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Toric Geometry

Organized by
Daniel Erman, Honolulu
Milena Hering, Edinburgh
Nathan Ilten, Burnaby
Hendrik Süß, Jena

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Abstract. Toric varieties provide a rich class of examples in algebraic geometry that benefit from deep and fruitful interactions with combinatorics. This workshop highlighted recent interactions between toric geometry and mirror symmetry, matroids, deformation theory and moduli spaces, and non-commutative geometry, as well as some exciting new developments within toric geometry itself.

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Introduction by the Organizers

The workshop *Toric geometry*, organized by Daniel Erman, Milena Hering, Nathan Ilten, Hendrik Süß had 48 in-person and 3 online participants. The participants came from all over the world and represented a diverse group along several axes, including gender, career stage, and subfields. This diversity was also reflected in the list of speakers, with some of the more junior speakers bringing especially fresh perspectives.

Toric varieties are special algebraic varieties that can be studied via combinatorial methods. Introduced by Demazure just over 50 years ago, toric geometry has grown from a niche subject to a vibrant research area with strong connections to numerous topics in algebraic geometry, commutative algebra, combinatorics, applied mathematics, and beyond. The combinatorial nature of toric varieties allows them to be a rich testing ground for the development of theories (e.g., the minimal model programme) and of conjectures in algebraic geometry. Furthermore, many interesting problems either have some kind of (sometimes hidden) intrinsic “toric” structure, or can be reduced to a problem in toric geometry.

This workshop brought together researchers from various areas in algebraic geometry and related fields who have recently made use of toric methods. Many of these areas are interconnected in their own right, but the focus of the workshop was on one particular aspect of the connective tissue of this research: toric geometry. Despite this focus, we emphasize that the perspective of the workshop was one looking outwards: how can toric methods lead to new results in and new connections between these other areas?

Recent years have seen major breakthroughs in the study of *free resolutions and derived categories* in toric geometry, due to new connections to symplectic geometry via homological mirror symmetry. Hicks gave an introduction to this connection, applying it to resolutions of toric subvarieties. Berkesch gave a combinatorial approach to these resolutions, and Hanlon built on these ideas by providing a basis for a category closely related to the derived category of a toric variety.

Mirror symmetry and/or symplectic geometry formed the focus of several additional talks, including those of Huang, Sabatini, and Tevelev. Tevelev used mirror symmetry to construct non-commutative resolutions of cyclic quotient singularities. Huang discussed how a connection with mirror symmetry provided a canonical noncommutative resolution for any toric variety (generalizing work of Špenko-Van den Bergh and Faber-Muller-Smith in the affine case). Sabatini reported about progress on classifying a symplectic generalization of Fano manifolds.

The study of *matroids* through toric techniques was discussed in the talks of Eur and Fink. Fink explained how both resolutions and multigraded commutative algebra (namely Cartwright-Sturmfels ideals) played an essential role in his main result, while Eur's work established cohomology vanishing results for wonderful varieties that were motivated by those from toric geometry.

Toric vector bundles were another central theme of the workshop. They played a key role in the aforementioned talks of Eur and Fink, and they were the focus of the talks of Altmann, Maclagan, Manon, and Smith. Altmann, Maclagan, and Manon provided several new ways of thinking about toric vector bundles, with each novel description suitable for applications to computing extensions of line bundles, tropical vector bundles, and Cox rings of projectivised toric vector bundles. Smith presented a new approach to computing cohomology of toric vector bundles, leading to an exciting new cohomology vanishing theorem.

Another common theme was *deformation theory and moduli spaces.* Corti posed a conjecture which would allow for the classification of smooth Fano varieties via zero-mutable log structures. Robins discussed recent progress on versal deformations of smooth toric varieties, and Rana leveraged toric techniques to investigate the boundary of the moduli space of Horikawa surfaces. Ablett's talk extended work of Gotzmann to the multigraded setting, allowing for explicit equations for Hilbert schemes in toric varieties.

In a related vein, *degenerations* played a role in many talks including those of Eur and Fink. Degenerations also featured implicitly in the talk of Sano, who discussed a formula for the Newton polytope of the Hurwitz form akin to that of the classical A-discriminant and resultant.

Finally, Adiprasito's talk proved the weighted version of Oda's Strong Factorization Conjecture, providing one of the highlights of the workshop.

The speaker list included a number of graduate student and postdocs, and longer talks by a number of junior participants brought fresh perspectives to the subject. On Tuesday evening, 19 speakers gave 5-minute introductions to their work. This included every junior participant (excluding Ablett and Robins, who gave longer talks) as well as many non-junior participants. This led to many lively discussions which continued throughout the week. Speakers giving 5-minute talks were:

(1) Diane Maclagan: *Gröbner fans of subschemes of toric varieties*
(2) Ben Wormleighton: *Five minutes on symplectic embeddings*
(3) Juliette Bruce: *Regularity of* $\mathcal{I}^p$
(4) Karin Schaller: *Seshadri constants on projective toric surfaces*
(5) Adrian Cook: *How can we detect (un-)stability of tangent sheaves of Gorenstein-Fano toric varieties?*
(6) Mahrud Sayrafi: *Geometry of Syzygies of Truncations*
(7) Jesse Huang: *Homotopy path algebras on CW complexes*
(8) Andrea Petracci: *On Deformations of monomial schemes*
(9) Sofia Garzon Mora: *Fine Polyhedral Adjunction*
(10) Sönke Rollenske: *Canonical rings of stable surfaces with* $K_X^2 = 1$ *and* $\chi(\mathcal{O}_X) = 3$
(11) Paul Görlach: *Intersection complexes from tautological systems*
(12) Achim Napame: *Prescription of singularities on stable sheaves*
(13) Andrew Hanlon: *Homological mirror symmetry for Batyrev pairs*
(14) Leonid Monin: *An example of Chow Quotient*
(15) Jeff Hicks: *Non Realizability of not a tropical curve*
(16) Leandro Meier: *Bounding minimal log discrepancy of complexity one T-varieties*
(17) Chris Eur: *Riemann-Roch (Mock?) vs FSEC*
(18) Alex Fink: *Tropical cellular resolutions*
(19) Simon Telen: *The Segre cubic of a pentagon*

A well-attended musical recital Thursday evening, followed by dancing, provided a chance for participants to relax and socialize.

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Workshop: Toric Geometry

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Abstracts

How do matroids behave like a smooth projective toric variety?

Christopher Eur

Let $\mathbb{K}$ be an algebraically closed field. Let us sample a few "positivity" properties that a smooth irreducible projective $\mathbb{K}$-variety X satisfies.

(H) When $\mathbb{K} = \mathbb{C}$, the singular cohomology ring $H^\bullet(X)$ satisfies the so-called "Kähler package" — namely, Poincaré duality, hard Lefschetz property, and Hodge–Riemann relations.

(S) For D an ample divisor on X, there is a large enough $m \gg 0$ such that we have $H^i(X, \mathcal{O}_X(mD)) = 0$ for all $i > 0$.

(K) When $\mathbb{K} = \mathbb{C}$, for D an ample divisor on X and K_X the canonical divisor of X, we have $H^i(X, \mathcal{O}_X(K_X + D)) = 0$ for all $i > 0$.

When X is a *toric* smooth projective variety, these positivity properties are known to be strengthened as follows.

(H′) The *Chow ring* $A^\bullet(X)$ satisfies the "Kähler package." Both the Chow ring of X and the K-ring of vector bundles on X admit a combinatorial description.

(D) For D a *nef* divisor on X, we have $H^i(X, \mathcal{O}_X(D)) = 0$ for all $i > 0$.

(BB) For D an ample divisor on X, we have $H^i(X, \mathcal{O}_X(K_X + D)) = 0$ for all $i > 0$.

Here, H stands for Hodge, S for Serre, K for Kodaira, D for Demazure, and BB for Batyrev–Borisov. We point to [6, Chapter 9] for a treatment of these facts. We report on various progress on the following question.

Question 1. *Do these strong versions of positivity for toric varieties hold for the wonderful varieties of linear subspaces introduced by De Concini and Procesi* [7]*?*

Let us begin by recalling the construction of wonderful varieties. Let $L \subseteq \mathbb{K}^n$ be a linear subspace of dimension r, not contained in a coordinate hyperplane. Let $\mathbb{P}L \subset \mathbb{P}^{n-1}_{\mathbb{K}}$ be the projectivization. Denote by $T = (\mathbb{K}^*)^n$ the standard torus, and by $\mathbb{P}T := T/\mathbb{K}^*$ its projectivization, i.e. the quotient by the diagonal $\mathbb{K}^*$.

Definition 2. *The* permutohedral variety X_n *is the sequential blow-up of* $\mathbb{P}^{n-1}$ *obtained by blowing-up the n coordinate points, then the strict transforms of the* $\binom{n}{2}$ *coordinate lines, and so forth. Let* $\pi : X_n \to \mathbb{P}^{n-1}$ *be the blow-down map.*

The wonderful variety W_L *of the subspace L is the strict transform of* $\mathbb{P}L$ *under the blow-up* π*. Its* boundary ∂W_L *is the complement* $W_L \setminus (W_L \cap \mathbb{P}T)$*, which is a simple normal crossing divisor on* W_L*.*

Note that by construction X_n is a smooth projective toric variety with $\mathbb{P}T$ as its open dense torus. We view X_n also as a T-variety via the surjection $T \to \mathbb{P}T$.

Recent remarkable developments in matroid theory showed that wonderful varieties have the property (H′) enjoyed by toric varieties: The Chow ring of W_L has a combinatorial description, depending only on the *matroid* that $L \subseteq \mathbb{K}^n$ defines

[7, 10], and this Chow ring of a matroid satisfies the Kähler package [1]. However, the other properties like (D) and (BB) remain less well-understood. Matt Larson conjectured the following during the 2023 BIRS workshop *Algebraic Aspects of Matroid Theory.*

Conjecture 3. *For any nef divisor D on X_n, we have $H^i(W_L, \mathcal{O}_{W_L}(D|_{W_L})) = 0$ for all $i > 0$, and if further $D|_{W_L}$ is big then $H^i(W_L, \mathcal{O}_{W_L}(K_{W_L} + D|_{W_L})) = 0$ for all $i > 0$.*[1]

We highlight three partial results towards the conjecture currently available.

Theorem 4. [9] *For a full-dimensional subcone $\mathcal{C}$ of the nef cone of X_n generated by the "simplicial generators" [2], the conjecture holds for any $D \in \mathcal{C}$.*

Theorem 5. [8] *The log-canonical divisor $K_{W_L} + \partial W_L$ satisfies*

$$H^i(W_L, \mathcal{O}_{W_L}(K_{W_L} + \partial W_L)) = 0 \text{ for all } i > 0.$$

Theorem 6. [4] *Under a mild hypothesis (that the matroid of L is connected), the anti-log-canonical divisor satisfies $(-1)^{r-1}\chi(W_L, \mathcal{O}_{W_L}(-K_{W_L} - \partial W_L)) \geq 0$.*

A strengthening of Theorem 6 would be that $H^i(W_L, \mathcal{O}_{W_L}(-K_{W_L} - \partial W_L)) = 0$ for all $i < \dim W_L$, but the validity of this stronger statement is open. Theorem 4 is an easy consequence of the result of Brion [5] on multiplicity-free subvarieties of flag varieties; [9] contains a version of Theorem 4 for all matroids, not necessarily realizable, whose proof requires further techniques. The other two theorems make use of a pair of T-equivariant vector bundles on X_n associated with L, known as "tautological bundles of matroids" introduced in [3]. We conclude by describing these vector bundles and by indicating how they are used.

Let $\underline{\mathbb{K}}^n$ denote the trivial vector bundle $X_n \times \mathbb{K}^n$, which is a T-equivariant bundle via the inverse action of T on $\mathbb{K}^n$, i.e. $t \cdot x = t^{-1}x = (t_1^{-1}x_1, \ldots, t_n^{-1}x_n)$.

Definition 7. *Let $\mathcal{S}_L$ to be the T-equivariant subbundle of $\underline{\mathbb{K}}^n$ whose fiber over a point $\overline{t} \in \mathbb{P}T \subset X_n$ is the linear subspace $t^{-1}L$. Define $\mathcal{Q}_L$ to be the quotient bundle $\underline{\mathbb{K}}^E/\mathcal{S}_L$.*

A key observation, already implicit in [11], is that the wonderful variety W_L is the vanishing locus of a section of $\mathcal{Q}_L$ [3, Theorem 7.10], so we have a resolution

$$0 \to \bigwedge^{n-r} \mathcal{Q}_L^\vee \to \cdots \to \bigwedge^2 \mathcal{Q}_L^\vee \to \mathcal{Q}_L^\vee \to \mathcal{O}_{X_n} \to \mathcal{O}_{W_L} \to 0.$$

Then, using the structure of the fibers in a toric morphism $X_n \to X_{n-1}$, one shows that the higher cohomologies of $\bigwedge^k \mathcal{Q}_L$ and $\bigwedge^k \mathcal{Q}_L^\vee$ vanish [8, Theorem 1.5], and thereby deduces Theorem 5. The approach for Theorem 6 is considerably more involved, using Kempf collapsing for pairs of bundles of the form $\mathcal{S}_L$; Alex Fink explained this in more detail during his talk in the workshop.

[1] After the talk, Jenia Tevelev communicated to us an example showing that one cannot relax the hypothesis of Conjecture 3 to allow replacing $D|_{W_L}$ by a nef divisor D' on W_L (that is, not necessarily pulled back from X_n).

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Resolutions in topology, algebra, and symplectic geometry

Jeff Hicks

Resolutions through mirror symmetry. Broadly speaking, a resolution is a method for replacing an object in mathematics with a sequence of simpler objects and gluing relations between them. Here are a few examples of resolutions that occur within different areas of mathematics.

Algebra. Given a module over a ring (or more generally a sheaf on a variety) we might replace the module with an exact sequence of "simpler" modules — for instance, free modules or line bundles. For example, if we consider the point $z = [1:1] \in \mathbb{P}^1$, the skyscraper sheaf $\mathcal{O}_z$ is resolved by the following exact sequence of line bundles:

$$\mathcal{O}_z \leftarrow \mathcal{O} \xleftarrow{x-y} \mathcal{O}(-1).$$

Properties of $\mathcal{O}_z$ can then be computed from the exact resolution instead.

Topology. Given a topological space X, we might try to glue together X as a sequence of mapping cones of simpler objects. The first example to consider is the circle, which can be presented as the mapping cone:

$$S^1 \cong \mathrm{cone}(\bullet \leftarrow (\bullet \sqcup \bullet)).$$

You can build any CW complex in this manner by iterating the mapping cone operation.

Differential geometry. A real smooth compact manifold M with metric can be understood via a *Morse function* $f : M \to \mathbb{R}$. This data provides a "handle decomposition" of M — a set of instructions for building the manifold in terms of the critical points of f. The function $\sin : S^1 \to \mathbb{R}$ the circle tells us to build the circle by starting with the minimum of the function f and then attaching it to an edge corresponding to the maximum.

Mirror Symmetry. These three stories are related via mirror symmetry, a proposed dictionary between symplectic and algebraic geometry. Working backward: Morse functions $f : M \to \mathbb{R}$ relate to resolutions of Lagrangian submanifolds in the Fukaya category $\mathrm{Fuk}(T^*M)$, which (when appropriately generalized) is derived equivalent to the category of coherent sheaves on a mirror space X. This strategy was employed in [1, 2] to produce results about the Rouquier dimension of the derived category of toric varieties.

Encoding topology via noncommutative neighborhoods. The above example suggests that resolutions in algebraic geometry can be frequently encoded in topology. One way to obtain this relation is through *cellular resolutions*. In the given example, the subcategory of line bundles $\mathrm{Pic}^{dg}(\mathbb{P}^n)$ generate $\mathrm{Coh}^{dg}(\mathbb{P}^n)$ and can be given the structure of a graded category. We then consider polyhedral complexes which are labeled by objects of $\mathrm{Pic}^{dg}(\mathbb{P}^n)$ along with monomial morphisms labeling each codimension 1 incidence between strata. We suggest the following alternative approach which has the topology of the resolution "baked in".

Definition 1 ([4]). *Let $\mathcal{C}$ be a dg-category over $\mathbb{C}$. A* noncommutative neighborhood *of an object $C \in \mathcal{C}$ is an augmented $\mathbb{C}$-algebra $\epsilon : R \to \mathbb{C}$ with a (derived) functor*

$$\mathcal{V}_C : \mathcal{C} \to \mathrm{Perf}(R)$$

satisfying the following properties:

- *$\mathcal{V}_C$ is a localization of categories*
- *$\mathcal{V}_C(C) \simeq \mathbb{C}$*
- *$\mathcal{V}_C$ is fully faithful on the object C, that is:*

$$\hom_{\mathcal{V}_C}(C, C) \simeq \hom_{\mathrm{Perf}(R)}(\mathbb{C}, \mathbb{C}).$$

The definition is designed to mimic the definition of an affine neighborhood $\operatorname{Spec} R$ of a point $x \in X$, where we let $\mathcal{C} = \mathrm{Coh}^{dg}(X)$, $\epsilon : R \to R/\mathfrak{m} \cong \mathbb{C}$, and $C = \mathcal{O}_x$. We say that this is a topological noncommutative neighborhood if there exists a manifold L so that R is the dg-algebra of chains on the based loop space of L. In computed examples the space L is usually aspherical (so that $\pi_k(L) = 0$ for all $k > 1$); in this setting $R = \mathbb{C}[\pi_1(L)]$. The notation $\mathcal{V}$ is inspired from *Viterbo restriction functor* from symplectic geometry associated with the Weinstein neighborhood of an exact Lagrangian submanifold.

Example 2. *Let X_Σ be any toric variety. Then we have an affine chart given by the big torus $(\mathbb{C}^*)^n \subset X_\Sigma$ containing the identity point of the torus. It follows that* *there is a localization of categories*

$$\mathrm{Coh}^{dg}(X_\Sigma) \to \mathrm{Coh}^{\mathrm{dg}}((\mathbb{C}^*)^{\mathrm{n}}) \simeq \mathrm{Perf}(\mathbb{C}[\mathbb{Z}^n]) \simeq \mathrm{Perf}(\mathbb{Z}[\pi_1(T^n)]).$$

This allows us to identify the torus from the resolution of L.

Example 3. *Let $H = V(x_0 + \cdots + x_n) \subset \mathbb{P}^n$. The functor*

$$\begin{aligned}
&(1) \qquad \mathcal{V} : \mathrm{Coh}^{dg}(\mathbb{P}^n) \simeq \mathrm{Perf}\left(\mathrm{End}\left(\left(\bigoplus_{i=1}^{n-1} \mathcal{O}(i)\right) \oplus L_{\mathcal{O}(n)}\mathcal{O}(n+1) \oplus \mathcal{O}(n)\right)\right)\\
&(2) \qquad \to \mathrm{Perf}(\mathrm{End}\left(L_{\mathcal{O}(n)}\mathcal{O}(n+1) \oplus \mathcal{O}(n)\right))\\
&(3) \qquad \to \mathrm{Perf}(\mathbb{C}[F_n])
\end{aligned}$$

is an example of a noncommutative neighborhood of $\mathcal{O}_H$, where $F_n = \pi_1(S^1 \vee \cdots \vee S^1)$ is the free group. In this example, we see that this noncommutative neighborhood is compatible with restriction functors induced by the inclusion of the coordinate hyperplanes $\mathbb{P}^{n-1} \to \mathbb{P}^n$.

