbf{X}]^W$ by the character of St , and he went on to prove a number of remarkable properties of the quotient

$$t(\lambda) = \frac{\text{ch } \mathbf{T}((p-1)\rho + \lambda)}{\text{ch } \text{St}} \in \mathbb{Z}[\mathbf{X}]^W.$$

Most striking among these is Sobaje’s *monotonicity theorem*: when $t(\lambda)$ is written in terms of the basis of W -orbits in $\mathbf{X}$, the coefficients are nonnegative and (weakly) decreasing with respect to an appropriate partial order. This raises the question:

Question 1. What is $t(\lambda)$ the character of?

The second question involves the geometry of the affine Grassmannian

$$\text{Gr} = \check{G}(\mathbb{C}((z)))/\check{G}(\mathbb{C}[[z]]),$$

where $\check{G}$ is the Langlands dual group to $\mathbf{G}$ over $\mathbb{C}$. The celebrated *geometric Satake equivalence* [5] relates perverse $\mathbb{k}$ -sheaves on Gr to representations of $\check{G}$.

The group $\mathbb{C}^\times$ acts on $L\check{G}$, $L^+\check{G}$, and on Gr by scaling the parameter z (this is called the *loop rotation action*). Let

$$\varpi = \{\zeta \in \mathbb{C}^\times \mid \zeta^p = 1\}$$

be the group of p -th roots of unity, and consider the fixed-point locus Gr^ϖ . More than a decade ago, Treumann predicted that a sheaf-theoretic version of “Smith localization” (see [7]) from Gr to Gr^ϖ should have representation-theoretic significance in the context of geometric Satake. To make this precise, Treumann defined in [10] a category $\text{Sm}(\text{Gr}^\varpi, \mathbb{k})$, called the *Smith category*, as well as a functor

$$\mathbf{Psm} : D^b(\text{Gr}, \mathbb{k}) \rightarrow \text{Sm}(\text{Gr}^\varpi, \mathbb{k}).$$

Treumann's vision was partly realized by Leslie–Lonergan [4]. A key insight of theirs is that the theory of *parity sheaves* [3] can be adapted to the Smith category: one obtains the category $\mathrm{SmPar}(\mathrm{Gr}^\varpi, \mathbb{k})$ of *Smith parity sheaves*. Inside Gr^ϖ , one can find the so-called *thin affine Grassmannian*

$$\mathrm{Gr}_{\mathrm{thin}} = \check{G}(\mathbb{C}((z^p)))/\check{G}(\mathbb{C}[[z^p]]) \subset \mathrm{Gr}^\varpi$$

as an open and closed subset (i.e., a union of connected components). One of Leslie–Lonergan's main results is that the functor

$$(\text{restrict to } \mathrm{Gr}_{\mathrm{thin}}) \circ \mathbf{Psm} : \mathrm{Parity}_{\mathrm{L}+\check{G}}(\mathrm{Gr}, \mathbb{k}) \rightarrow \mathrm{SmPar}_{\check{G}(\mathbb{C}[[z^p]])}(\mathrm{Gr}_{\mathrm{thin}}, \mathbb{k}).$$

corresponds under geometric Satake to the *Frobenius contraction functor* of Gros–Kaneda [2]. This work suggests the following question.

Question 2. What information lives on the components of Gr^ϖ *outside* of $\mathrm{Gr}_{\mathrm{thin}}$?

Finally, the third question is based on the insight of Riche–Williamson [6] that the Smith–Treumann localization functor and Smith parity sheaves make sense not only for $\check{G}(\mathbb{C}[[z]])$ -equivariant sheaves, but also for *Iwahori–Whittaker (IW) sheaves*, as considered in [1]. (As a stunning application of this idea, they gave in [6] a very short proof of the *tilting character formula*, valid for all p .)

Question 3. How are the two types of Smith–Treumann localization (namely, $\check{G}(\mathbb{C}[[z]])$ -equivariant, and IW) related? How are the results of Leslie–Lonergan related to those of Riche–Williamson?

Any answer to this question should involve the *Iwahori–Whittaker averaging functor*, denoted $\mathrm{Av}_{\mathrm{IW}}$. By [1], this functor is an equivalence of categories on Gr or on $\mathrm{Gr}_{\mathrm{thin}}$, but not on all of Gr^ϖ .

The remainder of this abstract is devoted to statements of results obtained in current work-in-progress with P. Sobaje. For $\lambda \in \mathbf{X}^+$, let Gr_λ be the corresponding $\check{G}(\mathbb{C}[[z]])$ -orbit. Let $\mathrm{Gr}_\lambda^\varpi = \mathrm{Gr}_\lambda \cap \mathrm{Gr}^\varpi$. By [6], this is a single $\check{G}(\mathbb{C}[[z^p]])$ -orbit. Let $\mathcal{E}(\lambda, \mathbb{k})$, resp. $\mathcal{E}^\varpi(\lambda, \mathbb{k})$, be the unique indecomposable parity sheaf whose support is $\overline{\mathrm{Gr}_\lambda}$, resp. $\overline{\mathrm{Gr}_\lambda^\varpi}$, normalized so that its restriction to Gr_λ , resp. $\mathrm{Gr}_\lambda^\varpi$, is perverse. Let $\mathcal{E}_{\mathrm{Sm}}^\varpi(\lambda, \mathbb{k})$ be the image of $\mathcal{E}^\varpi(\lambda, \mathbb{k})$ in the Smith category $\mathrm{Sm}(\mathrm{Gr}^\varpi, \mathbb{k})$.

Our first result can be seen as an answer to Question 1.

Theorem 1. *For $\lambda \in \mathbf{X}^+$, we have*

$$t(\lambda) = \sum_{\mu \in \mathbf{X}} (\mathrm{rank} \mathcal{E}_{\mathrm{Sm}}^\varpi(\lambda, \mathbb{k})|_{\mathrm{Gr}_{\mathrm{dom}(\mu)}^\varpi}) e^\mu,$$

where $\mathrm{dom}(\mu)$ is the unique dominant weight in the W -orbit of μ .

Our second result provides at least a partial answer to Question 3.

Theorem 2. *The following diagram commutes up to natural isomorphism:*

$$\begin{array}{ccc}
 \text{Parity}_{\mathbb{G}_m \ltimes \check{G}(\mathbb{C}[[z]])}(\text{Gr}, \mathbb{k}) & \xrightarrow[\text{[4]}]{\mathbf{Psm}} & \text{SmPar}_{\check{G}(\mathbb{C}[[z^p]])}(\text{Gr}^\varpi, \mathbb{k}) \\
 \mathcal{E}((p-1)\rho, \mathbb{k}) \star (-) \downarrow & & \downarrow \text{Av}_{\mathcal{IW}} \\
 \text{Parity}_{\mathbb{G}_m \ltimes \check{G}(\mathbb{C}[[z]])}(\text{Gr}, \mathbb{k}) & & \\
 \text{Av}_{\mathcal{IW}} \downarrow \text{[1]} & & \\
 \text{Parity}_{\mathbb{G}_m, \mathcal{IW}}(\text{Gr}, \mathbb{k}) & \xrightarrow[\text{[6]}]{\mathbf{Psm}} & \text{SmPar}_{\mathcal{IW}}(\text{Gr}^\varpi, \mathbb{k})
 \end{array}$$

It was shown in [6] that the bottom arrow in Theorem 2 sends indecomposable objects to indecomposable objects. Our last result describes how the output of the top arrow in Theorem 2 decomposes.

Theorem 3. *For $\lambda \in \mathbf{X}^+$, we have*

$$\mathbf{Psm}(\mathcal{E}(\lambda, \mathbb{k})) \cong \bigoplus_{\mu \in \mathbf{X}^+} \mathcal{E}_{\text{Sm}}^\varpi(\mu, \mathbb{k})^{\oplus a_{\mu, \lambda}}$$

where the multiplicities $a_{\mu, \lambda}$ are the same integers appearing in the decomposition

$$\mathbf{T}(\lambda) \otimes \text{St} = \bigoplus_{\mu \in \mathbf{X}^+} \mathbf{T}((p-1)\rho + \mu)^{\oplus a_{\mu, \lambda}}.$$

We hope that these results lay the groundwork for future study of Question 2.

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Symplectic singularities and vertex algebras

TOMOYUKI ARAKAWA

Although vertex algebras originally arose in two-dimensional chiral conformal field theory, it has recently been shown that they also appear in higher-dimensional quantum field theories, see e.g. [9, 12]. As a result [7, 10], it is expected that a large number of vertex algebras should arise from three-dimensional $\mathcal{N} = 4$ gauge theories or four-dimensional $\mathcal{N} = 2$ superconformal field theories, as *chiral quantizations* of various *symplectic singularities* [6].

Here a vertex algebra V is called a chiral quantization of a Poisson variety X if the associated variety X_V [1] of V is isomorphic to X .

Vertex algebras arising from higher-dimensional quantum field theories are generally more intricate than those associated with rational two-dimensional conformal field theories. Nevertheless, the following result shows that they still possess remarkably nice properties.

Theorem 1 ([4]). Let V be a chiral quantization of a symplectic singularity.

- (1) There exists only a finitely many simple ordinary V -modules.
- (2) The characters of simple ordinary representations satisfy a modular linear differential equation (MLDE).

Here a V -module M is called ordinary if its character $\chi_M(q) = q^{-c_V/24} \text{tr}_M(q^{L_0})$ makes sense, where c_V is the central charge of V .

Since the solution space of an MLDE is invariant under the natural action of $SL_2(\mathbb{Z})$, Theorem 1 establishes a certain modularity property of the character of ordinary V -modules. In fact, the character of a vertex algebra arising from four-dimensional $\mathcal{N} = 2$ superconformal field theories is often a *quasi-modular form*, as demonstrated in Theorem 3.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ the corresponding affine Kac-Moody algebra. The universal affine vertex algebra associated with $\mathfrak{g}$ at level k is given by $V^k(\mathfrak{g}) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K)} \mathbb{C}$. We have $X_{L_k(\mathfrak{g})} \cong \mathfrak{g}^*$.

Let $L_k(\mathfrak{g})$ be the simple quotient of $V^k(\mathfrak{g})$. Then $X_{L_k(\mathfrak{g})}$ is a G -invariant, conic subvariety of $X_{L_k(\mathfrak{g})} = \mathfrak{g}^*$.

Theorem 2 ([3]). Let $L_k(\mathfrak{g})$ be an admissible affine vertex algebra, so that $k +$

$h^\vee = p/q$, $p, q \in \mathbb{Z}_{\geq 1}$, $(p, q) = 1$, $p \geq \begin{cases} h^\vee & \text{if } (q, r^\vee) = 1 \\ h & \text{if } (q, r^\vee) = r^\vee. \end{cases}$ Then, there exists a nilpotent orbit $\mathbb{O}_q$ in $\mathfrak{g}^*$ depending only on the denominator q , such that

$$X_{L_k(\mathfrak{g})} \cong \overline{\mathbb{O}_q}.$$

Here, h , $h^\vee$ and $r^\vee$ denote the Coxeter number, the dual Coxeter number and the lacing number of $\mathfrak{g}$, respectively.

Thus, an admissible affine vertex algebra is a chiral quantization of a symplectic singularity if the nilpotent orbit closure $\overline{\mathbb{O}_q}$ is normal. Such a vertex algebra arises from a four-dimensional theory if it is *boundary admissible*, that is, if the numerator p is as small as possible [16].

The appearance of vertex algebras in three-dimensional $\mathcal{N} = 4$ gauge theories in particular implies that each Nakajima quiver variety admits a chiral quantization. In the case where the quiver is the Jordan quiver, we have the following result, which proves a conjecture in [8] for symmetric groups.

Theorem 3 ([5]). There exists a sheaf $\mathcal{V}$ of $\hbar$ -adic vertex algebras on $\mathrm{Hilb}^n(\mathbb{C}^2)$ such that $\mathrm{gr} \mathcal{V} \cong \pi_* \mathcal{O}_{J_\infty \mathrm{Hilb}^n(\mathbb{C}^2)_{\mathrm{super}}}$ and

$$X_{\Gamma(\mathrm{Hilb}^n(\mathbb{C}^2), \mathcal{V})^{\mathbb{C}^*}} \cong (\mathbb{C}^2)^n / S_n.$$

The normalized character of the vertex algebra $\Gamma(\mathrm{Hilb}^n(\mathbb{C}^2), \mathcal{V})^{\mathbb{C}^*}$ is a quasi-modular form of mixed weights, which is identified with MacMahon's multiple q -zeta values [13].

Here $\mathrm{Hilb}^n(\mathbb{C}^2)$ denotes the Hilbert scheme of points in $\mathbb{C}^2$, $\mathrm{Hilb}^n(\mathbb{C}^2)_{\mathrm{super}}$ is a certain super analogue of $\mathrm{Hilb}^n(\mathbb{C}^2)$ (see [5] for the details), and $J_\infty X$ denotes the arc space of X .

There is a family $\{S_G(\Sigma)\}$ of four-dimensional $\mathcal{N} = 2$ superconformal field theories known as the *theory of class $\mathcal{S}$* , parametrized by a pair consisting of a punctured Riemann surface Σ and a complex semisimple group G . The corresponding vertex algebra is called *chiral algebra of class $\mathcal{S}$* .

Theorem 4 ([2]). For any semisimple group G , there exists a unique family of vertex algebras satisfying the required conditions to be the chiral algebra of class $\mathcal{S}$ associated with genus zero punctured Riemann surfaces. Moreover, their associated varieties are isomorphic to genus zero Moore-Tachikawa varieties [15, 11].

In type A , the Moore-Tachikawa varieties are identified [11] with the Coulomb branches of three-dimensional gauge theories associated with star-shaped quivers, which are known to be symplectic singularities [17]. Thus, the genus-zero chiral algebras of class $\mathcal{S}$ provide chiral quantizations of symplectic singularities, at least in type A .

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Determinantal formulae for a family of modules for quantum affine $\mathfrak{sl}_n$

VYJAYANTHI CHARI

The study of finite-dimensional representations of a quantum affine algebra has been a central topic in representation theory for over three decades. The subject has deep connections to various fields, including integrable systems, algebraic geometry, and mathematical physics. More recently the connection with cluster algebras through the work of Hernandez and Leclerc has brought many new ideas to the subject. The work of Kashiwara and his collaborators has led to remarkable developments in the area and new tools are now available for the study of these representations.

In their papers, Hernandez and Leclerc identified a certain tensor subcategory denoted $\mathcal{F}_n$ of the category of finite-dimensional representations of the quantum affine algebra. They showed that there was an isomorphism between the Grothendieck ring of this category and an infinite rank cluster algebra. They conjectured, now a theorem that a cluster monomial corresponds to an irreducible representation whose tensor square is irreducible; such representations are called real. Moreover a cluster variable corresponded to an irreducible representation which is not isomorphic to a tensor product of nontrivial irreducible representations; such representations are called prime. They also conjectured the converse; namely all real representations in the category are cluster monomials and real prime representations are cluster variables. But this is only known to be true for very specific families of representations and is open in general. One of the reasons for this, is that it is highly nontrivial to prove that a module is prime or real.

We shall restrict our attention to a quantum affine algebra of type A_n . In this case the irreducible modules in the Hernandez–Leclerc subcategory are indexed by a

free abelian monoid $\mathcal{I}_n^+$ generated by elements $\omega_{i,j}$ where $i, j \in \mathbb{Z}$ and $0 \leq j - i \leq n + 1$. (This is a reformulation of the usual index set: the Drinfeld polynomials.) Associated with every element of this monoid one also has a standard or Weyl module. An important family of real modules which are known through the work of Duan-Li-Luo to be cluster monomials are the snake modules introduced by Mukhin and Young. These are indexed by elements of the form $\omega_{i_1, j_1} \cdots \omega_{i_r, j_r}$ with $i_1 < \cdots < i_r$ and $j_1 < \cdots < j_r$. These modules have many nice properties and their characters are explicitly known.

The index set for snake modules also defines a family of modules in the category $\mathcal{M}_N$ of finite length, complex smooth representations of $GL_N(F)$ where F is a non-Archimedean field; in that context they are called the ladder modules and have been studied by Gurevich, Lapid and Minguez. The irreducible representations in this world are also indexed by elements of $\mathcal{I}_n^+$; here the index set consists of the Zelevinsky multisegments. There is an associated notion of square irreducibility which is the analog of real modules in the quantum setting.

Loosely speaking, one can use an affine Schur Weyl duality to go between the category $\mathcal{F}_n$ and the Bernstein block in $\mathcal{M}_N$; the snake modules correspond to the ladder modules. Barbasch and Ciobataru explained the connection between $\mathcal{M}_N$ and the BGG-category $\mathcal{O}$ for $\mathfrak{gl}_r$. In particular the BGG-resolution of a finite-dimensional irreducible module of $\mathfrak{gl}_r$ gives a resolution of the irreducible ladder modules in terms of standard modules. Using an old result of Chari-Pressley, one can show that this leads to a resolution of the snake module by Weyl modules.

Lapid and Mínguez continued their study of smooth complex representations of $GL_N(F)$. They gave several equivalent definitions for an irreducible representation associated to a regular element to be square irreducible. A regular element is an element of the form $\omega_{i_1, j_1} \cdots \omega_{i_r, j_r}$ where $i_s \neq i_p$ and $j_s \neq j_p$ for all $1 \leq p \neq s \leq r$. They show that the property of square irreducibility also holds for certain non-regular representations.

In the quantum affine setting, however there are interesting representations coming from the connection with the cluster algebras which are not regular. In this talk we introduce a family of modules which we call alternating snake modules. The snake modules and the modules coming from the category $\mathcal{C}_1$ defined by Hernandez-Leclerc are both very special examples of alternating snake modules. A straightforward application of the results of Kashiwara et al, show that the modules are real. More interestingly, we can give necessary and sufficient conditions for an alternating snake module to be prime. We prove a unique factorization result; namely that an alternating snake module is isomorphic, uniquely (up to a permutation) to a tensor product of prime alternating snake modules. Further results include a presentation of these modules, analogous to the one given in the p -adic world for ladder modules.

We also prove a determinantal formula for these modules (under a mild condition). Namely we define a matrix with entries in the commutative Grothendieck ring $\mathcal{K}_0(\mathcal{F}_n)$ whose determinant is an alternating sum of classes of Weyl (standard) modules and equal to the class of the irreducible module. Under suitable conditions on the alternating snake (but still weaker than the condition that the corresponding Zelevinsky multisegments is regular) we show that the standard modules which occur with non-zero coefficients in the determinant are ± 1 .

Finally we give an application to the category $\mathcal{O}(\mathfrak{gl}_r)$; namely we are able to use our result to compute in $\mathcal{K}(\mathcal{O}(\mathfrak{gl}_r))$ the expression for certain infinite-dimensional irreducible modules in terms of the Verma modules.

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Combinatorial Seshadri stratifications on normal toric varieties

MARTINA COSTA CESARI

(joint work with R. Chirivì, X. Fang, P. Littelmann)

The theory of Seshadri stratifications was recently introduced by R. Chirivì, X. Fang and P. Littelmann in [1]. One of the aims of this theory is to construct a flat degeneration of an embedded projective variety $X \subseteq \mathbb{P}(V)$ into a union of toric varieties X_0 , using geometric data as subvarieties and vanishing order of functions. The degenerate variety X_0 is often more singular, but its combinatorial structure

makes it easier to understand, giving the possibility to obtain information on X via studying X_0 .

