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Non-Archimedean Geometry and Applications

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Abstract. The workshop focused on recent developments in non-Archimedean analytic geometry with various applications to other fields. The topics of the talks included foundational results on analytic spaces as well as applications to the local Langlands conjecture, p-adic cohomology theories, Shimura varieties, the non-Archimedean Simpson correspondence, and to questions on metrics in complex geometry, in the spirit of K-stability.

Mathematics Subject Classification (2020): 11Gxx, 14Gxx.

Introduction by the Organizers

The workshop on Non-Archimedean Analytic Geometry and Applications was organized by Johannes Nicaise (Leuven), Peter Scholze (Bonn), Michael Temkin (Jerusalem) and Annette Werner (Frankfurt).

Non-Archimedean geometry is a central area of geometry with particular relevance for Arithmetic Geometry and numerous applications to other fields. Among the crucial problems are the famous Langlands program and the huge field of p-adic Hodge theory. Non-Archimedean analytic spaces were first introduced by John Tate; later on Vladimir Berkovich and Roland Huber developed richer theories. Perfectoid geometry introduced by Peter Scholze has added a powerful and very influential new tool to attack many deep problems inside and outside arithmetic geometry. Condensed mathematics, pioneered by Dustin Clausen and Peter Scholze, impacts analytic geometry in general, including the classical field of complex geometry. This has led to a rich edifice of deep theories connecting characteristic zero and characteristic p geometry and also Archimedean and non-Archimedean geometry in an unexpected way.

The workshop had 48 on-site participants, and several online participants. Unfortunately, several particants were detained in their home country for political reasons and had to participate online. Altogether we had 18 one hour talks. A summary of the topics can be found below. Several participants explained work in progress or new conjectures or promising techniques to attack open conjectures. The workshop provided a lively platform to discuss these new ideas with other experts.

During the workshop we learned about progress in the geometric Langlands program. Thibaud van den Hove explained a version of the integral motivic Satake equivalence for quasi-split reductive groups. Eugen Hellmann talked about progress in the study of stacks of equivariant vector bundles on the Fargues-Fontaine curve related to Sen Theory.

Daxin Xu reported on progress in the p-adic Simpson correspondence by showing that semistable Higgs bundles of degree zero and rank two with non-zero nilpotent Higgs field are associated to representations of the fundamental group in this correspondence.

Several talks discussed applications to Shimura varieties. Lucas Gerth talked about a general relative theory of p-divisible groups with potential applications to moduli spaces. Mingjia Zhang reported on progress on the cohomology of Shimura varieties. More precisely, Nekovář–Scholl predicted actions of so-called plectic Galois groups (roughly, a product of several copies of absolute Galois groups of number fields) on the cohomology of some Shimura varieties. Restricting this conjecture to local Galois groups, it becomes possible to construct this action unconditionally, using the relation of the Shimura variety to the Igusa stack and the moduli stack of G-bundles on the Fargues–Fontaine curve. Toby Gee explained a modularity lifting theorem for abelian surfaces, building on a classicality criterion for weight two ordinary Siegel p-adic modular forms.

A few talks considered the interplay between complex geometry and non-Archimedean spaces. Walter Gubler gave a talk on a new approach to plurisubharmonic functions on Berkovich spaces generalizing results of Amaury Thuillier to higher dimension. Mattias Jonsson explained how to use non-Archimedean geometry over fields of residue characteristic zero to study degenerations in complex geometry, with applications to the K-stability conjectures on existence of metrics in complex geometry. Jérôme Poineau used Berkovich hybrid spaces to compactify complex algebraic varieties by attaching a non-Archimedean boundary.

Lucas Mann discussed an application of methods from non-Archimedean geometry to a problem in chromatic homotopy theory. Ben Heuer explained an exciting application of p-adic character varieties to anabelian geometry, outlining a possible new proof of Mochizuki's proof of Grothendieck's conjecture on hyperbolic curves.

Eva Viehmann talked on recent progress on moduli of truncated local G-shtukas, which employs ideas emported from the stacky approach to prismatic cohomology.

Katharina Hübner introduced the tame fundamental group of a rigid space as the fundamental group of the Galois category of finite tame covers. In analogy to the algebraic fundamental group, this group can be shown to be topologically finitely generated.

A series of talks discussed recent progress on deRham, p-adic and other cohomology theories. Hélène Esnault investigated vanishing in deRham cohomology under restriction to a dense open subset. Ferdinand Wagner explained results on q-Hodge filtrations and Habiro cohomology, building on q-de Rham cohomology by Bhatt and Scholze. Albert Vezzani explained a relative version of Hyodo-Kato cohomology using motivic tools. In Gabriel Dospinescu's talk new insights on the cohomology of the Drinfeld tower for GL_2 over the p-adic numbers were presented.

Juan Esteban Rodriguez Camargo used the formalism of analytic stacks of Clausen–Scholze to define a general analytic de Rham stack, whose theory of quasicoherent sheaves yields a formalism of analytic D-modules even on perfectoid spaces which do not possess a theory of differentials. This formalism gives another approach to relative Hyodo–Kato cohomology, and a new proof of the p-adic monodromy theorem (formerly Crew's conjecture).

The organizers made a specific effort to invite PhD students and postdocs. For many of them it was the first Oberwolfach workshop they attended. Seven talks were given by PhD students, postdocs and younger untenured faculty. The organizers also identified possible female invitees, thus ensuring that among the participants of the workshop there were eleven women mathematicians.

Workshop: Non-Archimedean Geometry and Applications

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Abstracts

On $\mathbb{C}_p$-representations of the fundamental group in the p-adic Simpson correspondence over curves

Daxin Xu

1. Deninger–Werner's theory

Let K be a finite extension of $\mathbb{Q}_p$ and $\mathbf{C}$ the p-adic completion of an algebraic closure $\overline{K}$ of K. Let X be a smooth projective curve over $\mathbf{C}$ (that can be defined over K).

Let k be the residue field of K and Z a smooth proper $\overline{k}$-curve. Recall that a vector bundle E on Z is *strongly semi-stable*, if $F_Z^{n,*}(E)$ is semi-stable for every integer $n \geq 0$. Let Y be a semi-stable $\mathcal{O}_K$-curve and $C = Y_{\mathbf{C}}$ its geometric generic fiber.

Definition 1 (Deninger–Werner [2])**.** (i) We denote by $\mathrm{VB}^{\mathrm{DW}}(Y_{\mathcal{O}_{\mathbf{C}}})$ the full subcategory of vector bundles E over $Y_{\mathcal{O}_{\mathbf{C}}}$ whose special fiber E_s is a strongly semi-stable bundle of degree zero on the normalization of each irreducible component of $Y_{\overline{k}}$.

(ii) Let $\mathrm{VB}^{\mathrm{DW}}(X)$ be the category of Higgs bundles which admit a model in $\mathrm{VB}^{\mathrm{DW}}(Y_{\mathcal{O}_{\mathbf{C}}})$ for a semi-stable model Y of X over a finite extension of K.

By parallel transport, Deninger and Werner defined a functor to the category of finite dimensional continuous $\mathbf{C}$-representations of the fundamental group $\pi_1(X)$ [2]:

$$\mathbb{V}^{\mathrm{DW}} : \mathrm{VB}^{\mathrm{DW}}(X) \to \mathrm{Rep}_{\mathbf{C}}(\pi_1(X)).$$

2. Parallel transport for Higgs bundles

The p-adic Simpson correspondence establishes an equivalence between v-topological vector bundles on X and Higgs bundles on X:

$$H_{\mathcal{X},\mathrm{Exp}} : \mathrm{VB}(X_v) \simeq \mathrm{HB}(X).$$

A Higgs bundle on X is a pair (E, θ) with an étale vector bundle E over X and a Higgs field $\theta : E \to E \otimes \Omega_X(-1)$. Here (-1) denotes the Tate twist. This equivalence depends on choices of a flat lift of X to $\mathrm{B}^+_{\mathrm{dR},2}$ and an exponential map $\mathrm{Exp} : \mathbf{C} \to 1 + \mathfrak{m}_{\mathbf{C}}$.

There are two approaches to this equivalence:

(i) The original approach is introduced by Faltings [3] and developed by Abbes–Gros and Tsuji [1]. It first establishes a “small” equivalence and then obtains the full equivalence by descent.

(ii) Another approach is based on the Hitchin fibration, recently studied by Heuer [4]. We can also geometrizes the above equivalence as an isomorphism of moduli stacks [5] using this idea.

The above functors can be fitted into the following diagram:

$$\begin{array}{ccccccc}
\nu^{-1}(\mathbb{L}) \otimes_{\mathbf{C}} \mathcal{O}_X & & \mathrm{VB}(X_v) & \xrightarrow{H_{\mathcal{X},\mathrm{Exp}}} & \mathrm{HB}(X) & & (E,0) \\
\uparrow & & \uparrow & \nearrow^{\mathbb{H}_{\mathcal{X},\mathrm{Exp}}} & \uparrow & & \uparrow \\
\mathbb{L} & & \mathrm{Rep}_{\mathbf{C}}(\pi_1(X)) & \xleftarrow[\mathbb{V}^{\mathrm{DW}}]{} & \mathrm{VB}^{\mathrm{DW}}(X) & & E
\end{array}$$

It is known that Higgs bundles in the image of $\mathbb{H}_{\mathcal{X},\mathrm{Exp}}$ are semi-stable of degree zero. Inspired by the result in the (complex) Simpson correspondence, one may ask the following question:

Question 2. *Which Higgs bundles lie in the essential image of* $\mathbb{H}_{\mathcal{X},\mathrm{Exp}}$*? Does every semi-stable Higgs bundle of degree zero lie in* $\mathbb{H}_{\mathcal{X},\mathrm{Exp}}$*?*

Next, we explain how to describle the essential image of $\mathbb{H}_{\mathcal{X},\mathrm{Exp}}$. Inspired by 1, we define:

Definition 3. (i) Let Y be a semi-stable $\mathcal{O}_K$-model of X. Let $\mathrm{HB}^{\mathrm{DW}}(Y_{\mathcal{O}_{\mathbf{C}}})$ be the categories of Higgs bundles (E, θ) with E belonging to $\mathrm{VB}^{\mathrm{DW}}(Y_{\mathcal{O}_{\mathbf{C}}})$ and a small Higgs field θ, i.e. $\theta(E) \subset p^{\alpha} E \otimes \Omega_Y^{\log}$ for some $\alpha \in \mathbb{Q}_{>\frac{1}{p-1}}$.

(ii) Let $\mathrm{HB}^{\mathrm{DW}}(X)$ be the category of Higgs bundles which admit a model in $\mathrm{HB}^{\mathrm{DW}}(Y_{\mathcal{O}_{\mathbf{C}}})$ for a semi-stable model Y of X over a finite extension of K.

The above definition is not enough for our purpose. We need to consider the pullback functoriality of $H_{\mathcal{X},\mathrm{Exp}}$. In the v-bundles side, it is given by the natural pullback in v-topology. In the Higgs bundles side, given $f : X' \to X$ a finite étale morphism of curves and $\mathcal{X}'$ a flat lift of X' to $\mathrm{B}^+_{\mathrm{dR},2}$, there is a "twisted pullback" functor [3, 6]:

$$f^{\circ}_{\mathcal{X}',\mathcal{X},\mathrm{Exp}} : \mathrm{HB}(X) \to \mathrm{HB}(X').$$

This functor is different to the usual pullback functor by a twisting action of line bundles on spectral curves, defined by $\mathcal{X}', \mathcal{X}, \mathrm{Exp}$. When the Higgs field is zero, we have $f^{\circ}(E, 0) = (f^*(E), 0)$.

We define $\mathrm{HB}^{\mathrm{pDW}}_{\mathcal{X},\mathrm{Exp}}(X)$ to be the category of Higgs bundles over X such that there exists a morphism $f : X' \to X$ and $\mathcal{X}'$ as above such that $f^{\circ}(E, \theta)$ belongs to $\mathrm{HB}^{\mathrm{DW}}(X')$.

