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Recent Trends in Algebraic Geometry

Organized by
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Abstract. Algebraic geometry has grown into a broad subject, with many different streams often advancing quite independently of each other. Nonetheless, important advances have often come from visionary applications of ideas in one part of the subject to another. This workshop brought together leaders and future leaders in different areas of the subject, centered on geometric methods or geometric problems. It also brought together groups from different regions of the globe, in order to bridge communities of different sorts, and help new ideas quickly spread throughout algebraic geometry. Some of the best freshly-minted algebraic geometers were deliberately invited, so that they could meet their peers from around the world and learn about different perspectives on the subject.

Mathematics Subject Classification (2020): 14-XX.

Introduction by the Organizers

The workshop *Recent Trends in Algebraic Geometry* was organised by Olivier Debarre (Paris), Gavril Farkas (Berlin), Rahul Pandharipande (Zurich), and Claire Voisin (Paris). There were 18 one-hour talks with a maximum of four talks a day, and an evening session of short presentations allowing young participants to introduce their current work (and themselves). The schedule deliberately left plenty of room for informal discussion and work in smaller groups. The extended abstracts give a detailed account of the broad variety of topics of the meeting. We focus on a representative sample here:

Samir Canning (with Larson, Payne, and Willwacher): Moduli spaces of curves with polynomial point counts

Canning discussed his recent breakthrough result on determining which moduli spaces M_g of curves of genus g have a polynomial point count over a finite field. It has been known for at least 20 years, due to work of Looijenga and Tommasi that the moduli spaces of curves of genus 3 and 4 have a polynomial point count. More generally, due to fundamental work of Deligne this point count is intimately related to the trace of the Frobenius endomorphism on the compactly supported cohomology of M_g. In particular, it follows that if one odd Euler characteristic $\chi_k(M_g)$ is not zero, then the point count $|M_g(\mathbb{F}_q)|$ is not polynomial. These Euler characteristics can be understood via Deligne's weight spectral sequence on M_g and it follows that if for all boundary strata $\overline{M}_\Gamma$ corresponding to a dual graph Γ of a stable curve of genus g, the invariant part of the cohomology of $\overline{M}_\Gamma$ is of Tate type, then the point count of M_g is polynomial. This methods is used to great effect to show that the point count of M_g is polynomial for $g \leq 8$.

To prove that for remaining cases the point count is not polynomial, Canning and his collaborators use the 11-th cohomology of $\overline{M}_{g,n}$, taking advantage of the fact that in genus 1 one has non-zero cohomology at the level of $M_{1,n}$ for $n \geq 11$. At the end of the day they manage to prove that $\chi_{11}(M_g) \neq 0$ for $g \geq 9$ but $g \neq 12$, whereas $\chi_{13}(M_{12}) = -6$, thus completing the proof of this impressive result.

Stefan Schreieder (with Engel and de Gaay Fortman): On the integral Hodge conjecture for abelian varieties.

Stefan Schreieder presented, for the first in front of any audience, a superb result on the rationality and the topology of algebraic varieties. A very important achievement of the 1970's was the Clemens-Griffiths theorem saying that a smooth cubic threefold X is not rational. Their argument involved the intermediate Jacobian of X, a 5-dimensional principally polarized abelian variety, that they proved not to be a isomorphic to a product of Jacobians of curves, while the rationality of X would imply that it is. In the same period, Artin and Mumford exhibited rationally connected threefolds that are not stably rational, (which means that their product with any projective space is not rational). The question whether the (very general) cubic threefold is stably rational or not remained open since. Together with Ph. Engel and O. de Gaay-Fortman, Schreieder established the stable irrationality of a very general cubic threefold over the complex numbers.

Another question that has been open for 50 years was whether an abelian variety over the complex numbers satisfies the integral Hodge conjecture, and more specifically, by looking at integral Hodge homology classes of degree 2, whether the minimal curve class of a principally polarized abelian variety is algebraic. The three authors prove that the minimal curve class is not algebraic on the intermediate Jacobian of a very general cubic threefold. This implies in turn its stable irrationality by previously known results.

Junliang Shen (with Maulik, Yin, and Zhang): The D-equivalence conjecture for K3$^{[n]}$-type.

Let X and X' be nonsingular, projective, complex Calabi-Yau varieties. The D-equivalence conjecture of Bondal-Orlov states that if X and X' are birational, then there is an equivalence of bounded derived categories

$$D^b(X) \simeq D^b(X')\,.$$

The conjecture, now almost 30 years old, has been settled in relatively few cases (the conjecture holds for Calabi-Yau 3-folds by Bridgeland's results and for hyperkähler 4-folds by work of Kawamata and Namikawa). Recently, Halpern-Leistner proved the D-equivalence conjecture for moduli spaces of stable sheaves on $K3$ surfaces. These moduli space are all hyperkähler of $K3^{[n]}$-type. The main result presented in Junliang Shen's lecture was the proof of the the D-equivalence conjecture for *all* hyper-Kahler varieties of $K3^{[n]}$-type.

His talk represented joint work with D. Maulik, Q. Yin, and R. Zhang. In fact, the results were stronger: Shen and his collaborators construct explicit Fourier-Mukai kernels using the earlier work of Markman on hyperholomorphic bundles. Moreover, a more general version of the D-equivalence conjecture is proven for bounded derived categories of twisted sheaves on X and X'.

Margarida Melo (with Brandt, Bruce, Chan, Moreland, and Wolfe): On the top weight rational cohomology of the moduli space of abelian varieties and universal Jacobians.

The talk of Melo, delivered at the very end of the conference displayed fascinating analogies with the one of Canning, at the very beginning of the conference and concerned the moduli space A_g of principally polarized abelian varieties of dimension g. The moduli space A_g being a quotient of a symmetric domain by the action of the symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$, its Euler characteristic and stable cohomology have been long understood due to fundamental work of Borel. In spite of that the full cohomology of A_g has only been known for $g \leq 3$ and it has been a long-standing open problem whether there is any odd degree cohomology class on A_g.

Melo and her collaborators determine the top degree cohomology of A_g for $g \leq 7$, in particular they show that odd degree cohomology appears both on A_5 (in degree 15) and on A_7 (in degrees 33 and 37). The method of proofs relies on the Deligne's weight spectral sequence and one fact that the boundary complex of a certain compactification of A_g can be identified with the link over the tropicalization of A_g. This reduces the problem of determining the top weight cohomology of A_g to highly sophisticated graph complex calculations that the authors manage to carry out in the range mentioned above.

Introductory talks from (and conversations with) younger (or not) participants.

On Tuesday evening, volunteering participants had the opportunity to present snapshots of their research in the form of five minute, one blackboard talks. The list of speakers was established by Cécile Gachet, and the session was moderated by

Dan Abramovich. The presentations, listed below, covered a similarly wide range of topics. This speakers included those younger participants in the workshop who did not have the opportunity to give a one-hour talk. As with previous years' young participants, we expect these researchers to quickly establish themselves as leaders in their areas. Here is a list of the presentations that were given. The first half featured:

Aitor Iribar Lopez: *The tautological ring of Mumford's partial compactification*
Matteo Verni: *Lefschetz defect in families*
Carolina Tamborini: *Abelian varieties with a G-action and curves*
Federico Moretti: *Degree of irrationality of low genus K3 surfaces*

After a short break, we resumed for the second half with:

Ignacio Barros: *Extremal divisors on moduli of K3 surfaces*
Alexandrou Theodosis: *Torsion and divisibility structure of higher Chow groups*
Charles Vial: *Splinters in positive characteristics*
Johannes Schmitt: *IMProofBench – a benchmark for AI mathematical reasoning*

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Workshop: Recent Trends in Algebraic Geometry

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Abstracts

Enumeration on the self-dual analogue of the Hilbert scheme of points

Joachim Jelisiejew
(joint work with Andrea Ricolfi, Reinier Schmiermann)

The Hilbert scheme of points $\mathrm{Hilb}_d(X)$ is typically highly singular for $\dim X > 2$, even if $X = \mathbb{A}^n_{\mathbf{k}}$. There are plenty of open questions about its geometry, see [2]. Two known classes of smooth points $[Z] \in \mathrm{Hilb}_d(\mathbb{A}^n)$ are

- local complete intersections Z, for any n,
- Gorenstein schemes Z, for $n = 3$, see [4].

A finite $\mathbf{k}$-scheme Z is *Gorenstein* if there exists an isomorphism $q\colon \omega_Z \to \mathcal{O}_Z$ of $\mathcal{O}_Z$-modules. Such an isomorphism can be interpreted as a symmetric bilinear form

$$q\colon \mathcal{O}_Z \times \mathcal{O}_Z \to \mathbf{k},$$

which satisfies $q(f_1 f_2,\, f_3) = q(f_1,\, f_2 f_3)$. We call it the *orientation* of Z. The locus $\mathrm{Hilb}_d^{\mathrm{Gor}}(\mathbb{A}^n) \subseteq \mathrm{Hilb}_d(\mathbb{A}^n)$ is open. It is thus interesting to look for (partial) compactifications of this locus other than the Hilbert scheme.
In enumerative geometry, the singularities of $\mathrm{Hilb}_d(\mathbb{A}^n)$ are avoided, for $n = 3, 4$, by introducing a virtual fundamental class and using virtual localization, for example see, [6, §11] for $n = 3$. To construct the virtual fundamental class, it is useful to present $\mathrm{Hilb}_d(\mathbb{A}^n)$ as the free quotient, for a fixed d-dimensional vector space V:

$$\left\{X_1, \ldots, X_n \in \mathrm{End}(V),\ v \in V \;\middle|\; \forall_{i,j}\ [X_i, X_j] = 0,\ \text{stability}\right\} / \,\mathrm{GL}(V)$$

where *stability* denotes the open condition $\mathbf{k}[X_1, \ldots, X_n]v = V$. It is only natural to ask if above one can replace $\mathrm{End}(V)$ by some classical subalgebra. We fix a full rank quadric Q on V and consider $\mathrm{Sym}_2 V \simeq \mathrm{End}(V)^{\mathrm{sym}} \subseteq \mathrm{End}(V)$, here $\mathrm{char}\,\mathbf{k} \neq 2$. We may form a free quotient by the orthogonal group $O(Q)$, keeping same commutativity and symmetry conditions

$$\left\{X_1, \ldots, X_n \in \mathrm{End}(V)^{\mathrm{sym}},\ v \in V \;\middle|\; \forall_{i,j}\ [X_i, X_j] = 0,\ \text{stability}\right\} / \,O(Q)$$

and the points of this quotient corresponds to pairs $([Z], q)$, where $Z \subseteq V^\vee$ is Gorenstein and q is its orientation. However, localization in this setup fails, because there are not enough torus fixed points. Again, it is desirable to construct a (partial) compactification which takes into account the orientation. The Iarrobino scheme is such a compactification.

Completed quadrics and Iarrobino scheme [3]

The variety $\mathrm{CQ}(V)$ of completed quadrics [5,8] is a wonderful compactification of the space of full rank quadrics on V. We have

$$\mathrm{CQ}(V)(\mathbf{k}) = \{[q_0], [q_1], \ldots\},$$

where $[q_0] \in \mathbb{P}\,\mathrm{Sym}_2(V)$ is the class of quadric of *any* nonzero rank, $q_1\colon \ker q_0 \to \operatorname{coker} q_0$ is a "residual" quadric, in particular $q_1^\top = q_1$, next, if $\ker q_1$ is nonzero, then $q_2\colon \ker q_1 \to \operatorname{coker} q_1$ a "next order residual" quadric and so on.

The geometry of $\mathrm{CQ}(V)$ is very rich and beautiful, however all available constructions are rather intricate [7, §10]. It is log-homogeneous [1] and admits [3] a remarkable subbundle $\mathcal{C}\mathrm{omp}_V \subseteq \mathrm{End}(V)_{\mathrm{CQ}(V)}$. An element $M \in \mathrm{End}(V)$ belongs to $(\mathcal{C}\mathrm{omp}_V)|_{[q_\bullet]}$ if the following are satisfied

1. (1) M preserves the flag $V \supseteq \ker q_0 \supseteq \ker q_1 \supseteq \ker q_2 \supseteq \ldots$,
2. (2) the induced operators $M|_{N_i}$ on $N_i := \frac{\ker q_i}{\ker q_{i+1}}$ are symmetric with respect to the full rank quadric $q_i|_{N_i}$ for every $i = -1, 0, \ldots$.

If the above holds, we say that M and $[q_\bullet]$ are *compatible*. For example, if q_0 has full rank, then a compatible M is an element symmetric with respect to q_0. On the other extreme, if q_0, q_1, ... are all rank one, then the symmetry condition becomes vacuous and M is compatible if and only if it preserves the flag. In this way, $\mathcal{C}\mathrm{omp}_V$ features a degeneration from symmetric to upper-triangular matrices.

The functor $\mathrm{CQ}(-)$ generalizes from vector spaces to arbitrary vector bundles. Let X be a $\mathbf{k}$-scheme. For any finite flat family $\operatorname{Spec}\mathcal{B} \to \operatorname{Spec}\mathcal{A}$ we define $\mathrm{CCQ}(\mathcal{A} \to \mathcal{B})$ as the functor of compatible pairs $M \in \mathcal{B}, [q_\bullet] \in \mathrm{CQ}(\mathcal{B})$. The *Iarrobino scheme of d points on X* $\mathrm{Iar}_d(X)$ is defined as $\mathrm{CCQ}(\mathcal{U} \to \mathrm{Hilb}_d(X))$, where $\mathcal{U}$ is the universal family. It follows that

$$\mathrm{Iar}_d(X)(\mathbf{k}) = \left\{ Z = Z_0 \supseteq Z_1 \supseteq \ldots, [q_\bullet] \mid q_i\colon \frac{I_{Z_i}}{I_{Z_{i+1}}} \to \left(\frac{I_{Z_i}}{I_{Z_{i+1}}}\right)^\vee \text{ iso. of } \mathcal{O}_Z\text{-mod} \right\},$$

where q_i are symmetric. The locus of oriented Gorenstein algebras $(Z \subseteq X, q)$ embeds as open subset of $\mathrm{Iar}_d(X)$, where $q_0 := q$. The scheme $\mathrm{Iar}_d(X)$ is projective over $\mathrm{Sym}_d(X)$, hence virtual localization is possible.

Virtually (in the popular sense) nothing is known about the geometry of $\mathrm{Iar}_d(X)$. We summarize below some observations from [3]:

- For C a smooth curve, $\mathrm{Iar}_d(C)$ is smooth. The natural map $\pi\colon \mathrm{Iar}_d(C) \to \mathrm{Hilb}_d(C)$ is flat, but not smooth. In comparison with $\mathrm{Hilb}_d(\mathbb{A}^1) \simeq \mathbb{A}^d$, the geometry of $\mathrm{Iar}_d(\mathbb{A}^1)$ is much more interesting, this scheme contains projective subschemes such as permutohedral toric varieties.
- Already $\mathrm{Iar}_d(\mathbb{A}^2)$ is not smooth, however it admits a perfect obstruction theory. It is plausible, but as of today open, whether $\mathrm{Iar}_d(S)$ admits a perfect obstruction theory for every smooth surface S.

The forgetful map $\pi\colon \mathrm{Iar}_d(-) \to \mathrm{Hilb}_d(-)$ a priori connects two different geometries, governed by GL_d and O_d, respectively. Despite this, the map behaves very regularly, at least in low dimensions. Namely, for $X = \mathbb{A}^1$, we have

$R\pi_*\mathcal{O}_{\mathrm{Iar}_d(X)} = \mathcal{O}_{\mathrm{Hilb}_d(X)}$. For $X = \mathbb{A}^2$, conjecturally, we have

$$R\pi_*\mathcal{O}^{\mathrm{vir}}_{\mathrm{Iar}_d(\mathbb{A}^2)} \simeq \mathcal{O}_{\mathrm{Hilb}_d(\mathbb{A}^2)},$$

where $\mathcal{O}^{\mathrm{vir}}$ is the virtual structure sheaf which lives in K-theory. The scheme $\mathrm{Iar}_d(\mathbb{A}^2)$ admits interesting bundles which may serve as insertions for the virtual counts. It is of interest to compare results of such counts with Haiman's theory of $\mathrm{Hilb}_d(\mathbb{A}^2)$.

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Moduli spaces of curves with polynomial point count

Samir Canning

(joint work with Hannah Larson, Sam Payne, Thomas Willwacher)

A fundamental counting question in algebraic geometry is to determine the number of curves of genus g over a finite field with q elements, up to geometric isomorphism. Let $\mathcal{M}_g$ be the moduli space of nonsingular curves of genus g. By [4, 6], the following point counts are known:

$$\#\mathcal{M}_2(\mathbb{F}_q) = q^3, \#\mathcal{M}_3(\mathbb{F}_q) = q^6 + q^5 + 1, \text{ and } \#\mathcal{M}_4(\mathbb{F}_q) = q^9 + q^8 + q^7 - q^6.$$

In each case, the count is polynomial in the size of the field. We prove that polynomiality occurs only for small genera.

Theorem ([3, Theorem 1.1])**.** *The count $\#\mathcal{M}_g(\mathbb{F}_q)$ is polynomial in q if and only if $g \leq 8$.*

The theorem is proven by a study of the cohomology of $\mathcal{M}_g$, which is related to the point count by the Grothendieck–Lefschetz trace formula:

$$\#\mathcal{M}_g(\mathbb{F}_q) = \sum_i (-1)^i \mathrm{tr}(\mathrm{Frob}_q^* | H^i_c(\mathcal{M}_{g,\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)).$$

Here, Frob_q^* is the map induced by Frobenius on the compactly supported ℓ-adic cohomology of $\mathcal{M}_{g,\overline{\mathbb{F}}_q}$, where ℓ is a prime such that $(\ell, q) = 1$.

By the Weil conjectures, the eigenvalues of Frob_q^* acting on $H_c^i(\mathcal{M}_{g,\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ are algebraic integers whose absolute values under any embedding of $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$ are $q^{k/2}$, where k is an integer such that $0 \leq k \leq i$. The numbers k are referred to as the weights, and they define a filtration

$$W_0 H_c^i(\mathcal{M}_g) \subset \cdots \subset W_i H_c^i(\mathcal{M}_g)$$

on the compactly supported cohomology. For example, the top compactly supported cohomology

$$H_c^{6g-6}(\mathcal{M}_g, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-3g+3)$$

is of weight $6g-6$. More generally, $\mathbb{Q}_\ell(-k)$ has weight $2k$. If $H_c^*(\mathcal{M}_g)$ is a sum of $\mathbb{Q}_\ell(-k)$ for possibly varying k, we say that $H_c^*(\mathcal{M}_g)$ is Tate. If $H_c^*(\mathcal{M}_g)$ is Tate, then $\#\mathcal{M}_g(\mathbb{F}_q)$ is polynomial in q, providing a method to prove polynomiality.

We also need a method to obstruct polynomiality. The weight k Euler characteristic is defined as

$$\chi_k(\mathcal{M}_g) = \sum_i (-1)^i \dim(\mathrm{gr}_k^W H_c^i(\mathcal{M}_g)).$$

Lemma. *If there is an odd k such that $\chi_k(\mathcal{M}_g) \neq 0$, then $\#\mathcal{M}_g(\mathbb{F}_q)$ is not polynomial in q.*

The weight filtration can be computed from a spectral sequence corresponding to the choice of a normal crossings compactification. The moduli space $\mathcal{M}_g$ has a canonical such compactification, the moduli space $\overline{\mathcal{M}}_g$ of stable curves. The boundary of $\overline{\mathcal{M}}_g$ is stratified by the topological type of stable curves, which is encoded in terms of stable graphs Γ of genus g. For each such stable graph, we set

$$\overline{\mathcal{M}}_\Gamma = \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)},$$

where $V(\Gamma)$ is the set of vertices of Γ, $g(v)$ is the genus assignment of the vertex v, and $n(v)$ is the number of incident half-edges to v.