Let V be a finite dimensional $\mathbb{K}$ -vector space, with $\mathbb{K}$ an algebraically closed field. Consider a collection of subvarieties $X_\sigma \subseteq X$, indexed by a finite set A . The set A inherits naturally a partial order given by inclusion of the subvarieties. Furthermore, let f_σ , with $\sigma \in A$, be a collection of homogeneous functions in $\text{Sym}(V^*)$ of degree larger or equal to one. The datum $(X_\sigma, f_\sigma)_{\sigma \in A}$ is called a Seshadri stratification if the following condition are fulfilled:

- if $\sigma < \tau$ is a cover relation in $\mathcal{A}$, namely if there is no $\sigma' \in \mathcal{A}$ such that $\sigma < \sigma' < \tau$ in $\mathcal{A}$, then X_σ is a prime divisor in X_τ ;
- there exists a unique maximal element $\tau_{\max} \in \mathcal{A}$ such that $X_{\tau_{\max}} = X$.
- the vanishing set of the restriction of f_τ to X_τ is the union of all X_σ with $\sigma < \tau$ a cover relation;
- f_τ vanishes on X_σ for $\sigma \not\leq \tau$.

Fix a total order $\leq^t$ on A refining the partial order given by inclusions. The existence of a Seshadri stratification on X implies the existence of a quasi-valuation with values in $\mathbb{Q}^A$. The quasi-valuation gives rise to a filtration on the homogeneous coordinate ring $\mathbb{K}[\hat{X}]$, where $\hat{X}$ is the cone over X in V , which allows for the construction of a semi-toric degeneration of X , such that the special fiber consists of a reduced union of equidimensional projective toric varieties, each corresponding to a maximal chain in A . The combinatorial structure of the degenerate fiber is entirely encoded by the image $\Gamma \subseteq \mathbb{Q}^A$ of the quasi-valuation, which decomposes into a finite collection of finitely generated monoids, one for each maximal chain. This collection, known as the associated fan of monoids, captures the full geometry of the degeneration.

In [2] we study the case when X is a normal toric variety. The rigid combinatorial structure inherent to toric varieties allows for a purely combinatorial approach to this theory. This approach reveals a strong connection between a particular class of triangulations of the polytope associated with the toric variety, the corresponding Seshadri stratifications, and the resulting degenerate varieties.

Let $T \simeq \mathbb{K}^n$ be a torus with character lattice M , where $\mathbb{K}$ is an algebraically closed field of characteristic zero. Given a full dimensional normal lattice polytope $P \subseteq M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$, we denote by $X_P \subseteq \mathbb{P}(V)$ the associated embedded toric variety. Note that $\hat{T} = \mathbb{K}^* \times T$ acts on $\hat{X}_P \subseteq V$ by $(\lambda, t) \cdot x = \lambda(t \cdot x)$.

Let A be the set of faces of P . For $\sigma \in A$ there is a unique T -orbits $O_\sigma \subseteq X_P$, denote by X_σ its Zariski-closure in X_P . For a collection of homogeneous T -eigenfunctions $f_\sigma \in \mathbb{K}[\hat{X}_P] \setminus \{0\}$, indexed by $\sigma \in A$, let μ_σ be the weight of f_σ and set $m_\sigma = \deg f_\sigma$. A *combinatorial Seshadri stratification* on X_P is a collection of such pairs $(X_\sigma, f_\sigma)_{\sigma \in A}$, fulfilling the following compatibility condition:

for all $\sigma \in A$, the rational weight $-\frac{\mu_\sigma}{m_\sigma}$ lies in the relative interior of the face σ .

The following result explains the connection of the definition of a combinatorial Seshadri stratification on a toric variety with the general definition of a Seshadri stratification:

Theorem 1. *Let $X_P \subseteq \mathbb{P}(V)$ be an embedded toric variety as above and denote by A the set of faces of the polytope P .*

Let X_σ , $\sigma \in A$, be the collection of T -orbit closures in X_P and let f_σ , $\sigma \in A$, be a collection of $\hat{T}$ -eigenfunctions $f_\sigma \in \mathbb{K}[\hat{X}_P]$ of degree $\deg f_\sigma \geq 1$. Denote by μ_σ the T -weight of f_σ . The following are equivalent:

- *the collection $(X_\sigma, f_\sigma)_{\sigma \in A}$ of T -orbit closures in X_P and homogeneous T -eigenfunctions defines a Seshadri stratification on X_P ,*
- *the collection $(X_\sigma, f_\sigma)_{\sigma \in A}$ of subvarieties and homogeneous functions defines a combinatorial Seshadri stratification.*

Given a combinatorial Seshadri stratification on X_P , for any fixed maximal chain $\mathfrak{C}$ in A , the set $\mathbb{B}_\mathfrak{C} = \{(m_\sigma, \mu_\sigma) \mid \sigma \in \mathfrak{C}\}$ turns out to be a basis of $\mathbb{Q} \oplus M_\mathbb{Q}$. Just using linear algebra, we construct for every maximal chain $\mathfrak{C} \subseteq A$ a valuation $\nu_\mathfrak{C} : \mathbb{K}[X_P] \setminus \{0\} \rightarrow \mathbb{Q}^\mathfrak{C}$ as follows: for a homogeneous T -eigenfunction $f \in \mathbb{K}[X_P]$ of degree m_f and weight μ_f , the valuation $\nu_\mathfrak{C}(f)$ is given by the coefficients of the expression of (m_f, μ_f) as a $\mathbb{Q}$ -linear combination of the basis $\mathbb{B}_\mathfrak{C}$. The quasi-valuation $\nu : \mathbb{K}[\hat{X}_P] \setminus \{0\} \rightarrow \mathbb{Q}^A$ associated to the combinatorial Seshadri stratification is defined by

$$\nu(f) = \min_{>^t} \{\nu_\mathfrak{C}(f) \mid \mathfrak{C} \text{ maximal chain in } A\} \subseteq \mathbb{Q}^A,$$

where, in order to define the minimum, a linearization $>^t$ of the partial order on A is fixed. As in the general case, the quasi valuation induces a filtration on the coordinate ring of X_P and a degeneration of this variety into a union of toric varieties.

The work in [2] answers to natural questions: are there many combinatorial Seshadri stratifications on X_P ? what is the structure of X_0 ? how is the geometry of X_P related to the geometry of X_0 ? how to determine explicitly $\nu(f)$, etc...

First of all one wants to classify all possible combinatorial Seshadri stratifications up to a natural equivalence relation, that is giving the same filtration on $\mathbb{K}[\hat{X}_P]$. A triangulation $\mathcal{T} = (\Delta_C)_{C \in \mathcal{F}(A)}$ of P indexed by chains of faces is, roughly speaking, a marking $\{\mathbf{v}_\sigma\}_{\sigma \in A}$ of the faces of P by rational points in the relative interior of the faces. The simplices of the triangulation are given by: for a chain of faces C the simplex Δ_C is the convex hull of points $\{\mathbf{v}_\sigma\}_{\sigma \in C}$. The following result classifies the combinatorial Seshadri stratifications:

Theorem 2. *There is a bijection between the set of equivalence classes of combinatorial Seshadri stratifications on X_P and triangulations $\mathcal{T}$ of P indexed by chains of faces in A .*

Another important result is that the quasi-valuation can be completely expressed in terms of the triangulation: let $f \in \mathbb{K}[\hat{X}_P] \setminus \{0\}$ be a homogeneous T -eigenfunction of degree m_f and weight μ_f , and let $\mathfrak{C} \subseteq A$ be a maximal chain.

We show that the following statements are equivalent:

- (i) $\nu(f) = \nu_\mathfrak{C}(f)$;
- (ii) $-\frac{\mu_f}{m_f} \in \Delta_\mathfrak{C}$;
- (iii) $\nu_\mathfrak{C}(f)$ has only non-negative entries.

The next task is to describe the structure of $\mathrm{gr}_\nu \mathbb{K}[\hat{X}_P]$.

A priori, one has a dependence on the choice of $\leq^t$. But, by the result above, it turns out that the relevant properties of ν only depend on the triangulation $\mathcal{T}$ and not on the choice of the linearization.

Let $S \subseteq \mathbb{Z} \oplus M$ be the weight monoid of the cone over the embedded toric variety. For a chain of faces $C \subseteq A$ let $K(\Delta_C) \subseteq \mathbb{R} \oplus M_{\mathbb{R}}$ be the cone over the simplex and set $S_C := S \cap K(\Delta_C)$. The union of the cones $K(\Delta_C)$ defines a fan of cones, where C is running over all chains of faces in A . In the same way, the union of the S_C defines a fan of monoids $S_{\mathcal{T}}$, with C running over all chains of faces in A . We show:

Theorem 3. *Denote by $\Gamma = \{\nu(f) \mid f \in \mathbb{K}[\hat{X}_P] \setminus \{0\}\} \subseteq \mathbb{Q}^A$ the image of the quasi-valuation ν .*

- (i) *Γ is a fan of monoids, isomorphic to $S_{\mathcal{T}}$. In particular, Γ depends, up to isomorphism, only on the triangulation $\mathcal{T}$ and is independent of the choice of the linearization $>^t$ of A .*
- (ii) *The associated graded algebra $\mathrm{gr}_\nu \mathbb{K}[\hat{X}_P]$ is isomorphic to the fan algebra $\mathbb{K}[\Gamma]$. In particular, the algebra $\mathrm{gr}_\nu \mathbb{K}[\hat{X}_P]$ depends only on the triangulation $\mathcal{T}$.*
- (iii) *The variety $X_0 = \mathrm{Proj}(\mathrm{gr}_\nu \mathbb{K}[\hat{X}_P])$ is reduced. It is the irredundant union of the toric varieties $\mathrm{Proj}(\mathbb{K}[S_{\mathfrak{C}}])$, where $\mathfrak{C}$ runs over of all maximal chains in A . The variety X_0 is equidimensional, i.e. all irreducible components of X_0 have same dimension as X_P .*

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From weight modules of $\widehat{\mathfrak{sl}}_2$ to modules of $u_q(\mathfrak{sl}_{2|1})$

THOMAS CREUTZIG

We are interested in a connection of representation categories of affine vertex algebras and quantum supergroups.

For this let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with Killing form κ normalized such that short roots have norm two. The affinization of $\mathfrak{g}$ is (we write $\mathfrak{g}[t, t^{-1}]$ for $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$)

$$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$$

with K central and non-vanishing commutation relations

$$[x \otimes t^a, y \otimes t^b] = [x, y] \otimes t^{a+b} + a \delta_{a+b,0} K \kappa(x, y). \quad [d, x \otimes t^a] = ax \otimes t^a.$$

The Cartan subalgebra $\mathfrak{h}$ extends to $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$. A $\widehat{\mathfrak{g}}$ -module M is said to be of level $k \in \mathbb{C}$ if K acts on M by multiplication with k . The affine vertex algebra $V^k(\mathfrak{g})$ is generated by fields $x(z)$ for $x \in \mathfrak{g}$ with OPEs

$$x(z)y(w) = \frac{k\kappa(x, y)}{(z-w)^2} + \frac{[x, y](w)}{(z-w)}$$

and as a $\widehat{\mathfrak{g}}$ -module $V^k(\mathfrak{g})$ is the vacuum Verma module at level k

$$V^k(\mathfrak{g}) \cong U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K \oplus \mathbb{C}d)} \mathbb{C}_k,$$

where $\mathbb{C}_k$ is the one-dimensional $\mathfrak{g}[t] \oplus \mathbb{C}K \oplus \mathbb{C}d$ -module on which K acts by multiplication with k and $\mathfrak{g}[t] \oplus \mathbb{C}d$ acts as zero. The simple quotient $L_k(\mathfrak{g})$ of $V^k(\mathfrak{g})$ inherits a vertex algebra structure. If k is a non-critical level, then $V^k(\mathfrak{g})$ and $L_k(\mathfrak{g})$ admit the Sugawara conformal vector denoted by $L(z) = \sum_{m \in \mathbb{Z}} L_m z^{-m-2}$. The generalized eigenvalues of L_0 are called conformal weights.

Modules of $V^k(\mathfrak{g})$ are in particular smooth modules of $\widehat{\mathfrak{g}}$ at level k and the experience is that the vertex algebra realizes categories of $\widehat{\mathfrak{g}}$ modules at level k with nice properties. Let

- (1) $V^k(\mathfrak{g})$ -wtmod the category of finitely generated smooth weight modules at level k with finite-dimensional weight spaces, that is objects M are finitely generated smooth $\widehat{\mathfrak{g}}$ -modules at level k such $\mathfrak{h}$ acts semisimply and so M is graded by conformal weight and $\mathfrak{h}$, that is

$$M = \bigoplus_{\lambda, \Delta} M_{\lambda, \Delta}$$

and $\dim M_{\lambda, \Delta} < \infty$ for any (λ, Δ) .

- (2) $V^k(\mathfrak{g})$ -wtmod_{KL} the full subcategory of $V^k(\mathfrak{g})$ -wtmod consisting of objects with finite-dimensional conformal weight spaces that are lower-bounded.
- (3) $L_k(\mathfrak{g})$ -wtmod the full subcategory of $V^k(\mathfrak{g})$ -wtmod consisting of objects that are $L_k(\mathfrak{g})$ -modules.
- (4) $L_k(\mathfrak{g})$ -wtmod_{KL} = $L_k(\mathfrak{g})$ -wtmod $\cap$ $V^k(\mathfrak{g})$ -wtmod_{KL}.

A vertex algebra is inspired from the physics of two-dimensional conformal field theory and this suggests that categories of modules of vertex algebras might be ribbon categories. This is usually hard to prove, except for nice categories like $L_k(\mathfrak{g})$ -wtmod_{KL}. In this case Kazhdan and Lusztig proved a ribbon equivalence with quantum group categories of $\mathfrak{g}$ [1, 2] for most levels. But that $L_k(\mathfrak{g})$ -wtmod_{KL} is a ribbon tensor category is a fairly recent result [3]. The categories $L_k(\mathfrak{g})$ -wtmod are much harder to study and results are scarce, in fact the only understood class of examples are the case of $\mathfrak{sl}_2$ at any admissible level. There and with collaborators I have recently established the following results

- (1) A complete classification of weight modules with T. Arakawa and K. Kawasetsu [4].
- (2) The existence of braided tensor structure [5] and then the ribbon structure with R. McRae and J. Yang [6].
- (3) A Verlinde formula that applies for these types of categories [7] and in particular proves the Verlinde conjecture for $L_k(\mathfrak{g})$ -wtmod of [8].

One would now like to see correspondences to quantum groups and with M. Rupert and S. Lentner we have developped a framework for this [9, 10]. Currently Simon Lenter and I, we are lifting this framework so that we can show that $L_k(\mathfrak{g})$ -wtmod for $\mathfrak{g} = \mathfrak{sl}_2$ and admissible levels corresponds to what I like to call a partial semisimplification of a quantum supergroup of $\mathfrak{sl}_{2|1} \oplus \mathfrak{sl}_2$. Here by a partial semisimplification the following is meant. Recall that the semisimplification of a category is the quotient category where one quotients by the idea of negligible morphisms. This category is semisimple but quite big and we prefer to work with the subcategory that is generated by the images of the simple objects of the original category. One can construct the category of weight modules of the quantum supergroup of $\mathfrak{sl}_{2|1}$ by starting with the one of the quantum group of the even subalgebra $\mathfrak{gl}_2$ and then gluing the modules of the positive parabolic borel with the negative one in a suitable way. This construction can be phrased in the language of Yetter-Drinfeld modules as well as purely categorical in terms of relative Drinfeld centers. By a partial semisimplification we mean the construction of $u_q^H(\mathfrak{sl}_{2|1})$ -wtmod where we replace $u_q^H(\mathfrak{gl}_2)$ -wtmod by its semisimplification.

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Okounkov’s conjecture via BPS Lie algebras

BEN DAVISON

(joint work with Tommaso Maria Botta)

This talk is about a recent proof [1] of Okounkov’s conjecture, which relates the graded dimensions of the Lie algebra $\mathfrak{g}_Q^{\text{MO}}$ generating the Maulik–Okounkov Yangian, introduced in [9], with the coefficients of Kac’s polynomials, introduced in

[7]. The proof proceeds by showing that the positive half $\mathfrak{n}_Q^{\text{MO},+}$ of the Maulik–Okounkov Lie algebra is isomorphic to a particular instance of a *BPS Lie algebra* $\mathfrak{g}_{\bar{Q},\bar{W}}$, as defined in [4].

Let Q be a quiver with vertices Q_0 and arrows Q_1 , and two maps $s, t: Q_1 \rightarrow Q_0$ taking an arrow to its source, or target, respectively. Given a field K , we denote by KQ the free path algebra of Q over K , with K -basis the paths in Q , including lazy paths 1_i of length 0 at each vertex $i \in Q_0$. If $\rho \in KQ\text{-mod}$ is a (finite-dimensional) KQ -module, we denote by $\dim_Q(\rho) := (\dim(1_i \rho))_{i \in Q_0}$ the *dimension vector* of ρ . A KQ -module ρ is called *absolutely indecomposable* if $\rho \otimes_K \bar{K}$ is an indecomposable $\bar{K}Q$ -module, with $\bar{K}$ the algebraic closure of K . Given a dimension vector $\mathbf{d} \in \mathbf{N}^{Q_0}$, the *Kac polynomial* [7] $\mathbf{a}_{Q,\mathbf{d}}(t) \in \mathbf{Z}[t]$ satisfies the condition that if $K = \mathbf{F}_q$ is the field of order q then $\mathbf{a}_{Q,\mathbf{d}}(q)$ is the number of isomorphism classes of absolutely indecomposable $\mathbf{d}$ -dimensional KQ -modules.

It was conjectured by Kac, and then proved by Hausel, Letellier and Rodriguez Villegas [6] that in fact $\mathbf{a}_{Q,\mathbf{d}}(t) \in \mathbf{N}[t]$. Additionally, it was conjectured by Kac and proved by Hausel [5] that $\mathbf{a}_{Q,\mathbf{d}}(0) = \dim(\mathfrak{g}_{Q,\mathbf{d}})$, where $\mathfrak{g}_Q$ denotes the Kac–Moody Lie algebra defined by the underlying graph Γ of Q . Since $\mathfrak{g}_Q$ admits a triangular decomposition $\mathfrak{g}_Q \cong \mathfrak{n}_Q^- \oplus \mathfrak{h}_Q \oplus \mathfrak{n}_Q^+$, this is equivalent to the statement that $\mathbf{a}_{Q,\mathbf{d}}(0) = \dim(\mathfrak{n}_{Q,\mathbf{d}}^+)$. These results lead to a question of Kac [7]: What is the Lie-theoretic meaning of the higher coefficients of Kac polynomials?

Given a dimension vector $\mathbf{d}$ we define $\mathbb{A}_{Q,\mathbf{d}} := \text{Hom}(\mathbb{C}^{\mathbf{d}(s(a))}, \mathbb{C}^{\mathbf{d}(t(a))})$, which is acted on by $\text{GL}_{\mathbf{d}} := \prod_{i \in Q_0} \text{GL}_{\mathbf{d}(i)}$ by change of basis. Let $\bar{Q}$ be the double of Q , obtained by setting $\bar{Q}_0 = Q_0$ and $\bar{Q}_1 = Q_1 \amalg Q_1^{\text{op}}$, where Q_1^{op} contains an arrow a^* for every arrow $a \in Q_1$, with the opposite orientation. There is an obvious identification $\mathbb{A}_{\bar{Q},\mathbf{d}} := \text{T}^* \mathbb{A}_{Q,\mathbf{d}}$ and this space carries the co-moment map $\mu_{Q,\mathbf{d}}: \mathbb{A}_{\bar{Q},\mathbf{d}} \rightarrow \mathfrak{gl}_{\mathbf{d}}$ taking $(\rho_a, \rho_{a^*})_{a \in Q_1}$ to $\sum_{a \in Q_1} [\rho_a, \rho_{a^*}]$. We define the *preprojective algebra* $\Pi_Q := \mathbb{C}\bar{Q} / \langle \sum_{a \in Q_1} [a, a^*] \rangle$, and let $\mathfrak{M}_{\mathbf{d}}(\Pi_Q)$ denote the stack of $\mathbf{d}$ -dimensional Π_Q -modules. There is an equivalence $\mathfrak{M}_{\mathbf{d}}(\Pi_Q) \cong \mu_{Q,\mathbf{d}}^{-1}(0) / \text{GL}_{\mathbf{d}}$.

