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Arbeitsgemeinschaft: Relative Langlands Duality

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Abstract. One of the fundamental properties of automorphic forms is that their *periods* – integrals against certain distinguished cycles or distributions – give special values of L-functions. The Langlands program posits that automorphic representations for a reductive group G correspond to (generalizations of) Galois representations into its Langlands dual group $\check{G}$. Periods and L-functions are specific ways to extract numerical invariants from the two sides of the Langlands program; in interesting cases, they match with one another.

Relative Langlands Duality is the systematic study of the manifestations of this matching at all "tiers" of the Langlands program (global, local, geometric, arithmetic, etc.). A key point is a symmetric conceptualization of both sides: periods arise from suitable Hamiltonian G-actions $G \circlearrowright M$ and L-functions from suitable Hamiltonian $\check{G}$-actions $\check{G} \circlearrowright \check{M}$.

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Introduction by the Organizers

The Arbeitsgemeinschaft on *Relative Langlands Duality*, organized by David Ben-Zvi (Austin), Yiannis Sakellaridis (Baltimore), and Akshay Venkatesh (Princeton) was attended by 47 participants in addition to the organizers, with mathematical backgrounds ranging from physics to analytic number theory.

One of the fundamental properties of automorphic forms is that their *periods* – integrals against certain distinguished cycles or distributions – give special values of L-functions. The Langlands program posits that automorphic forms for a reductive group G correspond to Galois representations into its Langlands dual group $\check{G}$, and period formulas can be expressed as a commutative diagram:

$$\begin{tikzcd}[column sep=small] \text{automorphic forms} \arrow[rr, leftrightarrow] \arrow[dr, "\text{period}"'] & & \text{Galois representations} \arrow[dl, "\text{L-function}"] \\ & \text{complex numbers.} & \end{tikzcd} \tag{1}$$

That is to say, "periods" and "L-functions" are specific ways to extract numerical invariants from the two sides of the Langlands program; and in interesting cases, they match with one another.

Relative Langlands Duality is the systematic study of the manifestations of this matching at all "tiers" of the Langlands program (global, local, geometric, arithmetic, etc.). A key point is a symmetric conceptualization of both sides: periods arise from suitable Hamiltonian G-actions $G \circlearrowright M$ and L-functions from suitable Hamiltonian $\check{G}$-actions $\check{G} \circlearrowright \check{M}$. Thus, (1) suggests a correspondence between such actions.

In this workshop we explored the relative form of the Langlands correspondence following the recent manuscript [1]. We discussed a special class of Hamiltonian actions of reductive groups called *hyperspherical varieties*, including the cotangent bundles of suitable spherical varieties, and described a duality

$$G \circlearrowright M \longleftrightarrow \check{M} \circlearrowleft \check{G}$$

between hyperspherical varieties for Langlands dual groups. The relative Langlands duality has a manifestation in each tier of the Langlands program, which all have the general form of Diagram 1: a measurement of automorphic objects for G associated to M matches a measurement of spectral objects for $\check{G}$ associated to $\check{M}$.

In order to organize all the different structures of the relative Langlands program in each tier and their interrelations we used the general metaphor provided by Topological Quantum Field Theory (TQFT). A TQFT is a collection of linear invariants attached to manifolds of different dimensions satisfying strong algebraic interrelations which encode in particular symmetries of these invariants. A key structure in TQFT is the notion of a boundary theory for a TQFT $\mathcal{Z}$, meaning a theory defined relative to $\mathcal{Z}$, and thus producing functionals on the invariants defined by $\mathcal{Z}$. The Langlands correspondence can be thought of as an equivalence of two TQFTs, one describing the theory of automorphic forms associated to G and one describing the theory of Langlands parameters into $\check{G}$. In this language the relative Langlands program concerns the matching of boundary theories for the dual TQFTs, a highly structured form of the matching of functionals such as periods and L-functions.

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Arbeitsgemeinschaft: Relative Langlands Duality

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Topological Quantum Field Theories

Dmitri Whitmore

Topological quantum field theories (TQFTs) originated from physics as the "zero energy" part of quantum field theories, where interesting structures still exist due to the presence of non-trivial topologies. In mathematics, TQFTs allow one to define interesting topological invariants. Within representation theory (including the Langlands programme), they provide a useful (albeit sometimes only metaphorical) organizational framework.

The main reference we follow is [1, D.1, D.2]. In addition, more detailed explanations of some definitions and examples appearing in this talk can be found in [2].

1. Basic Framework

The fundamental idea is to assign functorial (linear) invariants to the $((\infty, n)$-) category Bord_n of oriented manifolds of dimension at most n. This category has:

- $\mathrm{Obj}(\mathrm{Bord}_n) = \{\text{oriented 0-manifolds}\}$
- $\mathrm{Mor}(\mathrm{Bord}_n) = \{\text{bordisms between two 0-manifolds}\}$
- $2\text{-}\mathrm{Mor}(\mathrm{Bord}_n) = \{\text{bordisms between bordisms}\}$

  $\vdots$

Examples of objects include the point and the empty set. Examples of 1-morphisms include the interval $[0, 1] \in \mathrm{Mor}(*, *)$ and the circle $S^1 \in \mathrm{Mor}(\emptyset, \emptyset)$. An example of a 2-morphism is the 'pair of pants' $P \in \mathrm{Mor}(S^1 \sqcup S^1, S^1)$. All k-morphisms for $k > n$ are invertible, with $(n + 1)$-morphisms given by diffeomorphisms, $(n + 2)$-morphisms given by diffeomorphisms between diffeomorphisms, and so on.

Note that the manifolds comprising the data of Bord_n are manifolds with corners and have specified ingoing and outgoing boundaries. Composition of morphisms is defined by appropriately gluing allowing ingoing and outgoing boundaries, which is pictorially given by concatenation of diagrams. The category Bord_n has a symmetric monoidal structure given by disjoint union.

Definition 1.1. A TQFT Z is a symmetric monoidal functor:

$$Z : \mathrm{Bord}_n \to \mathcal{C}.$$

Here $\mathcal{C}$ is a typically a $\mathbb{C}$-linear category (given by some delooping of the complex numbers). This roughly means that $(n-1)$-$\mathrm{Mor}(\mathcal{C})$ has the structure of a $\mathbb{C}$-vector space, $(n-2)$-$\mathrm{Mor}(\mathcal{C})$ has the structure of a $\mathrm{Vect}_{\mathbb{C}}$-linear category, and so on. The symmetric monoidal structure then implies that

$$Z(\emptyset^n) = 1 \in \mathbb{C} = \mathrm{End}(\mathrm{Z}(\emptyset^{\mathrm{n}-1}))$$

with $\mathbb{C}$ thought of as the monoidal unit in

$$\mathrm{Vect}_{\mathbb{C}} = \mathrm{End}(Z(\emptyset^{n-2})),$$

and so on.

Thus, evaluating Z on a compact d-manifold M^d without boundary yields:

- a complex number
  $$Z(M^n) = \alpha \in \mathbb{C} = \mathrm{End}(Z(\emptyset^{n-1})) = \mathrm{End}(\mathbb{C})$$
  when $d = n$;
- a vector space
  $$Z(M^{n-1}) = V \in \mathrm{Vect}_{\mathbb{C}} = \mathrm{End}(Z(\emptyset^{n-2})) = \mathrm{End}(\mathrm{Vect}_{\mathbb{C}})$$
  when $d = n - 1$;
- a $\mathbb{C}$-linear category
  $$Z(M^{n-2}) \in 2\text{-}\mathrm{Vect}_{\mathbb{C}}$$
  when $d = n - 2$.

Remark 1.1. In practice, defining a TQFT Z in all dimensions can be challenging. One therefore often restricts to defining Z only within a specific range of dimensions.

2. Low-dimension examples

2.1. **1D TQFTs.** Let $Z : \mathrm{Bord}_1 \to \mathcal{C}$ be a 1-dimensional TQFT. Evaluating Z on a point with a choice of orientation gives rise to a pair of vector spaces $V = Z(*^+)$ and $W = Z(*^-)$. The map $Z(\supset) \in \mathrm{Hom}(V \otimes W, \mathbb{C})$ provides a perfect pairing between V and W, which can be seen by suitably deforming the interval and applying 'Zorro's lemma'. We see that V is finite-dimensional, an example of the general phenomenon that TQFTs impose strong finiteness and dualizability constraints on their outputs. The complex number $Z(S^1)$ is given by the dimension of V. If we in the non-derived setting (so that V is an object of the abelian category of vector spaces, instead of the derived category of vector spaces), then the data of $\dim V$ moreover completely classifies the TQFT up to isomorphism.

2.2. **2D TQFTs.** Suppose that $Z : \mathrm{Bord}_2 \to \mathcal{C}$ is a 2-dimensional TQFT. Evaluating on the circle now yields a vector space

$$Z(S^1) = A \in \mathrm{Vect}_{\mathbb{C}}.$$

This vector space has a commutative algebra structure arising from the pair of pants

$$Z(P) : A \otimes A \to A$$

Letting D (resp. D') denote the half-sphere with incoming (resp. outgoing) boundary a circle and empty outgoing (resp. incoming) boundary, we obtain maps

$$Z(D') = \mathbb{C} \to A$$

$$Z(D) : A \xrightarrow{\theta} \mathbb{C},$$

with $Z(D')(1)$ providing the unit of A. The map θ is called the trace and gives rise to a Frobenius algebra structure on A. The data of a 2-dimensional TQFT (evaluated on objects of dimension 1 and 2 and assumed to take values in abelian categories rather than the associated derived categories) turns out to be equivalent to giving a finite dimensional commutative Frobenius algebra.

3. Lagrangian TQFTs and finite group gauge theories

3.1. **Lagrangian TQFTs.** We often wish to factor our TQFT Z as a composition

$$\begin{tikzcd} \mathrm{Bord}_n \arrow[rr, "Z"] \arrow[dr, "\mathcal{F}_Z"'] & & \mathcal{C} \\ & \mathrm{Corr}_n \arrow[ur] & \end{tikzcd}$$

with Corr_n a category of spaces with morphisms given by correspondences, which we will take to be finite orbifolds. For our purposes, we will refer the data of Z together with such a factorization $\mathcal{F}_Z$ as a Lagrangian TQFT (an oversimplification by comparison to the actual definition of a Lagrangian QFT). Often we take $\mathcal{F}_Z = \mathrm{Map}(-, T)$ for some target space T. Physically, $\mathcal{F}_Z$ represents fields on M, which are linearized to obtain a TQFT through taking functions, sheaves, etc. Note that $\mathcal{F}_Z$ should not depend on the dimension of the input, while Z itself will. While not every TQFT arises as a Lagrangian TQFT, many important examples do. Moreover, the choice of factorization $\mathcal{F}_Z$ may not always be unique, which can be a source of interesting dualities in mathematics and physics.

3.2. **Finite group gauge theories.** A key source of examples of Lagrangian TQFTs are those arising from linearizing the moduli of G-local systems on a space. For a finite group G and manifold M, consider the finite orbifold

$$\mathrm{Loc}_G(M) = \mathrm{Hom}(\pi_1(M), G)/G = \mathrm{Map}(M, BG).$$

We define a TQFT Z_G^n (in codimensions 0 to 2) by linearizing $\mathrm{Loc}_G(M^d)$:

$$Z_G^n(M^n) = \#\mathrm{Loc}_G(M) = \sum_{x \in \mathrm{Loc}_G(M)} \frac{1}{\#\mathrm{Aut}(x)} \in \mathbb{C},$$
$$Z_G^n(M^{n-1}) = \mathrm{Fun}_{\mathbb{C}}(\mathrm{Loc}_G(M^{n-1})) \in \mathrm{Vect}_{\mathbb{C}},$$
$$Z_G^n(M^{n-2}) = \mathrm{Shv}_{\mathbb{C}}(\mathrm{Loc}_G(M^{n-2})) \in 2\text{-}\mathrm{Vect}_{\mathbb{C}}.$$

From a manifold M with boundary $\partial M = \partial^+ M \sqcup \partial^- M$, we obtain a correspondence:

$$\begin{tikzcd} & \mathrm{Loc}_G(M) \arrow[dl, "\pi^+"'] \arrow[dr, "\pi^-"] & \\ \mathrm{Loc}_G(\partial^+ M) & & \mathrm{Loc}_G(\partial^- M) \end{tikzcd}.$$

Morphisms are given by push-pull along such correspondences. For instance,

$$Z_G^n(M) = (\pi^-)_*(\pi^+)^* : Z_G^n(\partial^+ M) \to Z_G^n(\partial^- M).$$

3.3. **2D Case.** We now focus on the case when $n = 2$. We have:

$$Z_G^2(\mathrm{pt}) = \mathrm{Shv}_G(BG) = \mathrm{Rep}(G)$$
$$Z_G^2(S^1) = \mathbb{C}[G/G]$$
$$Z_G^2(\Sigma_g) = \frac{\#\{[x_1, y_1], \dots, [x_g, y_g] \in G^{2g} : [x_1, y_1] \cdots [x_g, y_g] = 1\}}{\#G},$$

so that, for instance,

$$Z_G^2(S^2) = \frac{1}{\#G}$$
$$Z_G^2(S^1 \times S^1) = \frac{1}{\#G} \sum_{x \in G} \#C_G(x) = \#\{\text{conjugacy classes}\}.$$

We saw already that $Z_G^2(S^1)$ has the structure of a Frobenius algebra. In this example, the Frobenius algebra structure on $\mathbb{C}[G/G]$ is given by convolution of class functions, with the trace (up to scalar) given by evaluating on the identity of G. This Frobenius algebra is semisimple, with a basis given by the irreducible characters of G. One can use the framework of 2-dimensional TQFTs to prove the following identities for every $g \geq 0$:

$$(\#G)^{2g-2} Z_G^2(\Sigma_g) = \sum_{\chi \text{ irred char.}} \chi(1)^{2-2g},$$

as a consequence of more general identities holding whenever the Frobenius algebra associated to a $2D$-TQFT is semisimple.

3.4. **Summary Table.** The following table summarises the output of Z_G^n for some small values of n.

	Z_G^2	Z_G^3	Z_G^4
Ξ^3		$\#\mathrm{Loc}_G(\Xi^3)$	$\mathbb{C}[\mathrm{Loc}_G(\Xi^3)]$
Σ^2	$\#\mathrm{Loc}_G(\Sigma^2)$	$\mathbb{C}[\mathrm{Loc}_G(\Sigma^2)]$	$\mathrm{Vect}(\mathrm{Loc}_G(\Sigma^2))$ (e.g. $S^2 \mapsto \mathrm{Rep}_G$)
S^1	$\mathbb{C}[G/G]$	$\mathrm{Vect}_G(G/G)$	
pt	$\mathrm{Rep}(G)$		

References

[1] David Ben-Zvi, Yiannis Sakellaridis, and Akshay Venkatesh, *Relative Langlands duality*, arXiv:2409.04677, 2024.

[2] Vladimir Fock, Andrey Marshakov, Florent Schaffhauser, Constantin Teleman, Richard Wentworth, *Five lectures on topological field theory*, in *Geometry and Quantization of Moduli Spaces*, pp. 109–164, Springer, 2016.

Structures in TQFTs

Surya Raghavendran

We discuss two methods for extracting lower dimensional TQFTs from higher dimensional ones: *circle compactification* and *interval compactification.*

1. Circle compactification

Recall that the outputs of a TQFT must satisfy a finiteness hypothesis.

Definition 1. Let $\mathcal{C}$ be a symmetric monoidal category. An object $F \in \mathcal{C}$ is **dualizable** if there exists $F^\vee \in \mathcal{C}$ together with morphisms

$$\mathrm{ev} : F \otimes F^\vee \to k, \qquad \mathrm{coev} : k \to F \otimes F^\vee$$

such that $(\mathrm{ev} \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \mathrm{coev}) = \mathrm{Id}$

Note that for a dualizable object $F \in \mathcal{C}$, we have that $\mathrm{End}(F) \cong F \otimes F^\vee$. The evaluation map then gives us a **trace** $\mathrm{End}(F) \to 1_{\mathcal{C}}$, and pre-composing with coevaluation gives us a canonical element, the **dimension**, $\dim F \in \mathrm{End}(1_{\mathcal{C}})$

Example 2. In the (oriented) bordism category Bord, every k-morphism M is a dualizable object. The coevaluation and evaluation maps are given by macaroni with M-shaped cross-sections. Composition is gluing along $M \sqcup \bar{M}$, which is the same as the cartesian product with S^1. Therefore we see that $\dim M = M \times S^1$. We can also consider the trace $\mathrm{tr}(f)$ where $f : M \to M$ is a $(k+1)$-morphism. An identical argument shows that $\mathrm{tr}(f) = M \times I/((x,0) \sim (f(x),1))$ the mapping torus.

Example 3. In the category of correspondences Corr, every object is dualizable with $X^\vee = X$; duality data is afforded by the diagonal maps. We see that $\dim X = X \times_{X\times X} X = \mathcal{L}X$ the loop space of X. Similarly, a map $f : X \to X$ defines an element of $\mathrm{Hom}_{\mathrm{Corr}}(\mathrm{pt}, X \times X)$; the trace $\mathrm{tr}(f)$ computes the intersection $\mathrm{graph}(f) \times_{X\times X} X$ which counts the fixed points of X.

Example 4. Consider an algebra A viewed as an object in the Morita category. A is self-dual with dualizability data coming from viewing A as a left or right $A \otimes A^{op}$-module. An argument similar to example 3 tells us that $\dim A = A \otimes_{A\otimes A^{op}} A$ which recovers the Hochschild homology of A.

Definition 5. Let Z be an n-dimensional TQFT. Its **circle compactification** is the $(n-1)$-dimensional TQFT given by $Z_{S^1}(-) = Z((-) \times S^1)$.

Since TQFTs are symmetric monoidal functors, the circle compactification is a TQFT which computes dimensions of the outputs of Z. This turns out to be a useful organizing perspective. Recall that there is a general philosophy that for G a reductive group, the representation theory of $G(\mathbb{F}_q)$ can be recovered by categorical traces of Frobenius. Example 2 suggests a cartoon for this as the value of a TQFT on a "mapping torus for Frobenius". Moreover, recall from the previous talk that many TQFTs arise by postcomposing a "Lagrangian" TFT Bord $\to$ Corr with a suitable choice of linearization. Assuming that this linearization is monoidal, we see that the trace of a linearization must agree with a linearization of a trace computed in Corr, i.e. the fixed points. This reflects the shape of statements such as the Grothendieck-Lefschetz trace formula.

2. Interval compactification

Given a way to treat an interval as if it were a closed manifold, we can mimic definition 5 to get an $(n-1)$-dimensional TQFT. *Boundary theories* will allow us to do this. Recall that an n-dimensional TQFT Z will assign to a k-manifold M with boundary, an object $Z(M) \in Z(\partial M)$. Morally, a *boundary theory* will give a collection of objects in $Z(\partial M)$ for every such M, that can be paired against $Z(M)$ to produce an invariant of lower category number. Moreover, these objects will satisfy a host of compatibility conditions, similar to the outputs of Z.

Let $\mathbb{1}$ denote the unit in the monoidal category of n-dimensional $\mathcal{C}$-valued TQFTs. Its value on every k-morphism is the identity k-morphism of the monoidal unit $1_{\mathcal{C}} \in \mathcal{C}$. Also recall that for any $m \leq n$ may restrict an n-dimensional TQFT Z along the inclusion of the subcategory $\mathrm{Bord}_m \to \mathrm{Bord}_n$ to get a truncated TQFT $\tau_{\leq m} Z$.

Definition 6. A **(left) boundary theory** for Z is an oplax natural transformation $\mathbb{1} \to \tau_{\leq n-1} Z$. The dual notion of a right boundary theory is defined identitically.

The cobordism hypothesis with singularities allows us to view a boundary theory $\mathcal{B}$ as providing an extension $Z^{\mathcal{B}}$ of Z to manifolds with "marked boundaries" - we may formally treat certain manifolds with boundary as if they were closed by marking the boundary. The extension is thought of as Z with the boundary theory imposed as a boundary condition.

Example 7. Consider an n-dimensional Lagrangian TFT given by $\mathrm{Map}((-), Y)$ and let X be a space with a map $X \to Y$. The space X determines a boundary theory: to any k-morphism M in Bord, the boundary theory assigns the correspondence

$$\begin{tikzcd} & \mathrm{Map}(M,X) \arrow[dl] \arrow[dr, "\pi^X"] & \\ \mathrm{pt} & & \mathrm{Map}(M,Y) \end{tikzcd}$$

To describe the corresponding extension of $\mathrm{Map}((-), Y)$ to bordisms with marked boundaries, let N be a $k+1$-morphism, with its boundary $\partial N = M$ marked. The extension assigns to M, the composition of correspondences

$$\begin{tikzcd} & \mathrm{Map}(M,X) \arrow[dl] \arrow[dr, "\pi^X"] & & \mathrm{Map}(N,Y) \arrow[dl] \arrow[dr] & \\ \mathrm{pt} & & \mathrm{Map}(M,Y) & & \mathrm{pt} \end{tikzcd}$$

The fiber product is given the space of twisted maps - maps $N \to Y$ whose restrictions to M lift to X.

Example 8. For a finite group G, recall 2d Dijkgraaf-Witten theory Z_G^2 from the last lecture. This admitted a description as a linearization of a Lagrangian TFT:

$$\begin{tikzcd} \mathrm{Bord}_2 \arrow[r, "\mathrm{Map}((-),BG)"] \arrow[rr, bend right, "Z_G^2"'] & \mathrm{Corr} \arrow[r, "\text{linearize}"] & \mathrm{Cat} \end{tikzcd}$$

Given a G-set X, we have a map $X/G \to BG$ to which we can apply the construction of example 7 - this boundary theory describes coupling the σ-model into X to G-gauge theory.

- $\Theta_X^2(\mathrm{pt}) \in \mathrm{Rep}(G)$ is obtained by linearizing the element of $\mathrm{Hom}_{\mathrm{Corr}}(\mathrm{pt}, BG)$ from 7 and yields the algebra of functions $\mathbb{C}[X]$. Special examples of X include the *Neumann* boundary theory $X = \mathrm{pt}$ and the *Dirichlet* boundary theory $X = G$.
- $\Theta_X^2(S^1) \in \mathbb{C}[\frac{G}{G}]$ is similarly obtained by linearizing an element of $\mathrm{Hom}_{\mathrm{Corr}}\left(\mathrm{pt}, \frac{G}{G}\right)$ and recovers the character of $\mathbb{C}[X]$ expressed via the Atiyah-Bott formula.
- Consider the interval I as a 1-morphism between the empty set and $\mathrm{pt} \sqcup \bar{\mathrm{pt}}$ in Bord. Since Θ_X^2 is oplax, there is a 2-cell

$$\begin{tikzcd} \mathrm{Vect} \arrow[r, "\Theta_X^2(\emptyset)"] \arrow[d, "\mathbb{1}(I)"] & \mathrm{Vect} \arrow[d, "Z_G^2(I)"] \\ \mathrm{Vect} \arrow[r, "\Theta_X^2(\mathrm{pt}\sqcup\bar{\mathrm{pt}})"'] \arrow[ur, Rightarrow] & \mathrm{Rep}(G\times G) \end{tikzcd}$$

  yielding a map $\mathbb{C}[X] \otimes \mathbb{C}[X] \to \mathbb{C}[G]$. This map recovers matrix elements of $\mathbb{C}[X]$.

Given a manifold with multiple boundary components, we can imagine marking different components of boundaries with different boundary theories. In particular, given a TQFT Z with a pair of a left and a right boundary theory we can extend Z to a functor $Z^{\mathcal{B}_L,\mathcal{B}_R}$ to a category of marked bordisms with two distinct markings of boundaries. An important 1-morphism in this category is the interval $\tilde{I}$ with each endpoint a boundary of different color.

Definition 9. Let Z be an n dimensional TQFT and let $\mathcal{B}_L, \mathcal{B}_R$ be left and right boundary theories respectively. The **interval compactification** $Z_{I,\mathcal{B}_L,\mathcal{B}_R}$ is the $(n-1)$-dimensional TQFT defined by the composition

$$\mathbb{1} \xrightarrow{\mathcal{B}_L} \tau_{n-1} Z \xrightarrow{\mathcal{B}_R} \mathbb{1}$$

Concretely, $Z_{I,\mathcal{B}_L,\mathcal{B}_R}(N) = Z^{\mathcal{B}_L,\mathcal{B}_R}(\tilde{I} \times N)$.

Example 10. In the setting of example 7, suppose we have two spaces $X_1, X_2 \to Y$, each of which determines a boundary theory. For M a k morphism, the interval compactification assigns the limit of

$$\begin{tikzcd}[column sep=small]
& \mathrm{Map}(M, X_1) \arrow[dl] \arrow[dr] & & \mathrm{Map}(M \times [0,1], Y) \arrow[dl] \arrow[dr] & & \mathrm{Map}(M, X_2) \arrow[dl] \arrow[dr] & \\
\mathrm{pt} & & \mathrm{Map}(M, Y) & & \mathrm{Map}(M, Y) & & \mathrm{pt}
\end{tikzcd}$$

Example 11. We consider the interval compactification of Z_G^2 associated to two boundary theories Θ_{X_1}, Θ_{X_2}. To a point, the interval compactification assigns the linearization of the limit in example 10. The latter is seen to be the balanced product $X_1 \times^G X_2$, and the linearization is the space of homomorphisms $\mathrm{Hom}_{\mathrm{Rep}\, G}(\mathbb{C}[X_1], \mathbb{C}[X_2])$. Accordingly, we see that interval compactifications against Neumann and Dirichlet boundary theories recover familiar construction.

- Taking $X_1 = \mathrm{pt}$, the interval compactification computes the invariants $\mathbb{C}[X_2]^G$. That is, compactifying against Neumann gauges the G-symmetry.
- Taking $X_1 = G$, the interval compactification recovers $\mathbb{C}[X_2]$ as an ordinary vector space. That is, compactifying against Dirichlet ungauges the G-symmetry.

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Unramified Langlands Duality

Siyan Daniel Li-Huerta

Let G be a split reductive group (say, over $\mathbb{Z}$), B be a Borel subgroup of G, and T a maximal torus in B. This induces a based root datum $(X^*(T), \Phi^+, X_*(T), \check{\Phi}^+)$. Since $(X_*(T), \check{\Phi}^+, X^*(T), \Phi^+)$ is also a based root datum, it induces another split reductive group $\check{G}$ (called the *Langlands dual* of G) along with a Borel subgroup $\check{B}$ of $\check{G}$ and a maximal torus $\check{T}$ in $\check{B}$.