The stratification by stable graphs gives rise to the weight spectral sequence

$$E_1^{j,k} = \bigoplus_{|E(\Gamma)|=j} (H^k(\overline{\mathcal{M}}_\Gamma) \otimes \det E(\Gamma))^{\mathrm{Aut}(\Gamma)} \Rightarrow H_c^{j+k}(\mathcal{M}_g), \tag{1}$$

which degenerates at E_2. The associated filtration on E_∞ is the weight filtration. Here, $E(\Gamma)$ is the set of edges of Γ, and $\det E(\Gamma)$ is the sign of the permutation representation of the automorphism group $\mathrm{Aut}(\Gamma)$ on the edge set.

When $g \leq 8$, we prove that every entry on the first page of the spectral sequence (1) is Tate, and thus so is $H_c^*(\mathcal{M}_g)$. It is not true that $H^*(\overline{\mathcal{M}}_\Gamma) \otimes \det E(\Gamma)$ is Tate, but the non-Tate part is not invariant under the $\mathrm{Aut}(\Gamma)$ action.

When $g \geq 9$, we analyze the generating function for the weight 11 Euler characteristic $\chi_{11}(\mathcal{M}_g)$, which was calculated by Payne and Willwacher [5]. The lower odd weight Euler characteristics vanish by theorems of Arbarello–Cornalba and

Bergström–Faber–Payne [1, 2]. Computer calculations show that $\chi_{11}(\mathcal{M}_g) \neq 0$ for $9 \leq g < 600$ and $g \neq 12$. Asymptotic analysis shows that $\chi_{11}(\mathcal{M}_g) \neq 0$ for $g \geq 600$. Nevertheless, $\chi_{11}(\mathcal{M}_{12}) = 0$. To handle the final case, we compute the thirteenth cohomology of $\overline{\mathcal{M}}_{g,n}$ for all g and n [3, Theorem 1.7].

Using the preceding theorem, we calculate that $\chi_{13}(\mathcal{M}_{12}) = -6$, finishing the proof of the main result.

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On minimal exponents of hypersurface singularities

Mircea Mustaţă

The goal of the talk was to give an overview of some recent work on the minimal exponent. After the definition in terms of b-functions and the characterization in terms of V-filtration, we discussed the connection with higher Du Bois and rational singularities, and stated two recent results from [1] and [2].

1. The setup. Let X be a smooth, irreducible, n-dimensional complex algebraic variety and Z a (nonempty) hypersurface in X, defined by $f \in \mathcal{O}_X(X)$. By a *log resolution* of (X, Z), we mean a proper morphism $\pi\colon Y \to X$, that is an isomorphism over $X \setminus Z$, with Y smooth, and such that $E := \pi^*(D)_{\rm red}$ is a simple normal crossing divisor.

2. Classical invariants. Let us recall briefly some well-known notions in birational geometry. For $\lambda \in \mathbf{Q}_{>0}$, the *multiplier ideal* $\mathcal{J}(\lambda Z)$ is given by

$$\mathcal{J}(\lambda Z) := \big\{g \in \mathcal{O}_X \mid \tfrac{|g|^2}{|f|^{2\lambda}} \text{ is locally integrable}\big\}.$$

Algebraically, this can be described in terms of a log resolution π of (X, Z):

$$\mathcal{J}(\lambda Z) = \pi_*\mathcal{O}_Y\big(K_{Y/X} - \lfloor \lambda\pi^*(Z)\rfloor\big).$$

The *log canonical threshold* $\mathrm{lct}(Z)$ is defined as

$$\min\big\{\lambda \in \mathbf{Q}_{>0} \mid \mathcal{J}(\lambda Z) \neq \mathcal{O}_X\big\}.$$

The algebraic and analytic descriptions of the multiplier ideals give corresponding descriptions for the log canonical threshold. The invariant we focus on is a refinement of the log canonical threshold.

3. The minimal exponent. By a theorem of Bernstein and Kashiwara, there is a nonzero polynomial $b(s) \in \mathbf{C}[s]$ such that

$$b(s)f^s \in \mathcal{D}_X[s] \cdot f^{s+1}, \tag{1}$$

where $\mathcal{D}_X$ is the sheaf of differential operators on X. The set of such $b(s)$ forms an ideal in $\mathbf{C}[s]$ and the monic generator of this ideal is the *b-function* $b_f(s)$.

It is a result of Kashiwara [5] that all roots of b_f are negative rational numbers. Furthermore, by a result of Lichtin and Kollár [6], the largest root of b_f is $-\mathrm{lct}(Z)$. On the other hand, by specializing $s = -1$ in (1), one can see that $b_f(-1) = 0$. The *minimal exponent* $\widetilde{\alpha}(Z)$ was defined by Saito as the negative of the largest root of $b_f(s)/(s+1)$, with the convention that it is ∞ if $b_f(s) = s+1$ (it is a theorem that this is the case if and only if Z is smooth). This invariant was systematically studied in the 80s by Varchenko, Steenbrink, Malgrange, Loeser, etc for isolated singularities. In that setting, it was known as *Arnold exponent*, and was defined using asymptotic expansions of integrals over vanishing cycles.

Note that $\mathrm{lct}(Z) = \min\left\{\widetilde{\alpha}(Z), 1\right\}$, hence the minimal exponent provides interesting information when $\mathrm{lct}(Z) = 1$. It is a result of Saito [11] that $\widetilde{\alpha}(Z) > 1$ if and only if Z has rational singularities.

Example. If $f = x_1^{a_1} + \ldots + x_n^{a_n}$, with $a_i \geq 2$ for all i, then $\widetilde{\alpha}(Z) = \sum_{i=1}^n \frac{1}{a_i}$.

The definition in terms of b-functions is not very useful for proving properties of the minimal exponent. A much more useful characterization was provided by Saito in [12] in terms of the V-filtration of Kashiwara and Malgrange. This allows relating the minimal exponent to the Hodge filtration on $\mathcal{O}_X[1/f]$ and, more generally, to the one on $\mathcal{O}_X[1/f]f^{-\alpha}$. Using this, several general properties of the minimal exponent (such as the behavior in families or under taking hyperplane sections) were proved in [9].

4. Higher singularities. The minimal exponent has been useful in describing some higher-order versions of Du Bois and rational singularities, as follows.

Theorem 1 ([8], [4]). The hypersurface Z has *p-Du Bois singularities* (that is, the canonical morphism $\Omega_Z^i \to \underline{\Omega}_Z^i$ is an isomorphism for all $i \leq p$, where $\underline{\Omega}_Z^i$ denotes the i-truncation of the Du Bois complex of Z) if and only if $\widetilde{\alpha}(Z) \geq p+1$.

Similarly, $\widetilde{\alpha}(Z)$ can be used to detect *p-rational singularities* in the sense of Friedman and Laza. This means that if $\mu\colon \widetilde{Z} \to Z$ is a resolution of singularities that is an isomorphism over $Z \setminus Z_{\mathrm{sing}}$, with $E = \mu^{-1}(Z_{\mathrm{sing}})_{\mathrm{red}}$ having simple normal crossings, then $\Omega_Z^i \to \mathbf{R}\mu_*\Omega_{\widetilde{Z}}^i(\log E)$ is an isomorphism for all $i \leq p$.

Theorem 2 ([3], [10]). The hypersurface Z has p-rational singularities if and only if $\widetilde{\alpha}(Z) > p+1$.

5. A birational description of the minimal exponent. We now give a description of $\widetilde{\alpha}(Z)$ in terms of a log resolution π of (X, Z), following [1]. More precisely, we describe when $\widetilde{\alpha}(Z) > \lambda$, for some $\lambda \in \mathbf{Q}_{\geq 0}$. We treat separately the cases when $\lambda \in \mathbf{Z}$ and $\lambda \notin \mathbf{Z}$.

Theorem 3. If p is a nonnegative integer, then $\widetilde{\alpha}(Z) > p$ if and only if the following conditions hold:

i) $R^q\pi_*\Omega_Y^i(\log E) = 0$ for all $q \geq 1$ and $i \leq p$.
ii) $\mathrm{codim}_Z(Z_{\mathrm{sing}}) \geq 2p$.

This result follows easily from the characterization of $(p-1)$-rational singularities in Theorem 2. The point is that the condition in ii) allows restating the vanishings in i) as similar vanishings for higher direct images of sheaves of log forms on a resolution of Z.

Theorem 4. Let p be a nonnegative integer and $\alpha \in (0,1) \cap \mathbf{Q}$.

i) If $\widetilde{\alpha}(Z) \geq p$, then the canonical map

$$R^q\pi_*\Omega_Y^{n-p}(\log E)\big(-E - \lfloor \alpha\pi^*(D)\rfloor\big) \to R^q\pi_*\Omega_Y^{n-p}(\log E)(-E) \tag{2}$$

is an isomorphism for all $q \neq p$ and it is injective for $q = p$.
ii) If $\widetilde{\alpha}(Z) > p$, then we have $\widetilde{\alpha}(Z) > p + \alpha$ if and only if the map in (2) is an isomorphism also for $q = p$.

Note that for $p = 0$, the assertion in ii) above says that $\widetilde{\alpha}(Z) > \alpha$ if and only if $\mathcal{J}(\alpha Z) = \mathcal{O}_X$.

6. A characterization via the poles of the Archimedean zeta function. Suppose, for simplicity, that $X = \mathbf{A}^n_{\mathbf{C}}$ and $f \in \mathbf{C}[x_1, \ldots, x_n]$. For every smooth function φ on X, with compact support, it is easy to see that the expression

$$Z_{f,\varphi}(s) = \int_{\mathbf{C}^n} |f(z)|^{2s}\varphi(z)dzd\overline{z}$$

is well-defined and holomorphic for $\mathrm{Re}(s) > 0$. It is an important result, first proved by Bernstein-Gelfand and Atiyah using the existence of log resolutions, and then reproved by Bernstein using the existence of b-functions, that $Z_{f,\varphi}$ extends meromorphically to $\mathbf{C}$. In this way, we can view Z_f as a meromorphic function on $\mathbf{C}$ with values in distributions. This is the *Archimedean zeta function* of f. It is a consequence of the approach via log resolutions that all poles of Z_f are negative rational numbers, with the largest pole being $-\mathrm{lct}(Z)$ (of course, this is related to the analytic description of the log canonical threshold).

It can be shown that, in general, Z_f has a pole of multiplicity ≥ 1 at every element of $\mathbf{Z}_{<0}$. A *nontrivial pole* of Z_f is a pole that is not in $\mathbf{Z}_{<0}$ or is in $\mathbf{Z}_{<0}$, but its multiplicity as a pole is ≥ 2. The following recent result of Davis, Lörincz, and Yang from [2] gives a characterization of the minimal exponent in terms of the poles of Z_f. The case when Z has isolated singularities was proved by Loeser in [7].

Theorem 5. The largest notrivial pole of Z_f is $-\widetilde{\alpha}(Z)$.

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Quadratic and real Donaldson–Thomas invariants for spin threefolds

Marc Levine

(joint work with Anneloes Viergever)

Enumerative geometry relies on the intersection theory furnished by the Chow ring/Chow groups, with input given by cycle classes, Chern classes of vector bundles, virtual fundamental classes .... Using the six-functor formalism for the motivic stable homotopy category, Déglise-Jin-Khan have developed an analogous intersection theory for an arbitrary motivic commutative ring spectrum. We will give an introduction to this theory and its applications to giving refinements to the Grothendieck-Witt ring of many known integer-valued invariants for enumerative problems.

1. Generalized cohomology and Borel-Moore homology for a motivic commutative ring spectrum

Consider the motivic stable homotopy category as a functor (B a noetherian scheme of finite Krull dimension, $\mathbf{Sch}_B$ = finite type separated B-schemes.)

$$\begin{aligned}\mathrm{SH}(-) :&\mathbf{Sch}_B^{\mathrm{op}} \to \textbf{Triangulated tensor categories}\\ &X \mapsto \mathrm{SH}(X)\\ &f : Y \to X \mapsto f^* : \mathrm{SH}(X) \to \mathrm{SH}(Y)\end{aligned}$$

with the additional structure of a *six-functor formalism* [5]. Déglise-Jin-Khan [2] use this to give, for each motivic commutative ring spectrum $\mathcal{E} \in \mathrm{SH}(B)$, the theories of twisted $\mathcal{E}$-cohomology and twisted $\mathcal{E}$-Borel Moore homology

$$(X, v \in \mathcal{K}(X)) \mapsto \mathcal{E}^{*,*}(X, v), \mathcal{E}^{\mathrm{B.M.}}_{*,*}(X, v)$$

Examples: k a field, $B = \operatorname{Spec} k$. We have motivic commutative ring spectra $\mathrm{EM}(\mathcal{K}^M_*), \mathrm{EM}(\mathcal{K}^{MW}_*), \mathrm{EM}(\mathcal{W}_*) \in \mathrm{SH}(k)$.

- The Chow groups

$$\mathrm{CH}^n(X) = \mathrm{EM}(\mathcal{K}^M_*)^{2n,n}(X), \mathrm{CH}_n(X) = \mathrm{EM}(\mathcal{K}^M_*)^{\mathrm{B.M.}}_{2n,n}(X)$$

- The Chow-Witt groups

$$\widetilde{\mathrm{CH}}^n(X, L) = \mathrm{EM}(\mathcal{K}^{MW}_*)^{2n,n}(X, L - \mathcal{O}), \widetilde{\mathrm{CH}}_n(X, L) = \mathrm{EM}(\mathcal{K}^M_*)^{\mathrm{B.M.}}_{2n,n}(X, L - \mathcal{O}_X)$$

- Witt-sheaf cohomology

$$H^n(X, \mathcal{W}(L)) = \mathrm{EM}(\mathcal{W}_*)^{n,0}(X, L - \mathcal{O}),$$
$$H^{\mathrm{B.M.}}_n(X, \mathcal{W}(L)) = \mathrm{EM}(\mathcal{W}_*)^{\mathrm{B.M.}}_{n,0}(X, L - \mathcal{O}_X).$$

2. Localization and virtual localization

One has Atiyah-Bott torus localization and the Bott residue theorem for the $T = \mathbb{G}_m^n$-equivariant Chow groups [3]. T-localization fails for Witt sheaf cohomology, but there is a replacement: N-localization, with N the normalizer of the diagonal torus in SL_2[7].

There are also "virtual localization theorems": A perfect obstruction theory ϕ on X gives a virtual fundament class $[X, \phi]^{vir} \in \mathrm{CH}_r(X)$, $r = \mathrm{rank}(\phi)$. This generalizes to a virtual fundamental class $[X, \phi]^{vir}_{\mathcal{W}} \in H^{\mathrm{B.M.}}_r(X, \mathcal{W}(\det \phi))$. There are equivariant versions $[X, \phi]^{vir}_G \in \mathrm{CH}^G_r(X)$, $[X, \phi]^{vir}_{G,\mathcal{W}} \in H^{\mathrm{B.M.}}_{G,r}(X, \mathcal{W}(\det \phi))$ if X has a G-action for which ϕ is G-linearized.

The Graber-Pandharipande virtual T-localization for the CH_*-virtual classes [4] shows how to compute $[X, \phi]^{vir}_T$ by restriction to X^T. I [8] showed how to generalize this to a similar looking virtual localization theory for $[X, \phi]^{vir}_{N,\mathcal{W}}$.

3. Degrees and orientations

To transform classes in $\mathrm{CH}_0(X)$ to integers, for $p : X \to \operatorname{Spec} k$ proper, one applies the degree map $\deg_k := p_* : \mathrm{CH}_0(X) \to \mathrm{CH}_0(\operatorname{Spec} k) = \mathbb{Z}$. Given a class $x \in H^{\mathrm{B.M.}}_0(X, \mathcal{W}(L))$, one needs an *orientation*, that is, an isomorphism $\rho : L \to M^{\otimes 2}$, to get a degree map $\deg^\rho_k : H^{\mathrm{B.M.}}_0(X, \mathcal{W}(L)) \to W(k)$ by

$$H^{\mathrm{B.M.}}_0(X, \mathcal{W}(L)) \xrightarrow{\rho} H^{\mathrm{B.M.}}_0(X, \mathcal{W}(M^{\otimes 2}))$$
$$\xrightarrow{\mathrm{can}} H^{\mathrm{B.M.}}_0(X, \mathcal{W}) \xrightarrow{p_*} H^{\mathrm{B.M.}}_0(\operatorname{Spec} k, \mathcal{W}) = W(k)$$

Proposition 1 (L. [6])**.** *Let X be a smooth projective threefold over k with a spin structure, i.e., an isomorphism $\tau : \omega_{X/k} \xrightarrow{\sim} M^{\otimes 2}$. Then for each $n \geq 1$, τ induces an orientation $\rho_n : \det \phi_{DT,n} \to L_n^{\otimes 2}$, where $\phi_{DT,n}$ is the Donaldson-Thomas perfect obstruction theory on* $\mathrm{Hilb}^n(X)$.

Definition 2. For X a smooth projective threefold with spin structure, taking the degree of $[\mathrm{Hilb}^n(X), \phi_{DT,n}]^{vir}_{\mathcal{W}} \in H_0^{\mathrm{B.M.}}(\mathrm{Hilb}^n(X), \mathcal{W}(\det \phi_{DT,n}))$ defines the nth quadratic D-T invariant

$$I_n^{\mathcal{W}}(X) := \deg_k^{\rho_n}([\mathrm{Hilb}^n(X), \phi_{DT,n}]^{vir}_{\mathcal{W}}) \in W(k).$$

4. Some examples

For X a smooth projective threefold, let $I_n(X) = \deg_k([\mathrm{Hilb}^n(X), \phi_{DT,n}]^{vir}_{\mathrm{CH}}) \in \mathbb{Z}$, and let $M(t)$ be the MacMahon function

$$M(t) := \prod_{n\geq 1}(1-t^n)^{-n} = 1 + \sum_{n\geq 1} P_n t^n$$

counting plane partitions.

We recall

Theorem 3 (MNOP [13], Behrend-Fantechi [1], Li [12], L.-Pandharipande [10])**.** *For X a smooth projective threefold over $\mathbb{C}$ we have*

$$1 + \sum_{n\geq 1} I_n(X)\cdot q^n = M(-q)^{\deg c_3(T_X\otimes K_X)}$$

MNOP=Maulik, Nekrasov, Okounkov, Pandharipande.

Two results for the Witt-sheaf versions:

Take $k=\mathbb{R}$, so $W(k) = W(\mathbb{R}) = \mathbb{Z}$ by the signature map, giving for X a smooth projective spin threefold over $\mathbb{R}$ the generating function

$$1 + \sum_{n\geq 1} I_n^{\mathcal{W}}(X) q^n \in \mathbb{Z}[[q]].$$

Theorem 4 (Viergever [14])**.** *For $k=\mathbb{R}$, $I_n^{\mathcal{W}}(\mathbb{P}^3) = 0$ for n odd and*

$$1 + \sum_{n\geq 1} I_n^{\mathcal{W}}(\mathbb{P}^3)\cdot q^n = M(-q^2)^{-10} \mod q^8,$$

that is, $I_2^{\mathcal{W}}(\mathbb{P}^3) = 10$, $I_4^{\mathcal{W}}(\mathbb{P}^3) = 25$, $I_6^{\mathcal{W}}(\mathbb{P}^3) = -50$.

Theorem 5 (L.-Viergever [11])**.** *For $k=\mathbb{R}$, $I_n^{\mathcal{W}}((\mathbb{P}^1)^3) = 0$ for n odd and*

$$1 + \sum_{n\geq 1} I_n^{\mathcal{W}}((\mathbb{P}^1)^3)\cdot q^n = M(q^2)^8$$

Both results use N-localization for the proof, relying heavily on computations used by MNOP [13] for the case of toric threefolds.

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Birational automorphism group of the Jacobian of a general Kummer quartic

Shigeru Mukai

The lattice $II_{1,17}(2^{+6})$ is reflective and its orthogonal group has a fundamental domain with 896+64 facets by Borcherds [4, §12]. This is the Picard lattice (with respect to the Beauville form) of the Jacobian symplectic 6-fold

$$\mathrm{Jac}^2|h| := \coprod_{D\in|h|} \mathrm{Jac}^2 D$$

of a very general Kummer quartic surface $(S, h) \subset (\mathbb{P}^3, \mathcal{O}(1))$.