Nakajima quiver varieties provide a parallel route around these singular and stacky spaces. Let $\mathbf{f} \in \mathbf{N}^{Q_0}$ be a dimension vector. We denote by $Q_{\mathbf{f}}$ the framed quiver, obtained by adding a single vertex ∞ to Q , and arrows $r_{i,1}, \dots, r_{i,\mathbf{f}_i}$ from ∞ to i for each $i \in Q_0$. We define $N_Q(\mathbf{f}, \mathbf{d}) := \left(\mu_{Q_{\mathbf{f}},(\mathbf{d},1)}^{-1}(0) \right)^{\text{st}} / \text{GL}_{\mathbf{d}}$, where “st” denotes the stable locus, with stability of a point representing a module ρ defined by the condition that $\Pi_{Q_{\mathbf{f}}} \cdot 1_{\infty} \cdot \rho = \rho$. Then $N_Q(\mathbf{f}, \mathbf{d})$ is a smooth and symplectic variety. We set $\mathbb{M}_{Q,\mathbf{f},\mathbf{d}} := \text{H}^*(N_Q(\mathbf{f}, \mathbf{d}), \mathbb{Q}[\dim(N_Q(\mathbf{f}, \mathbf{d}))])$, the cohomology of the constant perverse sheaf, and define $\mathbb{M}_{Q,\mathbf{f}} := \bigoplus_{\mathbf{d} \in \mathbf{N}^{Q_0}} \mathbb{M}_{Q,\mathbf{f},\mathbf{d}}$.

For example, if Q is the Jordan quiver with one vertex, labelled 0, and one loop, and $\mathbf{f} = 1$, then $N_Q(\mathbf{f}, \mathbf{d}) \cong \text{Hilb}_{\mathbf{d}_0}(\mathbb{A}^2)$, and by work of Nakajima and Grojnowski, $\mathbb{M}_{Q,\mathbf{f}}$ is an irreducible highest weight representation for a certain *generalised Kac–Moody Lie algebra* $\mathbf{heis}_{\infty}$. Note that $\mathbf{heis}_{\infty}$ carries a bigrading; it carries a cohomological grading, as well as a grading by $\mathbf{Z}^{Q_0} \cong \mathbf{Z}$. For a cohomologically graded vector space V we define $\chi_{q^{1/2}}(V) := \sum_{i \in \mathbf{Z}} (-1)^i \dim(V^i) q^{i/2}$. Then for every $d \in \mathbf{N}$ we have $\chi_{q^{1/2}}(\mathbf{heis}_{\infty,d}) = q^{-1} = \mathbf{a}_{Q,\mathbf{d}}(q^{-1})$. This is the first

indication that higher coefficients of Kac polynomials may have a Lie theoretic interpretation provided by the geometric representation theory of Nakajima quiver varieties.

Fix a decomposition $\mathbf{f} = \mathbf{f}' + \mathbf{f}''$, with $\mathbf{f}', \mathbf{f}'' \in \mathbf{N}^{Q_0} \setminus \{0\}$. For $i \in Q_0$ and $1 \leq m \leq \mathbf{f}'_i$, let $\mathbb{C}^*$ act on $\mathbb{A}_{\overline{Q}, (\mathbf{d}, 1)}$ by scaling $\rho(r_{i,m})$ with weight ± 1 , and $\rho(r_{i,m})$ with weight ∓ 1 , and by leaving all other components of $\mathbb{A}_{\overline{Q}, (\mathbf{d}, 1)}$ invariant. These two actions preserve $\mu_{Q, (\mathbf{f}, \mathbf{d})}^{-1}(0)$, and induce actions on $N_Q(\mathbf{f}, \mathbf{d})$. For both actions we have natural isomorphisms $N_Q(\mathbf{f}, \mathbf{d})^{\mathbb{C}^*} \cong \coprod_{\mathbf{d}' + \mathbf{d}'' = \mathbf{d}} N_Q(\mathbf{f}', \mathbf{d}') \times N_Q(\mathbf{f}'', \mathbf{d}'')$. Define

$$\text{Att}^{\pm} := \{\rho \in N_Q(\mathbf{f}, \mathbf{d}) \text{ such that } \lim_{z \rightarrow 0} (z \cdot \rho) \text{ exists}\}.$$

We would like to consider the correspondences $\iota: \text{Att}^{\pm} \rightarrow N_Q(\mathbf{f}, \mathbf{d})^{\mathbb{C}^*} \times N_Q(\mathbf{f}, \mathbf{d})$, sending $\rho \mapsto (\lim_{z \rightarrow 0} (z \cdot \rho), \rho)$, but either we give $\text{Att}^{\pm}$ the subspace topology, and ι is not continuous, or we break $\text{Att}^{\pm}$ up according to the components that modules flow to, and ι is not proper over $N_Q(\mathbf{f}, \mathbf{d})$. The problem is that of embedded $\mathbb{P}^1$ s, i.e. $\mathbb{C}^*$ -orbits that have a limit at zero and infinity.

The solution provided by Maulik and Okounkov goes via their stable envelope construction, which for Nakajima quiver varieties can be equivalently described in the following way: there is an $\mathbb{A}^1$ -family $N_Q(\mathbf{f}, \mathbf{d})_{\lambda}$ for $\lambda \in \mathbb{A}^1$ of $\mathbb{C}^*$ -equivariant varieties such that for $\lambda \neq 0$, the variety $N_Q(\mathbf{f}, \mathbf{d})_{\lambda}$ is affine, and $N_Q(\mathbf{f}, \mathbf{d})_0 = N_Q(\mathbf{f}, \mathbf{d})$. Since affine varieties cannot contain copies of $\mathbb{P}^1$, the above problem of embedded rational curves does not occur for them, and we define $L^{\pm}$ to be the limit of $i_{\lambda}(\text{Att}_{\lambda}^{\pm})$ as $\lambda \rightarrow 0$. Using $L^{\pm}$ as a correspondence gives us morphisms $\text{Stab}^{\pm}: \mathbb{M}_{Q, \mathbf{f}'}[u'] \otimes \mathbb{M}_{Q, \mathbf{f}''}[u''] \rightarrow \mathbb{M}_{Q, \mathbf{f}}^T$ where u', u'' are the equivariant parameters of the two-torus T that acts by scaling the first $\mathbf{f}'$ and last $\mathbf{f}''$ framing arrows separately. After localising with respect to equivariant parameters, these morphisms become invertible, so we may define the *R matrix*

$$R := (\text{Stab}^{+})^{-1} \circ \text{Stab}^{-}: \mathbb{M}_{Q, \mathbf{f}'} \otimes \mathbb{M}_{Q, \mathbf{f}''} \rightarrow \mathbb{M}_{Q, \mathbf{f}'} \otimes \mathbb{M}_{Q, \mathbf{f}''}[[u^{-1}]],$$

which we can expand in powers of u^{-1} as $1 + \hbar \mathbf{r} u^{-1} + \hbar O(u^{-2})$, with $\mathbf{r}$ defined to be the *classical r matrix*. Here, we have introduced a further torus acting on Nakajima quiver varieties, that scales the symplectic form with weight given by $\hbar$.

$\mathbf{Y}_Q^{\text{MO}} \subset \prod_{\mathbf{f}' \in \mathbf{N}^{Q_0}} \text{End}(\mathbb{M}_{Q, \mathbf{f}'})$ is the algebra generated by $\hbar^{-1}$ times the u^{-m} coefficients of all restricted traces of R , for $m \geq 1$, with respect to all endomorphisms of $\mathbb{M}_{Q, \mathbf{f}''}$, while $\mathfrak{g}_Q^{\text{MO}}$ is the subspace generated by the same recipe, but with m fixed to be 1. Maulik and Okounkov prove that $\mathfrak{g}_Q^{\text{MO}}$ is a $\mathbf{Z}^{Q_0}$ -graded, cohomologically graded Lie algebra, and there is a PBW isomorphism $\text{Sym}(\mathfrak{g}_Q^{\text{MO}}[u]) \rightarrow \mathbf{Y}_Q^{\text{MO}}$. Okounkov's conjecture states that for $\mathbf{d} \in \mathbf{N}^{Q_0}$ there is an equality $\chi_{q^{1/2}}(\mathfrak{g}_{Q, \mathbf{d}}^{\text{MO}}) = \mathbf{a}_{Q, \mathbf{d}}(q^{-1})$.

We set $\mathcal{A}_{\Pi_Q} = \bigoplus_{\mathbf{d} \in \mathbf{N}^{Q_0}} \mathcal{A}_{\Pi_Q, \mathbf{d}}$, where $\mathcal{A}_{\Pi_Q, \mathbf{d}} := H^{\text{BM}}(\mathfrak{M}_{\mathbf{d}}(\Pi_Q), \mathbf{Q}[\text{vir} \dim])$ is the Borel–Moore homology of the stack of Π_Q -modules, with a cohomological shift given by the (virtual) dimension of this stack. This is a $\mathbf{N}^{Q_0}$ -graded, cohomologically graded associative algebra, via the Hall algebra construction of Schiffmann and Vasserot [10]. Moreover, by dimensional reduction (which relates

this Hall algebra to the *critical* cohomological Hall algebras introduced by Kontsevich and Soibelman [8]) and the cohomological integrality theorem [4], there is a $\mathbf{N}^{Q_0}$ -graded, cohomologically graded subspace $\mathfrak{g}_{\tilde{Q}, \tilde{W}} \subset \mathcal{A}_{\Pi_Q}$, closed under the commutator Lie bracket, and a PBW isomorphism $\mathrm{Sym}(\mathfrak{g}_{\tilde{Q}, \tilde{W}}[u]) \rightarrow \mathcal{A}_{\Pi_Q}$. There are equalities $\chi_{q^{1/2}}(\mathfrak{g}_{\tilde{Q}, \tilde{W}, \mathbf{d}}) = \mathbf{a}_{Q, \mathbf{d}}(q^{-1})$ by [2].

This PBW theorem looks like the Maulik–Okounkov PBW theorem, except the cohomological Hall algebra $\mathcal{A}_{\Pi_Q}$, and the BPS Lie algebra $\mathfrak{g}_{\tilde{Q}, \tilde{W}}$, are $\mathbf{N}^{Q_0}$ -graded, and not $\mathbf{Z}^{Q_0}$ -graded. The distance between the Maulik–Okounkov Lie algebra and the BPS Lie algebra was reduced by the results of [3], where it was shown that there is a $\mathbf{Z}^{Q_0}$ -graded, cohomologically graded generalised Kac–Moody Lie algebra $\mathfrak{g}_{\Pi_Q}$ whose positive half is isomorphic to $\mathfrak{g}_{\tilde{Q}, \tilde{W}}$. Finally we show that $\mathfrak{g}_{\Pi_Q}$ is isomorphic to $\mathfrak{g}_Q^{\mathrm{MO}}$. As a corollary, this settles Okounkov’s conjecture.

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Non-commutative resolutions and canonical bases

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(joint work with Elijah Bodish, Vasily Krylov)

Let G be a complex reductive group. Consider the Springer resolution $\tilde{\mathcal{N}} \rightarrow \mathcal{N}$. The variety $\tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$ is called the *Steinberg variety*. Its equivariant K-theory $K^{G \times \mathbb{C}^\times}(\tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}})$ has an algebra structure, given by convolution. Kazhdan–Lusztig [9], and later Chriss–Ginzburg [7] identified this algebra with $\hat{\mathcal{H}}_{G^\vee}$ – the

affine Hecke algebra for the Langlands dual group $G^\vee$:

$$K^{G \times \mathbb{C}^\times}(\tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}) \simeq \hat{\mathcal{H}}_{G^\vee}.$$

Equivariant K-theory of the Springer variety is identified with the aspherical module over $\hat{\mathcal{H}}_{G^\vee}$:

$$K^{G \times \mathbb{C}^\times}(\tilde{\mathcal{N}}) \simeq M^{asph}.$$

Later Bezrukavnikov with collaborators categorified this and related isomorphisms in a series of papers [2, 3, 5, 4]. In particular, one has the following coherent categorification of the canonical Kazhdan–Lusztig basis in M^{asph} :

Theorem 1. ([5, 2]) There is a $G \times \mathbb{C}^\times$ -equivariant vector bundle $\mathcal{E}_{\mathcal{N}}$ on $\tilde{\mathcal{N}}$, such that the functor $\mathcal{F} \mapsto RHom(\mathcal{E}_{\mathcal{N}}, \mathcal{F})$ provides a derived equivalence

$$D^b(Coh^{G \times \mathbb{C}^\times}(\tilde{\mathcal{N}})) \simeq D^b(A - mod^{G \times \mathbb{C}^\times}),$$

where $A = RHom(\mathcal{E}_{\mathcal{N}}, \mathcal{E}_{\mathcal{N}}) = Hom(\mathcal{E}_{\mathcal{N}}, \mathcal{E}_{\mathcal{N}})$ is called the non-commutative resolution.

Classes of simple perverse coherent A -modules [1] (where we view A as a coherent sheaf of algebras on $\mathcal{N}$) form the Kazhdan–Lusztig canonical basis in $K_0(A - mod^{G \times \mathbb{C}^\times}) = K^{G \times \mathbb{C}^\times}(\tilde{\mathcal{N}}) = M^{asph}$.

This theorem, among other corollaries, suggests a way to construct canonical bases in equivariant K-theory of a resolution: construct a non-commutative resolution and consider the category of perverse coherent modules over it. This is an Artinian Noetherian category, hence classes of simple modules form a basis in the K-group.

An idea of the present work is implement the above suggestion for the case of affine Schubert varieties in type A. Let us fix some notations.

Consider the affine Grassmannian for GL_n , $Gr = GL_n((t))/GL_n[[t]]$. For a sequence of minuscule (fundamental) coweights $(\omega_{k_1}, \dots, \omega_{k_m})$ we have a resolution of singularities $\widetilde{Gr}^{\underline{k}} \rightarrow \overline{Gr}^\lambda$. Here $\lambda = \sum_i \omega_{k_i}$, $\overline{Gr}^\lambda$ is the affine Schubert variety, and $\widetilde{Gr}^{\underline{k}}$ is the convolution diagram.

In order to construct and describe the canonical basis in equivariant K-theory of this resolution, we need to formulate, in which terms we plan to do so. For that, recall the Cautis–Kamnitzer’s categorical geometric quantum affine skew Howe duality [6]. This is a statement, in particular implying that there is an action of the quantum affine $\dot{U}(\widehat{\mathfrak{gl}}_m)$ (Lusztig’s idempotent form) on the direct sum

$$(1) \quad \bigoplus_{k_1, \dots, k_m} K^{GL_n[[t]] \rtimes \mathbb{C}^\times}(\widetilde{Gr}^{\underline{k}}).$$

This is the structure we use to state our main result, which is an analog of Theorem 1 for the case of affine Grassmannian resolution:

Theorem 2.

- For any $\underline{k}$, $\widetilde{Gr}^{\underline{k}} \xrightarrow{\pi_{\underline{k}}} \overline{Gr}^{\lambda}$ admits a noncommutative resolution. That is, there is a $GL_n[[t]] \rtimes \mathbb{C}^{\times}$ -equivariant vector bundle $\mathcal{E}_{Gr}$ on $\widetilde{Gr}^{\underline{k}}$, such that $\mathcal{F} \mapsto R(\pi_{\underline{k}})_* R\mathcal{H}om(\mathcal{E}_{Gr}, \mathcal{F})$ induces a derived equivalence

$$D^b(Coh^{GL_n[[t]] \rtimes \mathbb{C}^{\times}}(\widetilde{Gr}^{\underline{k}})) \simeq D^b(\mathcal{A} - mod^{GL_n[[t]] \rtimes \mathbb{C}^{\times}}),$$

where $\mathcal{A} = R(\pi_{\underline{k}})_* R\mathcal{H}om(\mathcal{E}_{Gr}, \mathcal{E}_{\mathcal{N}})$ is coherent sheaf of algebras on $\overline{Gr}^{\lambda}$ (non-commutative resolution).

- Classes of simple perverse coherent $\mathcal{A}$ -modules form the global canonical (crystal) basis for the $\dot{U}(\hat{\mathfrak{gl}}_m)$ -action (1).

Let us note that for simplicity we stated our results for the case of module, as oppose to algebra. In fact, there is an analogous statement for the equivariant K-theory of $\widetilde{Gr}^{\underline{k}} \times_{\overline{Gr}^{\lambda}} \widetilde{Gr}^{\underline{k}}$ and analogous. The basis in this algebra is given by classes of simple perverse $(\mathcal{A} \otimes_{\mathcal{O}_{\overline{Gr}^{\lambda}}}^L \mathcal{A}^{opp})$ -modules.

Finally, let us note that there is a related result [8], which describes the basis of simple perverse coherent sheaves on the affine Grassmannian. The proof uses the Lusztig's correspondence, relating geometry of affine Grassmannian to geometry of Springer resolution in type A, and thus reducing this case to the situation of Theorem 1. Our proof follows the same strategy.

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p -supports of holonomic $\mathcal{D}$ -modules and q - $\mathcal{D}$ -modules

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based on arXiv:2504.19329

1. BITOUN'S THEOREM

Let X be a smooth connected variety over $\overline{\mathbb{Q}}$, and $\mathcal{D}_X$ be the sheaf of differential operators on X . Let M be a holonomic $\mathcal{D}_X$ -module. This data is defined over some finitely generated subring $R \subset \overline{\mathbb{Q}}$. For sufficiently large prime p , pick a maximal ideal $\mathfrak{p} \subset R$ lying over p and consider the reduction $M_{\mathfrak{p}}$ of M over the residue field $k_{\mathfrak{p}} = R/\mathfrak{p}$ (of characteristic p), a module over the sheaf $\mathcal{D}_{X_p}$.

Let Y be a smooth variety of dimension n over a field of characteristic p . Recall that the sheaf $\mathcal{D}_Y$ has p -center $\mathcal{Z}_Y$ locally generated by $\mathcal{O}_Y^p$ and $\partial_1^p, \dots, \partial_n^p$ (in local coordinates), and that $\mathcal{D}_Y$ is an Azumaya algebra over $\mathcal{Z}_Y$ of rank p^n . Canonically we have $\mathrm{Spec} \mathcal{Z}_Y = T^*Y^{(1)}$ where $Y^{(1)}$ is the Frobenius twist of Y . Thus, for every coherent $\mathcal{D}_Y$ -module M we may consider its support as a $\mathcal{Z}_Y$ -module,

$$\mathrm{supp}_p(M) \subset T^*Y^{(1)},$$

called the p -support of M (not to confuse with the singular support $\mathrm{supp}(\mathrm{gr}M)$). In general, this can be any closed subvariety of $T^*Y^{(1)}$.

In particular, in our case the module $M_{\mathfrak{p}}$ is finitely generated over $\mathcal{O}_{T^*X^{(1)}}$, so it has p -support

$$\mathrm{supp}_p(M_{\mathfrak{p}}) \subset T^*X^{(1)},$$

which is called the p -support of M at $\mathfrak{p}$.

Theorem 1.1. (*T. Bitoun [B], conjecture of M. Kontsevich*) *For sufficiently large p , $\mathrm{supp}_p(M_{\mathfrak{p}})$ is Lagrangian.*

Bitoun's proof of Theorem 1.1 is rather complicated. It was simplified by van den Bergh [vdB], but his proof still is not quite elementary. I will now give a short and elementary proof of this theorem.

2. PROOF OF BITOUN'S THEOREM

Step 1. The first step is the same as in previous proofs. It suffices to assume that X is affine. Embed X into $V = \mathbb{A}^n$, $i : X \hookrightarrow V$, and consider the holonomic $\mathcal{D}$ -module i_*M on V . We have a natural map

$$\pi : T^*V^{(1)} \rightarrow T^*X^{(1)},$$

and it is easy to see that

$$\mathrm{supp}_p(i_*M_{\mathfrak{p}}) = \pi^{-1}(\mathrm{supp}_p(M_{\mathfrak{p}})).$$

Thus, it suffices to assume that $X = V$ is an affine space.