Langlands duality predicts a relationship between

"automorphic" objects associated with G

and

"Galois" objects associated with $\check{G}$.

In this talk, we review Langlands duality in the following four settings:

	arithmetic	geometric
local	§1	§2
global	§3	§4

We only discuss the unramified situation. We also focus on the function field case, where one can pass from the geometric setting to the arithmetic one via *trace of Frobenius*.

1. Local arithmetic

Let F be a nonarchimedean local field, i.e. $\mathbb{F}_q(\!(z)\!)$ or a finite extension of $\mathbb{Q}_p$. Write $\mathcal{O}$ for the ring of integers of F, write $\mathbb{F}_q$ for its residue field, and write $|\cdot|$ for the absolute value on F that sends uniformizers to q^{-1}.

Suppose you want to study continuous $\mathbb{C}$-valued representations π of the topological group $G(F)$. Because $G(F)$ is totally disconnected but $\mathbb{C}$ has the Euclidean topology, one can show that, for every v in π, there exists a compact open subgroup K of $G(F)$ such that v is fixed by K (i.e. π is *smooth*).

In general, K might need to be very small in order for π^K to be nonzero. Let us only consider irreducible representations where π^K is nonzero for a very *large* (in fact, maximal!) compact open subgroup.

Definition 1. We say π is *unramified* if $\pi^{G(\mathcal{O})} \neq 0$.

Using the Haar measure on $G(F)$ that gives $G(\mathcal{O})$ volume 1, convolution endows the $\mathbb{C}$-vector space $\mathcal{H}_{G(\mathcal{O})} := C_c(G(\mathcal{O})\backslash G(F)/G(\mathcal{O}), \mathbb{C})$ with an algebra structure. Similarly, convolution endows $\pi^{G(\mathcal{O})}$ with an action of $\mathcal{H}_{G(\mathcal{O})}$.

Proposition 2. *The assignment $\pi \mapsto \pi^{G(\mathcal{O})}$ induces a bijection*

$$\{\textit{unramified irreducible representations } \pi \textit{ of } G(F)\} \cong \{\textit{irreducible } \mathcal{H}_{G(\mathcal{O})}\textit{-modules}\}.$$

This leads us to study the $\mathbb{C}$-algebra $\mathcal{H}_{G(\mathcal{O})}$. What is it?

Example 3. When $G = T$ is a torus, the commutativity of T implies that

$$T(\mathcal{O})\backslash T(F)/T(\mathcal{O}) = T(F)/T(\mathcal{O}) \cong X_*(T),$$

where the isomorphism sends μ in $X_*(T)$ to the image of $\mu(z)$ in $T(F)/T(\mathcal{O})$ for any uniformizer z of F. Therefore $\mathcal{H}_{T(\mathcal{O})}$ equals the group algebra

$$\mathbb{C}[X_*(T)] \cong \mathbb{C}[X^*(\check{T})],$$

where we use that $X^*(\check{T}) \cong X_*(T)$ by construction.

One can reduce the general case to Example 3 as follows. Write N for the unipotent radical of B, and write 2ρ for the sum of elements in Φ^+.

Theorem 4 (Satake). *The assignment $f(g) \mapsto |(2\check{\rho})(t)|^{1/2} \int_{N(F)} f(tn)\mathrm{d}n$ induces an algebra isomorphism $\mathcal{H}_{G(\mathcal{O})} \cong (\mathcal{H}_{T(\mathcal{O})})^W$, where W denotes the Weyl group.*

One can show that $\check{W}$ is naturally isomorphic to W. Combined with the Chevalley restriction theorem and algebraic Peter–Weyl theorem, Theorem 4 implies that

$$(\star) \qquad \mathcal{H}_{G(\mathcal{O})} \cong (\mathcal{H}_{T(\mathcal{O})})^W \cong \mathbb{C}[\check{T}]^{\check{W}} \cong \mathbb{C}[\check{T}/\!/\check{W}] \cong \mathbb{C}[\check{G}/\!/\check{G}] \cong K^0(\mathrm{Rep}^{\mathrm{fin}}_{\mathbb{C}}\, \check{G}).$$

Remark 5. Smoothness is a purely algebraic property of representations, and all our integrals are actually finite sums. Hence everything here holds after replacing $\mathbb{C}$ with an isomorphic field with a fixed square root of q (to evaluate $|(2\check{\rho})(t)|^{1/2}$).

2. Local geometric

The $F = \mathbb{F}_q(\!(z)\!)$ case of §1 suggests the following geometric analog. Let k be an algebraically closed field. We start by geometrizing $G(\mathcal{O})$ and $G(F)$.

Definition 6. For all k-algebras R, write $(L^+G)(R)$ for $G(R[\![z]\!])$ and $(LG)(R)$ for $G(R(\!(z)\!))$.

$\operatorname{Spec} k(\!(z)\!)$ is like a punctured disk, which deformation retracts to S^1, so LG is like a space of $\underline{\text{l}}$oops. $\operatorname{Spec} k[\![z]\!]$ is like a disk, so L^+G is like the subspace of loops that can be filled in. These loop spaces have the following geometric structure.

Proposition 7. *L^+G is an affine scheme, LG is an ind-affine scheme, and the étale quotient $\mathrm{Gr}_G \coloneqq LG/L^+G$ is an ind-projective scheme over k.*

Example 8. When $G = T$ is a torus, the isomorphism from Example 3 upgrades to show that Gr_T is the constant scheme $\underline{X_*(T)}$. In particular, there is a natural map $\underline{\langle 2\rho, -\rangle} : \mathrm{Gr}_T \to \underline{\mathbb{Z}}$.

According to Grothendieck's sheaves-functions dictionary, we should consider étale sheaves on $L^+G\backslash\mathrm{Gr}_G$. However, in general Gr_G is not smooth, so we should use *perverse sheaves*. We take $\overline{\mathbb{Q}}_\ell$ as our coefficients, where $\operatorname{char} k$ does not divide ℓ, and we fix a square root of q as in Remark 5.

We emulate the constant term integral from Theorem 4 using the maps

$$\begin{tikzcd} & \mathrm{Gr}_B \arrow[dl, "p"'] \arrow[dr, "q"] & & \\ \mathrm{Gr}_G & & \mathrm{Gr}_T \arrow[r, "\pi"] & \operatorname{Spec} k. \end{tikzcd}$$

Finally, we arrive at the following geometrization of Theorem 4. The following theorem has a long history but was ultimately proved by Mirković–Vilonen.

Theorem 9 (Mirković–Vilonen [1])**.** *The category of L^+G-equivariant perverse sheaves $\mathrm{Perv}(L^+G\backslash\mathrm{Gr}_G, \overline{\mathbb{Q}}_\ell)$ has a natural symmetric monoidal structure such that $\mathcal{F} \mapsto \pi_!(q_!p^*\mathcal{F}[\underline{\langle 2\rho, -\rangle}])$ is a fiber functor. This induces an exact tensor equivalence*

$$\mathcal{S} : \mathrm{Perv}(L^+G\backslash\mathrm{Gr}_G, \overline{\mathbb{Q}}_\ell) \cong \mathrm{Rep}^{\mathrm{fin}}_{\overline{\mathbb{Q}}_\ell} \check{G}.$$

Remark 10. When $k = \overline{\mathbb{F}}_q$, Theorem 9 is compatible with Theorem 4 as follows. For all dominant μ in $X^*(\check{T})$, write V_μ for the corresponding highest weight representation of $\check{G}$. Then the image of $[V_\mu]$ in $K^0(\mathrm{Rep}^{\mathrm{fin}}_{\overline{\mathbb{Q}}_\ell} \check{G})$ under $(\star)$ equals the function $(-1)^{\langle 2\rho, \mu\rangle} \operatorname{tr}(\mathrm{Frob}^*; \mathcal{S}^{-1}(V_\mu))$, where we use the fact that Theorem 9 also holds over $\mathbb{F}_q$ to get the Frobenius map.

3. Global arithmetic

Let Σ_0 be a geometrically connected smooth proper curve over $\mathbb{F}_q$, and write Σ for its base change to $\overline{\mathbb{F}}_q$. Write F for the function field of Σ_0, write $\mathbb{A}$ for its adele ring, and write $\mathbb{O}$ for the subring of integral adeles. Write Weil_{Σ_0} for the Weil group of Σ_0, so that there is a short exact sequence of topological groups

$$1 \longrightarrow \pi_1^{\text{ét}}(\Sigma) \longrightarrow \mathrm{Weil}_{\Sigma_0} \longrightarrow \mathrm{Frob}^{\mathbb{Z}} \longrightarrow 1.$$

Classically, Langlands duality studies $L^2(G(F)\backslash G(\mathbb{A})/G(\mathbb{O}))$ and tries to decompose it as a direct integral of $C_c(G(\mathbb{O})\backslash G(\mathbb{A})/G(\mathbb{O}),\mathbb{C})$-modules indexed by $\check{G}$-valued representations of Weil_{Σ_0}. Since compactly supported functions are dense in L^2, we might as well study $C_c(G(F)\backslash G(\mathbb{A})/G(\mathbb{O}),\overline{\mathbb{Q}}_\ell)$ instead. Moreover, spectral decompositions correspond to global sections of coherent sheaves, so we should try describing $C_c(G(F)\backslash G(\mathbb{A})/G(\mathbb{O}),\overline{\mathbb{Q}}_\ell)$ as the global sections of some coherent sheaf on a *space* of $\check{G}$-valued representations of Weil_{Σ_0} instead.

We will use the following families for our moduli problem, which correspond to

"$\check{G}$-valued representations of Weil_{Σ_0} whose restrictions
to $\pi_1^{\text{ét}}(\Sigma)$ are $\overline{\mathbb{Q}}_\ell$-locally finite and continuous"

under Tannaka duality.

Definition 11. For all derived $\overline{\mathbb{Q}}_\ell$-algebras A, write $\mathrm{LS}_{\check{G}}^{\mathrm{arith}}(A)$ for[1]

$$\left\{\text{right } t\text{-exact symmetric monoidal } D(\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}\check{G}) \to D(A)\otimes_{\overline{\mathbb{Q}}_\ell}\left[\mathrm{Ind}\, D_{\mathrm{lis}}(\Sigma,\overline{\mathbb{Q}}_\ell)\right]^{\mathrm{Frob}^*}\right\}.$$

Proposition 12 ([2])**.** $\mathrm{LS}_{\check{G}}^{\mathrm{arith}}$ *is a derived algebraic stack locally almost of finite type over* $\overline{\mathbb{Q}}_\ell$.

Let v be a closed point of Σ_0. Restricting to the decomposition group at v and using the Tannakian description yields a map

$$\mathrm{LS}_{\check{G}}^{\mathrm{arith}} \to \underline{\mathrm{Hom}}(*/\underline{\mathbb{Z}}, */\check{G}) \cong \underline{\mathrm{Hom}}(\underline{\mathbb{Z}},\check{G})/\check{G} \cong \check{G}/\check{G},$$

so $R\Gamma(\check{G}/\check{G},\mathcal{O}) = \overline{\mathbb{Q}}_\ell[\check{G}//\check{G}]$ acts on derived global sections of coherent sheaves on $\mathrm{LS}_{\check{G}}^{\mathrm{arith}}$ by pullback.

Conjecture 13 ([2])**.** *There is a canonical isomorphism*

$$C_c(G(F)\backslash G(\mathbb{A})/G(\mathbb{O}),\overline{\mathbb{Q}}_\ell) \cong R\Gamma(\mathrm{LS}_{\check{G}}^{\mathrm{arith}},\omega),$$

where ω denotes the dualizing sheaf, that is compatible with the $\mathcal{H}_{G(\mathcal{O}_v)} \cong \overline{\mathbb{Q}}_\ell[\check{G}//\check{G}]$-action under ($\star$) for all closed points v of Σ_0.

Remark 14. Any $\overline{\mathbb{Q}}_\ell^\times$-multiple of an isomorphism as in Conjecture 13 still satisfies the local compatibility condition. However, there should be a *canonical* such isomorphism; one could obtain it by combining Conjecture 21 for $k=\overline{\mathbb{F}}_q$, Remark 22, and Theorem 24. This is important for normalizing the numerical predictions of *relative Langlands duality.*

[1]When $\Sigma_0 = \mathbb{P}^1$, $\mathrm{Ind}\, D_{\mathrm{lis}}(\Sigma,\overline{\mathbb{Q}}_\ell)$ should be replaced by its left-completion. The same holds for Definition 19.

Example 15. When $G = T$ is a torus, every $\check{T}$-valued representation of Weil_{Σ_0} over $\overline{\mathbb{Q}}_\ell$ is irreducible, so $\mathrm{LS}^{\mathrm{arith}}_{\check{T}}$ has trivial derived structure. Therefore the Tannakian description and the commutativity of $\check{T}$ imply that

$$\mathrm{LS}^{\mathrm{arith}}_{\check{T}} \cong \underline{\mathrm{Hom}}_{\mathrm{grp}}(\mathrm{Weil}^{\mathrm{ab}}_{\Sigma_0}, \check{T})/\check{T} = \underline{\mathrm{Hom}}_{\mathrm{grp}}(F^\times\backslash\mathbb{A}^\times/\mathbb{O}^\times, \check{T})/\check{T},$$

where we use the isomorphism $\mathrm{Weil}^{\mathrm{ab}}_{\Sigma_0} \cong F^\times\backslash\mathbb{A}^\times/\mathbb{O}^\times$ from global class field theory that sends geometric Frobenii to uniformizers. This implies that $\omega \cong \mathcal{O}$, so we get

$$\begin{aligned} R\Gamma(\mathrm{LS}^{\mathrm{arith}}_{\check{T}}, \omega) &\cong \overline{\mathbb{Q}}_\ell[(F^\times\backslash\mathbb{A}^\times/\mathbb{O}^\times) \otimes_{\mathbb{Z}} X^*(\check{T})] \\ &\cong \overline{\mathbb{Q}}_\ell[(F^\times\backslash\mathbb{A}^\times/\mathbb{O}^\times) \otimes_{\mathbb{Z}} X_*(T)] \cong C_c(T(F)\backslash T(\mathbb{A})/T(\mathbb{O}), \overline{\mathbb{Q}}_\ell). \end{aligned}$$

The canonical isomorphism as in Remark 14 is the above map times $1 - q$.

We conclude with the following hint towards geometrizing this section.

Proposition 16 (Weil, Harder)**.** *There is a natural equivalence of groupoids*

$$G(F)\backslash G(\mathbb{A})/G(\mathbb{O}) \cong \{G\text{-bundles on } \Sigma_0\}.$$

4. Global geometric

Let k be an algebraically closed field, and let Σ be a connected smooth proper curve over k. Proposition 16 suggests the following geometrization of $G(F)\backslash G(\mathbb{A})/G(\mathbb{O})$.

Definition 17. For all k-algebras R, write $\mathrm{Bun}_G(R)$ for $\{G\text{-bundles on } \Sigma_R\}$.

Proposition 18. Bun_G *is a smooth algebraic stack over* k*.*

One interpretation of global geometric Langlands is that

"constructible sheaves on the space of nice G-equivariant coherent sheaves"

should be equivalent to

"coherent sheaves on the space of nice $\check{G}$-equivariant constructible sheaves."

We will make sense of the first line using Bun_G. To make sense of the second line, we will use the following space of $\check{G}$-local systems.

Definition 19. For all derived $\overline{\mathbb{Q}}_\ell$-algebras A, write $\mathrm{LS}^{\mathrm{restr}}_{\check{G}}(A)$ for

$$\Big\{\text{right } t\text{-exact symmetric monoidal } D(\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell} \check{G}) \to D(A) \otimes_{\overline{\mathbb{Q}}_\ell} \mathrm{Ind}\, D_{\mathrm{lis}}(\Sigma, \overline{\mathbb{Q}}_\ell)\Big\}.$$

Note that $\mathrm{LS}^{\mathrm{restr}}_{\check{G}}$ has the same "$\overline{\mathbb{Q}}_\ell$-locally finite" condition as $\mathrm{LS}^{\mathrm{arith}}_{\check{G}}$, which turns out to force the pointwise semisimplification of any family of $\check{G}$-local systems to be locally constant. Hence this condition <u>restr</u>icts the variation of our families.

Proposition 20 ([2])**.** $\mathrm{LS}^{\mathrm{restr}}_{\check{G}}$ *is a disjoint union of quotients by* $\check{G}$ *of affine formal derived schemes.*

Let v be a k-point of Σ. Taking fibers at v and using the Tannakian description yields a map $\mathrm{LS}^{\mathrm{restr}}_{\check{G}} \to */\check{G}$, so $\mathrm{Rep}^{\mathrm{fin}}_{\overline{\mathbb{Q}}_\ell} \check{G}$ acts on $\mathrm{Ind}\,\mathrm{Coh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$ by pullback. There is also an action of $\mathrm{Perv}(L^+G\backslash\mathrm{Gr}_G, \overline{\mathbb{Q}}_\ell)$ on $D(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ by convolving with sheaves on a *Hecke stack*.

Conjecture 21 ([2])**.** *There is a canonical equivalence*

$$D(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)_{\mathrm{Nilp}} \cong \left[\mathrm{Ind\,Coh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})\right]_{\mathrm{Nilp}}$$

that is compatible with the $\mathrm{Perv}(L^+G\backslash \mathrm{Gr}_G, \overline{\mathbb{Q}}_\ell) \cong \mathrm{Rep}^{\mathrm{fin}}_{\overline{\mathbb{Q}}_\ell} \check{G}$*-action under Theorem 9 for all k-points v of Σ.*

Remark 22. In fact, there is a recipe for the functor in Conjecture 21, and this recipe determines the functor uniquely.

Theorem 23 (Arinkin–Beraldo–Campbell–Chen–Færgeman–Gaitsgory–Lin–Raskin–Rozenblyum [4])**.** *When $k = \mathbb{C}$, Conjecture 21 is true.*

Theorem 24 (Arinkin–Gaitsgory–Kazhdan–Raskin–Rozenblyum–Varshavsky [3])**.** *When $k = \overline{\mathbb{F}}_q$ and Σ is defined over $\mathbb{F}_q$, the categorical trace*

$$\mathrm{tr}(\mathrm{Frob}_*; D(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)_{\mathrm{Nilp}})$$

is naturally isomorphic to the $\overline{\mathbb{Q}}_\ell$-vector space $C_c(G(F)\backslash G(\mathbb{A})/G(\mathbb{O}), \overline{\mathbb{Q}}_\ell)$. Consequently, Conjecture 21 implies Conjecture 13.

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Arithmetic topology and the Langlands program via TQFT

Arthur-César Le Bras

1. Number rings as 3-manifolds. Let F be a global field, i.e. a number field or a function field over a finite field $\mathbb{F}_q$, with ring of integers $\mathcal{O}_K$. To F we can attach the scheme X_F, which is $X_F = \mathrm{Spec}(\mathcal{O}_K)$ if K is a number field and the smooth projective geometrically connected curve X_F over $\mathbb{F}_q$ with field of meromorphic functions F, when F has positive characteristic. (The formulation is not uniform, since we are missing the archimedean places in the number field case.) It is a well-known analogy that X_F, while having Krull dimension 1, behaves like a 3-manifold from the point of view of the étale topology. This is for example justified by the following result of Artin-Verdier [2] and Deninger [3].

Theorem 1. [1] *Let $\mathcal{F}$ be a constructible étale sheaf of abelian groups on X_F. For any $r \in \mathbb{Z}$, the Yoneda pairing*

$$\mathrm{Ext}^r_{X_F}(\mathcal{F}, \mathbb{G}_m) \otimes H^{3-r}(X_F, \mathcal{F}) \to H^3(X_F, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}$$

is a perfect pairing of finite abelian groups (in particular, the cohomology of $\mathcal{F}$ vanishes in degrees > 3).

We can thus, ignoring again the archimedean places in the number field case, imagine X_F as a closed (i.e. compact, without boundary) non-oriented 3-manifold, with each closed point of X_F giving rise to a knot in this 3-manifold. This analogy can in fact be pushed quite far and is the subject of arithmetic topology (see e.g. [5]).

What if we consider instead the ring of S-integers of F, for a finite set of places S of F, containing the archimedean places when K is a number field? Geometrically, this amounts to replacing X_F by the non-empty open subscheme $X_{F,S}$ obtained by removing from X_F the closed points indexed by S. The duality result above remains valid for constructible étale sheaves on $X_{F,S}$, if one replaces cohomology by compactly supported cohomology. The scheme $X_{F,S}$ now behaves like a 3-manifold with boundary given by $\mathrm{Spec}(\prod_{x\in S} F_x)$, where for every $x \in S$, F_x denotes the fraction field of the completed local ring $\mathcal{O}_x$ of X_F at x. This boundary looks like a surface, by local Tate duality.

2. Spaces of automorphic forms: a puzzle. If F is a global field, S a finite set of places of F, and G is a redutive group over $\mathcal{O}_{F,S}$, one can form the $\mathbb{C}$-vector space

$$\mathcal{A}_G(X_{F,S}) = \varinjlim_{(K_x)_{x\in S}} \mathrm{Fun}(G(K)\backslash G(\mathbb{A}_F)/G(\mathbb{O}^S)\prod_x K_x, \mathbb{C})$$

(the colimit running over compact open subgroups K_x of $G(F_x)$) of automorphic functions for G unramified away from S (the precise meaning of $\mathrm{Fun}(-)$ entails some analytic considerations in the number field that we ignore for simplicity). If one takes the analogy between ring of S-integers in global fields and 3-manifolds, it is natural to wonder what the assignement $X_{F,S} \mapsto \mathcal{A}_G(X_{F,S})$ corresponds to on the side of 3-manifolds.

The standard way to attach vector spaces to 3-manifolds would be to consider singular cohomology. This is however not a good analogy. First, singular cohomology makes sense for manifolds of any dimension, while the theory of automorphic forms is something really specific to global fields. Second, the cohomology of a disjoint union of two 3-manifolds is the direct sum of their cohomologies. But if $F \to F'$ is a finite Galois extension of global fields, corresponding geometrically to a finite étale Galois cover on rings of S-integers, for a finite set of bad places S, then for G reductive, $\mathcal{A}_G(X_{F',S})$ looks more like a finite *tensor product* of copies of $\mathcal{A}_G(X_{F,S})$ (think, when $F = \mathbb{Q}$ and F' is a real quadratic extension, of spaces of modular forms versus spaces of Hilbert modular forms). Finally, for fixed F and

[1] The theorem as stated is a small lie: to make it correct as written in the number field, one needs to assume that K is totally imaginary. In general, one needs to correct the cohomology groups to take into account the contributions of the infinite places.

S, the cokernel of the natural map $\mathcal{A}_G(X_F) \to \mathcal{A}_G(X_{F,S})$ is *not* simply described in terms of the boundary $\prod_{x\in S} F_x$, as it would if we were looking at cohomology. These two first objections (and the third one as well, see § below) are formally more reminiscent of the properties of a TQFT.

3. A brief reminder on TQFT's. The notion of TQFT has been introduced in previous talks. Fix an integer $d \geq 1$ and a field of coefficients k. Classically, a (non-oriented) d-dimensional TQFT is a symmetric monoidal functor from the 1-category $\mathrm{Bord}^1(d)$ whose objects are closed $(d-1)$-manifolds and morphisms between closed $(d-1)$-manifolds are bordisms up to diffeomorphisms, with symmetric monoidal structure given by the disjoint union, to the category of k-vector spaces, with the tensor product of k-vector spaces. Note that if M is a closed d-manifold, we can see M as morphism in $\mathrm{Bord}^1(d)$ from the empty manifold of dimension $d-1$ to itself and so if Z is a d-dimensional TQFT, $Z(M)$ is an endomorphism of k, i.e. an element of k. In other words, a d-dimensional TQFT assigns numbers to closed d-manifolds, in a diffeomorphism-invariant manner, and the axioms of a TQFT are here to tell us how to compute these numbers by cutting M along $(d-1)$-dimensional closed submanifolds: if N is a closed $(d-1)$-manifold cutting M a closed d-manifold in two pieces M_0, M_1, $Z(M)$ is the image of $Z(M_0) \otimes Z(M_1) \in Z(N) \otimes_k Z(N)$ by the natural evaluation pairing $Z(N) \otimes_k Z(N) \to k$ coming from seeing $N \times [0,1]$ as a morphism from $N \sqcup N$ to the empty manifold.

If one tries to use this to simplify the geometric structure of M as much as possible, e.g. by triangulating it, one runs into the problem that one needs to cut M along non-closed manifolds, and thus one would like the TQFT to also assign something to manifolds of dimension smaller than $d-1$. This leads to the notion of an *extended* TQFT of dimension d: a symmetric monoidal functor from the (∞, d)-category $\mathrm{Bord}(d)$, whose objects are 0-dimensional manifolds, 1-morphisms bordisms between them, etc., with the symmetric monoidal structure given by the disjoint union, to a symmetric monoidal (∞, d)-category $\mathcal{C}$. (Under some assumptions on $\mathcal{C}$, the previous notion of TQFT is recovered by iterating $(d-1)$-times the process of passing to endomorphisms of the unit. From now, all TQFT's will be understood as extended.) In the arithmetic case, we saw manifolds of dimension $1, 2$ and 3 appear (finite, local, global fields), so the extended perspective will be relevant.

Remark 2. As a sample illustration of functoriality for a TQFT, note that if $i < d$, M is a closed i-manifold, f is a diffeomorphism of M and Z is a d-dimensional TQFT, the categorical trace of the endofunctor $Z(f)$ of $Z(M)$ is given by $Z(M_f)$, where M_f is the mapping torus of f (seen as an endomorphism in $\mathrm{Bord}(d)$ of the empty manifold of dimension i), an object of the category of $i+1$-iterated endomorphisms of the unit of $\mathcal{C}$.

4. The (heuristic) dictionary. Let G be a split reductive group (say over $\mathbb{Z}$, for simplicity), with Langlands dual $\check{G}$. We will illustrate how to think of the Langlands program as an equivalence of two 4-dimensional TQFT's, one attached to G, the automorphic TQFT $\mathcal{A}_G$, and one attached to $\check{G}$, the spectral TQFT $\mathcal{B}_{\check{G}}$. To avoid any ambiguity, let us stress that this is purely metaphorical. We first discuss closed arithmetic manifolds, then non-closed ones. The number in front of each item indicates the dimension.