Main Theorem *The birational automorphism group of* $\mathrm{Jac}^2|h|$ *is generated by* 864 *modified reflections with respect to the above* $896 - 32$ *facets and a group of order* 2^{10} *whose center is Rapagnetta's involution (*[13]*).*

1. Lattices

Let $(L, \langle\ ,\rangle)$ be a *lattice*, i.e., a free $\mathbb{Z}$-module with a non-degenerate symmetric bilinear form $\langle\ ,\rangle\colon L \times L \to \mathbb{Z}$.

1.1. **Reflective lattices.** We consider the hyperbolic case, namely, the signature is $(1, *)$ and the orthogonal group $O^+(L)$ of L preserving a positive cone C^+. The orthogonal transformation

$$r_m : x \to x - \frac{2\langle x, m\rangle}{\langle m^2\rangle} m$$

defined for a primitive element $m \in L$ with $\langle m^2 \rangle < 0$ is called a *reflection of* L if it preserves L, i.e., $r_m(L) \subset L$. $m \in L$ (or $-m$) is called the *center* of the reflection. A lattice L is called *reflective* if the subgroup generated by all reflections of L is of finite index in $O^+(L)$.

By Esselmann [7], reflective lattices exist only in the range $\operatorname{rank} L \leq 22$. For example, $U + D_{20}, U + D_{18}, U + D_{17}, \cdots$ are reflective lattices belonging to the Conway-Vinberg chain ([5] = [6, Chap. 28]).

1.2. **Notations and convention.** Throughout this abstract, we work over the complex number field $\mathbb{C}$.

- A_n, D_n and $E_{6,7,8}$ denote the negative definite root lattices of ADE-type, generated by (-2)-vectors.
- U denotes the hyperbolic lattice $\left(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$ of rank two.
- $\operatorname{Disc}(L) := \operatorname{Coker}[L \to \operatorname{Hom}(L, \mathbb{Z})]$ (with $\mathbb{Q}/2\mathbb{Z}$-valued quadratic form)
- $II_{a,b}(**)$ The genus of even lattices of signature (a, b) with discriminant type $**$. For example, $** = 2^{+2a}$ means that $\operatorname{Disc}(L)$ is a 2-elementary group of length $2a$ and the discriminant form is of even integral type ([15]).
- $G = L \rtimes R$ means the semi-direct product with normal subgroup L (in the left).

1.3. **Kummer lattice.** Let A be an abelian surface and $\operatorname{Km}(A)$ be its Kummer surface, i.e., the minimal resolution of the quotient $A/\pm 1_A$. The Picard lattice of $\operatorname{Km}(A)$ contains the lattice $16A_1$ as sub-lattice. The Kummer lattice **Kum** is the primitive hull of $16A_1$ in the Picard lattice. The isomorphism class of **Kum** does not depend on A. In fact, it is explicitly described by the Reed-Muller code $[16, 8, 4]$, the binary code of length 16, minimal weight 8 and dimension 4. The code is generated by four words $c_1 = (0000000011111111), c_2 = (0000111100001111)$ $c_3 = (0011001100110011)$ and $c_4 = (010101010101010101)$. Then we have,

$$\mathbf{Kum} = 16A_1 + \sum_{i=1}^{4} \mathbb{Z}\frac{c_i}{2} + \mathbb{Z}\frac{(1111111111111111)}{2}.$$

Kum belongs to $II_{0,16}(2^{+6})$ and hence $U + \mathbf{Kum}$ belongs to $II_{1,17}(2^{+6})$. By the uniqueness theorem of indefinite lattices (Nikulin [15]), $II_{1,17}(2^{+2a})$ consists of the unique lattice. Hence we use $U + \mathbf{Kum}$ and $II_{1,17}(2^{+6})$ interchangeably.

2. Known geometric realization ($\rho = 18$)

For $a = 0, 1, 2$, the lattice $II_{1,17}(2^{+2a})$ is realized as the Picard lattice of a K3 surface S, and gives an explicit description of the automorphism group $\operatorname{Aut}(S)$.

- $a = 0$ (The unimodular lattice) $II_{1,17} \simeq U + E_8 + E_8$ is realized by an elliptic K3 surface with 2 reducible fibers of (Kodaira) type $\tilde{E}_8$. In this case $\operatorname{Aut}(S)$ is finite (Vinberg [17]).
- $a = 1$ $II_{1,17}(2^{+2}) \simeq U + D_{16}$. S is the minimal resolution of the double $\mathbb{P}^1 \times \mathbb{P}^1$ studied by Horikawa, Dolgachev, Barth-Peters, M.-Namikawa etc. in 80's. The automorphism group $\operatorname{Aut}(S)$ is virtually $\mathbb{Z}$.

$a = 2$ $II_{1,17}(2^{+4}) \simeq U + D_8 + D_8$. S is the Kummer surface $\mathrm{Km}(E_1 \times E_2)$ of product type. The orthogonal group $O^+(II_{1,17}(2^{+4}))$ is the semi-direct product

$$\langle 24\ (-2)\text{−reflections}, 24\ (-4)\text{−reflections}\rangle \rtimes (2^4.\mathfrak{S}_{4,4}).$$

The centers of (-2)-reflections are represented by $\mathbb{P}^1$'s and all (-4)-reflections are geometrically realized by involutions after suitable modification.

Theorem. (Keum-Kondo [11]) *The automorphism group of* $\mathrm{Km}(E_1 \times E_2)$ *for general elliptic cuves* E_1, E_2 *is the semi-direct product*

$$\langle 24 \text{ modified } (-4)\text{−reflections}\rangle \rtimes 2^4.$$

$a = 3$ The next is our lattice $II_{1,17}(2^{+6}) \simeq U + \mathbf{Kum}$ but no more realized as Picard lattice of a K3 surface S since the sum of the rank and the length of discriminant group exceeds 22, the second Betti number of S.

But ...

3. Main theorem

... our lattice $U+\mathbf{Kum}$ is realized as the Picard lattice of a holomorphic symplectic 6-fold associated with a general Kummer quartic surface. More precisely, let $\bar{S} \subset \mathbb{P}^3$ be a Kummer quartic surface of a general curve C of genus 2. $\bar{S}$ is the image of the Jacobian surface $\mathrm{Jac}\, C$ by the linear system $|2\Theta|$. The minimal resolution S is the Kummer surface of $\mathrm{Jac}\, C$. We denote the hyperplane section class of $S \to \bar{S} \subset \mathbb{P}^3$ by h. Consider the (compactified) Jacobian fibration $\mathrm{Jac}^d\,|h| \to |h|$, whose fiber at $[D] \in |h|$ is $\mathrm{Jac}^d\, D$. In terms of standard notation of moduli of sheaves, $\mathrm{Jac}^d\,|h|$ is $M_S(0, h, d-2)$. By general theory, $\mathrm{Jac}^d\,|h|$ is a holomorphic symplectic 6-fold of deformation type $K3^{[3]}$. The birational class does not depend on d but only on its parity. When d is odd, $\mathrm{Jac}^d\,|h|$ is birationally equivalent to the Hilbert square $\mathrm{Km}(C)^{[2]}$.

Now we restrict ourselves to the even case, i.e., the symplectic 6-fold $\mathrm{Jac}^2\,|h|$. By general theory again, its Picard lattice (with respect to the Beauville-Bogomolov-Fujiki form) is the orthogonal complement of $(0, h, 0)$ in $U + \mathrm{Pic}(S)$. Since the orthogonal complement of h in $\mathrm{Pic}(S)$ is the Kummer lattice $\mathbf{Kum}$, we have $(0, h, 0)^\perp \simeq U + \mathbf{Kum}$.

Theorem. (Borcherds [4, p. 346]) *The orthogonal group* $O^+(II_{1,17}(2^{+6}))$ *is the semi-direct product*

$$\langle 64\ (-2)\text{−reflections}, 896\ (-4)\text{−reflections}\rangle \rtimes G.\mathfrak{A}_8,$$

where G is the extended extra-special 2*-group* $2^{1+8}.2$ *of order* 2^{10}*.*

The centers of all (-2)-reflections and 32 (-4)-reflections are represented by effective divisors.

Main Theorem. *The birational automorphism group of* $\mathrm{Jac}^2|h|$ *is the semi-direct product*

$$\langle 864 \text{ modified } (-4)\text{−reflections}\rangle \rtimes G,$$

and the central involution of G *is the Mongardi-Rapagnetta-Saccà involution.*

4. Sketch of proof

The proof is similar to the case of $\mathrm{Km}(E_1 \times E_2)$ in §2 and the case of general Jacobian Kummer surfaces in [12].

Firstly the centers of 64 (-2)-reflections are represented by effective irreducible divisors. 32 appear in the reducible fibers of the Lagrangian fibration $\mathrm{Jac}^2|h| \to |h| \simeq \mathbb{P}^3$ in pairs $(16\tilde{A}_1)$. The remaining 32 appear in the dual fibration $\mathrm{Jac}^2|\hat{h}| \to |\hat{h}| \simeq \mathbb{P}^{3,\vee}$ (another $16\tilde{A}_1$). (A Kummer quartic is self dual.)

The (in-)effectivity is subtle for the centers of 896 (-4)-reflections. The answer is that most of (-4)-centers are not but special 32 are represented by effective irreducible divisors!

We recall the general theory of MMP for $M_S(v)$ with $v = (r, *, s)$ from Bayer-Macri [2] and Hassett-Tschinkel [9]. Divisorial contractions are classified into the following three types modulo flops:

- (BN) (-2)-contraction induced from a rigid (or spherical) object, e.g., $(-2)\mathbb{P}^1$ on S. BN stands for Brill-Noether.
- (HC) The minimal resolution $S^{[n]} \to S^{(n)}$ of a symmetric product is typical. In this typical case the diagonal divisor (with Beauville norm $2-2n$) is contracted. HC stands for Hilbert-Chow.
- (LGU) The original one is the contraction from the moduli of Giesekser-semi-stable rank 2 sheaves to the moduli of Uhlenbeck-Yau compactification of μ-stable moduli. LGU stands for Li-Gieseker-Uhlenbeck.

In our case all 64 (-2)-divisors above are all (BN), and (HC) does not occur since $v = (0, h, 0)$ has divisibility 2. (LGU) happens for 32 (-4)-centers in the following way:

We need a preparation on $\mathrm{Jac}^0|h|$ instead of $\mathrm{Jac}^2|h|$. Consider the difference divisor $D - D$ in the abelian 3-fold $\mathrm{Jac}^0 D$. Moving $[D]$ over the linear system $|h| \simeq \mathbb{P}^3$, we obtain the difference divisor $\mathcal{D} - \mathcal{D}$ in $\mathrm{Jac}^0|h|$. We have a forgetful morphism $\mathcal{D} - \mathcal{D} \to S \times S$ by definition. Then this divisor $\mathcal{D} - \mathcal{D}$ is contracted to the 4-fold $S \times S$ by an extremal contraction of $\mathrm{Jac}^0|h|$ of (LGU)-type.

Now we return to $\mathrm{Jac}^2|h|$. Recall that the Kummer quartic $\bar{S} \subset \mathbb{P}^3$ has 16 tropes, that is, double conic plane sections. Let $t \subset S$ be the reduced part of one of them and consider the line bundle $\mathcal{O}_S(-t)$. Its tensor gives rise to an isomorphism from $\mathrm{Jac}^2|h|$ to $\mathrm{Jac}^0|h|$. Hence we have 16 divisorial contractions of (LGU)-type of $\mathrm{Jac}^2|h|$. We have another set of 16 divisorial contractions by duality, i.e., by changing h by $\hat{h}$. In total we obtain 32 contractible divisors with Beauville norm -4.

The remaining 864 (-4)-reflections, whose centers are ineffective, are realized by involutions after suitable modification, whose details we omit here.

5. Other realization

The realization of $U+\mathbf{Kum}$ in §3 is generalized to non-principally polarized abelian surface A of type $(1,d)$. The twice polarization descends to a polarization h of degree $4d$ on the Kummer surface $S = \mathrm{Kum}(A)$. By the same computation as in §3, the Picard lattice of $(4d+2)$-fold $M_S(0,h,0)$ contains $U+\mathbf{Kum}$. In the case $d=2$, $\mathrm{Kum}(A) \subset \mathbb{P}^5$ is described explicitly by Barth [1] (see also [8]). Hence $K3^{[5]}$-type Jacobian symplectic 10-folds $\mathrm{Jac}^4|h| = M_S(0,h,0)$ are also interesting.

$U+\mathbf{Kum}$ seems also realized in the Picard lattice of symplectic manifolds of (deformation) type OG10. There are two candidates of *pseudo Kummer surfaces*:

(A) 4-dimensional family of K3 surface $\bar{S}$ of degree 6 in $\mathbb{P}^4$ with spatial Heawood configuration SH of 15 nodes and 15 twisted cubics, where SH is the configuration $(15_7 - 15_7)$ of 15 points and 15 planes in the projective space $\mathbb{P}^3(\mathbb{F}_2)$ over the binary field.

(B) 4-dimensional family of double planes $\bar{S}$ with branch the union of six lines (see e.g., [18], [16]). The minimal resolution S has 21 $\mathbb{P}^1$'s with $(15_2 - 6_5)$ configuration.

It is interesting to study the birational automorphism group of $S^{[OG]}$ of these K3 surfaces S. Here $S^{[OG]}$ denotes originally the minimal resolution of the moduli space $M_S(2,0,-2)$ of semi-stable 2-bundles on S. But, more generally, it also denotes the minimal resolution of $M_S(2v)$ for v with $(v^2)=2$. For example, $(0,h,0)$, h being the pull-back of a line, seems a natural choice for v in the case (B).

In the case (A), the *sum* of a twisted cubic counted twice and the seven nodes on it gives the hyperplane section class h of $\bar{S} \subset \mathbb{P}^4$. Furthermore we have 15 such divisors in $|h|$. This is a very natural analogy of 16 tropes of a Kummer quartic surface. This kind of degree 6 divisor $2R + N_1 + \cdots + N_7$ is observed recently in our study of parabolic version of the description of general polarized K3 surface of genus 13 in [10].

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The cohomology ring and the derived category of Quot schemes on curves

Alina Marian
(joint work with Andrei Neguţ)

Let $V \to C$ be a rank r locally free sheaf over a smooth complex projective curve C. We consider the Grothendieck Quot scheme $\mathsf{Quot}_d(V)$ parameterizing rank 0 degree d quotients of V:

$$0 \to E \to V \to F \to 0, \quad \text{rank } F = 0, \quad \deg F = d.$$

We will denote by π, ρ the projections from the product $\mathsf{Quot}_d(V) \times C$ to the two factors. In any setting, the Quot scheme carries a universal short exact sequence

$$0 \to \mathcal{E} \to \rho^* V \to \mathcal{F} \to 0 \text{ on } \mathsf{Quot}_d(V) \times C,$$

and its deformation-obstruction complex is $\mathrm{Ext}^\bullet_\pi(\mathcal{E}, \mathcal{F})$. In the situation considered here, when the quotients F are supported at finitely many points of C, it is therefore clear that $\mathsf{Quot}_d(V)$ is a smooth projective variety of dimension rd.

In this report, based on joint work with Andrei Neguţ (cf. [MN1, MN2]), we give first a representation-theoretic characterization of the cohomology of the Quot scheme $\mathsf{Quot}_d(V)$ (with $\mathbb{Q}$ coefficients). Secondly, we provide a semiorthogonal decomposition of the derived category of $\mathsf{Quot}_d(V)$, which gives an effective approach for calculating the cohomology of natural tautological bundles over $\mathsf{Quot}_d(V)$.

A basic geometric object in our analysis is the nested Quot scheme

$$\mathsf{Quot}_{d,d+1}(V) \subset \mathsf{Quot}_d(V) \times \mathsf{Quot}_{d+1}(V),$$

$$\mathsf{Quot}_{d,d+1}(V) = \{(E \overset{\iota}{\hookrightarrow} V, \ \ E' \overset{\iota'}{\hookrightarrow} V) \text{ with } E' \overset{\kappa}{\hookrightarrow} E \text{ and } \iota' = \iota \circ \kappa\}. \tag{1}$$

It is endowed with maps

$$\begin{array}{ccccc} & & \mathsf{Quot}_{d,d+1}(V) & & \\ & \swarrow^{p_-} & \downarrow^{p_C} & \searrow^{p_+} & \\ \mathsf{Quot}_d(V) & & C & & \mathsf{Quot}_{d+1}(V) \end{array} \tag{2}$$

which remember $E \hookrightarrow V$, the support point of E/E', and $E' \hookrightarrow V$ respectively. The nested Quot scheme can be viewed as parametrizing points $E \subset V$ in $\mathsf{Quot}_d(V)$, along with non-zero morphisms $E \to \mathbb{C}_x$ for $x \in C$. We thus have the isomorphism

$$\begin{array}{ccc} \mathsf{Quot}_{d,d+1}(V) & \xrightarrow{\cong} & \mathbb{P}_{\mathsf{Quot}_d(V)\times C}(\mathcal{E}) \\ & {}_{p_- \times p_C}\searrow & \downarrow \\ & & \mathsf{Quot}_d \times C \end{array} \tag{3}$$

where the right-hand side is the projective bundle of one-dimensional quotients of the universal subsheaf $\mathcal{E} \to \mathsf{Quot}_d(V) \times C$. We note that $\mathcal{E}$ is a locally free sheaf of rank r. We consider the tautological sequence

$$0 \to \mathcal{G} \to (p_- \times p_C)^*(\mathcal{E}) \to \mathcal{L} \to 0 \text{ on } \mathbb{P}_{\mathsf{Quot}_d(V)\times C}(\mathcal{E}), \tag{4}$$

where $\mathcal{L}$ is the hyperplane line bundle of the projectivization, and the kernel $\mathcal{G}$ is a locally free sheaf of rank $r-1$. We write

$$c_1(\mathcal{L}) = \lambda.$$

We now consider the creation/annihilation operators

$$e_k = (p_+ \times p_C)_*(\lambda^k \cdot p_-^*) : H^*(\mathsf{Quot}_d(V)) \to H^*(\mathsf{Quot}_{d+1}(V) \times C), \tag{5}$$

$$f_k = (p_- \times p_C)_*(\lambda^k \cdot p_+^*) : H^*(\mathsf{Quot}_{d+1}(V)) \to H^*(\mathsf{Quot}_d(V) \times C). \tag{6}$$

for all $k, d \geq 0$. It is also natural to use the Chern classes of $\mathcal{G}$ in order to define the operators

$$a_k = (p_+ \times p_C)_*\Big(c_k(\mathcal{G}) \cdot p_-^*\Big) : H^*(\mathsf{Quot}_d(V)) \to H^*(\mathsf{Quot}_{d+1}(V) \times C). \tag{7}$$

for $k \in \{0, \dots, r-1\}$ and all $d \geq 0$. We may also consider iterated compositions, and for any $\gamma \in H^*(C^n)$ the operator

$$a_{k_1} \dots a_{k_n}(\gamma) : H^*(\mathsf{Quot}_d(V)) \to H^*(\mathsf{Quot}_{d+n}(V))$$

obtained as

$$\begin{aligned} H^*(\mathsf{Quot}_d(V)) &\xrightarrow{a_{k_1}\dots a_{k_n}} H^*(\mathsf{Quot}_{d+n}(V) \times C^n) \xrightarrow{\cdot \rho^*(\gamma)} \\ &\longrightarrow H^*(\mathsf{Quot}_{d+n}(V) \times C^n) \xrightarrow{\pi_*} H^*(\mathsf{Quot}_{d+n}(V)), \end{aligned} \tag{8}$$

where π, $\rho : \mathsf{Quot}_d(V) \times C^n \to \mathsf{Quot}_d(V), C^n$ denote the two projections. The following result is our main structure theorem for the cohomology of $\mathsf{Quot}_d(V)$.

Theorem 1. *(1) For any $k, k' \in \{0, \ldots, r-1\}$ and $d \geq 0$, we have*

$$a_k a_{k'} = a_{k'} a_k \tag{9}$$

as operators $H^*(\mathsf{Quot}_d(V)) \to H^*(\mathsf{Quot}_{d+2}(V) \times C \times C)$, with the operator denoted by a_k (respectively $a_{k'}$) acting on both sides in the first (respectively second) factor of $C \times C$.