Step 2. We have an action of $\mathrm{Sp}(V \oplus V^*)$ on $\mathcal{D}(V)$.

Lemma 2.1. (*Bernstein's lemma*) *For generic $g \in \mathrm{Sp}(V \oplus V^*)$, gM is finite over $\mathcal{O}(V)$, i.e., is a vector bundle with a flat connection.*

Proof. Consider the arithmetic singular support $\text{supp}_a(M) \subset V \oplus V^*$. This has dimension $\dim V$, so $\exists g \in \text{Sp}(V \oplus V^*)$ such that $g(\text{supp}_a(M)) \cap V = \emptyset$ by dimension count. Thus the map $g(\text{supp}_a(M)) \rightarrow V$ is finite, so gM is a finite $\mathcal{O}(V)$ -module.

Thus, we may assume that M is a finite $\mathcal{O}(V)$ -module.

Step 3. Suppose ∇ is a flat connection on a vector bundle E on an affine space V over a field of characteristic p . Let ∇_i be the covariant derivatives. Then ∇_i^p commute with functions, so they are endomorphisms C_i of E such that $[C_i, C_j] = 0$. These are called the p -curvature operators. Canonically, they define a form $C = \sum_i C_i dX_i$, $X_i = x_i^p$, i.e., a section of $\text{Fr}^* \Omega_{Y^{(1)}}^1$ when $\text{Fr} : Y \rightarrow Y^{(1)}$ is the Frobenius twist.

Moreover, the coefficients of the characteristic polynomials of C_j depend only on X_i , not on x_i , so we can define the spectral variety $S_\nabla \subset T^*Y^{(1)}$ such that the preimage $S_\nabla(y)$ of $y \in Y$ is the spectrum of the p -curvature of ∇ at y .

It is easy to prove the following lemma.

Lemma 2.2. *Let $M(\nabla)$ be the $\mathcal{D}$ -module defined by ∇ . Then $\text{supp}_p(M(\nabla)) = S_\nabla$.*

Step 4. It remains to prove the following proposition.

Proposition 2.3. *Let ∇ be a flat connection on a vector bundle E on V in characteristic p which lifts modulo p^2 and has rank $r < p$. Then S_∇ is Lagrangian.*

Proof. First consider the case when $\text{rank}(E) = 1$. It suffices to consider connections on the formal disk and show that $\frac{\partial C_i}{\partial x_j} = \frac{\partial C_j}{\partial x_i}$. So it is enough to treat the case $\dim V = 2$. Thus we have $\nabla_x = \partial_x + a(x, y)$, $\nabla_y = \partial_y + b(x, y)$. Hence the p -curvature of the connection ∇ is $A = \nabla_x^p = \partial_x^{p-1}a + a^p$, $B = \nabla_y^p = \partial_y^{p-1}b + b^p$.

Let $a = \sum a_{ij}x^i y^j$, $b = \sum b_{ij}x^i y^j$. So $a^p = \sum a_{ij}^p X^i Y^j$, $b^p = \sum b_{ij}^p X^i Y^j$, where $X = x^p$, $Y = y^p$. Since $\partial_y a = \partial_x b$, we have:

$$(1) \quad j a_{i-1,j} = i b_{i,j-1}.$$

Thus we need to show that $\partial_Y \partial_x^{p-1} a = \partial_X \partial_y^{p-1} b$, i.e.,

$$(2) \quad j a_{pi-1,pj} = i b_{pi,pj-1}.$$

But from (1) it follows that $p j a_{pi-1,pj} = p i b_{pi,pj-1} \pmod{p^2}$, which implies (1) modulo p .

Now consider the general case $\text{rank}(E) = r < p$. Pick a sufficiently generic point v in V near which the multiplicities of eigenvalues of C are constant. Then E can be written on the formal neighborhood of v as a direct sum of ∇ -invariant generalized eigenbundles E_λ , so that on each E_λ , C has only one eigenvalue λ .

So it suffices to assume that C has only one eigenvalue on a formal neighborhood of v . Then consider the connection $\det(\nabla)$ on $\Lambda^{\text{top}} E$, which has the connection form $r \cdot \lambda$. Since $r < p$, we have $r \neq 0$ in the ground field, so we are done by the rank 1 case.

3. GENERALIZATION TO THE q -CASE

Let $q \in k^\times$. Let $\mathcal{A}_q$ be the q -torus algebra with invertible generators x_i, p_i for $1 \leq i \leq n$ and relations:

$$x_i x_j = x_j x_i, \quad p_i p_j = p_j p_i, \quad p_i x_j = q^{\delta_{ij}} x_j p_i.$$

If q is a root of unity of order N , then $\mathcal{A}_q$ is an Azumaya algebra over its center Z_q , and we define the N -support $\text{supp}_N(M)$ of a finite $\mathcal{A}_q$ -module M as the support of M in the torus $\text{Spec}(Z_q)$. Note that Z_q has a natural Poisson structure coming from deformation of q .

Now suppose $k = \overline{\mathbb{Q}}$ and q is not a root of unity.

Definition 1. A finite $\mathcal{A}_q$ -module M is holonomic if $\text{GKdim}(M) = n$.

Let M be holonomic and $R, p, \mathfrak{p}, M_{\mathfrak{p}}$ be as above. Note that since $k_{\mathfrak{p}} = R/\mathfrak{p}$ is a finite field, q automatically projects to a root of unity $\overline{q} \in k_{\mathfrak{p}}$ of some order N . The following theorem generalizes Bitoun's theorem to the q -case and has a similar proof.

Theorem 3.1. For sufficiently large p , $\text{supp}_N(M_{\mathfrak{p}})$ is Lagrangian.

4. POTENTIAL APPLICATIONS

1. Let X be a conical symplectic singularity. It has a family of quantizations $A_{t,\lambda}$ defined by I. Losev, where $\lambda \in H^2(X) = H^2(\tilde{X}, \mathbb{C})$, where $\tilde{X}$ is a $\mathbb{Q}$ -factorial terminalization of X . The family of varieties $X_\lambda := \text{Spec} A_{0,\lambda}$ is the universal Poisson deformation of X defined by Namikawa. In characteristic p , $A_\lambda = A_{1,\lambda}$ has p -center $Z_\lambda \cong A_{0,\lambda^p - \lambda}$ over which A_λ is finite. For a finite module M over A_λ , we can consider $\text{supp}_p(M) \subset \text{Spec} Z_\lambda = X_{\lambda^p - \lambda}$.

Losev also defined (in another paper) a notion of a holonomic A_λ -module, which is a finite module M such that the support of $\text{gr}(M)$ is Lagrangian (both isotropic and coisotropic). The following conjecture generalizes Bitoun's theorem.

Conjecture 4.1. If M is holonomic, then $\text{supp}_p(M_{\mathfrak{p}})$ is Lagrangian for large p .

For Higgs branches, this conjecture should follow from the D -module case by quantum hamiltonian reduction.

2. The result in the q -case should imply results on Lagrangianity of p -supports for quantum cluster varieties in some cases.

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Singular support for G -categories and applications to W -algebras

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(joint work with Gurbir Dhillon)

The goal of this talk is threefold:

- First, we introduce a notion of singular support for categorical representations.
- Second, we discuss Whittaker coefficients of character sheaves.
- Third, we discuss applications to the classification of finite-dimensional representations of finite W -algebras.

Let us expand on the above three points.

Let G be a connected reductive group over an algebraically closed field k of characteristic zero. We are interested in categorical representations of G ; that is, module categories for $D(G)$, the monoidal category of D -modules on G under convolution.

By imitating the definition of the wavefront set of a smooth representation of p -adic groups, we define a notion of singular support for categorical representations of G . By definition, the singular support lives in $\mathfrak{g}^*$. Our first main result characterizes the singular support of G -categories with nilpotent singular support in terms of the vanishing of its Whittaker models. Moreover, we provide a number of natural examples of G -categories and compute their singular support.

Next, we study Whittaker coefficients of character sheaves on G . We show that if $\mathcal{F}$ is a character sheaf on G whose singular support is bounded by the closure of a nilpotent orbit $\mathbb{O}$, then the generalized Whittaker coefficients corresponding to a nilpotent element $e \in \mathbb{O}$ computes the microstalk of $\mathcal{F}$ at $(1, e) \in T^*(G/G)$.

Finally, we apply the above two results to give a categorical proof of the classification of finite-dimensional representations of finite W -algebras. This was proved by Losev-Ostrik for trivial central character and by Bezrukavnikov-Losev for an arbitrary central character. During the proof, we also explain how our methods recover a theorem of Etingof-Schedler realizing the Hochschild homology of W -algebras as the cohomology of the corresponding Springer fiber.

Birational maps to Grassmannians, representations, and poset polytopes

EVGENY FEIGIN

We fix two positive integers $k < n$ and consider the Grassmannian $\mathrm{Gr}(k, n)$ consisting of k dimensional subspaces in an n dimensional complex vector space. The classical Plücker embedding realizes $\mathrm{Gr}(k, n)$ as a smooth projective variety inside the projectivization of the k -th fundamental representation L_{ω_k} of the group SL_n . The coordinate ring of this embedding is known to be isomorphic to the direct sum over all non-negative integers m of dual irreducible representations $L_{m\omega_k}^*$. In other words, the modules $L_{m\omega_k}^*$ are identified with the spaces of sections of the

natural line bundles on the Grassmann varieties. It is natural to expect that algebraic and combinatorial properties of $L_{m\omega_k}^*$ can be applied to the study of the algebro-geometric properties of the Grassmannians; of course, the opposite direction is also very fruitful – many properties of the irreducible representations are explained in terms of the geometry of the Grassmann varieties. An example of this interplay is provided by the theory of flat toric degenerations. More precisely, one is interested in a family over an affine line whose general fiber is the classical Grassmannian and the special fiber is a projective toric variety. It was shown in [10, 8] that such a degeneration can be constructed using the Gelfand-Tsetlin polytopes, which give bases in the irreducible representations of SL_n via the chains of restrictions to smaller special linear groups. Yet another approach is based on the construction of monomial bases in the representation spaces. The elements of the bases are written as monomials in the root elements of the Lie algebra applied to the highest weight vector [7, 11]. The monomials are labeled by the integer points in certain convex polytopes, which are identified with the Newton polytopes of toric varieties showing up as the special fibers of the family.

Our goal is to extend the above mentioned results to the graph closures of the birational maps from the projective spaces to Grassmann varieties. More precisely, let $\mathbb{A}^N$ be the open dense Schubert cell inside $\mathrm{Gr}(k, n)$ with $N = k(n - k) = \dim \mathrm{Gr}(k, n)$. The standard embedding $\mathbb{A}^N \subset \mathbb{P}^N$ allows to define a birational map from $\mathbb{P}^N$ to the Grassmannian. We consider the subvariety $\mathbb{G}(k, n) \subset \mathbb{P}^N \times \mathrm{Gr}(k, n)$, which is the closure of the diagonally embedded $\mathbb{A}^N$. The varieties $\mathbb{G}(k, n)$ were studied in [3, 1] in connection with applications of algebraic geometry to theoretical problems of quantum chemistry. We study topological, algebro-geometric, representation-theoretic and combinatorial properties of the graph closures. Our main tool is the existence of the action of the abelian unipotent group $\mathbb{A}^N$ on $\mathbb{G}(k, n)$, which allows to use certain representations of this group for the description of the graph closures. Namely, we define a family $L_{m,M}$ of cyclic representations of the group $\mathbb{A}^N$ labeled by $m, M \in \mathbb{Z}_{\geq 0}$ and construct embeddings of the varieties $\mathbb{G}(k, n)$ to $\mathbb{P}(L_{m,M})$ (for positive values of the parameters). We define monomial bases of $L_{m,M}$ in terms of certain convex polytopes $X(m, M)$, generalizing the results of [7]. We show that the toric varieties, associated with $X(m, M)$ show up as special fibers of certain flat families, whose general fibers are the graph closures. Finally, we write down the set of quadratic relations, generating the defining ideal of $\mathbb{G}(k, n)$ and describe the topology of the varieties $\mathbb{G}(k, n)$.

The construction of $\mathbb{G}(k, n)$ is based on the existence of an open Schubert cell in the Grassmannians. It is natural to study the corresponding graph closures for a larger class of projective algebraic varieties, admitting an open dense cell. Unfortunately, there are very few results available [9, 6]. In particular, in [6] the case of the PBW degenerate flag varieties [4] is considered. The corresponding graph closure is described in terms of representation theory of the equioriented type A quiver [2]. However, the analogous problem for the classical complete flag

varieties is still completely open. The approach described above is not applicable in this situation, because the action of the abelian unipotent group is missing.

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Cluster algebras and shifted quantum groups

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Shifted quantum affine algebras arose [FT] in the study of quantized K -theoretic Coulomb branches of 3d $N = 4$ SUSY quiver gauge theories. A presentation of shifted quantum affine algebras by generators and relations was given by Finkelberg–Tsybaliuk.

Let $\mathfrak{g}$ be a simple complex finite-dimensional Lie algebra of rank n , and $\hat{\mathfrak{g}}$ the corresponding untwisted affine Kac-Moody algebra, central extension of the loop algebra $\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}]$. Drinfeld and Jimbo associated to each complex number $q \in \mathbb{C}^*$ the quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ which is a Hopf algebra. Shifted quantum affine algebras $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ can be seen as variations of $\mathcal{U}_q(\hat{\mathfrak{g}})$, but depending on an integral coweight μ of the underlying simple Lie algebra $\mathfrak{g}$. These coweights corresponding to shifts of formal power series in the Cartan-Drinfeld elements (that is quantum analogs of the $t^r h \in \mathcal{L}\mathfrak{g}$, with $r \in \mathbb{Z}$ and $h \in \mathfrak{h}$ in the Cartan subalgebra of $\mathfrak{g}$). In particular $\mathcal{U}_q^0(\hat{\mathfrak{g}})$ is a central extension of the ordinary quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$.

In [H1], we develop the representation theory of shifted quantum affine algebras. We establish several analogies with the representation theory of ordinary quantum

affine algebras, but also several new properties, in particular in relations to representations of the ordinary Borel subalgebra $\mathcal{U}_q(\hat{\mathfrak{b}}) \subset \mathcal{U}_q(\hat{\mathfrak{g}})$. Consider a category $\mathcal{O}_\mu$ of representations of $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ which is an analog of the ordinary category $\mathcal{O}$.

Theorem 1 [H1] *The simple representations $L(\Psi)$ in $\mathcal{O}_\mu$ are parametrized by n -tuples $\Psi = (\Psi_i(z))_{1 \leq i \leq n}$ of rational fractions regular at 0 with $\deg(\Psi_i(z)) = \alpha_i(\mu)$.*

For the moment, we assume that $\mathfrak{g}$ is of simply-laced type.

For $1 \leq i \leq n$ and $a \in \mathbb{C}^*$, let $\Psi_{i,a} = (1 - \delta_{i,j}za)_{j \in I}$ and $\omega_i^\vee$ a fundamental coweight. We have the following examples.

- $L_{i,a}^+ = L(\Psi_{i,a})$ is a one-dimensional representation of $\mathcal{U}_q^{\omega_i^\vee}(\mathfrak{g})$ called a positive prefundamental representation,

- $L_{i,a}^- = L(\Psi_{i,a}^{-1})$ is an infinite-dimensional representation of $\mathcal{U}_q^{-\omega_i^\vee}(\mathfrak{g})$ called a negative prefundamental representation

- $V_{i,a} = L(\Psi_{i,aq^{-1}}\Psi_{i,aq}^{-1})$ is a finite-dimensional fundamental representation of the ordinary quantum affine algebra (up to a twist).

Although individually each $\mathcal{U}_q^\mu(\hat{\mathfrak{g}})$ is not a Hopf algebra, one can define a ring structure on the sum of Grothendieck groups

$$K_0(\mathcal{O}) = \bigoplus_{\mu} K_0(\mathcal{O}_\mu)$$

from a procedure called fusion product. It contains the Grothendieck ring $K_0(\mathcal{C}^{sh})$ of finite-dimensional representations as a subring.

For example, for $\mathfrak{g} = sl_2$ we have the QQ -relation in $K_0(\mathcal{O})$:

$$[L_{1,a}^+][L_{1,a}^-] = 1 + [L(q^{-2})][L_{aq^{-2}}^-][L_{aq^2}^+],$$

with $L(q^{-2})$ one-dimensional representation associated to a constant ℓ -weight.

Theorem 2 [FH] *Analog QQ -relations hold in $K_0(\mathcal{O}^{sh})$ for general $\mathfrak{g}$, involving new elements $Q_{w(\omega_i),a} \in K(\mathcal{O}^{sh})$, where $1 \leq i \leq n$, w is a Weyl group element and $a \in \mathbb{C}^*$.*

The $Q_{w(\omega_i),a}$ will be called Q -variables. We have

$$Q_{\omega_i,a} = [L_{i,a}^+] \text{ and } Q_{-\omega_i,a} = [L_{i,aq^{-h}}^-],$$

with h the Coxeter number of $\mathfrak{g}$. In type A_2 , we have

$$Q_{s_1(\omega_1),a} = Q_{\omega_2-\omega_1,a} = L(\Psi_{2,aq^{-1}}\Psi_{1,aq^{-2}}^{-1}).$$

Recall that the cluster algebra $\mathcal{A}(Q)$ attached to a quiver Q is a commutative ring with a distinguished set of generators called cluster variables and obtained inductively by a procedure (mutations and exchange relations) from initial cluster variables. The cluster variables are grouped into overlapping subsets (the clusters). A cluster monomial is a monomial in cluster variables from a given cluster.

It is very useful to realize Grothendieck rings in terms of cluster algebras (see [H2] for a recent review).

In the following, we will restrict to subcategories $\mathcal{O}_{\mathbb{Z}}^{sh} \subset \mathcal{O}^{sh}$ and $\mathcal{C}_{\mathbb{Z}}^{sh} \subset \mathcal{C}^{sh}$ with an integrality conditions on possible simple constituents (the description of such subcategories is sufficient to understand the whole category).

Theorem 3 [HL, KKOP, 4] *The Grothendieck ring $K_0(\mathcal{C}_{\mathbb{Z}}^{sh})$ has a structure of a cluster algebra $\mathcal{A}(\Gamma_{\mathfrak{g}})$ for an explicit quiver $\Gamma_{\mathfrak{g}}$ with initial cluster variables which are classes of positive prefundamental representations. Moreover, all cluster monomials are classes simple objects.*

In the $\mathfrak{sl}_2$ -case, the quiver $\Gamma_{\mathfrak{sl}_2}$ is the infinite linear quiver :

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

The initial cluster is formed of classes of 1-dimensional positive prefundamental representations

$$\cdots \longrightarrow L_{1,q-2}^+ \longrightarrow L_{1,1}^+ \longrightarrow L_{1,q}^+ \longrightarrow \cdots$$

For $\mathfrak{g} = \mathfrak{sl}_3$, the quiver $\Gamma_{\mathfrak{sl}_3}$ and the initial cluster are

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\quad} & [L_{1,q-2}^+] & \xrightarrow{\quad} & [L_{1,1}^+] & \xrightarrow{\quad} & \cdots \\ & \swarrow & \nwarrow & \swarrow & \nwarrow & \swarrow & \\ \cdots & \longrightarrow & [L_{2,q-3}^+] & \longrightarrow & [L_{2,q-1}^+] & \longrightarrow & [L_{2,q}^+] \longrightarrow \cdots \end{array}$$

Theorem 4 [GHL1] *The Grothendieck ring $K_0(\mathcal{O}_{\mathbb{Z}}^{sh})$ is isomorphic to (a completion of) a cluster algebra $\mathcal{A}_{\Gamma'_{\infty}}$, with an explicit quiver Γ'_{∞} . The Q -variables are identified with cluster variables and the QQ -system above are identified with exchange relations.*

One of the crucial technical step in the proof of this Theorem is the convergence of certain g -vectors of cluster variables for certain infinite sequences of mutations, that we establish. This relies in part on a nice partition of the quiver in slices.