- (3, closed) If F is a global field, $\mathcal{A}_G(X_F)$ (with X_F as above) has already been specified, while $\mathcal{B}_{\check{G}}(X_F)$ is not known in general, but given when F is a function field by $\Gamma(\mathrm{LocSys}_{\check{G},X_F},\omega)$, the space of global sections of the dualizing sheaf on the stack of arithmetic $\check{G}$-local systems on X_F from [1].
- (2, closed) For X a smooth projective curve over $\overline{\mathbb{F}}_q$ or $\mathbb{C}$, $\mathcal{A}_G(X)$ is the category of sheaves on the stack $\mathrm{Bun}_{G,X}$ of G-bundles on X, where "sheaves" refer to ℓ-adic sheaves ($\ell \neq p$), D-modules or Betti sheaves depending on the situation, while $\mathcal{B}_{\check{G}}(X)$ is some modified version of the category of quasi-coherent sheaves on $\mathrm{LocSys}_{\check{G},X}$, the stack of restricted, de Rham or Betti $\check{G}$-local systems on X.
- (2, closed) The same applies for a local field F, using, so to say, the Fargues-Fontaine curve (or the twistor $\mathbb{P}^1$) attached to F in place of X in the previous item.
- (1, closed) The case of a punctured formal disk $\mathbb{D}^\times = \mathrm{Spec}(k(\!(t)\!))$ over $k = \mathbb{C}, \overline{\mathbb{F}}_p$ is the subject of the local geometric Langlands program. It indicates that $\mathcal{A}_G(\mathbb{D}^\times)$ is the 2-category of $D(LG)$-modules in $(\infty,1)$-categories (where LG is the loop group of G) while $\mathcal{B}_{\check{G}}(\mathbb{D}^\times)$ is the 2-category of sheaves of $(\infty,1)$-categories over $\mathrm{LocSys}_{\check{G},\mathbb{D}^\times}$.
- (1, closed) Interestingly, the case of finite fields seems unknown.
- (3, non-closed) If S is a non-empty finite set of places of a global field F, $\mathcal{A}_G(X_{F,S})$ is the vector space of automorphic functions for G unramified away from S, with its action of $\prod_{x\in S} G(F_x)$.
- (3, non-closed) If F is a local field with ring of integers $\mathcal{O}$, $\mathcal{A}_G(\mathrm{Spec}(F)) = \mathcal{C}_c^{\mathrm{sm}}(G(F)/G(\mathcal{O}),\mathbb{C})$ is the universal unramified representation of $G(F)$ and $\mathcal{B}_{\check{G}}(\mathrm{Spec}(F))$ is the structure sheaf on the stack of unramified L-parameters.
- (2, non-closed) If $\mathbb{D} = \mathrm{Spec}(k[\![t]\!])$ is the disk over $k = \mathbb{C}, \overline{\mathbb{F}_p}$, $\mathcal{A}_G(\mathbb{D}) = D(LG/L^+G)$ with its $D(LG)$-module structure coming from the left translation action and $\mathcal{B}_{\check{G}}(\mathbb{D}) = \mathrm{QCoh}(*/\check{G})$ seen as a sheaf of categories over $\mathrm{LocSys}_{\check{G},\mathbb{D}^\times}$ via the natural map $*/\check{G} \to \mathrm{LocSys}_{\check{G},\mathbb{D}^\times}$ given by the unramified, hence trivial, $\check{G}$-local system.

5. Some pages of the dictionary. We finish with several remarks illustrating the dictionnary above.

Ramification and gluing. Let F be a global field, x a place of F, $S = \{x\}$. One can cut X_F along $\mathrm{Spec}(F_x)$, getting the non-closed 3-manifolds $X_{F,S}$, $\mathrm{Spec}(\mathcal{O}_x)$. Consequently, $\mathcal{A}_G(X_F)$ should be the image of $\mathcal{A}_G(X_{F,S}) \otimes \mathcal{A}_G(\mathrm{Spec}(\mathcal{O}_x)) \in \mathcal{A}_G(\mathrm{Spec}(F_x)) \otimes \mathcal{A}_G(\mathrm{Spec}(F_x))$ under the pairing to the category of vector spaces coming from the self-duality of the category of smooth representations (or sheaves on the stack of G-bundles on the Fargues-Fontaine curve) under Bernstein–Zelevinsky duality. Unraveling it, this is nothing but the statement that the space of everywhere unramified automorphic functions is the unramified subspace of the $G(F_x)$-representation given by the space of automorphic functions which ramify at x.

Hecke action. In the Langlands program, the matching of objects on the Galois side and the automorphic side is compatible with important symmetries on both sides, the Hecke symmetries. It turns out that the compatibility with the Hecke action is also encoded in the equivalence of TQFT's, at least in the 2-dimensional case. Indeed, let M be a 2-dimensional closed manifold. Pick a point $x \in M$ and consider the 3-manifold N obtained by removing from $M \times [0,1]$ a small 3-disk around the point $(x, 1/2)$. We see N as a morphism from $M \sqcup S^2$ to M in the bordism category. If Z is a TQFT, we thus get a functor

$$Z(M) \otimes Z(S^2) \to Z(M),$$

an action of $Z(S^2)$ on $Z(M)$. If we pick the Betti incarnation of the spectral arithmetic TQFT $\mathcal{B}_{\check{G}}$, it makes good sense to evaluate it on the 2-sphere. Since $S^2 = D^2 \cup_{S^1} D^2$ and the 2-disk is contractible, we get[2]

$$\mathcal{B}_{\check{G}}(S^2) = \mathrm{QCoh}(*/\check{G} \times_{\check{G}/\check{G}} */\check{G}) = \mathrm{QCoh}(\check{\mathfrak{g}}[-1]/\check{G}).$$

On the automorphic side, we can take M to be a smooth projective complex curve X. But it does not make sense to evaluate the automorphic TQFT $\mathcal{A}_G$ on S^2, even in its Betti incarnation. However, we can replace S^2 by an algebraic analogue: the gluing of $\mathbb{D}$ with itself over $\mathbb{D}^\times$, and obtain, using that G-torsors on the disk trivialize, the category of sheaves on $L^+G\backslash LG/L^+G$, i.e. the derived Satake category[3], which geometric Satake shows to be equivalent to $\mathcal{B}_{\check{G}}(S^2)$ above. Moreover, this action of the derived Satake category on $\mathcal{A}_G(X) = D(\mathrm{Bun}_{G,X})$ is the Hecke action at x. We expect the same mechanism to apply one dimension higher (using instead the 3-sphere), for spaces of automorphic forms and their action of the (derived) Hecke algebra.

Categorical trace of Frobenius. Finally, we observe that in our list above, the first example in the function field case, resp. the third example in the non-archimedean equal characteristic case, behaves like the mapping torus of Frobenius on the second, resp. fourth example. The TQFT perspective suggests, following Remark 2, that the statements of the Langlands program in the former examples should be obtained as the categorical trace of Frobenius on the categories attached to the latter examples. This has indeed been verified in some cases: cf. [4] for the function field case, [6] for the local field case (under some tame ramification assumption, since nothing is known about local geometric Langlands beyond the tame case).

[2]One should rather consider ind-coherent sheaves, but we gloss over this point.

[3]We note in passing that this point of view also offers an intuitive explanation for the E_3-algebra structure on the derived Satake category: it comes from the fact that S^2 is an E_3-algebra object in the bordism category (as follows from embedding small 3-disks in a bigger 3-disk).

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Introduction to the relative Langlands program

Tasho Kaletha

1. Relative Langlands as a boundary theory in TQFT

Let Σ be a smooth projective geometrically connected $\mathbb{F}_q$-curve, F its field of rational functions. Consider the adeles $\mathbb{A} = \prod_v' [K_v : O_v]$ and the integral adeles $\mathbb{O} = \prod_v O_v$.

Recall from the talk of Le Bras that, using the analogy of arithmetic topology, $\Sigma \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ is to be considered as a closed 2-manifold, Σ is to be considered a closed 3-manifold, and for a place $v \in |\Sigma|$, F_v is to be considered of dimension 2. Starting with a reductive group G/O_F we can get the following shadow of a 4d TQFT

$$\begin{array}{rcl} \Sigma = \mathrm{Spec}(O_F) & \mapsto & \mathcal{C}(G(F)\backslash G(\mathbb{A})/G(\mathbb{O})) \\ \Sigma \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q & \mapsto & \mathrm{Shv}(\mathrm{Bun}_G) \\ F_v & \mapsto & \mathrm{Rep}(G(F_v)) \end{array}$$

The passage from $\Sigma \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ back down to Σ is via the descent structure, which on the values of TQFT is expressed by taking the trace of Frobenius. That is, we need to consider the category of Frobenius-equivariant sheaves on Bun_G, and for each such sheaf we obtain a function $\mathrm{Bun}_G(\mathbb{F}_q) \to \bar{\mathbb{Q}}_\ell$.

In arithmetic topology, we get the analog of a 3-manifold with boundary by taking a finite set of places S of Σ and considering $M = \Sigma - S$. Given a reductive group $G/(\Sigma - S)$ the space

$$\mathcal{A}^S([G]) = \varinjlim_{(K_v)_{v\in S}} \mathcal{C}(G(F)\backslash G(\mathbb{A})/\prod_{v\in S} K_v \times G(\mathbb{O}^S))$$

of automorphic forms that are unramified away from S can be seen as an object $Z(M) \in Z(\partial M) = \bigotimes_{v\in S} Z(F_v) = \bigotimes_{v\in S} \mathrm{Rep}(G(F_v)) = \mathrm{Rep}(G(F_S))$, i.e. a representation of $G(F_S) := \prod_{v\in S} G(F_v)$.

Now we consider how the "relative" part of the Langlands story, introduced by SV and BZSV, enriches the picture. It is reflected in an enrichment of TQFT by a "boundary theory", which formally has the following shape.

$$\begin{aligned}
\text{[vector in vector space] } M_4 &\mapsto Z(M) \in Z(\partial_1 M)\\
\text{[object in category] } M_3 &\mapsto Z(M) \in Z(\partial_1 M)\\
\text{[object in 2-category] } M_2 &\mapsto Z(M) \in Z(\partial_1 M)
\end{aligned}$$

A shadow of such a boundary theory is obtained from the G-variety X as follows:

$$\begin{aligned}
\Sigma &\mapsto \Theta_{\mathbf{1}_{X^\circ(\mathbb{O})}} \in \mathcal{C}(G(F)\backslash G(\mathbb{A})/G(\mathbb{O}))\\
(\Sigma \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q) &\mapsto (p_X)_!\bar{\mathbb{Q}}_\ell \in \mathrm{Shv}(\mathrm{Bun}_G)\\
F_v &\mapsto \mathcal{C}(X(F_v)) \in \mathrm{Rep}(G(F_v))
\end{aligned}$$

There is one more level we can go up: we can consider bordisms between bordisms. If N is a 2-manifold that is the boundary of two different 3-manifolds M_1 and M_2, and if L is a 4-manifold whose boundary is $M_1 \cup M_2$, then N will give us a category $Z(N)$ that will contain objects $Z(M_1)$ and $Z(M_2)$, and $Z(L) : Z(M_1) \to Z(M_2)$ will be a morphism in the category $Z(N)$.

We can apply this to the following situation: Consider a finite set of places S of F. We can let N be S, or equivalently the union of small neighborhoods of the elements of S in the 3-manifold associated to Σ. We can let M_1 be $\Sigma - S$ and M_2 be $N \times [0,1]$ with marked $N \times \{0\}$. Finally, L one can try to convince oneself that there is a bordism between M_1 and M_2, by deforming M_1. We had already agreed that $Z(N) = \mathrm{Rep}(G(F_S))$ and $Z(M_1) = \mathcal{A}^S([G]) \in Z(N)$. On the other hand, the datum of X provides $Z(M_2) = \mathcal{C}(X(F_S)) \in Z(N)$. Then $Z(L)$ becomes a morphism

$$\mathcal{C}(X(F_S)) = Z(M_2) \to Z(M_1) = \mathcal{A}^S([G])$$

in the category $Z(N) = \mathrm{Rep}(G(F_S))$. This morphism is

$$f_S \mapsto \Theta_{f_S \times \mathbf{1}_{X^\circ(\mathbb{O}^S)}}.$$

2. The Riemann Zeta function

Consider for a moment $F = \mathbb{Q}$. Riemann proved that the zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad \mathrm{Re}(s) > 1.$$

has analytic continuation to all of $\mathbb{C}$, with at simple pole at $s = 1$, and functional equation

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s).$$

The classical proof goes by showing that ζ is the Mellin transform of the theta function. Tate put Riemann's proof on adelic footing. More precisely, for a Schwartz function $f \in \mathcal{S}(\mathbb{A})$ and a continuous character $\chi : \mathbb{Q}^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ (called Hecke character), Tate considers the global zeta integral

$$Z(f,\chi) = \int_{\mathbb{A}^\times} f(x)\chi(x)d^*x,$$

where d^*x is a Haar measure on $\mathbb{A}^\times$. Consider now the character $|-|_{\mathbb{A}}^s$ and the adelic Gaussian $\Phi = \Phi_\infty \otimes \otimes_p \Phi_p$, where

$$\Phi_\infty(x) = e^{-\pi x^2}$$

and

$$\Phi_p = \mathbf{1}_{\mathbb{Z}_p}.$$

It is known that $\widehat{\Phi}_p = \Phi_p$, hence $\widehat{\Phi} = \Phi$.

$$Z(\Phi, |-|_{\mathbb{A}}^s) = \int_{\mathbb{A}^\times} \Phi(x)|x|_{\mathbb{A}}^s d^*x = \int_{\mathbb{A}^\times/\mathbb{Q}^\times} \Big(\sum_{a\in\mathbb{Q}^\times} \Phi(ax)\Big)|x|_{\mathbb{A}}^s d^*x = \langle \Theta_\Phi, |-|_{\mathbb{A}}^s\rangle_{\mathbb{G}_m},$$

where

$$\Theta_\Phi(x) = \sum_{a\in\mathbb{Q}} \Phi(ax)$$

is the Theta-series associated to the non-homogeneous affine $\mathbb{G}_m$-variety $\mathbb{A}^1$ and the test function $\Phi \in \mathcal{S}(\mathbb{A}^1(\mathbb{A}))$. We have again replaced summation over $\mathbb{Q}^\times$ with summation over $\mathbb{Q} = X(\mathbb{Q})$ and regularized the integral. As a function of $x \in \mathbb{A}^\times = \mathbb{G}_m(\mathbb{A})$ we have

$$\Theta_\Phi : \mathbb{G}_m(\mathbb{Q})\backslash\mathbb{G}_m(\mathbb{A}) \to \mathbb{C},$$

i.e. Θ_Φ is an automorphic form on $\mathbb{G}_m$.

Lemma 1.

$$Z(\Phi, |-|_{\mathbb{A}}^s) = \pi^{s/2}\Gamma(s/2)\zeta(s).$$

Corollary 2.

$$\pi^{s/2}\Gamma(s/2)\zeta(s) = \langle \Theta_\Phi, |-|_{\mathbb{A}}^s\rangle_{[G]},$$

where $G = \mathbb{G}_m$ and Φ is the global Gaussian function on $X(\mathbb{A})$ for the G-variety $X = \mathbb{A}^1$.

We shall now consider the case of a global function field, where the analogous function Θ_Φ has a geometric interpretation. For this, recall that Σ is a smooth projective geometrically connected curve over $\mathbb{F}_q$ and F is the function field of Σ. Let Pic_Σ denote the stack of line bundles on Σ.

Lemma 3. *(1) We have a bijection $Pic_\Sigma(\mathbb{F}_q) \cong F^\times\backslash\mathbb{A}^\times/\mathbb{O}^\times$.*

(2) If $a \in \mathbb{A}^\times$ corresponds to $\mathcal{L}$ under this bijection, then

$$\Theta_\Phi(a) = \#\mathcal{L}(\Sigma).$$

where $\Phi \in \mathcal{S}(\mathbb{A})$ is the global Gaussian from the zeta integrals in the previous subsection.

We can now consider the stack Pic_Σ^X classifying pairs $(\mathcal{L}, \sigma)$, where $\mathcal{L}$ is a line bundle on Σ and σ is a section of $\mathcal{L}$. Forgetting the section provides a morphism $p_X : \mathrm{Pic}_\Sigma^X \to \mathrm{Pic}_\Sigma$. The function Θ_Φ is associated by the sheaf-to-functions correspondence to the sheaf $(p_X)_! \bar{\mathbb{Q}}_\ell$. This sheaf is called the "period sheaf" and plays an important role in the global duality conjecture. We refer to §1.4 of the article of Sam Gunningham in these proceedings. For a more detailed discussion of the relation between the function Θ_X and the zeta function in the function-field setting, we refer to the article of Shilin Lai.

3. The Gross–Prasad period

3.1. **The period integral.** Let F be a global field and let V be a quadratic F-space, of dimension ≥ 2, and not isomorphic to the hyperbolic plane. Consider F as a quadratic space with the form x^2. Then we can form the orthogonal sum $V \oplus F$. We embed $H = \mathrm{SO}(V)$ diagonally into $G = \mathrm{SO}(V) \times \mathrm{SO}(V \oplus F)$.

Given a cuspidal tempered automorphic representation $\pi \subset L^2([G])$ and $\varphi \in \pi$ we can consider

$$\mathcal{P}_H(\varphi) = \int_{[H]} \varphi(h) dh.$$

Conjecture 4 (GP). *If the automorphic representation $\pi = \pi_V \boxtimes \pi_{V \oplus F}$ has a generic transfer under the standard representation of $\widehat{G}$, the following are equivalent.*

(1) There exists $\varphi \in \pi$ such that $\mathcal{P}_{[H]}(\varphi) \neq 0$.
(2) (a) $L(1/2, \pi_V \times \pi_{V \oplus F}) \neq 0$, and
(b) for every place v of F the local component π_v is H-distinguished, i.e. $Hom_{H(F_v)}(\pi_v, \mathbb{C}) \neq 0$.

3.2. **The period integral and theta series in the unramified setting.**

Lemma 5. *For any $f \in \mathcal{S}(X(\mathbb{A}))$ supported on $H(\mathbb{A}) \backslash G(\mathbb{A})$ and any $\varphi \in L^2_{cusp}([G])$ we have*

$$\langle \Theta_f, \varphi \rangle_{[G]} = \int_{H(\mathbb{A}) \backslash G(\mathbb{A})} f(g) \mathcal{P}_{[H]}(R_g \varphi) dg.$$

We now assume F is a function field and that V arises as $\Lambda \otimes_{O_F} F$ for an O_F-lattice Λ equipped with a quadratic form that has non-degenerate reductions modulo all primes.

Corollary 6. *Let $\varphi \in L^2_{cusp}([G])^{G(\mathbb{O})}$ be an everywhere unramified cuspform, and let $f = \mathbf{1}_{H(\mathbb{O}) \backslash G(\mathbb{O})}$. Then*

$$\mathcal{P}_{[H]}(\varphi) = \langle \Theta_f, \varphi \rangle_{[G]}.$$

3.3. **The local Ichino-Ikeda periods.** In this section F is a local field. Consider a tempered H-distinguished representation $\pi = \pi_V \boxtimes \pi_{V \oplus F}$ of $G = \mathrm{SO}(V) \times \mathrm{SO}(V \oplus F)$, which we recall means that $\mathrm{Hom}_{H(F)}(\pi, \mathbb{C}) \neq 0$. The representation

π is unitary, and we fix an invariant inner product $\langle -, - \rangle$ on π. Then we have the sesquilinear form

$$\mathcal{P} : \pi \times \pi \to \mathbb{C}, \qquad \mathcal{P}(v, w) \mapsto \int_{H(F)} \langle \pi(h)v, w \rangle dh.$$

The integral converges (this is not automatic, needs an argument, in general there is the concept of $X = H\backslash G$ being strongly-tempered which implies that matrix coefficients of tempered G-representations are integrable over H). The choices of inner product on π and Haar measure influence this definition, up to scalar.

Assume now that π is in addition unramified, i.e. $\pi^K \neq 0$ for some hyperspecial maximal compact $K \subset G(F)$. Pick $v \in \pi^K$ and normalize the scalar product so that $\langle v, v \rangle = 1$. Further normalize the Haar measure so that $\mathrm{vol}(K) = 1$.

Proposition 7 (Ichino–Ikeda)**.**

$$\mathcal{P}(v, v) = \Delta \cdot \frac{L(1/2, \pi_V \times \pi_{V \oplus F})}{L(1, \pi, Ad)}.$$

Here Δ is a constant defined by Ichino-Ikeda, depending only on G, but not on π, essentially a product of values of the zeta functions at positive even integers.

3.4. **The Ichino–Ikeda conjecture.** We now return to the setting of a global field F. We take a cuspidal automorphic representation $\pi = \pi_V \boxtimes \pi_{V \oplus F}$ of $G = \mathrm{SO}(V) \times \mathrm{SO}(V \oplus F)$. We assume for simplicity that it is tempered (Ichino–Ikeda state their conjecture without this assumption, but it is more complicated to state; moreover, due to the genericity assumptions that Ichino–Ikeda do make, the representation is expected to be tempered by the generalized Ramanujan conjecture). We further assume π occurs with multiplicity 1 in the cuspidal spectrum.

In order to state the conjecture we need to normalize the local period integrals; not individually, but at least their product. We normalize all unramified places as already discussed. In addition, we require that the product of the local Haar measures is the Tamagawa measure, and the product of all the local inner products is the Peterson inner product:

$$\langle \phi_1, \phi_2 \rangle = \int_{G(F)\backslash G(\mathbb{A})^1} \phi_1(g)\overline{\phi_2(g)} dg, \qquad \phi_1, \phi_2 \in \pi \subset \mathcal{A}_{\mathrm{cusp}}([G]).$$

where again the measure is the Tamagawa measure.

We write $\mathcal{S}_\pi$ for the centralizer of the conjectural global Arthur parameter of the cuspidal tempered automorphic representation π. In fact, given Arthur's book, one has an unconditional description of $\mathcal{S}_\pi$.

Conjecture 8 (Ichino–Ikeda, imprecise form)**.**

$$|\mathcal{P}(\phi)|^2 = |\mathcal{S}_\pi|^{-1} \prod_v{}' \mathcal{P}(\phi_v, \phi_v).$$

Given the unramified computation above, this can be expressed more precisely as follows.

Conjecture 9 (Ichino–Ikeda, precise form). *Choose a finite set of places S away from which all data is unramified. Then*

$$|\mathcal{P}(\phi)|^2 = |\mathcal{S}_\pi|^{-1}\Delta^S \frac{L^S(1/2, \pi_V \times \pi_{V\oplus F})}{L^S(1,\pi,Ad)} \prod_{v\in S} \mathcal{P}(\phi_v,\phi_v).$$

Cartan–Iwasawa decomposition for spherical varieties

Thibaud van den Hove

The goal of this talk was to introduce spherical varieties and some of the combinatorial objects used to study them. Throughout, let G and H be complex reductive groups, fix a maximal torus and Borel $T \subseteq B \subseteq G$, and let $U \subseteq B$ be the unipotent radical.

1. Spherical varieties

Definition 1. A normal connected G-variety $X/\mathbb{C}$ is *spherical* if it admits an open B-orbit.

Example 2. Here are some examples of spherical varieties:

(1) *The group case*: $G = H \times H$ acts on $X = H$ via left and right multiplication, where $H/\mathbb{C}$ is a reductive group. This is spherical by the big open cell.
(2) *The horospherical case*: $X = G/S$, where $S \subseteq G$ is a subgroup containing U. This is spherical by the Bruhat decomposition. This case also includes the example of flag varieties.
(3) *The symmetric case*: $X = G/S$, where $S = G^\theta$ is the fixed points of G under an involution $\theta\colon G \cong G$.
(4) Normal toric varieties agree with spherical varieties when $G = T$ is a torus.

Not all spherical varieties fit in the current framework of relative Langlands duality. Instead, the following assumption is often introduced.

Assumption 3. [BZSV24, Proposition 3.7.4] The B-stabilizers of points in the open B-orbit of the spherical variety X are connected.

This assumption implies that X has no *roots of type N*. Moreover, when X is smooth and affine, it ensures the Hamiltonian G-space given by the cotangent bundle $M = T^*X$ is *hyperspherical*.

2. Some structure theory

We now fix a complex spherical G-variety X.

Definition 4. Recall that a *nonzero rational B-eigenfunction on X* is an element $f \in \mathbb{C}(X)$ such that B acts on f via an eigencharacter $\chi_f\colon B \to \mathbb{G}_m$.

The eigencharacters that appear this way form a group Λ_X, called the *weight lattice* of X.

Its dual will be denoted $Q_X := \mathrm{Hom}_{\mathbb{Z}}(\Lambda_X, \mathbb{Z})$.

By [GN10, p98], there exists a torus $A_X \subseteq X$ with character lattice $X^*(A_X) \cong \Lambda_X$, and hence with cocharacter lattice $X_*(A_X) \cong Q_X$.

Now, let $\mathcal{V}_X$ denote the set of G-invariant valuations $\mathbb{C}(X) \to \mathbb{Z}$ which send the constant functions to 0. Then there is a natural map

$$\mathcal{V}_X \to Q_X : v \mapsto (\chi_f \mapsto v(f)).$$

This is well-defined as each eigencharacter χ_f determines the eigenfunction f up to a scalar, since B-invariant rational functions on f are constant by the existence of an open B-orbit.

By [Kno91, Corollary 2.8], the above map is injective, and its image is a cone of full rank in Q_X.

Definition 5. The subset $\mathcal{V}_X \subseteq Q_X$ is called the *valuation cone* of X.

Example 6. Here are some examples of weight lattices and valuation cones.

(1) When $X = G/B$ is the flag variety, it has an open U-orbit. Thus any rational B-eigenfunction is constant, so that $\Lambda_X = 0 = Q_X = \mathcal{V}_X$.
(2) In the horospherical case $X = G/U$, one can show that $\Lambda_X = X^*(T)$, and $\mathcal{V}_X = Q_X = X_*(T)$ [Pez10, Example 2.2.7, Corollary 3.2.1 (2)]
(3) In the group case $G = H \times H$ and $X = H$, the weight lattice $\Lambda_X = X^*(T_H)$ is the character lattice of a maximal torus $T_H \subseteq H$, and the valuation cone $\mathcal{V}_X \subseteq Q_X = X_*(T_H)$ agrees with the cone of anti-dominant cocharacters $X_*(T_H)^-$ (for a choice of Borel of H containing T_H) [Tim11, Theorem 26.2].

Finally, let us define embeddings of spherical varieties. We now assume that $X = G/S$ is a homogeneous spherical variety.