(2) For any $d \geq 0$, we have

$$H^*(\mathsf{Quot}_d(V)) = \bigoplus_{\gamma \in \mathbb{Q}\text{-basis of } H^*(C^d)_\Sigma}^{r > k_1 \geq \cdots \geq k_d \geq 0} \mathbb{Q} \cdot a_{k_1} \ldots a_{k_d}(\gamma)|0\rangle \tag{10}$$

where $H^*(C^d)_\Sigma$ denotes the space of coinvariants in $H^*(C^d)$ under the permutation of the i-th and j-th factors of C^d for any i and j such that $k_i = k_j$. Here $|0\rangle$ denotes the fundamental class of $\mathsf{Quot}_0(V) = point$.

We remark that the theorem offers a conceptual understanding of the Poincarë series of the Quot schemes $\mathsf{Quot}_d(V), d \geq 0$ calculated in [BGL, C].

In addition to the operators e_k, f_k of (5) and (6), we may also consider the operators of multiplication by universal classes

$$m_i = (\text{multiplication by } c_i(\mathcal{E})) \circ \pi^* : H^*(\mathsf{Quot}_d) \to H^*(\mathsf{Quot}_d \times C), \tag{11}$$

The algebra generated by the symbols

$$\Big\{ e_k, f_k, m_i \Big\}_{k \geq 0,\, i \in \{1, \ldots, r\}}$$

modulo their commutation relations is the shifted Yangian $Y_\hbar^r(\mathfrak{sl}_2)$ which thus acts on the cohomology $\mathsf{H}_{\mathsf{Quot}(V)} = \bigoplus_{d=0}^\infty H^*(\mathsf{Quot}_d(V))$ (Here $\hbar$ is a parameter appearing in the commutation relations.) We refer the reader to [MN1] for the full description of this algebra and its action on the cohomology of the Quot scheme.

We now describe a semiorthogonal decomposition of the derived category $\mathsf{D}_{\mathsf{Quot}_d(V)}$. Its blocks are labeled by compositions

$$\underline{d} = (d_0, d_1, \ldots, d_{r-1}), \quad d_0 + \cdots + d_{r-1} = d.$$

Associated with a composition, we consider the nested Quot scheme

$$\mathsf{Quot}_{\underline{d}}(V) = \Big\{ E^{(d)} \subset E^{(d-d_0)} \subset \cdots \subset E^{(d_{r-2}+d_{r-1})} \subset E^{(d_{r-1})} \subset V \Big\} \tag{12}$$

with length $V/E^{(i)} = i$. It is well known that $\mathsf{Quot}_{\underline{d}}(V)$ is a smooth fine moduli space of dimension rd, equipped with the universal flag of subbundles

$$\mathcal{E}^{(d)} \subset \mathcal{E}^{(d-d_0)} \subset \cdots \subset \mathcal{E}^{(d_{r-1})} \subset \rho^* V \text{ on } \mathsf{Quot}_{\underline{d}}(V) \times C.$$

We note the determinant line bundles

$$\mathcal{L}_1, \ldots, \mathcal{L}_r \to \mathsf{Quot}_{\underline{d}}(V) \tag{13}$$

given by

$$\mathcal{L}_i = \det \Big(R\pi_* \mathcal{E}^{(d_i + \cdots + d_{r-1})} - R\pi_* \mathcal{E}^{(d_{i-1} + \cdots + d_{r-1})} \Big), \ \ 1 \leq i \leq r. \tag{14}$$

(Note that $\mathcal{L}_r = (\det R\pi_*\mathcal{E}^{(d_{r-1})})^{-1}$.) We set

$$\mathcal{L}_{\underline{d}} = \bigotimes_{i=1}^{r} \mathcal{L}_i^{i-1}. \tag{15}$$

We also record the support map

$$s_{\underline{d}} : \mathsf{Quot}_{\underline{d}}(V) \to C^{(d_0)} \times C^{(d_1)} \times \cdots \times C^{(d_{r-1})} = C^{(\underline{d})} \tag{16}$$

which keeps track of the support points of the quotients $E^{(d)}/E^{(d-d_0)}, ..., V/E^{(d_{r-1})}$, and the morphism

$$t_{\underline{d}} : \mathsf{Quot}_{\underline{d}} \to \mathsf{Quot}_d$$

which only remembers the deepest sheaf $E^{(d)}$ in a flag. The following definition introduces the functors which are the building blocks of the derived category $\mathsf{D}_{\mathsf{Quot}_d}$.

Definition 2. For any composition $\underline{d}$ of d, we let

$$\widetilde{e}_{\underline{d}}^{\,red} : \mathsf{D}_{C^{(\underline{d})}} \to \mathsf{D}_{\mathsf{Quot}_d}$$

denote the string

$$\widetilde{e}_{\underline{d}}^{\,red} : \mathsf{D}_{C^{(\underline{d})}} \xrightarrow{Ls_{\underline{d}}^*} \mathsf{D}_{\mathsf{Quot}_{\underline{d}}} \xrightarrow{\otimes\mathcal{L}_{\underline{d}}} \mathsf{D}_{\mathsf{Quot}_{\underline{d}}} \xrightarrow{Rt_{\underline{d}*}} \mathsf{D}_{\mathsf{Quot}_d}. \tag{17}$$

It can be shown (cf. [MN2]) that the functors $\widetilde{e}_{\underline{d}}^{\,red}$ are suitably symmetrized compositions of elementary functors $\tilde{e}_i$ which lift to the derived category the cohomological operators (5). Following Definition (2), we have the following structural result for $\mathsf{D}_{\mathsf{Quot}_d(V)}$.

Theorem 3. *The functors $\widetilde{e}_{\underline{d}}^{\,\mathrm{red}}$ as $\underline{d} = (d_0, \ldots, d_{r-1})$ runs over compositions of d are fully faithful and semi-orthogonal, i.e. we have natural identifications*

$$\mathrm{Hom}_{\mathsf{Quot}_d}\left(\widetilde{e}_{\underline{d}}^{\,\mathrm{red}}(\gamma), \widetilde{e}_{\underline{d}'}^{\,\mathrm{red}}(\gamma')\right) \cong \begin{cases} 0 & \text{if } \underline{d} < \underline{d}' \text{ lexicographically} \\ \mathrm{Hom}_{C^{(\underline{d})}}(\gamma, \gamma') & \text{if } \underline{d} = \underline{d}' \end{cases} \tag{18}$$

The essential images of the functors $\widetilde{e}_{\underline{d}}^{\,\mathrm{red}}$ generate $\mathsf{D}_{\mathsf{Quot}_d}$ as a triangulated category.

We note that a semi-orthogonal decomposition of $\mathsf{D}_{\mathsf{Quot}_d(V)}$ indexed by the same data was obtained in [To] by a different method, using categorical wall-crossing for the framed one-loop quiver and the machinery of windows and categorical Hall prod- ucts. It would be interesting to match the decomposition [To] with our geometrically explicit construction.

If M is a line bundle on the curve C, let

$$M^{[d]} = \pi_*(\mathcal{F} \otimes \rho^* M)$$

be its corresponding tautological vector bundle of rank d on $\mathsf{Quot}_d(V)$. The semiorthogonal decomposition of Theorem 3 can be effectively used to calculate the cohomology of tautological bundles over the Quot scheme. As an example, we state the following result (cf. [MN2]).

Theorem 4. *We have* $\wedge^\ell M^{[d]} = \widetilde{e}^{\,red}_{(d-\ell,\ell,0,\dots,0)}(\mathcal{O}^{(d-\ell)} \boxtimes M^{(\ell)})$ *and*

$$H^\bullet\left(\mathsf{Quot}_d(V), \wedge^\ell M^{[d]}\right) \cong \wedge^\ell H^\bullet(C, V\otimes M) \bigotimes S^{d-\ell} H^\bullet(C,\mathcal{O}_C).$$

(Here we write $M^{(\ell)} \to C^{(\ell)}$ *for the descent of any line bundle* $M^\ell \to C^\ell$ *under the action of the symmetric group* S_ℓ*.)*

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Canonical decompositions for derived categories of G-surfaces and rationality

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(joint work with Alexey Elagin, Julia Schneider)

We work with smooth projective algebraic varieties over a perfect field $\mathbf{k}$. One of the most important questions about derived categories in algebraic geometry is: given a birational map $X \dashrightarrow Y$, how are $\mathcal{D}^b(X)$ and $\mathcal{D}^b(Y)$ related? A key observation in this direction is that if $f\colon Y \to X$ is a *derived contraction*, that is a morphism with $Rf_*\mathcal{O}_Y \simeq \mathcal{O}_X$ (e.g. a smooth blow up), then the derived pullback $Lf^*\colon \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ is fully faithful and we have a semiorthogonal decomposition

$$\mathcal{D}^b(Y) = \langle \mathcal{K}, Lf^*\mathcal{D}^b(X)\rangle, \quad \text{where } \mathcal{K} = \ker(Rf_*).$$

The kernel category $\mathcal{K}$ decomposes further in special cases (blow ups, Fano fibrations etc). Thus derived categories of birational varieties are expected to have common pieces. A specific particular case is the following:

Conjecture 1 (Kuznetsov)**.** Consider $X \subset \mathbb{P}^5$ a smooth complex cubic fourfold, so that $\mathcal{D}^b(X) = \langle \mathcal{A}_X, \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)\rangle$. Then

$$X \text{ is rational} \iff \mathcal{A}_X \simeq \mathcal{D}^b(S) \text{ for a surface } S.$$

The reasoning for the $\implies$ direction in Kuznetsov's conjecture is as follows. If semiorthogonal decompositions into indecomposable pieces were unique up to mutations, then every rational variety X would satisfy

$$\mathcal{D}^b(X) = \langle \mathcal{B}_1, \dots, \mathcal{B}_n\rangle \quad \text{with} \quad \mathcal{B}_j \subset \mathcal{D}^b(Z_j), \quad \dim Z_j \le \dim(X) - 2.$$

As $\mathcal{A}_X$ in Kuznetsov's conjecture is a K3 category, the only possibility for a blow up center is a K3 surface S.

However, uniqueness of decompositions into indecomposable pieces fails already in dimension two, so modifications are needed to make this strategy work.

Conjecture 2 (Kontsevich)**.** There should exist canonical semiorthogonal decompositions, well-defined up to mutations.

Canonical here can be understood in the following ways:

- Geometric: if $Y \to X$ is a blow up, Fano fibration etc, then canonical decompositions of $\mathcal{D}^b(Y)$ and $\mathcal{D}^b(X)$ are compatible.
- Matching quantum cohomology via the Gamma class and Iritani's blow up formulas.

Work in progress by Katzarkov, Kontsevich, Pantev and Yu constructs a canonical decomposition of quantum cohomology into elementary blocks called *atoms*. Motivated by this we make the following definition, for a fixed group G:

Definition 1. A G-atomic theory is an assignment for all smooth projective varieties with a G-action a mutation-equivalence class of G-invariant semiorthogonal decompositions, called atomic decompositions with the following property. For any G-equivariant derived contraction $f : Y \to X$ with rationally connected fibers $f\colon Y \to X$ atomic decomposition for $\mathcal{D}^b(X) = \langle \mathcal{A}_1, \dots, \mathcal{A}_n\rangle$, included into

$$\mathcal{D}^b(Y) = \langle \ker(Rf_*), Lf^*\mathcal{A}_1, \dots, Lf^*\mathcal{A}_n\rangle$$

can be refined to an atomic decomposition of $\mathcal{D}^b(Y)$.

For a group G we allow in particular the *geometric case* $k = \overline{\mathbf{k}}$, $G \subset \mathrm{Aut}(X/\mathbf{k})$ and the *arithmetic case* $G = \mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})$. Our main result is the following:

Theorem 1. *There is a canonical atomic theory for G-varieites of dimension ≤ 2.*

This proves Kontsevich' Conjecture in dimension 2 and improves an earlier result by Auel–Bernardara who introduced Kuznetsov component for geometrically rational surfaces in the arithmetic case. If $H \subset G$ is a subgroup, then the G-atomic theory we construct is compatible with the H-atomic theory, that is, G-atomic decompositions can be refined to H-atomic decompositions.

Example 2. In the simplest case $\mathbf{k} = \overline{\mathbf{k}}$ and $G = \{e\}$, for rational surfaces our result says the following. There are two types of minimal rational surfaces $\mathbb{P}^2$ and Hirzebruch surfaces $\mathbb{F}_n$, $n > 0$, $n \neq 1$, with atomic decompositions

$$\mathcal{D}^b(\mathbb{P}^2) = \langle \mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O}\rangle, \quad \mathcal{D}^b(\mathbb{F}_n) = \langle \mathcal{O}(-E-F), \mathcal{O}(-E), \mathcal{O}(-F), \mathcal{O}\rangle.$$

Here F is the fiber class and E is the negative section class. Now if X is any rational surface with minimal contractions $\pi_i\colon X \to X_i$, for $i = 1, 2$, then two exceptional collections on X induced from X_1 and X_2 by the blow up formula are both atomic decompositions, in particular they are mutation equivalent.

We refer to pieces of atomic semiorthogonal decompositions of $\mathcal{D}^b(X)$ as *atoms*; these pieces are independent of the choice an atomic decomposition as mutations simply permute the atoms of a decomposition. Conjecturally atomic decompositions we construct correspond to quantom cohomology atoms constructed by Katzarkov–Kontsevich–Pantev–Yu.

Sketch of proof of Theorem 1. There are three steps in the proof:

(1) Define atomic decompositions for minimal surfaces. The most important case is that of G-Mori fiber spaces, i.e. G-del Pezzo surfaces of rank one and G-conic bundles of rank two.
(2) Prove compatibility with Sarkisov links between G-Mori fibers spaces:

$$\begin{array}{ccc} X & \dashrightarrow & X' \\ {\scriptstyle Mfs}\downarrow & & \downarrow{\scriptstyle Mfs} \\ B & & B' \end{array}$$

By Iskovskikh' classification of Sarkisov links, there are only finitely many (but still a lot) things to check here.
(3) Decompose birational maps into Sarkisov links.

□

In the arithmetic case we deduce a complete birational classification of geometrically rational surfaces in terms of their atoms. In particular we have:

Corollary 1. *Let $X/\mathbf{k}$ be a surface. Then:*

$$X \text{ is rational} \iff \text{atoms of } \mathcal{D}^b(X) \text{ are } \mathcal{D}^b(L) \text{ for finite field extensions } L/\mathbf{k}.$$

This is the baby analog of Kuznetsov's Conjecture 1.

Proof. Given a birational map $\mathbb{P}^2 \dashrightarrow X$, atoms of X will come from blow up centers. Conversely, for the other direction we can assume X is minimal in which case the result follows by inspection. □

Example 3 (Châtelet surface)**.** A Châtelet surface surface $X \to \mathbb{P}^1$ is a conic bundle $y^2 - az^2 = P(x)$ with four singular fibers and a prescribed Galois action. It is known that X is irrational, but stably rational, more precisely by a result of Shepherd-Barron, $X \times \mathbb{P}^2$ is rational. It has atomic decomposition

$$\mathcal{D}^b(X) = \langle \mathcal{A}_X, \mathcal{O}(-h), \mathcal{O} \rangle$$

where h is the pullback of an ample generator from $\mathbb{P}^1$. Here $\mathcal{A}_X$ is not equivalent to any $\mathcal{D}^b(L)$ which is in agreement with Corollary 1.

Furthermore, it is easy to see that $\mathcal{A}_X \not\simeq \mathcal{D}^b(Z)$ for any curve Z. Thus if we could extend atomic theories to dimension three, $\mathcal{A}_X$ will be an atom of $\mathcal{D}^b(X \times \mathbb{P}^1)$, and $X \times \mathbb{P}^1$ would be forced to be irrational. We conjecture that this is the case.

Hurwitz–Brill–Noether, K3 surfaces and stability conditions

Andrés Rojas
(joint work with Gavril Farkas, Soheyla Feyzbakhsh)

Given a smooth projective curve C of genus g, the essential invariants to understand its extrinsic geometry are the *Brill–Noether loci*

$$W^r_d(C) := \left\{ L \in \mathrm{Pic}^d(C) : h^0(C, L) \geq r + 1 \right\}.$$

When the curve C is general in moduli, $W^r_d(C)$ is irreducible, of the expected dimension $\rho(g, r, d) := g - (r + 1)(g - d + r)$, and smooth away from $W^{r+1}_d(C)$, as described by a collection of important results dating back to the late 1970s and early 1980s [5–7]. One of these results, the *Gieseker–Petri theorem*, found a remarkable proof by Lazarsfeld [11] by specialization to curves on $K3$ surfaces:

Theorem 1 (Lazarsfeld)**.** *Let (X, H) be a polarized $K3$ surface with $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$, and let $g := \frac{H^2}{2} + 1$. Then:*

(1) A general curve $C \in |H|$ satisfies the Petri condition. In particular, $W^r_d(C)$ has the expected dimension $\rho(g, r, d)$.
(2) If $\rho(g, r, d) < 0$, then $W^r_d(C)$ is empty for every smooth curve $C \in |H|$.

In striking contrast to earlier proofs by degeneration, Lazarsfeld's result provides the first examples of smooth, Brill–Noether general curves.

Hurwitz–Brill–Noether theory is a much more recent development, and concerns the Brill–Noether theory of curves C equipped with a degree k morphism $C \xrightarrow{f} \mathbb{P}^1$. Assuming without loss of generality that $d \leq g - 1$, let us consider the number

$$\rho_k(g, r, d) := \max_{0 \leq \ell \leq r} \left(\rho(g, r - \ell, d) - \ell k \right).$$

Recent results of Pflueger [12] and Jensen–Ranganathan [8], via tropical geometry, and of H. Larson [10] and Larson–Larson–Vogt [9], via degeneration, describe the geometry of $W^r_d(C)$ for a general degree k cover $f : C \to \mathbb{P}^1$ of genus g. In particular, via the notion of *splitting type* first considered by H. Larson, there is a good understanding of the irreducible components of $W^r_d(C)$ and their dimensions; it turns out that $\dim W^r_d(C) = \rho_k(g, r, d)$.

In the work [4], we develop a new approach to Hurwitz–Brill–Noether theory via Bridgeland stability conditions on elliptic $K3$ surfaces. More precisely, let X be a *degree k elliptic $K3$ surface*, namely: $\mathrm{Pic}(X) = \mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot E$, where H is an ample class with $H^2 = 2g - 2$, and $|E|$ is an elliptic pencil satisfying $E \cdot H = k$. In this way, integral curves $C \in |H|$ are naturally endowed with the degree k map $f : C \hookrightarrow X \to \mathbb{P}^1$.

The following result may be thought of as a parallel, in the Hurwitz setting, of Lazarsfeld's theorem:

Theorem 2.

(1) A general curve $C \in |H|$ satisfies $\dim W^r_d(C) = \rho_k(g, r, d)$.
(2) If $\rho_k(g, r, d) < 0$, then $W^r_d(C)$ is empty for every smooth curve $C \in |H|$.

In particular, curves on elliptic $K3$ surfaces serve as the first known examples of smooth curves that behave generically from the viewpoint of Hurwitz–Brill–Noether theory. Indeed, genericity also extends to some important loci in $W^r_d(C)$, that can be defined for a degree k cover $f : C \to \mathbb{P}^1$.