Theorem 4 ([6])**.** *The functor* $\mathbb{H}_{\mathcal{X},\mathrm{Exp}}$ *induces an equivalence of categories:*

$$\mathrm{Rep}_{\mathbf{C}}(\pi_1(X)) \xrightarrow{\sim} \mathrm{HB}^{\mathrm{pDW}}_{\mathcal{X},\mathrm{Exp}}(X).$$

3. Case of nilpotent Higgs bundles of rank two (joint with Xinwen Zhu)

By Theorem 4, Question 2 reduces to ask whether every semi-stable of degree zero Higgs bundle lies in the category $\mathrm{HB}^{\mathrm{pDW}}_{\mathcal{X},\mathrm{Exp}}(X)$. It is known that every Higgs line bundle of degree zero lies in $\mathrm{HB}^{\mathrm{pDW}}_{\mathcal{X},\mathrm{Exp}}(X)$. Even in the Higgs field zero case, the question is widely open. Moreover, a positive answer to this question suggests that the category $\mathrm{HB}^{\mathrm{pDW}}_{\mathcal{X},\mathrm{Exp}}(X)$ should be independent of the choices of $\mathcal{X}, \mathrm{Exp}$ up to equivalences.

Here is our result in the case of nilpotent Higgs bundles of rank two.

Theorem 5 (X.-Zhu, in progress)**.** *Every semi-stable degree zero rank two Higgs bundles with non-zero nilpotent Higgs fields lies in* $\mathrm{HB}^{\mathrm{pDW}}_{\mathcal{X},\mathrm{Exp}}(X)$ *for some lift* $\mathcal{X}$ *and an exponential* Exp.

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The local Plectic Conjecture

Mingjia Zhang

(joint work with Tony Feng, Matteo Tamiozzo, in progress)

The Plectic Conjecture is raised by Jan Nekovář and Anthony Scholl [1], which predicts that the cohomology of Shimura varieties attached to a reductive group arising from Weil restriction of scalars has a larger symmetry (the so-called plectic structures). We discuss in this talk the local version of the conjecture and a proof of the local conjecture for abelian type Shimura varieties using the machinery of Fargues-Scholze local Langlands correspondence [2] and the Igusa stacks [3].

More precisely, let F be a totally real field of degree d and H/F be a reductive group. Let $\mathsf{G} := \mathrm{Res}^{\mathsf{F}}_{\mathbb{Q}}\,\mathsf{H}$ be the Weil restriction of scalars of H from F to $\mathbb{Q}$. Assume it is part of a Shimura datum (G, X) with Hodge cocharacter μ and reflex field E. Then the natural action of the Galois group $\Gamma_{\mathbb{Q}}$ on the cocharacter group $X_*(\mathsf{G})$ lifts naturally to the "plectic Galois group" $\Gamma^{\mathrm{plec}}_{\mathsf{F}/\mathbb{Q}} := \mathrm{Aut}_{\mathsf{F}}(\mathsf{F} \otimes \overline{\mathbb{Q}})$. We may define the plectic reflex Galois group $\Gamma^{[\mu]}_{\mathsf{F}/\mathbb{Q}}$ to be the stabilizer of the $\mathsf{G}(\overline{\mathbb{Q}})$-conjugacy class $[\mu]$ in $\Gamma^{\mathrm{plec}}_{\mathsf{F}/\mathbb{Q}}$.

Consider the Shimura variety attached to (G, X), i.e. a tower of algebraic varieties $\{\mathbf{Sh}_K\}_K$ over E for K running through neat compact open subgroups of $\mathsf{G}(\mathbb{A}_f)$. Let ℓ be a prime number and we fix an isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$. For an algebraic representation ξ of G with $\overline{\mathbb{Q}}_\ell$-coefficient, we denote by $\mathcal{L}_\xi$ the étale $\overline{\mathbb{Q}}_\ell$-local system attached to it. Write $R\Gamma(\mathbf{Sh}_\infty, \mathcal{L}_\xi)$ for the colimit of the cohomology groups $R\Gamma_{\text{ét}}(\mathbf{Sh}_{K,\overline{\mathbb{Q}}}, \mathcal{L}_\xi)$. The (global) Plectic Conjecture can be formulated as follows:

Conjecture 1 (Nekovář-Scholl). *The complex $R\Gamma(\mathbf{Sh}_\infty, \mathcal{L}_\xi)$ in $D^b(\Gamma_{\mathsf{E}}, \overline{\mathbb{Q}}_\ell)$ lifts canonically to an object in $D^b(\Gamma^{[\mu]}_{\mathsf{F}/\mathbb{Q}}, \overline{\mathbb{Q}}_\ell)$, compatibly with the Hecke actions.*

This conjecture is partly motivated by its application to semi-simplicity of the cohomology of Shimura varieties as Galois representations, see [4], [5]. It is still widely open, except for results for CM points [6], [7] and partial results for quaternionic Shimura varieties and Hilbert modular varieties [4], [1]. Other related results include the construction of partial Frobenii as predicted by the conjecture in the case of abelian type Shimura varieties [8], [9], the local version of the conjecture (see below) on the basic local of abelian type Shimura varieties [10], and a Lie algebra version for Hilbert modular varieties [11].

To state the local version of the conjecture, we fix a rational prime p and an isomorphism $\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_p$, this induces a p-adic place ν of E, we denote by E the completion E_ν. Write F_p for $\mathsf{F} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and we can similarly define the local plectic reflex Galois group $\Gamma^{[\mu]}_{\mathsf{F}_p/\mathbb{Q}_p}$ to be the stabilizer of $[\mu]$ (viewed as a cocharacter of $G := \mathsf{G}_{\mathbb{Q}_p}$ via ι) in $\mathrm{Aut}_{\mathsf{F}_p}(\mathsf{F} \otimes_{\mathbb{Q}} \mathbb{Q}_p)$. Restricting the cohomology of the Shimura varieties to the local Galois groups, we have the following local Plectic Conjecture.

Conjecture 2 (The local Plectic Conjecture). *The complex $R\Gamma(\mathbf{Sh}_\infty, \mathcal{L}_\xi)$ in $D^b(\Gamma_E, \overline{\mathbb{Q}}_\ell)$ lifts canonically to an object in $D^b(\Gamma^{[\mu]}_{\mathsf{F}_p/\mathbb{Q}_p}, \overline{\mathbb{Q}}_\ell)$, compatibly with the global conjecture via restriction along the natural map $\Gamma^{[\mu]}_{\mathsf{F}_p/\mathbb{Q}_p} \to \Gamma^{[\mu]}_{\mathsf{F}/\mathbb{Q}}$. In particular, when p splits completely in F, there exist partial Frobenii at p via the identification $\Gamma^{[\mu]}_{\mathsf{F}_p/\mathbb{Q}_p} \simeq \prod_{i=1}^d \Gamma^{[\mu_i]}_{\mathbb{Q}_p}$.*

The main result of our work in progress [12] is follows:

Theorem 3 (In progress). *For $p \neq \ell$ and abelian type Shimura data, the action of the local plectic reflex Galois group (in fact a Weil group version) on the cohomology of the corresponding Shimura varieties exists canonically.*

Let us explain briefly the idea of the proof. For simplicity of the formalism, we fix a torsion $\mathbb{Z}_\ell$-algebra Λ as the coefficient ring, and consider only the cohomology of Shimura varieties with constant coefficient. (The result for $\overline{\mathbb{Q}}_\ell$ can be obtained by taking an inverse limit of such and base-change.) Consider the v-stack Bun_G (over $\overline{\mathbb{F}}_p$) of G-bundles on the Fargues-Fontaine curve as in [2] and $D(\mathrm{Bun}_G)$, the derived category of étale sheaves on Bun_G with Λ-coefficients. Fargues–Scholze constructed Hecke operators, acting as endo-functors on $D(\mathrm{Bun}_G)$, whose essential image carry internally an action of the Weil group $W_{\mathbb{Q}_p}$ (or a subgroup of it). The main result (existence of abelian type Igusa stacks) of [3] leads to a formula of the cohomology of abelian type Shimura varieties in terms of these operators. Namely, we have the following:

Theorem 4 (In progress). *Let $p \neq \ell$ and (G, X) be an abelian type Shimura datum. Then there is a complex $\mathcal{F} \in D(\mathrm{Bun}_G)$, such that*

$$R\Gamma(\mathbf{Sh}_\infty, \Lambda) = i_1^* T_\mu(\mathcal{F})$$

as $G(\mathbb{Q}_p) \times W_E$*-representations. Here* T_μ *is a Hecke operator attached to the tilting representation of the Langlands dual group* $\widehat{G}$ *labelled by* μ*, and* $i_1 : \underline{BG(\mathbb{Q}_p)} \hookrightarrow \mathrm{Bun}_G$ *is the inclusion of the open stratum corresponding to the trivial* G*-bundle.*

Now the idea is straight-forward: Upon reconciling the way Fargues–Scholze constructed the Hecke operators (using a Beilinson-Drinfeld affine Grassmannian with the sheaf of degree-one Cartier divisors Div^1 as base), which made the action of the Weil group appear, we could construct a plectic version T_μ^{plec} of the Hecke operator (using a Beilinson-Drinfeld affine Grassmannian with a modification of Div^1 as base). Objects in its essential image carry internally a plectic reflex Weil group action. In fact, this part has already been worked out by Li-Huerta in [10], so we do not claim originality. The desired result then follows from the compatibility of the usual and plectic Hecke operators and an analogue of Theorem 4, replacing T_μ by T_μ^{plec}. It is also worth mentioning that with the abelian type Igusa stack, one can construct a plectic companion of the Shimura variety, whose cohomology realizes the lift of $R\Gamma(\mathbf{Sh}_\infty, \Lambda)$ to a complex with local plectic reflex Galois group action.

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Moduli spaces of analytic p-divisible groups

Lucas Gerth

We fix a prime number p. The objects of interest of our talk are analytic p-divisible groups in families. The base spaces for such families will be analytic adic spaces S over $\mathrm{Spa}(\mathbb{Q}_p)$. An analytic p-divisible S-group is then defined to be a smooth commutative adic group $\mathcal{G} \to S$ satisfying the following conditions:

(1) Multiplication by p defines a finite étale surjection $[p]\colon \mathcal{G} \to \mathcal{S}$, and
(2) $\mathcal{G}$ is of "p-topological torsion" [6, Def. 2.6].

Important examples of such groups are $\widehat{\mathbb{G}}_m = 1 + \mathbb{B}^\circ$, the open unit ball centered at 1, and p-topological torsion subgroups $\widehat{A}$ of abeloid varieties A [7]. For technical reasons, we restrict to a category of adic spaces S over $\mathbb{Q}_p$, that we call good adic spaces, for which Scholze's diamond functor is fully faithful, and that are stable under smooth maps. These include perfectoid spaces, as well as seminormal rigid spaces over non-archimedean fields K over $\mathbb{Q}_p$. For such an adic space S, we may consider the v-site of S [2, §8], consisting of locally spatial diamonds over S equipped with the v-topology, which we denote by S_v.

Our first result is a classification of analytic p-divisible groups over a good adic space S in terms of linear algebraic objects on S_v.

Theorem 1. *Let S be a good adic space over $\mathbb{Q}_p$. Then the following categories are canonically equivalent*

- *Analytic p-divisible groups $\mathcal{G} \to S$, and*
- *Tuples $(\mathbb{L}, E, f)$ where $\mathbb{L}$ is a $\mathbb{Z}_p$-local system on S_v, E is a vector bundle on $S_{\text{ét}}$ and $f\colon E \otimes_{\mathcal{O}_S} \mathcal{O}_{S_v} \to \mathbb{L}(-1) \otimes_{\mathbb{Z}_p} \mathcal{O}_{S_v}$ is a morphism of v-vector bundles.*

If $\mathcal{G}$ and $(\mathbb{L}, E, f)$ correspond to each-other, we have $\mathbb{L} = T_p\mathcal{G}$, $E = \mathrm{Lie}(\mathcal{G})$ and the map $f = f_{\mathcal{G}}$ fits in the following cartesian square of group diamonds

(2)
$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\log_{\mathcal{G}}} & \mathrm{Lie}(\mathcal{G}) \otimes_{\mathcal{O}_S} \mathcal{O}_{S_v} \\ \downarrow & & \downarrow f_{\mathcal{G}} \\ T_p\mathcal{G}(-1) \otimes_{\mathbb{Z}_p} \widehat{\mathbb{G}}_m & \xrightarrow[\mathrm{id} \otimes \log]{} & T_p\mathcal{G}(-1) \otimes_{\mathbb{Z}_p} \mathcal{O}_{S_v}. \end{array}$$

In the case $S = \mathrm{Spa}(K)$, for a non-archimedean field extension K of $\mathbb{Q}_p$, this recovers a theorem of Fargues [1, Théorème 0.1].