Local and Global Realizability Problems in Tropical Geometry. Based on the work of [2], we may expect the following:

Question 4. *Can we associate to tropical curve given by $\Sigma(1) \subset \Sigma_{\mathbb{P}^n}$ a nonpositively curved cusped manifold $L_{\Sigma(1)}$ whose boundary is a union of $(n+1)$ real tori of dimension $n-1$? Furthermore, what is the appropriate replacement $\mathcal{P}$ for $\mathrm{Perf}(\pi_1(L))$ when the space L has singularities so that we have a noncommutative neighborhood $\mathrm{Coh}^{dg}(X_\Sigma) \to \mathcal{P}$ for a line of $\mathbb{P}^n$ in general position?*

Evidence from symplectic geometry suggests that $\mathcal{P}$ will only be defined after picking an augmentation for the Legendrian link of the cusp locus of $L_{\Sigma(1)}$. This inspires the following more general question:

Question 5. *For a matroid M, does there exist a Legendrian Λ_M whose augmentations correspond to realizations of M?*

The Donaldson divisor theorem in symplectic geometry suggests that NC neighborhoods may arise via deformations.

Question 6. *For a large class of subvarieties $Y \subset X_\Sigma$ can we find:*

- *A deformation of categories $\mathcal{C}_t$ whose general fiber is isomorphic to* $\mathrm{Coh}^{dg}(X_\Sigma)$,
- *An object $C_0 \in \mathrm{Ob}(\mathcal{C}_0)$ deforming to $\mathcal{O}_Y$;*
- *A (possibly singular) nonpositively curved space space L_V; and*
- *A noncommutative neighborhood of C_0 given by $\mathcal{C}_0 \to \mathrm{Perf}(\mathbb{C}[\pi_1(L_V)])$ (where the latter category may be appropriately modified to account for the singularities of L_V)?*

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Cellular resolutions of normal toric embeddings

Christine Berkesch

(joint work with Lauren Cranton Heller, Greg Smith, Jay Yang)

The construction of families of explicit free resolutions is a central, long-standing challenge in the field of commutative algebra, as such examples are key to understanding the geometry of syzygies. In [3], cellular resolutions are constructed in the sense of [7, Definition 4.3] without a finiteness assumption on the underlying complex. We generalize this construction via a stratification function.

Definition 1. *For a lattice $L \subset \mathbb{Z}^n$, a* stratification *is a pair of a map $\psi\colon \mathbb{R}L \to \mathbb{Z}^n$ and a cell structure on the torus $\mathbb{R}L/L$ satisfying the following three conditions.*

(1) *The map ψ is constant on open cells of the cell structure Ψ on $\mathbb{R}L$ induced by the cell structure of the torus $\mathbb{R}L/L$;*
(2) *The map ψ is continuous with the standard topology on $\mathbb{R}L$ and the Alexandrov topology on $\mathbb{Z}^n$ with poset structure induced by the component-wise order; and*
(3) *For all points p and q in $\mathbb{R}L$ such that $p - q \in L \subseteq \mathbb{Z}^n$, we have $\psi(p) - \psi(q) = p - q$.*

Fix a stratification function $\psi\colon \mathbb{R}L \to \mathbb{Z}^n$. Then ψ is constant within each open cell $\sigma \in \Psi$ by Definition 1.(1). Denote by $\psi(\sigma)$ the value $\psi(\boldsymbol{a})$ for $\boldsymbol{a} \in \sigma$. Then ψ can be used to assign the Laurent monomial $\boldsymbol{x}^{\psi(\sigma)}$, making a labeled cell complex from Ψ. To define a cellular complex supported on Ψ from ψ, let $S = \Bbbk[x_1, x_2, \ldots, x_n] = \Bbbk[\mathbb{N}^n] \subset \Bbbk[\mathbb{Z}^n]$, where $\Bbbk$ is an algebraically closed field.

Definition 2. *For an open cell σ in Ψ, let $S\sigma$ denote the free $\mathbb{Z}^n$-graded S-module with generator σ in degree $\psi(\sigma)$. The* cellular complex F_ψ *with underlying support Ψ is the free $\mathbb{Z}^n$-graded S-complex where*

$$(F_\psi)_i = \bigoplus_{\substack{\sigma\in\Psi \\ \dim\sigma = i}} S\sigma = \bigoplus_{\substack{\sigma\in\Psi \\ \dim\sigma = i}} S\big(-\psi(\sigma)\big)$$

and differential given by

$$\partial\sigma = \sum_{\substack{\tau\in\Psi \\ \dim\tau = i}} \epsilon(\sigma,\sigma')\frac{\boldsymbol{x}^{\psi(\sigma)}}{\boldsymbol{x}^{\psi(\sigma')}}\sigma' \,.$$

Finally, define the module associated to ψ by

$$M_\psi \coloneqq S \cdot \left\{ \boldsymbol{x}^{\psi(\boldsymbol{a})} \;\middle|\; \boldsymbol{a} \in \mathbb{R}L \right\} \subseteq \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}] = \Bbbk[\mathbb{Z}^n].$$

Define the $\mathbb{Z}^n$-graded group algebra

$$S[L] \coloneqq S[y^{\boldsymbol{a}} \mid \boldsymbol{a} \in L] \subseteq S[y_1^{\pm 1}, y_2^{\pm 1}, \dots, y_n^{\pm 1}],$$

where $\deg(\boldsymbol{x}^{\boldsymbol{a}} \boldsymbol{y}^{\boldsymbol{b}}) = \boldsymbol{a} + \boldsymbol{b} \in \mathbb{Z}^n$. As in [3, Section 3], there is an $S[L]$-module structure on S given by the isomorphism $S \cong S[L] / \langle \boldsymbol{y}^{\boldsymbol{a}} - 1 \mid \boldsymbol{a} \in L \rangle$ so that applying the functor $- \otimes_{S[L]} S$ to F_ψ replaces each copy of $S[L]$ corresponding to a coset of cells σ with a copy of S twisted by the image $\overline{\psi(\sigma)}$ of $\psi(\sigma)$ under the quotient map $\eta \colon \mathbb{Z}^n \to \mathbb{Z}^n / L$:

$$F_{\psi,i} \otimes_{S[L]} S = \bigoplus_{\substack{\sigma \in \Psi / L \\ \dim \sigma = i}} S(-\overline{\psi(\sigma)}).$$

Theorem 3. *Let $L \subset \mathbb{Z}^n$ be a lattice and $\psi \colon \mathbb{R}L \to \mathbb{Z}^n$ a stratification. If*

$$\Psi_{\leq \boldsymbol{u}} \coloneqq \overline{\left\{ \boldsymbol{a} \in \mathbb{R}L \;\middle|\; \psi(\boldsymbol{a}) \leq \boldsymbol{u} \right\}}$$

is contractable or empty for each $\boldsymbol{u} \in \mathbb{Z}^n$, then the S-complex F_ψ is a $\mathbb{Z}^n$-graded cellular resolution of the S-module M_ψ and $F_\psi \otimes_{S[L]} S$ is a $\mathbb{Z}^n/L$-graded resolution of the S-module $M_\psi \otimes_{S[L]} S$.

To connect this work to toric embeddings, given a lattice $L \subset \mathbb{Z}^n$,

$$I_L \coloneqq \left\langle \boldsymbol{x}^{\boldsymbol{u}} - \boldsymbol{x}^{\boldsymbol{v}} \;\middle|\; \boldsymbol{u} - \boldsymbol{v} \in L \right\rangle \subseteq S$$

is the *toric ideal* determined by L. If L defines a toric embedding of smooth projective toric varieties $\varphi \colon Y \hookrightarrow X$, then S is the Cox ring of X and I_L defines the image of Y in X.

The normalization of S/I_L can be resolved using Theorem 3 by using the *ceiling stratification*, which is defined by the coordinate-wise ceiling function

$$\psi \colon \mathbb{R}L \to \mathbb{Z}^n \quad \text{given by} \quad \psi(\boldsymbol{v}) = \lceil \boldsymbol{v} \rceil, \tag{1}$$

where the cell complex structure Ψ on $\mathbb{R}L \subset \mathbb{R}^n$ is determined by the intersection of $\mathbb{R}L$ with the integral translates of the coordinate hyperplanes in $\mathbb{R}^n$.

Theorem 4. *If $\psi = \lceil \boldsymbol{v} \rceil$ and Ψ is given by the hyperplanes in $\mathbb{R}L$ parallel to the coordinate axes of $\mathbb{R}^n$, then the complex $F_\psi \otimes_{S[L]} S$ is a resolution of the normalization $\overline{S/I_L}$ whose length is equal to the rank of the lattice L.*

Example 5. *When $X = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)\big)$ is the second Hirzebruch surface, whose fan has rays $(1,0), (0,1), (-1,2), (0,-1)$, and Y is the identity point in the torus of X, then $S = \Bbbk[x_0, x_1, x_2, x_3]$, the toric ideal is $I_L = \langle x_0 - x_2, x_1 x_2^2 - x_3 \rangle$, and Figure 1 shows the image of ψ on one fundamental domain of $\mathbb{R}L/L$. The* *resolution of $\overline{S/I_L}$ constructed in Theorem 4 is*

$$\begin{matrix} S(0,0) \\ \oplus \\ S(1,-1) \end{matrix} \xleftarrow{\begin{bmatrix} x_3 & x_3 & -x_1x_2 & -x_0x_1 & x_2-x_0 \\ -x_2 & -x_0 & 1 & 1 & 0 \end{bmatrix}} \begin{matrix} S(0,-1)^2 \\ \oplus \\ S(1,-1)^2 \\ \oplus \\ S(-1,0) \end{matrix} \xleftarrow{\begin{bmatrix} -x_0 & 1 & 0 \\ x_2 & -1 & 0 \\ 0 & -x_0 & 1 \\ 0 & x_2 & -1 \\ -x_3 & 0 & x_1 \end{bmatrix}} \begin{matrix} S(-1,-1) \\ \oplus \\ S(0,-1) \\ \oplus \\ S(1,-1) \end{matrix} \leftarrow 0\,.$$

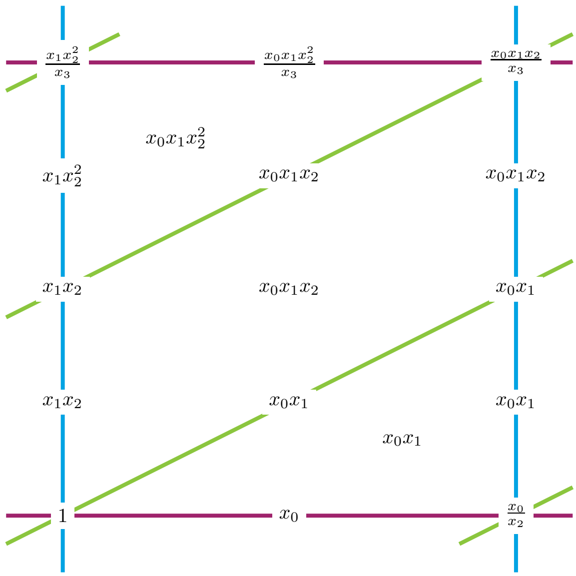

FIGURE 1. The monomial labeling on the fundamental domain of $\mathbb{R}L/L$ in $\mathbb{R}L$ that is northeast of the origin in $\mathbb{R}L \cong \mathbb{R}^2$ for Example 5. Colored lines indicate the intersection of $\mathbb{R}L$ with the four types of coordinate planes in $\mathbb{Z}^4$, two of which coincide in the horizontal red lines.

Theorem 4 generalizes the resolution of the diagonal of a unimodular toric variety in [2, Section 6]. Further, the ceiling stratification resolution of Theorem 4 sheafifies the resolution of Hanlon–Hicks–Lazarev [6], and thus the minimal free resolution of $\overline{S/I_L}$ studied by Brown–Erman [4] embeds into that $F_\psi \otimes_{S[L]} S$. Other choices of stratification yield the resolutions of the diagonal of Anderson [1] and in some cases, Favero–Huang [5].

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Singular Log Structures and Smoothings of toric Fano 3-folds

Alessio Corti
(joint work with Tim Gräfnitz, Helge Ruddat)

I report on joint work with Tim Gräfnitz and Helge Ruddat [CGR25].

Motivation and context. Our motivation is to prove a criterion for smoothing a toric Fano 3-fold. In fact, more generally, we wish for criteria for deforming a toric Fano variety to a mildly (terminal, klt, ...) singular one.

Sufficient conditions for the existence of a smoothing are given in [CHP24], but we wish for necessary and sufficient conditions.

Method. We propose to construct smoothings by the following steps:

(1) Let $P \subset N$ be a 3-dimensional lattice polytope. We assume that $0 \in P$ is a strictly interior point, and that the vertices of P are primitive lattice vectors. We want to deform the toric Fano variety X_P whose fan is the spanning fan of P.
(2) Let $Q = P^\star \subset M$ be the polar of P. Denote by $\star Q \subset M$ the central subdivision. Then $\star Q$ is the moment complex of a reducible Fano variety X, which, by a standard construction in toric geometry, is a degeneration of X_P. Instead of deforming X_P, we deform X.
(3) Endow X with a (singular) log structure, thus promoting it to a log scheme $X^\dagger$.
(4) Construct a log resolution $f \colon Y^\dagger \to X^\dagger$.
(5) Deform $Y^\dagger$.

In summary, in this talk I talk about a class of singular log structures, called *zero-mutable log structures*, defined in [CGR25]. We conjecture that these log structures admit log crepant log resolutions. The conjecture implies an optimal criterion for smoothing Gorenstein toric Fano 3-folds.

N.B. There are other approaches to smoothing, see the recent work [F25].

What is a log structure, Alessio? In this talk I don't give the textbook definition of log structure because that would take up the whole time. Instead, I try to give some feeling for the log structures that we actually work with.

A special case. A key special caise is when $X = \cup X_i$ is a (simple) normal crossing scheme with irreducible components X_i. There is a naturally defined line bundle

$\mathcal{LS}$ on $\operatorname{Sing} X$ such that for all i,j and $D = X_i \cap X_j$

$$\mathcal{LS}_{|D} = (N_D X_i) \otimes (N_D X_j) \otimes \mathcal{O}_D(\sum_{k\neq i,j} X_k)$$

A (smooth) log structure is a never-vanishing section $s \in \Gamma(\mathcal{LS})$. A *singular* log structure is a section $s \in \Gamma(\mathcal{LS})$ that is allowed to vanish somewhere. The locus $Z(s)$ where s vanishes is the *singular locus* of the log structure.

More generally, Helge Ruddat and I [CR23] have a notion of *generic toroidal crossing (gtc) space*, a stratified scheme

$$X = \coprod X^\star_\tau$$

that at the generic point of each stratum looks like a toroidal crossing scheme. (Decorated with additional data that satisfies conditions that we don't discuss here.) There is a sheaf $\mathcal{LS}$ on X, constructed by gluing line bundles $\mathcal{L}_\rho$ on the closures of the codimension-one strata $X_\rho = \overline{X^\star_\rho}$. The gluing is specified by isomorphisms that exist on codimension-two strata $X_\omega = \overline{X^\star_\omega}$:

$$\otimes_{\omega\leq\rho} \mathcal{L}^{k_\rho}_{\rho|X_\omega} \cong \mathcal{O}_{X_\omega}$$

Just as before, a smooth log structure is a never-vanishing section $s \in \Gamma(\mathcal{LS})$. A *singular* log structure is a section $s \in \Gamma(\mathcal{LS})$ that is allowed to vanish somewhere. The locus $Z(s)$ where s vanishes is the *singular locus* of the log structure.

(In the paper [CR23] we don't actually go as far as spelling out how to make a category out of these log structures: we don't spell out the sort of data that give a morphism $f^\dagger\colon X^\dagger \to Y^\dagger$ in terms of these sections of $\mathcal{LS}$.)

Zero-mutable log structures and their resolutions.

Definition 1. *Let $L \cong \mathbb{Z}^2$ be a rank-2 lattice. A* log datum *on L is a finite set*

$$S = \{(e_i, \mu_i) \mid i \in I\}$$

of pairs (e_i, μ_i) where

(i) For all $i \in I$, $e_i \in L$, and we write $e_i = \ell_i u_i$ with u_i primitive and $\ell_i \in \mathbb{N}_+$. We assume that the e_i are pairwise distinct.

(ii) For all $i \in I$, $\mu_i \vdash \ell_i$ is a partition. We write

$$\mu_i = (\ell_{i,1} \geq \ell_{i,2} \geq \cdots \geq \ell_{i,k(i)} \geq 0) \quad \textit{where} \quad \ell_i = \sum_{k=1}^{k(i)} \ell_{i,k}$$

(iii) The datum is subject to the condition: $\sum_{i\in I} e_i = 0$; equivalently, $\sum_{i\in I} \ell_i u_i = 0$.

Let S be a log datum on L, let $M = L \oplus \mathbb{Z}$, and write $u = (0,1) \in M$. Consider the fan Σ in M with maximal cones the cones $\sigma_i = \langle u_i, u_{i+1}, u\rangle_+$, $i \in I$. We also denote by $\rho_i = \langle u_i, u\rangle_+$ the *walls* of the fan (i.e., the codimension-1 cones), and by $\omega = \langle u\rangle_+$ the (unique) *joint* (i.e., the codimension-2 cone). This fan is the moment polyhedral complex of an affine 3-fold that we denote by X_S or simply X. Denoting by $k[\Sigma]$ the Stanley–Reisner ring of the fan, we have that $X = \operatorname{Spec} k[\Sigma]$. Note that X is reducible with irreducible components $X_i = \operatorname{Spec} k[\sigma_i \cap M]$ intersecting along the surfaces:

$$D_i = \operatorname{Spec} k[\rho_i \cap M] = X_{i-1} \cap X_i$$

Finally note that $k[\omega \cap M] = k[u]$, and the surfaces D_i intersect along the curve $\mathbb{A}^1_u = \operatorname{Spec} k[u]$, the affine line with coordinate u.

It can be seen that the set $\mathrm{LS}_{k^\dagger}(X)$ of log structures on X over $k^\dagger$ (compatible with the gtc structure) is the set of data consisting of:

(1) For all $i \in I$, a function $f_i \in k[\rho_i \cap M]^\times$ subject to the condition:
(2) $\prod_{i\in I}(f_i|k[u])^{e_i} = 1$.

The functions $f_i \in k[\sigma_i \cap M]$ are called *wall functions*. The condition is called the *joint compatibility condition.*

Definition 2. *Let L be a rank-2 lattice and $S = \{(e_i, \mu_i) \mid i \in I\}$ a log datum for L. A log structure given by wall functions f_i is* subordinated to S *if for all i*

$$f_i = \prod_{k=1}^{k(i)} f_{i,k}$$

where for all i, k:

(i) $Z_{i,k} = (f_{i,k} = 0) \subset D_i$ is a smooth curve;
(ii) $f_{i,k}|k[u] = u^{\ell_{i,k}}$.