More precisely, for a fixed ℓ with $\max\{0, r+2-k\} \leq \ell \leq r$, there are unique integers $e, m_1 \geq 0$ and $m_2 > 0$ satisfying

$$r + 1 = m_1(e + 2) + m_2(e + 1), \quad r + 1 - \ell = m_1 + m_2.$$

Then one can define the following locus in $W^r_d(C) \setminus W^{r+1}_d(C)$:

$$V^r_{d,\ell}(C, f) := \left\{ L \in \mathrm{Pic}^d(C) : \ f_*L = \mathcal{O}(e + 1)^{\oplus m_1} \oplus \mathcal{O}(e)^{\oplus m_2} \oplus N \right\},$$

where N means a direct sum of line bundles of negative degree. For appropriate values of ℓ, by [9, 10] the closures $\overline{V^r_{d,\ell}(C, f)}$ are known to describe the irreducible components of $W^r_d(C)$ for a general $f : C \to \mathbb{P}^1$. We prove the following:

Theorem 3. *Let X be a general degree k elliptic $K3$ surface as above. For $C \in |H|$ general, the loci $V^r_{d,\ell}(C, f)$ are smooth of the expected dimension $\rho(g, r - \ell, d) - \ell k$.*

The main technique to establish these results is wall-crossing with respect to stability conditions on the bounded derived category $\mathcal{D}(X)$. Roughly speaking Bridgeland stability, introduced in [2, 3], is a generalization of the usual notions of stability for coherent sheaves. It replaces $\mathrm{Coh}(X)$ by other appropriate abelian subcategories of $\mathcal{D}(X)$.

Associated to any polarization $H' \in \mathrm{Pic}(X)_{\mathbb{Q}}$ there is a ray of Bridgeland stability conditions σ_w $(w > 0)$ with a fixed abelian subcategory $\mathrm{Coh}^0(X) \subset \mathcal{D}(X)$. Stability conditions in this ray enjoy remarkable properties:

- For $w \gg 0$, σ_w-stability is equivalent to Gieseker stability for sheaves.
- Stability of objects varies along a locally finite wall and chamber structure.
- For σ_w lying outside a wall, there is a moduli space of σ_w-stable objects of a fixed Chern character.

If we consider the moduli space $\mathcal{M}$ parametrizing H'-Gieseker stable sheaves (or equivalently, σ_w-stable objects in $\mathrm{Coh}^0(X)$ for $w \gg 0$) of Chern character $(0, H, 1 + d - g)$, then $\mathcal{M}$ is naturally a relative degree d compactified Jacobian for curves in the linear system $|H|$. By understanding how σ_w-stability of objects in $\mathcal{M}$ varies as w decreases (via *wall-crossing* techniques), one can infer constraints on the geometry of the Brill–Noether loci $W^r_d(C)$ for $C \in |H|$. Whereas this philosophy can be easily applied for $K3$ surfaces of Picard rank one (see [1] for a reproof of Lazarsfeld's theorem), performing wall-crossing for $K3$ surfaces of higher Picard rank is usually very challenging.

In the problem at hand, we show that if the polarization is chosen carefully (namely, $H' = E + \epsilon H$ with $\epsilon \in \mathbb{Q}_{>0}$ small enough), then all possible destabilizations for objects of $\mathcal{M}$ along the ray σ_w involve line bundles of the form $\mathcal{O}_X(t_i E)$. This is part of a more general phenomenon, where other Chern characters apart from $(0, H, 1 + d - g)$ can be considered, that describes wall-crossing in a similar fashion. For a given object, the information provided by wall-crossing can then be encoded in a collection of integers, that we call its *Bridgeland stability type*.

After constructing moduli spaces of objects with a fixed Bridgeland stability type and exploiting their algebro-geometric properties (expected dimension, smooth, irreducibility), we obtain strong constraints from which our results follow.

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Asymptotic directions in the moduli space of curves

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(joint work with Paola Frediani, Gian Pietro Pirola)

The local geometry of the Torelli locus inside $\mathcal{A}_g$ is governed by the second fundamental form, which, at a non-hyperelliptic point, is a linear map $II : I_2 \to Sym^2 H^0(K_C^{\otimes 2})$, where I_2 is the vector space of quadrics containing the canonical curve (for an explicit description see [1], [2] and [3])

Our goal is to study the base locus of the image of II as a linear system of quadrics in $\mathbb{P}H^1(T_C) \cong \mathbb{P}^{3g-4}$ (see [4]). We call *asymptotic direction* a nonzero tangent vector $\zeta \in H^1(T_C)$ such that $II(Q)(\zeta \odot \zeta) = 0$ for all $Q \in I_2$.

Since a tangent direction to a totally geodesic subvariety is clearly asymptotic, this study is related to the Coleman-Oort conjecture according to which for g sufficiently high there should not exist special (or Shimura) subvarieties of $\mathcal{A}_g$ generically contained in the Torelli locus. We recall that special subvarieties of

$\mathcal{A}_g$ are totally geodesic and in the last years the study of the second fundamental form has been used to attack this problem, starting from [3].

For dimension reasons the intersection of the quadrics in $II(I_2)$ is non empty for $g \leq 9$ (II is injective, see [2]). In fact for $g \leq 7$ there are examples of special subvarieties of $\mathcal{A}_g$ generically contained in the Torelli locus but not for higher genus (see for example [5],[7], [6], [8]).

On the other hand for high values of g one could expect that this intersection would be empty. One of our results is that instead for all g there are examples of asymptotic directions depending on the geometry of the curve, in particular for any trigonal and bielliptic curve.

For our computation we develop a new technique to calculate $II(Q)(\zeta \odot \zeta)$ on certain tangent vectors by computing residues of some meromorphic forms, using the Hodge Gaussian maps introduced in [1]. Our technique works for ζ of not maximal rank, where the rank is the one of the associated infinitesimal variation fo Hodge structures $\cup\zeta : H^0(K_C) \to H^1(\mathcal{O}_C)$.

We denote by $C^{(d)}$ the symmetric product of C and we recall that the image of the natural map $\mathbb{P}T_{C^{(d)}} \to \mathbb{P}H^1(T_C)$ is given by classes of elements in $H^(C; T_C)$ in the annihilator of $H^0(C, 2K_C(-D))$ where D is an effective divisor of degree at most d hence it is clearly of rank at most d. When $d = 1$ the image is the bicanonical curve and the tangent vectors corresponding to such points are called Schiffer variations. One of the main results we obtain by application of these ideas is the following theorem.

Theorem 1. *Let C be a non-hyperelliptic curve of genus $g \geq 4$ and $d < Cliff(C)$ a positive integer, then:*

1. *the locus in $\mathbb{P}(H^1(T_C))$ of tangent directions of rank at most d is equal to the image of $\mathbb{P}T_{C^{(d)}}$*
2. *any tangent directions of rank at most d is not asymptotic.*

Notice that the first part of the above result can be seen as a generalization of the generic Torelli theorem of Griffiths.

In the case where the rank of ζ is equal to the Clifford index of the curve, we give sufficient conditions ensuring that the infinitesimal deformation ζ is not asymptotic. This allows us to determine all asymptotic directions of rank 1.

Theorem 2.

1. *If C is trigonal of genus $g \geq 8$, or of genus $g = 6, 7$ and Maroni degree 2, then rank one asymptotic directions are exactly the Schiffer variations in the ramification points of the g^1_3.*
2. *On a smooth plane quintic there are no rank one asymptotic directions.*

For trigonal curves of genus $g = 5$, or $g = 6, 7$ and Maroni degree 1, we show that there exist further asymptotic directions. We describe these asymptotic directions and we give the explicit equations of the trigonal curves admitting such asymptotic directions.

For bielliptic curves we show the following:

Theorem 3. *On any bielliptic curve of genus at least 5 there exist linear combinations of two Schiffer variations that are asymptotic of rank 2.*

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The D-equivalence conjecture for K3$^{[n]}$-type

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(joint work with Davesh Maulik, Qizheng Yin, Ruxuan Zhang)

Throughout, we work over the complex numbers $\mathbb{C}$. The purpose of my lecture is to present a proof of the D-equivalence conjecture for hyper-Kähler varieties of $K3^{[n]}$-type, *i.e.*, hyper-Kähler varieties that are deformation equivalent to the Hilbert scheme of n points on a $K3$ surface. Our method actually proves a stronger version, involving *arbitrary* Brauer classes.

Theorem 1 ([2])**.** *Let $X \dashrightarrow X'$ be a birational map of hyper-Kähler varieties of $K3^{[n]}$-type, which induces an identification of the Brauer groups*

$$\mathrm{Br}(X) = \mathrm{Br}(X').$$

Then for any class $\alpha \in \mathrm{Br}(X)$, we have a derived equivalence

$$D^b(X, \alpha) \simeq D^b(X', \alpha')$$

with $\alpha' \in \mathrm{Br}(X')$ the class corresponding to α.

When $\alpha = 0$, this result confirms in the case of $K3^{[n]}$-type the D-equivalence conjecture of Bondal–Orlov, which generally predicts that any birational projective nonsingular Calabi–Yau varieties are derived equivalent. The D-equivalence conjecture is known for Calabi–Yau 3-folds by Bridgeland and for hyper-Kähler fourfolds by Kawamata and Namikawa over 20 years ago. Very few examples are known in higher dimensions due to the complexity of birational geometry. Recently, Halpern–Leistner established the D-equivalence conjecture for nonsingular moduli of stable sheaves on a $K3$ surface using window techniques, confirming this conjecture for infinitely many families of higher dimensional examples. Since these moduli spaces are hyper-Kähler, Theorem 1 gives a new proof of Halpern–Leistner's result, but the construction of the equivalences seem to be quite different; it is interesting to explore the connection between these two methods.

Our proof of Theorem 1 relies on a class of projectively hyperholomorphic bundles by Markman [1]; Markman's work generalizes the earlier work of Buskin in his proof of the Shafarevich conjecture for rational Hodge isometries between $K3$ surfaces.

Now let S be a $K3$ surface, and let M be a 2-dimensional nonsingular fine moduli space of stable vector bundles on S; then M is also a $K3$ surface with a universal vector bundle

$$\mathcal{U} \rightsquigarrow M \times S.$$

Conjugating the Bridgeland–King–Reid correspondence, this gives rise to a vector bundle

$$\mathcal{U}^{[n]} \rightsquigarrow M^{[n]} \times S^{[n]}$$

of rank $n!\mathrm{rk}(\mathcal{U})$ on the product of the Hilbert schemes of points, which further induces a derived equivalence

$$\mathrm{FM}_{\mathcal{U}^{[n]}} : D^b(M^{[n]}) \xrightarrow{\simeq} D^b(S^{[n]}). \tag{1}$$

A key result proven by Markman in [1] is that the Fourier–Mukai kernel $\mathcal{U}^{[n]}$ is *projectively hyperholomorphic*. Hence it can be deformed as a twisted vector bundle along diagonal twistor paths.

Ideally, one hopes that the Fourier–Mukai equivalence (1) is deformed to a Fourier–Mukai equivalence

$$D^b(X, \alpha) \xrightarrow{\simeq} D^b(X', \alpha')$$

which immediately solves our problem. However, this is not possible: even if we consider the very special case that the birational transform is an identity $X \xrightarrow{\mathrm{id}} X' = X$ with trivial Brauer classes $\alpha = \alpha' = 0$, the derived equivalence $D^b(X) \xrightarrow{\simeq} D^v(X)$ should be induced by the structure sheaf of the diagonal $\Delta_X \subset X \times X$; the Fourier–Mukai kernel is a torsion sheaf, which is not locally free.

Instead, our strategy is to deform (1) to two derived equivalences

$$D^b(X, \alpha) \xrightarrow{\simeq} D^b(Y, \beta), \quad D^b(X', \alpha') \xrightarrow{\simeq} D^b(Y, \beta), \tag{2}$$

so that each equivalence is induced by a (projectively hyperholomorphic) bundle, and the composition recovers the desired equivalence of Theorem 1.

Although achieving (2) for a general birational map $X \dashrightarrow X'$ seems still difficult, we prove that we can decompose the birational map $X \dashrightarrow X'$ into a sequence of birational maps of hyper-Kähler varieties of $K3^{[n]}$-type

$$X = X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \cdots \dashrightarrow X_N = X'$$

such that for each $1 \leq i \leq N$, there exists a hyper-Kähler variety Y_i of $K3^{[n]}$-type with a Brauer class $\beta_i \in \mathrm{Br}(Y_i)$ achieving (2):

$$D^b(X_{i-1}, \alpha_{i-1}) \xrightarrow{\simeq} D^b(Y_i, \beta_i), \quad D^b(X_i, \alpha_i) \xrightarrow{\simeq} D^b(Y_i, \beta_i).$$

Here all the classes $\alpha_i \in \mathrm{Br}(X_i)$ are induced by the original one $\alpha \in \mathrm{Br}(X)$, and the number N measures the "distance" between the Kähler cones of X, X' respectively in the birational Kähler cone of X. Eventually, the desired derived equivalence of Theorem 1 is realized as a composition of $2N$ Fourier–Mukai transforms, each of which is induced by a projectively hyperholomorphic bundle deformed from $\mathcal{U}^{[n]}$ for a suitable choice of S and M.

We note that in Theorem 1 the Brauer class α can be chosen arbitrarily; the bundles we constructed in the proof in fact gives a nontrivial bound for the period–index problem — a long standing question measuring the difference of two important invariants associated with a Brauer class α: the period $\mathrm{per}(\alpha)$ and the index $\mathrm{ind}(\alpha)$. Pursuing this idea further, we obtain the following theorem.

Theorem 2 ([3]). *For any hyper-Kähler variety X of $K3^{[n]}$-type, there exists an integer N_X such that*

$$\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^{\dim(X)}$$

for all $\alpha \in \mathrm{Br}(X)$ with $\mathrm{per}(\alpha)$ corpime to N_X.

The famous period-index conjecture predicts that $N_X = 1$ and the exponent can be obtained to be $\dim(X) - 1$ for any X; this conjectual bound is close to and slightly stronger than the one we obtained in Theorem 2. However, Huybrechts [4] recently conjectured that the exponent for a hyper-Kähler variety X can be reduced to $\frac{1}{2}\dim(X)$, which is much stronger than what we can obtain for the moment from Markman's projectively hyperholomorphic bundles.

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Chen ranks formulae

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(joint work with Gavril Farkas, Claudiu Raicu, and Alexandru Suciu)

Chen ranks are fundamental invariants in geometric group theory, derived from the lower central series of the maximal metabelian quotient of a finitely–generated group. Their effective computation, however, requires tools from algebraic geometry, a mathematical field with different methods and perspectives.

The setup is as follows. Let G be a finitely generated group, and consider the complex vector space

$$V^\vee := H^1(G, \mathbb{C})$$

whose dimension is $n = b_1(G)$. Define also

$$K^\perp := \ker\left(\bigwedge^2 H^1(G,\mathbb{C}) \xrightarrow{\cup_G} H^2(G,\mathbb{C})\right)$$

the kernel of the cup–product map. Note that $K^\perp$ is the algebraic orthogonal of a subspace $K \subseteq \bigwedge^2 V = H_1(G,\mathbb{C})$, which coincides with the image of the dual of the map $\cup_G$. Let $S = \mathrm{Sym}(V)$ denote the symmetric algebra of V.

The homology $W(G)$ of the complex of graded S–modules

$$K \otimes S \to V \otimes S(1) \to S(2) \tag{1}$$

naturally induced by the Koszul complex is an infinitesimal version of the Alexander invariant and is called the *Koszul module* of the group G. It was shown in [5] that, if the group is *1–formal* in the sense of rational homotopy theory, then the Chen ranks of the group coincide with the dimensions of the graded pieces of $W(G)$ (up to a shift by two in degrees).

Note that the complex (1) can be defined for any n–dimensional complex vector space V and any subspace $K \subseteq \bigwedge^2 V$ not necessarily associated to a group. In that case, the homology $W(V,K)$ of the complex (1) is called the *Koszul module* of the pair (V,K), see [6].

We are immediately lead to the following fundamental question:

Question. Given a pair (V,K) as above, can we effectively compute the Hilbert series of the Koszul module $W(V,K)$?

We can answer this question under certain hypotheses on the support of the Koszul module. Recall from [6] that the set-theoretic support $\mathcal{R}(V,K)$ of the Koszul module $W(V,K)$ is the cone in $V^\vee$ defined by

$$\mathcal{R}(V,K) := \left\{a \in V^\vee : \text{there exists } b \in V^\vee \setminus \mathbb{C}\cdot a \text{ such that } a \wedge b \in K^\perp\right\}.$$

This set, called the *resonance* of the pair (V,K), naturally carries a scheme structure defined by the annihilator of $W(V,K)$. As such, it is the affine cone over a projective scheme

$$\mathbb{R}(V,K) \subseteq \mathbb{P}(V^\vee),$$

called the *projectivized resonance scheme.* Set–theoretically, the projectivized resonance is closely related to the intersection

$$\mathrm{Gr}_2(V^\vee) \cap \mathbb{P}(K^\perp) \subseteq \mathbb{P}(\textstyle\bigwedge^2 V^\vee)$$

via the incidence variety, see for example [4]. Specifically, if

$$\Xi := \{(x,L) \in \mathbb{P}(V^\vee) \times \mathrm{Gr}_2(V^\vee) : \; x \in L\}$$

denotes the incidence, and $\mathrm{pr}_1 : \Xi \to \mathbb{P}(V^\vee)$, $\mathrm{pr}_2 : \Xi \to \mathrm{Gr}_2(V^\vee)$ are the natural projection maps, then we have the following set–theoretical identification

$$\mathbb{R}(V,K) = \mathrm{pr}_1(\mathrm{pr}_2^{-1}(\mathrm{Gr}_2(V^\vee) \cap \mathbb{P}(K^\perp))).$$

In particular, $\mathrm{Gr}_2(V^\vee) \cap \mathbb{P}(K^\perp) = \emptyset$ if and only if $\mathbb{R}(V,K) = \emptyset$, and $\mathrm{Gr}_2(V^\vee) \cap \mathbb{P}(K^\perp)$ is finite if and only if $\mathbb{R}(V,K)$ is a union of disjoint projective lines i.e. is non–empty of minimal dimension.

Our first answer to the main question is given by the following:

Theorem 1 (see [1], [2]). If $\mathcal{R}(V,K) = \{0\}$, then $W_q(V,K) = 0$ for all $q \geq n-3$.

Note that, since the resonance is the support of the Koszul module, a trivial resonance locus corresponds to a Koszul module of finite length. However, the theorem above provides an explicit and optimal vanishing bound. In algebraic geometry, this has direct applications to the study of syzygies of generic canonical curves, via an explicit version of Hermite reciprocity, as shown in [1].

When the resonance is non-trivial, the first result is the following:

Theorem 2 (see [3]). Assume that the scheme-theoretic intersection $\mathrm{Gr}_2(V^\vee) \cap \mathbb{P}(K^\perp)$ is finite of length ℓ. Then

$$\dim W_q(V,K) = (q+1)\ell \text{ for all } q \geq n-3.$$

The proofs of Theorems 1 and 2 are similar, relying on a careful analysis of the hypercohomology spectral sequence associated to a twisted Koszul complex, in which Bott's theorem is used to compute the cohomology of vector bundles on Grassmannians, see [1], [2], [3].

To analyze the case where the resonance loci have higher-dimensional components, we use a condition we call *strong–isotropy*, see [4]. Specifically, we say that resonance $\mathcal{R}(V,K)$ is *strongly isotropic* if the following three conditions hold true:

- The resonance is *linear*, i.e. $\mathcal{R}(V,K) = \overline{V}_1^\vee \cup \cdots \cup \overline{V}_k^\vee$, where each $\overline{V}_t^\vee \subset V^\vee$ is a linear subspace and $\overline{V}_t^\vee \not\subseteq \overline{V}_s^\vee$ for all t, s.
- The components are *isotropic*, i.e., $\bigwedge^2 \overline{V}_t^\vee \subseteq K^\perp$ for all t.
- The components are *separable*, i.e., for each t, if $\langle \overline{V}_t^\vee \rangle$ denotes the ideal generated by $\overline{V}_t^\vee$ in the exterior algebra of $V^\vee$, then
$$\langle \overline{V}_t^\vee \rangle \cap K^\perp \subseteq \bigwedge^2 \overline{V}_t^\vee .$$

One consequence of strong–isotropy is that the scheme-theoretic intersection $\mathrm{Gr}_2(V^\vee) \cap \mathbb{P}(K^\perp)$ is the disjoint union of the sub-Grassmannians $\mathrm{Gr}_2(\overline{V}_t^\vee)$, see [3]. In this setting, we prove:

Theorem 3 (see [3]). Assume that the resonance of the pair (V,K) is strongly isotropic, and write $\mathcal{R}(V,K) = \overline{V}_1^\vee \cup \cdots \cup \overline{V}_k^\vee$. Then

$$\dim W_q(V,K) = \sum_{t=1}^{k} (q+1) \binom{q + \dim(\overline{V}_t^\vee)}{q+2} \quad \text{for all } q \geq n-3. \tag{2}$$

One of the applications of Theorem 3 is to (certain) hyperplane arrangement groups for which an asymptotic formula of type (2) was conjectured. Note that an asymptotic version of Theorem 3 appears already in [4], and the version stated here strengthens that result by providing an explicit effective bound.