Definition 7. An *embedding (or partial compactification)* of X is a normal G-variety $\overline{X}$ containing X as an open dense G-subvariety. It is said to be *simple* if it has a unique closed G-orbit, and *toroidal (or without colors)* if no B-stable divisor in X contains a G-orbit of $\overline{X}$ in its closure.

Now, fix a toroidal compactification $X \subseteq \overline{X}$. Each B-stable prime divisor of $\overline{X}$ defines a valuation $v_D \in \mathcal{V}_X$ [Kno91, Lemma 2.4 ff.]. For each G-orbit $Y \subseteq \overline{X}$, let $\mathcal{C}_Y(\overline{X})$ be the cone in Q_X generated by $\{v_D \,|\, D \subseteq \overline{X}$ a B-stable prime divisor, $D \supseteq Y\}$. Let $\mathcal{F}(\overline{X})$ be the fan in Q_X defined as the union of the cones $\mathcal{C}_Y(\overline{X})$, for all G-orbits $Y \subseteq \overline{X}$.

Recall that a fan $\mathcal{F}$ in Q_X is a finite set of cones in Q_X. We say it is an *allowable strictly convex fan*, if each of its elements is a strictly convex cone which is generated by finitely many elements in $\mathcal{V}_X$, that each face of a cone in $\mathcal{F}$ belongs to $\mathcal{F}$, and that any $v \in \mathcal{V}_X$ belongs to the interior of at most one cone in $\mathcal{F}$.

Theorem 8. [GN10, Theorem 8.2.2] *The map $\overline{X} \mapsto \mathcal{F}(\overline{X})$ induces a bijection between isomorphism classes of toroidal embeddings of X, and the set of allowable strictly convex fans in Q_X.*

This is the main ingredient in the proof of Theorem 9 below, for which we refer to [GN10].

3. Cartan–Iwasawa decomposition

Let us again assume that $X = G/S$ is homogeneous. We moreover write $\mathcal{K} = \mathbb{C}(\!(t)\!)$ for the complex Laurent series, and $\mathcal{O} = \mathbb{C}[\![t]\!]$ for the complex power series.

Recall the torus $A_X \subseteq X$ such that $X_*(A_X) = Q_X$. Then we have a map

$$\mathcal{V}_X \subseteq Q_X \to A_X(K) \to X(K).$$

Moreover, the G-action on X induces a $G(\mathcal{O})$-action on $X(\mathcal{K})$.

Theorem 9 ([GN10, Theorem 8.2.9])**.** *The map above induces a bijection*

$$\mathcal{V}_X \cong G(\mathcal{O})\backslash X(\mathcal{K}).$$

This theorem generalizes well-known decompositions of reductive groups.

Example 10.

(1) In the group case $G = H \times H$ and $X = H$, we have

$$G(\mathcal{O})\backslash X(\mathcal{K}) \cong H(\mathcal{O})\backslash H(\mathcal{K})/H(\mathcal{O})$$

and $\mathcal{V}_X \cong X_*(T_H)^-$. Thus we recover the *Cartan decomposition*

$$H(\mathcal{K}) = \bigsqcup_{\mu \in X_*(T_H)^-} H(\mathcal{O})t^\mu H(\mathcal{O}).$$

(2) In the horospherical case $X = G/U$, we can rewrite

$$G(\mathcal{O})\backslash X(\mathcal{K}) \cong U(\mathcal{K})\backslash G(\mathcal{K})/G(\mathcal{O}),$$

and we have $\mathcal{V}_X = X_*(T)$. Thus we recover the *Iwasawa decomposition*

$$G(\mathcal{K}) = \bigsqcup_{\nu \in X_*(T)} U(\mathcal{K})t^\nu G(\mathcal{O}).$$

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The notion of the unramified Plancherel formula and examples, including the Macdonald formula

Jessica Fintzen

Let F be a non-archimedean local field with ring of integers $\mathcal{O}$ whose residue field $\mathbb{F}_q$ contains q elements. Let G be a connected split reductive group over F, with complex Langlands dual group $\check{G}$. We also view G as defined over $\mathcal{O}$ as split reductive group, so that we can write $G(\mathcal{O})$ for a maximal compact subgroup of $G(F)$. We denote by X a "nice" spherical G-variety and by $C_c(X(F))^{G(\mathcal{O})}$ the space of compactly supported complex valued functions on $X(F)$ that are fixed under the $G(\mathcal{O})$-action. (We keep "nice" intentionally vague here. It should be whatever is needed for the below conjecture to hold.)

We started the talk by recalling the Plancherel formula for (split) reductive groups, which we then restricted to spherical representations, and then re-interpreted in the setting where we view G as a spherical variety for the group $G \times G$. This was meant as a motivation and first example of the following informal conjecture due to [1], which was the main focus of the talk:

Conjecture 1 ([1, §9])**.** *There is an isomorphism* $\mathrm{Iso} : C_c(X(F))^{G(\mathcal{O})} \to \mathbb{C}[Z]$ *for some variety* Z *over* $\check{G}/\!\!/\check{G}$

- *that sends the characteristic function* $1_{X(\mathcal{O})}$ *of* $X(\mathcal{O})$ *in* $C_c(X(F))^{G(\mathcal{O})}$ *to the constant function 1 on* Z*, i.e., the identity of* $\mathbb{C}[Z]$*,*
- *that preserves the inner product on the two spaces, where the inner product on* $C_c(X(F))^{G(\mathcal{O})}$ *is given using a measure that is normalized so that the volume of* $X(\mathcal{O})$ *is* $\dfrac{|X(\mathbb{F}_q)|}{|G(\mathbb{F}_q)|} \cdot q^{\dim G - \dim X}$*, and the inner product on* $\mathbb{C}[Z]$ *of two elements* f *and* g *is given by* $\int f\bar{g}d\mu$ *for some (to be determined) measure* μ *on* $Z(\mathbb{C})$*,*
- *such that for every finite dimensional algebraic representation* V *of* $\check{G}$ *we have*
$$\mathrm{Iso}(T_V(f)) = \chi_V \cdot \mathrm{Iso}(f) \quad \text{for all } f \in C_c(X(F))^{G(\mathcal{O})},$$
where T_V *denotes the element of the Hecke algebra* $C_c(G(\mathcal{O})\backslash G(F)/G(\mathcal{O}))$ *attached to* V *via the Satake isomorphism that we recalled during the talk, and* χ_V *denotes the character of* V*. In order to define* $T_V(f)$ *as* $\int_{G(F)} T_V(g) \cdot g.fdg$ *we normalize the action of* $G(F)$ *on* $C_c(X(F))$ *to be unitary.*

In summary, the conjecture predicts an isomorphism

$$(C_c(X(F))^{G(\mathcal{O})}, 1_{X(\mathcal{O})}, \langle\_,\_\rangle) \simeq (\mathbb{C}[Z], 1, \langle\_,\_\rangle_\mu)$$

that intertwintes the action of Hecke operators with multiplication by the corresponding characters. Moreover, Ben-Zvi, Sakellaridis and Venkatesh predict that

$$Z = \check{G}_X /\!\!/ \check{G}_X,$$

where $\check{G}_X$ denotes the dual group of X, and

$$\mu = \frac{1}{\det(1 - q^{-1/2} g|_{V_X})} \cdot (\text{Haar measure on } \check{G}_X^{(1)}),$$

where $\check{G}_X^{(1)}$ denotes a maximal compact subgroup of $\check{G}_X$ and V_X is a certain representation of $\check{G}_X$, see [1, §9] for details.

This conjecture is known in many cases due to [3, 4], see also [1, §9]

In the talk, we showed that the conjecture holds in the following three examples:

(1) $G = \mathbb{G}_m$ and $X = \mathbb{A}_1$, where $Z = \mathbb{G}_m$ and μ is as conjectured for $\check{G}_X = \mathbb{G}_m$ and $V_X = T^*\mathbb{A}^1$. Here $\mathbb{A}_1(F) = F$ is equipped with the additive Haar measure and, as said above, the action of $\mathbb{G}_m$ on it is normalized to be unitary, i.e., $g \in G(F) = F^\times$ sends $f \in C_c(F)$ to $g.f$ with $(g.f)(x) = f(g^{-1}x)|g^{-1}|^{1/2}$.

(2) G a split reductive group with maximal split torus A that is contained in a Borel subgroup B whose unipotent radical is U, and $X = G/U$. In this case $Z = \check{A}$ and the realization of the Satake isomorphism via a bi-module structure on $C_c(G(\mathcal{O})\backslash G(F)\ U(F))$ was used to check the conjecture.

(3) $G = G' \times G'$ and $X = G'$ for a split reductive group G'. This example is the example we started with where $Z = G' /\!\!/ G'$ and the measure μ is the Plancherel measure that was explicitly calculated by Macdonald, see [2].

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Unramified Plancherel formula for spherical varieties

Yi Shan

We recall the example of Hecke's integral and then state the numerical unramified Plancherel conjecture for spherical varieties. Our main reference is [1, §9]. In this talk, F always denotes a non-archimedean local field, with integer ring $\mathcal{O}$ and uniformizer ϖ such that the residue field $\mathcal{O}/\varpi\mathcal{O}$ is isomorphic to $\mathbb{F}_q$.

1. Hecke's integral

Let $G = \mathrm{PGL}_2$ and $X = \mathrm{PGL}_2/\mathbb{G}_\mathrm{m} \simeq \mathbb{P}^1 \times \mathbb{P}^1 - \Delta$. Let $\mathcal{B} := G(\mathcal{O})\backslash G(F)$ be the Bruhat-Tits tree of PGL_2, the left coset $G(\mathcal{O})$ corresponding to $v_0 \in \mathcal{B}$, then the F-points of X can be described as (oriented) straight lines in $\mathcal{B}$.

Decompose $\mathcal{B}$ as a disjoint union $\mathcal{B} = \bigsqcup_{n=0}^{\infty} \mathcal{B}_n$, where $\mathcal{B}_n$ is the set of vertices of the Bruhat-Tits tree having distance n to v_0. The $G(\mathcal{O})$-orbits of $X(F)$ are X_n, $n \geq 0$, where X_n consists of straight lines with distance n to v_0.

For any natural number $n \geq 0$, the Hecke operator $T_n := \mathbf{1}_{G(\mathcal{O})\begin{pmatrix} \varpi^n & \\ & 1 \end{pmatrix}G(\mathcal{O})} \in \mathcal{C}_c^\infty(G(\mathcal{O})\backslash G(F)/G(\mathcal{O}))$ acts on the basic function $\mathbf{1}_{X(\mathcal{O})} = \mathbf{1}_{X_0}$ by:

$$T_n \mathbf{1}_{X_0} = \mathbf{1}_{\{\text{lines passing through some vertex in } \mathcal{B}_n\}} = \sum_{i=0}^{n} \mathbf{1}_{X_i}.$$

Under the normalization of the measure on $X(F)$ as in [1, (9.3)], we normalize the Hecke action to be unitary, *i.e.* $T_n \mathbf{1}_{X_0} = q^{-n/2} \sum_{i=0}^{n} \mathbf{1}_{X_i}$. Similarly, for $m \geq 1$ we have $T_n \mathbf{1}_{X_n} = q^{-n/2} \mathbf{1}_{X_{n+m}}$.

In this case, we have $\langle T_n \mathbf{1}_{X(\mathcal{O})}, \mathbf{1}_{X(\mathcal{O})} \rangle = q^{-n/2}$, and it can be checked that

$$\int_{S^1} z^n \cdot \frac{(1-z^2)(1-z^{-2})}{2(1-q^{-1/2}z)^2(1-q^{-1/2}z^{-1})^2} \frac{dz}{2\pi i z} = q^{-n/2} = \langle T_n \mathbf{1}_{X(\mathcal{O})}, \mathbf{1}_{X(\mathcal{O})} \rangle.$$

2. Numerical unramified Plancherel conjecture

Let G be a split connected reductive group over F, with dual group $G^\vee$ defined over $k = \overline{\mathbb{Q}_\ell} \simeq \mathbb{C}, \ell \neq p$, and X a spherical G-variety satisfying that its cotangent bundle T^*X is hyperspherical. We make some normalization:

- The $G(F)$-eigenmeasure on $X(F)$ satisfies that
$$\operatorname{vol} X(\mathcal{O}) = \frac{|X(\mathbb{F}_q)|}{|G(\mathbb{F}_q)|} q^{\dim G - \dim X},$$
- and the action of Hecke operators is unitary.

Conjecture 2.1. [1, Proposition 9.2.1] *There is an isomorphism*

$$\left(\mathcal{C}_c^\infty(X(F))^{G(\mathcal{O})}, \mathbf{1}_{X(\mathcal{O}),\langle -,- \rangle_{\mathrm{L}^2(X(F))}}\right) \simeq (\mathbb{C}[Z], 1, \langle -,- \rangle_\mu)$$

intertwining the action of Hecke operators with multiplication, where

- $G_X^\vee \supset A_X^\vee$ *is the dual group of the spherical G-variety X,*
- $\check{G}_X^{(1)} \subset G_X^\vee$ *is a compact form, and* $\check{A}_X^{(1)} = A_X^\vee \cap \check{G}_X^{(1)}$,
- $Z = G_X^\vee // G_X^\vee \simeq A_X^\vee // W_X$,
- $V_X = S_X \oplus [\check{\mathfrak{g}}_X^\perp \cap \check{\mathfrak{g}}_e]$ *is a* graded *self-dual representation of $G_X^\vee$ defined in* [1, §4.5],
- *and the unramified Plancherel measure is*
$$\mu = \frac{dg}{\det(1 - q^{-1/2} g|_{V_X})},$$
where dg is the Haar measure on $\check{G}_X^{(1)}$.

Remark 2.2. Via the isomorphism $G_X^\vee//G_X^\vee \simeq A_X^\vee//W_X$, we can write the unramified Plancherel measure μ as

$$\mu = \frac{1}{|W_X|}\frac{\det(1-t|_{\mathfrak{g}_X^\vee/\mathfrak{a}_X^\vee})}{\det(1-q^{-1/2}t|_{V_X})}dt,$$

where dt is the Haar measure on $\check{A}_X^{(1)}$.

Remark 2.3. The Hamiltonian $G^\vee$-space $M^\vee = G^\vee \overset{G_X^\vee}{\times} V_X$ is the dual of the hyperspherical Hamiltonian G-space $M = T^*X$.

Example 2.4. We now use the numerical unramified Plancherel conjecture to write down the Plancherel measure μ for each of the following aforementioned 4 cases, which matches our calculations.

(1) (Tate's thesis) In this case, $G_X^\vee = G^\vee = \mathbb{G}_\mathrm{m}$ and $V_X = S_X = T^*(\mathrm{Std}) = \mathrm{Std}\oplus\mathrm{Std}^\vee$, where Std is the standard 1-dimensional representation of $\mathbb{G}_\mathrm{m}$. Hence the Plancherel measure μ should be
$$\frac{dt}{(1-q^{-1/2}t)(1-q^{-1/2}t^{-1})}.$$
(2) (Horospherical) In this case, $G_X = A^\vee$ is the dual maximal torus of $G^\vee$, and $V_X = S_X = \mathfrak{a}^\vee$, thus μ should be some nonzero multiple of dt.
(3) (Group case) In this case, $G_X^\vee$ is the anti-diagonal $H^\vee$ in $G^\vee = H^\vee \times H^\vee$, and $V_X = \mathfrak{h}^\vee$ with weight 2, thus μ should be
$$\frac{1}{|W|}\prod_{\alpha^\vee\in\Phi(H^\vee,A_H^\vee)}\frac{1-\alpha^\vee(t)}{1-q^{-1}\alpha^\vee(t)}dt = \frac{1}{|W|}\frac{dt}{|c(t)|^2},$$
which recovers Macdonald's formula for unramified Plancherel measure [2].
(4) (Hecke's integral) In this case, $G_X^\vee = G^\vee = \mathrm{SL}_2$, and $V_X = S_X = \mathrm{Std}\oplus\mathrm{Std}$, where Std is the standard 2-dimensional irreducible representation of SL_2. Hence the Plancherel measure μ should be $\frac{(1-t^2)(1-t^{-2})}{2(1-q^{-1/2}t)^2(1-q^{-1/2}t^{-1})^2}dt$.

By a direct calculation, one has the following result:

Proposition 2.5. *Let V, W be algebraic representations of $G^\vee$ with associated Hecke operators T_V, T_W. The multiplicities in the q-series*

$$\langle T_V.\mathbf{1}_{X(\mathcal{O})}, T_W.\mathbf{1}_{X(\mathcal{O})}\rangle = \sum_{i\geq 0} m_i q^{-i/2}$$

satisfy that m_i is equal to the dimension of weight i subspace of

$$\mathrm{Hom}_{\mathbb{C}[V_X]}(\mathbb{C}[V_X]\otimes V, \mathbb{C}[V_X]\otimes W)^{G_X^\vee}.$$

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Some sheaf theoretic background

Joakim Færgeman

The goal of this talk is introduce sheaf theories on various exotic spaces. By 'exotic', we mean spaces that are possibly of very infinite type or very derived. Sheaf theories on such spaces naturally occur in the local and global geometric Langlands program.

There are two main sheaf theories, topological sheaf theory and coherent sheaf theory. Topological sheaf theories usually appears on the geometric side of the Langlands correspondence, while the coherent usually appears on the spectral side.

Topological

By a topological sheaf theory on a finite type scheme over a field k, we mean one of the following three settings:

- (Betti): Suppose $k = \mathbb{C}$ is the complex numbers. In this case, we can consider the dg category of sheaves of $\mathbb{C}$-vector spaces on the underlying analytic topology of X.
- (de Rham): Suppose k has characteristic zero. In this case, we can consider the category of D-modules on X.
- (étale): For arbitrary k and for a prime ℓ different from the characteristic of k, we can consider the category of ℓ-adic sheaves on X.

In any of these cases, we denote by $\mathrm{Shv}(X)$ the resulting sheaf theory. The goal is to extend $\mathrm{Shv}(-)$ to spaces of 'infinite type', such as the loop group of a reductive group. We do this in steps.

Suppose that S is an arbitrary scheme (possibly not of finite type). We define $\mathrm{Shv}(S)$ as follows:

$$\mathrm{Shv}(S) := \mathrm{colim}_{S \to T} \mathrm{Shv}(T).$$

Here, the colimit is taken over all maps $S \to T$, where T is a scheme of finite type, and the transition functors are given by !-pullback.

Suppose that Y is an ind-scheme. That is, Y can be written as a union of closed subschemes Y_i. Then we define:

$$\mathrm{Shv}(Y) := \mathrm{colim}_i \mathrm{Shv}(Y_i).$$

Here, the transition functors are given by $*$-pushforward along the closed embeddings. If the ind-schemes satisfy certain nice properties (such as 'placidity'), then the sheaf theories satisfy nice functorial properties: one has !-pullback and $*$-pushforward.

Coherent

Let $S = \mathrm{Spec}(A)$ be an affine derived scheme (almost) of finite type. Then we may consider two forms of coherent sheaf theories. One the one hand, we have the usual category $\mathrm{QCoh}(S) = A{-}\mathrm{mod}$ of quasi-coherent sheaves on S. On the other hand, we may consider the slightly larger category $\mathrm{IndCoh}(S) := \mathrm{Ind}(\mathrm{Coh}(S))$ of ind-coherent sheaves. The goal is to extend these sheaf theories to derived algebraic stacks

Suppose that Y is a derived stack (almost) of finite type. We define:

$$\mathrm{QCoh}(Y) := \lim_{S\to Y}\mathrm{QCoh}(S);$$

$$\mathrm{IndCoh}(Y) := \lim_{S\to Y}\mathrm{IndCoh}(S).$$

Here, both limits are over all derived affine schemes mapping to Y. In the first colimit, the transition functors are given by $*$-pullback, and in the second, they are given by !-pullback.

The following example conveys the relationship between the topological and coherent sheaf theories. Let $\mathbb{G}_m$ be the multiplicative group and let $B\mathbb{G}_m$ be its classifying stack. Then

$$\mathrm{Shv}(B\mathbb{G}_m) \simeq k[\epsilon]\text{−mod}, \ |\epsilon| = -1.$$

Derived geometric Satake

Ko Aoki

Consider a pinned connected reductive group G over $\mathbf{C}$ and a smooth projective curve X over $\mathbf{C}$. Then the (de Rham) *geometric Langlands conjecture*, which is now a theorem, is an equivalence between linear presentable ∞-categories

$$\mathrm{DMod}(\mathrm{Bun}_G) \simeq \mathrm{IndCoh}_{\mathrm{nil}}(\mathrm{LS}_{G^\vee}),$$

where X is implicit in the notation.

To obtain the statement of derived geometric Satake, we take the *raviolo* R instead of the curve X. Here, the "raviolo" refers to the stack $D \amalg_{D^\times} D$, where D and $D^\times$ are the disk and punctured disk, i.e., the spectra of $O = \mathbf{C}[\![T]\!]$ and $F = \mathbf{C}(\!(T)\!)$, respectively. On the automorphic side, Bun_G is the stack classifying two principal G-bundles on D identified on $D^\times$, which is equivalent to $G_O\backslash G_F/G_O$, where G_O and G_F denote the positive loop group and loop group, respectively. Here, the quotient G_F/G_O is known as the *affine Grassmannian*, denoted Gr_G. On the spectral side, we are led to compute the mapping stack $\mathrm{Map}(S^2, */G^\vee)$, where S^2 denotes the homotopy type of the 2-sphere. It is

$$\mathrm{Map}(S^2, */G^\vee) \simeq */G^\vee \times_{\mathrm{Map}(S^1,*/G^\vee)} */G^\vee \simeq */G^\vee \times_{G^\vee/G^\vee} */G^\vee.$$

Therefore, we expect an equivalence

$$\mathrm{Shv}_{\mathrm{con}}(\mathrm{Gr}_G)^{G_O} \simeq \mathrm{Coh}((* \times_{G^\vee} *)/G^\vee).$$

We note that, due to the renormalization issue, taking Ind does *not* precisely recover the equivalence for $X = R$; see [1].

We then modify the right-hand side using Koszul duality. For a vector bundle V, recall that we have

$$\mathrm{Coh}(* \times_V *) \simeq \mathrm{Perf}(V^*[2]).$$

Using the identification $*\times_{G^\vee} * \simeq *\times_{\mathfrak{g}^\vee} *$ together with this duality, the equivalence we expect can be written as

$$\mathrm{Shv}_{\mathrm{con}}(\mathrm{Gr}_G)^{G_O} \simeq \mathrm{Perf}(\mathfrak{g}^{\vee,*}[2]/G^\vee).$$

The *derived geometric Satake equivalence* of Bezrukavnikov–Finkelberg [2] says that this is indeed an equivalence of *monoidal* linear ∞-categories. We now recall the monoidal structures on both sides:

- On the left-hand side, since
$$G_O\backslash \mathrm{Gr}_G \simeq G_O\backslash G_F/G_O \simeq */G_O \times_{*/G_F} */G_O,$$
we see that it has the convolution monoidal structure.
- On the right-hand side, we can use the underlying monoidal structure of the usual tensor product operations, which is symmetric monoidal, of perfect complexes.

We first relate this equivalence to others. First, this equivalence is compatible with the equivalence
$$\mathrm{Shv}_{\mathrm{con}}(*)^{G_O} \simeq \mathrm{Perf}(\mathfrak{g}^{\vee,*}[2]/\!/G^\vee),$$
which comes from the identification of the group cohomology of $G(\mathbf{C})$ as the ring of symmetric functions. On the spectral side, this corresponds to restricting to the *Kostant section* $\mathfrak{t}/\!/W \simeq \mathfrak{g}^{\vee,*}/\!/G^\vee \to \mathfrak{g}^{\vee,*} \to \mathfrak{g}^{\vee,*}/G^\vee$.

Also, by taking the hearts of the t-structures on both sides, we recover the *abelian geometric Satake equivalence* of Mirković–Vilonen [4], which we have already discussed in the previous lectures. It states the equivalence of monoidal abelian categories
$$\mathrm{Perv}(\mathrm{Gr}_G)^{G_O} \simeq \mathrm{Rep}(G^\vee).$$
Here, "geometric" refers to the sheaf-theoretic nature of the equivalence.

We next examine further structures in the geometric Satake equivalences. The usual proof of derived geometric Satake uses abelian geometric Satake as an input. So let us see how it works. We want to identify the left-hand side with the category of representations via *Tannaka duality.* However, Tannaka duality is a statement about *symmetric* monoidal categories. So we seek to equip the automorphic side with a symmetric monoidal structure. Here, an additional structure comes into play. By varying the position of the raviolo along a curve, we obtain an additional structure known as the *factorization* (or *fusion*) structure. From this, we can show that the convolution monoidal structure on $\mathrm{Perv}(\mathrm{Gr}_G)^{G_O}$ naturally upgrades to a symmetric monoidal structure. However, with this symmetric monoidal structure, the abelian geometric Satake functor does *not* refine to a symmetric monoidal equivalence. So in [4], they adjusted this symmetric monoidal structure explicitly. (We note that on the derived level, Nocera [5] upgraded this factorization-plus-convolution structure to an E_3-monoidal structure in the Betti setting. We also note that Campbell–Raskin [3] proved derived geometric Satake factorizably.)

Another approach to resolving the incompatibility of symmetric monoidal structures is to use *shearing*, which is to be explained in detail in the next lecture. Basically, when a linear presentable ∞-category has an action of $\mathbb{G}_m$, this operation yields another linear presentable ∞-category with $\mathbb{G}_m$-action. We consider the $\mathbb{G}_m$-action on the stack $\mathfrak{g}^{\vee,*}/G^\vee$ coming from the squaring action on $\mathfrak{g}^{\vee,*}$ and the adjoint action via $2\rho\colon \mathbb{G}_m \to G^\vee$. Then shearing gives us the correct spectral side. This shearing approach is also more suitable when we instead start from a reductive group defined over $\mathbf{F}_q$ and consider the compatibility of geometric Satake with the Frobenius action. This approach is taken in the relative Langlands duality paper (and hence in this workshop).

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The local geometric conjecture

Tom Gannon

1. Restating Derived Geometric Satake Via Shearing

Let Vect_{gr} denote the derived category of graded vector spaces. There's an interesting equivalence of categories $(-)^{\nearrow} : \mathrm{Vect}_{gr} \to \mathrm{Vect}_{gr}$ called *shearing*: intuitively, if you have a complex of graded vector spaces, you have two gradings: the actual grading and the cohomological grading. Shearing changes the cohomological grading so that things that were in cohomological degree 0 and was in grading d goes to things in cohomological degree d and grading d. It's not too difficult to show that this equivalence is *monoidal*:

Proposition: There is a monoidal equivalence $(-)^{\nearrow} : \mathrm{Vect}_{gr} \to \mathrm{Vect}_{gr}$.