For $\mathfrak{g} = \mathfrak{sl}_2$, the new quiver Γ'_{∞} is obtained from the older quiver Γ_{∞} by inverting the direction of one arrow :

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

The initial variables are positive/negative prefundamental representations :

$$\cdots \longrightarrow L_{1,q}^+ \longrightarrow L_{1,1}^+ \longleftarrow L_{1,q-2}^- \longrightarrow L_{1,q-4}^- \longrightarrow \cdots$$

For $\mathfrak{g} = \mathfrak{sl}_3$, the quiver $\Gamma'_{\mathfrak{sl}_3}$ is also built from the periodic quiver $\Gamma_{\mathfrak{sl}_3}$, except that it contains 3 quadrilaterals :

$$\begin{array}{ccccccc} \cdots & \bullet & \xrightarrow{\quad} & \bullet & \xleftarrow{\quad} & \bullet & \xrightarrow{\quad} & \bullet & \xleftarrow{\quad} & \bullet & \xrightarrow{\quad} & \cdots \\ & \swarrow & & \nwarrow & \swarrow & & \nwarrow & \swarrow & & \nwarrow & \swarrow & \\ \cdots & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \cdots \end{array}$$

The initial cluster variables are the $[L_{1,q^{2r}}^+]$, $[L_{2,q^{2r-1}}^+]$, $[L_{1,q^{2s}}^+]$, $[L_{2,q^{2s-1}}^+]$, with $r \geq 0$, $s \leq -2$, as well as $[Q_{s_1(\omega_1),q^{-1}}]$ and $[Q_{s_1(\omega_1),q^{-3}}]$.

In general, we conjecture the following, that we establish for $\mathfrak{g} = \mathfrak{sl}_2$ in [GHL1].

Conjecture [GHL1] *All cluster monomials in $\mathcal{A}_{\Gamma_\infty}$ correspond to classes of simple objects in $\mathcal{O}$ through our isomorphism.*

In [GHL2] we investigate the non-simply laced types. There are several important differences, for example the nice partition of the quiver into slices does not work here, and we need difference methods to establish the stabilization of g -vectors.

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Linear degenerations of Schubert varieties via quiver Grassmannians

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The fruitful investigation of *linear degenerations of flag varieties* using quiver Grassmannians took place in the last fifteen years, appearing in several papers by Feigin, Finkelberg, Cerulli Irelli, Reineke, Fang, Fourier [1, 2, 3, 4, 5]. The cited works characterise several geometric and combinatorial aspects of linear degenerations of flag varieties, such as their defining equations, cellular decompositions and - making use of rank tuples - flatness, irreducibility and normality. Here, we consider the Schubert varieties X_w in the flag variety Fl_{n+1} . For a permutation w in the symmetric group S_{n+1} , these are defined as:

$$X_w = \{V \in Fl_{n+1} : \dim(F_p \cap V_q) \geq \#\{k \leq q : w(k) \leq p\} \text{ for } 1 \leq p, q \leq n+1\}.$$

We provide a construction for realising smooth Schubert varieties as quiver Grassmannians and desingularizing non-smooth Schubert varieties. We then exploit this construction to define linear degenerations of Schubert varieties, giving a combinatorial description of the correspondence between their isomorphism classes and the orbits of certain quiver representations.

First, consider the following quiver with relations (Γ, I) and its representation M :

$$M : \begin{array}{ccccccc} \bullet & \xrightarrow{\text{id}} & \bullet & \xrightarrow{\text{id}} & \dots & \xrightarrow{\text{id}} & \bullet \\ \downarrow \iota_{2,1} & \circlearrowleft & \downarrow \iota_{2,1} & \circlearrowleft & \downarrow \iota_{2,1} & \circlearrowleft & \downarrow \iota_{2,1} \\ \mathbb{C}^2 & \xrightarrow{\text{id}} & \mathbb{C}^2 & \xrightarrow{\text{id}} & \dots & \xrightarrow{\text{id}} & \mathbb{C}^2 \\ \downarrow \iota_{3,2} & \circlearrowleft & \downarrow \iota_{3,2} & \circlearrowleft & \downarrow \iota_{3,2} & \circlearrowleft & \downarrow \iota_{3,2} \\ \dots & \xrightarrow{\text{id}} & \dots & \xrightarrow{\text{id}} & \dots & \xrightarrow{\text{id}} & \dots \\ \downarrow \iota_{n+1,n} & \circlearrowleft & \downarrow \iota_{n+1,n} & \circlearrowleft & \downarrow \iota_{n+1,n} & \circlearrowleft & \downarrow \iota_{n+1,n} \\ \mathbb{C}^{n+1} & \xrightarrow{\text{id}} & \mathbb{C}^{n+1} & \xrightarrow{\text{id}} & \dots & \xrightarrow{\text{id}} & \mathbb{C}^{n+1} \end{array},$$

whose vertical linear maps $\iota_{i+1,i}$ are fixed as the standard inclusion of $\mathbb{C}^i$ into $\mathbb{C}^{i+1}$.

Proposition 1. [6, Corollary 4.6] *Given (Γ, I) and M as above, the quiver Grassmannian $\text{Gr}_{\mathbf{e}}(M)$ is a smooth and irreducible projective variety for any dimension vector $\mathbf{e}$. Its dimension is given by the Euler-Ringel form of (Γ, I) : $\langle \mathbf{e}, \mathbf{dim}(M) - \mathbf{e} \rangle$.*

By appropriately interpreting the intersection conditions that define a (fixed) Schubert variety X_w , we can define two dimension vectors - $\mathbf{r}^w$ and $\mathbf{e}^w$ - for (Γ, I) with the following properties (see [6] for the precise definitions):

Theorem 1. [6, Theorem 5.18] The quiver Grassmannian $\text{Gr}_{\mathbf{r}^w}(M)$ is isomorphic to certain Bott-Samelson resolutions of X_w .

Theorem 2. [6, Theorem 6.4] Given a smooth Schubert variety X_w , the quiver Grassmannian $\text{Gr}_{\mathbf{e}^w}(M)$ is isomorphic to X_w .

Now, Theorems 1 and 2 allow us to define *linear degenerations of Schubert varieties*, denoted by X_w^f , as the quiver Grassmannian consisting of all subrepresentations of M^f :

$$\begin{array}{ccccccc}
 \bullet & \xrightarrow{f_1^1} & \bullet & \xrightarrow{f_1^2} & \dots & \xrightarrow{f_1^{n-1}} & \bullet \\
 \downarrow \iota_{2,1} & \circlearrowleft & \downarrow \iota_{2,1} & \circlearrowleft & & \downarrow \iota_{2,1} & \circlearrowleft & \downarrow \iota_{2,1} \\
 \mathbb{C}^2 & \xrightarrow{f_2^1} & \mathbb{C}^2 & \xrightarrow{f_2^2} & \dots & \xrightarrow{f_2^{n-1}} & \mathbb{C}^2 \\
 \downarrow \iota_{3,2} & \circlearrowleft & \downarrow \iota_{3,2} & \circlearrowleft & & \downarrow \iota_{3,2} & \circlearrowleft & \downarrow \iota_{3,2} \\
 \vdots & \xrightarrow{f_i^1} & \vdots & \xrightarrow{f_i^j} & \dots & \xrightarrow{f_i^{n-1}} & \vdots \\
 \downarrow \iota_{n+1,n} & \circlearrowleft & \downarrow \iota_{n+1,n} & \circlearrowleft & & \downarrow \iota_{n+1,n} & \circlearrowleft & \downarrow \iota_{n+1,n} \\
 \mathbb{C}^{n+1} & \xrightarrow{f_{n+1}^1} & \mathbb{C}^{n+1} & \xrightarrow{f_{n+1}^2} & \dots & \xrightarrow{f_{n+1}^{n-1}} & \mathbb{C}^{n+1}
 \end{array}$$

of the chosen, appropriate dimension vector. If we denote

$$V \in \prod_{i,j \in (\Gamma, I)_0} \text{Gr}(e_{i,j}^w, \mathbb{C}^i)$$

and

$$f = (f_{n+1}^1, \dots, f_{n+1}^{n-1}) \in \prod_{j=1}^{n-1} U_{n+1},$$

where U_{n+1} is the subset of Mat_{n+1} consisting of upper-triangular matrices and each f_i^j is the restriction of f_{n+1}^j to $\mathbb{C}^i$, then we can give the following:

Definition 1. The *universal linear degeneration of the Schubert variety* X_w is the variety defined as

$$Y = \{(f, V) : f_\alpha(V_{s(\alpha)}) \subseteq V_{t(\alpha)} \text{ for all } \alpha \in (\Gamma, I)_1\}.$$

Then, for a fixed $f \in \prod_{j=1}^{n-1} U_{n+1}$ and π the projection map $\pi : Y \rightarrow \prod_{j=1}^{n-1} U_{n+1}$, the f -linear degenerate Schubert variety X_w^f is the fibre $\pi^{-1}(f) = \text{Gr}_{e^w}(M^f)$.

The first natural question we consider is about the representation type of (Γ, I) - that is, whether there are finitely many isomorphism classes of indecomposable representations of (Γ, I) - when we restrict to the representations determined by a tuple $f \in \prod_{j=1}^{n-1} U_{n+1}$ and to certain isomorphisms. In order to do so, we consider the orbits under the action defined as

$$(1) \quad h \cdot M^f = (h_2 f_{n+1}^1 h_1^{-1}, h_3 f_{n+1}^2 h_2^{-1}, \dots, h_n f_{n+1}^{n-1} h_{n-1}^{-1}),$$

for some $h \in \prod_{j=1}^{n-1} B_{n+1}$, where B_{n+1} is the Borel subgroup of invertible upper-triangular matrices inside GL_{n+1} , and M^f is the (Γ, I) -representation determined by f . We notice then that our action can be regarded as an expansion of the one considered in [7] by Miller and Sturmfels in the context of matrix Schubert varieties. There, the orbits are described in terms of ranks of certain submatrices, namely the north-west ranks. We adapt this parametrisation to our action by defining the south-west arrays, denoted by $\mathbf{s}$, and showing the following:

Theorem 3. Two representations M^f, M^g . for $f, g \in \prod_{j=1}^{n-1} U_{n+1}$, are in the same orbit under the action given in (1) if and only if $\mathbf{s}^f = \mathbf{s}^g$. Furthermore, M^f lies in the (Zariski) closure of the orbit of M^g if and only if $\mathbf{s}^f \leq \mathbf{s}^g$.

Making use of Theorem 3, some properties of the linear degenerations of Schubert varieties can be proven almost immediately. For instance, by defining an appropriate grading on the coefficient quiver of (Γ, I) and then applying [8, Theorem 1], we obtain that the Euler characteristic of $\mathrm{Gr}_{e^w}(M^f)$ is equal to the number of fixed points in $\mathrm{Gr}_{e^w}(M^f)$ under a certain action of the torus $\mathbb{C}^*$ on $\mathrm{Gr}_{e^w}(M^f)$.

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Cluster structures on schemes of bands

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(joint work with Luca Francone)

Let G denote a simple and simply-connected algebraic group of type A, D, E over an algebraically closed field K of characteristic 0. Let B be a Borel subgroup of G with maximal torus T , and let U be its unipotent radical. Let B^- be the opposite Borel subgroup of B with respect to T , and U^- its unipotent radical. Let $W = N(T)/T$ denote the Weyl group, and let $c \in W$ be a Coxeter element. Choose a representative $\bar{c}$ of c in $N(T)$.

We introduce new geometric objects, called (G, c) -bands. A (G, c) -band over K is a sequence $b = (g(s))_{s \in \mathbb{Z}}$ of elements $g(s) \in G$ such that

$$g(s)g(s+1)^{-1} \in U(c^{-1})\bar{c}, \quad (s \in \mathbb{Z}).$$

If $G = SL(n)$ and $c = c_{st} = s_1 s_2 \cdots s_{n-1}$ is the standard Coxeter element, then an $(SL(n), c_{st})$ -band can be represented by an infinite matrix $B \in \mathrm{Mat}_{\infty, n}(K)$ such that every submatrix of B consisting of n consecutive rows has determinant 1.

We show that in general, the (G, c) -bands over K are the K -rational points of an infinite-dimensional affine integral scheme $B(G, c)$, whose ring of regular functions $R(G, c)$ is a unique factorization domain. By construction there are natural projection maps:

$$\pi_s : B(G, c) \longrightarrow G, \quad (s \in \mathbb{Z}).$$

At the level of K -points, the map π_s sends a band $b = (g(t))_{t \in \mathbb{Z}}$ to $g(s)$. We show that the dual maps $\pi_s^* : K[G] \rightarrow R(G, c)$ are injective, thus we get infinitely many subalgebras $\pi_s^*(K[G])$ of $R(G, c)$ isomorphic to $K[G]$.

Let $\Delta_{u(\varpi_i), v(\varpi_i)} \in K[G]$ denote the generalized minor associated with the fundamental weight ϖ_i and $u, v \in W$. Let $\Delta_{u(\varpi_i), v(\varpi_i)}^{(s)} := \pi_s^*(\Delta_{u(\varpi_i), v(\varpi_i)}) \in R(G, c)$. Let m_i be the smallest integer k such that $c^k(\varpi_i) = w_0(\varpi_i)$. The regular functions $\Delta_{u(\varpi_i), v(\varpi_i)}^{(s)}$ satisfy relations of the form:

$$\Delta_{c^k(\varpi_i), v(\varpi_i)}^{(s)} = \Delta_{c^{k-1}(\varpi_i), v(\varpi_i)}^{(s+1)}, \quad (1 \leq k \leq m_i, v \in W).$$

In [2] a cluster algebra $\mathcal{A}$ associated with G was introduced. More precisely, $\mathcal{A}$ is the algebra $\mathcal{A}_{w_0}$ of [2, §3] attached to the longest element w_0 of W . Moreover, a family of initial seeds of $\mathcal{A}$ whose quivers Γ_c correspond to Coxeter elements c of W was described. Given a Coxeter element c , let $\tilde{c} := w_0 c^{-1} w_0$ denote its dual Coxeter element. Our first main result states that there is an algebra isomorphism $K \otimes_{\mathbb{Z}} \mathcal{A} \rightarrow R(G, c)$ such that all cluster variables of the initial seed of $\mathcal{A}$ with quiver $\Gamma_{\tilde{c}}$ are mapped to regular functions of the form $\Delta_{c^k(\varpi_i), \tilde{c}^l(\varpi_i)}^{(s)}$ for suitable integers k, l, s .

It is easy to see that the action of G on itself by right translation lifts to a diagonal free action of G on $B(G, c)$. This induces a linear action of G on the algebra $R(G, c)$, which we can restrict to an action of the unipotent subgroup U . Our second main result states that the subalgebra $R(G, c)^U$ of U -invariant functions is isomorphic to $K \otimes \mathcal{B}$, where $\mathcal{B}$ is an upper cluster algebra with a distinguished initial cluster corresponding to the subset of regular functions $\Delta_{\varpi_i, \varpi_i}^{(s)}$ ($s \in \mathbb{Z}$, $1 \leq i \leq \text{rk}(G)$). The algebra $R(G, c)^U$ contains also the G -invariant functions $\theta_{i,k}^{(s)}$ defined at the level of points $b = (g(t))_{t \in \mathbb{Z}}$ by

$$\theta_{i,k}^{(s)}(b) = \Delta_{\varpi_i, \varpi_i}(g(s)g(s+k)^{-1}), \quad (s \in \mathbb{Z}, 1 \leq i \leq \text{rk}(G), k \in \mathbb{Z}_{\geq 1}).$$

Our third main result states that the subalgebra $R(G, c)^G$ of G -invariant functions is isomorphic to $K \otimes \mathcal{C}$, where $\mathcal{C}$ is a cluster algebra with a family of distinguished clusters corresponding to subsets of functions of the form $\theta_{i,k}^{(s)}$ for suitable s, i, k .

The three above cluster algebras $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are not new. They all arise as cluster structures in the Grothendieck rings of certain categories of representations of quantum affine algebras. Finding a geometric model for these Grothendieck rings was our initial motivation for introducing and studying (G, c) -bands.

More precisely, let $U_q(\hat{\mathfrak{g}})$ denote the quantum affinization of the Lie algebra $\mathfrak{g}$ of $G(\mathbb{C})$. (Here we fix $q \in \mathbb{C}^*$, not a root of 1.) The cluster algebra $\mathcal{C}$ is isomorphic to the Grothendieck ring of the category $\mathcal{C}_{\mathbb{Z}}$ of finite-dimensional $U_q(\hat{\mathfrak{g}})$ -modules

introduced in [5], and further studied in [6, 10, 8, 9]. The cluster algebra $\mathcal{B}$ is isomorphic to the Grothendieck ring of the subcategory $\mathcal{O}_{\mathbb{Z}}^+$ of the Hernandez-Jimbo category $\mathcal{O}$ of the Borel subalgebra of $U_q(\widehat{\mathfrak{g}})$ introduced and studied in [7]. Finally, the cluster algebra $\mathcal{A}$ recently appeared in connection with the subcategory $\mathcal{O}_{\mathbb{Z}}^{\text{shift}}$ of the Hernandez category $\mathcal{O}^{\text{shift}}$ of representations of the Finkelberg-Tsybaliuk shifted quantum affine algebras [3, 4, 2].

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Harish-Chandra center for affine Kac-Moody algebras in positive characteristic

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(joint work with Gurbir Dhillon)

Computing the centers in the universal enveloping algebras is a basic question in Lie representation theory. For the semisimple Lie algebras over the complex numbers this has been done by Harish-Chandra, while in the same setting in positive (not too small) characteristic a solution was given by Veldkamp. Namely, consider a connected reductive group G over an algebraically closed field $\mathbb{F}$ of characteristic p . For simplicity we assume that p is larger than the Coxeter numbers of all of the (almost) simple normal subgroups of G . Let $\mathfrak{g}$ denote the Lie algebra of G . Consider the universal enveloping algebra $U(\mathfrak{g})$. Veldkamp proved that the center of $U(\mathfrak{g})$ is generated by two subalgebras: its “Harish-Chandra center” and the “ p -center”. The Harish-Chandra center, by definition, is the subalgebra of invariants for the G -action on $U(\mathfrak{g})$ induced by the adjoint action of G on $\mathfrak{g}$. Its description is completely analogous to Harish-Chandra’s, hence the name. The p -center is a new phenomenon that appears in characteristic p , this is a subalgebra identified

with the symmetric algebra $S(\mathfrak{g}^{(1)})$, where the superscript indicates the Frobenius twist.

For the affine Lie algebras the description of the center is known over the complex numbers (and, slightly more generally, over characteristic 0 fields), it is due to B. Feigin and E. Frenkel in the early 90's. Namely, we consider the affine Lie algebra $\hat{\mathfrak{g}}$ and let κ denote a level, an element of the base field. We can consider the central reduction $U_\kappa(\hat{\mathfrak{g}})$ and its usual completion $\tilde{U}_\kappa(\hat{\mathfrak{g}})$ that acts on the smooth representations. The result is more convenient to state when $\mathfrak{g}$ is simple. If κ is different from $-h^\vee$, where $h^\vee$ is the dual Coxeter number, then the center consists of scalars. Otherwise – in which case κ is called the *critical level* – it is the algebra of functions on the space of opers $\mathrm{Op}_{G^L}(D^\times)$, where $D^\times$ is the punctured disc, i.e., the spectrum of the algebra of Laurent series, and G^L is the Langlands dual group. This algebra is a suitable completion of the algebra of polynomials in infinitely many variables.