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On the integral Hodge conjecture for abelian varieties

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(joint work with Phil Engel, Olivier de Gaay Fortman)

Clemens and Griffiths [CG72] proved that a rationally connected threefold is irrational if its intermediate Jacobian is, as principally polarized abelian variety, not isomorphic to a product of Jacobians of curves. They applied this criteria to the intermediate Jacobian of a smooth cubic threefold, thereby establishing the irrationality of such cubics. The question whether smooth cubic threefolds are stably rational remained open.

By Matsusaka's criterion, one may reformulate the Clemens–Griffiths criterion by saying that a rationally connected threefold Y is irrational if the minimal curve class $[\Theta_Y]^{g-1}/(g-1)! \in H_2(JY,\mathbb{Z})$ of its intermediate Jacobian (JY,Θ_Y) is not represented by an effective curve. In [Voi17], Voisin generalized this criterion by proving that Y does not admit a decomposition of the diagonal (hence is in particular not stably rational, nor retract rational nor $\mathbb{A}^1$-connected, see e.g. [LS24] and the references therein), if the minimal curve class of its intermediate Jacobian is not algebraic, i.e. not represented by a $\mathbb{Z}$-linear combination of curves. Voisin showed in [Voi17] that for many codimension 3 loci in the moduli space of cubic threefolds, the criterion fails and the associated cubic has in fact a decomposition of the diagonal.

In many cases of interest, JY is a Prym variety, in which case $2[\Theta_Y]^{g-1}/(g-1)!$ is always algebraic. If, in addition, JY has Picard rank one, then Voisin's criterion essentially boils down to the question whether JY contains a curve whose cohomology class is some odd multiple of the minimal class. The case of cubic threefolds has been a classical open problem until now. In fact, the validity of the integral Hodge conjecture for abelian varieties has been an open question until now; the problem goes back at least to Barton and Clemens, see [BC77, p. 66].

In this talk, we presented the following result.

Theorem 1. *Let $Y \subset \mathbb{P}^4_{\mathbb{C}}$ be a very general cubic hypersurface. Then the homology class of any curve $C \subset JY$ on its intermediate Jacobian JY is an even multiple of the minimal class $[\Theta_Y]^4/4!$.*

By [Voi17], the theorem implies:

Corollary 2. *Very general cubic threefolds $Y \subset \mathbb{P}^4_{\mathbb{C}}$ do not admit a decomposition of the diagonal, hence they are neither stably nor retract rational, nor $\mathbb{A}^1$-connected.*

Our obstruction also works for very general principally polarized abelian varieties of dimension $g \geq 4$. For such abelian varieties, we show that the homology class of any curve is an even multiple of the minimal class.

1. Motivation of the obstruction. If (X, Θ) is a principal polarized abelian variety and $f\colon JC \to X$ is a morphism from a Jacobian of a curve C such that $f_*[C] = m \cdot [\Theta^{g-1}]/(g-1)!$, then the composition

$$X \xrightarrow{f^\vee} JC \xrightarrow{f} X$$

is multiplication by m, where $f^\vee$ denotes the dual of f and where we use the principal polarizations on X and JC to identify those abelian varieties canonically with their duals. Assume now that m is odd and let $\Lambda := \mathbb{Z}_{(2)}$. Then m is invertible in Λ and so $\frac{1}{m} f^\vee_*$ splits f_*. Hence, we get a canonical decomposition

$$H_1(JC, \Lambda) = f^\vee_* H_1(X, \Lambda) \oplus \ker(f_*).$$

Since $(f^\vee)^*\Theta_C = m \cdot \Theta$, we have $f^\vee_* H_1(X, \Lambda) \cong H_1(X, \Lambda)$. It thus suffices to show that $H_1(JC, \Lambda)$ does not have $H_1(X, \Lambda)$ as a direct factor. Of course, it is impossible to prove this directly on the level of Λ-modules for a single X, but we will be able to do so for very general X after spreading out everything to families, at which point monodromy operators are at our disposal.

We now outline how to prove this for the intermediate Jacobian of a cubic threefold; the case of very general principally polarized abelian varieties of dimension at least 4 is similar. We start with the Segre cubic threefold

$$Y_0 = \left\{ \sum_{i=0}^{5} x_i = \sum_{i=0}^{5} x_i^3 = 0 \right\} \subset \mathbb{P}^5_{\mathbb{C}},$$

which is the unique cubic threefold with 10 nodes. We denote its set of nodes by S and let

$$Y \to \mathrm{Def}_{Y_0} = \Delta^S = \Delta^{10}$$

be the universal deformation, which is smooth over the punctured polydisc $(\Delta^*)^S$ and such that the node $s \in S$ does not smooth out over the coordinate hyperplane $\{t_s = 0\}$. Fix a base point $t \in (\Delta^*)^S$ and fix for $s \in S$ a vanishing cycle

$$\alpha_s \in H_3(Y_t, \mathbb{Z}(-1)) = H_1(JY_t, \mathbb{Z}),$$

unique up to a sign. The collection of these classes yields an integral realization of the R_{10} matroid, see [Gwe04]. Moreover, the above vanishing cycles are contained in W_{-2} of the corresponding limit mixed Hodge structure. The principal polarization on JY_t identifies $W_{-2}H_1(JY_t,\mathbb{Z})$ to the dual of $U \coloneqq \mathrm{gr}_0^W H_1(JY_t,\mathbb{Z})$. We thus get a realization

$$S \longrightarrow U^*, \qquad s \mapsto y_s$$

of $\underline{R}_{10}$, where y_s denotes the linear form on U that is induced by the vanishing cycles α_s. The nilpotent operator $N_s = T_s - \mathrm{id}$ identifies via the given principal polarization to a monodromy bilinear form B_s on U, which by the Picard–Lefschetz formula identifies to $B_s = y_s^2$, i.e. to $x \otimes y \mapsto y_s(x)y_s(y)$.

Consider now a family of nodal curves $C \to \Delta^S$, smooth over $(\Delta^*)^S$, and with regular total space. Then the dual graph $G \coloneqq \Gamma(C_0)$ is an S-colored graph, i.e. there is a decomposition $E(G) = \bigsqcup_{s\in S} E_s$ of its set of edges, by declaring that an edge has color s if the corresponding node does not smooth out over the general fibre of $\{t_s = 0\}$. (The regularity assumption on C implies that nodes over different coordinate hyperplanes never specialize to the same node on the special fibre, so that this definition is well-defined.) Note further that there is a canonical isomorphism

$$\mathrm{gr}_0^W H_1(JC_t,\mathbb{Z}) \cong H_1(G,\mathbb{Z}).$$

The monodromy about $\{t_s = 0\}$ induces as before a monodromy bilinear form Q_s on the above free $\mathbb{Z}$-module and the Picard–Lefschetz formula implies $Q_s = \sum_{e\in E_s} x_e^2$, where $x_e \in H_1(G,\mathbb{Z})^*$ is the linear form induced by the edge e (together with the choice of an orientation).

To motivate our obstruction, we make now the strong assumption that for $\Lambda = \mathbb{Z}_{(2)}$, we can find a family of curves as above such that for general $b \in \Delta^S$, $H_1(JC_b,\Lambda)$ contains $H_1(X_b,\Lambda)$ as a direct summand and this extends to a decomposition of the corresponding local system on $(\Delta^*)^S$. This implies that the monodromy operators respect the direct sum decomposition. We thus get a direct sum decomposition

$$H_1(G,\Lambda) \cong \mathrm{gr}_0^W H_1(C_t,\Lambda) = U_\Lambda \oplus U'$$

for some $U' \subset H_1(G,\Lambda)$. Moreover, the above decomposition is orthogonal with respect to the monodromy quadratic form Q_s and we have $Q_s|_{U_\Lambda} = B_s$ for all $s \in S$.

If the above family of curves does not exist on the nose but only after a base change of the form $\Delta^S \to \Delta^S$, $(t_s) \mapsto (t_s^d)$ (which is still a very strong assumption), then we will get a Q_s-orthogonal decomposition as above, but with $Q_s|_{U_\Lambda} = d \cdot B_s$ for all $s \in S$. This motivates the following definition:

Definition 1. Let $(\underline{R}, S)$ be a regular matroid with integral realization $S \to U^*$, $s \mapsto y_s$. Let Λ be a ring. A *quadratic Λ-splitting of level d* of $(\underline{R}, S)$ in a graph G is an S-coloring $E = \sqcup_{s\in S} E_s$ of the edges of G together with an embedding $U_\Lambda \hookrightarrow H_1(G,\Lambda)$ which induces a decomposition

$$H_1(G,\Lambda) = U_\Lambda \oplus U', \tag{1}$$

for some $U' \subset H_1(G, \Lambda)$, such that for all $s \in S$ the following holds for the s-th diagonal quadratic form $Q_s = \sum_{e \in E_s} x_e^2$ on $H_1(G, \Lambda)$:

(1) the decomposition (1) is orthogonal with respect to Q_s;
(2) the restriction of Q_s to U_Λ agrees with $d \cdot B_s = d \cdot y_s^2$.

2. Outline. In order to prove Theorem 1, we proceed as follows:

- We show that $\underline{R}_{10}$ does not admit a quadratic $\mathbb{Z}_{(2)}$-splitting of level 1 into a graph. To this end, we introduce Albanese graphs associated to matroids and reduce the problem to the computation of what we call "solutions of the Albanese graph", which are certain collections of s-colored 1-chains b_s that satisfy linear relations that are dictated by the matroid. We then show that all solutions are even (i.e. they contain an even number of edges of each color), while the existence of an splitting as above implies that there are odd solutions. Technically speaking, the problem reduces to showing that a certain sparse 160×160 matrix over $\mathbb{F}_2$ has the same rank as a matrix of rank 160×170 that is given by adding 10 additional columns.
- We prove a technical theorem that allows us to essentially reduce a quadratic Λ-splitting of level d into a graph to one of level 1. (The actual statement proven in [EGFS25] is slightly different and involves the aforementioned solutions in Albanese graphs; morally it allows us to go from level d to level 1.)
- We show that if $X \to \Delta^S$ is a matroidal degeneration of principally polarized abelian varieties, associated to a regular matroid $(\underline{R}, S)$ without loops and with totally unimodular realization $S \to U^*$, such that the very general fibre contains a curve whose cohomology class is an odd multiple of the minimal class, then $(\underline{R}, S)$ admits a quadratic Λ-splitting of some level d into some S-colored graph G.

In this talk the method is mostly described in the case of the intermediate Jacobian of very general cubics, where the problem is 2-local and so we worked with $\mathbb{Z}_{(2)}$-coefficients. In [EGFS25], the general case of $\mathbb{Z}_{(\ell)}$-coefficients and arbitrary regular matroids $(\underline{R}, S)$ will be considered. Moreover, we show in [EGFS25] the following statement which generalizes the first item above:

Theorem 2. *Let $(\underline{R}, S)$ be a regular matroid. Then $(\underline{R}, S)$ admits a $\mathbb{Z}_{(2)}$-splitting of some level $d \geq 1$ into some graph if and only if $(\underline{R}, S)$ is cographic.*

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Counting maps to an elliptic curve in several ways

Aitor Iribar López
(joint work with Rahul Pandharipande, Hsian-Hua Tseng.)

We report on joint work with R. Pandharipande and H.-H. Tseng on the enumerative geometry of the space of maps $C \to E$, when C is a genus g curve, E is an elliptic curve, and both curves move in moduli. Three approaches to this problem are discussed. They connect to 3 open problems, related to intersection theory on the moduli space of g-dimensional principally polarized abelian varieties $\mathcal{A}_g$, the Gromov-Witten theory of the universal elliptic curve $\mathcal{E} \longrightarrow \overline{\mathcal{M}}_{1,1}$ and the $g = 1$ Gromov-Witten theory of the Hilbert scheme of points in $\mathbb{C}^2$.

1. Cycles on $\mathcal{A}_g$

G. van der Geer introduces in [8] the *tautological ring* $\mathsf{R}^*(\mathcal{A}_g)$ as the subring of $\mathsf{CH}^*(\mathcal{A}_g)$ generated by the Chern classes of the Hodge bundle:

$$\lambda_i = c_i(\mathbb{E}) \in \mathsf{CH}^i(\mathcal{A}_g)\,.$$

The only relations are $\lambda_g = 0$ and $c(\mathbb{E} \oplus \mathbb{E}^\vee) = 1$, but most geometrically defined cycles are not known to be tautological. The relation between geometric cycles and tautological classes is governed by the projection operator defined by S. Canning, S. Molcho, D. Oprea and R. Pandharipande [1]:

$$\text{taut} : \mathsf{CH}^*(\mathcal{A}_g) \longrightarrow \mathsf{R}^*(\mathcal{A}_g)\,. \qquad (1)$$

The following basic question is introduced in [3]:

Question 1. *Is the tautological projection* (1) *a ring homomorphism?*

An abelian variety of dimension g can have Picard number greater than 1 in two ways: through real multiplication and through abelian subvarieties. We focus on the latter. Fix a positive integer d and consider the *Noether-Lefschetz locus*:

$$\mathsf{NL}_{g,d} = \{(X,\theta) \in \mathcal{A}_g \mid \text{there is a map } E \to X \text{ such that } \deg(f^*\theta) = d\}\,.$$

It gives rise to special cycles $[\mathsf{NL}_{g,d}] \in \mathsf{CH}^{g-1}(\mathcal{A}_g)$. We can also consider cycles given by the fundamental class of the Jacobian locus $[\mathcal{J}_g] \in \mathsf{CH}_{3g-3}(\mathcal{A}_g)$. We will consider their relation to that tautological projection (1).

2. Maps to a moving elliptic curve

Let $\pi : \mathcal{E} \longrightarrow \overline{\mathcal{M}}_{1,1}$ be the universal family, with section 0, and let $\overline{\mathcal{M}}_{g,1}(\pi, d)$ be the moduli space of relative maps to the fibers of π. It is a Deligne–Mumford stack with a virtual class $[\overline{\mathcal{M}}_{g,1}(\pi, d)]^{\mathrm{vir}}$ of expected dimension $2g$. Consider the diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,1}(\pi,d) & \xrightarrow{\mathrm{ev}} & \mathcal{E} \\ \Big\downarrow \mathrm{ft} & & \\ \overline{\mathcal{M}}_{g,1} & & \end{array} .$$

Gromov-Witten invariants of π are defined by the above correspondence:

$$\langle \Lambda \tau_k(0) \rangle_{g,1}^{\pi,d} := \int_{[\overline{\mathcal{M}}_{g,1}(\pi,d)]^{\mathrm{vir}}} \mathrm{ft}^* \Lambda \cdot \psi_1^k \, \mathrm{ev}^*[0] \,. \tag{2}$$

Determining these invariants in general is an open question since degeneration of the target elliptic curve to a rational curve is not allowed.

Question 2. *How to compute the Gromov-Witten invariants* $\langle \Lambda \tau_k(0) \rangle_{g,1}^{\pi,d}$*?*

3. Maps to the Hilbert scheme of points

Consider the Hilbert scheme of d points on the plane $\mathsf{Hilb}^d(\mathbb{C}^2)$, which carries an action of the torus $T = (\mathbb{C}^*)^2$. For any partition μ of d, there is a cohomology class

$$|\mu\rangle \in H_T^{2d-2l(\mu)}(\mathsf{Hilb}^d(\mathbb{C}^2)) \,,$$

and these form a basis of the T-equivariant cohomology over $H_T^*(pt)$. The divisor class $D = -\big|(1, 2^{d-2})\big\rangle$ is the first Chern class of the universal quotient, and generates H^2. A curve class β_n is defined by the condition that

$$\int_{\beta_n} D = n \,.$$

The Deligne–Mumford stack $\overline{\mathcal{M}}_{g,r}(\mathsf{Hilb}^d(\mathbb{C}^2), \beta_n)$ parametrizes space of stable maps to $\mathsf{Hilb}^d(\mathbb{C}^2)$ in the curve class β_n. It is equivariantly proper, carries a T-equivariant virtual class and evaluation maps

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{g,r}(\mathsf{Hilb}^d(\mathbb{C}^2), \beta_n) \longrightarrow \mathsf{Hilb}^d(\mathbb{C}^2) \,, \; i = 1, \ldots, r \,.$$

Gromov-Witten invariants of $\mathsf{Hilb}^d(\mathbb{C}^2)$ are defined by equivariant integration:

$$\langle \mu_1, \ldots, \mu_r \rangle_g^{\mathsf{Hilb}^d(\mathbb{C}^2)} = \sum_{n \geq 0} q^n \int_{[\overline{\mathcal{M}}_{g,r}(\mathsf{Hilb}^d(\mathbb{C}^2),\beta_n)]^{\mathrm{vir}}} \prod_{i=1}^r \mathrm{ev}_i^*(|\mu_i\rangle) \in \mathbb{Q}(t_1, t_2, q) \,. \tag{3}$$

Question 3. *Provide a method to compute the invariants* (3) *for all* g.

For $g = 0$, this was carried out by R. Pandharipande and A. Okounkov [5]. In this case, the theory is governed by the operator of quantum multiplication by D:

$$\mathsf{M}_D = D\star : H_T^*(\mathsf{Hilb}^d(\mathbb{C}^2))[[q]] \longrightarrow H_T^*(\mathsf{Hilb}^d(\mathbb{C}^2))[[q]] \,. \tag{4}$$

For $g > 0$, no closed expressions were known. In [6], Pandharipande asks for a closed expression for $\langle D \rangle_1^{\mathsf{Hilb}^d(\mathbb{C}^2)}$, which was later conjectured by R. Pandharipande and H.-H. Tseng.

4. Results

In the talk, the following triple equivalence between particular instances of Questions 1, 2 and 3 was discussed:

Theorem 1 ([3,4])**.** *Consider the following statements:*

(1) The projection (1) *is an homomorphism when applied to* $[\widetilde{\mathsf{NL}}_{g,d}]$ *and* $[\mathcal{J}_g]$*:*

$$\mathrm{taut}([\widetilde{\mathsf{NL}}_{g,d}] \cdot [\mathcal{J}_g]) = \mathrm{taut}([\widetilde{\mathsf{NL}}_{g,d}]) \cdot \mathrm{taut}([\mathcal{J}_g])\,.$$

(2) We can evaluate any Gromov-Witten invariant (2) *with a* λ_g *insertion:*

$$\langle \lambda_g \Lambda \tau_k(0) \rangle_{g,1}^{\pi,d} = \frac{g\,\sigma_{2g-1}(d)}{6|B_{2g}|} \int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \lambda_{g-1} \Lambda \psi_1^k\,.$$

(3) The invariant $\langle D \rangle_1^{\mathsf{Hilb}^d(\mathbb{C}^2)}$ *has the following closed expression:*

$$\langle D \rangle_1^{\mathsf{Hilb}^d(\mathbb{C}^2)} = \frac{-1}{24} \frac{(t_1+t_2)^2}{t_1 t_2} \left(\mathsf{Tr}_d + \sum_{e=1}^{d-1} \sigma_{-1}(d-e) \mathsf{Tr}_e \right)\,,$$

where $(t_1+t_2)\mathsf{Tr}_d \in \mathbb{Q}(t_1,t_2,q)$ *is the trace of the operator* M_D *in* (4).