Let's explain an upshot of this. First, a warm up: if we take the category of modules for $k[t]$ where t is placed in grading two and cohomological degree zero, we get an equivalence:

$$k[t]\mathrm{-mod}(\mathrm{Vect}_{gr}) \xrightarrow{\sim} k[t]^{\nearrow}\mathrm{-mod}(\mathrm{Vect}_{gr})$$

even though if I erase the word grading there is no equivalence of categories taking $k[t]$ to $k[t]^{\nearrow}$!

To see how this applies in our setting, let $M^\vee$ be an arbitrary *affine* Hamiltonian $G^\vee$-space. Recall that this in particular means that $G^\vee$ acts on $M^\vee$ and we have a 'moment map' i.e. a $G^\vee$-equivariant map $M^\vee \to \check{\mathfrak{g}}^*$. Since $M^\vee$ is affine, this moment map is totally determined by a $G^\vee$-equivariant ring map $\mathrm{Sym}(\check{\mathfrak{g}}) \to \mathcal{O}(M^\vee)$. Thus, any affine Hamiltonian $G^\vee$-space gives what's sometimes called an *algebra object* $\mathcal{O}(M^\vee) \in \mathrm{Sym}(\check{\mathfrak{g}})\mathrm{-mod}(\mathrm{Rep}(G^\vee)) = \mathrm{QCoh}(\check{\mathfrak{g}}^*/G)$. Notice that this category is very close to the derived geometric Satake equivalence of the last talk. There is just the pesky shift. However, shearing *does* give an equivalence

$$\mathrm{QCoh}(\check{\mathfrak{g}}^*/G \times \mathbb{G}_m) = \mathrm{Sym}(\check{\mathfrak{g}})\text{−mod}(\mathrm{Rep}(G^\vee)_{\mathrm{gr}})$$
$$\xrightarrow{(-)^{\nearrow}} \mathrm{Sym}(\check{\mathfrak{g}}[-2])\text{−mod}(\mathrm{Rep}(G^\vee)_{\mathrm{gr}}) \approx^{1} \mathrm{QCoh}(\check{\mathfrak{g}}^*[2]/G \times \mathbb{G}_m)$$

so the right hand side of this equivalence, if we forget the grading, is close to a category we actually understand like humans: it's the category of graded $\mathrm{Sym}(\check{\mathfrak{g}})$-modules with a compatible $G^\vee$-representation on each grading. With this, we can first restate the derived geometric Satake equivalence [1]:

Theorem: There is a monoidal equivalence of categories

$$\mathcal{H}_G \cong \mathrm{QCoh}(\check{\mathfrak{g}}^*[2]/G) \cong \mathrm{Sym}(\check{\mathfrak{g}})^{\nearrow}\text{−mod}(\mathrm{Rep}(G^\vee))$$

which is compatible with the natural embeddings of $\mathrm{Rep}(G^\vee)$ via the geometric Satake equivalence of [2] and the functor $V \mapsto \mathrm{Sym}(\check{\mathfrak{g}}) \otimes V$.

Therefore, intuitively, if you are brave enough to pretend that cohomological degree in a derived category is a grading, you can pretend that derived geometric Satake gives an equivalence with the category that affine Hamiltonian $G^\vee$-spaces naturally are algebra objects of.

2. Local Geometric Langlands

Now let's go more into what the relative Langlands duality heurstics suggest for one dimensional objects.

2.1. **Local Geometric Langlands - A Side.** A heuristic motivated by the adage that the Langlands program is an equivalence of two four dimensional arithmetic (topological?) field theories, we feed in a circle (really, a punctured disk[2]) $\mathring{\mathbb{D}}$ into this theory onto both sides of our posed equivalence and see what comes up. The '$\mathcal{A}_G$' topological theory is reasonably easy to describe heuristically. Just as feeding in something like $\mathcal{A}_G(\mathrm{Spec}(\mathbb{Q}_p))$ gives the 1-category of smooth representations of $G(\mathbb{Q}_p)$, we expect that $\mathcal{A}_G(\mathring{\mathbb{D}})$ to be a category of 'representations' of $G(\mathring{\mathbb{D}})$. But what kind of representations? Well, in keeping with our 'fully extended' topological field theory philosophy, we expect $\mathcal{A}_G(\mathring{\mathbb{D}})$ to be a 2-category, or more precisely an $(\infty, 2)$-category. Therefore, a reasonable interpretation on the A-side is categories with an action of $G(\mathring{\mathbb{D}})$. Making this precise is requires some work (like, the foundations of sheaf theory in the setting I'm discussing below is open) but for now I'll just say that it's possible to define a category, which I'll write as $\mathrm{Shv}(G(\mathring{\mathbb{D}}))$, which can be reasonably called the category of $\mathcal{D}$-modules on

[1]Due to technical issues, we do not discuss here, in derived algebraic geometry, the tautological equivalence $\mathrm{QCoh}(\mathrm{Spec}(A)) \simeq A$−mod only holds if A lies in nonnegative cohomological degrees. However, we believe this heuristic equivalence is still useful; we do not know if the literal equivalence of categories written here holds.

[2]At first approximation, you can take $\mathbb{D}$ to be the functor $A \mapsto \mathrm{Spec}(A) \times \mathrm{Spec}(\mathbb{C}[[t]])$, but it's not literally this: rather, as a functor Ring $\to$ Set, it's $A \mapsto A((t))$, which contains but is usually different than $\mathrm{Spec}(\mathbb{C}[[t]])(A) := \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{C}[[t]], A)$

$G(\mathring{\mathbb{D}})$. Moreover, this category has a monoidal structure, so it makes sense to talk about 'module categories' for that category. Therefore, this hypothetical/heuristic topological field theory $\mathcal{A}_G(\mathring{\mathbb{D}})$ has the property that

$$\mathcal{A}_G(\mathring{\mathbb{D}}) = \mathrm{Shv}(G(\mathring{\mathbb{D}}))\text{−modcat}.$$

2.2. **The Local Geometric Conjecture.** Now, the philosophy of 'relative Langlands duality' states that, to many Hamiltonian G-spaces M, one can construct boundary conditions associated to M in both $\mathcal{A}_G$ and $\mathcal{B}_G$ and, moreover, there should exist some Hamiltonian $G^\vee$-space $M^\vee$ such that the equivalence $\mathcal{A}_G \simeq \mathcal{B}_{G^\vee}$ should match the boundary conditions of M and $M^\vee$. Informally, (in the sense that the details or even definition of the sheaf theory is not defined, in general), if $M = T^*(X)$ the boundary condition $\mathcal{A}_{G \dashrightarrow M}$ corresponding to $G \dashrightarrow M$ when evaluated at $\mathring{\mathbb{D}}$ gives $\mathrm{Shv}(X(\mathring{\mathbb{D}})) \in \mathcal{A}_G(\mathring{\mathbb{D}}) = \mathrm{Shv}(G(\mathring{\mathbb{D}}))\text{−modcat}$. This is difficult to define and the technical foundations of what I'm writing aren't known. What's more known (but not totally known) how to define is the unramified setting:

$$\langle \mathcal{A}_G(\mathbb{D}), \mathcal{A}_{G\dashrightarrow M}(\mathbb{D})\rangle \simeq \mathrm{Shv}(G(\mathbb{D})\backslash G(\mathring{\mathbb{D}})) \otimes_{\mathrm{Shv}(G(\mathring{\mathbb{D}}))} \mathrm{Shv}(X(\mathring{\mathbb{D}})) \simeq \mathrm{Shv}^{G(\mathbb{D}))}(X(\mathring{\mathbb{D}})).$$

Notice also that the right hand side admits a 'convolution action' of sheaves in the equivariant Hecke category, defined completely analogously to the action of the Hecke *algebra* on $L^2(G(\mathcal{O})\backslash X(F))$.

Under Langlands duality, the boundary condition associated to such an M is supposed to correspond to some $?_{G^\vee \dashrightarrow M^\vee} \in \mathcal{B}_{M^\vee}(\mathbb{D})$. There are heuristic expectations about what this is, it's hard and people have some good sense of what it should be but don't have a totally precise answer. On the other hand, we have a conjectural guess as to what the pairing of this hypothetical 'dual boundary' condition with $\mathcal{B}(\mathbb{D})$ is. I do not know why this is the case (and I will need to ask people) but the thing to write is

$$\langle \mathcal{B}_{G^\vee}(\mathbb{D}), ?_{G^\vee\dashrightarrow M^\vee}\rangle \simeq \mathrm{QCoh}(M^\vee)^{G^\vee,\nearrow}$$

if $M = T^*X$. The explanation I just gave is not the most motivating, but there is one thing about it which suggests it might be a good 'match' under Langlands duality. Specifically, since $M^\vee$ is a Hamiltonian $G^\vee$-space, the fact that the moment map $\mu : M^\vee \to \check{\mathfrak{g}}$ is $G^\vee$-equivariant gives you a *monoidal* functor

$$\begin{aligned}\mathrm{Sym}(\check{\mathfrak{g}})\text{−mod}(\mathrm{Rep}(G^\vee)) \xrightarrow{X\mapsto \mathcal{O}(M^\vee)\otimes_{\mathrm{Sym}(\check{\mathfrak{g}})} X} &= \mathcal{O}(M^\vee)\text{−mod}(\mathrm{Rep}(G^\vee)) \\ &= \mathrm{QCoh}(M^\vee/G)\end{aligned}$$

but there's one issue: the category on the left hand side of this functor is *not* the derived Satake equivalence codomain. However, we know how to fix that: we *shear* according to the grading of $\mathrm{Sym}(\mathfrak{g})$! The construction of $M^\vee$ has a grading that we haven't talked about much. However, it does have it, and so we can use it to make our *local geometric conjecture* for $M = T^*(X)$ mostly precise, see [3, Conjecture 7.5.1]:

Conjecture: There is an equivalence of categories $\mathrm{Shv}^{G(\mathbb{D}))}(X(\mathring{\mathbb{D}})) \simeq \mathrm{QCoh}(M^\vee)^{G^\vee, \nearrow}$ that is equivariant with respect to the $\mathcal{H}_G \cong \mathrm{QCoh}(\check{\mathfrak{g}}^*)^{G^\vee, \nearrow}$-module structure also subject to some conditions discussed in [3, Conjecture 7.5.1].

Remark: Recall that we have defined a vector space V_X and a dual group $G_X^\vee \subseteq G^\vee$ so that $M^\vee = G^\vee \times^{G_X^\vee} V_X$. Observe that

$$\mathrm{QCoh}(M^\vee)^G = \mathrm{QCoh}(G^\vee \backslash M^\vee) = \mathrm{QCoh}(G^\vee \backslash (G^\vee \times^{G_X^\vee} V_X))$$

$$= \mathrm{QCoh}(G_X^\vee \backslash V_X) = \mathrm{QCoh}(V_X)^{G_X^\vee}$$

so this naturally gives an equivalent formulation of the above conjecture.

3. Examples and Computations

3.1. **Full Equivalences as Examples.** Let's now give some examples where this equivalence is known.

Example 1: Let's first work with $G \dashrightarrow M := *$. In this case, $G^\vee \dashrightarrow M^\vee = G^\vee \dashrightarrow T^*(G^\vee/_\psi N^\vee)$ where $N^\vee$ is the dual unipotent radical of the Borel. Now, one of the foundational results about Kostant sections (see [5], say) is that the $G^\vee$-action on $T^*(G^\vee/_\psi N^\vee)$ is free and that the composite map $T^*(G^\vee/_\psi N^\vee) \xrightarrow{\text{moment map}} \check{\mathfrak{g}}^* \to \check{\mathfrak{g}}^*//G^\vee := \mathrm{Spec}(\mathrm{Sym}(\check{\mathfrak{g}})^{G^\vee})$ induces an isomorphism

$$G^\vee \backslash T^*(G^\vee/_\psi N^\vee) \xrightarrow{\sim} \check{\mathfrak{g}}^*//G^\vee.$$

On the other hand, applying $M = *$ to an equation above gives $\mathrm{Shv}_{G_\mathcal{O}}(*)$. Now, whatever $\mathrm{Shv}_{G_\mathcal{O}}$ means, the pro-unipotence of $\ker(G(\mathbb{D}) \xrightarrow{t=0} G)$ implies that the forgetful functor $\mathrm{Shv}_{G_\mathcal{O}}(X(\mathbb{D})) \to \mathrm{Shv}_G(X(\mathbb{D}))$ is fully faithful provided the latter category is defined, and that if $X = *$ this forgetful functor is in fact an equivalence of categories. Thus

$$\mathrm{Shv}^{G(\mathbb{D})}(*) = \mathrm{Shv}(*/G).$$

Now, this category contains a constant sheaf as a generator, but this generator is not compact, as explained in for example [4]. But this is a common 'renormalization' issue in studying equivariant sheaves: our solution here will be to re-define $\mathrm{Shv}_{G_\mathcal{O}}$ to implicitly renormalize so that the constant sheaf is compact. This is the first of many technical issues that arise in trying to make Shv precise in any setting.

This example also explains the need for shearing: under our renormalization, $\mathrm{Shv}_{G_\mathcal{O}}(*) = \mathrm{Shv}_G(*)$ is the category of modules for the *derived* ring $H_G^*(*)$ which, even in the case when $G = \mathbb{G}_m$, the category of modules for $H_G^*(*)$ is not equivalent to the category of modules for the ordinary ring: for example, $H_T^*(*)-\mathrm{mod}$ contains no compact object which might reasonably be called the 'skyscraper' at the point $1 \in \check{\mathfrak{t}}^*$ since, on all compact objects of $H_T^*(*)-\mathrm{mod}$ the action of any object in positive cohomological degree is nilpotent. A potential fix for this is hoping for a graded lift of both categories appearing in the derived geometric Satake equivalence.

Example 2: Recall that there is an expectation $G\times G \dashrightarrow G$ has dual $G^\vee \times G^\vee \dashrightarrow G^\vee$ up to twisting one of the actions by the Chevalley involution. In this case, the local conjecture reads

$$\mathrm{Shv}_{G(\mathbb{D})}(G(\mathring{\mathbb{D}})/G(\mathbb{D})) = \mathrm{Shv}_{G(\mathbb{D})\times G(\mathbb{D})}(G(\mathring{\mathbb{D}})) = \mathrm{QCoh}_{G^\vee\times G^\vee}(T^*(G^\vee))$$

$$\cong \mathrm{QCoh}^{G^\vee\times G^\vee}(G^\vee \times \check{\mathfrak{g}}^*)^{\nearrow} \cong \mathrm{QCoh}^{G^\vee}(G^\vee\backslash(G^\vee \times \check{\mathfrak{g}}^*))^{\nearrow} = \mathrm{QCoh}^{G^\vee}(\check{\mathfrak{g}}^*)^{\nearrow}$$

so the local conjecture in this case recovers the derived geometric Satake equivalence discussed above.

3.2. **The Basic Object.** To any $G \dashrightarrow X$, there is a natural vector $L^2_{G(\mathbb{Z}_p)}(X(\mathbb{Q}_p))$: it's the indicator function on $X(\mathbb{Z}_p)$. In the heuristics given by topological field theory, since we are viewing $\mathrm{Spec}(\mathbb{Z}_p)$ as a bordism from the empty set to $\mathrm{Spec}(\mathbb{Q}_p)$, we can think of this indicator function as

$$\mathcal{A}_{G,M}(\mathbb{Z}_p) \in \mathcal{A}_{G,M}(\mathbb{Q}_p) = L^2(X(\mathbb{Q}_p)) \in \mathcal{A}_{G,M}(\mathbb{Q}_p) = \mathrm{Rep}(G(\mathbb{Q}_p))$$

where the first 'inclusion' comes from the fact that $\mathbb{Z}_p$ is a bordism and the second inclusion comes from the fact that $\mathcal{A}_{G,M}$ is a boundary condition. There is a categorified version of this:

$$\mathcal{A}_{G,M}(\mathbb{D}) \in \mathcal{A}_{G,M}(\mathring{\mathbb{D}}) = \mathrm{Shv}^{G(\mathbb{D})}(X(\mathring{\mathbb{D}})) \in \mathrm{Shv}(G(\mathring{\mathbb{D}}))\text{−modcat}$$

and the categorification of the indicator function is $\delta_X := i_*(\omega_{X_\mathcal{O}})$. Probably less susprisingly, the corresponding 'distinguished object' on the B side is $\mathcal{O}(M^\vee)$, and one of the desirada of the local conjecture. Taking endomorphisms, we see that

$$\underline{\mathrm{End}}_{\mathrm{QCoh}(M^\vee)^{G^\vee}}(\mathcal{O}(M^\vee)) = O(M^\vee)^{G^\vee}$$

should match with

$$\begin{aligned}\underline{\mathrm{End}}_{\mathrm{Shv}^{G_\mathcal{O}}(X_F)}(\delta_X) &= \underline{\mathrm{End}}_{\mathrm{Shv}^{G_\mathcal{O}}(X_\mathcal{O})}(\omega_X)\\ &= \text{`equivariant Borel-Moore homology of } X_\mathcal{O}\text{'}\end{aligned}$$

after shearing, which in practice means 'treating cohomological degree like it's a grading.' Let's check this explicitly in examples:

Conjecture: The derived geometric Satake equivalence takes $\delta_X \in \mathrm{Shv}_{G(\mathbb{D})}(G(\mathring{\mathbb{D}})/G(\mathbb{D}))$ to the shearing of $\mathcal{O}(M^\vee) \in \mathrm{QCoh}(\check{\mathfrak{g}}^*)^{G^\vee\times\mathbb{G}_m}$.

Example: The $\mathbb{G}_m$-variety $\mathbb{G}_m \dashrightarrow T^*\mathbb{A}^1$ is self dual. Note that $\mathcal{O}(M^\vee)^{\mathbb{G}_m} = k[x_1, \xi_{-1}]^{\mathbb{G}_m} = k[x_1\xi_{-1}]$ is a polynomial algebra in one variable. We claim that the equivariant Borel-Moroe homology of $X_\mathcal{O}$ is too. Here is a hack proof. Since $\mathbb{A}^1_\mathcal{O}$ is the set of loops into $\mathbb{A}^1$, its contractible (as a $\mathbb{G}_m$-space and so the equivariant Borel-Moore homology is $H^*_{\mathbb{G}_m}(\mathrm{pt})$, a formal algebra which is indeed polynomials in one variable after equipping it with a grading and shearing.

Example: It's known that $\mathrm{SL}_2 \dashrightarrow \mathbb{A}^2$ is dual to $\mathrm{PGL}_2 \dashrightarrow T^*(\mathrm{PGL}_2/\mathbb{G}_m)$. Observe that, by analogous arguments to the above, its equivariant Borel-Moore homology is $H^{\mathrm{PGL}_2}_{\mathrm{BM}}(\mathrm{PGL}_2/\mathbb{G}_m) = H^{\mathbb{G}_m}_{\mathrm{BM}}(\mathrm{pt})$. On the other hand,

$$\begin{aligned}\mathcal{O}(M^\vee)^{\mathrm{SL}_2} &= \mathcal{O}(T^*(\mathbb{A}^2))^{\mathrm{SL}_2} = \mathcal{O}(T^*(\mathbb{A}^2 \setminus 0))^{\mathrm{SL}_2}\\ &= \mathcal{O}(T^*(\mathrm{SL}_2/U))^{\mathrm{SL}_2} = \mathcal{O}(\mathrm{SL}_2 \times^U (\mathfrak{g}/\mathfrak{u})^*)^{\mathrm{SL}_2}\end{aligned}$$

$$\mathcal{O}((\mathfrak{g}/\mathfrak{u})^*)^U = \mathrm{Sym}(\mathfrak{g}/\mathfrak{u})^U = \mathrm{Sym}(\mathfrak{t})$$

is also a polynomial algebra; this is written so that it manifestly generalizes to $G \dashrightarrow T^*(G/T)$ and $G^\vee \dashrightarrow \mathrm{Spec}(\mathcal{O}(T^*(G/U))$ which is a Hamiltonian G-space that typically isn't smooth.

Remark: In fact, there is an expected upgrade to the above Borel-Moore isomorphism given in [3, Conjecture 8.1.8] which, informally speaking, removes the $G^\vee$-invariance of the functions on $\mathcal{O}(M^\vee)$. To state it precisely, recall to any affine Hamiltonian $G^\vee$-space $M^\vee$, the moment map $M^\vee \xrightarrow{\mu} \check{\mathfrak{g}}^*$ gives rise to adjunctions

$$\mu^* : \mathrm{QCoh}(\check{\mathfrak{g}}^*)^{G^\vee} \leftrightarrow \mathrm{QCoh}(M^\vee)^{G^\vee} : \mu_*$$

and so, in particular, you can recover the ring of functions on $M^\vee$ *as an algebra object of* $\mathrm{QCoh}(\check{\mathfrak{g}}^*)^{G^\vee}$ via the formula $\mu_*(\mathcal{O}(M^\vee))$. Now, since the local conjecture respects pointings and is $\mathcal{H}_G \cong \mathrm{QCoh}(\check{\mathfrak{g}}^*)^{G^\vee,\nearrow}$-linear, these properties imply that the adjunction above corresponds to

$$\mathrm{act}_{\delta_X} : \mathcal{H}_G \leftrightarrow \mathrm{Shv}_{G(\mathbb{D})}(X(\mathring{\mathbb{D}})) : \mathrm{act}^{\delta_X}_R$$

where δ_X is the basic object, $\mathrm{act}(\mathcal{F}) := \mathcal{F} \star \delta_X$, and $\mathrm{act}^R_{\delta_X}$ is its right adjoint. The object $\mathrm{act}^R_{\delta_X}(\delta_X)$ is sometimes called the *internal endomorphisms* of δ, the *Plancherel algebra* (in the notation of [3, Definition 8.1.4], or the *relative Coulomb branch algebra.* [3, Conjecture 8.1.8] states that the derived geometric Satake functor takes the Plancharel algebra to $\mathcal{O}(M^\vee)$. It's immediately implied by the local conjecture but is more tractible given the sheaf category isn't really defined.

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Spherical varieties

Kalyani Kansal

Let G be a split connected reductive group over a field k of characteristic 0. Let B be a maximal Borel subgroup of G and A be its maximal abelian quotient. Let $X^\bullet(A)$ be the group of characters of A and $X_\bullet(A)$ the group of co-characters, both written additively. We have a short exact sequence

$$1 \to N \to B \to A \to 1,$$

where N is the unipotent radical for B.

1. Review of the first talk on spherical varieties

Recall that a spherical variety for G is a normal variety X with a G-action such that B has a dense orbit. Let $x_0 \in X$ be a point of this dense B-orbit and H its stabilizer. We assume H is connected. We set up our running example for this talk:

Example 1.1. *Consider the action of $G = \mathrm{PGL}_2$ on $\mathbb{P}^1 \times \mathbb{P}^1$ given by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot ([x_1 : x_2], [y_1 : y_2]) = [ax_1 + bx_2 : cx_1 : dx_2], [ay_1 + by_2 : cy_1 : dy_2].$$

The stabilizer of $([1 : 0], [0 : 1])$ under this action is the group H of diagonal matrices. Let $\mathrm{diag}(\mathbb{P}^1)$ be the diagonal embedding of $\mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$. The group action induces an isomorphism

$$\mathrm{PGL}_2 / H \cong \mathbb{P}^1 \times \mathbb{P}^1 - \mathrm{diag}(\mathbb{P}^1).$$

The G-orbits in $\mathbb{P}^1 \times \mathbb{P}^1$ are $\mathbb{P}^1 \times \mathbb{P}^1 - \mathrm{diag}(\mathbb{P}^1)$ and $\mathrm{diag}(\mathbb{P}^1)$. Let B be the group of upper triangular matrices. The B-orbits in $\mathbb{P}^1 \times \mathbb{P}^1 - \mathrm{diag}(\mathbb{P}^1)$ are: $\{[x : 1], [y : 1] | x \neq y\}$, $D_1 = \mathbb{P}^1 \times [1 : 0] - \{[1 : 0], [1 : 0]\}$ and $D_2 = [1 : 0] \times \mathbb{P}^1 - \{[1 : 0], [1 : 0]\}$. The B-orbit $\{[x : 1], [y : 1] | x \neq y\}$ is dense open in $\mathbb{P}^1 \times \mathbb{P}^1 - \mathrm{diag}(\mathbb{P}^1)$ and $\mathbb{P}^1 \times \mathbb{P}^1$, which are both spherical varieties.

The action of $b \in B$ on X induces an action on the rational functions $k(X)$ of X via $f \mapsto (p \in X \mapsto f(b^{-1}(p)))$. The N-invariants of $k(X)$ admit an action by A which results in a decomposition

$$k(X)^N = \oplus_{\chi \in X^\bullet(A)} V_\chi,$$

where each V_χ is an eigenspace for A-action with eigenvalue given by χ. Each V_χ is either 0 or 1-dimensional. Let $\chi(X) \subset X^\bullet(A)$ be the group of characters χ for which $V_\chi \neq 0$. Let $\mathfrak{a} = X_\bullet(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ and let $\mathfrak{a}_X = \chi(X)^* \otimes_{\mathbb{Z}} \mathbb{Q}$, where $\chi(X)^*$ is the $\mathbb{Z}$-linear dual of $\chi(X)$. Clearly, $\mathfrak{a}$ surjects onto $\mathfrak{a}_X$. Finally, recall that there exists a map

$$\rho\colon \{\text{valuations on } k(X)\} \to \mathfrak{a}_X$$

induced by mapping a valuation v to the function on $\chi(X)$ that takes χ to $v(f_\chi)$, where f_χ is any generator of V_χ. This is well-defined because, by definition, a valuation on $k(X)$ restricts to 0 on $k^\times$. The map ρ induces an injection

$$\{G\text{-invariant valuations on } k(X)\} \hookrightarrow \mathfrak{a}_X.$$

Denote the cone generated by the image $\mathcal{V}$.

The action of A on the GIT quotient of Bx_0 by N factors through a quotient A_X, which in turn acts faithfully on this quotient. We have $X^\bullet(A_X) = \chi(X)$. Let $P_X \subset B$ be the parabolic stabilizing Bx_0, and let N_X be the unipotent radical of P_X and L_X the Levi quotient.