In this talk, I am not giving the formal definition of zero-mutable log structure. To do that one first needs to define mutations. By definition, a zero-mutable log structure is a log structure subordinated to a log datum that can be mutated to a trivial log datum. We conjecture that zero-mutable log structures always have log resolutions.

A simple example The simplest example of a zero-mutable log structure is the A_1 log structure. Consider the vectors in $L = \mathbb{Z}^2$:

$$e_1 = (1,0), \quad e_2 = (0,2), \quad e_3 = (-1,-2)$$

The A_1 *log datum* is the 0-mutable log datum

$$S = \Big\{\big(e_1, (1)\big), \big(e_2, (1^2)\big), \big(e_3, (1)\big)\Big\}$$

Let now $L = \mathbb{Z}^2$ and S be the A_1 log datum. Writing

$$x = x^{(1,0,0)}, \quad y = x^{(-1,-2,0)}, \quad z = x^{(0,1,0)}, \quad w = x^{(0,-1,0)}, \quad u = x^{(0,0,1)}$$

in $M = L \oplus \mathbb{Z}$, we have

$$X = \begin{cases} xy - w^2 & = 0 \\ zw & = 0 \end{cases} \subset \mathbb{A}^5_{x,y,z,w,u}$$

and:

$$D_1 = \mathbb{A}^2_{x,u}, \quad D_2 = \mathbb{A}^2_{z,u}, \quad D_3 = \mathbb{A}^2_{y,u}$$

and $X = X_1 \cup X_2 \cup X_3$ where:

$$X_1 = \mathbb{A}^3_{x,z,u}, \quad X_2 = \mathbb{A}^3_{y,z,u}, \quad X_3 = (xy - w^2 = 0) \subset \mathbb{A}^4_{x,y,w,u}$$

The A_1 log structure is given by the wall functions:

$$f_1(x,u) = u, \quad f_2(z,u) = u^2 + a_1uz + a_2z^2, \quad f_3(y,u) = u$$

where a_1, a_2 are general constants; in particular, most importantly, f_2 has 2 distinct roots on $\mathbb{P}^1$.

Here are some concluding remarks. The generic log structure on X, namely the one with $f_2 = u^2 + az$, is not smoothable. Gross and Siebert already observed this and called this phenomenon the "denominator problem." We can interpret this fact as follows. The (log) deformation problem of the A_1 log structure is badly behaved: it has two irreducible components, one consisting of smoothings and the other of deformations to the generic log structure on X. However, the deformation problem of the A_1 singularity is perfectly well behaved. Everywhere you go, there will be good deformation problems doing good things and bad deformation problems doing bad things, but it takes a log structure to make a good deformation problem do bad things.

The solution for us is to take a log resolution $f^\dagger \colon Y^\dagger \to X^\dagger$ and deform $Y^\dagger$ instead.

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Birational King's Conjecture

Andrew Hanlon

(joint work with Matthew R. Ballard, Christine Berkesch, Michael K. Brown, Lauren Cranton Heller, Daniel Erman, David Favero, Sheel Ganatra, Jesse Huang)

The bounded derived category of coherent sheaves, $D(X)$, on a complex algebraic variety X is a powerful invariant that serves as a universal repository for homological algebra on X. A foundational result of Beilinson [2] shows that $D(\mathbb{P}^n)$ is generated by the Serre twisting sheaves $\mathcal{O}(-n), \dots, \mathcal{O}(-1), \mathcal{O}$. The computations

$$\mathrm{RHom}(\mathcal{O}(-j), \mathcal{O}(-j)) \cong \mathbb{C} \tag{1}$$

$$\mathrm{RHom}(\mathcal{O}(-j), \mathcal{O}(-i)) = 0 \text{ for } i > j \tag{2}$$

$$\mathrm{RHom}^d(\mathcal{O}(-j), \mathcal{O}(-i)) = 0 \text{ for } d > 0 \tag{3}$$

make the line bundles $\mathcal{O}(-n), \dots, \mathcal{O}(-1), \mathcal{O}$ a full (generation) strong (3) exceptional (1, 2) collection, which we will abbreviate to FSEC. The presence of a FSEC of line bundles is a statement on the computability and finiteness of $D(\mathbb{P}^n)$.

King's conjecture [10] posited that Beilinson's result extends to smooth projective toric varieties. More precisely, he conjectured that if X is a smooth projective toric variety then $D(X)$ has a FSEC of line bundles. Unfortunately, King's conjecture is false [8] (see also [11, 4]) and derived categories of toric varieties can be more complicated. Kawamata [9] proved that $D(X)$ has a full exceptional collection of objects, but it is desirable to have a more explicit and computable description.

In an influential Oberwolfach report [3], Bondal proposed that $D(X)$ for a toric X should be described in terms of a stratification on the real torus $M_{\mathbb{R}}/M$ where M is the character lattice of X. If Σ is a fan for X, then every ray $\rho \in \Sigma(1)$ induces a function $f_\rho \colon M_{\mathbb{R}}/M \to \mathbb{R}/\mathbb{Z}$ given by pairing with the primitive generator u_ρ of ρ, that is, $f_\rho(m) = \langle m, u_\rho \rangle$. The *Bondal stratification* is the stratification induced by the toric hyperplane arrangement of the level sets $f_\rho^{-1}(0)$ for all $\rho \in \Sigma(1)$. Moreover, for every point $\theta \in M_{\mathbb{R}}/M$, we can assign a line bundle $\mathcal{O}_X(D_\theta)$ given by

$$D_\theta = \sum_{\rho \in \Sigma(1)} \lfloor -\langle \theta, u_\rho \rangle \rfloor D_\rho \tag{4}$$

Note that (4) is well-defined up to linear equivalence and is constant on strata of the Bondal stratificaton. We call

$$\Theta = \{\mathcal{O}_X(D_\theta) : \theta \in M_{\mathbb{R}}/M\} \tag{5}$$

the *Bondal-Thomsen collection* of X to recognize Bondal's geometric definition and earlier work of Thomsen [12] realizing Θ as the summands of the pushforward of the structure sheaf under the toric Frobenius morphism.

When $X = \mathbb{P}^n$, the Bondal-Thomsen collection coincides with the Beilinson collection. Although one cannot hope for Θ to be or contain a FSEC in general, Bondal claimed that Θ generates $D(X)$, which was proved in [5, 6]. Moreover, [6] gave an explicit short resolution of the diagonal for any smooth toric X using products of elements in Θ built geometrically from the Bondal stratification and generalizing Beilinson's resolution for $\mathbb{P}^n$. Although the resolution of the diagonal gives a concrete description of $D(X)$, we can go further.

A curious feature of the Bondal stratification, the Bondal-Thomsen collection, and the construction from [6] is that they only depend on $\Sigma(1)$. The rays of the fan do not determine X but rather a finite set of toric varieties parameterized by the secondary fan. We let

$$X_1 = X, \dots, X_k$$

be the toric varieties corresponding to the top-dimensional cones Γ_i of the secondary fan. For example, when $X = \mathbb{P}^n$, k is 1. When X is the Hirzebruch surface of type 3, $k = 2$ and $X_2 = \mathbb{P}(1, 1, 3)$ is a weighted projective space. As illustrated in this latter example, the X_i may not all be smooth, but there is always an associated smooth toric Deligne-Mumford stack $\mathcal{X}_i$.

Now, let $\widetilde{\mathcal{X}}$ be any smooth toric Deligne-Mumford stack with proper birational toric morphisms $\pi_i\colon \widetilde{\mathcal{X}} \to \mathcal{X}_i$. These conditions imply that the derived pullbacks π_i^* are fully faithful and their adjoints π_{i*} are localizations. We define the *Cox category* $D_{\mathrm{Cox}}(X) \subset D(\widetilde{\mathcal{X}})$ to be the full subcategory generated by all of the $\pi_i^* D(\mathcal{X}_i)$. Further, one can show that for any $\theta \in M_{\mathbb{R}}/M$, D_θ is anti-effective and thus $-D_\theta \in \Gamma_i$ for some i. Thus, for every $\theta \in M_{\mathbb{R}}/M$, we can define $\mathcal{O}_{\mathrm{Cox}}(D_\theta) = \pi_i^* \mathcal{O}_{\mathcal{X}_i}(D_\theta)$ where i is such that $-D_\theta$ lies in Γ_i. By varying θ as in (5), we obtain a Bondal-Thomsen collection $\Theta \subset D_{\mathrm{Cox}}(X)$.

The Cox category was introduced in [1] where we prove a birational realization of King's conjecture:

Theorem. *If X is a smooth projective toric variety, Θ is a full strong exceptional collection for $D_{\mathrm{Cox}}(X)$*

Note that since each derived pushforward π_{i*} preserves elements of Θ, this computable and explicit description of the Cox category can be used to uniformly describe the derived categories of the $\mathcal{X}_i$. In addition, we show that the resolution of the diagonal from [6] lifts to a resolution of the diagonal of the Cox category providing an explanation for its dependence only on $\Sigma(1)$.

As an outline and the details of the proof are both given in [1], we will spend the remainder of this note on the combinatorial structure of the FSEC. In [1], it is shown that

$$\mathrm{Hom}(\mathcal{O}_{\mathrm{Cox}}(D_\theta), \mathcal{O}_{\mathrm{Cox}}(D_\varphi)) = \mathbb{C}\langle P_\theta \cap (M - \varphi)\rangle \tag{6}$$

where

$$P_\theta = \{x \in M_{\mathbb{R}} : \langle x, u_\rho\rangle \geq \lfloor -\langle \theta, u_\rho\rangle \rfloor \text{ for all } \rho \in \Sigma(1)\}$$

is the polytope of the nef divisor $-D_\theta$ on the corresponding toric stack in the secondary fan.

This gives a purely combinatorial description of the generators and morphisms for the Cox category as follows. Given some collection of primitive integral vectors u_ρ indexed by a set $\Sigma(1)$, we call a lattice polytope P in $M_{\mathbb{R}}$ a *Bondal-Thomsen polytope* if the defining half-spaces of P are all of the form $\langle x, u_\rho\rangle \geq a_\rho$ for $\rho \in \Sigma(1)$ allowing for some to be virtual and there exists a point $\theta \in M_{\mathbb{R}}$ such that

$$a_\rho + 1 > \langle -\theta, u_\rho\rangle \geq a_\rho$$

for all $\rho \in \Sigma(1)$. More colloquially, Bondal-Thomsen polytopes are those with normal vectors u_ρ containing a point that is as integrally close as possible to all the facets, including virtual ones. This is equivalent to $P = P_\theta$ from above. Whenever the set of integral vectors contains a real basis, there are finitely many Bondal-Thomsen polytopes and these polytopes determine the Cox category precisely through (6). Given the role of the Bondal-Thomsen collection in derived categories of toric varieties, we expect these polytopes to have interesting combinatorial structure.

Question. *What are the combinatorial properties and significance of the Bondal-Thomsen polytopes? For example, can they and their lattice points be enumerated* *or these numbers estimated? How does this finite set of polytopes fit more generally in the theory of lattice polytopes?*

One result towards this vague question, giving an upper bound on how big of a denominator one needs to allow in rational θ to obtain all of the Bondal-Thomsen polytopes, appears in [7].

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Gotzmann's persistence theorem for smooth projective toric varieties

Patience Ablett

The Hilbert scheme $\mathrm{hilb}_P(\mathbb{P}^n)$ is a widely studied object in algebraic geometry. From an algebraic perspective this scheme parameterises homogeneous saturated ideals with a given Hilbert polynomial. In [1] Haiman and Sturmfels extended these ideas to the multigraded setting, where the Hilbert scheme parameterises homogeneous ideals with a given Hilbert function in a polynomial ring graded by some abelian group. A case of particular interest is when the multigraded ring in question is the Cox ring, denoted $\mathrm{cox}(X)$, of a smooth projective toric variety X. In this case the Hilbert function of a homogeneous ideal in $\mathrm{cox}(X)$ eventually agrees with a polynomial for degrees sufficiently far into the nef cone of X. We can therefore define $\mathrm{hilb}_P(X)$ in an analogous manner to $\mathrm{hilb}_P(\mathbb{P}^n)$. We consider the parameter space of homogeneous ideals in $\mathrm{cox}(X)$ which are saturated with respect to the irrelevant ideal of X and have Hilbert polynomial P.

It is natural to ask what properties of standard-graded Hilbert schemes have an extension to the smooth projective toric variety case. Of particular interest are two theorems of Gotzmann, regularity and persistence, which are used in the explicit construction of the Hilbert scheme. Gotzmann's regularity theorem gives a bound on the Castelnuovo-Mumford regularity of a standard-graded ideal as introduced in [4]. Gotzmann's persistence theorem can be informally restated in the following way. For a homogeneous ideal I in a standard-graded polynomial ring, a Hilbert polynomial $P(t)$, and sufficiently large $d \in \mathbb{N}$, checking that $H_I(d) = P(d)$ and $H_I(d+1) = P(d+1)$ guarantees that $P_I(t) = P(t)$. Here, $H_I(d)$ denotes the Hilbert function of S/I and $P_I(t)$ denotes the associated Hilbert polynomial. The surprising aspect here is that by checking the value of $H_I(d)$ in just two points we have identified the polynomial $P_I(t)$, as opposed to the expected $\deg(P_I) + 1$ points. Combining this result with Gotzmann's regularity theorem allows us to obtain explicit equations for the Hilbert scheme $\mathrm{hilb}_P(\mathbb{P}^n)$.

Maclagan and Smith define Castelnuovo-Mumford regularity for the multigraded case [2] and generalise Gotzmann's regularity theorem [3] to any smooth projective toric variety. Their generalisation recovers Gotzmann's original result when $X = \mathbb{P}^n$. In this talk we discuss a generalisation of our earlier informal restatement of Gotzmann's persistence theorem to any smooth projective toric variety. We begin with the case of the product of projective spaces, before extending to the more general setting.

We introduce some terminology that will allow us to state Theorem 1. Let X be a smooth projective toric variety of Picard rank s. For a homogeneous ideal $J \subseteq \mathrm{cox}(X)$, we have an associated Hilbert polynomial $P_J(t_1, \dots, t_s)$. We say that $P(t_1, \dots, t_s) \in \mathbb{Q}[t_1, \dots, t_s]$ is a Hilbert polynomial on $\mathrm{cox}(X)$ if there exists some ideal $J \subseteq \mathrm{cox}(X)$, homogeneous with respect to the $\mathbb{Z}^s$-grading, such that $P_J = P$.

Theorem 1. *Let S be the Cox ring of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$, let $I \subseteq S$ be an ideal, homogeneous with respect to the $\mathbb{Z}^s$-grading, and let $P(t_1, \dots, t_s) \in \mathbb{Q}[t_1, \dots, t_s]$ be a Hilbert polynomial on S. Then there exists a point $(d_1, \dots, d_s) \in \mathbb{N}^s$ such that if $H_I(b_1, \dots, b_s) = P(b_1, \dots, b_s)$ for all points $(b_1, \dots, b_s) \in \mathbb{N}^s$ with $b_i \in \{d_i, d_i + 1\}$, then $P_I = P$.*

Theorem 1 says that for an ideal I in the Cox ring of the product of s projective spaces we can confirm $P_I(t_1, \dots, t_s) = P(t_1, \dots, t_s)$ simply by checking $H_I(b_1, \dots, b_s)$ at the vertices of an s-dimensional hypercube in $\mathbb{N}^s$. This is therefore a natural s-dimensional generalisation of our earlier informal restatement of Gotzmann's original persistence result. Note that the proof of Theorem 1 is constructive, allowing us to find appropriate $(d_1, \dots, d_s)$ for a given I and P. As outlined in [3](Algorithm 6.3), $P_I(t_1, \dots, t_s)$ can have as many as $\binom{n_1+\cdots+n_s+s}{s}$ coefficients, so we would naively expect to check $H_I(d_1, \dots, d_s)$ in this many points to verify $P_I(t_1, \dots, t_s)$. Even when setting $n_i = 1$ for all i we have $\binom{n_1+\cdots+n_s+s}{s} \geq 2^s$ for all $s \in \mathbb{Z}_{\geq 1}$. Further, $\binom{n_1+\cdots+n_s+s}{s}$ grows significantly faster than 2^s as s increases, meaning our theorem is a significant improvement.

We can extend Theorem 1 to more general smooth projective toric varieties. In particular, our result still depends solely on the Picard rank of the variety.

Theorem 2. *Let X be a smooth projective toric variety of Picard rank s. Let $R = cox(X)$, with a $\mathbb{Z}^s$-grading resulting from the identification $pic(X) \cong \mathbb{Z}^s$. Let $J \subseteq R$ be an ideal, homogeneous with respect to the $\mathbb{Z}^s$-grading. Let $P(t_1, \ldots, t_s)$ be a Hilbert polynomial on R. Then there exists at most 2^s points $(r_1, \ldots, r_s) \in \mathbb{N}^s$ such that checking $H_J(r_1, \ldots, r_s) = P(r_1, \ldots, r_s)$ for all of these points guarantees that $P_J = P$.*

The 2^s points of Theorem 2 form a zonotope, generalising the hypercube seen in Theorem 1. Again, the Hilbert polynomial of a subscheme of a d-dimensional smooth projective toric variety X of Picard rank s can have up to $\binom{s+d}{d}$ coefficients. This means that naively we have to check $\binom{s+d}{d}$ points to find this polynomial. Theorem 2 ensures that we only have to check 2^s points to find the Hilbert polynomial for any smooth projective toric variety. For $d \geq s$, we have $\binom{s+d}{d} \geq 2^s$ for all $s \geq 1$, and the binomial coefficient also grows faster as s increases. The key contribution is that the complexity of finding the Hilbert polynomial no longer depends on the dimension of X, and only depends on the Picard rank.

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Oda's strong factorization conjecture (and relatives)

Karim Adiprasito

Consider two smooth, birational toric varieties; or in other words, consider two smooth simplicial fans with the same support. What can be said about their relation? Oda conjectured conjectured that the two varieties were related by a sequence of smooth blowups and blowdowns (this is known as the *weak factorization conjecture*) and stronger that these operations could be reordered so that the blowups precede the blowdowns. This is the *strong factorization conjecture*, and it is still open, in contrast to the weak conjecture that was resolved by Morelli and independently Wlodarzyk [Mor96, Wło97].

In another direction of relaxation, one can relax the location of the blowups, and instead allow for blowups (and blowdowns) at rationally smooth points. This could be called the *rational strong factorization conjecture*. Further, one can investigate this question entirely in the combinatorial category, and one arrives at the *Alexander conjecture* [Ale30].

We have the following result:

Theorem 1 (Adiprasito-Pak, [AP24])**.** *The rational strong factorization conjecture, and in particular the Alexander conjecture, are true.*

We presented a proof of this result. The algorithm developped by the authors is not immediately easy to modify to restrict to smooth blowups and blowdowns. Still, it seems worthwhile to try. Indeed, in the weak factorization conjecture, the rational version was known before, see for instance [Pac91] and was smoothing was a comparatively easy step. We proposed one possible route, that leads to a reasonable conjecture that, if true, would imply the strong factorization conjecture. To state this conjecture, it is useful to remind ourselves of the Knudsen-Mumford-Waterman theorem.