Then, (*a*)*,* (*b*) *and* (*c*) *are equivalent, and moreover, all of them are true.*

Some parts of the proof, which depend on work of R. Pandharipande, H.-H. Tseng [7] and F. Greer, C. Lian [2] were explained. In the last part of the talk, the following result was discussed, which answers Question 3 for $g = 1$:

Theorem 2 ([4])**.** *For every d, there is a matrix* W_d *of size* $|\,\mathrm{Part}(d)\,| \times |\,\mathrm{Part(d)}\,|$ *with coefficients in* $\mathbb{Q}(t_1,t_2)[[q]]$ *such that, if* $\det(\mathsf{W}_d)$ *is not identically* 0*, then all the Gromov-Witten invariants* (3) *for* $g = 1$ *can be reconstructed from the knowledge of* M_D *and* $\langle D \rangle_1^{\mathsf{Hilb}^d(\mathbb{C}^2)}$*. For* $d \le 7$*,* $\det(\mathsf{W}_d) \neq 0$.

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Chow groups with twisted coefficients

Burt Totaro

It is natural to ask whether we can define the Chow group of algebraic cycles with twisted coefficients, as we can do for cohomology. Rost gave such a definition. Namely, he defined the Chow groups of a scheme X with coefficients in a locally constant étale sheaf E, assuming that E is torsion of exponent invertible in the base field k [4, Remarks 1.11 and 2.5]. We generalize the definition so that E need not be torsion (and, in characteristic $p > 0$, p need not act invertibly on E) [5]. It was not obvious how to make such a definition, because Chow groups are defined in terms of the Zariski topology, not the étale topology.

Definition 1. Let X be a separated scheme of finite type over a field k. Let E be a locally constant étale sheaf on X, and let i be an integer. The *twisted Chow group* $CH_i(X, E)$ is defined to be the cokernel of the residue map on étale cohomology groups of fields:

$$\oplus_{x \in X_{(i+1)}} H^1_{\text{et}}(k(x), E(1)) \to \oplus_{x \in X_{(i)}} H^0_{\text{et}}(k(x), E).$$

Here we interpret $\mathbf{Z}(1)$ as $G_m[-1]$, a shift of the multiplicative group in the derived category of étale sheaves over a field, as in Voevodsky's theory of motivic cohomology [2, Theorem 4.1]. Define $E(1)$ as the derived tensor product $E \otimes^L_{\mathbf{Z}} G_m[-1]$.

As we want, for the constant sheaf $\mathbf{Z}_X$, the twisted Chow group $CH_i(X, \mathbf{Z}_X)$ is the usual Chow group $CH_i X$.

Thanks to Rost's work on cycle modules, twisted Chow groups have essentially all the formal properties of the usual Chow groups: proper pushforward, flat pullback, localization sequence, homotopy invariance, products on smooth varieties, and pullback by arbitrary morphisms of smooth varieties [5, section 1]. Also, for X smooth over k, there are cycle maps from twisted Chow groups to twisted étale motivic cohomology, and (when $k = \mathbf{C}$) to ordinary cohomology with twisted coefficients [5, Theorems 6.1 and 7.1].

For a connected scheme X, we can think of a locally constant étale sheaf on X as a representation of the étale fundamental group of X. When this is a permutation representation, twisted Chow groups coincide with the usual Chow groups of a suitable covering space of X (in the topological sense), perhaps not connected [5, Lemma 2.1]. Note that every representation of a finite group G over the rationals $\mathbf{Q}$ is a summand of a permutation representation (such as the regular representation $\mathbf{Q}G$). As a result, twisted Chow groups tensored with the rationals are completely understood in terms of the usual Chow groups of covering spaces of X. Rational twisted Chow groups should be a useful formalism; but the more novel problem is to try to understand twisted Chow groups integrally or with finite coefficients.

A key feature of twisted Chow groups is that they are always generated by the usual Chow groups of covering spaces of X, in the following sense [5, Theorem 8.1].

Theorem 2. *Let X be a k-scheme of finite type, G a finite group, $Y \to X$ an étale G-torsor (so $X = Y/G$), and E a $\mathbf{Z}G$-module. Then $CH_i(X, E)$ is generated by the images of the homomorphisms*

$$E^H \otimes_{\mathbf{Z}} CH_i(Y/H) \to CH_i(Y/H, E) \to CH_i(X, E)$$

over all subgroups H of G, where the last map is the transfer or pushforward.

More strongly, we do not need to use all subgroups of G. For each element $x \in E$, let G_x be the centralizer of x in G. Then $CH_i(X, E)$ is generated by the elements $\mathrm{tr}_{G_x}^G(xy)$ for all $x \in E$ and all $y \in CH_i(Y/G_x)$.

The case of codimension-1 cycles in Theorem 2 reproves Merkurjev–Scavia's 2024 computation of Serre's "negligible" subgroup of finite group cohomology in degree 2 [3, Theorem 1.3, Corollary 4.2], [5, Corollary 8.3]. That was the key to their construction of the first known field with a Galois representation over $\mathbf{F}_p$ that does not lift to $\mathbf{Z}/p^2$ [3, Theorem 1.4].

Although Theorem 2 gives explicit generators for twisted Chow groups in terms of the usual Chow groups, it is not clear how to describe the relations. There is a similar theory for which we understand the relations. Namely, Heller–Voineagu–Østvær defined *twisted motivic cohomology* with coefficients in a locally constant sheaf E; this is a bigraded theory, $H^i_{\mathrm{M}}(X, E(j))$ [1, section 5.2], [5, section 4]. In one way, these behave better than twisted Chow groups: for any short exact sequence $0 \to A \to B \to C \to 0$ of locally constant étale sheaves on a scheme X with A coflasque, there is a long exact sequence of twisted motivic cohomology [5, Lemma 4.1]:

$$\cdots \to H^{2i-1}_{\mathrm{M}}(X, C(i)) \to H^{2i}_{\mathrm{M}}(X, A(i)) \to H^{2i}_{\mathrm{M}}(X, B(i)) \to H^{2i}_{\mathrm{M}}(X, C(i)) \to 0.$$

(For a finite group G, a $\mathbf{Z}G$-module A is *coflasque* if $H^1(H, A) = 0$ for every subgroup H of G.) For twisted Chow groups, we only have an exact sequence

$$CH_i(X, A) \to CH_i(X, B) \to CH_i(X, C) \to 0$$

under the stronger assumption that A is a summand of a permutation module over $\mathbf{Z}$ [5, Lemma 2.3, Theorem 14.1].

Using the exact sequence above for twisted motivic cohomology, we can always describe $H^{2i}_{\mathrm{M}}(X, E(i))$ as the cokernel of a homomorphism between the usual Chow groups of suitable finite covering spaces of X [5, Remark 4.2]. The problem of finding analogous relations for twisted Chow groups remains open, in view of [5, Remark 14.2]:

Theorem 3. *For a locally constant étale sheaf E on a smooth k-scheme X, there is a natural surjection*

$$H^{2i}_{\mathrm{M}}(X, E(i)) \to CH^i(X, E);$$

but this is not always an isomorphism, even for $i = 1$.

The counterexample involves an étale sheaf E with structure group $\mathbf{Z}/2 \times \mathbf{Z}/2$.

One positive result on twisted Chow groups is that $CH^1(X, E)$ injects into twisted étale motivic cohomology $H^2_{\text{et}}(X, E(1))$, whereas $H^2_{\text{M}}(X, E(1))$ need not [5, Theorems 10.1 and 14.1]. This suggests that twisted Chow groups form a meaningful intermediary between twisted motivic cohomology and twisted étale motivic cohomology:

$$H^{2i}_{\text{M}}(X, E(i)) \to CH^i(X, E) \to H^{2i}_{\text{et}}(X, E(i)).$$

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The group scheme of symmetries of a Lagrangian fibration

Mark Andrea A. de Cataldo

(joint work with Yoonjoo Kim, Christian Schnell)

I report on on-going joint work with Yoonjoo Kim at Columbia Univeristy and Christian Schell at Stony Brook University.

Let $f : X \to B$ be a Lagrangian fibration (for ease of exposition, X is nonsingular Kähler equipped with a holomorphic symplectic form σ, f is proper holomorphic with connected fibers onto B a complex manifold and the general fiber is Lagrangian wrt to σ). The case when f is projective of quasi-projective varieties is a special case. In our arguments, we need to work analyticlly, even in an algebraic set-up, in which case, the results, not the proofs, are in the algebraic context.

It is well-known that if we restrict f over the open subset B^{nc} of non-critical values of f, then the resulting fibration is one of compact complex tori and one has the relative Albanese torus fibration $Alb(X^{nc}/B^{nc})$ acting on X^{nc}/B^{nc}, turning the latter into a torsor over the former (Liouville-Arnold Theorem).

We prove that there is a smooth commutative group analytic space P/B that acts on X/B and that embeds in a locally closed fashion into the non flat group space $Aut(X/B)/B$ and hence also in the relative Douady space of $X \times_B X$ over B. The group space P/B does not have connected fibers over B, but it does over B^{nc}. The union P^o/B of the connected components P^o_b of the identity of each fiber P_b as b ranges in B is a a smooth commutative group space with connected fibers, and its construction, as a quotient of the total space of a vector bundle by the étalé space of a constructible sheaf of finitely generated free abelian groups, says that P^o/B deserves the name of relative Albanese. Note however, that X/B

is not a torsor for P/B, nor for P^o/B. Both group objects are not proper over B in general.

Earlier work by other authors, established the result for Lagrangian fibrations with reduced fibers. The novelty here is that the result holds in complete generality, without assumptions on the fibers.

The proof consists of a careful analysis of the natural, yet very transcendental, holomorphic B-map form the total space of the cotangent bundle to the group object $Aut(X/B)$, map obtained by considering the Hamiltonian vector fields induced by the symplectic structure.

We then use the group space P/B to give a precise form of the Decomposition Theorem for the direct image complex Rf_*Q_X , in terms of certain direct summands that we name Ngô strings. Each of this string is a direct sum of shifted intersection complexes supported on certain subspaces of B (called supports) with coefficient local systems arising from the group scheme and from the constructible sheaf $R^{2n}f_*Q_X$, where n is the relative dimension of f.

The total space of the fibration can be assumed to be normal (instead of smooth) (and then Rf_*Q_X is to be replaced by Rf_*IC_X, IC_X the intersection cohomology complex of X), but at present, we need B smooth. While B may be expected to be smooth if X is smooth, if X is singular, B can be singular. It would be nice to remove the smoothness assumption on B in our work.

The decomposition into Ngô strings should also be studied in detail for special Lagrangian fibrations arising in geometry, where it could lead to insight in their cohomological properties.

IMProofBench: Building a benchmark for AI mathematical reasoning through collaborative problem creation

Johannes Schmitt

(joint work with Tim Gehrunger, Jeremy Feusi, Gergely Bérczi)

The rapid development of Artificial Intelligence based on Large Language Models (such as ChatGPT) has created an urgent need for rigorous evaluation of their mathematical capabilities. A benchmark is a standardized test suite that measures performance on representative tasks. Current mathematical benchmarks suffer from significant limitations: those focusing on unique numerical answers [1, 2] fail to test proof generation and are vulnerable to shortcuts. On the other hand, benchmarks requiring formally verified proofs in systems like Lean or Coq [3] exclude much of contemporary research mathematics since the prerequisite basics from these mathematical theories have not yet been formalized. This work introduces IMProofBench, a new benchmark designed to evaluate AI systems on research-level mathematical proof generation, addressing these gaps through a focus on long-form arguments that meet the standards of peer-reviewed mathematics.

The project was first presented during a 10-minute talk in the Tuesday evening short talks session, followed by a one-hour collaborative work session on Thursday evening. The initial presentation began with an interactive quiz demonstrating current AI capabilities. Three questions were posed:

- **(A)** What is the minimal integer whose square is between 5 and 17?
- **(B)** For $S = \{1, \ldots, 8\}$, how many maps $\circ : S \times S \to S$ make $(S, \circ)$ into a group?
- **(C)** What is the étale fundamental group of Spec $\mathbb{C}[x, y]/(xy - 1)$?

ChatGPT-o3 incorrectly answered (A) with 3 instead of -4, but correctly solved both (B) with the formula $\sum_{|G|=8} 8!/|\mathrm{Aut}(G)| = 22080$ and (C) as $\hat{\mathbb{Z}}$. This exercise illustrated both the surprising competence and characteristic failure modes of current systems, motivating the need for systematic evaluation.

IMProofBench addresses fundamental issues with existing benchmarks. Unlike numerical-answer formats, proof-based evaluation directly targets hallucination and logical gaps–even when final answers are correct, flawed reasoning is penalized. The benchmark maintains a private problem repository to prevent training contamination and overfitting, also addressing concerns about inadvertently accelerating AI capabilities. Each problem consists of a proof-requiring main question paired with automatically verifiable subquestions, enabling both deep evaluation and efficient testing.

The Thursday evening work session attracted approximately 20 participants who collaboratively created and tested problems. The session demonstrated strong engagement across career stages, from graduate students to senior faculty. Many participants initially submitted questions suitable for advanced oral examinations before progressing to research-oriented problems. Over 60 submissions were generated during the session, with remote support from collaborators who manually entered questions into ChatGPT and other AI systems that have access to web search and computational tools, capabilities not yet available through our direct API integration.

The interactive testing revealed nuanced AI performance patterns. While several participants reported solution attempts that exhibited deeply flawed arguments, there were also some reported successes, such as calculating coefficients in tautological relations on universal Jacobians (in response to a question by Q. Yin). Participants expressed both surprise at specific achievements and reassurance that fundamental mathematical reasoning remains beyond current capabilities. Several of them noted that the exercise clarified the gap between pattern matching and genuine mathematical understanding.

The benchmark's design philosophy prioritizes authentic mathematical practice: problems should reflect actual work at PhD level and above, require genuine insight beyond routine algorithm application, and generate consequences that can be checked mechanically. There is a plan for a dual evaluation structure – human-graded proofs complemented by automated subquestion checking – which balances thoroughness with scalability. Initial focus areas include algebraic geometry and related fields, with plans to expand coverage through domain-specific editors.

Next steps consist of reviewing submitted problems and providing detailed feedback to contributors, followed by systematic evaluation across frontier models (e.g.

ChatGPT o3, Claude Opus 4, Gemini 2.5 Pro). All contributors will be invited to participate in grading AI responses and have the option of co-authorship on the resulting publication targeted for the International Conference on Learning Representations (ICLR) 2026.

The collaborative format at Oberwolfach, bringing together domain experts who could contribute sophisticated problems while directly experiencing AI capabilities, suggests a productive model for developing evaluation frameworks across mathematical disciplines. The direct engagement with AI systems helped participants calibrate their intuitions about current capabilities while contributing to a community resource for tracking progress in automated mathematical reasoning.

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A counterexample to the log canonical Beauville–Bogomolov decomposition

Zsolt Patakfalvi

(joint work with Fabio Bernasconi, Stefano Filipazzi, Nikolaos Tsakanikas)

We work over the field $\mathbb{C}$ of complex numbers.

Smooth projective varieties with torsion canonical divisor are one of the fundamental classes of varieties studied in birational geometry. The Beauville–Bogomolov decomposition provides a structure theorem for these varieties. It asserts that a smooth K-trivial variety can be decomposed, possibly after an étale cover, into a product of abelian varieties, strict Calabi–Yau varieties, and irreducible holomorphic symplectic varieties [Bog74, Bea83]. From this one can deduce the statement that is sometimes called the weak Beauville–Bogomolov decomposition: the Albanese morphism of a variety with trivial canonical divisor is isotrivial. This was first established by Calabi [Cal57]. The isotriviality of the Albanese morphism has been extended by Cao [Cao19] to the case of smooth projective varieties with nef anti-canonical divisor. This result has later been applied by Cao and Höring [CH19] to establish a Beauville–Bogomolov type decomposition of their universal cover.

From the perspective of the classification theory of varieties, a decomposition theorem for smooth projective varieties with trivial canonical class is not sufficient. One would need variants allowing mild singularities (e.g., klt or log canonical). In fact, such varieties are one of the main building blocks of the final outputs of the Minimal Modeal Program (MMP), singular Fano varieties and canonically polarized varieties being the other two.

Following this motivation, in the series of articles [GKP16, Dru18, GGK19, HP19], an analog of the Beauville–Bogomolov decomposition for projective varieties with klt singularities and numerically trivial canonical class has been obtained. This result has been partially extended to the case of klt pairs with nef anti-canonical class in [CCM21, PZ19, MW21], and it can be summarized by the following structure theorem.

Theorem 1 (Decomposition theorem for klt pairs with nef anti-canonical divisor). [GKP16, Dru18, GGK19, HP19, CCM21, PZ19, MW21] *Let* (X, Δ) *be a projective klt pair such that* $-(K_X + \Delta)$ *is nef. The following statements hold:*

(1) the Albanese morphism $\mathrm{alb}_X \colon (X, \Delta) \to \mathrm{Alb}_X$ *is an isotrivial morphism for the pair* (X, Δ) *and has connected fibers;*

(2) if $K_X + \Delta \equiv 0$*, then there exists a finite quasi-étale cover* $\gamma \colon Y \to X$ *such that*

$$(Y, \gamma^*\Delta) = (F, \Delta_F) \times A \times \prod Y_i \times \prod Z_i,$$

where F *is a rationally connected variety and* $\Delta_F = (\gamma^*\Delta)|_F$*,* A *is an abelian variety, the* Y_i *are singular strict Calabi–Yau varieties and the* Z_i *are irreducible symplectic varieties; and*

(3) if X *is smooth and* $\Delta = 0$*, then the universal cover* Y *of* X *admits a decomposition*

$$Y = F \times \mathbb{C}^q \times \prod Y_i \times \prod Z_i,$$

where F*,* Z_i *and* Y_i *are as above and smooth.*

Note that the decomposition of the universal cover of a klt pair with nef anti-canonical class is still an open problem; see [MW21, Conjecture 1.5]. Note also that in point (3) the splitting does not happen on a finite cover, as instead in point (2), but only on the universal one. This can be traced back to the fact that, for any polarization, the polarized automorphism group is finite in the K-trivial klt case, but it is infinite in general in the $-K_X$ nef case.

Varieties and pairs with log canonical singularities form the largest class of varieties for which the MMP is expected to hold. It is thus a central question to decide whether Theorem 1 holds for log canonical pairs. However, one could then wonder which point of Theorem 1 should generalize to the log canonical case. As the polarized automorphism groups can be positive dimensional in the log canonical case, even in the K-trivial case (e.g., take $(\mathbb{P}^1, \{0\} + \{\infty\})$ with $\mathcal{O}_{\mathbb{P}^1}(1)$ as polarization), one does not expect point (2) to extend to the log canonical case. Hence, even in the K-trivial log canonical case, the best one could hope for is that a decomposition as in point (3) of Theorem 1 holds on the universal cover.

Thus, we are primarily interested in the following question:

Question 1. Given a log canonical variety (or pair) X with $K_X \sim_{\mathbb{Q}} 0$, does the universal cover Y of X_{reg} admit a decomposition

$$Y = F_{\mathrm{reg}} \times \mathbb{C}^q \times \prod Y_{i,\mathrm{reg}} \times \prod Z_{i,\mathrm{reg}},$$

where F, Y_i and Z_i are the singular versions of the factors of point (1)refitm:smooth of Theorem 1?

Our main theorem is the following:

Theorem 2. *The answer to Question 1 is negative.*

In fact, Theorem 2 is an immediate corollary of the following more precise theorem, which states that the Albanese morphism fails, in general, to be isotrivial.

Theorem 3. *For every integer $d \geq 4$, there exists a projective log canonical variety X of dimension d such that the following hold, where $\mathrm{alb}_X : X \to \mathrm{Alb}_X$ is the Albanese morphism of X:*

(1) $K_X \sim 0$;
(2) Alb_X *is an elliptic curve;*
(3) *every fiber of* alb_X *is birational to exactly finitely many other fibers; and*
(4) *the natural map* $\pi_1(X_{\mathrm{reg}}) \to \pi_1(\mathrm{Alb}_X)$ *is an isomorphism.*

Therefore, any quasi-étale cover of X is induced by an étale cover of Alb_X. In particular, the universal cover of X_{reg} admits a fibration to $\mathbb{C}$ where any fiber is birational to exactly countably many other fibers.

In the case of pairs, we also construct a counterexample to the Beauville–Bogomolov decomposition where the pair has plt singularities.