2. Combinatorial data

Our next order of business is to classify embeddings $G/H \hookrightarrow X$ of spherical varieties as open dense G-orbits. Turns out, one can cover such embeddings by finite many subembeddings $G/H \hookrightarrow X_i$ such that X_i is open in X for each i, $\cup_i X_i = X$ and each embedding $G/H \hookrightarrow X_i$ is simple, i.e., X_i has a unique (non-empty) closed G-orbit. Thus, one can proceed by first classifying simple embeddings (which we will do in terms of *colored cones*) and then combining these to get not-necessarily-simple embeddings (in terms of *colored fans*).

Proposition 2.1 ([4]). *Let $G/H \hookrightarrow X$ be a simple embedding and Y the unique closed G-orbit of X. Let X° be the complement in X of the union of all B-stable divisors $D \subset X$ such that D does not contain Y. Then*

- $X^\circ = \{x \in X | Y \subset \overline{Bx_0}\}$.
- *X° is an open affine in X.*
- $X = GX^\circ$.

Example 2.2. *In our running example above, consider the embedding* $\mathrm{PGL}_2/H \hookrightarrow \mathrm{PGL}_2/H$, *where* $Y = X^\circ = \mathrm{PGL}_2/H$. *For the embedding* $\mathrm{PGL}_2/H \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$, $Y = \mathrm{diag}(\mathbb{P}^1)$ *and* $X^\circ = \{[x:1],[y:1]\}$.

The upshot of the proposition above is that for simple embeddings, describing X is equivalent to describing the B-stable and affine X°. A regular function f on Bx_0 extends to X° iff for all prime divisors $D \subset X^\circ - Bx_0$, $v_D(f) \geq 0$ for the associated valuation v_D. Such prime divisors are either G-stable or B-stable but not G-stable, and each one of the latter kind is necessarily the closure of a prime divisor in G/H.

Definition 2.3. *Let $\Delta(G/H)$ be the set of prime divisors of G/H that are B-stable but not G-stable. Call such prime divisors colors.*

Thus, one can verify that X° is determined by a subset $\mathcal{D} \subset \Delta(G/H)$ along with the set of G-stable prime divisors of X (that can be viewed as a finite subset of $\mathcal{V}$). This turns out to be equivalent to the data of a subset $\mathcal{D} \subset \Delta(G/H)$ along with the data of a cone in $\mathfrak{a}_X$ generated by a finite subset of $\mathcal{V}$ and $\rho(\mathcal{D})$. The term *colored cone* refers to pairs $(\mathcal{C}, \mathcal{D})$ with $\mathcal{C}$ a strictly convex cone in $\mathfrak{a}_X$ and $\mathcal{D} \subset \Delta(G/H)$, that can possibly come from a suitable X°. We immediately get a bijection between the set of simple embeddings and colored cones.

Example 2.4. *In our running example, let $\alpha \in X^\bullet(A)$ denote the simple root given by mapping*

$$\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \mapsto t.$$

One has an equality $\mathfrak{a} = \mathfrak{a}_X = \mathbb{Q}\alpha^\vee$. Further $\rho(D_1) = \rho(D_2) = \alpha^\vee/2$ and $\mathcal{V} = \mathbb{Q}_{\leq 0}\alpha^\vee$. There are only two colored cones in this setting, namely $(\varnothing, \varnothing)$ and $(\mathcal{V}, \varnothing)$, corresponding to the embeddings $\mathrm{id}\colon \mathrm{PGL}_2/H \hookrightarrow \mathrm{PGL}_2/H$ *and* $\mathrm{PGL}_2/H \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ *respectively.*

A *colored fan* $\mathcal{F}$ is a combinatorial apparatus comprising of a finite set of colored cones $\{(\mathcal{C}_i, \mathcal{D}_i)\}_i$ satisfying some conditions allowing it to encode infomation on how the simple embeddings fit together to give an embedding. An embedding $G/H \hookrightarrow X$ is complete (i.e. X is proper) if for the associated fan, $\mathcal{V} \subset \cup_i \mathcal{C}_i$.

Next, we will construct a based root datum from a spherical variety X and use it to construct a reductive group G_X. Here are the steps in the recipe specified in [3]:

(1) Designate $\chi(X)$ to be the weight lattice.
(2) The set $\mathcal{V} \subset \mathfrak{a}_X$ is a Weyl chamber for a finite reflection group, by [1, 2]. Designate $\mathcal{V} \subset \mathfrak{a}_X$ to be the antidominant Weyl chamber. (Thus, the root datum remembers some data related to compactifications of the spherical variety.)
(3) Use $\mathcal{V}$ to determine a set Σ_X (not the final set) of simple roots, and use this to construct the Weyl group W_X.
(4) We assume that "there are no roots of type N". Renormalize Σ_X to get the set Δ_X of simple roots (the final set) for the based root datum. We omit the recipe for renormalization, but the process specifies for each $\gamma \in \Delta_X$, an *associated root* of G.

The based root datum abstractly gives a reductive group G_X with a choice of Borel. We note without details that the antidominance of $\mathcal{V}$ and renormalization of Σ_X to get Δ_X is related to constructing some compatibility between G_X and G.

3. A relation between G_X and G

Let $G^\vee$, $G_X^\vee$, $A^\vee$ and $A_X^\vee$ denote respectively the Langlands dual groups of G, G_X, A and A_X. The inclusion $\chi(X) \hookrightarrow X^\bullet(A)$ induces maps $A \twoheadrightarrow A_X$ and $A_X^\vee \hookrightarrow A^\vee$. (The injectivity of the second map is related to the assumption of connectedness on H.) Let $\mathfrak{g}_X^\vee$ and $\mathfrak{g}^\vee$ denote respectively the Lie algebras of $G_X^\vee$ and $G^\vee$.

Definition 3.1. *A map $G_X^\vee \to G^\vee$ is distinguished if*

- *it extends the map $A_X^\vee \hookrightarrow A^\vee$ above, and*
- *for all $\gamma \in \Delta_X$, the corresponding root space of $\mathfrak{g}_X^\vee$ maps into the root space in $\mathfrak{g}^\vee$ corresponding to its associated root.*

Let $\rho_{L_X} \in X^\bullet(A)$ be half the sum of the positive roots of L_X. It can alternately be viewed as a co-character $\mathbb{G}_m \to A^\vee$ via the identification $X^\bullet(A) = X_\bullet(A^\vee)$.

Theorem 3.2 ([5, 3])**.** *There exists a map $G_X^\vee \times \mathrm{SL}_2 \to G^\vee$ such that the restriction to $G_X^\vee$ is distinguished and the restriction to SL_2 has weight $2\rho_{L_X}$, i.e. the map $\mathrm{SL}_2 \to G^\vee$ induces a map from the diagonal torus $\cong \mathbb{G}_m$ of SL_2 to $A^\vee \subset G^\vee$ given by $2\rho_{L_X}$.*

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Hyperspherical varieties

Dmitry Kubrak

The goal of this talk was to introduce *hyperspherical varieties*, which can be considered as a Hamiltonian version of spherical varieties. The main result discussed during the talk was the structure theorem that identifies any hyperspherical variety with the so-called *Whittaker induction* of a symplectic representation. The construction of the dual hyperspherical variety in the polarized case was also outlined.

Throughout the talk let $\mathbb{F}$ be an algebraically closed field of characteristic 0. All actions of algebraic groups on varieties will be right actions, unless stated otherwise.

1. Hamiltonian induction and reduction

Let $\alpha\colon H \to G$ be a map of algebraic groups over $\mathbb{F}$. Given a variety X with an action of H one can consider the corresponding *induction*, given by the formula

$$\mathrm{ind}_H^G(X) := X \times^H G := (X \times G)/H, \quad \text{where } h \circ (x,g) = (xh, \alpha(h)^{-1}g).$$

This produces a stack (in the case $H \to G$ is a subgroup it is still a variety) with an action of G. When the map α is given by the projection $p\colon G \to \operatorname{Spec}\mathbb{F}$ from the trivial group, we get the quotient X/G, which one can also call the *reduction*

$$\mathrm{red}_G(X) := \mathrm{ind}_G^{\operatorname{Spec}\mathbb{F}}(X) \simeq X/G.$$

Note that $\mathrm{ind}_H^G(X) \simeq \mathrm{red}_H(X \times G)$.

Similar constructions make sense for *Hamiltonian G-spaces.* Recall that a Hamiltonian G-space is a smooth symplectic variety X endowed with an action of G by symplectomorphisms, and endowed with an extra datum given by a moment map $\mu\colon X \to \mathfrak{g}^*$. A sanity check, as usual, is that Hamiltonian induction/reduction should be compatible with classical induction/reduction when applied to the cotangent bundle T^*X of a variety X with H-action. Skipping some details, the resulting formulas are given by

$$\text{h-red}_G(X) := X/\!\!/\!\!/_0 G := \mu^{-1}(0)/G,$$

for the the *Hamiltonian reduction*, and

$$\text{h-ind}_H^G(X) := \text{h-red}_H(X \times T^*G) \simeq (X \times T^*G)/\!\!/\!\!/ H$$

for the *Hamiltonian induction.* If X was graded[1], then naturally so is $\text{h-ind}_H^G(X)$.

Identifying $T^*G \simeq \mathfrak{g}^* \times G$, one can also rewrite $\text{h-ind}_H^G(X) \simeq X \times_{\mathfrak{h}^*} \mathfrak{g}^* \times^H G$; in particular there is a natural projection $\text{h-ind}_H^G(X) \to G/H$, which is a fiber bundle with the fiber given by $X \times_{\mathfrak{h}^*} \mathfrak{g}^*$. In the case X is a symplectic H-representation this endows $\text{h-ind}_H^G(X)$ with the structure of an affine bundle over G/H. See [1, Section 3.3] for more details.

2. Whittaker induction

We will need a slight *twisted* variant of the Hamiltonian induction construction, namely the *Whittaker induction.* It takes as an input a homomorphism $H \times \mathrm{SL}_2 \to G$ (rather than $H \to G$) and defines an operation

$$\text{wh-ind}_H^G \colon \{\text{(graded) Hamiltonian } H\text{-spaces}\} \longrightarrow \{\text{(graded) Hamiltonian } G\text{-spaces}\}.$$

Let $H \subset G$ be a subgroup, and let $H \times \mathrm{SL}_2 \to G$ be a homomorphism which extends the above embedding. Let us fix a G-equivariant identification $\mathfrak{g} \simeq \mathfrak{g}^*$. We also have a map $\mathfrak{sl}_2 \to \mathfrak{g}$, providing us with elements $(e, h, f) \in \mathfrak{g} \simeq \mathfrak{g}^*$. Consider the cocharacter ϖ given by the composition $\mathbb{G}_m \to \mathrm{SL}_2 \to G$; one has $\mathrm{Ad}(\varpi(t)) \circ f = t^{-2} f$. One has that $H \subset G$ is a subgroup of the centralizer of (the image of) $\mathfrak{sl}_2 \to \mathfrak{g}$.

We can decompose

$$\mathfrak{g} \simeq \mathfrak{j} \oplus \overline{\mathfrak{u}} \oplus \mathfrak{u}^0 \oplus \mathfrak{u},$$

where $\mathfrak{j} \subset \mathfrak{g}$ is the Lie algebra of the centralizer of $\mathfrak{sl}_2$ and $\overline{\mathfrak{u}} \oplus \mathfrak{u}^0 \oplus \mathfrak{u}$ is the sum of all nontrivial $\mathfrak{sl}_2$-subrepresentations, decomposed into the sum of negative, zero, and positive weight spaces for the left adjoint action of ϖ; in particular $f \in \overline{\mathfrak{u}}$. Subspaces $\mathfrak{u}$ and $\overline{\mathfrak{u}}$ are Lie subalgebras graded by the $\mathbb{G}_m$-action induced by ϖ; we put $U, \overline{U} := \exp(\mathfrak{u}), \exp(\overline{\mathfrak{u}}) \subset G$ be the corresponding unipotent subgroups.

We let $\mathfrak{u}_+ \subset \mathfrak{u}$ be the Lie subalgebra spanned by elements with grading ≥ 2. Further, for simplicity, we will assume that $\mathfrak{u}_+ = \mathfrak{u}$ (this is equivalent to $-1 \in \mathrm{SL}_2$ being central in G) and refer to [1, Section 3.4.1] for the general definition.

We will still denote by f the image of $f \in \mathfrak{g}^*$ under the restriction map $\mathfrak{g}^* \to \mathfrak{u}^*$. Given a Hamiltonian H-space X we can define a Hamiltonian HU-space $\widetilde{X}$ whose underlying space is X, the action of H and the corresponding moment map are as before, the action of U is trivial, but the corresponding moment map $X \to \mathfrak{u}^*$

[1]Namely, had a $\mathbb{G}_{\mathrm{gr}}$-action that rescales the symplectic form and moment map by the square character, and that commutes with the H-action.

sends everything to $f \in \mathfrak{u}^*$. The authors of [1] define **Whittaker induction** of X to be

$$\text{wh-ind}_H^G(X) := \text{h-ind}_{HU}^G(\widetilde{X}).$$

As discussed in [1, Sections 3.4.5, 3.4.6], if X is graded, there is also a natural grading on $\text{wh-ind}_H^G(X)$. In the case when X is a symplectic H-representation, there is an explicit formula for $\text{wh-ind}_H^G(X)$ as a vector bundle over G/H: namely,

$$\text{wh-ind}_H^G(X) \simeq V \times^H G, \quad \text{with } V := [X \oplus (\mathfrak{h}^\perp \cap \mathfrak{g}^{*,e})],$$

and where $\mathfrak{g}^{*,e}$ is the annulator of $e \in \mathfrak{g}$. In particular, when H and X are trivial, we can identify $\text{wh-ind}_H^G(X) \simeq \mathfrak{g}^{*,e} \times G$.

3. Hyperspherical Hamiltonian spaces

Definition 3.1. *A (smooth) graded irreducible Hamiltonian space M is called **hyperspherical** is*

(1) M is affine;
(2) the field $\mathbb{F}(M)^G$ of G-invariant rational functions on M is commutative with respect to the Poisson bracket;
(3) the image $\mu(M)$ of the moment map intersects non-trivially the nilpotent cone of $\mathfrak{g}^*$;
(4) the stabilizer in G of a generic point in G is connected;
(5) $\mathbb{G}_{\mathrm{gr}}$-action on M is "neutral".

To explain what condition (5) means, one should first establish some consequences of the other four. Namely, as shown by the authors in [1, Section 3.5], following the work Losev [2], under the first three assumptions above, there is a unique closed $G \times \mathbb{G}_{\mathrm{gr}}$-orbit M_0 in M; it's image under the moment map is necessarily contained in the nil-cone of $\mathfrak{g}^*$. Let $x \in M_0(\mathbb{F})$ be a point, and let $H := \mathrm{Stab}_x \subset G$. Let $f := \mu(x) \in \mathfrak{g}^*$; it is a nilpotent element. Let $S := (T_xM_0)^\perp/(T_xM_0)^\perp \cap T_xM_0$ be the fiber of *symplectic normal bundle* to the orbit M_0. The $\mathbb{G}_{\mathrm{gr}}$-action on M is called *neutral* if

(1) one can complete f to a map $\alpha\colon \mathrm{SL}_2 \to G$, so that the action of corresponding $\mathbb{G}_m \subset \mathrm{SL}_2 \xrightarrow{\alpha} G$ agrees with $\mathbb{G}_{\mathrm{gr}}$-action when restricted to M_0;
(2) the subgroup $\mathbb{G}_m = \{(\alpha(t)^{-1}, t) \subset G \times \mathbb{G}_{\mathrm{gr}}\}$ (which fixes x) acts on the fiber S by simple rescaling.

One can show that the map α is unique if it exists, and that it commutes with $H \subset G$, thus giving a map $H \times \mathrm{SL}_2 \to G$.

The authors prove the following important structural result for hyperspherical varieties:

Theorem 3.2 ([1, 3.6.1])**.** *Let M be a hyperspherical G-variety. Let $x \in M_0$ be a point; let $H \times \mathrm{SL}_2 \to G$ and S be as above. Then there is a unique $G \times \mathbb{G}_{\mathrm{gr}}$-isomorphism*

$$M \simeq \text{wh-ind}_H^G(S)$$

which carries the base-point of Whittaker induction to x, and is an isomorphism on the fibers of correponding symplectic normal bundles.

To paraphrase, a hyperspherical variety is uniquely encoded by the data of a subgroup $H \subset G$, a map $H \times \mathrm{SL}_2 \to G$ and a certain symplectic H-representation S. Let us also note that if we want a Whittaker induction $\text{wh-ind}_H^G(S)$ to be hyperspherical then the quotient G/HU should be spherical (but more assumptions are also needed).

4. Polarization

Definition 4.1. *In the setting of Theorem 3.2, a* ***polarization*** *of M is a Lagrangian H-stable decomposition*

$$S \simeq S^+ \oplus S^-.$$

Denote $X := S^+ \times^{HU} G$ where the action of U on S^+ is trivial. One can view M as a so-called *Whittaker-twisted cotangent bundle* over X, denoted $T^*(X, \Psi)$; unfortunately we do not have time to explain this in detail, so let us refer the reader to [1, Sections 3.2, 3.7]. Let us nevertheless mention the following addendum to Theorem 3.2, which shows the importance of considering the variety X:

Proposition 4.2 ([1, 3.7.4])**.** *Let M be a polarized hyperspherical variety. Then X is a spherical G-variety and B-stabilizers of points in the open B-orbit are connected. Moreover, if X satisfies these assumptions, then the Whittaker induction $\text{wh-ind}_H^G(S)$ is hyperspherical.*

Now, the general idea for the construction of the dual hyperspherical variety is to use Theorem 3.2: namely, given data $H \subset G$, $H \times \mathrm{SL}_2 \to G$ and S associate to it a subgroup $\check{H} \subset \check{G}$ of the Langlands dual of G, a map $\check{H} \times \mathrm{SL}_2 \to G$ and a representation $\check{S}$ of $\check{H}$. In the polarizable case, one can take the first two pieces of the dual data to be the subgroup $G_X^\vee \subset \check{G}$ and the map $G_X^\vee \times \mathrm{SL}_2 \to \check{G}$ constructed by Knopp, and which appeared in Kalyani's talk. In [1, Sections 4.3, 4.4] the authors also provide a natural candidate for what the representation $\check{S}$ could be; in this construction the highest weights of $\check{S}$ are defined explicitly by certain "colours" of the variety X.

References

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[2] Losev, I., "Algebraic Hamiltonian actions". Mathematische Zeitschrift 263, no. 3 (2009): 685-723.

Periods and L-functions

Shilin Lai

Let Ξ be a smooth projective curve defined over $\mathbb{F}_q$, then we can form its function field $F = \mathbb{F}_q(\Xi)$, integral adeles $\mathbb{O}$, and adeles $\mathbb{A}$. We fix an isomorphism of coefficient fields $\mathbb{C} \cong \bar{\mathbb{Q}}_\ell$ for convenience.

Let G be a reductive group over $\mathbb{F}_q$. The classical (unramified) global Langlands conjecture is roughly a picture of the form

$$\begin{Bmatrix} \text{Unramified automorphic} \\ \text{representations of } G(\mathbb{A}) \end{Bmatrix} \longleftrightarrow \{\check{G}\text{-local systems on } \Xi\}$$

In particular, there are L-functions defined on both sides, and they are supposed to agree. The definition of L-functions on either side depends on additional data, and one part of the relative Langlands framework is to clarify this. The picture is now roughly

$$\begin{tikzcd} \begin{Bmatrix} \text{Unramified automorphic} \\ \text{representations of } G(\mathbb{A}) \end{Bmatrix} \arrow[r, leftrightarrow] \arrow[d, squiggly, "X"] & \{\check{G}\text{-local systems on } \Xi\} \arrow[d, squiggly, "\check{X}"] \\ \text{Period} \arrow[r, leftrightarrow, "\approx"] & \text{Values of } L\text{-function} \end{tikzcd}$$

This talk will explain the constructions of period functions and L-functions. We will also illustrate the connection in the Iwasawa–Tate case, which was explained in Kaletha's talk from two days ago. The following talk will categorify both sides into sheaves.

1. Automorphic side

The A-side TQFT $\mathcal{A}_G$ evaluated on the 3-dimensional object Ξ gives the vector space of functions on $\mathrm{Bun}_G(\mathbb{F}_q)$. A boundary condition gives an element of this space. Let X be any G-variety, then the boundary theory $\mathcal{A}_{G,X}$ produces the following element.

Definition 1.1. The *theta series* for X is the function $\Theta_X : \mathrm{Bun}_G(\mathbb{F}_q) \to \mathbb{C}$ defined by

$$\Theta_X(g) := \sum_{x \in X(F)} \mathbf{1}_{X(\mathbb{O})}(xg)$$

for all $g \in G(\mathbb{A})$. Here, $\mathbf{1}_{X(\mathbb{O})}$ is the indicator function of the open compact set $X(\mathbb{O}) \subseteq X(\mathbb{A})$.

Remark 1.2. In general, the presence of a non-trivial $\mathbb{G}_{\mathrm{gr}}$-action introduces an additional twist by a square root of the canonical bundle, cf. [1, (10.6)].

Example 1.3. In the Iwasawa–Tate case, $G = \mathbb{G}_m$ acts on $X = \mathbb{A}^1$ by scaling, so

$$\Theta_X(a) = \sum_{x \in F} \mathbf{1}_{\mathbb{O}}(ax) = \#\{x \in F \mid x \in a^{-1}\mathbb{O}\}.$$

This is exactly the number of global sections of the line bundle attached to a.

This example is a part of the following geometric reinterpretation of the theta series, which follows by the same type of computation.

Lemma 1.4. *Let $\mathcal{G}$ be the G-bundle attached to $g \in G(\mathbb{A})$, then $\Theta_X(g)$ counts the number of X-sections of the X-bundle associated to $\mathcal{G}$.*

Given a function $\varphi : \mathrm{Bun}_G(\mathbb{F}_q) \to \mathbb{C}$, we can pair it against the theta series

$$\langle \Theta_X, \varphi \rangle = \int_{G(F)\backslash G(\mathbb{A})} \Theta_X(g)\varphi(g)\, dg = \sum_{x \in \mathrm{Bun}_G(\mathbb{F}_q)} \frac{1}{\#\mathrm{Aut}_x} \Theta_X(x)\varphi(x)$$

We call this the *period* of φ. In other words, $\langle \Theta_X, - \rangle$ is the period functional on automoprhic forms.

Example 1.5. If $X = H\backslash G$ is homogeneous, then in our everywhere unramified setting, an easy computation shows that

$$\langle \Theta_X, \varphi \rangle = \int_{H(F)\backslash H(\mathbb{A})} \varphi(h)\, dh$$

is the usual period.

2. Spectral side

Let Ξ be a $\check{G}$-local system. Its L-function depends on the additional datum of an algebraic representation

$$r : \check{G} \to \mathrm{GL}_n .$$

This determines a rank n local system $r \circ \Xi$ on Ξ.

Definition 2.1. Let $\mathcal{L}$ be a local system on Ξ. Its L-function is defined to be

$$L(s, \mathcal{L}) = \det \left(1 - q^{-s}\mathrm{Frob}|\mathrm{R}\Gamma(\Xi_{/\bar{\mathbb{F}}_q}, \mathcal{L})\right)^{-1}.$$

The determinant is graded, namely the odd degree terms carry an exponent of -1.

Since Ξ is a smooth curve, the cohomology is supported in degrees $[0, 2]$, and really the only interesting degree is 1. Let $V = \mathrm{H}^1(\Xi_{/\bar{\mathbb{F}}_q}, \mathcal{L})$, then the corresponding term in the above product is

$$\det(1 - q^{-s}\mathrm{Frob}|V) = \mathrm{Tr}(q^{-s}\mathrm{Frob}| \wedge^\bullet V).$$

Again, the trace is a graded trace.

Classically, L-functions are usually defined as an Euler product over the places of a global field. In the function field case, this connection is given by the Grothendieck–Lefschetz trace formula.

Theorem 2.2. *There is a factorization*

$$L(s, \mathcal{L}) = \prod_{v \in |\Xi|} L(s, \mathcal{L}_v),$$

where each term is defined by

$$L(s, \mathcal{L}_v) = \det(1 - (q^{-s}\mathrm{Frob})^{\deg v}|\mathcal{L}_v)^{-1}.$$

Example 2.3. Let $\mathbf{1}$ be the constant local system of rank 1, then for each v,

$$L(s, \mathbf{1}_v) = \frac{1}{1 - q^{-s \deg v}}.$$

The global L-function is equal to

$$L(s, \mathbf{1}) = \frac{P(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})} = \prod_{v \in |\Xi|} L(s, \mathbf{1}_v),$$

where $P(T)$ is some polynomial of degree $2g$, $g = \mathrm{genus}(\Xi)$. Note that this is a meromorphic function of $s \in \mathbb{C}$, and it has simple poles at $s = 0$ and $s = 1$. In this case, Poincaré duality implies the functional equation $L(s, \mathbf{1}) = L(1 - s, \mathbf{1})$.

3. Numerical relations

In the Tate–Iwasawa setting of $G = \mathbb{G}_m$ acting on $X = \mathbb{A}^1$, we will compute the period

$$\langle \Theta_X, q^{-s \deg} \rangle,$$

where $\deg : \mathrm{Pic}_\Xi(\mathbb{F}_q) \to \mathbb{Z}$ is the degree function. This was exactly the computation in Tate's thesis that Kaletha explained previously, but we will say it in a slightly different way.

By definition,

$$\begin{aligned}
\langle \Theta_X, q^{-s \deg} \rangle &= \sum_{\mathcal{L} \in \mathrm{Pic}_\Xi(\mathbb{F}_q)} q^{-s \deg(\mathcal{L})} \Theta_X(\mathcal{L}) \\
&= \sum_{\mathcal{L} \in \mathrm{Pic}_\Xi(\mathbb{F}_q)} q^{-s \deg(\mathcal{L})} \# \mathrm{H}^0(\mathcal{L}) \\
&\text{“}=\text{”} \sum_{D \in \mathrm{Div}^+ \Xi} q^{-s \deg D}.
\end{aligned}$$

For the final equality, the zero section contributes a divergent term, which also appeared in Tate's thesis. It can be handled by a suitable regulariation procedure.

An effective divisor is a formal linear combination of closed points of Ξ with non-negative integer coefficients, so the final sum above factors

$$\begin{aligned}
\sum_{D \in \mathrm{Div}^+ \Xi} q^{-s \deg D} &= \prod_{v \in |\Xi|} \sum_{n \geq 0} q^{-s \cdot n \deg v} \\
&= \prod_{v \in |\Xi|} \frac{1}{1 - q^{-s \deg v}}.
\end{aligned}$$

The final expression is the Euler product of an L-function, so we get an equality

$$\langle \Theta_X, q^{-s \deg} \rangle = L(s, \mathbf{1}).$$

In the remaining talks, we will see a categorified version of this numerical equality.