Theorem 1 can be used to single out the class of dualizable analytic p-divisible groups. Namely, these are the groups $\mathcal{G} \to S$ whose associated map $f = f_{\mathcal{G}}$ extends to a short exact sequence of v-vector bundles

(3)
$$0 \longrightarrow \mathrm{Lie}(\mathcal{G}) \otimes \mathcal{O}_{S_v} \xrightarrow{f} T_p\mathcal{G}(-1) \otimes \mathcal{O}_{S_v} \xrightarrow{g} \omega \otimes \mathcal{O}_{S_v}(-1) \longrightarrow 0,$$

for a vector bundle ω on $S_{\text{ét}}$. The main examples of dualizable groups are groups of good reduction, that is, $\mathcal{G} = \mathfrak{G}_\eta$ for a p-divisible group $\mathfrak{G}$ over a formal model $\mathcal{S}$ of S over $\mathbb{Z}_p$. When $S = \mathrm{Spa}(C)$, for a complete and algebraically closed field $C/\mathbb{Q}_p$, a celebrated result of Scholze–Weinstein [3, Thm. B] states, in our terminology, that any dualizable analytic p-divisible group $\mathcal{G} \to \mathrm{Spa}(C)$ has good reduction. However, this need not hold over arbitrary bases S and already fails over $S = \mathrm{Spa}(C, C^+)$, where $C^+ \subset C$ is an open and bounded valuation ring of rank ≥ 2.

Next, we develop a Dieudonné theory for analytic p-divisible groups over perfectoid bases. This generalizes a construction of Scholze–Weinstein [3, Prop. 5.1.6], which we recover as the case $S = \mathrm{Spa}(C)$.

Theorem 4. *Let $\mathcal{G}$ be an analytic p-divisible group over a perfectoid space S over $\mathbb{Q}_p$. Then there exists a functorial coherent sheaf $\mathcal{E}(\mathcal{G})$ on the Fargues–Fontaine curve X_S, together with a modification, natural in $\mathcal{G}$ and S*

$$0 \longrightarrow T_p\mathcal{G} \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_S} \longrightarrow \mathcal{E}(\mathcal{G}) \longrightarrow i_* \mathrm{Lie}(\mathcal{G}) \longrightarrow 0. \tag{5}$$

Moreover, $\mathcal{E}(\mathcal{G})$ is a vector bundle if and only if $\mathcal{G}$ is dualizable.

This is compatible with prismatic Dieudonné theory ([4, App. to Lecture 17], [5]) as follows: Let $S = \mathrm{Spa}(R, R^+)$ be affinoid perfectoid and let $\mathfrak{G}$ be a p-divisible group over R^+ with generic fiber $\mathcal{G} \to S$. Let (M, φ_M) be the (covariant) prismatic Dieudonné module of $\mathfrak{G}$, a Breuil–Kisin–Fargues module over $\mathrm{A_{inf}}(R^+)$. Let $\mathcal{E}(M, \varphi_M)$ denote the corresponding vector bundle on the Fargues–Fontaine curve, which comes with a modification of $T_p\mathcal{G} \otimes_{\mathbb{Z}_p} \mathcal{O}_{X_S}$. Then we have a natural isomorphism

$$\mathcal{E}(\mathcal{G}) = \mathcal{E}(M, \varphi_M), \tag{6}$$

compatible with the modifications.

We conclude with applications to moduli spaces. We introduce some notations. Let $\breve{\mathbb{Z}}_p = W(\overline{\mathbb{F}}_p)$ and $\breve{\mathbb{Q}}_p = \breve{\mathbb{Z}}_p[\frac{1}{p}]$. We define the universal cover of an analytic p-divisible group $\mathcal{G}$ to be the inverse limit diamond

$$\widetilde{\mathcal{G}} = \varprojlim_{[p]} \mathcal{G}. \tag{7}$$

By Theorem 1, analytic p-divisible groups satisfy v-descent along covers of perfectoid spaces. We can be more precise. Let G_0 be a p-divisible group over $\overline{\mathbb{F}}_p$ and denote by $\mathcal{M}_{G_0}$ its associated Rapoport–Zink space, a smooth formal scheme over $\breve{\mathbb{Z}}_p$ parametrizing deformations (G, ρ) of G_0 up to quasi-isogenies [8]. Let $\mathcal{G}_0 \to \mathrm{Spa}(\breve{\mathbb{Q}}_p)$ be the generic fiber of a fixed deformation of G_0. Consider the v-sheaf $\mathcal{M}_{\mathcal{G}_0}$ taking a good adic space S over $\breve{\mathbb{Q}}_p$ to the set of isomorphism classes of pairs $(\mathcal{G}, \beta)$, where $\mathcal{G} \to S$ is a dualizable analytic p-divisible group and β is an isomorphism

$$\beta \colon \widetilde{\mathcal{G}} \xrightarrow{\cong} \widetilde{\mathcal{G}}_0 \times_{\mathrm{Spa}(\breve{\mathbb{Q}}_p)} S.$$

Our last result is the following.

Theorem 8. *There is a natural isomorphism*

$$\mathcal{M}_{\mathcal{G}_0} = (\mathcal{M}_{G_0})_\eta. \tag{9}$$

More generally, by adding Polarizations, Endomorphisms, and Level structures, we obtain a description of the local Shimura varieties of Scholze–Weinstein of EL and PEL type ([4, Lecture 24]) as moduli spaces of analytic p-divisible groups with extra structures.

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On the vanishing in cohomology of the restriction map to the generic point

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(joint work with Mark Kisin and Alexander Petrov, in progress)

1. Introduction

Let X be an irreducible smooth projective complex algebraic variety. Grothendieck's generalized Hodge conjecture (GGHC) [2] predicts that if $H \subset H^i(X, \mathbb{Q})$ is a Hodge substructure, with corner piece of the Hodge filtration lying in $H^{i-c,c}$, then there is a codimension $\geq c$ subscheme $Z \subset X$ such that the restriction homomorphism

$$H^i(X, \mathbb{Q}) \to H^i(X \setminus Z, \mathbb{Q})$$

in Betti cohomology dies.

While the notion of Hodge structures is analytic, more precisely harmonic, there is one case where GGHC can be expressed purely algebraically. This is when $H^{i,0}(X) = H^0(X, \Omega^i_X) = 0$ and $H = H^i(X, \mathbb{Q})$. In this case $c \geq 1$ and by the comparison isomomorphism between Betti and de Rham cohomology, the conjecture just predicts that there is a dense open $U \subset X$ such that the restriction homomorphism

$$H^i(X) \to H^i(U)$$

in de Rham cohomology dies.

As de Rham cohomology fulfils base change, this vanishing is equivalent to the one for de Rham cohomology of a field $K \subset \mathbb{C}$ of finite type over which X is defined. As usual, it enables one to consider the problem mod p, so over a finite field, for almost all or simply for many ps, or to restrict X over K to K_p a p-adic completion and try to think with modern p-adic methods.

Before doing this, let us emphasize that we know a positive answer to GGHC only in two cases: $i = 1$, then $H^0(X, \Omega^1) = 0$ implies $H^1(X, \mathcal{O}_X) = 0$ either by Hodge duality (analytic) of Hard Lefschetz (algebraic) which in turn implies that $H^1_{dR}(X) = 0$. And $i = 2$: the same argument yields then $H^2(X, \mathcal{O}_X) = 0$ which in turn by the exponential sequence and GAGA implies that $H^2(X, \mathbb{Q})$ is spanned by the Néron-Severi group, which is a finite dimensional $\mathbb{Q}$-vectorspace. So U can be taken to be the complement of the union of its generators.

2. mod p for many p's

Theorem 1. *Let S be a smooth affine scheme over $\mathbb{Z}$, let X_S/S be a smooth proper scheme over S. The following holds true.*

1) *If $H^0(X_S, \Omega^i_{X_S/S})/\text{torsion} = 0$ then there is a dense open $S^\circ \subset S$ such that for all closed points $s \in S^\circ$, for any dense affine open $U_s \subset X_s$, the restriction homomorphism $H^i(X_s) \to H^i(U_s)$ in de Rham cohomology dies.*
2) *If there is a dense set of closed points $s \in S$ such that for each such s, there is a dense affine open $V_s \subset X_s$ such that the restriction homomorphism $H^i(X_s) \to H^i(V_s)$ in de Rham cohomology dies, then*

$$H^0(X_S, \Omega^i_{X_S/S})/\text{torsion} = 0.$$

We may remark at this point that the formulation of Theorem 1 2) is mimicked from the one for the p-curvature conjecture [5], as coupled with GGHC it predicts:

> If there is a dense set of closed points $s \in S$ such that for each such s, there is a dense affine open $V_s \subset X_s$ such that the restriction homomorphism $H^i(X_s) \to H^i(V_s)$ in de Rham cohomology dies, then there is a dense open $U \subset X_S \times_S \mathrm{Spec}(\mathbb{C})$ such that the restriction homomorphism $H^i(X_S \times_S \mathrm{Spec}(\mathbb{C})) \to H^i(U)$ in de Rham cohomology dies.

Here $\mathrm{Spec}(\mathbb{C}) \to S$ is a generic point. So unlike for the p-curvature conjecture in general, and as the p-curvature conjecture in the particular case of a Gauß-Manin connection of a smooth proper family, the the prediction does not request all closed points of a dense $S^\circ \subset S$, only a dense subset.

Sketch of proof. The proof of 1) uses the Cartier isomomorphism, the proof of 2) uses in addition Deligne-Illusie's Hodge-to-de Rham degeneration [1]. □

3. Over $\mathbb{Z}_p$: p-derived complete de Rham cohomology

3.1. **In p-derived complete de Rham cohomology.** Let U be a smooth scheme over $\mathbb{Z}_p$. *p-derived complete de Rham cohomology* is defined as

$$H^i(\hat{U}) := R^i\lim_n(\Omega^\bullet_{U/\mathbb{Z}_p} \otimes^L_{\mathbb{Z}_p} \mathbb{Z}_p/p^n).$$

As $U/\mathbb{Z}_p$ is smooth, we can forget the ^{L}. Thus $H^i(\hat{U})$ is an extension of the classical limit $\lim_n H^i(U_n)$, where $U_n = U \bmod p^n$, with $R^1\lim H^{i-1}(U_n) \xrightarrow{\text{Bockstein } \cong} R^1\lim_n H^i(U)[p^n]$, yielding the exact sequence

$$0 \to R^1\lim_n H^i(U)[p^n] \to H^i(\hat{U}) \to \lim_n H^i(U_n) \to 0. \tag{1}$$

As a $\mathbb{Z}_p$-module, $H^i(\hat{U})$ is endowed with the p-adic topology, compatibly with (1). The classical limit is then the separated quotient of $H^i(\hat{U})$ and the kernel the p-adic closure of 0.

Theorem 2. *Let X be a smooth proper scheme over $\mathbb{Z}_p$. If $H^0(X_1, \Omega^i_{X_1}) = 0$ then for any dense affine open $U \subset X$ smooth over $\mathbb{Z}_p$, the restriction homomorphism*

$$H^i(\hat{X}) \to \lim_n H^i(U_n)$$

in the separated quotient of $H^i(\hat{U})$ dies.

Theorem 2 goes in the direction of GGHC. At the same time it says that *any dense affine* kills the restriction homomorphism in the separated quotient of $H^i(\hat{U})$, while in algebraic de Rham cohomology the predicted affine in GGHC should be "*small enough.*" For example for $X_\circ = \mathbb{P}^1 \times \mathbb{P}^1$ and $U_\circ = X \setminus$ Diagonal, $H^2(\hat{X}_\circ) = NS \otimes_{\mathbb{Z}} \mathbb{Z}_p$, where the Néron-Severi group NS is equal to $\mathbb{Z}^2$, and one computes

Lemma 3.

$$\mathrm{Im}(H^2(X_\circ)) \subset R^1\lim_n H^i(U_\circ)[p^n] \subset H^2(\hat{U}_\circ)$$

is not torsion.

Given Theorem 2 one could ask the following.