Theorem 2 (Knudsen-Mumford-Waterman, [KKMS73])**.** *Given any lattice polytope P, there exists a positive integer n such that the dilation nP has a unimodular triangulation.*

Let us now consider a general polytope d-dimensional polytope P, and a set of points V that lies in P and contains its vertices. Assume V is totally ordered in some way, i.e. we have a list $(v_1, \cdots v_m)$ enumerating the vertices of V.

The *placing triangulation* associated to this total order is obtained as follows: T_0 is the lexicographically least d-dimensional simplex of the order.

And T_i is obtained from the T_{i-1} by considering the minimal vertex of v of V not in T_{i-1}. Consider further the minimal subcomplex H_i of ∂T_{i-1} so that if $v * H_i$ denotes the cone over H_i with apex v, the support of

$$v * H_i \cup T_{i-1}$$

is convex. Define then $T_i := v * H_i \cup T_{i-1}$. This process terminates with a triangulation of P. The following conjecture implies the strong factorization conjecture:

Conjecture 3 (Refined Knudsen-Mumford-Waterman)**.** *Consider a lattice polytope P. There is a positive integer n and a total order on the lattice points of nP so that the placing triangulation of nP given by the order is unimodular.*

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Toric sheaves and polytopes

Klaus Altmann
(joint work with Andreas Hochenegger and Frederik Witt)

Over a smooth projective toric variety X we describe and construct *toric sheaves*, that is, reflexive sheaves with a linearised action of the torus, in terms of polytope-valued *Weil decorations*. As an application, we discuss (i) the universal extension of nef line bundles and (ii) sheaf cohomology. The talk is based on [3].

Divisors and virtual polytopes. Let k be an algebraically closed field of characteristic zero, and $X = \mathbb{P}(\Delta)$ be the toric variety over k defined by the ample and smooth lattice polytope Δ. We denote T its torus, $\Sigma = \mathcal{N}(\Delta)$ its underlying fan normal to Δ, and M its character lattice.

We first consider toric sheaves of rank one, i.e., line bundles $\mathcal{O}(D) = \mathcal{O}_X(D) \subseteq \mathrm{k}[M]$ given by a *toric divisor* D in $\mathrm{Div}_T = \mathrm{Div}_T(X)$, the free abelian subgroup generated by the closures D_ρ of the torus orbits corresponding to the rays ρ of Σ.

Further, let $\mathrm{Pol}^+ = \mathrm{Pol}^+(\Sigma)$ be the semigroup of *compatible polytopes*, namely the lattice polytopes $\nabla \subseteq M_\mathbb{R}$ whose normal fan $\mathcal{N}(\nabla)$ is refined by Σ so that the associated line bundle

$$\mathcal{O}(\nabla) := \mathcal{O}_X\big(\mathrm{div}(\nabla)\big) \subseteq \mathrm{k}[M]$$

given by the toric divisor $\mathrm{div}(\nabla) = -\sum_{\rho\in\Sigma(1)} \min\langle\nabla,\rho\rangle D_\rho$ is nef. The resulting map $\mathrm{Pol}^+ \to \mathrm{Div}_T$ extends to the Grothendieck group $\mathrm{Pol} \to \mathrm{Div}_T$ where it defines a group isomorphism for X is projective. We can therefore write any toric divisor D as the formal difference $\nabla^+ - \nabla^-$ of two compatible polytopes.

Theorem [1]**,** [4]**.** The cohomology in degree 0 is given by

$$\mathrm{H}^i(X, \nabla^+ - \nabla^-)_0 = \widetilde{\mathrm{H}}^{i-1}\big(\nabla^- \setminus \nabla^+, \mathrm{k}\big),$$

where $\widetilde{\mathrm{H}}$ denotes reduced singular cohomology.

Corollary. The global sections of $\mathcal{O}(\nabla^+ - \nabla^-)$ correspond to the lattice points of

$$(\nabla^+ : \nabla^-) := \{u \in M_\mathbb{R} \mid u + \nabla_- \subseteq \nabla_+\}.$$

Remark. Passing to a "common denominator" of two given elements inside the Grothendieck group of Pol^+ allows us to extend the usual binary relations and operations such as $\subseteq$ and $\cap$. In particular, $\nabla\colon \mathrm{Div}_T \to \mathrm{Pol}$, $\nabla(D) = \nabla^+ - \nabla^-$ defines a meet semi-lattice isomorphism between the semi-lattices $(\mathrm{Div}, \wedge, \leq)$ (with $D \wedge D' = \sum \min(a_\rho, a'_\rho) D_\rho$) and $(\mathrm{Pol}, \cap, \subseteq)$; see also the table below.

Div_T	invertible sheaves $\subseteq \mathrm{k}[M]$	Pol
$D \leq D'$	$\mathcal{O}(D) \subseteq \mathcal{O}(D')$	$\nabla(D) \subseteq \nabla(D')$
$D + D'$	$\mathcal{O}(D) \cdot \mathcal{O}(D') = \mathcal{O}(D) \otimes \mathcal{O}(D')$	$\nabla(D) + \nabla(D')$
$-D$	$\mathcal{O}(D)^{-1} = \mathcal{H}om_\mathcal{O}(\mathcal{O}(D), \mathcal{O})$	$-\nabla(D)$
$D \wedge D'$	$\mathcal{O}(D) \cap \mathcal{O}(D')$	$\nabla(D) \cap \nabla(D')$

Weil decorations. Now consider a toric sheaf $\mathcal{E}$ of arbitrary rank. Let $j\colon T \hookrightarrow X$ be the open embedding of the torus, and E be the torus invariant sections of $j_*\mathcal{E}|_T$ over T. We can then consider $\mathcal{E}$ as a subsheaf of $j_*\mathcal{E}|_T = E \otimes_{\mathrm{k}} \mathrm{k}[M]$. For $e \in E \setminus \{0\}$ we define the subsheaf

$$\mathcal{E}(e) := \mathcal{E} \cap \big(\mathrm{k} \cdot e \otimes (j_T)_* \mathcal{O}_T\big) \subseteq \mathcal{E}$$

which is reflexive and of rank 1. The isomorphic subsheaf $\mathcal{L}(e)$ of $(j_T)_*\mathcal{O}_T = \mathrm{k}[M]$ resulting via

$$\begin{tikzcd} \mathcal{E}(e) \arrow[r, hook] \arrow[d, "\cdot 1/e"', "\cong"] & e \cdot \mathrm{k}[M] \arrow[d, "\cong"', "\cdot 1/e"] \\ \mathcal{L}(e) \arrow[r, hook] & \mathrm{k}[M] \end{tikzcd}$$

induces a well-defined Weil divisor $D(e)$ with $\mathcal{O}(D(e)) = \mathcal{L}(e)$. Adding formally the divisor $D(0) = (\infty)$ yields the *Weil decoration*

$$\mathcal{D} = \mathcal{D}_{\mathcal{E}}\colon E \to \mathrm{Div}_T(X), \quad e \mapsto D(e)$$

that actually determines $\mathcal{E}$ and satisfies

$$\mathcal{D}(e) \geq \mathcal{D}(e') \wedge \mathcal{D}(e'')$$

for all $e \in \mathrm{span}\{e', e''\}$. We usually tacitly identify toric divisors with (virtual) polytopes and consider $\mathcal{D}$ to be Pol-valued.

One can prove that the image of $\mathcal{D}_{\mathcal{E}}$ is finite. Pooling together the elements with equal divisor yields a stratification $\{\mathbb{S}\}$ of E defining a finite join semi-lattice; here, $\mathbb{S}' \leq \mathbb{S}$ if and only if $\mathbb{S}'$ is in $\overline{\mathbb{S}}$, the closure of $\mathbb{S}$. In fact, the Weil decoration defines an anti-semi-lattice isomorphism onto its image, that is,

$$\mathcal{D}_{\mathcal{E}}(\mathbb{S}' \vee \mathbb{S}) = \mathcal{D}_{\mathcal{E}}(\mathbb{S}') \wedge \mathcal{D}_{\mathcal{E}}(\mathbb{S}).$$

Remark. We can also consider Weil decorations for reflexive sheaves over non-toric varieties. These are now maps $\mathcal{D}\colon \mathcal{E}_\eta \to \mathrm{Div}(X)$ defined on the generic stalk in the same way as before. In particular, $\mathcal{D}_{\mathcal{E}}(e + e') \geq \mathcal{D}_{\mathcal{E}}(e) \wedge \mathcal{D}_{\mathcal{E}}(e')$ still holds. Yet, $\mathcal{E}_\eta$ is a $K(X)$-vector space; for $f \in K(X)$ we have $\mathcal{D}(f \cdot e) = \mathrm{div}(f) + \mathcal{D}(e)$. However, we can always choose a *framing* E, a k-vector space inside $\mathcal{E}_\eta$ with $\mathcal{E}_\eta = E \otimes K(X)$. The image of $\mathcal{D}|_E$ is finite giving again a stratification on E.

Next, toric morphisms $\lambda\colon \mathcal{E} \to \mathcal{F}$ translate into linear maps $\lambda\colon E \to F$ between the spaces of invariant sections together with the condition $\mathcal{D}_{\mathcal{E}}(e) \leq \mathcal{D}_{\mathcal{F}}(\lambda(e))$ for all $e \in E$. Regarding $D(e)$ as a virtual polytope $\nabla(e)$, $\mathcal{E} \to \mathcal{F}$ is surjective if and only if $E \to F$ is surjective, and for all f, $\nabla_{\mathcal{F}}(f) = \bigcup_{\lambda(e)=f} \nabla_{\mathcal{E}}(e)$.

Universal extensions. As a first application of Weil decorations, we revisit the toric universal extension sheaf of nef line bundles [2].

For $\nabla^+, \nabla^- \in \mathrm{Pol}^+$ we consider the *virtual intersection* $Q := \nabla^+ \cap \nabla^-$. Making ∇^+, ∇^- and Q simultaneously nef yields the connected components C_ν of $\nabla^+ \setminus \nabla^-$

as well as the *compatible* polytopes $\nabla_\nu := Q \cup C_\nu$ in Pol^+. By the theorem above

$$\mathrm{Ext}^1(\nabla^-, \nabla^+)_0 = \mathrm{H}^1\big(X, \nabla^+ - \nabla^-\big)\big)_0 = \widetilde{\mathrm{H}}^0(\nabla^- \setminus \nabla^+) = \bigoplus_{\nu=0}^{s} \mathrm{k}[\nabla_\nu] \big/ \textstyle\sum[\nabla_\nu].$$

This yields the exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(\nabla^-, \nabla^+)_0^\vee \xrightarrow{\ \iota\ } \textstyle\bigoplus_{\nu=0}^{s} \mathrm{k}[\nabla^\nu] \xrightarrow{(1\ldots1)} \mathrm{k}\cdot e_- \longrightarrow 0.$$

Translated to toric sheaves this induces the exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(\nabla^-, \nabla^+)_0^\vee \otimes\ \mathcal{O}(\nabla^+) \longrightarrow \mathcal{E}(\nabla^-, \nabla^+) \xrightarrow{\ 1\ } \mathcal{O}(\nabla^-) \longrightarrow 0\ ,$$

where $\mathcal{E}(\nabla^-, \nabla^+)$ is the desired toric universal extension. It is defined by the Weil decoration sending the strata

$$\mathbb{S}_\nu := \mathrm{k}[\nabla_\nu] \setminus \{0\} \mapsto \nabla_\nu, \quad \mathbb{S}_+ = \ker(1, \ldots, 1) \setminus \{0\} \mapsto \nabla^+ \quad \text{and} \quad \eta \mapsto Q,$$

where η stands for the generic stratum.

Cohomology of toric sheaves. Choosing Δ sufficiently ample we assume that the Weil decoration $\mathcal{D}^+$ of $\mathcal{E}^+ = \mathcal{E} \otimes \mathcal{O}(\Delta)$ is amply, i.e., $\mathrm{int}(\mathrm{Pol}^+)$-valued. We then define a constructible subsheaf $\mathcal{F}(\mathcal{E}) \subseteq \underline{E}$ on $\Delta \subseteq M_\mathbb{R}$ by

$$\mathcal{F}(\mathcal{E})(U) := \{e \in E \mid U \subseteq \mathcal{D}^+(e)\} \subseteq E \quad \text{for } U \subseteq \Delta \text{ open.}$$

Theorem [3]**.** The cohomology in degree 0 is given by

$$\mathrm{H}^\ell(X, \mathcal{E})_0 = \mathrm{H}^\ell\big(\Delta, \mathcal{F}(\mathcal{E})\big).$$

Furthermore, we have the spectral sequence

$$E_1^{-\ell,q} := \bigoplus_{\mathbb{S}<\mathbb{T}} \bigoplus_{\mathbb{S}=\mathbb{S}_0<\ldots<\mathbb{S}_\ell=\mathbb{T}} \overline{\mathbb{S}} \otimes \widetilde{\mathrm{H}}^{q-1}\big(\Delta \setminus \mathrm{int}_\Delta\, \mathcal{D}^+(\mathbb{T})\big) \Rightarrow \mathrm{H}^{q-\ell}\big(\Delta, \mathcal{F}(\mathcal{E})\big).$$

In particular, this shows that every nefly decorated toric sheaf $\mathcal{E}$, that is, $\mathcal{D}(e)$ is nef for all $e \in E \setminus \{0\}$, is automatically acyclic.

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Cryptomorphisms of toric vector bundles

Diane Maclagan
(joint work with Bivas Khan)

The goal of this talk, which was based on [KM], was to present several different descriptions of toric vector bundles. The word "cryptomorphism" comes from matroid theory, and means a "complicated isomorphism".

A toric vector bundle $\mathcal{F}$ on a toric variety X_Σ with dense torus $T \cong (K^*)^n$ is a T-equivariant vector bundle on X_Σ.

Klyachko [Kly89] gave an equivalence of categories between the category of toric vector bundles with equivariant morphisms and a certain category of filtered vector spaces, which allows classification to be reduced to combinatorics and linear algebra.

We summarize this in the affine case. Consider an affine toric variety $U_\sigma = \operatorname{Spec}(R)$, where $R = K[\sigma^\vee \cap M]$). A toric vector bundle $\mathcal{F}$ on U_σ is the sheafification $\tilde{P}$ for a locally free R-module P. The T-equivariance implies that P is $M \cong \mathbb{Z}^n$-graded, and free:

$$P = \oplus_{i=1}^m R(\alpha_i)$$

for some $\alpha_i \in M$. Set $E = \oplus_{i=1}^m K$, which we may regard as the fiber over the identity of the torus. Evaluating a section in P at the identity of the torus gives an element of E, and Klyachko's formulation records the subspaces obtained by evaluating M-homogeneous components of P. It is only necessary to record the subspaces when σ is a ray of the fan of Σ, when they are indexed by $\mathbb{Z}$, and we write $E^i(j) \subseteq E$ for the subspace indexed by j on the ith ray.

Example 1. *An important example of a toric vector bundle is the tangent bundle on $\mathbb{P}^2$, which is toric with rays spanned by $\mathbf{v}_0 = (-1,-1)$, $\mathbf{v}_1 = (1,0)$, and $\mathbf{v}_2 = (0,1)$. In this case $E = K^2$, and the filtrations are:*

$$E^i(j) = \begin{cases} E & j \leq 0 \\ \operatorname{span}(\mathbf{v}_i) & j = 1 \\ \{\mathbf{0}\} & j \geq 2 \end{cases}$$

for $0 \leq i \leq 2$.

Polymatroids and matroids.

Definition 2. *A polymatroid on a set $\mathcal{H}$ is a function $2^{\mathcal{H}} \to \mathbb{N}$ satisfying*

(1) $\operatorname{rk}(\emptyset) = 0$,

(2) $\operatorname{rk}(A) \leq \operatorname{rk}(B)$ *when $A, B \in \mathcal{H}$ with $A \subseteq B$, and*

(3) $\operatorname{rk}(A \cap B) + \operatorname{rk}(A \cup B) \leq \operatorname{rk}(A) + \operatorname{rk}(B)$ *for all $A, B \in \mathcal{H}$.*

Example 3. *When $\mathcal{H} = \{0,1,2\}$, the function*

$$\operatorname{rk}(\emptyset) = 0, \operatorname{rk}(\{0\}) = \operatorname{rk}(\{1\}) = \operatorname{rk}(\{2\}) = 1,$$
$$\operatorname{rk}(\{0,1\}) = \operatorname{rk}(\{0,2\}) = \operatorname{rk}(\{1,2\}) = \operatorname{rk}(\{0,1,2\}) = 2$$

is a polymatroid.

A collection of subspaces $\mathcal{H}$ gives a polymatroid, which we call realisable, if

$$\mathrm{rk}(A) = \dim\left(\sum_{V \in A} V\right).$$

Definition 4. *The polymatroid of $\mathcal{F}$ is the realisable polymatroid $\mathcal{H} = \{\cap_i E^i(j_i)\}$, where the intersections range over all rays of Σ..*

A flat of a polymatroid is $A \subseteq \mathcal{H}$ maximal for a given rank. In the realizable case flats are the subspaces spanned by the given subspaces. The collection of all flats forms a poset under inclusion, with smallest element $\emptyset$, and largest element $\mathcal{H}$.

A toric vector bundle can be described by giving a realisation of a polymatroid, and for each ray of the fan a descending filtration of flats of the polymatroid indexed by $\mathbb{Z}$ satisfying certain compatibility constraints. This description separates the combinatorial from the linear algebra aspects of toric vector bundles, and is the foundation of the tropical approach in [KM].

Example 5. *In the case of the tangent bundle on $\mathbb{P}^2$, the polymatroid is the one of Example 3, which has realisation given by the subspaces*

$$\mathrm{span}((-1,-1)), \mathrm{span}((1,0)), \mathrm{span}((0,1)).$$

The flats of this polymatroid are:

$$\emptyset, \{0\}, \{1\}, \{2\}, \{0,1,2\}.$$

The chains of flats are $\emptyset \subseteq \{i\} \subseteq \{0,1,2\}$ for $i = 0,1,2$, with the indexing as in Example 1.

Cox module. Since $\mathcal{F}$ is a locally free sheaf on X_Σ, by work of Cox [Cox95] we have $\mathcal{F} = \tilde{P}$ for a module P over the Cox ring $S = K[x_1, \ldots, x_s]$ of X_Σ (where $s = |\Sigma(1)|$. The T-equivariance of $\mathcal{F}$ implies that P is $\mathbb{Z}^s$-graded. A presentation of P can be derived from a choice of surjection

$$\oplus_{i=1}^m \mathcal{O}(D_{\mathbf{w}_i}) \twoheadrightarrow \mathcal{F}, \tag{1}$$

where the $D_{\mathbf{w}_i}$ are torus-invariant divisors, as we now describe.