One can then ask what happens with the center of $\tilde{U}_\kappa(\hat{\mathfrak{g}})$ (or the center of the corresponding affine vertex algebra) if the characteristic of the base field is positive. For the critical level (and p sufficiently large) this question has been answered in a recent preprint of Arakawa, Topley and Villareal, where they established an analog of Veldkamp's theorem. In our work with Dhillon, we look at describing the Harish-Chandra center, i.e., the invariants of the loop group $G((t))$ in the subalgebra $\hat{U}_\kappa(\hat{\mathfrak{g}}) \subset \tilde{U}_\kappa(\hat{\mathfrak{g}})$ of all elements of finite PBW degree in the case when $\kappa \neq -h^\vee$.

Our results are as follows. Assume that $p > h$, the Coxeter number. When $\kappa \in \mathbb{F}_p$, we show that the Harish-Chandra center is contained in the p -center, the situation polar opposite to what happens for semisimple Lie algebras. For $\kappa \notin \mathbb{F}_p$, we show that the Harish-Chandra center is identified with the algebra of functions on $\mathrm{Op}_{G^L}(D^{\times,(1)})$. This hints at an intriguing connection between the representations of the affine algebra associated to the discs D at non-integral levels and the representations of the affine algebra associated to the disc $D^{(1)}$ at the critical level. Important ingredients in our construction are explicit descriptions of the Harish-Chandra centers for $G = \mathbb{G}_m$ and SL_2 .

The periplectic Lie superalgebra, KLR algebras and categorification

JONAS NEHME

(joint work with Catharina Stroppel)

This is based on [6]. In [8, 2], a categorification of tensor powers $V^{\otimes d}$ of the natural representation V of $\mathfrak{gl}(n)$ was constructed using category $\mathcal{O}$ for $\mathfrak{gl}(d)$. The action of $\mathfrak{gl}(n)$ on $V^{\otimes d}$ is then categorified by the action of translation functors on category $\mathcal{O}$.

Our goal is to present a similar story for the periplectic Lie superalgebra $\mathfrak{p}(n)$, i.e. we replace category $\mathcal{O}$ for $\mathfrak{gl}(d)$ by $\mathrm{rep}(\mathfrak{p}(n))$, the category of finite-dimensional representations of $\mathfrak{p}(n)$, and want to answer the following question: What does this categorify?

We approach this question via Schur–Weyl duality for the periplectic Lie superalgebra $\mathfrak{p}(n)$. In this case, the counterpart is given by the super Brauer category $\mathbf{sBr}$ and there is a full monoidal functor

$$\mathbf{sBr} \rightarrow \mathbf{rep}(\mathfrak{p}(n)).$$

In [1], it was shown that the endofunctor $_ \otimes V$ for the natural representation V of $\mathfrak{p}(n)$ decomposes into $_ \otimes V = \bigoplus_{i \in \mathbb{Z}} \Theta_i$, where Θ_i are the projections onto the generalized eigenspace for some *fake Casimir*.

We introduce an idempotent version of $\mathbf{sBr}$ to encapsulate the decomposition of $_ \otimes V$ diagrammatically. This arises as a cyclotomic quotient of a more general KLR algebra, which we call *electric KLR category*. We denote this by $\mathbf{sR}$, and it is a $\mathbb{C}$ -linear monoidal supercategory with objects generated by $a \in \mathbb{Z}$ and morphisms generated by

$$\begin{array}{ccc} \begin{array}{c} b \quad a \\ \diagdown \quad \diagup \\ a \quad b \end{array} : a \otimes b \rightarrow b \otimes a, & \begin{array}{c} a \\ \bullet \\ a \end{array} : a \rightarrow a, \\ \begin{array}{c} a+1 \quad a \\ \frown \end{array} : \mathbf{1} \rightarrow (a+1) \otimes a, & \begin{array}{c} \frown \\ a \quad a+1 \end{array} : a \otimes (a+1) \rightarrow \mathbf{1}. \end{array}$$

The first two are even morphisms, while the cup and cap are odd morphisms. We impose certain relations which are similar to the relations of the KLR algebra. In fact, this electric KLR category $\mathbf{sR}$ is filtered by the number of propagating strands and its associated graded is the usual KLR algebra from [3, 7]. The additional generators (the cup and the cap) come from the fact that $V \cong \Pi V^*$ for $\mathfrak{p}(n)$, where Π is the parity change functor.

This category can be graded via

$$\begin{aligned} \deg \left(\begin{array}{c} a \\ \bullet \\ a \end{array} \right) &= 2, \quad \deg \left(\begin{array}{c} a+1 \quad a \\ \frown \end{array} \right) = -\epsilon, \quad \deg \left(\begin{array}{c} \frown \\ a \quad a+1 \end{array} \right) = \epsilon, \\ \deg \left(\begin{array}{c} b \quad a \\ \diagdown \quad \diagup \\ a \quad b \end{array} \right) &= b_{a,b} := \begin{cases} -2 & \text{if } b = a, a+1, \\ 4 \operatorname{sgn}(b-a)(-1)^{b-a} & \text{otherwise.} \end{cases} \end{aligned}$$

Fixing the dot to be of degree 2, the degrees of all the crossings are, in fact, forced. In particular, there is no way around the ± 4 appearing generically for the degree of crossings.

We introduce a level 1 cyclotomic quotient $\mathbf{sR}^{\text{cycl}}$ of $\mathbf{sR}$ and prove the following theorem.

Theorem 1. There is an equivalence of $\mathbf{sR}$ -module categories

$$\mathbf{sR}^{\text{cycl}} \cong \mathbf{Kar}(\mathbf{sBr}).$$

Via proving a basis theorem for $\mathbf{sR}^{\text{cycl}}$ we obtain the following:

Theorem 2. The (locally unital) superalgebra $\mathbf{sR}^{\text{cycl}}$ is an upper-finite based quasi-hereditary superalgebra with special idempotents indexed by partitions.

This in particular implies that $K_0(\text{sR}^{\text{cycl}}\text{-proj})$ has $\mathbb{Q}(q)$ -basis $[\Delta(\lambda)]$ (the standard objects) indexed by partitions λ .

We then compute relations in $K_0(\text{sR}\text{-proj})$ and obtain:

Theorem 3. We have $K_0(\text{sR}\text{-proj}) \cong \mathfrak{el}_q^\epsilon$ as $\mathbb{Q}(q)$ -algebras.

The algebra $\mathfrak{el}_q^\epsilon$ appearing in the above theorem is the *quantum electric Lie algebra*. It is a quantized version of the electric Lie algebra which was first considered in [4] in the context of electrical networks. It is the $\mathbb{Q}(q)$ -algebra generated by the elements $\mathcal{E}_i$ for $i \in \mathbb{Z}$ with relations

$$\begin{aligned} \mathcal{E}_i \mathcal{E}_j &= q^{b_{ij}} \mathcal{E}_j \mathcal{E}_i \quad \text{if } |i - j| > 1, \\ q^3 \mathcal{E}_i^2 \mathcal{E}_{i+1} - [2] \mathcal{E}_i \mathcal{E}_{i+1} \mathcal{E}_i + q^{-3} \mathcal{E}_{i+1} \mathcal{E}_i^2 &= -q^\epsilon [2] \mathcal{E}_i, \\ q^{-3} \mathcal{E}_i^2 \mathcal{E}_{i-1} - [2] \mathcal{E}_i \mathcal{E}_{i-1} \mathcal{E}_i + q^3 \mathcal{E}_{i-1} \mathcal{E}_i^2 &= -q^\epsilon [2] \mathcal{E}_i, \end{aligned}$$

Observe that this is a filtered algebra and its associated graded is the positive part of $\mathfrak{gl}(\infty)$ (up to some q -rescaling). We realize this as a coideal in some Hopf algebra which is reminiscent of a quantized universal enveloping algebra. For this, we can define a natural representation W and a Hecke algebra action on $W^{\otimes d}$, which allows us to define a Fock space $\mathcal{F}$.

We obtain the following commutative diagram:

$$\begin{array}{ccc} K_0(\text{sR}^{\text{cycl}}\text{-proj}) & \hookrightarrow & K_0(\text{sR}\text{-proj}) \\ \downarrow \sim & & \downarrow \sim \\ \mathcal{F} & \hookrightarrow & \mathfrak{el}_q^\epsilon \end{array}$$

A similar version holds for right-modules over sR . For this we need to replace $\mathcal{F}$ by the dual Fock space $\mathcal{F}^*$ and $\mathfrak{el}_q^\epsilon$ by $\mathfrak{el}_{q^{-1}}^\epsilon$. These two versions are then related by a *bar involution*.

The Fock space $\mathcal{F}$ is not an irreducible representation of $\mathfrak{el}_q^\epsilon$. There exists a filtration $\mathcal{F} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \mathcal{F}_2 \supset \dots$ such that $\mathcal{F}_i / \mathcal{F}_{i+1}$ is an irreducible representation of $\mathfrak{el}_q^\epsilon$. These quotients are categorified by $\text{proj}(\mathfrak{p}(n))$ for the various n , see [5] for details.

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Hall algebras of quivers via coherent sheaves and D-modules

TUDOR PĂDURARIU

(joint work with Yukinobu Toda)

The talk was based on the papers [4, 5] and on the forthcoming work [6].

For a quiver $Q = (I, E)$ and dimension vector $d \in \mathbb{N}^I$, consider the Kac polynomial

$$A_{Q,d}(t) \in \mathbb{Z}[t].$$

For an overview of the Kac polynomial, see [7]. Recall that, for a finite field k with q elements, the value $A_{Q,d}(q)$ equals the number of absolutely indecomposable representations of Q over the finite field k . For finite type quivers, the Kac polynomial $A_{Q,d}$ is constant. However, for general quivers, $A_{Q,d}$ may not be constant, and it is known that its coefficients are natural numbers. The constant coefficient of $A_{Q,d}$ equals the dimension of the graded d piece for an associate Kac-Moody Lie algebra of Q . One might then ask the following questions:

- (1) Do the coefficients of $A_{Q,d}$ have a description in terms of a graded Lie algebra associated to Q ?
- (2) Do the coefficients of $A_{Q,d}$ have a geometric description, for example as Betti numbers of a natural variety associated to Q ?

An answer for the first question is the BPS Lie algebra

$$\mathfrak{g}_{\text{BPS}} := \bigoplus_{d \in \mathbb{N}^I} \mathfrak{g}_{\text{BPS}}[d]$$

of an associated quiver with potential $(\tilde{Q}, \tilde{W})$ corresponding to Q . The degree zero part of $\mathfrak{g}_{\text{BPS}}$ recovers the Kac-Moody Lie algebra associated to Q . Before the above answer was found, Okounkov conjectured as an answer the Lie algebra $\mathfrak{g}_{\text{MO}}$ introduced in his joint work with Maulik [3]. This conjecture was proved by Botta-Davison [1], who showed that the positive half of $\mathfrak{g}_{\text{MO}}$ is isomorphic to the BPS Lie algebra $\mathfrak{g}_{\text{BPS}}$.

For the second question, a first suggestion may be the BPS cohomology of the good moduli space $J(d)$ of representation of the Jacobi algebra $\text{Jac}(\tilde{Q}, \tilde{W})$:

$$\mathfrak{g}_{\text{BPS}}[d] := H^\bullet(J(d), \varphi).$$

Here, $\varphi := \varphi_{\text{Tr } \tilde{W}} \text{IC}_{\tilde{X}(d)}$ is a perverse sheaf on $J(d)$, $\tilde{X}(d)$ is the coarse space of dimension d representations of the tripled quiver $\tilde{Q}$, and

$$\text{Tr } \tilde{W}: \tilde{X}(d) \rightarrow \mathbb{A}_{\mathbb{C}}^1$$

is a regular function induced by the potential. One may attempt to find a simpler description of $H^\bullet(J(d), \varphi)$ in terms of singular cohomology or intersection cohomology of an associated variety to Q , but it is not clear if this is possible.

In [4] and [5], we propose using a noncommutative space:

$$\mathbb{T}(d)_w,$$

called a *BPS category*, as answer to Question 2. We briefly recall its construction. Let $\mathcal{X}(d)$ be the stack of dimension d representations of Q , let $\mathcal{P}(d) = T^*\mathcal{X}(d)$ be the stack of dimension d representations of the preprojective algebra of Q , and let $\mathcal{Y}(d)$ be the stack of dimension d representations of the doubled quiver of Q . Consider the closed immersion:

$$i: \mathcal{P}(d) \hookrightarrow \mathcal{Y}(d).$$

Note that there is an orthogonal decomposition of the derived category of coherent sheaves

$$D^b(\mathcal{P}(d)) := \bigoplus_{w \in \mathbb{Z}} D^b(\mathcal{P}(d))_w,$$

where $D^b(\mathcal{P}(d))_w \subset D^b(\mathcal{P}(d))$ is the subcategory of complexes on which the diagonal $\mathbb{G}_m$ acts with weight $w \in \mathbb{Z}$. We consider a category

$$\mathbb{T}(d)'_w \subset D^b(\mathcal{Y}(d))_w$$

of complexes such that, for any $\lambda: B\mathbb{G}_m \rightarrow \mathcal{P}(d)$, its λ -weights are in a specific interval of width determined by the λ -positive weights on the space of dimension d representations of the tripled quiver $\tilde{Q}$. For a precise definition, see [4]. Then

$$\mathbb{T}(d)_w \subset D^b(\mathcal{P}(d))_w$$

is the subcategory of complexes such that $i_*F \in \mathbb{T}(d)'_w$. We also introduce the category $\mathbb{T}(d)^{\text{reg}}_w \subset D^b(\mathcal{P}(d)^{\text{reg}})_w$ for the stack $\mathcal{P}(d)^{\text{reg}}$ obtained from $\mathcal{P}(d)$ by removing a trivial derived direction.

For a dimension vector $d \in \mathbb{N}^I$, denote by $\underline{d}$ the sum of its coordinates. Let $P(d)$ be the good moduli space of the classical truncation of $\mathcal{P}(d)$. The following combines the main results in [4, 5]:

Theorem 1. Let Q be a quiver and let $d \in \mathbb{N}^I$.

(a) Assume that $\gcd(w, \underline{d}) = 1$. Then there is an isomorphism of $\mathbb{Q}$ -vector spaces:

$$K_0^{\text{top}}(\mathbb{T}(d)_w)_{\mathbb{Q}} \xrightarrow{\sim} H^\bullet(J(d), \varphi) =: \mathfrak{g}_{\text{BPS}}[d].$$

(b) Assume that Q has enough arrows, see [4] for precise hypotheses. Then $\mathbb{T}(d)^{\text{reg}}_w$ is a regular dg-category, proper over $P(d)$, with trivial relative Serre functor over $P(d)$.

In particular, one recovers the coefficients of the Kac polynomial as dimension of topological K-theory for certain dg-categories which, under some hypotheses on the arrows of Q , are twisted noncommutative crepant resolutions of singularities of $P(d)$. Notice that, in general, $P(d)$ has no (geometric) crepant resolutions of singularities.

Next, it is an interesting question to construct geometrically the Lie algebra $\mathfrak{g}_{\text{MO}}$, or its positive part $\mathfrak{g}_{\text{BPS}}$, or quantum groups associated to them. Further, it is interesting to categorify these algebras. Recall that Ringel constructed quantum groups associated to halves of Kac-Moody Lie algebras using Hall algebras, which can be categorified (following Lusztig) using categories of perverse sheaves on the moduli stacks of representations $\mathcal{X}(d)$. In this direction, one may consider the following monoidal category (we ignore certain twists appearing in its definition):

$$\text{HA}(Q)_0 := \bigoplus_{d \in \mathbb{N}^I} \mathbb{T}(d)_0,$$

where the monoidal structure is induced by the Hall product. We expect to extend the results of [5] to show that:

$$K_0^{\text{top}}(\text{HA}(Q)_0) \xrightarrow{\sim} \text{Sym}(\mathfrak{g}_{\text{BPS}}).$$

In particular, we expect that equivariant versions of $\text{HA}(Q)_0$ are related to quantum groups associated to BPS Lie algebras. For $\mu \in \mathbb{Q}$, there are also monoidal categories $\text{HA}(Q)_\mu$ which we expect to compare with slope subalgebras of the BPS Lie algebra.

In [6], we provide evidence for considering categories $\mathbb{T}(d)_w$ as coherent versions, or classical limits, of dg-categories of twisted D-modules by a fractional power of the canonical line bundle

$$\text{D-mod}_{\text{coh}}(\mathcal{X}(d))_{w/d}.$$

We conjecture that the (algebraic or topological) K-theory of $\text{D-mod}_{\text{coh}}(\mathcal{X}(d))_{w/d}$ and $\mathbb{T}(d)_w$ are isomorphic under a specialization map. In particular, the coefficients of the Kac polynomial have an interpretation as dimensions of topological K-theory of categories of twisted D-modules on $\mathcal{X}(d)$. In particular, if we consider a slope zero Hall category, namely the monoidal category:

$$\text{D-HA}(Q)_0 := \bigoplus_{d \in \mathbb{N}^I} \text{D-mod}_{\text{coh}}(\mathcal{X}(d)),$$

then we conjecture the following:

Conjecture 1. There is an isomorphism $K_0^{\text{top}}(\text{D-HA}(Q)_0) \xrightarrow{\sim} \text{Sym}(\mathfrak{g}_{\text{BPS}})$.

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Common triangular bases for cluster algebras

FAN QIN

Roughly speaking, a cluster algebra $\mathbf{A}$ is an algebra endowed with clusters of generators. Cluster monomials are monomials of the generators from the same cluster, which can be computed by recursive algorithms. Fomin and Zelevinsky introduced these algebras in [2] with the following ambitious expectations:

- Many coordinate rings arising from Lie theory are cluster algebras.
- Moreover they possess analogs of the dual canonical bases $\mathbf{B}^*$ of quantized enveloping algebras $U_q(\mathfrak{n})$. These bases further contain all the cluster monomials.

By [1], cluster algebras have natural quantizations under mild conditions. We will consider quantum cluster algebras defined over the base ring $\mathbb{Z}[v^\pm]$ or $\mathbb{C}(v)$, where v is a formal quantum parameter.

From now on, assume that a quantum cluster algebra $\mathbf{A}$ satisfies a technical condition, called the existence of a green to red sequence [7]. Equivalently, $\mathbf{A}$ possesses an injective-reachable sequence [8], or its Donaldson-Thomas transformation can be realized by a finite mutation sequence. This assumption is met by almost all known $\mathbf{A}$ arising from representation theory or (higher) Teichmüller theory.

In [8], the common triangular basis $\mathbf{L}$ for a quantum cluster algebra $\mathbf{A}$ was introduced under the above assumption. $\mathbf{L}$ is defined to satisfy strong properties, at the cost of its existence being unknown in general. Nevertheless, there is no known example where $\mathbf{L}$ fails to exist.

$\mathbf{L}$ is a Kazhdan-Lusztig type basis containing all cluster monomials, which can be determined as follows [10]. Choose any cluster $\mathbf{s}$ and embed $\mathbf{A}$ into the associated non-commutative Laurent polynomial ring $\mathcal{LP}(\mathbf{s})$. Following [8], we construct a distinguished linearly independent subset $\mathbf{I}(\mathbf{s})$ of $\mathbf{A}$, whose elements are sorted by an (unbounded) partial order (the dominance order). Then a unique linearly independent subset $\mathbf{L}(\mathbf{s})$ of the completion $\widehat{\mathcal{LP}}(\mathbf{s})$ can be obtained using an algorithm similar to that for computing Kazhdan-Lusztig polynomials.

Definition 1. A basis $\mathbf{L}$ of $\mathbf{A}$ is called a common triangular basis if it equals $\mathbf{L}(\mathbf{s})$ for any $\mathbf{s}$.

It follows from the definition that the common triangular basis $\mathbf{L}$ is unique if it exists.

For any generalized Cartan matrix C and weyl group element w , the integral form of the quantized coordinate ring $\mathbb{Z}[v^\pm][N^w]$ of the unipotent cells N^w is a quantum cluster algebra, see [3][5][4].