Problem 4 (p-GGHS)**.** Let X be a smooth proper scheme over $\mathbb{Z}_p$. If $H^0(X_1, \Omega^i_{X_1}) = 0$, is it the case that there is a open dense $U \subset X$ smooth over $\mathbb{Z}_p$ with $U_1 \subset X_1$ dense such that the restriction homomorphism

$$H^i(\hat{X}) \to H^i(\hat{U})$$

dies?

Theorem 2 says that the p-adic closure of 0 is where the subtlety of p-GGHC is hidden. It raises immediately another problem: if we had a positive answer to p-GGHC, would this imply GGHC? This is to say: is it the case that the kernel of $H^i(U) \to H^i(\hat{U})$ is torsion? As this kernel is easily described to be the submodule of strongly p-divisible elements, i.e. of those $x \in H^i(U)$, such that there are $x_n \in H^i(U),\ n \in \mathbb{N}$ with $x = x_0,\ x_n = px_{n+1}$, the problem can be formulated as follows.

Problem 5. Let U be a good smooth affine scheme over $\mathbb{Z}_p$. Are the strongly p-divisible elements in $H^i(\hat{U})$ all torsion?

Here "good" means that U is the complement of the normal crossings smooth compactification relative to $\mathbb{Z}_p$. Indeed, M. D'Addezio computed that e.g. for $p \geq 3$, $H^1(U)$ possesses non-torsion strongly p-divisible elements, where U is the complement in $\mathbb{A}^1$ of a degree 2 integral point which ramifies mod p.

Sketch of proof of Theorem 2. The vanishing in $H^i(U_1)$ is the content of Theorem 1 1). We illustrate the proof by showing how to go from U_1 to U_2. We want to show

$$0 = \mathrm{Im} H^0(X_1, \mathcal{H}^i_{X_3}) \subset H^0(X_1, \mathcal{H}^i_{X_2})$$

where $\mathcal{H}^i_{X_m}$ is the de Rham cohomology Zariski sheaf. We write the exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^\bullet_{X_2} & \longrightarrow & \Omega^\bullet_{X_3} & \longrightarrow & \Omega^\bullet_{X_1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{=} & & \\ 0 & \longrightarrow & \Omega^\bullet_{X_1} & \longrightarrow & \Omega^\bullet_{X_2} & \longrightarrow & \Omega^\bullet_{X_1} & \longrightarrow & 0 \end{array}$$

inducing the exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{H}^i_{X_2}/\delta_{1,2}\mathcal{H}^{i-1}_{X_1} & \longrightarrow & \mathcal{H}^i_{X_3} & \longrightarrow & \mathcal{H}^i_{X_1} \\ & & \downarrow & & \downarrow & & \downarrow{=} \\ 0 & \longrightarrow & \mathcal{H}^i_{X_1}/\delta_{1,1}\mathcal{H}^{i-1}_{X_1} & \longrightarrow & \mathcal{H}^i_{X_2} & \longrightarrow & \mathcal{H}^i_{X_1} \end{array}$$

where $\delta_{m,n}$ is the Bockstein. Combined with the Cartier isomorphism $\mathcal{H}^i_{X_1} \cong \Omega^i_{X_1}$, $\delta_{1,1} : \mathcal{H}^{i-1}_{X_1} \to \mathcal{H}^i_{X_1}$ is easily computed to be the Kähler differential $d : \Omega^i_{X_1} \to \Omega^{i+1}_{X_1}$. Thus by our vanishing assumption, the composite

$$\mathcal{H}^i_{X_2}/\delta_{1,2}\mathcal{H}^{i-1}_{X_1} \to \mathcal{H}^i_{X_1}/\delta_{1,1}\mathcal{H}^{i-1}_{X_1} = \Omega^i_{X_1}/d\Omega^{i-1}_{X_1} \to d\Omega^i_{X_1}$$

dies after taking H^0. It follows

$$0 = H^0(X_1, \mathcal{H}^i_{X_1}) = \mathrm{Im} H^0(X_1, \mathcal{H}^i_{X_2}/\delta_{1,2}\mathcal{H}^{i-1}_{X_1}) \subset H^0(X_1, \mathcal{H}^i_{X_2}).$$

This finishes the proof.

The precise computation of $\mathcal{H}^i_{X_m}$ and of their Bockstein in [3] and [4] allow to generalize the argument for all pairs $(2n-1, n)$ in place of $(3, 2)$. □

4. Over $\mathbb{Z}_p[[u]]$: prismatic cohomology

Let us write $H^i_\Delta(\hat{U})$ for prismatic cohomology of a smooth affine U over $\mathbb{Z}_p$. With precisely the same assumptions as in Theorem 2, one should be able to replace the p-adic separated quotient of $H^i(\hat{U})$ with the $\mathfrak{m}$-adic separated quotient of $H^i_\Delta(\hat{U})$. We have gone as far as computing the vanishing of the restriction homomorphism $H^i_\Delta(\hat{X}) \to \lim_n H^i_{\Delta/(I^2,p^n)}(\hat{U})$ where I is the prismatic ideal. Peter

Scholze explained to us that our proof of Theorem 2 generalizes well to prismatic cohomology. We haven't yet checked the details.

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q-Hodge filtrations & Habiro cohomology

Ferdinand Wagner

q-de Rham cohomology, constructed by Bhatt–Scholze [2], is a cohomology theory for smooth schemes over $\mathbb{Z}$ with coefficients in the power series ring $\mathbb{Z}[[q-1]]$. After reduction modulo $(q-1)$ it recovers de Rham cohomology and after completion at any prime p it recovers prismatic cohomology relative to the p-adic q-de Rham prism $(\mathbb{Z}_p[[q-1]], [p]_q)$. This makes q-de Rham cohomology the only known nontrivial case in which prismatic cohomology for various primes combines into a global object.

In this talk we discussed the following two questions:

Question A. Does the Hodge filtration on de Rham cohomology admit a q-deformation as well?

Question B. Can q-de Rham cohomology be descended from $\mathbb{Z}[[q-1]]$ to the *Habiro ring*

$$\mathcal{H} \overset{\text{def}}{=} \lim_{m\in\mathbb{N}} \mathbb{Z}[q]^{\wedge}_{(q^m-1)}\,?$$

q-Hodge filtrations. Surprisingly, the answer to Question A turns out to be *no* in general. However, canonical "q-Hodge filtrations" always exist away from finitely many primes. More precisely, we'll show in upcoming work that for any smooth $\mathbb{Z}$-algebra R such that all primes $p \leq \dim(R/\mathbb{Z})$ are invertible in R, the q-de Rham complex $q\text{-}\Omega_R$ can be equipped with a canonical q-deformation of the Hodge filtration.

Moreover, it turns out that Question B has a positive answer whenever Question A has:

Theorem/Construction 1. *Let us denote by* $\mathrm{Sm}_{\mathbb{Z}}^{q\text{-Hdg}}$ *the category of pairs* $(R, \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\Omega_R)$*, where* R *is smooth and* $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\Omega_R$ *is a* q*-deformation of the Hodge filtration (subject to a few natural conditions). Given such a pair, define the* q*-Hodge complex*

$$q\text{-}\mathrm{Hdg}_{(R,\mathrm{fil}^\star_{q\text{-Hdg}})} \overset{\mathrm{def}}{=} \operatorname{colim}\left(\mathrm{fil}^0_{q\text{-Hdg}}\, q\text{-}\Omega_R \xrightarrow{(q-1)} \mathrm{fil}^1_{q\text{-Hdg}}\, q\text{-}\Omega_R \xrightarrow{(q-1)} \cdots\right)^\wedge_{(q-1)} .$$

Then the functor $q\text{-}\mathrm{Hdg}_{(-)} \colon \mathrm{Sm}_{\mathbb{Z}}^{q\text{-Hdg}} \to \mathcal{D}(\mathbb{Z}[[q-1]])$ *factors canonically and non-trivially through a functor*

$$q\text{-}\mathcal{H}\mathrm{dg}_{(-)} \colon \mathrm{Sm}_{\mathbb{Z}}^{q\text{-Hdg}} \longrightarrow \mathcal{D}(\mathcal{H})$$

valued in the derived ∞*-category of the Habiro ring.*

Since $q\text{-}\Omega_{(-)} \simeq \mathrm{L}\eta_{(q-1)}(q\text{-}\mathrm{Hdg}_{(-)})$, this also implies that the q-de Rham complex itself descends to the Habiro ring, with the descent given by $\mathrm{L}\eta_{(q-1)}(q\text{-}\mathcal{H}\mathrm{dg}_{(-)})$.

Relation to $\mathrm{TC}^-(-/\mathrm{ku})$. Another rich source of examples of q-deformations of the Hodge filtration comes from a surprising connection to homotopy theory. In [1], Antieau showed that for any quasisyntomic ring R, there exists a *motivic filtration* $\mathrm{fil}^\star_{\mathrm{mot}}\, \mathrm{HC}^-(R)$ on the negative cyclic homology R, whose associated graded is the completion of the Hodge filtration on the derived de Rham complex of R (up to shift):

$$\mathrm{gr}^\star_{\mathrm{mot}}\, \mathrm{HC}^-(R) \simeq \mathrm{fil}^\star_{\mathrm{Hdg}}\, \widehat{\mathrm{dR}}_R[2\star] .$$

In their seminal work [4], Hahn–Raksit–Wilson show that Antieau's motivic filtration is an instance of a much more general formalism which they call the *even filtration*. The even filtration can also be applied in purely homotopical situations, such as for $\mathrm{TC}^-(-/\mathrm{ku})$: topological negative cyclic homology relative to the complex K-theory spectrum ku. This leads to a q-de Rham analogue of Antieau's result.

Theorem/Construction 2. *Let* R *be quasisyntomic and* $2 \in R^\times$*. Suppose that* R *admits a lift to an* $\mathbb{E}_2$*-ring spectrum* $\mathbb{S}_R$ *satisfying* $R \simeq \mathbb{S}_R \otimes \mathbb{Z}$*. Then the derived* q*-de Rham complex of* R *can be equipped with a* q*-deformation* $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_R$ *of the Hodge filtration in such a way that the associated graded of the even filtration on* $\mathrm{TC}^-(\mathrm{ku} \otimes \mathbb{S}_R/\mathrm{ku})$ *is the completion of the filtration* $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_R$ *(up to shift):*

$$\mathrm{gr}^\star_{\mathrm{ev}}\, \mathrm{TC}^-(\mathrm{ku} \otimes \mathbb{S}_R/\mathrm{ku}) \simeq \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_R[2\star] .$$

In the case where $R = \mathbb{Z}[x]$ with spherical lift $\mathbb{S}[x]$, the theorem was first shown in unpublished work of Raksit.

Using the periodic complex K-theory spectrum KU in place of ku, one can recover the q-Hodge complex as well, and using a bit of genuine equivariant homotopy theory, also the descent to the Habiro ring can be recovered.

Refined TC^-. If R is a $\mathbb{Q}$-algebra, a spherical lift $\mathbb{S}_R$ as in Theorem/Construction 2 exists tautologically (just take R itself). However, the result is not very interesting in this case. A related issue is that for $\mathbb{Q}$-algebras the q-de Rham complex can be defined, but it will just be $\Omega^*_R[[q-1]]$. In particular, for varieties over $\mathbb{Q}$, there can be no comparison from q-de Rham cohomology to non-rational cohomology theories, like étale cohomology with torsion coefficients.

We expect that this issue can be fixed as follows: Based on Efimov's theorem on the rigidity of localising motives [3], Efimov–Scholze construct a refinement of $\mathrm{TC}^-(-/k)$ for any complex orientable $\mathbb{E}_\infty$-ring spectrum k (such as $k = \mathrm{ku}$ or $k = \mathrm{KU}$). The refinement $\mathrm{TC}^{-,\mathrm{ref}}(-/k)$ is valued in the ∞-category of nuclear ind-modules over $k[[t]]$, where $t \in \pi_{-2}(k)$ is any complex orientation.