For a torus invariant divisor $\sum a_i D_i$ we write $\mathbf{a} \in \mathbb{Z}^s$ for the coefficient vector, which we use to denote the grading. The surjection (1) induces a linear map $\phi\colon K^m \to E \cong K^r$ by taking the fiber over the identity. A *circuit* of ϕ is an element of $\ker(\phi)$ of minimal support $\mathrm{supp}(\mathbf{c}) = \{i : c_i \neq 0\}$. The support of a circuit is a circuit of the matroid corresponding to ϕ. For a collection of vectors $\{\mathbf{d}_1, \ldots, \mathbf{d}_l\}$ we write $\min(\mathbf{d}_i)$ for the vector with $\min(\mathbf{d}_i)_j = \min((\mathbf{d}_i)_j)$. For a circuit $\mathbf{c}$, we define $\mathbf{u}_{\mathbf{c},i} = \mathbf{d}_{\mathbf{w}_i} - \min(\mathbf{d}_{\mathbf{w}_l} : c_l \neq 0)$.

Theorem 6. *Given a surjection*

$$\oplus_{i=1}^m \mathcal{O}(D_{\mathbf{w}_i}) \twoheadrightarrow \mathcal{F},$$

set

$$P = \oplus_{i=1}^m S(\mathbf{d}_{\mathbf{w}_i})/R,$$

where R is the submodule of $\oplus S(\mathbf{d}_{\mathbf{w}_i})$ generated by

$$\sum_{i=1}^{m} c_i \mathbf{x}^{\mathbf{u}_{\mathbf{c},i}} \mathbf{e}_i$$

for circuits $\mathbf{c} \in \mathbb{Z}^s$. Then $\mathcal{F} = \widetilde{P}$.

Example 7. *For the tangent bundle on $\mathbb{P}^2$, we have an equivariant surjection coming from the Euler sequence:*

$$\mathcal{O}(D_0) \oplus \mathcal{O}(D_1) \oplus \mathcal{O}(D_2) \twoheadrightarrow T\mathbb{P}^2 \to 0.$$

The Cox ring of $\mathbb{P}^2$ is $S = \mathbb{C}[x_0, x_1, x_2]$, and we have $T\mathbb{P}^2 = \widetilde{P}$ for

$$P = S(D_0) \oplus S(D_1) \oplus S(D_1)/\langle x_0\mathbf{e}_0 + x_1\mathbf{e}_1 + x_2\mathbf{e}_2\rangle.$$

Restrictions of toric vector bundles. An advantage of toric vector bundles is that their invariants can be computed using the combinatorics. To extend this advantage to an arbitrary vector bundle $\mathcal{E}$ on a variety Y, one method would be to choose a good embedding $i\colon Y \to X_\Sigma$ of Y into a toric variety X_Σ for which there is a toric vector bundle $\mathcal{F}$ on X_Σ with $\mathcal{E} = i^*\mathcal{F}$.

Question 8. *For which vector bundles is this possible?*

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Locally trivial deformations of toric varieties

Sharon Robins

(joint work with Nathan Ilten)

Deformation theory is a vital tool for understanding the local structure of a moduli space around a fixed object X. A systematic approach to studying infinitesimal deformations of X involves defining a functor, Def_X , that associates, for every local Artin ring, the set of deformations over that ring up to equivalence. Despite a theoretical understanding of Def_X , explicit computations with examples are challenging. This work investigates the deformation theory of a smooth complete toric varieties $X = X_\Sigma$ using the combinatorics of its defining fan Σ.

The study of deformation of smooth complete toric varieties was initiated by N. Ilten [2]. It has been shown that smooth complete toric varieties may have obstructions to their deformations ([3]). However, the homogeneous first-order deformations of smooth complete toric varieties are unobstructed ([5]). This positions smooth complete toric varieties in a unique middle ground within deformation theory: they are not as well behaved as Calabi-Yau varieties (which have unobstructed deformations) but they are also not as poorly behaved as varieties that follow Murphy's law [4] (which can have very bad singularities in their deformation spaces).

Let $X = X_\Sigma$ be a smooth complete toric variety with the corresponding fan Σ with character lattice M and dual lattice N. Given a ray ρ, the primitive lattice generator is denoted by n_ρ and evaluation of n_ρ at $u \in M$ is denoted by $\rho(u)$. For a given ray ρ and $u \in M$, we define the simplicial complex

$$V_{\rho,u} := \bigcup_{\sigma\in\Sigma} \operatorname{Conv}\left\{ n_{\rho'} \;\middle|\; \begin{matrix} \rho'(u) < 0 & \text{if } \rho' \neq \rho \\ \rho'(u) < -1 & \text{if } \rho' = \rho \end{matrix} \right\}_{\rho'\in\Sigma(1)\cap\sigma} \subseteq N \otimes \mathbb{R}.$$

As shown in [2, 3], the sheaf cohomology of the tangent sheaf $mathcalT_X$ satisfy the isomorphisms

$$H^k(X, \mathcal{T}_X) \cong \bigoplus_{\rho,u} \widetilde{H}^{k-1}(V_{\rho,u}, \mathbb{K}),$$

for every $k \geq 1$. This approach allows us to compute many examples where either $H^1(X, \mathcal{T}_X)$ or $H^2(X, \mathcal{T}_X)$ vanishes. As a consequence, we obtain the unobstructedness result for toric varieties of lower Picard rank.
Let X be a smooth complete toric variety of Picard rank one or two. Then X has unobstructed deformations. Furthermore, X is rigid unless it is the projectivization of a direct sum of line bundles on $\mathbb{P}^1$ such that the largest and smallest degrees differ by at least two. For smooth complete toric surfaces, $H^2(X, \mathcal{T}_X)$ is always zero ([2, Corollary 1.5]). However, in general, one cannot expect either $H^1(X, \mathcal{T}_X)$ or $H^2(X, \mathcal{T}_X)$ to vanish for all higher-dimensional toric varieties. Thus, to better understand obstructions, a detailed study of the deformation functor Def_X is required. Using the same simplicial complexes $V_{\rho,u}$, we extend the correspondences between sheaf cohomology of the tangent sheaf and simplicial cohomology in a functorial way by defining a combinatorial deformation functor $\operatorname{Def}_\Sigma$.

Theorem 1 ([1, Corollary 5.1.5])**.** *Let $X = X_\Sigma$ be a complete toric variety smooth in codimension 2 and $\mathbb{Q}$-factorial in codimension 3. Then the functor Def_X of deformations of X is isomorphic to the combinatorial deformation functor $\operatorname{Def}_{\widetilde{\Sigma}}$, where $\widetilde{\Sigma}$ be a any simplcial subfan of Σ containing all three dimensional cones of Σ.*

For smooth complete toric varieties of Picard rank 3 it is possible for both $H^1(X, \mathcal{T}_X)$ and $H^2(X, \mathcal{T}_X)$ to be non-zero. We obtain the following reduction, which will appear in future work.

Theorem 2. *Let X be a smooth complete toric variety of Picard rank three. Then X is unobstructed unless it is a projective bundle over the Hirzebruch surface $\mathbb{F}_r = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(r))$.*

In dimensions 3, we classify all obstructed cases among smooth complete toric varieties of Picard rank three. The minimal obstruction in those case are always quadratic or cubic. For some specific examples, we are also able to completely compute the hull.

Theorem 3. *There are toric varieties with the following hulls:*

(1) $\mathbb{K}[[t_1, t_2, t_3, t_4]]/(t_3^2 t_4)$ *(*[1, Example 6.4.2]*)*
(2) $\mathbb{K}[[t_1, \dots, t_7]]/(t_1 t_4 + t_2 t_6)$ *(*[1, Example 6.4.4]*)*
(3) $\mathbb{K}[[t_1, \dots, t_{2r-1}]]/\langle t_1 \rangle \cdot \langle t_2, t_4, \dots, t_{2r-2} \rangle$ *for* $r \geq 2$ *(*[1, Example 6.4.5]*)*

Thus, there exist examples where the hull has a generically non-reduced component, or is irreducible but singular at the origin, or has a pair of irreducible components whose difference in dimension is arbitrarily large. None of these phenomena had previously been observed for deformation spaces of smooth toric varieties.
In all the obstructed examples for which we were able to compute the hull, we observe that the deformation space is equal to its tangent cone. It is natural to ask the following questions:

Question 4. *Is the deformation space of a smooth complete toric variety determined by the tangent cone?*

Question 5. *Is there any relation between the number of irreducible components of the deformation space and the number of distinct simplicial complexes that correspond to first-order deformations?*

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Positive monotone Hamiltonian spaces and reflexive GKM graphs

Silvia Sabatini

Given a compact symplectic manifold (M, ω), we say that it is *positive monotone* if there exists $\lambda > 0$ such that $c_1 = \lambda[\omega]$, where c_1 denotes the first Chern class of the tangent bundle of M endowed with an almost complex structure compatible with ω. Therefore the symplectic form can be rescaled so that

$$c_1 = [\omega]\,, \tag{1}$$

condition that will be henceforth assumed. If (M, ω) is a Kähler manifold with Kähler form satisfying (1), then it can be proved that (M, ω) is indeed a Fano variety. Hence positive monotone symplectic manifolds can be regarded as a natural generalization of Fano varieties, the latter giving a large pool of examples. Then the following is a natural

Question: How far is a positive monotone symplectic manifold from being (homotopy equivalent/homeomorphic/diffeomorphic/symplectomorphic to) a Fano variety?

We recall that all Fano varieties are simply connected, their Todd genus is one, for each n the set of n-dimensional Fano varieties forms a bounded family and their volume $c_1^n[M]$ is bounded above by a constant that only depends on n (see [8]). In dimension 12 and higher, Fine and Panov [4] construct positive monotone symplectic manifolds that are not simply connected, thus implying that not all positive monotone symplectic manifolds are homotopy equivalent to Fano varieties.

Our goal is the study of positive monotone symplectic manifolds in the presence of *Hamiltonian torus actions*, namely (effective) symplectic actions of a compact torus T that admit a moment map $\psi\colon M \to \mathfrak{t}^*$.
Their analysis and the methods used strongly depend on the topology of the fixed point set M^T, the dimension of the manifold $\dim(M)$, as well as the *complexity* of the action (defined as $\dim(M)/2 - \dim(T)$). Note that such methods, which rely solely on the symplectic structure and the existence of the action, will generally differ from those used to study Fano varieties. However, they can also be employed to classify Fano varieties when these are endowed with a holomorphic action of a complex torus.

Mirroring facts that hold for Fano varieties we ask the following
Questions:

(A) Under which conditions on the complexity and/or the topology of the fixed point set are they *simply connected* and their *Todd genus is one*?
(B) Are there *finitely many* of them up to (equivariant) homotopy equivalence/ diffeomorphism/symplectimorphism or complex cobordism?
(C) Is a *classification* (up to a suitable notion of equivalence) possible?

First steps toward answering the above questions.
Henceforth we assume that the Hamiltonian T-space is positive monotone with $c_1 = [\omega]$.

(A) In [3], Fine and Panov conjecture that all positive monotone Hamiltonian S^1-spaces of dimension 6 are diffeomorphic to Fano varieties. In [9], Lindsay and Panov prove that such spaces are all simply connected and have Todd genus one; the same result holds in arbitrary dimension, if the complexity is assumed to be one [10]. The methods used in these two papers are very different: In the first, where the authors deal with complexity two spaces, “hard symplectic topology” (inter alia Seiberg-Witten theory) as well as equivariant methods are used. In the second, the properties of the Duistermaat-Heckman function play an essential role.

Questions (B) and (C) have been completely answered in [2], if the complexity is one and the space is *tall*, namely no reduced space is a point. Here the authors analyze the invariants of complexity one tall Hamiltonian T-spaces in the Karshon-Tolman classification to deduce that in the positive monotone case all such invariants “come” from smooth toric Fano varieties. It follows that all such spaces are equivariantly symplectomorphic to Fano varieties endowed with suitable torus actions and that there are finitely many of them up to equivariant symplectomorphism.

(Smooth) toric Fano varieties have a well known combinatorial object associated to them, namely a (smooth) *reflexive polytope.* In the symplectic setting this could be indeed defined as the moment map image of a positive monotone symplectic toric manifold satisfying (1), suitably translated so that the unique interior point is the origin.

Let Δ be a smooth reflexive polytope in $\mathbb{R}^n$ with lattice $\mathbb{Z}^n$. Consider the set E of edges of Δ and let $m(e)$ be the *magnitude* of E, namely the *affine length* of $e \in E$ (the length measured with respect to the lattice $\mathbb{Z}^n$). It turns out that this integer corresponds to the symplectic area of the sphere S_e^2 that is mapped through the moment map to e. We call the collection $\cup_{e\in E}\{S_e^2\}$ of these spheres the *toric one skeleton* of the corresponding symplectic toric manifold.

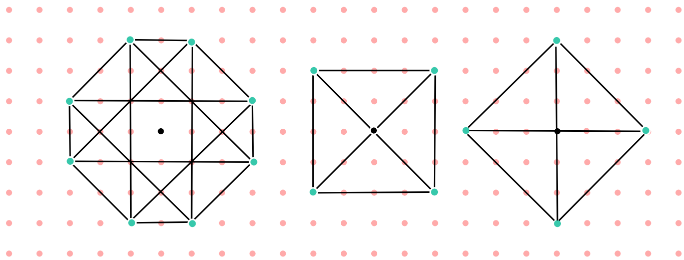
In [7] the authors prove that

(*) $\sum_{e\in E} m(e)$ *only depends on the f-vector, or equivalently on the h-vector of* Δ.

We recall that the h-vector of Δ corresponds to the vector of even Betti numbers of the manifold M so that $\psi(M) = \Delta$. The above phenomenon is just a manifestation of a much broader fact regarding Hamiltonian torus actions with isolated fixed points. In this case, the combinatorial object of interest is a multigraph, where the edges should be thought of as the images of the spheres corresponding to the *toric one skeleton*, the Poincaré dual to the union of these spheres being the Chern class c_{n-1} (see [6, 7]). In this multigraph the edges are directed and each of them is labeled by the weight of the T action at the initial point of the edge, which corresponds to a fixed point. A very special case of the above is that of **Hamiltonian GKM spaces**, where the weights at each fixed point are pairwise linearly independent; the corresponding graph is called *GKM graph.* If the corresponding manifold is positive monotone, then the GKM graph is called a **reflexive GKM graph**. The examples in the picture correspond to the GKM graphs of some special coadjoint orbits endowed with a symplectic structure satisfying $c_1 = [\omega]$ (for more details see [7]).

The following strategy was used in [6, 1, 5] to classify all the possible labeled multigraphs (or reflexive GKM graphs) arising from a manifold of a given dimension and given Betti numbers:

- From a generalization of (*), the sum of the magnitudes is constant. Since each of them is a positive integer, *the set of possible magnitudes is finite.*
- The magnitudes satisfy more complicated relations; for instance each of them must be divisible by the *index* of the manifold.
- From the set of magnitudes satisfying these extra relations, in [6] the authors developed an algorithm and a computer program that compute the set of possible weights.
- The weights determine many equivariant topological invariants: the (equivariant) Chern numbers, Chern classes and cohomology ring, as well as the reflexive GKM graph.



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Collapsing tautological bundles

Alex Fink

(joint work with Andrew Berget)

This talk is based on [3].

Let $[L] \in \mathrm{Gr}(r, k^n)$ be the point of the Grassmannian representing a linear subspace $L \subseteq k^n$. The torus orbit closure $\overline{T \cdot [L]}$ is normal [14] and projective, so it is the toric variety $X_{P(L)}$ of its moment polytope $P(L)$. Such $P(L)$ are examples of *matroid base polytopes* $P(M)$ [8], polytopes in $\mathbb{R}^n$ whose vertices lie in $\{0,1\}^n$ and whose edges are in directions $e_i - e_j$. The condition on edges is equivalent to the existence of a toric surjection $X_n \twoheadrightarrow X_{P(M)}$ from the permutahedral toric variety.

Subdivisions of a matroid polytope into matroid polytopes have arisen multiple times in the algebraic geometry literature. For David Speyer in 2005 [11] they arose as the objects classifying tropical linear spaces. Speyer conjectured there a tight upper bound on the number of faces in each dimension in a matroid polytope subdivision. Later [12, 7] he constructed a matroid function $h(M) \in \mathbb{Z}[t]$ and reduced his original conjecture to showing that the coefficients of $h(M)$ were of alternating sign, which he proved (using Kawamata–Viehweg vanishing) when all matroids involved are representable over $\mathbb{C}$. The speaker with Shaw and Speyer [6] further reduced the conjecture to showing that the leading coefficient $\omega(M)$ of $h(M)$ was of the expected sign. It is this last conjecture that is proved for all matroids in [3].

Let f_L be the composite map $X_n \twoheadrightarrow X_{P(L)} \hookrightarrow \mathrm{Gr}(r, \mathbb{C}^n)$, and let $\mathcal{S}_L$ and $\mathcal{Q}_L$ be the pullbacks along f_L of the tautological sub- and quotient bundles $\mathcal{S}$, $\mathcal{Q}$ on $\mathrm{Gr}(r, \mathbb{C}^n)$. When the matroid M has no representation L, there still exist equivariant K-classes $[\mathcal{S}_M], [\mathcal{Q}_M] \in K_T^0(X_n)$ which play the role of the classes of the bundles $\mathcal{S}_L$ and $\mathcal{Q}_L$ [2], although they are not the classes of any line bundles with the hoped-for positivity properties. The original definition of $h(M)$ is equivalent to

$$h(M) = (-1)^{\operatorname{codim} P(M)} \sum_k (1-t)^k \, \chi\left(\bigwedge^k [\mathcal{S}_M] \otimes \bigwedge^k [\mathcal{Q}_M]^\vee\right);$$

observe that all the operations on vector bundles invoked here do descend to K-theory. Thus if M is connected of rank r we have

$$\omega(M) = -\chi\left(\bigwedge^{n-r} [\mathcal{Q}_M^\vee] \otimes \bigwedge^r [\mathcal{Q}_M^\vee]\right).$$

A new proof of Speyer's conjecture for the $\mathbb{C}$-representable case is given in [3]. This proof exposes a simplicial complex, the titular external activity complex of a pair of matroids, which we use to make the connection between our apparatus for general matroids and the definition of $h(M)$. As this suggests we may replace the two instances of M in the definition of $\omega(M)$ by a pair of matroids M_1 and M_2.

Our $\mathbb{C}$-representable proof is based on two key geometric transformations. We start with the vector bundle $\mathcal{S}_{L_1} \oplus \mathcal{S}_{L_2}$ sitting inside the trivial bundle $\underline{\mathbb{C}^{2n}}$. First we perform a *Kempf collapsing* [9], replacing this bundle by the union $\widehat{Y}_{L_1,L_2}$ of all its fibres within a single copy of $\mathbb{C}^{2n}$. We then take a Gröbner degeneration of the collapsing, arriving at the Stanley–Reisner scheme of the external activity complex.