Remark 3.1. We now make two important remarks about the above example.

(1) A better formulation of the equality is that

$$\langle \Theta_X, q^{-s\deg} \rangle = L(0, |\cdot|^s)$$

for any fixed $s \in \mathbb{C}$. Here, the point of evaluation on the right hand side is determined by the $\mathbb{G}_{\mathrm{gr}}$-action on the dual variety $\check{M} = T^*\mathbb{A}^1$, and it is independent of the automorphic representation.

We get an equality of L-functions since we are allowed to twist the automorphic form by central character. This is not possible in the Gross–Prasad case, and the period integral only sees one particular L-value (namely the central one) instead of the entire L-function.

(2) The Riemann–Roch theorem gives the identity

$$\Theta_X(x) = q^{-(g-1)}|x|^{-1}\Theta_X(\mathfrak{d}x^{-1}),$$

where $\mathfrak{d}$ is the canonical divisor. Replacing x by $\mathfrak{d}^{\frac{1}{2}}x$ removes the extra $\mathfrak{d}$ from the formula, and this is exactly the correction alluded to in Remark 1.2 if we used the scaling $\mathbb{G}_{\mathrm{gr}}$-action.

The Riemann–Roch theorem can be derived from the Poisson summation formula. The Fourier transform can be interpreted as switching between the two different ways of polarizing the Hamiltonian G-space $M = T^*X$, and the above identity states that the theta series attached to both polarizations agree. In Kaletha's talk, we saw that this additional symmetry (of M) leads to the functional equation.

Periods and L-sheaves

Rok Gregoric

If the Langlands conjecture is concisely expressed as the equivalence between two 4-dimensional topological quantum field theories $\mathcal{A}_G$ and $\mathcal{B}_{\check{G}}$, the relative Langlands conjecture concerns the induced duality between *boundary theories*

$$\mathcal{A}_{G,X} \in \mathcal{A}_G, \qquad \mathcal{B}_{\check{G},\check{X}} \in \mathcal{B}_{\check{G}},$$

for certain G-stacks X and $\check{G}$-stacks $\check{X}$. The *unramifield global setting* corresponds to evaluating these field theories on a "2-manifold" – a fixed connected smooth algebraic curve Σ over the base field $\mathbb{F}$ – which produces pairs of categories[1] and objects

$$\mathcal{A}_{G,X}(\Sigma) \in \mathcal{A}_G(\Sigma), \qquad \mathcal{B}_{\check{G},\check{X}}(\Sigma) \in \mathcal{B}_{\check{G}}(\Sigma).$$

The goal of this talk is to specify these objects more explicitly.

[1]We will always understand categories to mean $(\infty,1)$-categories, or more precisely k-linear $(\infty,1)$-categories, often referred to as DG categories in the Geometric Langlands literature.

1. The categories – non-linear sigma models

The field theories $\mathcal{A}_G$ and $\mathcal{B}_{\check{G}}$ are obtained roughly as non-linear sigma models. That is to say, they may be factored as the composite functors

$$\mathrm{Corr} \xrightarrow{\mathrm{Map}(-,T)} \mathrm{Stacks} \xrightarrow{\mathrm{lin.}} \mathrm{Cat}$$

of the mapping stack into a fixed "target stack" T, followed up by a linearization functor – usually some flavor of sheaf theory. Specifically we have[2]

$$\mathcal{A}_G(\Sigma) \simeq \mathrm{SHV}(\mathrm{Bun}_G), \qquad \mathcal{B}_{\check{G}}(\Sigma) \simeq \mathrm{QC}^!(\mathrm{Loc}_{\check{G}}),$$

where the mapping stack and linearization are given on the *automorphic side* by

(Lin) Bun_G, the classifying stack of algebraic G-bundles on Σ.

(Stk) SHV, a "topological" sheaf theory: D-modules, topological sheaves, or étale sheaves respectively, depending on whether we are considering the de Rham, Betti, or étale version of Langlands.

and on the *spectral side* by

(Lin) $\mathrm{Loc}_{\check{G}}$, the classifying stack of local systems of $\check{G}$-bundles on Σ, separately defined in each of the de Rham, Betti, or étale settings.

(Stk) $\mathrm{QC}^!$, the ind-coherent sheaves in the sense of Gaitsgory-Rosenblyum.

To see that these are indeed instances of the sigma-model construction, we can express the stacks in question as mapping stacks

$$\mathrm{Bun}_G \simeq \mathrm{Map}(\Sigma, \mathrm{B}G), \qquad \mathrm{Loc}_{\check{G}} \simeq \mathrm{Map}(\Sigma_{\mathrm{top}}, \mathrm{B}\check{G}), \tag{1.1}$$

where the "target stacks" are the classifying stacks $\mathrm{B}G$ and $\mathrm{B}\check{G}$, and the stack Σ_{top} is a "topological realization" of the algebraic curve Σ, taken to mean the de Rham space Σ_{dR} in the de Rham setting and the Betti space Σ_{B} in the Betti setting. In the étale setting, we *do not* define $\mathrm{Loc}_{\check{G}}$ as a mapping stack, but rather take it to be the moduli stack of tempered $\check{G}$-local systems on Σ of [AGKRRV].

2. The boundary theory objects – period and L-sheaves

For a Hamiltonian G-space M with a mirror $\check{G}$-space $\check{M}$, the relative Langlands conjecture posits the existence of corresponding boundary theories to $\mathcal{A}_G$ and $\mathcal{B}_{\check{G}}$ which are exchanged under Langlands duality. We restrict[3] to the untwisted polarized case, which is to say we assume that $M = T^*X$ and $\check{M} = T^*\check{X}$ for an affine G-scheme X and affine $\check{G}$-scheme $\check{X}$. In this case, the output of boundary theories $\mathcal{A}_{G,X} \in \mathcal{A}_G$ and $\mathcal{B}_{\check{G},\check{X}} \in \mathcal{B}_{\check{G}}$ consist of the respective objects

$$\mathcal{P}_X \in \mathrm{SHV}(\mathrm{Bun}_G), \qquad \mathcal{L}_{\check{X}} \in \mathrm{QC}^!(\mathrm{Loc}_{\check{G}}),$$

[2]Outisde the de Rham setting, the automorphic side needs to be further corrected to certain kind of "automorphic sheaves" $\mathrm{AUT}(\mathrm{Bun}_G) \subseteq \mathrm{SHV}(\mathrm{Bun}_G)$, which admit a spectral decomposition with respect to the Hecke action. But we ignore such finer points here. This, and the spectral Beilinson projector, will be discussed in future lectures.

[3]This is to simplify the discussion, but is rather restrictive, since a Hamiltonian space and its mirror are seldom both polarized. See the paper for discussion of how to extend the constructions discussed here beyond the untwisted polarized case.

which we wish to specify. Recall that the data of the G-variety X and $\check{G}$-variety $\check{X}$ is equivalent to quotient stack maps$X/G \to \mathrm{B}G$ and $\check{X}/\check{G} \to \mathrm{B}\check{G}$.Using these, we look to (1.1) define auxiliary stacks

$$\mathrm{Bun}_G^X := \mathrm{Map}(\Sigma, X/G), \qquad \mathrm{Loc}_{\check{G}}^{\check{X}} := \mathrm{Map}(\Sigma_{\mathrm{top}}, \check{X}/\check{G}),$$

along with canonical maps$\mathrm{Bun}_G^X \xrightarrow{\pi} \mathrm{Bun}_G$ and $\mathrm{Loc}_{\check{G}}^{\check{X}} \xrightarrow{\check{\pi}} \mathrm{Loc}_{\check{G}}$.These give rise to pushforward functors in each respective "linearizing" sheaf theory, and we define the *period sheaf of* X and *L-sheaf of* $\check{X}$ as

$$\mathcal{P}_X := \pi_! k_{\mathrm{Bun}_G^X}, \qquad \mathcal{L}_{\check{X}} := \check{\pi}_* \omega_{\mathrm{Loc}_{\check{G}}^{\check{X}}}$$

where the constant sheaf $k_{\mathrm{Bun}_G^X}$ and the dualizing sheaf $\omega_{\mathrm{Loc}_{\check{G}}^{\check{X}}}$ are the respective monoidal units of $\mathrm{SHV}(\mathrm{Bun}_G^X)$ and $\mathrm{QC}^!(\mathrm{Loc}_{\check{G}}^{\check{X}})$ for $\otimes_k$ and $\otimes^!$ respectively.

As a first example, consider a homogeneous G-space $X = G/H$. We have

$$\mathrm{Bun}_G^{G/H} \simeq \mathrm{Map}\big(\Sigma, \frac{G/H}{G}\big) \simeq \mathrm{Map}(\Sigma, \mathrm{pt}/H) \simeq \mathrm{Bun}_H,$$

and likewise on the spectral side for a homogeneous $\check{G}$-space $\check{X} = \check{G}/\check{H}$

$$\mathrm{Loc}_{\check{G}}^{\check{G}/\check{H}} \simeq \mathrm{Loc}_{\check{H}}$$

In the étale setting, where we have so far refrained from describing the stack $\mathrm{Loc}_{\check{G}}^{\check{X}}$, we take this as the definition. That is to say, the stack $\mathrm{Loc}_{\check{G}}^{\check{X}}$ is, in the étale setting, only considered[4] in the homogeneous case $\check{X} = \check{G}/\check{H}$.

Remark 2.1. In the above discussion, we are suppressing a number of finer points which are required to make everything correct. These omissions, which we continue to suppress after this remark, are of three sorts:

- On the automorphic side, the definition of the stack Bun_G^X needs to be modified slightly to take into account a choice of spin structure on the curve Σ. This uses an auxiliary $\mathbf{G}_m$-action on X that had implicitly been part of the data.
- On the spectral side, the definition of the L-sheaf $\mathcal{L}_{\check{X}}$ should be sheared, with respect to a grading induced from an auxiliary $\mathbf{G}_m$-action on $\check{X}$.

[4]Though the paper indeed only defines the auxiliary stack $\mathrm{Loc}_{\check{G}}^{\check{X}}$ in the homogeneous case, the author of this note would like to suggest a Tannakian definition that may at least be stated without such a restriction. We suggest that the functor of points $\mathrm{Loc}_{\check{G}}^{\check{X}} : \mathrm{CAlg}_k \to \mathrm{Grpd}_\infty$ should send a k-algebra R to the ∞-groupoid of right t-exact symmetric monoidal small colimit preserving k-linear functors $\mathrm{QC}(\check{X}/\check{G}) \to \mathrm{QLisse}(\Sigma) \otimes_{\mathrm{Vect}_k} \mathrm{Mod}_R$, where $\mathrm{QLisse}(X)$ denotes the quasi-lisse sheaves on Σ in the sense of [AGKRRV]. For $\check{X} = \check{G}/\check{H}$, this recovers the definition of tempered $\check{H}$-local systems on Σ from [AGKRRV], and thus agrees with $\mathrm{Loc}_{\check{G}}^{\check{G}/\check{H}} \simeq \mathrm{Loc}_{\check{H}}$. In fact, this Tannakian approach could also work to define the stack of twisted Langlands parameters $\mathrm{Loc}_{\check{G}}^{\check{X}}$ in the de Rham and Betti contexts as well, by replacing $\mathrm{QLisse}(X)$ with the respective categories of D-modules and Betti sheaves on Σ respectively.

- Even then, the sheaves $\mathcal{P}_X$ and $\mathcal{L}_{\check{X}}$ obtained in this way are the unnormalized versions. Before the relative Langlands conjecture can possibly be suggested to interchange them, additional renormalizations are required, producing variants called $\mathcal{P}_X^{\mathrm{nor}}$ and $\mathcal{L}_{\check{X}}^{\mathrm{nor}}$.

All this is carefully discussed in [BZSV24]; see there for details.

3. Unpacking the stacks Bun_G^X and $\mathrm{Loc}_{\check{G}}^{\check{X}}$

To better understand the stacks Bun_G^X and $\mathrm{Loc}_{\check{G}}^{\check{X}}$, let us describe the fibers of the structure maps π and $\check{\pi}$ to Bun_G and $\mathrm{Loc}_{\check{G}}$ respectively. On the automorphic side, an R-point of Bun_G corresponds to a G-bundle E over the base-change $\Sigma_R = \Sigma \times_{\mathrm{Spec}(\mathbb{F})} \mathrm{Spec}(R)$. The fiber of the map π then fits into the pullback square

$$\begin{tikzcd} \mathrm{Bun}_G^X \arrow[d, "\pi"'] & \Gamma(\Sigma_R; E \times^G X) \arrow[l] \arrow[d] \\ \mathrm{Bun}_G & \mathrm{Spec}(R), \arrow[l, "\eta_E"] \end{tikzcd}$$

exhibiting its fiber over this R-point of Bun_G with the sections of the associated X-bundle of the G-bundle $E \to \Sigma_R$. If we similarly identify on the spectral side an R-point of $\mathrm{Loc}_{\check{G}}$ with a $\check{G}$-local system ρ over Σ_R, then we find a Cartesian square as before

$$\begin{tikzcd} \mathrm{Loc}_{\check{G}}^{\check{X}} \arrow[d, "\check{\pi}"'] & \Gamma_\nabla(\Sigma_R; \rho \times^{\check{G}} \check{X}) \arrow[l] \arrow[d] \\ \mathrm{Loc}_{\check{G}} & \mathrm{Spec}(R), \arrow[l, "\eta_\rho"] \end{tikzcd}$$

where Γ_∇ denotes the "flat sections" of the associated $\check{X}$-bundle of the $\check{G}$-local system ρ.

- *In the de Rham case*, where a local system is given by an algebraic bundle $\check{E} \to \Sigma_R$ together with a flat connection ∇, this can be understood as the "horizontal sections" of the associated bundle $\check{E} \times^{\check{G}} \check{X}$, i.e. $\check{X}$-valued solutions s of the "parallel transport equation" $\nabla s = 0$ over Σ_R.
- *In the Betti case*, we may conversely identify a $\check{G}$-local system on Σ_R with its monodromy representation, that is to say, with (the conjugacy class of) a group homomorphism $\rho : \pi_1(\Sigma) \to \check{G}_R$. Through the $\check{G}$-action on $\check{X}$, this defines a $\pi_1(\Sigma)$-action on the base-change $\check{X}_R$. The flat sections are identified with the derived fixed-points of this action
$$\Gamma_\nabla(\Sigma_R; \rho \times^{\check{G}} \check{X}) \simeq (\check{X}_R)^{\pi_1(\Sigma)}.$$

The latter fixed-point perspective can be used to obtain an explicit quotient stack expression for $\mathrm{Loc}_{\check{G}}^{\check{X}}$ in the Betti case.Note that the affine scheme

$$\mathrm{Hom}_{\mathrm{Grp}}(\pi_1(\Sigma), \check{G}) = \mathrm{Spec}(R_{\mathrm{uni}})$$

supports the universal representation $\rho : \pi_1(\Sigma) \to \check{G}_{R_{\mathrm{uni}}}$, in terms of which the classifying stack of G-local systems on Σ is expressed as

$$\mathrm{Loc}_{\check{G}} \simeq \mathrm{Spec}(R_{\mathrm{uni}})/\check{G}.$$

When combined with pullback square above, this exhibits the stack of $\check{X}$-twisted Langlands parameters as a global quotient stack

$$\mathrm{Loc}_{\check{G}}^{\check{X}} \simeq (\check{X}_{R_{\mathrm{uni}}})^{\pi_1(\Sigma)}/\check{G}.$$

If the genus g of Σ is ≥ 2 and the reductive group G is semisimple, then the presentation for $\pi_1(\Sigma)$ as the quotient of the free group on $2g$ generators along a single commutator relation allows us to identify the base-change with the fiber

$$\check{X}_{R_{\mathrm{uni}}} = \mathrm{fib}(\check{X}^{2g} \to \check{X}),$$

showing that $\mathrm{Loc}_{\check{G}}^{\check{X}}$ is quite an explicit and well-behaved algebraic stack.

4. Period sheaves and period functions

Let us now specialize to the finite case, i.e. étale Langlands over $\mathbb{F} = \overline{\mathbf{F}}_q$ and with coefficients in $k = \overline{\mathbf{Q}}_\ell$. We assume that the algebraic curve in question is base-changed as $\Sigma_{\overline{\mathbf{F}}_q}$ from a curve Σ over the finite field $\mathbf{F}_q$. Let F denote the function field of Σ. Under the Beauville-Laszlo-type adelic description of the groupoid of G-bundles on Σ

$$\mathrm{Bun}_G(\mathbf{F}_q) \simeq G(F)\backslash G(\mathbb{A})/G(\mathbb{O}),$$

the sheaf-function-correspondence allows us to extract an automorphic function $[\mathscr{F}] \in \mathcal{C}_c(G(F)\backslash G(\mathbb{A})/G(\mathbb{O}), k)$ from a Weil sheaf of k-vector spaces $\mathscr{F}$ on Bun_G. Its value at a point $[g] \in G(F)\backslash G(\mathbf{A})/G(\mathbf{O})$, corresponding to a G-bundle $E \to \Sigma$, is given by the trace of the Frobenius endomorphism on the geometric fiber

$$[\mathscr{F}]([g]) = \mathrm{tr}(\eta_E^*\mathscr{F} \xrightarrow{\mathrm{Frob}} \eta_E^*\mathscr{F}),$$

where $\eta : \mathrm{Spec}(\mathbb{F}) \to \mathrm{Bun}_G$ classifies the base-changed G-bundle $E_{\overline{\mathbf{F}}_q} \to \Sigma_{\overline{\mathbf{F}}_q}$.By a base-change formula for étale sheaves and the identification of the fiber of π over η_E with the sections of the associated X-bundle $E \times^G X$ over $\Sigma_{\overline{\mathbf{F}}_q}$, we find the relevant geometric fiber of the period sheaf $\mathcal{P}_X$ to be

$$\eta_E^*\mathcal{P}_X \simeq \eta_E^*\pi_!(k_{\mathrm{Bun}_G^X}) \simeq \Gamma_c(\Gamma(\Sigma_{\overline{\mathbf{F}}_q}; E_{\overline{\mathbf{F}}_q} \times^G X); k).$$

Its Weil sheaf structure amounts to the Frobenius action on this space of sections, and so the value of the corresponding automorphic function is the cardinality

$$[\mathcal{P}_X]([g]) = \#\Gamma(\Sigma; E \times^G X). \tag{4.1}$$

By unpacking how the coset $[g] \in G(F)\backslash G(\mathbb{A})/G(\mathbb{O})$ encodes the G-bundle E over Σ, we can identify the global sections of the associated X-bundle of E with the intersection

$$\Gamma(\Sigma; E \times^G X) \simeq X(F) \cap g^{-1}X(\mathbb{O})$$

inside the adelic values $X(\mathbb{A})$. In particular, we have

$$[\mathcal{P}_X]([g]) = \#(X(F) \cap g^{-1}X(\mathbb{O})) = \sum_{x \in X(F)} \mathbf{1}_{X(\mathbb{O})}(xg) = \Theta_X([g]),$$

recovering the definition of the theta-function Θ_X associated to X. That is to say, the image of the period sheaf $\mathcal{P}_X$ under the sheaf-function-correspondence is the Theta function Θ_X. In this sense, $\mathcal{P}_X$ categorifies Θ_X.

Finally, let us specialize this to the Iwasawa-Tate case. That is to say, take $G = \mathbf{G}_m$ and $X = \mathbf{A}^1$ with its scaling action. In that case there is a canonical identification

$$\mathrm{Bun}_{\mathbf{G}_m}(\mathbf{F}_q) \simeq \mathrm{Pic}(\Sigma)$$

with the Picard group of line bundles on Σ, which sends a $\mathbf{G}_m$-bundle $E \to \Sigma$ to the associated $\mathbf{A}^1$-bundle $L := E \times^{\mathbf{G}_m} \mathbf{A}^1$ over Σ. The formula (4.1) therefore expresses the value of the function, corresponding to the period sheaf, at the coset $[g] \in F^\times \backslash \mathbb{A}^\times / \mathbb{O}^\times$ corresponding to the line bundle $L \in \mathrm{Pic}(\Sigma)$, with

$$[\mathcal{P}_{\mathbf{A}^1}]([g]) = \#\Gamma(\Sigma; L) = q^{h^0(\Sigma;L)}.$$

Just as we have seen that period sheaves are related to theta functions, later talks will discuss how L-sheaves are related to L-functions.

References

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Global Geometric Duality

Sam Gunningham

The goal of this talk was to understand the following conjecture.

Conjecture 1 (Global Geometric Duality, [BZSV24] Conjecture 12.1.1)**.** *The spectral projection of the period sheaf and the L-sheaf are exchanged under the geometric Langlands correspondence:*

$$\mathrm{AUT}(\mathrm{Bun}_G) \xleftrightarrow{\sim} QC^!(\mathrm{Loc}_{G^\vee})$$
$$(\mathcal{P}_{X,\Psi})^{\mathrm{spec}} \longleftrightarrow \mathcal{L}_{X^\vee,\Psi^\vee}$$

1. Generalities

1.1. **The setup.** For concreteness we will work exclusively in the *Betti* setting of geometric Langlands in this note (the de Rham and étale settings are largely analogous). Let G and $G^\vee$ be Langlands dual complex reductive groups, and $M = T^*(X, \Psi)$ and $M^\vee = T^*(X^\vee, \Psi^\vee)$ be dual polarized hyperspherical varieties for G and $G^\vee$. Let Σ be a compact Riemann surface of genus g equipped with a choice of spin structure $K^{\frac{1}{2}}$.

Throughout this note, we work in the setting of derived categories (that is, differential-graded, or $\mathbb{C}$-linear stable ∞-cateogories), suppressing the derived nature for the sake of brevity. For example, a "vector space" really means a cochain complex, a "sheaf" really a complex of sheaves, etc.

1.2. **The automorphic category.** Let Bun_G denote the analytic stack of holomorphic G-bundles on Σ. The automorphic category $\mathrm{AUT}(\mathrm{Bun}_G)$ consists of sheaves of $\mathbb{C}$-vector spaces (in the complex analytic topology) whose microsupport is contained inside the global nilpotent cone $\mathcal{N} \subseteq T^* \mathrm{Bun}_G$. Informally, the *microsupport* (also known as *singular support*) of a sheaf is a subset of the cotangent bundle which measures the directions in which the sheaf is locally constant (slightly more precisely: the codirections in which the sheaf is not locally constant). For example, a sheaf has microsupport contained in the zero-section if and only if it is locally constant.

The *Betti spectral projection functor*

$$(-)^{\mathrm{spec}} : \mathrm{SHV}(\mathrm{Bun}_G) \to \mathrm{AUT}(\mathrm{Bun}_G)$$

is the left adjoint to the inclusion of automorphic sheaves into the category of all Betti sheaves.

Example 1 ($G = \mathbb{G}_m$ - running example)**.** *The stack* $\mathrm{Bun}_{\mathbb{G}_m}$ *decomposes as a product:*

$$\mathrm{Bun}_{\mathbb{G}_m} \cong \mathbb{Z} \times \mathrm{Pic}^0 \times B\mathbb{G}_m.$$

Moreover, as the global nilpotent cone is just the zero section, the automorphic category consists of locally constant sheaves. Thus the automorphic category is a tensor product

$$\mathrm{AUT}(\mathrm{Bun}_{\mathbb{G}_m}) \cong \mathrm{SHV}(\mathbb{Z}) \otimes \mathrm{SHV}_0(\mathrm{Pic}_0) \otimes \mathrm{SHV}(B\mathbb{G}_m).$$

Each tensor factor can be computed explicitly:

- $\mathrm{SHV}(\mathbb{Z}) = \mathrm{Vect}_{\mathbb{Z}}$*, the category of* $\mathbb{Z}$*-graded vector spaces;*
- $\mathrm{SHV}_0(\mathrm{Pic}^0) \cong \mathbb{C}[\pi_1(\mathrm{Pic}^0)] - \mathrm{mod}$ *(note that* $\pi_1(\mathrm{Pic}^0) \cong H_1(\Sigma; \mathbb{Z})$*);*
- $\mathrm{SHV}(B\mathbb{G}_m) \cong H^*(B\mathbb{G}_m; \mathbb{C}) - \mathrm{mod} \cong \mathbb{C}[\beta] - \mathrm{mod}$, *where* β *is in degree 2;*

1.3. **The spectral category.** Let $\mathrm{Loc}_{G^\vee}$ denote the derived moduli stack of Betti $G^\vee$-local systems on Σ. On the spectral side of the geometric Langlands correspondence, we have the category $\mathrm{QC}^!(\mathrm{Loc}_{G^\vee})$ of category of ind-coherent sheaves.