We expect that the formalism of even filtrations can be extended to this setting. If this is the case, we can construct new cohomology theories for smooth varieties over $\mathbb{Q}$ as follows:

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{ku}}(X) &\overset{\mathrm{def}}{=} \mathrm{gr}^0_{\mathrm{ev}}\, \mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku} \otimes X/\mathrm{ku})\,, \\ \mathrm{R}\Gamma_{\mathrm{KU}}(X) &\overset{\mathrm{def}}{=} \mathrm{gr}^0_{\mathrm{ev}}\, \mathrm{TC}^{-,\mathrm{ref}}(\mathrm{KU} \otimes X/\mathrm{KU})\,. \end{aligned}$$

We note that $\mathrm{R}\Gamma_{\mathrm{ku}}(X)$ comes naturally equipped with a filtration, given by the shifted graded pieces $\mathrm{gr}^\star_{\mathrm{ev}}\, \mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku} \otimes X/\mathrm{ku})[-2\star]$ for $\star \geq 1$ (over KU everything will be periodic, so we won't get an interesting filtration in this case). We should think of $\mathrm{R}\Gamma_{\mathrm{ku}}(X)$ with its filtration as a version of "q-Hodge filtered q-de Rham cohomology of X". But in contrast to the "naive" q-de Rham cohomology of X, which would just be $\mathrm{R}\Gamma(X, \Omega^*_X[[q-1]])$, the new theory $\mathrm{R}\Gamma_{\mathrm{ku}}(X)$ should be non-trivial modulo any prime p, and so it could have interesting comparison maps to étale cohomology with torsion coefficients.

Similarly, $\mathrm{R}\Gamma_{\mathrm{KU}}(X)$ should be thought of as a version of "q-Hodge cohomology" of X. We expect that $\mathrm{R}\Gamma_{\mathrm{KU}}(X)$ admits a natural descent along $\mathcal{H} \to \mathbb{Z}[[q-1]]$, either given by a version of Theorem/Construction 1, or via genuine equivariant homotopy theory, as remarked after Theorem/Construction 2.

The upshot is that Questions A and B should have a non-trivial positive answer for varieties over $\mathbb{Q}$!

In joint work with Samuel Meyer, we compute the values of these new cohomology theories on a point, i.e. $\mathrm{R}\Gamma_{\mathrm{ku}}(\operatorname{Spec}\mathbb{Q})$ and $\mathrm{R}\Gamma_{\mathrm{KU}}(\operatorname{Spec}\mathbb{Q})$. For simplicity, let us state the result with ku and KU replaced by $\mathrm{KU}_p^\wedge$ for an arbitrary prime p.

Theorem 3 (joint with Samuel Meyer)**.** *Let $Z \subseteq \operatorname{Spa}(\mathbb{Z}_p[[q-1]])$ be the union of the locally closed subsets $\operatorname{Spa}(\mathbb{F}_p((q-1)), \mathbb{F}_p[[q-1]])$ and $\operatorname{Spa}(\mathbb{Q}_p(\zeta_{p^n}), \mathbb{Z}_p[\zeta_{p^n}])$ for all $n \geq 0$. Then*

$$\pi_*\, \mathrm{TC}^{-,\mathrm{ref}}\big(\mathrm{KU}_p^\wedge \otimes \mathbb{Q}/\mathrm{KU}_p^\wedge\big) \cong \mathcal{O}(Z^\dagger)[\beta^{\pm 1}]\,,$$

where $Z^\dagger$ denotes the overconvergent neighbourhood of Z and $\beta \in \pi_2(\mathrm{KU})$ is the Bott element.

We note that in particular $\mathcal{O}(Z^\dagger)/p \neq 0$, so even though we started with a rational input, $\mathrm{TC}^{-,\mathrm{ref}}$ spits out a non-rational answer, which is exactly what we need in order to get a comparison to étale cohomology with torsion coefficients.

We wish to pursue such a comparison in future work. It should also be interesting to see how the Habiro-descent of $\mathrm{R}\Gamma_{\mathrm{KU}}(X)$ compares to Peter Scholze's recent construction of *Habiro cohomology* [6].

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D-modules on arc-stacks

Juan Esteban Rodríguez Camargo

(joint work with Johannes Anschütz, Guido Bosco, Arthur-César Le Bras, Peter Scholze)

1. Introduction

The theory of D-modules has been of extreme importance in modern representation theory, algebraic and arithmetic geometry. The recent proof of the geometric Langlands conjecture [1, 2, 3, 4, 5] is a perfect example of the scope of the theory. Nowadays, the preferred way to introduce D-modules is via algebraic geometry itself, namely, via a construction called the *(algebraic) de Rham stack* due to Simpson [20]. Thanks to this point of view, and to the general theory of abstract six functor formalisms [11], one can built up the basic framework of six operations for D-modules on stacks; this is the minimum set up required for running the general machinery of geometric Langlands.

On the other hand, in classical analytic geometry, there has not been a way to systematically develop a general theory of D-modules satisfying the good categorical and geometric properties of their algebraic counterpart. Thanks to the theory of Analytic Geometry and Condensed Mathematics of Clausen and Scholze, we currently have the tools to develop such a formalism in great generality.

In the work [14], the author introduced the analytic de Rham stack in p-adic geometry, an object that captures a very well behaved theory of analytic D-modules together with their six operations. This theory of D-modules contains and extends the theory of D-cap modules of Ardakov and Wadsley [6]. The construction of analytic D-modules is done via the so called *analytic de Rham stack*, this construction has several geometric and categorical advantages that we do not see via the classical approaches (eg. the construction of six operations is a routine verification, and they are automatically compatible with general six operations for analytic stacks).

The theory of the analytic de Rham stack can be developed over a general *normed base*, for example, in [17] Scholze uses the complex variant of the analytic de Rham stack to set up the language for the geometrization of the real local Langlands correspondence.

One of the most astonishing properties of the analytic de Rham stack is that it satisfies very strong descent results, similar as étale or motivic sheaves on v or arc-stacks [16, 18]. One of the main goals of our work is to develop the foundations for a theory of the analytic de Rham stack (in p-adic geometry) that can be even evaluated in "topological objects" such as (a slight modified variant of) arc-stacks. For example, using our theory, we can talk about analytic D-modules of objects such as perfectoid spaces, Fargues-Fontaine curves, Bun_G, etc. We expect this presentation of the theory of the analytic de Rham stack to play an important role in the geometrization of the locally analytic p-adic Langlands correspondence.

2. Main constructions

Let us discuss some of the main constructions that lead to the definition of the analytic de Rham stack. Let $\mathbb{Q}_{p,\square}$ be the analytic ring of solid p-adic rational numbers, that is, $D(\mathbb{Q}_{p,\square})$ is the derived category of solid $\mathbb{Q}_p$-vector spaces. In order to construct the analytic de Rham stack we need to decide the category of analytic stacks that we use. This boils down to pick an appropriate category of analytic rings that will model the spaces we want to study. Motivated from their spectral properties, we chose the following category of analytic rings[1]:

Definition 2.1. A solid $\mathbb{Q}_p$-algebra is called *Gelfand* if it belongs in the full subcategory of solid $\mathbb{Q}_p$-algebras generated under colimits by Banach rings. We let GelfRing be the category of Gelfand $\mathbb{Q}_p$-algebras.

Let us summarize some of the nice features of Gelfand rings that convinced us to use them to model our theory of analytic geometry:

(1) Let A be a Gelfand ring, one can construct an underlying Berkovich space $\mathcal{M}(A)$ consisting on rank 1-multiplicative valuations seen as maps of condensed sets
$$|-|_x\colon A \to \mathbb{R}_{\geq 0}$$
such that $|p| = 1/2$.

(2) Let A be a Gelfand ring, then A has an underlying solid ring $A^{\leq 1}$ of elements of spectral norm ≤ 1. The underlying set of $A^{\leq 1}$ consists of those elements $a \in A$ such that $|a|_x \leq 1$ for all $x \in \mathcal{M}(A)$. However, $A^{\leq 1}$ has a non trivial solid structure.

[1]The official definition of Gelfand rings is slightly more technical, but all the main examples of Gelfand rings are as in Definition 2.1

(3) Given A a Gelfand ring, its uniform completion is the ring $A^u = (A^{\leq 1})^{\wedge}_p[\frac{1}{p}]$ where $(A^{\leq 1})^{\wedge}_p$ is the p-completion. The uniform completion of a Gelfand ring is always a Banach $\mathbb{Q}_p$-algebra.

(4) Gelfand rings have a *spectral radical* or †*-nil radical* $\mathrm{Nil}^{\dagger}(A) = A^{\leq 0}$, whose underlying set consists on those elements $a \in A$ such that $|a|_x = 0$ for all $x \in \mathcal{M}(A)$. The spectral radical of A is also the same as the kernel of the map $A \to A^u$. We define the *spectral reduction* or †*-reduction* of A to be $A^{\dagger-\mathrm{red}} := A/\mathrm{Nil}^{\dagger}(A)$.

(5) Let A be a Gelfand ring, then we have homeomorphisms of Berkovich spaces

$$\mathcal{M}(A) = \mathcal{M}(A^{\dagger-\mathrm{red}}) = \mathcal{M}(A^u).$$

Remark 2.2. The motivation of the name "Gelfand ring" comes from the following fact: given a bounded ring A (as introduced in [14]) one can construct the Berkovich spectrum $\mathcal{M}(A)$, and construct a pairing of **sets**

$$\langle -, - \rangle \colon A(*) \times \mathcal{M}(A) \to \mathbb{R}_{\geq 0}$$

mapping $(a, x) \mapsto |a|_x$. This map is nothing but an avatar of the Gelfand transform of Banach algebras over $\mathbb{C}$. The pairing $\langle -, - \rangle$ extends to a map of condensed sets $A \times \mathcal{M}(A) \to \mathbb{R}_{\geq 0}$ if and only if A is a Gelfand ring. In other words, the Gelfand transform of A is continuous if and only if A is a Gelfand ring.

The category of analytic stacks that we study is the following:

Definition 2.3. A *Gelfand stack* is a functor

$$X \colon \mathrm{GelfRing} \to \mathrm{Ani}$$

with values in anima satisfying descent for the !-topology of analytic rings. We let GelfStk denote be the category of Gelfand stacks.

The category GelfStk contains a large variety of objects that we study in p-adic geometry: it contains a generalization of Berkovich spaces (called *derived Berkovich spaces*) given by glueing Gelfand rings along their Berkovich spectrum. It also contains classifying stacks of the form BG^{sm} and BG^{la} where G is a p-adic Lie group, and G^{sm} (resp. G^{la}) is its incarnation using locally constant $\mathbb{Q}_p$-valued functions (resp. G^{la} is its incarnation using locally analytic functions). This allows us to study representation theory of p-adic Lie groups using analytic stacks.

The category GelfRing has a basis consisting on the so-called *nil-perfectoid rings* $\mathrm{PerfdRing}^{\mathrm{nil}} \subset \mathrm{GelfRing}$. A Gelfand ring A is called nil-perfectoid if and only if its spectral reduction $A^{\dagger-\mathrm{red}}$ is a classical perfectoid Banach $\mathbb{Q}_p$-algebra. Nil-perfectoid rings are the analytic analogue of semiperfectoid rings in integral p-adic Hodge theory [7] and prismatic theory [8].

To end the introduction of the main players of our theory, let us recall the definition of arc-stacks.

Definition 2.4. Let PerfdRing be the category of perfectoid $\mathbb{Q}_p$-algebras. An arc-stack over $\mathbb{Q}_p$ is a functor

$$X \colon \mathrm{PerfdRing} \to \mathrm{Ani}$$

that satisfies descent for the arc-topology. We let arcStk be the category of arc-stacks over $\mathbb{Q}_p$.

The following is an approximation of the main descent theorem for the analytic de Rham stack. See Remark 2.7 for the appropriate technical changes needed to get a correct version.

Theorem 2.5. *There is a perfectoidization functor*

$$(-)^{\diamond}\colon \mathrm{GelfStk} \to \mathrm{arcStk}$$

sending a Gelfand stack Y to its restriction $Y^{\diamondsuit}$ to a sheaf on perfectoid rings along the inclusion $\mathrm{PerfdRing} \subset \mathrm{GelfRing}$*. The functor $(-)^{\diamond}$ admits a colimit preserving right adjoint, called the de Rham stack,*

$$(-)^{\mathrm{dR}}\colon \mathrm{arcStk} \to \mathrm{GelfStk}$$

that sends an arc-stack X to the Gelfand stack whose value at a nil-perfectoid ring $A \in \mathrm{PerfdRing}^{\mathrm{nil}}$ is given by $X^{\mathrm{dR}}(A) = X(A^u)$. Moreover, for X a derived Berkovich space, its de Rham stack $X^{\mathrm{dR}} := (X^{\diamond})^{\mathrm{dR}}$ agrees with the analytic de Rham stack of [14]*.*

Remark 2.6. Theorem 2.5 might looks as a definition in a first glance. It's non-trivial content is the fact that the de Rham stack functor $(-)^{\mathrm{dR}}$ commutes with colimits: this is another way to say that the functor $X \mapsto X^{\mathrm{dR}}$ satisfies arc-descent.