Theorem 1 ([8][9]). The dual canonical basis $\mathbf{B}^*$ of a quantized coordinate ring $\mathbb{Z}[v^\pm][N^w]$ is the common triangular basis $\mathbf{L}$.

Therefore, $\mathbf{L}$ provide analogs of the dual canonical bases for quantum cluster algebras. They meet the expectation of Fomin and Zelevinsky by the following result.

Theorem 2 ([11]). For almost all known quantum cluster algebras $\mathbf{A}$ arising from Lie theory, they possess the common triangular bases $\mathbf{L}$.

In addition, when the generalized Cartan matrix C is symmetric, $\mathbf{L}$ has non-negative structure constants. Furthermore, there exists a monoidal category $\mathcal{M}$ such that, possibly after mild change (change the quantization and the *frozen part* or, equivalently, the *q-center*), $\mathbf{A}$ becomes isomorphic to the Grothendieck ring $K_0(\mathcal{M})$, such that $\mathbf{L}$ corresponds to the set of the isoclasses of the simple objects.

In a forthcoming work, for unpunctured surfaces Σ , we will show that the associated quantum cluster algebras $\mathbf{A}(\Sigma)$ possess the common triangular bases $\mathbf{L}$, whose structure constants are non-negative.

Recall that, when C is of finite type, the quantized enveloping algebra $U_q(\mathfrak{n})$ has a PBW basis, which consists of monomials of the root vectors. Moreover, the root vectors could be computed by the braid group action on the quantum group $U_q(\mathfrak{g})$.

In general, cluster algebras do not have PBW type bases, because they are not polynomial rings. However, PBW type bases do exist in special cases. Let $\overline{\text{dBS}}$ denote a double Bott-Samelson variety, which has a partial compactification $\overline{\text{dBS}}$ such that $\overline{\text{dBS}}$ is an affine space, see [13]. The associated quantum cluster algebra $\mathbf{A}(\overline{\text{dBS}})$, understood as the integral form of the quantized coordinate ring of $\overline{\text{dBS}}$ has the following property.

Theorem 3 ([11][12]). The quantum cluster algebra $\mathbf{A}(\overline{\text{dBS}})$ has the standard basis $\mathbf{M}$, which consists of monomials of the distinguished cluster variables called the fundamental variables.

In addition, its common triangular basis $\mathbf{L}$ is the Kazhdan–Lusztig type basis with respect to $\mathbf{M}$ whose elements are sorted by the lexicographical order.

Moreover, when C is of finite type, the fundamental variables can be computed using the braid group action introduced in [6], analogous to how the root vectors of $U_q(\mathfrak{n})$ are computed via a braid group action.

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Local intersection cohomology of varieties of complexes

MARKUS REINEKE

(joint work with Xin Fang)

Let V_* be a finite dimensional graded $\mathbb{C}$ -vector space. We denote by

$$\mathrm{Com}(V_*) = \{V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} V_n \mid f_{i+1} \circ f_i = 0 \text{ for all } i\} \subset \bigoplus_i \mathrm{Hom}(V_i, V_{i+1})$$

the *variety of complexes* on V_* , on which the group $G(V_*) = \prod_i \mathrm{GL}(V_i)$ acts by change of basis with orbits

$$\mathcal{O}(\mathbf{r}, \mathbf{h}) = \{f_* \mid \mathrm{rank}(f_i) = r_i, \dim H^i(V_*, f_*) = h_i \text{ for all } i\},$$

thus $\dim V_i = h_i + r_{i+1} + r_i$ for all i .

The irreducible components of $\mathrm{Com}(V_*)$ are the orbit closures $\overline{\mathcal{O}(\mathbf{r}, \mathbf{h})}$ for which the *support* $\Omega = \Omega(\mathbf{r}, \mathbf{h}) = \{i \mid h_i \neq 0\}$ is *sparse*, that is, $i \notin \Omega$ or $i + 1 \notin \Omega$ for all i .

These irreducible affine varieties are known to be normal, Cohen-Macaulay, with rational singularities, Frobenius-split, and admitting open embeddings into $\mathrm{SL}(\bigoplus_i V_i)$ -Schubert varieties. Our main result is of quantitative nature, describing the Poincaré polynomials of stalks of intersection cohomology complexes on the irreducible components in terms of shifted sums of products of q -binomial coefficients:

Theorem 1. If $f_* \in \overline{\mathcal{O}(\mathbf{r}, \mathbf{h})}$ is a point in an irreducible component of $\mathrm{Com}(V_*)$ with support Ω , given by ranks $\mathrm{rank}(f_i) = r_i - k_i$, then $\sum_i \dim \mathcal{H}_{f_*}^{2i}(\mathbf{IC}(\overline{\mathcal{O}(\mathbf{r}, \mathbf{h})}))q^i$

$$= \sum_{(0 \leq t_i \leq k_{i-1}, k_i)_{i \in \Omega}} q^{\sum_{i \in \Omega} (h_i + t_i)t_i} \prod_{i \in \Omega} \begin{bmatrix} k_i \\ k_i - t_i \end{bmatrix} \begin{bmatrix} k_{i-1} \\ t_i \end{bmatrix} \prod_{i \notin \Omega} \begin{bmatrix} k_{i-1} + k_i \\ k_i \end{bmatrix}.$$

To derive this, we turn to quantized enveloping algebras and canonical bases and their geometric realization [2]. Let $\mathcal{U}^+ = \mathcal{U}_v^+(\mathfrak{sl}_{n+1})$ be the quantized enveloping algebra of the Lie algebra $\mathfrak{sl}_{n+1}$, which is a $\mathbb{Q}(v)$ -algebra with generators E_i for $i = 1, \dots, n$ and relations $E_i E_j = E_j E_i$ if $|i - j| \geq 2$, as well as

$$E_i^2 E_j - (v + v^{-1})E_i E_j E_i + E_j E_i^2 = 0 \text{ if } |i - j| = 1.$$

It carries a canonical $\mathbb{Q}$ -algebra involution given by $\overline{E_i} = E_i$ for all i and $\overline{v} = v^{-1}$.

The algebra $\mathcal{U}^+$ admits a PBW type linear basis $B_{\mathbf{w}}$ associated to the reduced decomposition $\mathbf{w} = (n, n - 1, n, \dots, 1, \dots, n)$ of the longest Weyl group element. Using this, we can define the canonical basis: it is the unique linear basis $\mathcal{B}$ of $\mathcal{U}^+$

such that $\bar{b} = b$ for all $b \in \mathcal{B}$, such that $\mathbb{Z}[v^{-1}]\mathcal{B} = \mathbb{Z}[v^{-1}]B_{\mathbf{w}} =: \mathcal{L}$, and such that $\mathcal{B} = B_{\mathbf{w}} \bmod v^{-1}\mathcal{L}$.

This basis can be interpreted geometrically: the elements of $\mathcal{B}$ (denoted $\mathcal{E}_{\mathcal{O}}$) and the elements of $B_{\mathbf{w}}$ (denoted $E_{\mathcal{O}}$) are in bijection with the orbits $\mathcal{O}$ of $G(V_*)$ on $\bigoplus_i \operatorname{Hom}(V_i, V_{i+1})$ for various choices of $V_1, \dots, V_n$. The base change coefficients between these bases are then completely described in terms of local intersection cohomology of orbit closures:

$$[\mathcal{E}_{\mathcal{O}} : E_{\mathcal{O}'}] = \begin{cases} 0 & , \quad \mathcal{O}' \not\subset \overline{\mathcal{O}}, \\ v^{\cdots} \sum_i \dim \mathcal{H}_{f_*}^i(\mathbf{IC}(\overline{\mathcal{O}})) v^i & , \quad f \in \mathcal{O}' \subset \overline{\mathcal{O}} \end{cases}$$

Note that the canonical basis is only known completely for $n = 1, 2, 3$; in general there exist criteria for a monomial in divided powers $E_i^{(a)}$ to belong to $\mathcal{B}$.

We construct the elements $\mathcal{E}_{\mathcal{O}}$ explicitly for $\overline{\mathcal{O}}$ an irreducible component of $\operatorname{Com}(V_*)$, in which case (using $E_{i,i+1} = E_i E_{i+1} - v^{-1} E_{i+1} E_i$)

$$E_{\mathcal{O}(\mathbf{r}, \mathbf{h})} = E_n^{(h_n)} E_{n-1,n}^{(r_{n-1})} E_{n-1}^{(h_{n-1})} \cdots E_{1,2}^{(r_1)} E_1^{(h_1)}.$$

To this end, we define (see [1] for the details) certain monomials $V_{i,j}$ in $E_i, \dots, E_j$, and certain non-monomial elements C_i in E_{i-1}, E_i, E_{i+1} . If the support of an irreducible component is $\Omega = \{i_1 < i_2 < \dots < i_s\}$, we prove that

$$V_{i_s+1,n} \cdot C_{i_s} \cdot V_{i_{s-1}+1,i_s-1} \cdot C_{i_{s-1}} \cdot \dots \cdot C_{i_2} \cdot V_{i_1+1,i_2-1} \cdot C_{i_1} \cdot V_{1,i_1-1}$$

belongs to $\mathcal{B}$ by a direct calculation.

We conjecture that our method can be generalized to provide explicit formulas for all $\mathcal{E}_{\mathcal{O}(\mathbf{r}, \mathbf{h})}$ in terms of certain Catalan combinatorics.

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Tensor triangulated geometry for representations of supergroups

VERA SERGANOVA

(joint work with Julia Pevtsova, Alex Sherman)

Let G be a complex quasireductive supergroup, meaning that G_0 has connected component of the identity a reductive group. Then the category $\operatorname{Rep} G$ of all G -modules (possibly infinite dimensional) has enough projectives. We say a supergroup E is *elementary quasitoral* if E_0 is a central, toral subgroup and E is generated by its odd part. Such supergroups can be thought of as analogues of elementary abelian p -groups in modular representation theory of finite groups. It is a famous theorem of Chouinard that if G is a finite group, then a G -module M over a field k of characteristic p is projective if and only if its restriction to every elementary abelian p -subgroup of G is projective. For supergroups over the complex numbers we have:

Theorem 1. A G -module M (either finite or infinite dimensional) is projective if and only if its restriction to every elementary quasitoral subgroup of G is projective.

In the modular setting one has a notion of a rank variety which in particular allows to check projectivity via restrictions to subalgebras of the form $k[x]/x^p$ going back to the work of Dade and Carlson. We have a similar tool in the super setting. Let $\mathfrak{g}$ be the Lie superalgebra of G . We say that $x \in \mathfrak{g}_1$ is *homological* if $x^2 := \frac{1}{2}[x, x]$ is semisimple in $\mathfrak{g}_0$ (and thus acts semisimply on any G -module). Write $\mathfrak{g}^{hom}$ for the set of homological elements in $\mathfrak{g}_1$. Then for $x \in \mathfrak{g}^{hom}$ we have a symmetric monoidal functor:

$$DS_x : \text{Rep } G \rightarrow \text{SVec}$$

given by restriction to the subgroup generated by x followed by semisimplification. More explicitly:

$$DS_x M = \frac{\ker(x : M^{x^2} \rightarrow M^{x^2})}{\text{im}(x : M^{x^2} \rightarrow M^{x^2})},$$

where M^{x^2} are the invariants of M under x^2 .

Theorem 2. Let M be a G -module, and for a field extension k of $\mathbb{C}$ write $M_k = M \otimes_{\mathbb{C}} k$ for the G_k -module obtained by extension of scalars. The following are equivalent:

- (1) M is projective;
- (2) $DS_x M_k = 0$ for all field extensions $\mathbb{C} \subset k$ and all $x \in (\mathfrak{g} \otimes k)^{hom}$;

Note that Theorem 1 implies Theorem 2 once one has shown Theorem 2 for elementary quasitoral supergroups. The latter fact follows from the representation theory of a finite-dimensional Grassmann algebra.

The underpinnings of the approach to Theorems 1 and 2 come from [4], where Sylow subgroups and analogues of p -groups (which we call *cohesive* supergroups) were defined.

The cohomological support variety of a finite-dimensional G -module was first studied in [2] motivated by the classical theory developed for finite groups and extended to many other contexts. A well-known deficiency of the cohomological construction is that it is not known to be amenable to the tensor product property. With the help of Theorem 1 and the DS_x functor we are able to resolve this problem.

We recall that the cohomological support variety of G is given by the GIT quotient $\mathfrak{g}_1//G_0$, since $\text{Spec } H^*(\mathfrak{g}, \mathfrak{g}_1, \mathbb{C}) \cong \text{Spec } \mathbb{C}[\mathfrak{g}_1]^{G_0}$ as shown in [2]. Invariant theory tells us that the closed points of the GIT quotient are in natural bijection with closed orbits of G_0 on $\mathfrak{g}_1$, and we prove that every closed orbit lies in $\mathfrak{g}^{hom}$. Thus we will view a closed point of $\mathfrak{g}_1//G_0$ as an element of $\mathfrak{g}^{hom}$ (up to G_0 -conjugacy).

In what follows, write $\mathcal{V}_G^{coh}(M) \subset \mathfrak{g}_1//G_0$ for the cohomological support variety of a finite-dimensional G -module M .

Theorem 3. For a finite-dimensional G -module M , we have

$$\mathcal{V}_G^{coh}(M) = \{x \in \mathfrak{g}_1 // G_0 \mid DS_x M \neq 0\}.$$

In particular, since DS_x is monoidal, we have

$$\mathcal{V}_G^{coh}(M_1 \otimes M_2) = \mathcal{V}_G^{coh}(M_1) \cap \mathcal{V}_G^{coh}(M_2).$$

In [1] Balmer introduced the spectrum of an essentially small tensor triangulated category rendering, in particular, an elegant way to classify the lattice of thick tensor ideals. As the category of G -modules is highly non-semisimple, the Balmer spectrum of its stable category (of finite dimensional) modules provides a geometric invariant to capture the global structure of $\text{Rep } G$ as a tensor category. In some classical representation theoretic cases, such as representations of finite groups or finite group schemes, the Balmer spectrum has been shown to be essentially the spectrum of the cohomology ring with the correspondence given by the cohomological variety.

It is known that for representations of supergroups the cohomological support variety does not give the Balmer spectrum unless G is, for instance, elementary quasitoral. The reason for this is the existence of non-closed homological orbits of G_0 acting on $\mathfrak{g}_1$. All elements of $\mathfrak{g}^{hom}$ are needed to detect projectivity of a module, and many are lost in the GIT quotient $\mathfrak{g}_1 // G_0$.

A natural solution is to replace the GIT quotient by a stacky quotient. Hence, we consider the set-theoretic quotient:

$$\mathbb{Y}_G = (\mathfrak{g}^{hom} \setminus \{0\}) / (G_0 \times \mathbb{G}_m),$$

where $\mathbb{G}_m$ acts by scaling. Explicitly, the points of $(\mathfrak{g}^{hom} \setminus \{0\}) / (G_0 \times \mathbb{G}_m)$ are nonzero homological elements up to conjugacy by G_0 and scaling.

For any G -module M , we define its *rank support* as

$$\text{supp}_G^{rk}(M) := (\{u \in \mathfrak{g}^{hom} : DS_u M \neq 0\} \setminus \{0\}) / (G_0 \times \mathbb{G}_m).$$

Conjecture. The Balmer spectrum of the stable category of finite dimensional G -modules is homeomorphic to $\mathbb{Y}_G$.

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Monoidal categorification of genus zero Skein algebras

PENG SHAN

(joint work with Dylan G.L. Allegretti and Hyun Kyu Kim)

1. CHARACTER VARIETIES AND SKEIN ALGEBRAS

Let S be an oriented smooth surface of genus g with n boundary components such that $2 - 2g - n < 0$. Let $\pi_1(S)$ be its fundamental group. The SL_2 -character variety $\mathcal{M}_{\mathrm{flat}}(S, \mathrm{SL}_2)$ of S is the affine GIT quotient of the representation space $\mathrm{Hom}(\pi_1(S), \mathrm{SL}_2)$ by the conjugation action of SL_2 on the target. Each homotopy class of a loop $\gamma \in S$ yields a regular function on $\mathcal{M}_{\mathrm{flat}}(S, \mathrm{SL}_2)$ whose value on the conjugacy class of a representation $\rho : \pi_1(S) \rightarrow \mathrm{SL}_2$ is the trace $\mathrm{Tr}(\rho(\gamma))$.

Fix loops $\gamma_1, \dots, \gamma_n$ freely homotopic to the boundary components. Given an n -tuple $\lambda = (\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^*)^n$, we may also consider the relative character variety $\mathcal{M}_{\mathrm{flat}}^\lambda(S, \mathrm{SL}_2)$. It is the closed subscheme in $\mathcal{M}_{\mathrm{flat}}(S, \mathrm{SL}_2)$ cut out by the equations $\mathrm{Tr}_{\gamma_i} = \lambda_i + \lambda_i^{-1}$.

The Kauffman bracket skein algebra of S is a noncommutative deformation of the algebra of regular functions on $\mathcal{M}_{\mathrm{flat}}(S, \mathrm{SL}_2)$. Let $\mathcal{L}_A(S)$ be the $\mathbb{C}[A^{\pm 1}]$ -module generated by isotopy classes of framed links in $S \times [0, 1]$, equipped with the algebra structure given by superposition. The skein algebra $\mathrm{Sk}_A(S)$ is defined as the quotient of $\mathcal{L}_A(S)$ by the skein relations. There is also a relative version $\mathrm{Sk}_{A, \lambda}(S)$ given by the quotient of $\mathrm{Sk}_A(S) \otimes_{\mathbb{C}[A^{\pm 1}]} \mathbb{C}[A^{\pm 1}, \lambda_1^{\pm 1}, \dots, \lambda_n^{\pm 1}]$ by evaluating the classes of γ_i at $-(\lambda_i + \lambda_i^{-1})$ for $1 \leq i \leq n$. It quantizes the algebra of regular functions on $\mathcal{M}_{\mathrm{flat}}^\lambda(S, \mathrm{SL}_2)$.

2. K-THEORETIC COULOMB BRANCHES

Braverman–Finkelberg–Nakajima [3] proposed a mathematical definition for the Coulomb branches of a 3-dimensional $\mathcal{N} = 4$ supersymmetric gauge theory. We will use a K -theoretic version of their construction. More precisely, let $\mathcal{K} = \mathbb{C}((z))$ and $\mathcal{O} = \mathbb{C}[[z]]$. Given a complex reductive group G and a G -representation N , by considering the loop group $G_{\mathcal{K}}$ acting on $N_{\mathcal{K}}$, and $G_{\mathcal{O}}$ acting on $N_{\mathcal{O}}$, one can form a (infinite) vector bundle $\mathcal{T}_{G, N} = G_{\mathcal{K}} \otimes_{G_{\mathcal{O}}} N_{\mathcal{O}}$ over the affine Grassmannian $G_{\mathcal{K}}/G_{\mathcal{O}}$. The variety of triples is the closed ind-subscheme in $\mathcal{T}_{G, N}$ given by

$$\mathcal{R}_{G, N} = \{[g, n] \in \mathcal{T}_{G, N} : gn \in N_{\mathcal{O}}\}.$$

It carries an action of $G(\mathcal{O})$ via left multiplication on the first factor. It also have a $\mathbb{C}^\times$ loop rotation action given by $t[g(z), n(z)] = [g(tz), t^{\frac{1}{2}}n(tz)]$ for $t \in \mathbb{C}^\times$. If in addition N carries an action by a torus F commuting with G , there is also an F -action on $\mathcal{R}_{G, N}$ by acting on the second factor.