Example 2 ($G = \mathbb{G}_m$ - running example)**.** *We have an isomorphism*

$$\mathrm{Loc}_{\mathbb{G}_m} = B\mathbb{G}_m \times H^1(\Sigma; \mathbb{G}_m) \times \mathbb{A}^1[-1]$$

Thus, as on automorphic side, the spectral category splits as a tensor product (note that the first two factors are smooth, so $\mathrm{QC}^! = \mathrm{QC}$*):*

$$\mathrm{QC}^!(\mathrm{Loc}_{\mathbb{G}_m}) \cong \mathrm{QC}(B\mathbb{G}_m) \otimes \mathrm{QC}(H^1(\Sigma;\mathbb{G}_m)) \otimes \mathrm{QC}^!(\mathbb{A}^1[-1]). \tag{1.1}$$

As before each factor can be computed explicitly:

- $\mathrm{QC}(B\mathbb{G}_m) \cong \mathrm{Rep}(\mathbb{G}_m) \cong \mathrm{Vect}_{\mathbb{Z}} \cong \mathrm{SHV}(\mathbb{Z})$*;*
- $\mathrm{QC}(H^1(\Sigma;\mathbb{G}_m)) \cong \mathbb{C}[H_1(\Sigma;\mathbb{Z})]-\mathrm{mod} \cong \mathrm{SHV}_0(\mathrm{Pic}^0)$*;*
- $\mathrm{QC}^!(\mathbb{A}^1[-1]) \cong \mathbb{C}[\beta]-\mathrm{mod} \cong \mathrm{SHV}(B\mathbb{G}_m)$*.*

These observations yield the geometric Langlands correspondence in this case (Betti geometric class field theory): $\mathrm{AUT}(\mathrm{Bun}_{\mathbb{G}_m}) \cong \mathrm{QC}^!(\mathrm{Loc}_{\mathbb{G}_m})$*.*

1.4. **Periods and L-sheaves.** Recall that we have stacks $\pi : \mathrm{Bun}_G^X \to \mathrm{Bun}_G$ and $\pi^\vee : \mathrm{Loc}_{G^\vee}^{X^\vee} \to \mathrm{Loc}_{G^\vee}$. A point of Bun_G^X (respectively $\mathrm{Loc}_{G^\vee}^{X^\vee}$) is given by a G-bundle (respectively $G^\vee$-local system) together with a section of the associated X-bundle (respectively, a flat section of the associated $X^\vee$-bundle). Then we define:

$$\mathcal{P}_{X,\Psi} := \pi_!(\underline{\mathbb{C}}^{\Psi}_{\mathrm{Bun}_G^X}) \qquad \mathcal{L}_{X^\vee,\Psi^\vee} := \pi^\vee_*(\omega^{\Psi^\vee}_{\mathrm{Loc}_{G^\vee}^{X^\vee}})\,/\!\!/.$$

Here, $\underline{\mathbb{C}}^{\Psi}_{\mathrm{Bun}_G^X}$ is just the constant sheaf in the case when Ψ is non-trivial, and in general is twisted by a suitable version of the Artin–Schreier local system. Similarly, $\omega^{\Psi^\vee}_{\mathrm{Loc}_{G^\vee}^{X^\vee}}$ is the (Grothendieck-Serre) dualizing complex when when $\Psi^\vee$ is trivial, and otherwise must be suitably twisted. The symbol $/\!\!/$ denotes *shearing* with respect to a certain grading (see [BZSV24], 6.1).

1.5. **Periods of automorphic sheaves and L-values.** The purpose of this section is to explain the geometric analogue of the idea that period integrals correspond to values of L-functions.

By adjunction, for any automorphic sheaf $\mathcal{F} \in \mathrm{AUT}(\mathrm{Bun}_G)$, we have

$$\mathrm{Hom}_{\mathrm{AUT}(\mathrm{Bun}_G)}(\mathcal{P}^{\mathrm{spec}}_{X,\Psi}, \mathcal{F}) \cong \mathrm{Hom}_{\mathrm{SHV}(\mathrm{Bun}_G)}(\mathcal{P}_{X,\Psi}, \mathcal{F}) \cong H^*(\mathrm{Bun}_G^X; \pi^!(\mathcal{F})).$$

This can be thought of as the geometric avatar of a period integral.

Now suppose that $\mathcal{F} = \mathcal{F}_\rho$ is a Hecke eigensheaf associated to a $G^\vee$ local system $\rho \in \mathrm{Loc}_{G^\vee}$. By definition, $\mathcal{F}_\rho$ corresponds to the object $\delta_\rho := i_{\rho*}(\mathcal{O}_{\mathrm{pt}})$ in $\mathrm{QC}^!(\mathrm{Loc}_{G^\vee})$ under the global geometric Langlands correspondence. In nice cases (say, ρ is a smooth point of $\mathrm{Loc}_{G^\vee}$ and there is an isolated fixed point $\{x\} = X^{\vee\rho}$), we can understand this hom space explicitly as a certain (suitably twisted and sheared) symmetric algebra on $H^1(\Sigma; (T_x X^\vee)_\rho)$. This can be understood as the geometric avatar of the corresponding L-value. The nice conditions mentioned above correspond to cases where the L-function has no poles.

2. Special cases of the conjecture

2.1. **The Whittaker Case.** Let $X = G/U$ and $\Psi \to X$ the $\mathbb{G}_a$-bundle associated to a generic additive character $\psi : U \to \mathbb{G}_a$. The associated period sheaf $\mathsf{W} := \mathcal{P}_{G/U,\Psi}$ is called the *Whittaker sheaf.*

In this case, the dual hyperspherical variety is simply $M^\vee = \mathrm{pt} = T^*(\mathrm{pt}, 0)$. It follows that the L-sheaf is given by the (Grothendieck-Serre) dualizing complex $\omega \in \mathrm{QC}^!(\mathrm{Loc}_{G^\vee})$. In this case, as $\mathrm{Loc}_{G^\vee}$ is a quasi-smooth symplectic stack of dimension 0, there is an isomorphism $\omega_{\mathrm{Loc}_{G^\vee}} \cong \Xi(\mathcal{O}_{\mathrm{Loc}_{G^\vee}})$ (here, Ξ is the natural fully faithful functor from quasi-coherent to ind-coherent sheaves). In other words, the functor $\mathrm{Hom}_{\mathrm{AUT}(\mathrm{Bun}_G)}(\mathsf{W}, -)$ corresponds to the global sections functor on the spectral side. In particular, if $\mathcal{F}_\rho$ is a Hecke-eigensheaf corresponding to $\rho \in \mathrm{Loc}_{G^\vee}$, we have $\mathrm{Hom}(\mathsf{W}, \mathcal{F}_\rho) \cong R\Gamma(i_*\mathcal{O}_{pt}) \cong \mathbb{C}$.

Example 3 ($G = \mathbb{G}_m$ (running example))**.** . *Here $G/U = G$ and Ψ is trivial. Thus, the Whittaker sheaf is given by $i_!(\underline{\mathbb{C}}_{\mathrm{pt}})$ where $i : \mathrm{pt} \to \mathrm{Bun}_{\mathbb{G}_m}$ corresponds to the trivial bundle. In this case, we can compute the spectral projection explicitly. Indeed, the map $\mathrm{pt} \to \mathrm{Pic}^0$ is homotopy equivalent to the universal cover $\widetilde{\mathrm{Pic}^0} \cong H^1(\Sigma; \mathcal{O}_\Sigma) \to \mathrm{Pic}_0$, and the compactly supported pushforward of the constant sheaf under this map is identified with the regular local system, that is $\mathbb{C}[\pi_1(\mathrm{Pic}^0)]$ as a module over itself. As expected, this indeed corresponds to the structure sheaf under the identification $\mathrm{SHV}_0(\mathrm{Pic}^0) \cong \mathrm{QC}(H^1(\Sigma; \mathbb{G}_m))$.*

2.2. **The spectral Whittaker/Atiyah–Bott case.** Reversing the roles from the previous case, we consider $M = pt$ and $M^\vee = T^*(G^\vee/U^\vee, \Psi^\vee)$. As $\mathrm{Bun}_G^X = \mathrm{Bun}_G$, we see that the period sheaf is identified with the constant sheaf $\underline{\mathbb{C}}_{\mathrm{Bun}_G} \in \mathrm{AUT}(\mathrm{Bun}_G)$. The L-sheaf in this case is referred to as the *spectral Whittaker sheaf*, $s\mathsf{W} \in \mathrm{QC}^!(\mathrm{Loc}_{G^\vee})$. The spectral Whittaker sheaf knows about the cohomology of Bun_G (famously computed by Atiyah and Bott [AB83]):

$$H^*(\mathrm{Bun}_G) \cong \mathrm{End}_{\mathrm{SHV}(\mathrm{Bun}_G)}(\underline{\mathbb{C}}_{\mathrm{Bun}_G}) \cong \mathrm{End}_{\mathrm{QC}^!}(s\mathsf{W}).$$

Example 4 ($G = \mathbb{G}_m$ - running example)**.** *In this case, the spectral Whittaker sheaf is given by the skyscraper sheaf $i_*\mathcal{O}_{\mathrm{pt}}$, where $i : \mathrm{pt} \to \mathrm{Loc}_{\mathbb{G}_m}$ corresponds to the trivial local system. As usual, this object splits as a pure tensor with respect to the decomposition (1.1), and the middle tensor factor is given by the skyscraper sheaf at the identity element $1 \in H^1(\Sigma; \mathbb{G}_m)$. As expected, this object indeed corresponds to the trivial local system under the identification $\mathrm{QC}(H^1(\Sigma, \mathbb{G}_m)) \cong \mathrm{SHV}_0(\mathrm{Pic}^0)$.*

2.3. **Tate's thesis.** Let $G = G^\vee = \mathbb{G}_m$ and $X = X^\vee = \mathbb{A}^1$ (and $\Psi, \Psi^\vee$ trivial). The fiber of $\pi : \mathrm{Bun}_{\mathbb{G}_m}^{\mathbb{A}^1} \to \mathrm{Bun}_{\mathbb{G}_m}$ over a line bundle $L \in \mathrm{Bun}_{\mathbb{G}_m}$ is the space of sections $H^0(L \otimes K^{\frac{1}{2}})$. By Riemann-Roch, $\dim H^0(L \otimes K^{\frac{1}{2}}) = 0$ if $\deg(L) < -g+1$ and $\dim H^0(L \otimes K^{\frac{1}{2}}) = d$ if $d > g-1$. It follows that the period sheaf is just a shift of the constant sheaf on the connected components $\mathrm{Bun}_{\mathbb{G}_m}^d$ for $d \notin [-(g-1), g-1]$. On the other hand, if $d \in [-(g-1), g-1]$, $\dim H^0(L \otimes K^{\frac{1}{2}})$ varies locally in L, and thus the the period sheaf to the those components is not locally constant. To compute the spectral projection explicitly appears to be much more involved.

On the spectral side, the fiber of $\pi^\vee$ over a local system $\rho \in \mathrm{Loc}_{\mathbb{G}_m}$ is given by the cohomology $H^*(\Sigma; \mathbb{C}_\rho)$. If ρ is non-trivial, then $H^0(\Sigma; \mathbb{C}_\rho) = H^2(\Sigma; \mathbb{C}_\rho) = 0$, and $H^1(\Sigma; \mathbb{C}_\rho) = \mathbb{C}^{2g-2}$ (note that every local system is trivial if $g = 0$). It follows that the restriction of the L-sheaf to the locus of non-trivial local systems is straightforward to compute (this corresponds to the "nice" cases mentioned in 1.5, where the L-function has no poles). In general, it is a more involved problem to understand the behavior of the L-sheaf in a neighborhood of the trivial local system. These questions have been explored more thoroughly in [FW24].

2.4. **Functoriality via relative Langlands.** Finally, we mention one further class of examples. Suppose that $G = G_1 \times G_2$ is a product of reductive groups (and thus $G^\vee \cong G_1^\vee \times G_2^\vee$). Via the theory of integral transforms for sheaves, objects of the automorphic category for $G = G_1 \times G_2$ may be regarded as functors $\mathrm{AUT}(\mathrm{Bun}_{G_1}) \to \mathrm{AUT}(\mathrm{Bun}_{G_2})$, and there is an analogous interpretation the spectral side. In this way, period and L-sheaves give rise to instances of Langlands functoriality. Some particular examples of this phenomenon:

- The group case: $G_1 = G_2 = X$. Conjecture 1 in this case implies the miraculous duality of Drinfeld–Gaitsgory [DG11].
- The Eisenstein case: $G_1 = M$, $G_2 = L$, and $X = M/U_P$, where P is a parabolic subgroup of M with unipotent radical U_P, and $L = P/U_P$. This does not quite fit into the theory of hyperspherical duality as M/U_P is not affine; nevertheless, the expectation is that the dual boundary condition should be given by the analogous data for the dual parabolic $P^\vee$ of $M^\vee$. Then Conjecture 1 should correspond to the identification of Eisenstein and constant term functors under Langlands duality.
- Gan–Gross–Prasad-type examples. Let $G_1 = SO_{2n}$, $G_2 = SO_{2n+1}$, and $X = G/SO_{2n}$. The dual data here is: $G_1^\vee = SO_{2n}$, $G_2^\vee = Sp_{2n}$ and $X^\vee = \mathrm{std} \otimes \mathrm{std}$, the tensor product of the standard representations of the two factors. On the one side, we the corresponding integral transform is associated to the inclusion $SO_{2n} \hookrightarrow SO_{2n+1}$, whereas there is no such homomorphism between SO_{2n} and Sp_{2n} on the dual side.

References

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Global Conjectures

Chen-wei (Milton) Lin

Fix a smooth projective curve $\Sigma_{/\mathbb{F}_q}$ of genus g, with function field $F = \mathbb{F}_q(\Sigma)$. Let $k = \bar{\mathbb{Q}}_\ell$ and write $\mathrm{SHV}(-)$ for ind-constructible l-adic sheaves. Throughout, we work with distinguished split forms of hyperspherical dual pairs [2, Ch. 5.3]

$$(M \circlearrowleft G \times \mathbb{G}_{\mathrm{gr}})_{/\mathbb{F}_q} \text{ and } \big(\check{G} \times \mathbb{G}_{\mathrm{gr}} \circlearrowright \check{M}\big)_{/k}$$

We will also denote $b_G := (g-1)\dim G$, a constant which appears throughout.

1. Overview

We have discussed how hyperspherical dual pairs induce boundary theories (at dimension 2) by defining period and L-sheaves. Now we discuss numerical conjectures coming out of the framework, which bears the shape:

$$\text{“}M\text{-period of } \mathcal{F}_\phi/f_\phi = L\text{-function } \delta_\phi/\phi \text{ on } \check{M}\text{”}$$

where $\mathcal{F}_\phi$ and δ_ϕ are corresponding test objects under the geometric Langlands correspondence. In section 2, we spell this out by supposing matching boundary theories and taking trace. This yields the Geometric Numerical Conjecture, Conjecture 1. We picture this process as:

$$\begin{array}{lll} \text{dim} & A\text{-side} & B\text{-side} \\ & & \\ 2 : \Sigma_{\bar{\mathbb{F}}_q} & \mathcal{P}_M \in \mathrm{SHV}(\mathrm{Bun}_{G,\bar{\mathbb{F}}_q}) & \mathcal{L}_{\check{M}} \in \mathrm{Ind\,Coh}_{\mathcal{N}}(\mathrm{Loc}^{\mathrm{res}}_{\check{G},\bar{\mathbb{F}}_q}) \\ \quad\big\downarrow{\scriptstyle \mathrm{tr}} & & \\ 3 : \Sigma_{\mathbb{F}_q} & P_M \in \mathrm{Fct}(\mathrm{Bun}_G(\mathbb{F}_q)) & L_{\check{M}} \in \Gamma(\mathrm{Loc}^{\mathrm{arith}}_{\check{G}}, \omega^{\mathrm{mer}}) \end{array}$$

See [1] for further explanation of objects on the right-hand side, and the meaning of the subscript $\mathcal{N}$. In section 3 we state a parallel[1] Tempered Numerical Conjecture, Conjecture 2, and show how this unifies three classical examples.

2. Geometric Numerical Conjecture

Our test objects are cuspidal Hecke eigensheaves: sheaves $\mathcal{F}_\phi \in \mathrm{SHV}_{\mathcal{N}}(\mathrm{Bun}_G)$ that satisfy Hecke equivariance properties with respect to a parameter ϕ; see [4, Ch. 3], [5], and whose geometric constant term all vanishes. In particular, the corresponding trace $f_\phi := \mathrm{tr}\,\mathcal{F}_\phi$ is compactly supported, [3, Ch. 4]. Under the geometric Langlands correspondence $\mathcal{F}_\phi$ corresponds to a δ sheaf, $\delta_\phi \in \mathrm{Ind\,Coh}_{\mathcal{N}}(\mathrm{Loc}^{\mathrm{res}}_{\check{G},\bar{\mathbb{F}}_q})$. Our geometric objects thus live in the following diagram:

$$\begin{array}{lcl} \mathcal{P}_X \in \mathrm{SHV}(\mathrm{Bun}_{G,\bar{\mathbb{F}}_q}) & & \\ \quad\quad \big\uparrow \big\downarrow {\scriptstyle (-)^{\mathrm{spec}}} & & \\ \mathcal{F}_\phi \in \mathrm{SHV}_{\mathcal{N}}(\mathrm{Bun}_{G,\bar{\mathbb{F}}_q}) & \simeq^{\mathbb{L}} & \mathrm{Ind\,Coh}_{\mathcal{N}}(\mathrm{Loc}^{\mathrm{res}}_{\check{G},\bar{\mathbb{F}}_q}) \ni \delta_\phi \end{array}$$

[1] At the moment, this is not a consequence of the geometric version.

where a) $\mathbb{L}$ is the conjectured geometric Langlands equivalence in [1, Conj. 21.2.7], and b) $(-)^{\mathrm{spec}}$ is *Beilinson's spectral projector*, conjecturally right adjoint of inclusion of nilpotent sheaves. Granted both conjectures, and matching of spectrally projected period sheaf $(\mathcal{P}_X^{\mathrm{norm}})^{\mathrm{spec}}$ with L-sheaf, $\mathcal{L}_{\check{X}}$, [2, Conj 12.1.1] we have

$$\begin{aligned}\underline{\mathrm{Hom}}_{\mathrm{SHV}(\mathrm{Bun}_G)}(\mathcal{F}_\phi, \mathcal{P}_X^{\mathrm{norm}}) &\simeq \underline{\mathrm{Hom}}_{\mathrm{SHV}_{\mathcal{N}}(\mathrm{Bun}_{G,\bar{\mathbb{F}}_q})}(\mathcal{F}_\phi, (\mathcal{P}_X^{\mathrm{norm}})^{\mathrm{spec}}) \\ &\simeq \underline{\mathrm{Hom}}_{\mathrm{Ind\,Coh}_{\mathcal{N}}(\mathrm{Loc}^{\mathrm{res}}_{\check{G},\bar{\mathbb{F}}_q})}(\delta_\phi, \mathcal{L}_{\check{X}}^{\mathrm{norm}})\end{aligned}$$

By lemma [2, Lem 2.6.1], which computes the trace of a hom complex, we have

Conjecture 1. [2, 14.31] Geometric Numerical Conjecture. Under the above notations, for a cuspidal Hecke eigensheaf $\mathcal{F}_\phi$ with trace f_ϕ:

$$\sum_{x \in \mathrm{Bun}_G(\mathbb{F}_q)} f_\phi(x) P_X^{*,\mathrm{norm}}(x) = \underbrace{q^{-b_G/2}}_{\text{constant}} \sum_{x \in \check{X}^\phi} L^{\mathrm{norm}}(1, \phi^d, (T^\vee)^{\not{/}})$$

A few new terms are appearing in this equation, which we elaborate: $P_X^{*,\mathrm{norm}}$ is the trace of $*$-period sheaf - the (naïve) Verdier dual of the !-period sheaf. It differs from P_X^{norm} in Conjecture 2, see [2, 14.8] for a fascinating discussion.

Next, following [8]: d is an involution of pinning of G, characterized by:

(1) negating the pinning on the simple roots
(2) acting on torus by $t \mapsto w_G(t^{-1})$, where w_G is a lift of the longest Weyl element which takes B to B^-.

In the case of SL_n the duality involution is given by

$$g \mapsto \mathrm{ad}(w_G)({}^t g^{-1})$$

ϕ^d thus denotes the induced Galois representation where action Γ_F on $\check{X}$ via $\check{G} \times \mathbb{G}_{\mathrm{gr}}$ precomposed by $(-)^d$.

Lastly, the sheared L-function can be computed as follows: Given V a $\Gamma_F \times \mathbb{G}_m$ module under action ϕ, with $V = \bigoplus_{i \in \mathbb{Z}} V^{(i)}$ where $V^{(i)}$ are sub Γ_F-modules of weight i under $\mathbb{G}_{\mathrm{gr}}$ action, we define

$$L^{\mathrm{norm}}(s, \phi, V^{\not{/}}) := \prod_i L^{\mathrm{norm}}(s, V^{(i)})$$

the latter terms are Artin L-function of Γ_F normalized according to [2, Ch. 11].

3. Tempered Numerical Conjecture

We choose $k \simeq \mathbb{C}$ to make statements of automorphic forms. We encourage readers to omit the "norm" decoration in first reading.

Conjecture 2. [2, 14.2.1] Our input:

- an everywhere unramified tempered automorphic representation π, with corresponding extended L-parameter of [2, Ch. 2.6] provided by [6]

$$\phi : \Gamma_F \to \check{G} \times \mathbb{G}_m$$

from Weil group Γ_F to extended dual group whose projection onto $\mathbb{G}_m$ is a square root of cyclotomic character, $\varpi^{1/2}$[2].
- The classical scheme of fixed points

$$\check{X}^{\phi} := \{x \in X \,:\, \phi(\gamma)x = x \;\forall \gamma \in \Gamma_F\}$$

is a finite set and reduced.

We can *choose* $f \in \pi$ a spherical vector such that $f^d = \bar{f}$ [3] such that for *all* hyperspherical pair with (twisted) polarizations,

$$(G, M = T^*(X, \Psi)_{/\mathbb{F}_q} \quad (\check{G}, \check{M} = T^*\check{X})_{/k}$$

such that if f is cupsidal,

$$P_X^{\mathrm{norm}}(f) = \underbrace{q^{-b_G/2}}_{\text{constant term}} \cdot \sum_{x \in \check{X}^{\phi}} L^{\mathrm{norm}}(0, T_x \check{X}^{\not{\!\phi}}) \quad b_G := (g-1)\dim G$$

Here $P_X^{\mathrm{norm}}(f) = \int_{g \in G(F)\backslash G(\mathbb{A})} P_X^{\mathrm{norm}}(g) f(g)\, dg$, where

$$P_X^{\mathrm{norm}}(g) = \underbrace{c}_{\text{constant}} \sum_{x \in X(F)} \underbrace{\eta(g)|\partial^{1/2}|^{\gamma}}_{\text{volume character}} 1_{X(\mathbb{O})}(x \cdot (g, \underbrace{\partial^{1/2}}_{\text{torus}})), \quad (g, \partial^{1/2}) \in G \times \mathbb{G}_m$$

For the explanation of the under-braced normalizing terms, see [2, 10.3-10.4]. Here $\partial^{1/2} \in \mathbb{G}_m(\mathbb{A})$ is a torus element tracking $\mathbb{G}_{\mathrm{gr}}$ action[4] Without the torus action, constant, and character coming from volume form, this has the more familiar form:

$$P'_X(g) := \sum_{x \in X(F)} 1_{X(\mathbb{O})}(xg),$$

which would be helpful and sufficient to keep in mind for the upcoming discussion.

Remark 1. The conjecture has many extensions by relaxing various hypotheses; we choose the simplest case here to illustrate the basic aspects.

Example 1. For any G, we can plug in the Whittaker pair.

$$T^*_{\psi}(N\backslash G) \circlearrowleft G \quad \check{G} \circlearrowright *$$

where $\psi : N \to \mathbb{G}_a$ a non-degenerate character of N.

On the A-side: By unfolding,

$$P_X^{\mathrm{norm}}(f) = \underbrace{c}_{\text{constant}} \int_{N(F)\backslash N(\mathbb{A})} \psi(n) f(n \underbrace{a_0}_{\text{torus}}) d'n$$

where $a_0 \in T(\mathbb{A})$ is an explicit torus element encoding $\mathbb{G}_{\mathrm{gr}}$ action.

On the B-side: $\check{X}^{\phi} = \{*\}$, and the L-value is 1. The right hand is

$$\underbrace{q^{-b_G/2}}_{\text{constant}}$$

That the two sides coincide normalizes our choice of f.

[2]This is fixed upon the identification of square root of q from $k \simeq \mathbb{C}$.

[3]where f^d is f precomposed by the duality involution of G.

[4]one can take $\partial^{1/2} = \prod_v \varpi_v^{n_v/2}$ where $\sum_v n_v = 2g-2$, where ϖ_v is uniformizer.

Example 2. $G = \mathbb{G}_m$. π is equivalent to an everywhere unramified unitary idèle class character, χ. We input dual Iwasawa-Tate pair

$$(T^*\mathbb{A}^1 \circlearrowleft \mathbb{G}_m)_{/\mathbb{F}_q}, \quad (\mathbb{G}_m \circlearrowright T^*\mathbb{A}^1)_{/k}$$

We choose $f \in \pi$ corresponding $q^{-b_G/2}\chi$, as forced by normalization in Example 1.

On the A-side:

$$\begin{aligned} P_X^{\mathrm{norm}}(f) &= \underbrace{c_0}_{\text{constant}} \int_{g \in G(F)\backslash G(\mathbb{A})} \underbrace{|(g)|^{1/2}}_{\text{volume character}} \quad P_X(g)\chi(g)\, dg \\ &= \underbrace{c_1}_{\text{constant}} \int_{G(\mathbb{A})} \underbrace{|g|^{1/2}}_{\text{volume character}} \quad 1_{\mathbb{O}}(g)\chi(g)\, dg \end{aligned}$$

where the second equality follows by unfolding, as done in Tate's thesis.

On the B-side: $\check{X}^\phi = \{0\}$ and the tangent space has weight 1 under $\mathbb{G}_{\mathrm{gr}}$. The right-hand side is

$$q^{-b_G/2} L^{\mathrm{norm}}(1/2, \chi)$$

The two sides match by Tate's thesis: factoring the global expression locally.

Example 3. $G = H \times H$. Let $\pi \times \pi^*$ be a cupsidal automorphic representation of $G(\mathbb{A})$. Its corresponding L-parameter has the shape

$$(\phi, \phi^d) : \Gamma_F \to \left(\check{H} \times \check{H}\right) \times \mathbb{G}_{\mathrm{gr}}$$

$$\gamma \mapsto (\phi(\gamma), \phi^d(\gamma), \varpi^{1/2}(\gamma))$$

In this example, the right-hand side admits multiple fixed points. We input dual group-case pair

$$H \circlearrowleft H \times H \qquad \check{H} \times \check{H} \circlearrowright \check{H}$$

$$h \cdot (x, y) := x^{-1} h y, \quad (x, y) \cdot h := x h (y^{-1})^d$$

On the A-side: By unfolding, Fubini's theorem, and the fact that f is unramified,

$$P_X^{\mathrm{norm}}(f) = \int_{[H]} |f(x)|^2 \, dx$$

On the B-side: $\check{X}^\phi := \left\{x \in \check{H} \,:\, (\gamma_1, \gamma_2) \cdot x = \gamma_1 x (\gamma_2^{-1})^d\right\} = Z_\phi(\check{H})$. Thus, as $\check{\mathfrak{h}}$ has weight 2 under the $\mathbb{G}_{\mathrm{gr}}$ action, the right hand side is

$$|Z_\phi \check{H}||L^{\mathrm{norm}}(0, \mathrm{ad}, \check{\mathfrak{h}}^{/\!\!/}) = |Z_\phi \check{H}| \cdot L^{\mathrm{norm}}(1, \mathrm{ad}, \check{\mathfrak{h}})$$

That the two sides match is equivalent to the conjecture of Lapid-Mao, [7].

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