Remark 2.7. **Theorem 2.5 is not true as stated**. In order to get the correct version one has to slightly modify both categories of arc and Gelfand stacks:

(1) In one hand, to get a clean statement, it is better to work only with *sheaves* for the arc and !-topology and not with *hypersheaves*.
(2) On the other hand, we have to work with perfectoid and Gelfand rings that are *separable* and of *quasi-finite dimension*.
   i. The separability property only means that the uniform completion is a separable Banach $\mathbb{Q}_p$-algebra (that is, there is a countable dense subspace), equivalently, that it is ω_1-compact as Banach ring; this is relevant to have a basis of nil-perfectoid rings in Gelfand rings.
   ii. The quasi-finite dimensionality condition means that the underlying diamond of the ring is quasi-proétale over a finite dimensional affine space over $\mathbb{Q}_p$. This is important to guarantee that the formation of the de Rham stack satisfies arc-descent.

This slightly more restricted framework does not affect the main applications to p-adic and arithmetic geometry.

3. First applications

In this last section we mention without further detail some applications of this new perspective on the analytic de Rham stack. Let X be an arc-stack over $\mathbb{F}_p$, one can construct Fargues-Fontaine de Rham stacks by taking

$$\mathrm{FF}_X^{\mathrm{dR}} := (X \times \mathrm{Spd}\mathbb{Q}_p)^{\mathrm{dR}}/\varphi_X^{\mathbb{Z}}.$$

When X is a rigid variety over a non-archimedean field of characteristic p, sheaves on $\mathrm{FF}_X^{\mathrm{dR}}$ are a generalization of overconvergent F-isocrystals and rigid cohomology of X. If instead X is a rigid variety over a field of characteristic zero, sheaves on $\mathrm{FF}_X^{\mathrm{dR}}$ give rise to a generalization of Hyodo-Kato cohomology. In the simplest example of $X = \mathrm{Spa}\mathbb{Q}_p$, we already recover both Crew's conjecture and Tsuzuki's theorem, giving a new and independent proof of the local p-adic monodromy theorem (see [12]):

Theorem 3.1 (local p-adic monodromy theorem)**.** *After fixing the pseudo-uniformizer $p \in \mathbb{Q}_p$, there is an equivalence of vector bundles*

$$\mathrm{VB}(\mathrm{FF}^{\mathrm{dR}}_{\mathrm{Spd}\mathbb{Q}_p}) \cong (\varphi, N, \mathrm{Gal}^{sm}_{\mathbb{Q}_p}) - \mathrm{VB}(\mathbb{Q}_p^{un})$$

where $\mathbb{Q}_p^{un}$ is the maximal unramified extension of $\mathbb{Q}_p$.

We conclude by mentioning other relevant applications of analytic de Rham stacks:

i. One can apply the machinery of Fargues-Scholze [10] to the de Rham stack of $\mathrm{Bun}_G^{\mathrm{dR}}$. This would give a concrete geometric realization over $\mathbb{Q}_p$ of the motivic local Langlands correspondence of [19].

ii. Applying the de Rham stack to the Hodge-Tate period map of infinite level Shimura varieties [15], one can give geometric interpretations to the theory of Pan of locally analytic completed cohomology [13]. In particular, it is this perspective what explains the Eichler-Shimura theory of [9] as a two step process: first a locally analytic Beilinson-Bernstein localization on flag varieties, second a pullback of sheaves along the de Rham stack of the Hodge-Tate period map.

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Modularity lifting theorems for abelian surfaces

Toby Gee

(joint work with George Boxer, Frank Calegari and Vincent Pilloni)

Given an abelian surface $A/\mathbf{Q}$, we let $\overline{\rho}_{A,p}$ denote the mod p Galois representation associated to its p-torsion points $A[p]$. If $A/\mathbf{Q}$ admits a polarization of degree prime to p, then we can and do view $\overline{\rho}_{A,p}$ as a representation

$$\overline{\rho}_{A,p} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\mathbf{F}_p).$$

The main theorem of [2] is as follows.

Theorem 1. *Let $A/\mathbf{Q}$ be an abelian surface with a polarization of degree prime to 3. Suppose the following hold:*

(1) The mod 3 representation

$$\overline{\rho}_{A,3} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\mathbf{F}_3)$$

is surjective.

(2) The representation $\overline{\rho}_{A,3}|_{G_{\mathbf{Q}_2}}$ is unramified, and the characteristic polynomial of $\overline{\rho}_{A,3}(\mathrm{Frob}_2)$ is not $(x^2 \pm x + 2)^2$.

(3) A has good ordinary reduction at 3, and the characteristic polynomial of Frob_3 does not have repeated roots.

Then A is modular.

More precisely, there exists a cuspidal automorphic representation π of $\mathrm{GL}_4/\mathbf{Q}$ (the transfer of a cuspidal automorphic representation of $\mathrm{GSp}_4/\mathbf{Q}$ of weight 2) such that $L(s, H^1(A)) = L(s, \pi)$. Consequently, $L(s, H^1(A))$ has a holomorphic continuation to $\mathbf{C}$ and satisfies the expected functional equation.

The proof of Theorem 1 follows Wiles's strategy for proving the modularity of semistable elliptic curves, and in particular, we make use of an analogue of the 3-5 switch used by Wiles [5] to prove residual modularity. This switch exploited the rationality of certain twists of the modular curve $X(5)/\mathbf{Q}$. In our case, we use a rational moduli space of abelian surfaces to carry out a 2-3 switch.

More precisely, given an abelian surface $A/\mathbf{Q}$ as in Theorem 1, we consider the moduli space $P(A[3])/\mathbf{Q}$ of genus-two curves X equipped with a symplectic isomorphism $\mathrm{Jac}(X)[3] \simeq A[3]$ and a fixed rational Weierstrass point. We showed in our earlier work [1, Thm 10.2.1] that $P(A[3])$ is rational. This allows us to find another abelian surface $B/\mathbf{Q}$ with $B[3] \simeq A[3]$ such that the map

$$\overline{\rho}_{B,2} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\mathbf{F}_2) \simeq S_6$$

has image isomorphic to S_5 (due to the rational Weierstrass point). Condition (2) of Theorem 1 ensures that we can find such a B which furthermore has good ordinary reduction at 2. We can also arrange that there is a real quadratic field $F^+/\mathbf{Q}$ such that $\overline{\rho}_{B,2}(\mathrm{Gal}(\overline{\mathbf{Q}}/F^+)) \simeq A_5$.

A consideration of the representation theory of A_5 in characteristic 2 allows us to write $\overline{\rho}_{B,2} = \mathrm{Sym}^3 \overline{r}$, where $\overline{r} : \mathrm{Gal}(\overline{\mathbf{Q}}/F^+) \to \mathrm{GL}_2(\overline{\mathbf{F}}_2)$. By a theorem of Tate, the composite $\mathrm{Gal}(\overline{\mathbf{Q}}/F^+) \to A_5 \hookrightarrow \mathrm{PGL}_2(\mathbf{C})$ lifts to an odd Artin representation $\mathrm{Gal}(\overline{\mathbf{Q}}/F^+) \to \mathrm{GL}_2(\mathbf{C})$. The odd Artin conjecture over totally real fields is known (due to Pilloni–Stroh and Sasaki), and by Kim–Shahidi's symmetric cube functoriality, we are able to deduce the modularity of $\overline{\rho}_{B,2}|_{\mathrm{Gal}(\overline{\mathbf{Q}}/F^+)}$ and thus (by solvable base change) the modularity of $\overline{\rho}_{B,2}$, i.e., of $B[2]$.

It remains to use modularity lifting theorems to pass from the modularity of $B[2]$ to the modularity of B, and thus to the modularity of $A[3] \simeq B[3]$, and finally to the modularity of A. We do this by proving the following (imprecisely stated) modularity lifting theorem.

Theorem 2. *Suppose that $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\overline{\mathbf{Z}}_p)$ is unramified at all but finitely many primes and de Rham at p, with Hodge–Tate weights $\{0, 0, 1, 1\}$. Suppose furthermore that:*

a) $\overline{\rho} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GSp}_4(\overline{\mathbf{F}}_p)$ is modular.
b) $\overline{\rho}(\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}))$ is large.
c) ρ is pure.
d) $\rho|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)}$ is ordinary and p-distinguished.

Then ρ is modular.

Hypothesis (a) of Theorem 2 is responsible for assumption (1) in Theorem 1, while the more serious hypothesis (d) corresponds to assumption (3) there.

We prove Theorem 2 in two steps. Firstly, we use the Taylor–Wiles method to show that ρ is p-adically modular, in the sense that it contributes to a Hida family of Siegel modular forms. This is relatively standard, although the need to consider the primes $p = 2, 3$ causes some pain. Secondly, we prove a classicality criterion for weight 2 ordinary Siegel p-adic modular forms. This is the technical heart of the paper; the proof is a generalization of a part of Lue Pan's remarkable work [3] from GL_2 to GSp_4, building upon work of Rodríguez Camargo [4].

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Motivic Satake for ramified groups

Thibaud van den Hove

1. The motivic Satake equivalence for ramified groups

The geometric Satake equivalence is a cornerstone of the geometric Langlands program and geometric representation theory. It yields an equivalence between equivariant perverse sheaves on the affine Grassmannian and representations of the Langlands dual group, and is essentially the only geometric way to make this dual group appear. First proved by Mirkovic–Vilonen [MV07], many variants have since appeared, e.g., for different sheaf theories, for ramified groups [Zhu15], and for groups over mixed characteristic local fields [Zhu17].

Thus, a natural question is whether one can construct a version of the Satake equivalence, which generalizes all these variants, and is in particular independent of the choice of cohomology theory, i.e., motivic. For split reductive groups in equal characteristic, this was achieved by Richarz–Scholbach for motives with rational coefficients [RS21], and in joint work of the author with Cass and Scholbach for motives with integral coefficients [CvdHS22]. The main theorem of this talk was an extension of this integral motivic Satake equivalence to quasi-split reductive groups, over arbitrary non-archimedean local fields [vdH24a].

Theorem 1. *Let F be a non-archimedean local field with ring of integers $\mathcal{O}$, finite residue field $k \cong \mathbb{F}_q$, and inertia group I. Let G/F be a quasi-split reductive group, and $\mathcal{G}/\mathcal{O}$ a very special parahoric model. Then there is a monoidal equivalence*

$$\left(\mathrm{MTM}_{L^+\mathcal{G}}(\mathrm{Gr}_{\mathcal{G}}, \mathbb{Z}[\frac{1}{p}]), \star\right) \cong \left(\mathrm{Rep}_{\widehat{G}^I}(\mathrm{MTM}(\mathrm{Spec}(\overline{k}), \mathbb{Z}[\frac{1}{p}])), \otimes\right),$$

where $L^+\mathcal{G}$ and $\mathrm{Gr}_{\mathcal{G}}$ denote the positive loop group and twisted affine Grassmannian of $\mathcal{G}$, defined over $\overline{k}$.

Here, MTM denotes the abelian category of mixed Tate motives, which arises as the heart of a t-structure on DTM. This is a subcategory of the category of étale motives DM, on which we can define a t-structure, unconditional on the standard conjecture on algebraic cycles.