We convey a free resolution through these transformations and observe that $\omega(M)$ remains recoverable in each setting. The vector bundle $\bigwedge^{n-r}[\mathcal{Q}_{L_1}^\vee] \otimes \bigwedge^r [\mathcal{Q}_{L_2}^\vee]$, which yields $\omega(M)$ when $L_1 = L_2 = L$ represents M, is a bigraded component of a module in the Koszul complex resolving $\mathcal{S}_{L_1} \oplus \mathcal{S}_{L_2}$. When the Koszul complex is pushed forward along the Kempf collapsing map, it remains a resolution. This uses Binglin Li's computation [10] of the multidegree of the image of $\widehat{Y}_{L_1,L_2}$ in $(\mathbb{P}^1)^n$, which is multiplicity-free; Brion's theorem [4] that multiplicity-free varieties are normal with rational singularities; and to finish Weyman's geometric technique for syzygies [13].

Next, $\widehat{Y}_{L_1,L_2}$ has the same multigraded generic initial ideal as $\widehat{Y}_{D(L_1,L_2),\mathbb{C}\mathbf{1}}$ where $D(L_1, L_2)$ is an auxiliary linear space (whose matroid is the *diagonal Dilworth truncation*) and $\mathbb{C}\mathbf{1}$ is a generic line. Conca, de Negri, and Gorla's results [5] on

Cartwright–Sturmfels star ideals show that the Gröbner degenerations of $\widehat{Y}_{L_1,L_2}$ and $\widehat{Y}_{D(L_1,L_2),\mathbb{C}\mathbf{1}}$ have the same multigraded Betti table. Ardila and Boocher [1] explicitly constructed a free resolution of the latter Gröbner degeneration, whose Betti table matches the pushforward of our Koszul complex. The ultimate consequence is that $\omega(M)$ can be recovered from the free resolution of the external activity complex.

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A-resultants, Hurwitz forms, and energy functionals of toric varieties

Yuji Sano

This note focuses on the weight polytopes of the multivariate resultants and the Hurwitz forms as their variants in the context of toric geometry, following the framework of Gelfand–Kapranov–Zelevinsky [3].

Let $A = \{\omega_0, \cdots, \omega_N\} \subset \mathbb{Z}^n$ be a point configuration generating the lattice $M_{\mathbb{Z}} := \mathbb{Z}^n$. A point $\omega_i \in A$ corresponds to the monomial $t^{\omega_i} := \prod_{i=1}^n t_j^{\omega_{ij}}$ where $\omega_j := (\omega_{1j}, \cdots, \omega_{nj})^\top$. Let $\mathbb{C}^A := \{\sum_{j=0}^N a_j t^{\omega_j} \mid \omega_i \in A\}$. Then, *A*-**resultant** R_A is the defining polynomial of the closure of the hypersurface

$$\nabla_A := \left\{ (f_1, \cdots, f_{n+1}) \in \prod_{i=1}^{n+1} \mathbb{C}^A \,\middle|\, \bigcap_{i=1}^{n+1} (f_i = 0) \neq \emptyset \right\}.$$

This coincides with the Chow form that is the defining polynomial of the hypersurface in the Grassmannian

$$Z_{N-n-1}(X) := \overline{\{L \in \mathbb{G}(N-n-1, \mathbb{P}^N) \mid L \cap X \neq \emptyset\}}$$

of the toric variety

$$X_A = \overline{\left\{ [t^{\omega_0} : \ldots : t^{\omega_N}] \in \mathbb{P}^N \,\middle|\, t = (t_1, \ldots, t_n) \in (\mathbb{C}^\times)^n \right\}}$$

whose momentum polytope is the convex hull Q of A.

One of the main results of [3] is the combinatorial characterization of the weight polytope $\mathbb{W}(R_A)$ of R_A. It states the following:

Theorem. [4, 3] *The weight polytope* $\mathbb{W}(R_A)$ *coincides with* $\mathrm{SecPoly}(Q, A)$*.*

Here, the polytope $\mathrm{SecPoly}(Q, A)$ is called **the secondary polytope** that is the convex polytope whose vertices η_T are calculated with respect to regular triangulations T of (Q, A):

$$\begin{aligned} \eta_{T,n} &:= (\eta_{T,n}(\omega_0), \cdots, \eta_T(\omega_N)) \in \mathbb{Z}^{N+1}, \\ \eta_{T,n}(\omega_i) &:= \sum_{\omega_i \in \sigma : n\text{-simplex}} \mathrm{Vol}_{\mathbb{Z}}(\sigma). \end{aligned}$$

Recently, as a variant of R_A, Sturmfels named the **Hurwitz form** Hu_A in [7] the defining polynomial of

$$Z_{N-n}(X_A) := \overline{\{L \in \mathbb{G}(N-n, \mathbb{P}^N) \mid \sharp(L \cap X_A) < \deg(X_A)\}}.$$

In Kähler geometry, Paul [5] discovered that the resultants and the Hurwitz forms appear in Mabuchi's K-energy in the context of the existence problem of canonical Kähler metrics on polarized manifolds.

With this motivation, in [6], we obtained a counterpart to the secondary polytopes for the Hurwitz forms of toric varieties to the secondary polytopes of the resultants. For a triangulation T of Q, we define the corresponding vector

$$\begin{aligned} \xi_T &:= n\eta_{T,n} - \eta_{T,n-1}, \\ \eta_{T,n-1}(\omega_i) &:= \sum_{\omega_i \in \sigma : (n-1)\text{-simplex}} \mathrm{Vol}_{\mathbb{Z}}(\sigma). \end{aligned}$$

Theorem. [6] *The weight polytope* $\mathbb{W}(\mathrm{Hu}_{X_A})$ *coincides with the convex hull of the vectors* ξ_T *for all regular triangulations* T *of* (Q, A)*.*

Our proof uses an alternative method to [3] that employs some recent results [2, 5, 1] from Kähler geometry. These connections suggest further questions related to K-stability, which I hope to explore in future work.

On the other hand, there is a pure combinatorial approach to compute $\mathbb{W}(\mathrm{Hu}_A)$. The Cayley Trick allows one to reinterpret the Hurwitz form (and also the resultant) as a discriminant. More precisely, the Hurwitz form coincides with the discriminant $\mathbb{W}(\Delta_{X \times \mathbb{P}^{n-1}})$ of $X \times \mathbb{P}^{n-1}$ under the Segre embedding $\mathbb{P}^N \times \mathbb{P}^{n-1} \hookrightarrow$

$\mathbb{P}^{(N+1)n-1}$. Hence, one can compute $\mathbb{W}(\Delta_{X\times\mathbb{P}^{n-1}})$ instead of $\mathbb{W}(\mathrm{Hu}_A)$, following [3].

Problem. *Can we (re)-prove the coincidence between* $\mathbb{W}(\Delta_{X\times\mathbb{P}^{n-1}})$ *and* $\mathbb{W}(\mathrm{Hu}_A)$ *in a combinatorial way?*

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Standard stable Horikawa surfaces

Julie Rana
(joint work with Sönke Rollenske)

Smooth, minimal surfaces with invariants on the Noether line were completely described by Horikawa and so are known as Horikawa surfaces. When the volume K^2 is a multiple of 8, the moduli space of these surfaces consists of two disconnected components. We show that the closures of these two components intersect in a divisor parametrizing explicitly described semi-smooth surfaces. This is the content of the following theorem.

Theorem 1. *Let* $\mathfrak{H}^{\mathrm{I}}_{8\ell}$ *and* $\mathfrak{H}^{II}_{8\ell}$ *denote the two components of the moduli spaces described by Horikawa in* [4]*. The intersection of* $\overline{\mathfrak{H}^{\mathrm{I}}}_{8\ell}$ *with* $\overline{\mathfrak{H}^{\mathrm{II}}}_{8\ell}$ *for* $(\ell > 1)$ *contains a divisor* $\mathfrak{D}$ *parametrising explicitly described non-normal (but semi-smooth) surfaces. The intersection is not normal crossing at the general point of* $\mathfrak{D}$.

For any value of K^2 along the Noether line, our methods give rise to new, generically non-reduced, irreducible components of increasing dimensions in the same connected component of the classical moduli space of Horikawa surfaces:

Theorem 2. *Let $k \geq 5$. The connected component of $\overline{\mathfrak{H}}_{2k}$ containing classical Horikawa surfaces contains*

$$\overline{\mathfrak{H}}_{2k} \supset \mathfrak{H}_{2k} \cup \bigcup_{\substack{k > m > \frac{k+4}{2} \\ m \equiv k \bmod 2}} \overline{\mathfrak{H}}_{2k}^{(m)},$$

where the $\overline{\mathfrak{H}}_{2k}^{(m)}$ are generically non-reduced, irreducible components of dimension $5k + 4m + 19 > \dim \mathfrak{H}_{2k}$.

Figure 1 illustrates these theorems in the case that $K^2 = 32$.

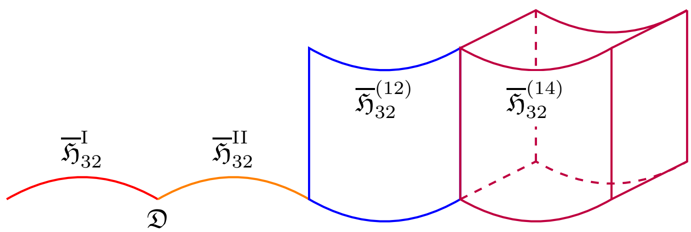

FIGURE 1. Standard components in $\overline{\mathfrak{H}}_{32}$

1. Background

Horikawa surfaces are all double covers of Hirzebruch surfaces $\mathbb{F}_m$. These are ruled surfaces with $\mathrm{Pic}(\mathbb{F}_m)$ generated by the class of a fiber Γ and the so-called "infinity section" σ_∞ with $\sigma_\infty^2 = -m$. The Hirzebruch surface has a useful description as the toric variety $(\mathbb{C}[t_0, t_1, x_0, x_1] \backslash V(I))/((\mathbb{C}^*)^2)$ with irrelevant ideal $I = (t_0, t_1) \cap (x_0, x_1)$ and weights are given by the matrix

$$\begin{pmatrix} t_0 & t_1 & x_0 & x_1 \\ 1 & 0 & \alpha & \alpha - m \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

for any choice of $\alpha \in \mathbb{Z}$. Under this representation, the fibers of $\mathbb{F}_m$ are given by zeros of linear polynomials in t_0 and t_1, while the positive section σ_0 is given by $\{x_0 = 0\}$, and the negative section σ_∞ by $\{x_1 = 0\}$.

Smooth surfaces of general type with $K^2 = 2p_g - 4$ have invariants on the so-called "Noether line", and were fully classified by Horikawa in [4]. They are now known as Horikawa surfaces. For example, when $K^2 = 32$, Gieseker's moduli space $\mathfrak{H}_{32}$ consists of two connected components, each of dimension 140, which we denote by $\mathfrak{H}_{32}^{(0)}$ and $\mathfrak{H}_{32}^{(10)}$. Surfaces in the first component are generically double covers of $\mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{F}_0$ branched over a smooth curve in the linear system $|6\sigma_\infty + 20\Gamma|$. This component also contains higher-codimension strata parametrising double covers of $\mathbb{F}_2$, $\mathbb{F}_4$, $\mathbb{F}_6$, and $\mathbb{F}_8$. The component $\mathfrak{H}_{32}^{(10)}$ consists of type (10) surfaces; these are double covers of $\mathbb{F}_{10}$ branched over a smooth reducible curve: σ_∞ and a smooth curve in the linear system $|5\sigma_\infty + 50\Gamma|$.

It's an open question, and the spark of our initial interest in this project, as to whether surfaces in the two components are diffeomorphic [2].

2. A Novel Approach to Horikawa Surfaces

Inspired by [1], we realize Horikawa surfaces as hypersurfaces in toric ambient threefolds. For example, the general surface in $\mathfrak{H}_{32}^{(0)}$ is defined by a polynomial of the form $z^2 - f(t_0, t_1, x_0, x_1)$ of bidegree $(-40, 6)$ inside the threefold

$$(\mathbb{C}[t_0, t_1, x_0, x_1, z] \backslash V(I))/((\mathbb{C}^*)^2)$$

where the weights are given by the matrix

$$\begin{pmatrix} t_0 & t_1 & x_0 & x_1 & z \\ 1 & 0 & -10 & -10 & -20 \\ 0 & 1 & 1 & 1 & 3 \end{pmatrix}$$

with irrelevant ideal $I = (t_0, t_1) \cap (x_0, x_1, z)$.

To obtain the canonical surfaces desired by Horikawa, the branch divisor can have no more than ADE singularities. Our crucial insight is that because we are interested in *KSBA-stable* surfaces, we may relax this condition. To this end, we define a *standard stable Horikawa surface of type* (m) to be a surface with ample canonical class and semi log canonical singularities which is constructed as a double cover of $\mathbb{F}_m$ branched over a curve in the linear system $6\sigma_\infty + 2a\Gamma|$. Observe that a standard stable Horikawa surface of type (m) is KSBA-stable, has invariants on the Noether line, and satisfies $K^2 = 4a - 6m - 8$. When $2a \geq 5m$, the general branch divisor is smooth, and these surfaces are precisely Horikawa's original surfaces. When $5m > 2a > 4m+4$, the general branch divisor is reducible, non-reduced, and singular; in particular the general such surface has $2(a - 2m)$ pinch points.

3. Connecting the components

The divisor $\mathfrak{D}$ described in Theorem 1 parametrizes standard stable Horikawa surfaces with $2(a - 2m)$ pinch points. We illustrate this for $K^2 = 32$. To begin, we degenerate a smooth type (0) surface to a surface in the boundary of the type (10) component as follows: define a family T_λ of surfaces over $\mathbb{A}^1_\lambda$ as the complete intersection

$$T_\lambda = (z^2 - f(t_0, t_1, z, x_0, x_1) = 0) \cap (\lambda y_0 = t_1^5 x_0 - t_0^5 x_1)$$

in the fourfold

$$(\mathbb{C}[t_0, t_1, y_0, x_0, x_1, z] \backslash V(I))/((\mathbb{C}^*)^2)$$

where the weights are given by the matrix

$$\begin{pmatrix} t_0 & t_1 & y_0 & x_0 & x_1 & z \\ 1 & 0 & -5 & -10 & -10 & -20 \\ 0 & 0 & 1 & 1 & 1 & 3 \end{pmatrix}$$

with irrelevant ideal $I = (t_0, t_1) \cap (x_0, x_1, z)$. Here, f is a polynomial of bidegree $(-40, 6)$. When λ is nonzero, we eliminate y_0 and obtain a general surface in $\mathfrak{H}_{32}^{(0)}$, as described above. When $\lambda = 0$, we can define a new variable of bidegree $(-15, 1)$, namely $y_1 = \frac{x_0}{t_0^5} = \frac{x_1}{t_1^5}$, and observe that the bidegree of f forces each term in f to be divisible by y_1^2. Thus, the special fiber is indeed a double cover of $\mathbb{F}_{10}$ branched over $2\sigma_\infty + B$ where $B \in |4\sigma_\infty + 50\Gamma|$.

Note that the singular surface just describe is easily deformed to a smooth surface of type (10) by again deforming the branch divisor in a similar manner.

More generally, we define such a family for general Horikawa spaces consisting of two connected components. See [5, Section 4] for complete details.

4. Final remarks

The construction given in the previous section provides a set-theoretic description of the divisor $\mathfrak{D}$. To obtain a scheme-theoretic description of $\mathfrak{D}$, we study deformations of these surfaces using the general framework described in [3]. We also exploit the notion of a standard stable Horikawa surface, together with our novel description of Horikawa surfaces as hypersurfaces inside toric threefolds, to realize new components of the KSBA-moduli spaces containing Horikawa's original components, leading to Theorem 2.

In keeping with the theme of this workshop, our project exemplifies the powerful insights that may be gained by reframing classical objects using modern language of toric geometry.

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Birational Coherent Constructible Correspondence

Jesse Huang

(joint work with David Favero)

In recent years, continuous progress has been made in understanding the birational geometry of toric varieties and multigraded syzygies using homological mirror symmetry (HMS), notably the novel discovery of birationally uniform resolutions of the diagonal coherent sheaf of toric varieties by line bundles in the *Bondal-Thomsen collection* [2, 6, 16, 8, 13], and the subsequent proof of a reformulated version of King's Conjecture [4] based on these resolutions. These recent results are heavily inspired by a microlocal sheaf-theoretic version of HMS called the Coherent Constructible Correspondence (CCC) originally suggested by Bondal [3].

Although CCC is proved for toric varieties [10, 11, 12, 18], how the correspondence resonates with various guiding principles of mirror symmetry remains a subtle problem: classically, mirror symmetry predicts a mirror map, which identifies the (globally undefined) “stringy” Kähler moduli space of a toric variety X_Σ and the (globally defined) complex moduli of coefficients of a superpotential $W_{\Sigma(1)}$ that only depends on rays, whose regular fiber tropicalizes, over various chamber regions far away from the GKZ discriminant locus, to different polyhedral complexes dual to simplicial subdivisions of Σ using $\Sigma(1)$, as explained in [14]. The pullback of the mirror map that identifies these moduli parameters is meant to induce an equivalence of certain *constructible cosheaves of categories* which enhances the stalkwise homological mirror symmetry.

It is fair to say that, although the ultimate mathematical statement of the above has not been made, the idea has been taken as a guiding principle in many works on toric mirror symmetry [1, 5, 7, 15]. We would like to adapt this principle into the context of CCC, which means the A-side geometry must be promoted to a global family of mirrors. Moreover, this family must simultaneously capture not only the derived category in each phase and the combined Cox category of all phases (see [4]), but also wall crossing functors among them, monodromy functors around discriminant loci, semiorthogonal components of various types along Mori trees etc., *all simultaneously along a single conical Lagrangian* over a real moduli parameter space separated into various regions, rather than a finite overlapping collection of conical Lagrangians over the same torus $M_{\mathbb{R}}/M$ with stop removal functors from their union. Previous attempts to build this global family use nice description of window categories on the B-side in very limited cases [19, 17]. The Bondal-Thomsen collection, however, is an intrinsic A-side construction using a natural cubic stratification of the real vector space $\mathbb{R}^{\Sigma(1)}$, from which multigraded modules over the Cox ring can be identified as representations of exit paths, making it a natural candidate to use to construct a conical Lagrangian $\mathbb{L}_+ := \mathbb{L}_\Theta \subset T^*\mathbb{R}^{\Sigma(1)}/M$ that best approximates the desired continuous family of mirrors.

This work in preparation [9] studies properties of the cosheaf of microlocal sheaf categories $\mu Sh^w_{\mathbb{L}_\Theta}$ and wrapped kernels of various structural functors forming the cosheaf, and prove that $\mu Sh^w_{\mathbb{L}_\Theta}$ provides a desired birationally uniform enhancement of the results in FLTZ and Kuwagaki which naturally complements the B-side results in recent work [4]. We summarize our results in this report.