The equivariant K -theory $K^{G(\mathcal{O})}(\mathcal{R}_{G, N})$ admits an algebra structure by convolution product. It is commutative, whose spectrum is by definition the BFN

Coulomb branch. It admits a natural quantization given by turning on the equivariance with respect to the loop rotation $K^{G(\mathcal{O}) \rtimes \mathbb{C}^\times}(R_{G,N})$. Finally, adding equivariance with respect to the flavor group F yields a deformation of this algebra

$$\mathcal{A} = K^{G(\mathcal{O}) \rtimes \mathbb{C}^\times \times F}(R_{G,N}).$$

3. MAIN RESULTS

Consider the genus zero surface $S = S_{0,n+2}$ with $n+2$ boundary components. Inspired by physical ideas of Gaiotto, Moore and Neitzke, in [1], Allegretti and I described a group $\tilde{G} = G \times F$ and representation N of $\tilde{G}$ associated to such a surface S , and conjectured an isomorphism between the relative skein algebra attached to S and the K-theoretic Coulomb branch $\mathcal{A}$ associated with $\tilde{G}$ and N . The main result of this talk is a proof of this conjecture [2]. The precise statement is the following.

Let $G = G_1 \times \cdots \times G_{n-1}$ where $G_i = \mathrm{SL}_2$ for $i = 1, \dots, n-1$. Set $N_j = \mathbb{C}^2$ for $j = 0, \dots, n$. Then there is an action of G_i on $N_j \otimes N_{j+1}$ where an element of G_i acts by matrix multiplication on a tensor factor whose index is equal to i and trivially on a tensor factor whose index is not equal to i . By taking the direct sum, we get an action of G_i on the vector space

$$N = \bigoplus_{j=0}^{n-1} N_j \otimes N_{j+1}.$$

Thus we get an action of G on this vector space. Similarly, for $i = 1, \dots, n$, we let F_i act on $N_j \otimes N_{j+1}$ by scalar multiplication if $j = i-1$ and trivially if $j \neq i-1$. We let $F_0 \cong \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{C}^* \right\}$ act on $N_j \otimes N_{j+1}$ by matrix multiplication on the first tensor factor if $j = 0$ and trivially if $j \neq 0$, and we let $F_{n+1} \cong \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{C}^* \right\}$ act on $N_j \otimes N_{j+1}$ by matrix multiplication on the second tensor factor if $j = n-1$ and trivially if $j \neq n-1$. By taking the direct sum, we get an action of F_i on N and hence an action of F on N .

Theorem 1 ([2]). Let $\tilde{G} = G \times F$ and N be as above, and identify

$$\mathbf{k} := \mathbb{C}[A^{\pm 1}, \lambda_i^{\pm 1}] \cong K^{\mathbb{C}^\times \times F}(\mathrm{pt}).$$

Then there is a $\mathbf{k}$ -algebra isomorphism

$$\mathrm{Sk}_{A,\lambda}(S) \cong K^{G \rtimes \mathbb{C}^\times \times F}(\mathcal{R}_{G,N})$$

from the skein algebra to the quantized Coulomb branch.

Recall that by definition $K^{G \rtimes \mathbb{C}^\times \times F}(\mathcal{R}_{G,N})$ is the Grothendieck group of the equivariant derived category $D^b \mathrm{Coh}^{G \rtimes \mathbb{C}^\times \times F}(\mathcal{R}_{G,N})$, which admits a monoidal structure given by convolution. Cautis and Williams [4] constructed a t-structure on this equivariant derived category. The heart of this t-structure is an abelian monoidal category called the category of Koszul-perverse coherent sheaves. Our results imply that this abelian monoidal category provides a monoidal categorification of the skein algebra $\mathrm{Sk}_{A,\lambda}(S)$. Moreover, the classes of simple objects in this

category yields a positive basis of $\mathrm{Sk}_{A,\lambda}(S)$. This confirms partially a conjecture by Thurston [5]. We plan to explore the relation between this new bases coming from sheaf theory and those bases defined in topological terms by Thurston in the future.

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Representations of shifted affine quantum groups and Coulomb branches

ERIC VASSEROT

(joint work with Michela Varagnolo)

Given an arbitrary Cartan matrix $\mathbf{c}$, a mathematical definition of the Coulomb branch of a 3D, $\mathcal{N}=4$ quiver gauge theory associated with two I -graded vector spaces V and W was given by Nakajima and Weekes in [6]. It was proved in [6] that the quantization of the Coulomb branch is a truncated shifted Yangian, and the fixed point set of some $\mathbb{C}^\times$ -action on the space of triples (i.e., the BFN space) associated with the Coulomb branch was computed. Some consequences for the module category of the truncated shifted Yangian were also discussed. A related construction in physics in the context of 4D, $\mathcal{N}=2$ quiver gauge theory was considered by Kimura and Pestun in [4].

In this paper we consider the Coulomb branch with symmetrizers of the 3D, $\mathcal{N}=4$ quiver gauge theory associated with the Cartan matrix $\mathbf{c}$. We relate it to a truncated shifted quantum loop group of type $\mathbf{c}$, generalizing the work of Finkelberg-Tsybaliuk [2] in the symmetric case. The BFN space is infinite dimensional. We use the formalism of Cautis-Williams [1] to represent it by an ind-geometric derived ∞ -stack. We then prove a version of the Segal-Thomason localization theorem which relates the Coulomb branch to the K-theory and Borel-Moore homology of the fixed point subset of the $\mathbb{C}^\times$ -action on the BFN space.

This result yields an equivalence from the integral category $\mathcal{O}$ of the truncated shifted quantum loop group to the category of nilpotent modules of a new version of quiver Hecke algebras, which we call an integral $\mathbb{Z}$ -quiver Hecke algebra. This quiver Hecke algebra is attached to the symmetric Cartan matrix $\underline{\mathbf{c}}$ obtained by unfolding $\mathbf{c}$, and depends also on a grading given by $\mathbf{c}$ (and not by $\underline{\mathbf{c}}$ only). While

the presence of this unfolding was already observed in [6], the role of the integral $\mathbb{Z}$ -quiver Hecke algebra is new and important. For symmetric $\mathbf{c}$ the integral $\mathbb{Z}$ -quiver Hecke algebra coincides with the parity quiver Hecke algebra of type $\mathbf{c}$ considered in [3]. For non symmetric $\mathbf{c}$, the definition of the integral $\mathbb{Z}$ -quiver Hecke algebra differs from the definition of the parity quiver Hecke algebra of $\underline{\mathbf{C}}$.

This new quiver Hecke algebra allows us to decategorify the integral category $\mathcal{O}$ of the truncated shifted quantum loop group in term of a finite dimensional module over the simple Lie algebra whose Cartan matrix is $\underline{\mathbf{C}}$. This finite-dimensional module is not generally known. We provide a few conditions it satisfies and compute it in type B_2 . We also give a (partly conjectural) combinatorial rule to compute this representation. This rule uses a crystal which generalizes Nakajima's monomial crystal.

Notably, the integral $\mathbb{Z}$ -quiver Hecke algebra admits a cohomological grading, as it is a convolution algebra in Borel-Moore homology. Consequently, our equivalence of categories yields a grading on the integral category $\mathcal{O}$. Furthermore, although we focus on finite types, many of our results extend naturally to the case of symmetrizable generalized Cartan matrices. We will return to this elsewhere.

Another motivation for this work comes from [8], where we provide a geometrization of (shifted) quantum loop groups of arbitrary types via the critical K-theory of quiver varieties, generalizing Nakajima's work on symmetric types in [5]. Quiver varieties are the 3D mirror duals of Coulomb branches. We aim to better understand the relationships between these two constructions.

First, we describe quiver Hecke algebras modeled over spaces of $\mathbb{Z}$ -flags, i.e., spaces of sequence of finite-dimensional vector spaces labeled by $\mathbb{Z}$. We compare them with the tensor product algebras introduced by Webster. Next, we fix a non-symmetric Cartan matrix $\mathbf{c}$ whose Dynkin diagram a folded Dynkin diagram of a symmetric Cartan matrix $\underline{\mathbf{C}}$. We introduce the integral $\mathbb{Z}$ -quiver Hecke algebras of type $\mathbf{c}$, which are modeled on $\mathbb{Z}$ -quiver Hecke algebras of type $\underline{\mathbf{C}}$, with an additional integrality condition that generalizes the parity quiver Hecke algebras from [3] in the symmetric case. We then prove that the module categories of integral $\mathbb{Z}$ -quiver Hecke algebras are quotients of the module categories of tensor product algebras of type $\underline{\mathbf{C}}$. Let $\mathfrak{g}$ be complex simple Lie algebra of $\mathbf{c}$, and $\underline{\mathfrak{g}}$ the complex simple Lie algebra of $\underline{\mathbf{C}}$. Subsequently, we decategorify the integral $\mathbb{Z}$ -quiver Hecke algebras ${}^0\mathcal{T}_{\mu}^{\rho}$ by weight subspaces in some $\underline{\mathfrak{g}}$ -modules. While we do not explicitly compute these modules, we discuss their connections to [3] in the symmetric case and we compute them in certain specific scenarios, such as the generic case.

Next, we introduce the BFN space with symmetrizers $\mathcal{R}$, following [6]. To facilitate the application of K-theory later, we employ a variation of the formalism from [1], which uses ind-tamely presented ∞ -stacks of ind-geometric type. We then describe the fixed point locus of certain automorphisms. Finally, we introduce the Coulomb branches of 4D, $N=2$ quiver gauge theories with symmetrizers $\mathcal{A}_{\mu,R}^{\lambda}$.

Finally, we introduce shifted quantum groups $\mathbf{U}_{\mu,R}$ and their integral category $\mathcal{O}$. Next, we introduce truncated shifted quantum groups and their module category ${}^0\mathcal{O}^{\rho} = \bigoplus_{\mu} {}^0\mathcal{O}_{\mu}^{\rho}$, along with the surjective algebra homomorphism

$\Phi : \mathbf{U}_{\mu,R} \otimes R_{T_W} \rightarrow \mathcal{A}_{\mu,R}^\lambda$ that maps to Coulomb branches with symmetrizers. We then prove a localization theorem for Coulomb branches, employing techniques similar to those in [7], which identify the Coulomb branch $\mathcal{A}_\mu^\lambda$ with the integral $\mathbb{Z}$ -quiver Hecke algebras ${}^0\widetilde{\mathcal{T}}_\mu^\rho$ after suitable completions. Leveraging the localization theorem, we establish a connection between the truncated shifted category $\mathcal{O}$ and integral $\mathbb{Z}$ -quiver Hecke algebras, and we discuss a few implications at the decategorified level. The main result is the following.

Theorem 1.

- (1) ${}^0\mathcal{O}_\mu^\rho$ is equivalent to a category of nilpotent modules over the integral $\mathbb{Z}$ -quiver Hecke algebra ${}^0\mathcal{T}_\mu^\rho$.
- (2) There is a representation of $\underline{\mathfrak{g}}$ in $K({}^0\mathcal{O}^\rho)$ and an embedding of $K({}^0\mathcal{O}^\rho)$ into a tensor product of fundamental modules of $\underline{\mathfrak{g}}$ which takes the simple modules into the dual canonical basis.

The representation of $\underline{\mathfrak{g}}$ in the Grothendieck group $K({}^0\mathcal{O}^\rho)$ is not known in general. We define a crystal of type $\underline{\mathcal{C}}$ which is, conjecturally, isomorphic to the crystal of the $\underline{\mathfrak{g}}$ -module $K({}^0\mathcal{O}^\rho)$. This yields a combinatorial rule to compute the ℓ -highest weight of all simple modules in ${}^0\mathcal{O}^\rho$ which holds true in type B_2 . We will come back to this elsewhere.

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Dimer models and tropical Lagrangian coamoebae

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(joint work with Christopher Kuo)

We introduce and study a class of combinatorial objects, which we call tropical Lagrangian coamoebae, associated to free resolutions of modules over Laurent polynomial rings. Explicitly, the tropical Lagrangian coamoeba $T(F^\bullet)$ associated to a free resolution $F^\bullet$ in n variables is a simplicial complex in the real n -torus which encodes the discrete information in $F^\bullet$. We show that $T(F^\bullet)$ is the support of a certain constructible sheaf $C(F^\bullet)$ which is mirror to M . We expect that $C(F^\bullet)$ is a minimal representative of such mirrors in the appropriate sense, and conceptually its construction tries to codify how one builds such a minimal mirror.

One of our main sources of interest in this construction is the theory of dimer models. Dimer models have been a continuing source of interest in combinatorics, geometry, and mathematical physics for several decades. Recall that dimer models deal with perfect matchings on bipartite graphs. For an edge-weighted graph $\Gamma \subset T^2$ (and the associated periodic graph in $\mathbb{R}^2$), the dimer model was shown by Kenyon-Okounkov-Sheffield (following classical work of Kasteleyn and Temperley-Fisher) to be solvable in terms of a matrix-valued Laurent polynomial, the Kasteleyn operator $K(z, w)$.

The spectral curve $\det K(z, w) = 0$ in particular plays a key role in this theory. This curve is the support of the spectral data of $K(z, w)$ — the coherent sheaf corresponding to its cokernel — from which the edge weights on Γ can be recovered up to gauge equivalence. The resulting interplay between graphical combinatorics and spectral data turns out to be of interest from a wide range of perspectives, including real algebraic geometry (Kenyon-Okounkov), string theory (Feng-He-Kennaway-Vafa), and integrable systems and cluster algebras (Goncharov-Kenyon, Fock-Marshakov).

In prior work with Treumann-Zaslow we have shown that the passage to spectral data is a mirror map: an edge-weighted graph $\Gamma \subset T^2$ determines a Lagrangian brane in T^*T^2 whose coamoeba is approximated by Γ , and whose mirror coherent sheaf is the spectral data of $K(z, w)$. But from the perspective of this coherent sheaf, the matrix $K(z, w)$ is the data of a free resolution, and the graph Γ is the associated tropical Lagrangian coamoeba. One summary of the present work is thus that the constructions just recalled admit a robust generalization, where $K(z, w)$ is replaced by any free resolution of any coherent sheaf on an algebraic torus of any dimension, and where the graph Γ is replaced by a more general simplicial complex.

Our constructions also provide a symplectic counterpart to the tropical coamoebae or brane brick models of Futaki-Ueda and Franco-Lee-Seong-Vafa. These generalize dimer models in T^2 in a different direction, extending two key observations of Feng-He-Kennaway-Vafa. First, the coamoeba of a sufficiently nice curve $\Sigma \subset (\mathbb{C}^\times)^2$ retracts onto a bipartite graph Γ of which Σ is a spectral curve. And second, the adjacencies of the faces of Γ determine the intersections of vanishing

cycles of the equation defining Σ . The works above argue more generally that the coamoeba of a sufficiently nice hypersurface in $(\mathbb{C}^\times)^n$ retracts onto a codimension-one polyhedral complex (called a tropical coamoeba or, when $n = 3$, a brane brick model), and that this complex divides T^n into chambers whose adjacencies continue to encode intersections of vanishing cycles.

We thus have two different generalizations of dimer models in T^2 , the one just recalled and the one we provide. But this is to be expected: a curve in $(\mathbb{C}^\times)^2$ is both a complex hypersurface and — by applying a hyperkähler rotation — a Lagrangian submanifold. In particular, bipartite graphs in T^2 are degenerations of both hypersurface coamoebae and Lagrangian coamoebae. In higher dimensions, however, these classes of coamoebae no longer overlap. Studying their combinatorial approximations thus leads to two generalizations of bipartite graphs in T^2 , each extending different features of the dimer model. The one studied in by Futaki-Ueda and successors extends the way adjacencies of faces of graphs in T^2 encode intersections of vanishing cycles. The one studied here instead extends the way edge weightings on graphs in T^2 parametrize spectral data.

Metaplectic affine Hecke categories and Soergel bimodules

ZHIWEI YUN

(joint work with Gurbir Dhillon, Yau-Wing Li, Xinwen Zhu)

Let k be an algebraically closed field, and write $K := k((t))$. Let G be a connected, reductive, and quasi-split group over K . The set of K -points of G are naturally the k -points of a group ind-scheme over k , the loop group LG . We write $I \subset LG$ for an Iwahori subgroup. We consider a Kac–Moody central extension $1 \rightarrow \mathbb{G}_m \rightarrow \widetilde{LG} \rightarrow LG \rightarrow 1$. Write $\widetilde{I} \subset \widetilde{LG}$ for the preimage of I . Let I^+ be the pro-unipotent radical of $\widetilde{I}$, and let $\widetilde{A} = \widetilde{I}/I^+$ be the quotient torus.

The sheaves we consider have field coefficients E , which can be of arbitrary characteristic not equal to $\text{char}(k)$. Let χ be a character sheaf (rank one tamely ramified E -local system) on $\widetilde{A}$. Our main object of interest is the monoidal category of bi- $(\widetilde{I}, \chi)$ -monodromic sheaves on $\widetilde{LG}$, which we denote by ${}_\chi \mathcal{M}_\chi$. Concretely, ${}_\chi \mathcal{M}_\chi$ is the full subcategory of sheaves on the stack

$$Z = I^+ \backslash \widetilde{LG} / I^+$$

generated under colimits by $(\widetilde{A}, \chi)$ -equivariant objects under both the left and right translations by $\widetilde{A}$. We call ${}_\chi \mathcal{M}_\chi$ a *monodromic affine Hecke category*.

Our main result says that the monodromic affine Hecke categories are determined by Weyl group combinatorics, following the paradigm of Soergel bimodules. More precisely, ${}_\chi \mathcal{M}_\chi$ can be reconstructed from the following quadruple:

$$(\widetilde{W}_\chi, \widetilde{W}_\chi^\circ, S_\chi, \mathfrak{f}_\chi).$$

Here $\widetilde{W}_\chi$ is the stabilizer of χ under the action of the extended affine Weyl group $\widetilde{W}$ of G ; $(\widetilde{W}_\chi^\circ, S_\chi)$ is a Coxeter subgroup of $\widetilde{W}_\chi$ generated by reflections given by

“ χ -relevant affine coroots”; $\mathfrak{f}_\chi$ is the formal completion of at χ of the moduli space of rank one local systems on $\tilde{A}$, together with its natural action of $\widetilde{W}_\chi$.

Some highlights of our results: (1) we allow nontrivial monodromy along the Kac-Moody center (metaplectic cases); (2) we allow modular coefficients; (3) we give a combinatorial description of all blocks.

Here are a couple of applications of our main result.

Given a rank one local system χ_c on the Kac-Moody center $\mathbb{G}_m$ of $\widetilde{LG}$, we have the notion of a *metaplectic Langlands dual group* $H^\vee$ over the coefficient field E . It is a split reductive group whose maximal torus is isogenous to that of G , and whose Weyl group is isomorphic to that of G . For example, when $G = \mathrm{Sp}_{2n}$ and χ_c has order two, then $H^\vee \cong \mathrm{Sp}_{2n}$.

The following result generalizes (and uses) the derived Satake equivalence of Bezrukavnikov–Finkelberg.

Theorem 1 (metaplectic derived Satake equivalence). Suppose the coefficient field E has characteristic zero. There is an equivalence of monoidal categories

$$D_{\chi_c}(L^+G \backslash LG / L^+G) \simeq \mathrm{IndCoh}_{\mathrm{nilp}}((\mathrm{pt}/H^\vee) \times_{\mathfrak{h}^\vee/H^\vee} (\mathrm{pt}/H^\vee)).$$

Next, let $\mathrm{char}(k) = 0$ and G is split and almost simple. Let κ and $\check{\kappa}$ be dual levels for G and $\check{G}$. We have:

Theorem 2 (Conjectured by Gaitsgory). There are t -exact monoidal equivalences

$$\begin{aligned} D\text{-mod}_\kappa(I \backslash LG / I) &\simeq D\text{-mod}_{-\check{\kappa}}(\check{I} \backslash \check{L}\check{G} / \check{I}); \\ D\text{-mod}_\kappa(L^+G \backslash LG / L^+G) &\simeq D\text{-mod}_{-\check{\kappa}}(L^+\check{G} \backslash \check{L}\check{G} / L^+\check{G}). \end{aligned}$$

Here, $D\text{-mod}(\cdots)$ means the derived category of D -modules.

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