On the other hand, there is a monoidal functor

$$\text{gr-}\mathbb{Z}[\frac{1}{p}]\text{-}\operatorname{Mod} \to \operatorname{MTM}(\operatorname{Spec}(\overline{k}), \mathbb{Z}[\frac{1}{p}]).$$

One can equip the inertia-invariants $\widehat{G}^I/\operatorname{Spec}(\mathbb{Z}[\frac{1}{p}])$ with a natural $\mathbb{G}_m$-action, so that its global sections $\mathcal{O}_{\widehat{G}^I}$ form a Hopf algebra in gr-$\mathbb{Z}[\frac{1}{p}]$-Mod. Considering its image $\mathcal{O}_{\widehat{G}^I}$ in $\operatorname{MTM}(\operatorname{Spec}(\overline{k}), \mathbb{Z}[\frac{1}{p}])$, we define

$$\operatorname{Rep}_{\widehat{G}^I}(\operatorname{MTM}(\operatorname{Spec}(\overline{k}), \mathbb{Z}[\frac{1}{p}])) := \operatorname{coMod}_{\mathcal{O}_{\widehat{G}^I}}(\operatorname{MTM}(\operatorname{Spec}(\overline{k}), \mathbb{Z}[\frac{1}{p}])),$$

which is the category of representations of $\widehat{G}^I$, internally in $\operatorname{MTM}(\operatorname{Spec}(\overline{k}), \mathbb{Z}[\frac{1}{p}])$.

2. The integral Satake isomorphism

For unramified G, the classical Satake isomorphism relates the spherical Hecke algebra of G with the ring of functions on $\widehat{G}$, at least with $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-coefficients. To describe the spherical Hecke algebra with $\mathbb{Z}$-coefficients, one needs to use the Vinberg monoid $V_{\widehat{G},\rho_{\mathrm{adj}}} \xrightarrow{d} \mathbb{A}^1$ [Zhu24]. By taking traces of Frobenius from Theorem 1, one can upgrade this to general quasi-split groups.

Theorem 2. *Let G/F and $\mathcal{G}/\mathcal{O}$ be as in Theorem 1, and*

$$\mathcal{H}_{\mathcal{G}} := C_c(\mathcal{G}(\mathcal{O})\backslash G(F)/\mathcal{G}(\mathcal{O}), \mathbb{Z})$$

the corresponding spherical Hecke algebra. Then there is a canonical isomorphism

$$\mathcal{H}_{\mathcal{G}} \cong \mathbb{Z}[V^I_{\widehat{G},\rho_{\mathrm{adj}}|d=q}]^{c_\phi(\widehat{G}^I)},$$

where $\phi \in \operatorname{Gal}(\overline{k}/k)$ is a geometric Frobenius, and c_ϕ denotes the ϕ-twisted conjugation action.

The proof of Theorem 2 is similar to the proof of [Zhu24], by taking the trace of Frobenius of the equivalence in Theorem 1. However, here we can see the advantage of using motives. Namely, [Zhu24] uses the ℓ-adic Satake equivalence, and hence the functions on $\mathcal{G}(\mathcal{O})\backslash G(F)/\mathcal{G}(\mathcal{O})$ obtained by taking trace of Frobenius a priori take value in $\overline{\mathbb{Q}}_\ell$. Thus, a subcategory of equivariant perverse sheaves on the affine Grassmannian carefully has to be singled out, for which the resulting functions actually take value in $\mathbb{Z}$. This is also where the restriction to unramified G in [Zhu24] appears. Instead, when using motives, the trace of Frobenius functions on $\mathcal{G}(\mathcal{O})\backslash G(F)/\mathcal{G}(\mathcal{O})$ automatically take value in $\mathbb{Z}[\frac{1}{p}]$, and then it is not difficult to determine when they even take value in $\mathbb{Z}$.

Theorem 2 can be simplified by passing to $\mathbb{Z}[q^{\pm\frac{1}{2}}]$-coefficients, and then we get

$$\mathcal{H}_{\mathcal{G}} \otimes \mathbb{Z}[q^{\pm\frac{1}{2}}] \cong \mathbb{Z}[q^{\pm\frac{1}{2}}][\widehat{G}^I]^{c_\phi(\widehat{G}^I)}.$$

3. The stack of spherical Langlands parameters

Now, let G/F be a quasi-split reductive group as above, and consider the stack $\mathrm{Loc}_{{}^cG} = \mathrm{Loc}^{\square}_{{}^cG}/\widehat{G}$ of local Langlands parameters, defined over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{p}])$, as introduced in [Zhu21]. Let $\sigma \in \mathrm{Gal}(\overline{F}/F)$ be an arithmetic Frobenius. When G is unramified, there is the closed substack of unramified Langlands parameters $\mathrm{Loc}^{\mathrm{unr}}_{{}^cG} := \widehat{G}q^{-1}\sigma/\widehat{G} \subseteq \mathrm{Loc}_{{}^cG}$, whose global sections agree with the spherical Hecke algebra by the Satake isomorphism. Although this substack does not exist for general G, we have the following generalization [vdH24b]:

Definition 3. The *stack of spherical Langlands parameters* is the quotient stack

$$\mathrm{Loc}^{\mathrm{sph}}_{{}^cG} := \widehat{G}^I q^{-1}\sigma/\widehat{G}^I,$$

defined over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{p}])$

Indeed, the global sections of $\mathrm{Loc}^{\mathrm{sph}}_{{}^cG}$ agree with the spherical Hecke algebra of G by Theorem 2. It can moreover be related to the full stack $\mathrm{Loc}_{{}^cG}$:

Proposition 4. *There is a natural monomorphism* $\mathrm{Loc}^{\mathrm{sph}}_{{}^cG} \to \mathrm{Loc}_{{}^cG}$*, which is fiberwise over* $\mathrm{Spec}(\mathbb{Z}[\frac{1}{p}])$ *a closed immersion.*

4. Eichler–Shimura congruence relations

The Eichler–Shimura relations for the modular curve can be used to relate Fourier coefficients of modular forms to Frobenius eigenvalues of Galois representations. These relations have been conjecturally generalized by Blasius–Ragowski, at least for Shimura varieties with good reduction. They have recently been proven for Hodge type Shimura varieties by Daniels–van Hoften–Kim–Zhang [DvHKZ24]. Our final result generalizes the Eichler–Shimura congruence relation, by removing the assumptions on the ramification [vdH24b]:

Theorem 5. *Let* $\mathrm{Sh}_K(\mathbb{G}, \mathbb{X})$ *be a Shimura variety of Hodge type, defined over a completion* E *of the reflex field* $\mathbb{E}$ *at a place lying over* p*, with Iwahori-level* $\mathcal{I}$ *at* p*. Let* ℓ *be a prime number, prime to* $|\pi_0(Z(G))|$*. Then the action of inertia on* $\mathrm{H}^i(\mathrm{Sh}_K(\mathbb{G}, \mathbb{X})_{\overline{E}}, \overline{\mathbb{Z}}_\ell)$ *is unipotent, whereas the action of any lift of Frobenius on* $\mathrm{R}\Gamma(\mathrm{Sh}_K(\mathbb{G}, \mathbb{X})_{\overline{E}}, \overline{\mathbb{Z}}_\ell)$ *satisfies an explicit Hecke polynomial with coefficients in the Iwahori–Hecke algebra* $\mathcal{H}_{\mathcal{I}} \otimes \overline{\mathbb{Z}}_\ell$*.*

The proof follows the same strategy as [DvHKZ24], by using the construction of Igusa stacks and the spectral action from Fargues–Scholze, but uses the stack $\mathrm{Loc}^{\mathrm{sph}}_{{}^cG}$ of spherical Langlands parameters, instead of the stack of unramified Langlands parameters. In case the local Shimura group is not quasi-split, one also needs to use transfer morphisms as defined by Haines.

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Psh and pluriharmonic functions on Berkovich spaces

Walter Gubler

(joint work with Joe Rabinoff)

Let k be a non-archimedean field which might be trivially valued.

Thuillier introduced in his thesis [5] harmonic and subharmonic functions on smooth strictly k'-analytic curves for non-trivially valued non-archimedean fields k' and he developed a potential theory with analogous properties as in the case of Riemann surfaces. In the complex case, plurisubharmonic functions generalize subharmonic functions to higher dimensions. In the non-archimedean case, there are global approaches to psh functions (or more precisely to psh metrics on line bundles) on projective varieties by Zhang [6], and later by Boucksom–Jonsson [1] who developed a beautiful pluripotential theory for smooth projective varieties over a trivially valued field. Zhang's approach is used in Arakelov theory to extend arithmetic intersection numbers to semipositive metrized line bundles and Boucksom–Jonsson's pluripotential theory has applications to mirror symmetry and to the Calabi–Yau problem.

Inspired by the complex analytic approach to plurisubharmonic functions, we introduce in [4] a local approach to plurisubharmonic functions working for any k-analytic space X. We report on this approach here.

Definition 1. A function $u\colon X \to [-\infty, \infty)$ is called *classically psh* if u is upper semicontinuous and has the property that for every non-trivially valued non-archimedean field extension k' of k and for every k'-analytic map $f\colon Y \to X_{k'}$

from a strictly k'-analytic smooth curve Y, the pull-back of u to Y is subharmonic in the sense of Thuillier.

Classically psh functions have the expected properties [4, Proposition 4.2]:

(1) They form a sheaf PSH on X.
(2) For an analytic function f on X, we have $\log|f| \in \mathrm{PSH}(X)$.
(3) For $u_1, u_2 \in \mathrm{PSH}(X)$ and $\lambda_1, \lambda_2 \geq 0$, we have $\lambda_1 u_1 + \lambda_2 u_2, \max\{u_1, u_2\} \in \mathrm{PSH}(X)$.
(4) The infimum of a decreasing net in $\mathrm{PSH}(X)$ is classically psh.
(5) For $u\colon X \to [-\infty, \infty)$ and a non-archimedean field extension k'/k, we have $u \in \mathrm{PSH}(X)$ if and only if the pull-back of u to $X_{k'}$ is classically psh.
(6) Classically psh functions are functorial with respect to analytic maps.
(7) For a smooth strictly k-analytic curve over a non-trivially valued field k, classically psh functions agree with subharmonic functions.

We show in [4, Theorem 4.3], based on [3], that psh functions from the global approaches of Zhang and Boucksom–Jonsson are classically psh.

The next result is the *local maximum principle* from [4, Theorem 6.1]. We denote by ∂X the boundary of the k-analytic space X in the sense of Berkovich.

Theorem 2. *If a classically psh function u on X takes its maximum at $x \in X \backslash \partial X$, then u is locally constant at x.*

The proof uses as a basic tool that on a connected Berkovich space, two points can be connected by overconvergent compact analytic curves. This allows to reduce to the local maximum principle for subharmonic functions on smooth strictly k-analytic curves proved by Thuillier [5, Proposition 3.1.11].

For a k-affinoid space X, Ducros [2, Lemme 3.1] has shown that there are k_i-affinoid spaces X_i over non-archimedean field extensions k_i/k such that

$$\partial X = \bigcup_{i=1}^{r} X_i.$$

If X has pure dimension d, then X_i is of pure dimension $d-1$. Proceeding inductively, we get $\partial X_i = \bigcup_j X_{ij}$ for k_{ij}-analytic spaces X_{ij} over non-archimedean field extensions k_{ij}/k_i and so on, leading to a stratification of the boundary ∂X.

For the next result, we recall that the Shilov boundary $\Gamma(X)$ of the affinoid space X is the smallest subset where $|f|$ takes its maximum for every analytic function f.

Proposition 3. *If X is a k-affinoid space of pure dimension d, then the zero-dimensional strata $X_{i_1,i_2,\ldots,i_d}$ of the above stratification are precisely the points of the Shilov boundary of X.*

This result, shown in [4, Lemma 6.3], leads to the following *global maximum principle* [4, Theorem 6.4].

Theorem 4. *Let X be a k-affinoid space. Then any classically psh function u takes its maximum at the Shilov boundary $\Gamma(X)$.*

Indeed, by the local maximum principle, the function u takes its maximum at ∂X. By passing to irreducible components, we may assume that X is of pure dimension d. Then the proof follows by induction on d using Proposition 3.

Definition 5. A real valued function u on a k-analytic space X is called *pluriharmonic* if u and $-u$ are classically psh.

It is clear that the pluriharmonic functions on X form a real vector space $\mathrm{PH}(X)$. The properties of classically psh functions induce corresponding properties for pluriharmonic functions [4, Proposition 7.2].

The following finiteness result is from [4, Theorem 7.4].

Theorem 6. *Let X be a quasi-compact k-analytic space. Then $\mathrm{PH}(X)$ is a finite dimensional real vector space.*

To prove this, we cover X by finitely many k-affinoid spaces. This reduces the claim to the affinoid case. Then the global maximum principle from Theorem 4 shows that any pluriharmonic function is determined by the restriction to the Shilov boundary $\Gamma(X)$. Since $\Gamma(X)$ is finite, this proves the theorem.