1. HHL resolution revisited

Consider a toric GIT problem $\mathbb{A}^{\Sigma(1)}//G$ with $G = \mathbb{G}_m^k$ acting linearly, where the weights of the action span the character lattice $\widehat{G}$. This gives a short exact sequence

$$0 \to \ker q \to \mathbb{Z}^{\Sigma(1)} \xrightarrow{q} \widehat{G} \to 0$$

Write $M := \ker q$. Let $\Theta \subset \widehat{G}$ be the Bondal-Thomsen collection. Recall that, the terms in the Hanlon-Hicks-Lazarev resolution of the diagonal are given by image of the composition of maps along the first and second rows in the following diagram:

$$\begin{tikzcd} M_{\mathbb{R}} \arrow[r, hook, "i"] & \mathbb{R}^{\Sigma(1)} \arrow[r, "-\lfloor\cdot\rfloor"] \arrow[d, "(-1)(\cdot)"] & \mathbb{Z}^{\Sigma(1)} \arrow[r, "q"] & \widehat{G} \\ & \mathbb{R}^{\Sigma(1)} \arrow[r, "-\lfloor\cdot\rfloor"] & \mathbb{Z}^{\Sigma(1)} \arrow[r, "q"] & \widehat{G} \end{tikzcd} \tag{1}$$

where the first row is the map considered by Bondal in [3], and the differentials are given by the exit paths in the CW refinement of $\mathbb{T} = M_{\mathbb{R}}/M$ with a chosen orientation, whose cells are labelled by $\Theta \times \Theta$.

The HHL resolution can be understood directly as giving a description of the wrapped diagonal sheaf that can be read off directly from the conical Lagrangian $\Lambda_\Theta \cup -\Lambda_\Theta$ along the diagonal torus $\mathbb{T}_\Delta \overset{\Delta}{\hookrightarrow} \mathbb{T} \times \mathbb{T}$, where Λ_Θ is the union of microsupports of Bondal-Thomsen strata sheaves. First, the constant sheaf $k_{\mathbb{T}_\Delta}$ can be resolved by a complex of stalk corepresentatives with respect to $\Lambda_\Theta \cup -\Lambda_\Theta$. The right-wrapped diagonal sheaf $w_R^{\Lambda_\Theta \cup -\Lambda_\Theta} \Delta_* k_{\mathbb{T}_\Delta}$ is then a complex of stalk corepresentatives that goes to the HHL resolution after applying a mirror functor.

2. Key lemma: wrapped diagonal sheaf

Notice that each ordered tuple of signs $s \in \{+,-\}^{|\Sigma(1)|}$ induces a cube stratification of $\mathbb{R}^{\Sigma(1)}$ by translations of $[0,1)^{s_+} \times (-1,0]^{s_-}$, each providing a collection Θ_s of cubes intersecting $M_{\mathbb{R}} = q_{\mathbb{R}}^{-1}(0)$. When $s_+ = \emptyset$ (or $s_- = \emptyset$), the vertices of these cubes are precisely $q^{-1}(\Theta)$ (or $q^{-1}(-\Theta)$). When s consists of mixed signs, it gives the Bondal-Thomsen collection for a different GIT problem $s^*(q) : \mathbb{Z}^{\Sigma(1)} \to \widehat{G}$. (Although this does not look like a natural operation algebraically, considering all signs turns out helpful in getting a CW stratification to simplify the structure along the diagonal). We define

$$\mathbb{L}_s = \bigcup_{\widetilde{d} \in q^{-1}(\Theta_s)} ss(Q_s(\widetilde{d}))/\ker q \subset T^*(\mathbb{R}^{\Sigma(1)}/\ker q)$$

where $Q_s(\widetilde{d})$ is the closed orthant sheaf at $\widetilde{d}$ in the direction prescribed by s, and consider the diagonal

$$\mathbb{R}^{\Sigma(1)}/M \overset{\Delta}{\hookrightarrow} \prod_s (\mathbb{R}^{\Sigma(1)}/M)_s = (\mathbb{R}^{\Sigma(1)}/M)^{2^{|\Sigma(1)|}}$$

The product skeleton $\prod_s \mathbb{L}_s$ restricts to $\mathbb{L} = \bigcup_s \mathbb{L}_s$ along this diagonal $(\mathbb{R}^{\Sigma(1)}/M)_\Delta$. On a sufficiently small closed ϵ-neighborhood of the zero fiber $q_{\mathbb{R}}^{-1}(\overline{B}_\epsilon)$, the Lagrangian $\mathbb{L}$ is a CW stratification skeleton. Furthermore, right wrapping the constant sheaf on $q_{\mathbb{R}}^{-1}(\overline{B}_\epsilon)$ to $\mathbb{L}$ gives the constant sheaf on $\mathbb{R}^{\Sigma(1)}/M$ by noncharacteristic deformation lemma as $\epsilon \to \infty$, since the microsupports of the isotoped sheaves never intersect $\mathbb{L}^\infty$.

This allows one to write down an explicit presentation of the wrapped diagonal in $\prod_s (\mathbb{R}^{\Sigma(1)}/M)_s$ with respect to $\prod_s \mathbb{L}_s$ in terms of a complex consisting of products of sheaves of the form $Q_s(\widetilde{d})$. The wrapped diagonal with respect to $\mathbb{L}_+ \times \mathbb{L}_-$ can then be obtained by pulling back the complex to the factor $(\mathbb{R}^{\Sigma(1)}/M)_+ \times (\mathbb{R}^{\Sigma(1)}/M)_-$. This sketches the proof of the following theorem

Lemma 1. *The right-wrapped diagonal sheaf to $\mathbb{L}_+ \times \mathbb{L}_-$ is quasi-isomorphic to an explicit complex of box-product Bondal-Thomsen orthant sheaves labeled by the cellular decomposition and entrance path data on $q_{\mathbb{R}}^{-1}(\overline{B}_\epsilon)$ prescibed by $\mathbb{L}|_{q_{\mathbb{R}}^{-1}(\overline{B}_\epsilon)}$.*

This gives an immediate corollary

Corollary 2. *The category $Sh^\diamond(\mathbb{L}_+)$ is compactly generated by $\bigoplus_{d\in\Theta} \pi_! Q^\circ(\widetilde{d})$ where $\pi : \mathbb{R}^{\Sigma(1)} \to \mathbb{R}^{\Sigma(1)}/M$ is the covering map and $\widetilde{d}$ is an arbitrary lift of d.*

Moreover, one can now use the diagonal to study local and global properties of the sheaf of categories $\mu Sh^w_{\mathbb{L}_+}$.

3. Main Results

We summarize our main results into one theorem.

Theorem 3. *The cosheaf of categories $\mu Sh^w_{\mathbb{L}_+}$ has the following list of properties:*

1. *The cosheaf $q_{\mathbb{R}*}\mu Sh^w_{\mathbb{L}_+}$ is constructible with respect to a stratification on $\widehat{G}_{\mathbb{R}}$ induced by Θ-translations of the GKZ fan. On any* ***deep*** *chamber C, the cosheaf of categories $q_{\mathbb{R}*}\mu Sh^w_{\mathbb{L}_+|_C}$ is locally constant, whose stalk at each character lattice point in C is the wrapped constructible sheaf category $Sh^w(\mathbb{L}_{\mathbf{\Sigma}_C})$, where $\mathbb{L}_{\mathbf{\Sigma}_C}$ is the FLTZ skeleton mirror to the GIT quotient stack $X_{\mathbf{\Sigma}_C}$.*
2. *The extension of sections from a small ϵ-neighborhood of any fiber to the global section*
$$w_L j_{x,\epsilon!} : \mu Sh^w_{\mathbb{L}_+}(\mathbb{L}_+|_{q_{\mathbb{R}}^{-1}(B_{x,\epsilon})}) \to \mu Sh^w_{\mathbb{L}_+}(\mathbb{L}_+)$$
is fully-faithful. In particular, this together with (1) implies that extension of sections from any deep chamber C is fully faithful.
3. *(Corollary 2) The global section $\mu Sh^w_{\mathbb{L}_+}(\mathbb{L}_+)$ is equivalent to the homotopy category of Bondal-Thomsen monads K_Θ.*
4. *By (2), there is a window skeleton $\mathbb{W}_C = \bigcup_{F\in Sh^w(\mathbb{L}_+|_C)} ss(w_L j_{x,C!}F)$ associated to each C, with $Sh^w(\mathbb{W}_C) = Sh^w(\mathbb{L}_{\mathbf{\Sigma}_C}) = D^b(X_{\mathbf{\Sigma}_C})$. We have*
$$\mathbb{L}_+ = \bigcup_C \mathbb{W}_C, \text{ with } Sh^w(\mathbb{W}_C) \xhookrightarrow{inc} Sh^w(\mathbb{L}_+).$$

The sheaf of categories $\mu Sh^w_{\mathbb{L}_+}$ carries information about all birational models and transformations among them. Our result naturally complements the B-side described in [4] which glues $D^b(X_{\mathbf{\Sigma}_C})$ into K_Θ, hence "birational CCC".

It would be interesting to investigate the relation between the jumping locus of $q_*\mu Sh^w_{\mathbb{L}_+}$ and the GKZ discriminant locus.

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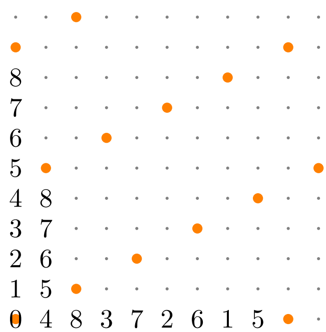
An exercise in homological mirror symmetry

Jenia Tevelev
(joint work with Yanki Lekili)

There will be two players in this talk. The first one is an affine toric surface X, aka cyclic quotient singularity $\mathbb{A}^2/\mu_r$, where the primitive root of unity $\zeta \in \mu_r$ acts on $\mathbb{A}^2$ with weights (ζ, ζ^a), and a and r are coprime. The second player is an r-dimensional algebra R with a basis $\{w_i\}$ for $i \in \mathbb{Z}_r$ and multiplication table

$$w_j w_i = \begin{cases} w_{j+i} & \text{if a certain condition is satisfied,} \\ 0 & \text{otherwise.} \end{cases}$$

To explain the condition, let $b = a^{-1} \mod r$ and let $\gamma : \mathbb{Z}^2 \to$ $mathbbZ_r$ be a homomorphism $(i,j) \mapsto j - bi \mod r$. We plot points in the lattice $\Gamma = \mathrm{Ker}(\gamma)$ in orange. Consider the biggest Young diagram in the first quadrant with the bottom left corner at $(0,0)$ that does not contain orange dots in its interior. We fill every box of this Young diagram with the number $\gamma(i,j) \in \mathbb{Z}_r$, where (i,j) is the bottom left corner of the box. For example, if $r = 9$ and $a = 2$ then $b = 5$ and we get

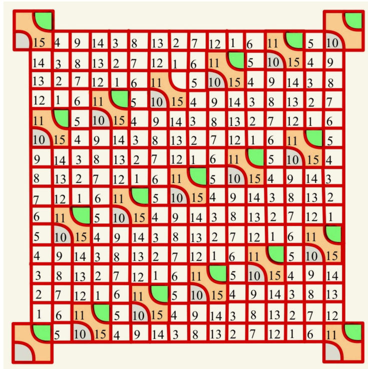


We locate the box filled with j (resp., with i) in the bottom row (resp., left column) of the Young diagram. If the smallest rectangle containing these boxes is contained in the Young diagram, then $w_j w_i = w_{j+i}$. Otherwise, $w_j w_i = 0$.

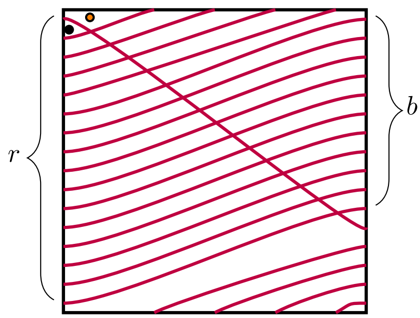
In our example, the non-trivial products are $w_4 w_1 = w_5$, $w_4 w_2 = w_6$, $w_4 w_3 = w_7$, and $w_4^2 = w_8$. The unit of R is w_0. The reader can check that R is associative. It is not commutative with two exceptions, $a = b = r - 1$ and $a = b = 1$.

In the talk, I will use homological mirror symmetry to explain why X and R have the same category of singularities in the sense of Buchweitz and Orlov. This is a byproduct of an investigation in [1], where we describe explicitly how the deformation space of X (given by the classical Kollár–Shepherd-Barron correspondence) embeds into the deformation space of R (using a construction from [2]).

The main idea is to transform the multiplication table of R into a union of curves in $\mathbb{R}^2$, as illustrated here for $r = 16$, $a = 3$, $b = 11$:

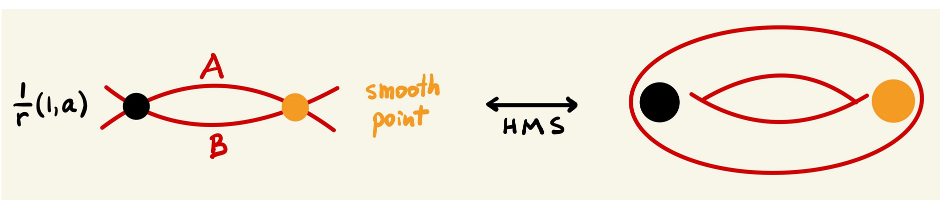


Taking the quotient of $\mathbb{R}^2$ by the lattice $\mathbb{Z}^2$ of orange dots, and adding punctures in the orange and gray region, gives a 2-punctured torus and an exact Lagrangian $\mathbb{K}$ on it (we represent a 2-torus as a rectangle with opposite sides identified.)



It is straightforward to check that R, as defined above, is an endomorphism algebra of $\mathbb{K}$ in the Fukaya category of the 2-punctured torus.

On the other hand, one can compactify the affine toric surface X by a projective (non-toric) surface $\bar{X}$ in such a way that toric coordinate axes of X meet in an additional smooth point of $\bar{X}$ and their union E is an anticanonical divisor of $\bar{X}$.



Homological mirror symmetry for E, established by Lekili and Polishchuk, is an equivalence between the perfect derived category of E and the derived Fukaya category of the 2-punctured torus. Under this equivalence, the Lagrangian $\mathbb{K}$ corresponds to the restriction to E of a remarkable vector bundle F on $\bar{X}$ introduced by Kawamata as the maximal iterated extension of the ideal sheaf $\mathcal{O}(-A)$ by itself. Consequently, homological mirror symmetry gives an isomorphism of the endomorphism algebra R of $\mathbb{K}$ in the Fukaya category with the endomorphism algebra of $F|_E$ on E. The latter can be easily seen to be isomorphic to the algebra of endomorphisms of F on $\bar{X}$, which is known as the Kalck–Karmazyn algebra.

Using the results of Karmazyn, Kuznetsov and Shinder on categorical absorption of singularities, we have a semi-orthogonal decomposition of the bounded derived category of $\bar{X}$ into the bounded derived category of R and a subcategory of the perfect derived category of $\bar{X}$. It follows easily that X and R have the same singularity category, as originally proved by Kalck and Karmazyn.

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Cox rings of projectivized toric vector bundles

Christopher Manon

I'll describe some results on the Cox rings of toric vector bundles obtained by utilizing tools from computational commutative algebra and representation theory.

Let X be a smooth, projective variety with finitely generated Picard group $\mathrm{Pic}(X)$ over an algebraically closed field k. Recall that the *Cox ring* of X is the sum of the global section spaces taken over all line bundles $\mathcal{L} \in \mathrm{Pic}(X)$, with product given by the multiplication of sections:

$$\mathrm{Cox}(X) = \bigoplus_{\mathcal{L} \in \mathrm{Pic}(X)} H^0(X, \mathcal{L}).$$

The space X is said to be a *Mori dream space* when $\mathrm{Cox}(X)$ is finitely generated over k. A notable example given by the normal toric variety X_Σ associated to a smooth, projective fan Σ. In this case $\mathrm{Cox}(X_\Sigma)$ is a polynomial ring on a set of generators indexed by the rays $\rho \in \Sigma(1)$.

A *toric vector bundle* $\mathcal{E}$ over the toric variety X_Σ is a vector bundle in the usual sense, equipped with a linear action by the torus T, making the projection map $\pi : \mathcal{E} \to X_\Sigma$ a T-map. We let r be the rank of $\mathcal{E}$. The associated *projectivized* toric vector bundle $\mathbb{P}\mathcal{E}$ is defined as the bundle of rank 1 quotients. The space $\mathbb{P}\mathcal{E}$ is smooth and projective with a free Picard group of finite rank. As a sort of linear thickening of X_Σ, one might expect that $\mathbb{P}\mathcal{E}$ exhibits similar behavior to its toric base. This makes it reasonable to ask: when is $\mathbb{P}\mathcal{E}$ a Mori dream space? This question originates in a paper of Hering, Payne, and Mustaţă [6].

The first work along these lines appears in papers by Hausen and Süß [5], where torus quotient techniques are used to show that projectivizations of tangent bundles and bundles with rank 2 are always Mori dream spaces. The latter result was also obtained by González [3]. Non-examples were produced in a paper by González, Hering, Payne, and Süß [4] using conditions on the Klyachko filtrations of $\mathcal{E}$ ([8]). In particular these authors use the quotient construction pioneered by Hausen and Süß to show that the Cox ring of $\mathbb{P}\mathcal{E}$ can be realized as a polynomial ring over the Cox ring of a certain blow-up of projective space, in the case that the Klyachko filtrations of $\mathcal{E}$ have one intermediary step. This blow-up can be arranged to be spaces known to not be a Mori dream space by work of Castravet and Tevelev [1].

In joint work with Kiumars Kaveh, we introduce combinatorial and computational techniques to the study of $\mathrm{Cox}(\mathbb{P}\mathcal{E})$. The goal is to organize the Mori dream space condition as a wall crossing phenomenon in an appropriate fan. To prepare, we recast the Klyachko data of $\mathcal{E}$ as a pair (L, D), where $L \subset \mathrm{k}[y_1, \ldots, y_m]$ is a linear ideal of height $m-r$, and D is a structured $n \times m$ matrix, where $n = |\Sigma(1)|$. We refer to D as a "diagram" if each row w_i of D is a point on the tropical variety $\mathrm{Trop}(L)$, and if for every face $\sigma \in \Sigma$, all of the rows corresponding to the rays of σ lie in a distinguished subspace of $\mathrm{Trop}(L)$ called an *apartment* ([7, Definition 4.8]). The set of diagrams is denoted $\Delta(\Sigma, L)$. We then have an over-parametrizations of toric vector bundles on X_Σ.

Theorem 1. *Every integral point $D \in \Delta(\Sigma, L) \cap \mathbb{Z}^{n\times m}$ determines a toric vector bundle $\mathcal{E}_{L,D}$ on X_Σ. Moreover, every toric vector bundle over X_Σ can be realized in this way for some L and D.*

Each row w_i of an integral diagram D determines a graded filtration of the quotient ring $\mathrm{k}[y_1, \ldots, y_m]/L$. The pieces of this filtration are the spaces

$$F_r^{w_i}(d) = \{\sum C_\alpha y^\alpha \mid \deg(y^\alpha) = d, \langle w_i, \alpha\rangle \geq r\}.$$

These spaces then fit together to give an expression for the Cox ring of $\mathbb{P}\mathcal{E}_{L,D}$:

$$\mathrm{Cox}(\mathbb{P}\mathcal{E}_{L,D}) = \bigoplus_{r_1,\ldots,r_n,d} F_{r_1}^{w_1}(d) \cap \cdots \cap F_{r_n}^{w_n}(d).$$

As a result, the Cox ring of $\mathbb{P}\mathcal{E}_{L,D}$ is expressed as an iterated Rees algebra of a polynomial ring. In [7], this expression is used to define a procedure which constructs a finite generating set of $\mathrm{Cox}(\mathbb{P}\mathcal{E}_{L,D})$, provided one exists. The same expression is used to give a necessary and sufficient condition for a given set $\mathcal{B} \subset \mathrm{Cox}(\mathbb{P}\mathcal{E}_{L,D})$ to be a generating set, phrased in terms of the primeness of a certain set of ideals ([7, Proposition 5.8]).