We note that there exists an earlier example [EIM23, Example 6.3], whose Albanese morphism is isotrivial, but does not split after a finite étale base change. This shows that there is no log canonical Beauville–Bogomolov decomposition as in point (2) of Theorem 1. However, this example really only exploits the fact that the polarized automorphism groups are positive dimensional. In particular, the fibration does become split when one passes to the quasi-étale universal cover, and hence it does not give a counterexample to Question 1.

We also treat the case of open varieties. By [Kaw81, Fuj24], the quasi-Albanese morphism of a quasi-projective variety with logarithmic Kodaira dimension 0 is a dominant morphism with irreducible general fibers. We show that this is optimal.

Theorem 4. *For every integer $d \geq 4$, there exists a smooth quasi-projective variety U of dimension d and logarithmic Kodaira dimension $\overline{\kappa}(U) = 0$ such that the following hold:*

(1) *the quasi-Albanese morphism* $\mathrm{alb}_U \colon U \to \mathbb{G}_m$ *is flat with irreducible fibers; and*
(2) *every fiber of* alb_U *is birational to exactly finitely many other fibers.*

We remark that the variety in Theorem 3 is obtained by base change of a suitable compactification of the variety in Theorem 4.

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Semi-orthogonal decompositions of Fano moduli spaces

Jenia Tevelev

(joint work with Elias Sink, Sebastián Torres)

Let C be a smooth complex projective curve of genus $g \geq 2$, and let $\mathcal{SU}_C(2, \Lambda_0)$ (resp. $\mathcal{SU}_C(2, \Lambda)$) denote the coarse moduli space of semistable rank-2 vector bundles on C with a fixed determinant of even (resp. odd) degree. Both are Fano varieties of dimension $3g-3$ and Picard number 1, but whereas $\mathcal{SU}_C(2, \Lambda)$ is smooth, $\mathcal{SU}_C(2, \Lambda_0)$ has Gorenstein rational singularities.

Belmans–Galkin–Mukhopadhyay [2, Conjecture A] and Narasimhan [10, Conjecture 1.1] conjectured that $D^b(\mathcal{SU}_C(2, \Lambda))$ admits a semiorthogonal decomposition (SOD) with components $D^b(\mathrm{Sym}^k C)$. This was proved in [16,17]; see also [19].

Theorem 1 (Tevelev–Torres)**.** *$D^b(\mathcal{SU}_C(2,\Lambda))$ admits a semiorthogonal decomposition with blocks $D^b(\mathrm{Sym}^k C)$ (two copies for $k < g-1$ and one copy for $k = g-1$).*

Many geometric resolutions of the singularities of $\mathcal{SU}_C(2,\Lambda_0)$ have appeared in the literature. On the other hand, Kuznetsov [8] defines a noncommutative resolution of singularities of a projective variety X to be a smooth proper triangulated category $\mathcal{D}$ equipped with an adjoint pair of functors

$$f_* : \mathcal{D} \to D^b(X), \qquad f^* : \mathrm{Perf}(X) \to \mathcal{D},$$

such that $f_* \circ f^* \cong \mathrm{Id}$. Bondal and Orlov [3] conjectured that every variety admits a minimal noncommutative resolution; see also [6].

Padurariu and Toda [12] introduced, in a general moduli-theoretic framework, the notion of *quasi-BPS categories*. In our context, these are the subcategories $\mathbb{B}_0$ and $\mathbb{B}_1$ of the derived category of the stack of semistable rank-2 vector bundles [12, Section 3.4]. Moreover, $\mathbb{B}_0$ provides a noncommutative resolution of $\mathcal{SU}_C(2,\Lambda_0)$, whereas $\mathbb{B}_1$ may be regarded as a Brauer-twisted noncommutative resolution.

Theorem 2 (Sink–Tevelev)**.** *The categories $\mathbb{B}_0$ and $\mathbb{B}_1$ admit semiorthogonal decompositions with components $D^b(\mathrm{Sym}^k C)$: four blocks for $k < g-1$ and two blocks for $k = g-1$. Blocks with even k (resp. odd k) contribute to $\mathbb{B}_0$ (resp. $\mathbb{B}_1$).*

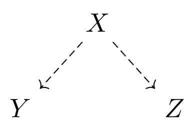
In the even-genus case, Theorem 2 verifies a conjecture of Belmans [1]: here $\mathbb{B}_0$ is a strongly crepant noncommutative resolution of $\mathcal{SU}_C(2,\Lambda_0)$ in the sense of [8], categorifying the intersection cohomology of $\mathcal{SU}_C(2,\Lambda_0)$.

The categories $D^b(\mathcal{SU}_C(2,\Lambda))$, (resp., $\mathbb{B}_0$, and $\mathbb{B}_1$) are studied via their admissible embeddings in $D^b(M)$, where M denotes the Thaddeus moduli space of stable pairs (F,s); here F is a rank-2 vector bundle on C with fixed determinant of degree $2g-1$ (resp. $2g$), and $s \in H^0(C,F)$ is a nonzero section (see [18]). Moreover, the derived categories $D^b(\mathrm{Sym}^{2k} C)$ embed into $D^b(M)$ via explicit Fourier–Mukai functors whose kernels are tensor bundles twisted by various line bundles.

In the odd-degree case, the moduli spaces of stable bundles and of stable pairs are birational. In contrast, when $\deg(\Lambda_0) = 2g$, the morphism $M \to \mathcal{SU}_C(2,\Lambda_0)$ has generic fiber $\mathbb{P}^1$. Although M is always rational, the rationality of $\mathcal{SU}_C(2,\Lambda_0)$ remains a longstanding open problem, going back to the early works of Tyurin and Newstead. In light of the Kuznetsov rationality proposal [9], Theorem 2 suggests that any weak factorization of a hypothetical birational map $\mathcal{SU}_C(2,\Lambda) \dashrightarrow \mathbb{P}^{3g-3}$ should involve blow-ups and blow-downs of $\mathrm{Sym}^{2k} C$ for $2k \le g-1$.

There is a large body of conjectures predicting explicit semiorthogonal decompositions of Fano varieties, often based on analyses of Hodge diamonds and on the additivity of Hochschild homology in semiorthogonal decompositions. Such predictions are challenging to prove. One general approach is to analyze the two-ray game determined by two extremal contractions of a Fano variety:

(1)



In our situation, $X = M$, $Y = \mathbb{P}^{3g-3}$ (resp. $\mathbb{P}^{3g-2}$ in the even-degree case), and $Z = \mathcal{SU}_C(2,\Lambda)$ (resp. $\mathcal{SU}_C(2,\Lambda_0)$). We begin with a known semiorthogonal decomposition of $D^b(Y)$ and extend it to one of $D^b(X)$, which in our case is straightforward. The hard part is to mutate this decomposition of $D^b(X)$ to one compatible with a semiorthogonal decomposition of $D^b(Z)$:

Conjecture 3. *Let* (1) *be the extremal contractions of smooth Fano varieties. Then there exist semiorthogonal decompositions of $D^b(X)$ that are compatible with pullbacks along these maps:*

$$D^b(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_s, \mathcal{P}_1, \ldots, \mathcal{P}_r \rangle = \langle Q_1, \ldots, Q_u, \mathcal{B}_1, \ldots, \mathcal{B}_t \rangle,$$

$$D^b(Y) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_s \rangle, \qquad D^b(Z) = \langle \mathcal{B}_1, \ldots, \mathcal{B}_t \rangle.$$

Moreover, the two decompositions of $D^b(X)$ are related by a mutation.

Our approach uses weaving patterns of [16], allowing for tight control of the Fourier–Mukai kernels for the various functors $D^b(\mathrm{Sym}^k C) \to D^b(M)$ embedding the blocks on different stages of the mutation. From the perspective of homological mirror symmetry for Fano manifolds, such mutations should exist in general, and we can predict a description of the corresponding braid (recall that mutation gives rise to a braid group action on the set of semiorthogonal decompositions):

Conjecture 4. *The braid appearing in Conjecture 3 can be computed as the monodromy of the eigenvalues of $c_1(X)$ acting on the small quantum cohomology ring $\mathbb{QH}^*(X,\mathbb{C})$, as the quantum parameter τ varies along a path in the ample cone of X (with a small B-field perturbation iB to avoid collisions of eigenvalues). As the path approaches the walls of the ample cone, the eigenvalues cluster into groups reflecting the structure of the birational contractions $Y \dashleftarrow X \dashrightarrow Z$.*

We do not claim originality for these conjectures, which align with existing literature (see, e.g., [4–7, 13]). Currently, the only way to test Conjecture 4 is by computing and comparing the two braids; see [15] for numerous examples. It would be valuable to accumulate further evidence by realizing other pairs of Fano varieties Y and Z via a common Fano variety X, such as toric Fano varieties, maximal flag varieties, Fano threefolds, and various moduli-theoretic Fano spaces.

One motivation for Theorem 1 was the categorification of the Muñoz result [11] that the quantum spectrum of $\mathcal{SU}_C(2,\Lambda)$ consists of eigenvalues 8λ, where

$$\lambda = 1-g,\ (2-g)i,\ 3-g, \ldots, g-3,\ (g-2)i,\ g-1,$$

and the eigenspace corresponding to 8λ is isomorphic to $H^*(\mathrm{Sym}^{\,g-1-|\lambda|}(C),\mathbb{C})$. We therefore conjecture:

Conjecture 5. *The semiorthogonal decomposition of $\mathcal{SU}_C(2,\Lambda)$ from Theorem 1 is atomic, i.e., compatible with the orthogonal decomposition of $H^*(\mathcal{SU}_C(2,\Lambda),\mathbb{C})$ into generalized eigenspaces of quantum multiplication by c_1. The precise notion of compatibility is as in the Sanda–Shamoto conjecture* [13].

Following a suggestion of Padurariu and Toda, we prove Conjecture 3 in the setting of the Hecke correspondence

$$\begin{array}{ccc} & \mathcal{H} & \\ \swarrow^{\pi_0} & & \searrow^{\pi} \\ \mathcal{SU}_C(2,\Lambda_0) & & \mathcal{SU}_C(2,\Lambda) \end{array}$$

Here $\mathcal{H}$ denotes the moduli space of stable parabolic bundles (at a fixed point $q \in C$) with a fixed odd determinant. We obtain semiorthogonal decompositions

$$\langle \mathbb{B}_0, \mathbb{B}_1 \rangle = D^b(\mathcal{H}) = \langle D^b(\mathcal{SU}_C(2,\Lambda)),\ D^b(\mathcal{SU}_C(2,\Lambda)) \otimes \mathcal{O}_\pi(1) \rangle$$

compatible with the morphisms π_0 and π. Here $\mathbb{B}_0$ and $\mathbb{B}_1$ are the quasi-BPS categories for $\mathcal{SU}_C(2,\Lambda_0)$, and the second decomposition reflects that π is a $\mathbb{P}^1$-bundle. We construct a mutation in $D^b(\mathcal{H})$ relating the decompositions of Theorems 1 and 2, which we call the *Hecke Braid*.

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On the top weight rational cohomology of the moduli space of abelian varieties and universal Jacobians

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(joint work with Madeleine Brandt, Juliette Bruce, Melody Chan, Gwyneth Moreland, Corey Wolfe)

Constructing compactifications of moduli spaces often requires the use of combinatorial data, which is associated with degeneration data of the original objects.

In the last few years, tropical methods have been applied quite successfully in understanding the combinatorics behind a number of compactifications of moduli spaces, in particular by endowing them with a tropical modular interpretation. Consequently, one can study different properties of these (compactified) spaces by studying their tropical counterparts.

In the talk, I will focus on the moduli space $\mathcal{A}_g$ of abelian varieties of dimension g. In particular, I will explain joint work with Madeleine Brandt, Juliette Bruce, Melody Chan, Gwyneth Moreland and Corey Wolfe where we apply the tropical understanding of the classical toroidal compactifications of $\mathcal{A}_g$ to compute, for small values of g, the top weight rational cohomology of A_g. We also show that our results can be used to study the stable cohomology of the Satake compactification of $\mathcal{A}_g$ and the cohomology of $\mathrm{GL}_g(\mathbb{Z})$.

1. The top-weight cohomology of $\mathcal{A}_g$

The moduli stack $\mathcal{A}_g$ of (principally polarized) abelian varieties of dimension g is a smooth stack of dimension $d = \binom{g+1}{2}$, and it is one of the most studied moduli spaces in algebraic geometry. However, the full cohomology ring of $\mathcal{A}_g$ is known only up to $g = 3$: the cases when $g \leq 2$ are classically known, and the case when $g = 3$ is the work of Hain [Hai02].

Since $\mathcal{A}_g$ is not proper and its coarse moduli space A_g is a complex algebraic variety, its rational cohomology groups of $\mathcal{A}_g$ admit a weight filtration in the sense of mixed Hodge theory, with graded pieces $\mathrm{Gr}_j^W H^\bullet(\mathcal{A}_g;\mathbb{Q})$ which may appear for j from 0 to $2d$, so the piece of weight $j = 2d$ is known as the *top-weight rational cohomology* of $\mathcal{A}_g$. This report presents joint work of the author with Madeleine Brandt, Juliette Bruce, Melody Chan, Gwyneth Moreland and Corey Wolf in [BBCMMW24] concerning the study of the top-weight rational cohomology of $\mathcal{A}_g$, $Gr_{2d}^W H^\bullet(\mathcal{A}_g;\mathbb{Q})$. In detail, our computations allow us to determine $Gr_{2d}^W H^\bullet(\mathcal{A}_g;\mathbb{Q})$

for $g \leq 7$, as described below: The top-weight rational cohomology of $\mathcal{A}_g$ for $2 \leq g \leq 7$, is

$$\begin{aligned}
&\mathrm{Gr}^W_6 H^k(\mathcal{A}_2;\mathbb{Q}) = 0\\
&\mathrm{Gr}^W_{12} H^k(\mathcal{A}_3;\mathbb{Q}) = \begin{cases}\mathbb{Q} \text{ if } k=6,\\ 0 \text{ else,}\end{cases}\\
&\mathrm{Gr}^W_{20} H^k(\mathcal{A}_4;\mathbb{Q}) = 0\\
&\mathrm{Gr}^W_{30} H^k(\mathcal{A}_5;\mathbb{Q}) = \begin{cases}\mathbb{Q} \text{ if } k=15,20,\\ 0 \text{ else,}\end{cases}\\
&\mathrm{Gr}^W_{42} H^k(\mathcal{A}_6;\mathbb{Q}) = \begin{cases}\mathbb{Q} \text{ if } k=30,\\ 0 \text{ else,}\end{cases}\\
&\mathrm{Gr}^W_{56} H^k(\mathcal{A}_7;\mathbb{Q}) = \begin{cases}\mathbb{Q} \text{ if } k=28,33,37,42\\ 0 \text{ else.}\end{cases}
\end{aligned}$$

The cases $g = 2, 3$ and 4 were already known: the whole cohomology ring of $\mathcal{A}_2$ was classically computed by Igusa, the whole cohomology ring of $\mathcal{A}_3$ was computed by Hain in [Hai02] and the case $g = 4$ is contained in the work of Hulek and Tommasi in [HK12]. We notice that our results exhibit for the first time the existence of nonzero odd cohomology of $\mathcal{A}_g$ (for $g = 5, 7$), answering an open question of Grushevsky (see [Gru09, Open Problem 7]).

Our techniques for studying $\mathcal{A}_g$ are analogous to those employed in [CGP21] for $\mathcal{M}_g$: there exist well behaved compactifications of $\mathcal{A}_g$ associated to combinatorial data that has a tropical modular interpretation. The moduli spaces parametrizing such tropical varieties are generalized cone complexes whose homology can be identified with the compactly supported cohomology of $\mathcal{A}_g$ and computed using combinatorial techniques.

In detail, the moduli spaces $\mathcal{A}_g$ admit toroidal compactifications $\overline{\mathcal{A}_g}^{\Sigma}$, which are proper Deligne–Mumford stacks. The compactifications $\overline{\mathcal{A}_g}^{\Sigma}$ are associated to admissible decompositions Σ of Ω_g^{rt}, the rational closure of the cone of positive definite quadratic forms in g variables. The same data was also used to construct the moduli space $A_g^{\mathrm{trop},\Sigma}$ of tropical abelian varieties of dimension g in the category of generalized cone complexes in joint work of the author with Brannetti and Viviani in [BMV11].

Notice that a similar relation has been shown to hold more generally for toroidal compactifications of locally symmetric varieties in the recent preprint [ABBCV] by Assaf, Brandt, Bruce, Chan and Vlad.

Then for any admissible decomposition Σ of Ω_g^{rt} and for each $i \geq 0$, and writing $LA_g^{\mathrm{trop},\Sigma}$ for the link of the cone point of $A_g^{\mathrm{trop},\Sigma}$, the following canonical identification holds:

$$\widetilde{H}_{i-1}(LA_g^{\mathrm{trop},\Sigma};\mathbb{Q}) \cong \mathrm{Gr}^W_{2d} H^{2d-i}(\mathcal{A}_g;\mathbb{Q}).$$

This identification follows from applying the generalization to Deligne–Mumford stacks, spelled out in [CGP21], of Deligne's comparison theorems in mixed Hodge theory using the fact that there exist admissible decompositions Σ for which $\overline{\mathcal{A}_g}^{\Sigma}$ is a smooth simple normal crossings compactification of $\mathcal{A}_g$ whose boundary complex is identified with $LA_g^{\mathrm{trop},\Sigma}$ and that the homeomorphism type of $LA_g^{\mathrm{trop},\Sigma}$ is independent of Σ.

Therefore, in order to compute the topology of $A_g^{\mathrm{trop},\Sigma}$, we consider the *perfect* or *first Voronoi* toroidal compactification $\overline{\mathcal{A}_g}^{\mathrm{P}}$ and its tropical version $A_g^{\mathrm{trop},\mathrm{P}}$, associated to the *perfect cone decomposition*, as it is very well known and enjoys interesting combinatorial properties. We then identify the homology of the link of $A_g^{\mathrm{trop},\mathrm{P}}$ with the homology of a complex, that we call the *perfect chain complex* $P_\bullet^{(g)}$, using the framework of cellular chain complexes of symmetric CW-complexes due to Allcock-Corey-Payne [ACP22].

To compute the homology of the complex $P_\bullet^{(g)}$ we use a related complex $V_\bullet^{(g)}$, called the Voronoi complex. This complex which was introduced in [EVGS13, LS78] to compute the cohomology of the modular groups $\mathrm{GL}_g(\mathbb{Z})$ and $\mathrm{SL}_g(\mathbb{Z})$ and we show that it relates to $P_\bullet^{(g)}$ as they both sit in the following exact sequence

$$0 \longrightarrow P_\bullet^{(g-1)} \longrightarrow P_\bullet^{(g)} \xrightarrow{\pi} V_\bullet^{(g)} \longrightarrow 0.$$

We can therefore compute the homology of $P_\bullet^{(g)}$ by using the results in the homology of $V_\bullet^{(g)}$ in [EVGS13, LS78], which are based on the existence of lists of perfect forms for $g \leq 7$ obtained by Jaquet in [Jaq93].

Our main results on $H^*(\mathcal{A}_g;\mathbb{Q}) \cong H^*(\mathrm{Sp}_{2g}(\mathbb{Z});\mathbb{Q})$ are also related to the stable cohomology of Satake compactifications. More precisely, the classes we find in this paper seem to relate to interesting generators of the stable cohomology ring of $\mathcal{A}_g^{\mathrm{Sat}}$, defined by Charney and Lee.

Finally, our results also highlight the connection between $H^*(\mathrm{Sp}_{2g}(\mathbb{Z});\mathbb{Q})$ and $H^*(\mathrm{GL}_g(\mathbb{Z});\mathbb{Q})$, that is, for all k,

$$H^{\binom{g}{2}-k}(\mathrm{GL}_g(\mathbb{Z});\mathbb{Q}) \cong H_{k+g-1}(V^{(g)})$$

and

$$H_{k-1}(P^{(g)}) \cong \mathrm{Gr}^W_{g^2+g} H^{g^2+g-k}(\mathcal{A}_g;\mathbb{Q}) \twoheadleftarrow H^{g^2+g-k}(\mathcal{A}_g;\mathbb{Q}).$$

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