For any linear form $\ell = \sum C_i y_i \in L$ there is an associated *linear Cox equation* $g = \sum C_i x^{\mathbf{d_i}} Y_i$. The linear Cox equations always define a subset of the relations which hold in the Cox ring. We say a bundle $\mathcal{E}_{L,D}$ is *CI* (for "complete intersection") if there is a set $\ell_1, \ldots, \ell_{m-r} \in L$ so that the corresponding linear Cox equations $g_1, \ldots, g_{m-r}$ present $\mathrm{Cox}(\mathbb{P}\mathcal{E}_{L,D})$. The following is [7, Proposition 6.17].

Theorem 2. *Let Σ, L, D be as above, then there is a fan $F(\Sigma, L)$ supported on $\Delta(\Sigma, L)$ along with a distinguished subfan $F^\star(\Sigma, L) \subset F(\Sigma, L)$ such that $\mathcal{E}_{L,D}$ is CI if and only if $D \in |F^\star(\Sigma, L)|$.*

All previous known classes of Mori dream space projectivized toric vector bundles belong to the class of CI bundles. In contrast with previous known examples, CI bundles may have many steps in their Klyachko filtrations.

Now I will discuss joint work with Courtney George, introducing techniques from representation stability into the study of $\mathrm{Cox}(\mathbb{P}\mathcal{E})$ [2]. Suppose we have two toric vector bundles $\mathcal{E}, \mathcal{F}$ over X_Σ, where both $\mathbb{P}\mathcal{E}$ and $\mathbb{P}\mathcal{F}$ are Mori dream spaces, is the same true for $\mathbb{P}(\mathcal{E} \oplus \mathcal{F})$? We can even dial down our expectations, and ask this question when $\mathcal{E} = \mathcal{F}$, unfortunately the answer can be "no." In particular, a non-example is given by the tangent bundle $\mathcal{T}Z$ of a certain toric $3-$fold introduced in [4].

Regardless, we are still able to characterize when there is a positive answer to this question. To do so, we introduce the *flag bundle* $\mathcal{FL}_I(\mathcal{E})$ for a dimension set $I \subset [r-1]$. We rephrase the ℓ-fold sum $\mathcal{E} \oplus \cdots \oplus \mathcal{E}$ as the tensor product $\mathrm{k}^\ell \otimes \mathcal{E}$. The following is [2, Theorem 1.3].

Theorem 3. *The following are equivalent:*

(1) $\mathbb{P}(V \otimes \mathcal{E})$ *is a Mori dream space for all* V *with* $\dim(V) \leq \ell$.
(2) $\mathcal{FL}_I(\mathcal{E})$ *is a Mori dream space for all* I *with* $\max(I) \leq \ell$.

The link between these two classes of spaces is provided by invariant theory. This theorem has a number of corollaries. Let $\mathcal{FL}(\mathcal{E})$ denote the full flag bundle, ie the case $I = [r-1]$.

Corollary 4. *The following are equivalent:*

(1) $\mathbb{P}(V \otimes \mathcal{E})$ *is a Mori dream space for all* V *with* $\dim(V) < r$.
(2) $\mathbb{P}(V \otimes \mathcal{E})$ *is a Mori dream space for all* V.
(3) $\mathcal{FL}(\mathcal{E})$ *is a Mori dream space.*
(4) $\mathcal{FL}(V \otimes \mathcal{E})$ *is a Mori dream space for all* V.
(5) $\mathcal{FL}(\mathcal{E}^\vee)$ *is a Mori dream space.*

Some examples of a bundle $\mathcal{E}$ with these properties are any rank 2 bundle, the tangent bundle of a product of projective spaces, and any irreducible bundle of rank n on $\mathbb{P}^n$. The full flag bundle $\mathcal{FL}(\mathcal{E})$ should correspond to an interesting blow-up of the full flag variety under the quotient construction of Hausen and Süß in these cases. It would be interesting to have a characterization of those fans Σ whose associated toric variety has tangent bundle with these properties.

Notably, the full flag bundle $\mathcal{FL}(\mathcal{E})$ exhibits much more "functorial" Mori dream space behavior than the projectivization. The first part of the corollary can be strengthened in the sense that, provided $\mathcal{E}$ satisfies the conditions above, the degree necessary to generate $\mathrm{Cox}(\mathbb{P}(V \otimes \mathcal{E}))$ stabilizes at the case $\dim(V) = r$. It would be interesting to have a presentation of $\mathrm{Cox}(\mathbb{P}(V \otimes \mathcal{E}))$ which is functorial in V.

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TBD: Toric Bundles Duh!

Gregory G. Smith
(joint work with Michael Perlman)

How does the projectivization of a (toric) vector bundle behave like a smooth projective toric variety? Inspired by Christopher Eur's talk at this workshop, we focus on two intertwined challenges for this class of projective varieties: (1) formulating an effective form of Fujita vanishing [4, Theorem 1.4.35] or generalizing Demazure vanishing [1, Theorem 9.2.3], and (2) strengthening Kawamata–Viehweg vanishing [4, Theorem 4.3.1] or extending Mustaţă vanishing [1, Theorem 9.3.7]. The goal is to better understand the cohomology of torus-equivariant vector bundles.

To elaborate, consider the smooth d-dimensional projective toric variety X. Enumerate the irreducible torus-invariant divisors $D_1, D_2, \dots, D_n$ on X and let Σ be the fan associated to X. A *toric vector bundle* $\mathcal{E}$ is a locally-free $\mathcal{O}_X$-module of rank r with a torus action such that the canonical map $\operatorname{Spec}\big(\operatorname{Sym}(\mathcal{E})\big) \to X$ is equivariant and the torus acts linearly on the fibres. According to the equivalence of categories [3, Theorem 2.2.1], a toric vector bundle $\mathcal{E}$ corresponds to compatible filtrations $\mathbb{C}^r \supseteq \cdots \supseteq E^j(k) \supseteq E^j(k+1) \supseteq \cdots \supseteq 0$ where $j \in [n]$ and $k \in \mathbb{Z}$. For each character $\mathbf{u} \in M \cong \mathbb{Z}^d$ and any $i \in \mathbb{Z}$, the related strategy for computing cohomology identifies a $\mathbb{C}$-complex $C(\mathcal{E}, \mathbf{u})$ such that $\mathrm{H}^i\big(C(\mathcal{E}, \mathbf{u})\big) = H^i(X, \mathcal{E})_{\mathbf{u}}$ as shown in [3, Theorem 4.1.1]. Another approach uses the Cox ring of X, namely the $\operatorname{Pic}(X)$-graded polynomial ring $S := \mathbb{C}[x_1, x_2, \dots, x_n]$ with $\deg(x_i) = \mathcal{O}_X(D_i)$. Assume that the square-free monomial ideal B is the irrelevant ideal of X and let P be an graded S-module such that $\mathcal{E} = \widetilde{P}$. For any positive integer i, the map $\operatorname{Ext}_S^{i+1}(S/B^{[m]}, P) \hookrightarrow \bigoplus_{\mathcal{L} \in \operatorname{Pic}(X)} H^i(X, \mathcal{E} \otimes \mathcal{L})$ is an isomorphism in all sufficiently positive degrees; see [2, Theorem 0.2]. However, neither of these methods identify the toric vector bundles $\mathcal{E}$ with vanishing higher cohomology.

We address this significant shortcoming by simultaneously generalizing these earlier techniques. Set $\lambda_j := \max\{k \in \mathbb{Z} \mid E^j(k) \neq 0\}$, for any $j \in [n]$, and choose an integer m such that $m > \max\{\lambda_j(\mathcal{E}) \mid j \in [n]\}$. For each $\sigma \in \Sigma$, set

$\mathcal{E}|_{U_\sigma} \cong \bigoplus_{j=1}^{r} \mathcal{O}_X(\mathbf{u}_{\sigma,j})$. For any integer i, we construct an explicit free S-complex

$$\check{\mathrm{C}}^i(\mathcal{E}, m) := \bigoplus_{\sigma \in \Sigma(d-i)} \bigoplus_{j=1}^{r} S(\mathbf{u}_{\sigma,j} - m\,\mathbf{1}_{\widehat{\sigma}})$$

such that $\mathrm{H}^i\big(\check{\mathrm{C}}(\mathcal{E}, \mathbf{u})\big) = \bigoplus_{\mathcal{L}} H^i(X, \mathcal{E} \otimes \mathcal{L})$ is an isomorphism for all sufficiently positive line bundles $\mathcal{L}$. Notably, the colimit (over m) of the complexes $\check{\mathrm{C}}(\mathcal{E}, m)$ is essentially the Čech complex arising from the torus-equivariant open covering. Curiously, the cohomology group $\mathrm{H}^0\big(\check{\mathrm{C}}(\mathcal{E}, \mathbf{u})\big)$ is independent of m, thereby distinguishing a special presentation (or parliament of polytopes) for $\mathcal{E}$.

Building on these new computation tools, we describe toric vector bundles $\mathcal{E}$ with no higher cohomology. Following [1, Definition 5.1.5], a *primitive collection* is a subset $\{j_1, j_2, \dots, j_\ell\} \subseteq [n]$ that indexes a minimal nonface in the fan Σ. Its relation is $\mathbf{v}_{j_1} + \mathbf{v}_{j_2} + \cdots + \mathbf{v}_{j_\ell} = \sum_{i \in \tau(1)} c_i\,\mathbf{v}_i$ where $\mathbf{v}_j$ is the primitive lattice generator of the jth ray in Σ and τ is the smallest cone in Σ containing the vector on the left side of this equation. The divisor $D := a_1 D_1 + a_2 D_2 + \cdots a_n D_n$ is nef if and only if, for all primitive collections, we have $a_{j_1} + a_{j_2} + \cdots + a_{j_\ell} \geq \sum_{i \in \tau(1)} c_i\, a_i$; see [1, Theorem 6.4.9]. For any $j \in [n]$, set $\mu_j := \max\{k \in \mathbb{Z} \mid E^j(k) \cong \mathbb{C}^r\}$. When the torus-equivariant divisor D satisfies

$$a_{j_1} + \mu_{j_1} + a_{j_2} + \mu_{j_2} + \cdots + a_{j_\ell} + \mu_{j_\ell} \geq \sum_{i \in \tau(1)} c_i(a_i + \lambda_j)$$

for any primitive collection, we establish that $H^i\big(X, \mathcal{E}(D)\big) = 0$ for all $i \geq 1$.

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Reporter: Leandro Meier

Participants

Patience Ablett
Mathematics Institute
University of Warwick
Gibbet Hill Road
Coventry CV4 7AL
UNITED KINGDOM

Prof. Dr. Karim Adiprasito
Jussieu Institute of Mathematics – Paris
Rive Gauche (IMJ-PRG)
Sorbonne Université
4 place Jussieu
P. O. Box 247
75252 Paris Cedex 5
FRANCE

Prof. Dr. Klaus Altmann
Institut für Mathematik
Freie Universität Berlin
Königin-Luise-Str. 24-26
14195 Berlin
GERMANY

Prof. Dr. Victor V. Batyrev
Fachbereich V – Mathematik
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Prof. Dr. Christine Berkesch
Department of Mathematics
University of Minnesota
127 Vincent Hall
206 Church Street S. E.
Minneapolis, MN 55455
UNITED STATES

Prof. Dr. Lev A. Borisov
Department of Mathematics
Rutgers University
Hill Center, Busch Campus
110 Frelinghuysen Road
Piscataway, NJ 08854-8019
UNITED STATES

Prof. Dr. Michel Brion
Laboratoire de Mathématiques
Université de Grenoble Alpes
Institut Fourier, Bureau 43C
100 rue des Maths
38610 Gières Cedex
FRANCE

Dr. Juliette Bruce
Department of Mathematics
Dartmouth College
29 N. Main Street, 6188 Kemeny Hall
Hanover NH, 03755-3551
UNITED STATES

Adrian Cook
School of Mathematics
University of Edinburgh
James Clerk Maxwell Bldg.
Edinburgh EH9 3JZ
UNITED KINGDOM

Prof. Dr. Alessio Corti
Department of Mathematics
Imperial College London
Huxley Building
180 Queen's Gate
London SW7 2AZ
UNITED KINGDOM

Dr. Lauren Cranton Heller
Department of Mathematics
University of Nebraska, Lincoln
Lincoln NE 68588
UNITED STATES

Prof. Dr. Sandra Di Rocco
Department of Mathematics
Royal Institute of Technology
Lindstedtsvägen 25
100 44 Stockholm
SWEDEN

Daniel Erman
Department of Mathematics
University of Hawaii at Manoa
2565 McCarthy Mall (Keller Hall 401A)
Honolulu, Hawaii 96822
UNITED STATES

Prof. Dr. Christopher Eur
Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, PA 15213
UNITED STATES

Prof. Dr. Alex Fink
School of Mathematical Sciences
Queen Mary University of London
Mile End Road
London E1 4NS
UNITED KINGDOM

Sofia Garzon Mora
Institut für Mathematik
Freie Universität Berlin
Arnimallee 3
14195 Berlin
GERMANY

Paul Görlach
Institut f. Algebra & Geometrie
Otto-von-Guericke-Universität
Magdeburg
Postfach 4120
39016 Magdeburg
GERMANY

Prof. Dr. Christian Haase
Institut für Mathematik
Freie Universität Berlin
Arnimallee 3
14195 Berlin
GERMANY

Andrew Hanlon
Dartmouth College
27 N. Main Street
Hanover NH 03755-3551
UNITED STATES

Prof. Dr. Jürgen Hausen
Fachbereich Mathematik
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Dr. Milena Hering
School of Mathematics
The University of Edinburgh
James Clerk Maxwell Building
Peter Guthrie Tait Road
Edinburgh EH9 3FD
UNITED KINGDOM

Dr. Liana Heuberger
Dept. of Mathematical Sciences
University of Bath
Claverton Down
Bath BA2 7AY
UNITED KINGDOM

Dr. Jeff Hicks
University of St. Andrews
North Haugh
St. Andrews Fife KY16 9SS
UNITED KINGDOM

Dr. Jesse Huang
Department of Pure Mathematics
University of Waterloo
200 University Avenue West
Waterloo N2L3G1
CANADA

Dr. Nathan Ilten
Department of Mathematics
Simon Fraser University
8888 University Drive
Burnaby BC V5A 1S6
CANADA

Prof. Dr. Kiumars Kaveh
Department of Mathematics
University of Pittsburgh
301 Thackery Hall
Pittsburgh PA 15260
UNITED STATES

Dr. Tsung-Ju Lee
Department of Mathematics, National
Cheng Kung University
No. 1, Dasyue Rd.
70101 Tainan
TAIWAN

Prof. Dr. Diane Maclagan
Mathematics Institute
University of Warwick
Gibbet Hill Road
Coventry CV4 7AL
UNITED KINGDOM

Prof. Dr. Christopher Manon
Department of Mathematics
University of Kentucky
715 Patterson Office Tower
Lexington, KY 40506-0027
UNITED STATES

Leandro Meier
Mathematisches Institut
Friedrich-Schiller-Universität
Ernst Abbe Platz 2
07743 Jena
GERMANY

Dr. Leonid Monin
EPFL
1015 Lausanne
SWITZERLAND

Dr. Achim Napame
IMECC-UNICAMP
Rua Sérgio Buarque de Holanda, 651
Campinas CEP 13083-859
BRAZIL

Prof. Dr. Benjamin Nill
Fakultät für Mathematik
Institut für Algebra und Geometrie
Otto-von-Guericke-Universität
Magdeburg
Postfach 4120
39016 Magdeburg
GERMANY

Dr. Andrea Petracci
Dipartimento di Matematica
Università di Bologna
Piazza di Porta S. Donato, 5
40126 Bologna
ITALY

Dr. Julie Rana
Department of Mathematics
Lawrence University
711 E. Boldt Way
Appleton WI 54911
UNITED STATES

Sharon Robins
Dept. of Mathematics and Statistics
Simon Fraser University
Burnaby BC V5A 1S6
CANADA

Prof. Dr. Sönke Rollenske
FB 12 / Mathematik und Informatik
Universität Marburg
Hans-Meerwein-Straße 6
35043 Marburg
GERMANY

Dr. Silvia Sabatini
Mathematisches Institut
Universität zu Köln
Weyertal 86 – 90
50931 Köln
GERMANY

Prof. Dr. Yuji Sano
Department of Applied Mathematics
Fukuoka University
Jonan-ku, Nanakuma 8-19-1
Fukuoka 814-01
JAPAN

Dr. Mahrud Sayrafi
Fields Institute
222 College Street
Toronto, Ontario M5T 3J1
CANADA

Dr. Karin Schaller
Institut für Mathematik
Goethe-Universität Frankfurt
Robert-Mayerstr. 6-8
Postfach 111932
60325 Frankfurt am Main
GERMANY

Prof. Dr. Henry K. Schenck
Department of Mathematics
Auburn University
Auburn, AL 36849
UNITED STATES

Prof. Dr. Frank-Olaf Schreyer
Fachbereich Mathematik und Informatik
Universität des Saarlandes
Campus E2 4
66123 Saarbrücken
GERMANY

Prof. Dr. Gregory G. Smith
Department of Mathematics and
Statistics
Queen's University
Jeffery Hall
Kingston ON K7L 3N6
CANADA

Prof. Dr. Frank Sottile
Department of Mathematics
Texas A & M University
College Station, TX 77843-3368
UNITED STATES

Prof. Dr. Hendrik Süß
Mathematisches Institut
Universität Jena
Ernst-Abbe-Platz 2-4
07743 Jena
GERMANY

Dr. Bernard Teissier
IMJ - PRG
Bâtiment Sophie Germain
Case 7012
8, Place Aurélie Nemours
75205 Paris Cedex 13
FRANCE

Simon Telen
Max-Planck-Institut für
Mathematik in den Naturwissenschaften
Inselstraße 22
04103 Leipzig
GERMANY

Prof. Dr. Jenia Tevelev
Department of Mathematics
University of Massachusetts
Lederle Graduate Research Tower
710 North Pleasant Street
Amherst, MA 01003-9305
UNITED STATES

Prof. Dr. Milena Wrobel
Institut für Mathematik
Carl-von-Ossietzky-Universität
Oldenburg
Ammerländer Heerstraße 114-118
26129 Oldenburg
GERMANY

Dr. Ben Wormleighton
Department of Mathematics
Washington University in St. Louis
Campus Box 1146
One Brookings Drive
St. Louis, MO 63130-4899
UNITED STATES