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Relative Hyodo–Kato cohomology

Alberto Vezzani
(joint work with Fabrizio Andreatta, Federico Binda, in progress)

Let K be a non-archimedean field, whose residue field k is perfect of characteristic $p > 0$. The Hyodo–Kato cohomology $R\Gamma_{\mathrm{HK}}(\mathcal{X})$ is a p-adic cohomology theory that can be defined for *unipotent* rigid analytic varieties which, as envisaged by Fontaine and Jannsen [9, 13], has the following fundamental properties:

(1) It takes values in $\mathcal{D}_{\varphi,N}(K_0)$ i.e., in the (infinity derived) category of (φ, N)-modules over $K_0 = W(k)[1/p]$;
(2) it is *motivic*, in the sense that it has étale descent, it is ball-invariant, and $H^1_{\mathrm{HK}}(\mathbb{G}_m)$ is one-dimensional; moreover, the monodromy on $H^1_{\mathrm{HK}}(\mathbb{G}_m)$ is non-trivial;

(3) it extends rigid cohomology, i.e. whenever $\mathcal{X}$ has a smooth formal model $\mathfrak{X}$, then $H^*_{\mathrm{HK}}(\mathcal{X}) \cong H^*_{\mathrm{rig}}(\mathfrak{X}_k)$ with monodromy $N = 0$;
(4) it compares to the de Rham cohomology, i.e. if $\mathrm{char}K = 0$, then there is a canonical equivalence of K-vector spaces (the Hyodo–Kato isomorphism) $H^*_{\mathrm{HK}}(X) \otimes_{K_0} K \simeq H^*_{\mathrm{dR}}(X)$.

Remark 1. Any rigid variety which is étale locally of semi-stable (even, poly-stable) reduction is "unipotent". By alteration results [5, 20] any variety becomes unipotent up to a base change along a finite field extension K'/K.

Remark 2. It can be shown [3] that the properties 1-3 determine this cohomology theory uniquely (up to twisting each monodromy map by a universal constant).

Contrarily to the de Rham and étale p-adic cohomologies, Hyodo–Kato does not admit a relative version. We then plan to offer a candidate for a relative Hyodo–Kato cohomology, so that the analogous properties 1-4 listed above are still valid. Note that over a base, it is not completely clear what the category of coefficients should be: e.g. in case $\mathfrak{S}$ is a smooth formal scheme over $\mathcal{O}_K$, a natural choice would be $\mathcal{D}_{\varphi,N}(W(\mathfrak{S}_k^{\mathrm{Perf}})[1/p])$ but this category does not admit any obvious functor to $\mathrm{QCoh}(\mathfrak{S}_K)$ where the de Rham cohomology takes its values[1].

Classically, there are (at least) three different approaches to defining the Hyodo–Kato cohomology:

- via log-geometry, by computing $H^*_{\mathrm{logrig}}(\mathfrak{X}_k)$ via a de Rham–Witt complex (see [2, 4, 7, 10, 12]);
- via nearby cycles functors, by putting $H^*_{\mathrm{HK}}(X) = H^*_{\mathrm{rig}}(\Psi X)$ for a suitably defined ΨX, which carries an extra action by a monodromy operator (see [15, 16]);
- via the de Rham–Fargues–Fontaine cohomology [14], and the identification between D-modules over the Fargues–Fontaine curve and (φ, N)-modules (cfr. the talk by Rodríguez Camargo).

The second approach can be generalized in families, e.g. by using some motivic techiniques, as follows.

Theorem 3 ([1]). *Let $\mathfrak{S}$ be a p-adic formal scheme. From now on, we let $\mathcal{S}$ be its (analytic) generic fiber.*

(1) The canonical motivic[2] functor $\mathrm{DA}(\mathfrak{S}_k) \simeq \mathrm{FDA}(\mathfrak{S}) \xrightarrow{\xi} \mathrm{RigDA}(\mathcal{S})$ exhibits the full subcategory closed under colimits $\mathrm{RigDA}^{\mathfrak{S}-uni}(\mathcal{S})$ generated by its image as the category of modules $\mathrm{Mod}_{\chi\mathbb{1}}\mathrm{DA}(\mathfrak{S}_k)$, where χ is the right adjoint to ξ.
(2) For any compact motive $M \in \mathrm{RigDA}(\mathfrak{S}_\eta)$, there is a rig-étale cover $\mathfrak{S}' \to \mathfrak{S}$ such that $M_{\mathcal{S}'}$ lies in $\mathrm{RigDA}^{\mathfrak{S}'-uni}(\mathcal{S}')$.

[1] We denote by QCoh the infinity category of solid quasi-coherent sheaves following Clausen–Scholze [18].

[2] DA resp. FDA resp. RigDA is the infinity category of étale algebraic resp. formal resp. adic motives with rational coefficients, see [1].

(3) *If $\mathfrak{S}$ has semi-stable reduction, then along the natural stratification $\bigsqcup U_i$ of $\mathfrak{S}_k$ we have isormorphisms $\chi\mathbb{1}|_{U_i} \simeq (\mathbb{1} \oplus \mathbb{1}(-1)[-1])^{\otimes 1+c_i}$ with c_i being the codimension of U_i. In particular (see* [3]*) there is an equivalence $M \mapsto \Psi M$, $\mathrm{Mod}_{\chi\mathbb{1}}\mathrm{DA}(U_i) \simeq \mathrm{DA}(U_i)_{\underline{N}}$ where $\underline{N}$ stands for* codim$U_i + 1$ *commuting ind-nilpotent operators of the form $\Psi M \to \Psi M(-1)$.*

Which leads to the following (cfr. [3] for the absolute case):

Definition 4. Let $\mathrm{RigCoeff}(\mathfrak{S}_k)$ be the category of coefficients for relative rigid cohomology over $\mathfrak{S}_k$, i.e. the category $(\lim_{\mathfrak{T}} \mathrm{QCoh}(\mathcal{T}))^{h\varphi}$ as $\mathfrak{T}$ varies among frames over $\mathfrak{S}_k$ (see [8]). We can define the relative Hyodo–Kato cohomology $R\Gamma_{\mathrm{HK}}(-/\mathfrak{S})$ for $\mathfrak{S}$-unipotent motives as the composite:

$$\mathrm{RigDA}^{\mathfrak{S}-uni}(\mathcal{S}) \simeq \mathrm{Mod}_{\chi\mathbb{1}}\mathrm{DA}(\mathfrak{S}_k) \xrightarrow{R\Gamma_{\mathrm{rig}}} \mathrm{Mod}_{\chi\mathbb{1}}\mathrm{RigCoeff}(\mathfrak{S}_k)^{\mathrm{op}} =: \mathrm{HKCoeff}(\mathfrak{S})^{\mathrm{op}}.$$

The (relative version of the) Hyodo–Kato isomorphism can be deduced formally.

Theorem 5. *By specializing at the frame $\mathfrak{S}$, we get a natural morphism*

$$\mathcal{E}\colon \mathrm{HKCoeff}(\mathfrak{S}) \to \mathrm{Mod}_{\chi\mathbb{1}}\mathrm{QCoh}(\mathcal{S}) \xrightarrow{\varepsilon^*} \mathrm{QCoh}(\mathcal{S})$$

where ε is the canonical augmentation $\xi\chi\mathbb{1} \to \mathbb{1}$. There is a canonical equivalence $\mathcal{E}\left(R\Gamma_{\mathrm{HK}}(\mathcal{X}/\mathfrak{S})\right) \cong R\Gamma_{\mathrm{dR}}(\mathcal{X}/\mathcal{S})$.

In light of Theorem 3(3), one can give a better description of the Hyodo–Kato coefficients, and even specify in which category of "constructible" coefficients we land in.

Corollary 6. *Let $\mathfrak{S}$ be of semi-stable reduction and M be a compact motive in $\mathrm{RigDA}^{\mathfrak{S}-uni}(\mathcal{S})$. There exists a stratification $\mathfrak{S}_k = \bigsqcup V_i$, refining the natural one, for which $R\Gamma_{\mathrm{dR}}(M|_{]V_i[})$ has a natural structure of a $(\varphi, \underline{N})$-isocrystal over V_i, where $\underline{N} = (N_1, \ldots, N_r)$ are* codim$V_i + 1$ *commuting nilpotent operators of the form $F \to F(-1)$.*

Proof. We may and do refine the natural stratification of $\mathfrak{S}_k$ into one for which $\Psi M|_{V_i} \in \mathrm{DA}(V_i)_{\underline{N}}$ lies in $\mathrm{DA}(V_i)^{\mathrm{dual}}_{\underline{N}}$, hence the Hyodo–Kato cohomology restricts to the datum of a $(\varphi, \underline{N})$-isocrystal over V_i. We may even assume that each tube $]V_i[$ is Stein, so that the functor ε^* corresponds to the functor "forgetting the monodromy operators" (there are no non-trivial elements in $\mathrm{Ext}^1(\mathbb{1}, \mathbb{1}(1))$). □

Remark 7. There is an ℓ-adic version of the realizations above, giving rise to a "relative Weil–Deligne" realization functor for rigid varieties, which compares to the relative ℓ-adic étale cohomology of the generic fibers via an "ℓ-adic Hyodo–Kato isomorphism" $\mathcal{E}(R\Gamma_{\text{ét}}(\Psi\mathcal{X}/\mathfrak{S}, \mathbb{Q}_\ell)) \simeq R\Gamma_{\text{ét}}(\mathcal{X}/\mathcal{S})$. Note that in this case, the functor $\mathcal{E}\colon \mathrm{Sh}(\mathfrak{S}, \mathbb{Q}_\ell)_{\underline{N}} \to \mathrm{Sh}(\mathcal{S}, \mathbb{Q}_\ell)$ is **not** described as "forgetting the monodromy": even in the absolute case, it corresponds to embedding representations of $\mathbb{Z}_\ell(1) \rtimes \mathrm{Gal}(k)$ as Galois representations of the local field. The ℓ-adic version of Corollary 6 gives in particular a relative version of Grothendieck's monodromy theorem and some constructibility results of the ℓ-adic cohomology with geometric coefficients (cfr. [11]).

Remark 8. Following Drinfeld [6], the category of $(\varphi, \underline{N})$-isocrystals over $\mathfrak{S}_k$ can be described as $\mathrm{Perf}(\mathcal{W}_{\mathbb{Q}_p}/\mathcal{G})^{h\varphi}$ with $\mathcal{W}$ a weakly terminal semi-perfect frame over $\mathfrak{S}_k$ and $\mathcal{G}$ a groupoid acting on it, e.g. in case $\mathfrak{S}_k$ is smooth, then $\mathcal{W} = W(\mathfrak{S}_k^{\mathrm{Perf}})$ and the functor $\mathcal{E}\colon \mathrm{Perf}(\mathcal{W}_{\mathbb{Q}_p}/\mathcal{G})^{h\varphi}_{\underline{N}} \to \mathrm{Perf}(\mathcal{S})$ then becomes analogous to the absolute base change functor $\mathcal{E}\colon \underline{\mathcal{D}}_{\varphi,N}(K_0) \to \mathcal{D}(K)$.

Using the universal property of our Hyodo–Kato coefficients (with respect to extending rigid cohomology to unipotent motives), we also obtain a compatibility result with the (relative) "log-rigid" approach à la Shiho [19], as follows.

Theorem 9. *Let $\mathfrak{S}$ be a p-adic formal scheme with semi-stable reduction. Endow it with the standard compactifying log-structure, and let* $\mathrm{logDA}^{uni}(\mathfrak{S})$ *be the subcategory of* $\mathrm{logDA}(\mathfrak{S})$ *spanned under colimits by the image of* $\mathrm{DA}(\underline{\mathfrak{S}}) \to \mathrm{logDA}(\mathfrak{S})$.

1. *(see [17]) The natural functor* $\mathrm{logDA}^{uni}(\mathfrak{S}) \to \mathrm{RigDA}^{\mathfrak{S}-uni}(\mathcal{S})$ *is an equivalence.*
2. *There is a log-rigid realization*
$$\mathrm{logDA}(\mathfrak{S}) \to \mathrm{logRigCoeff}^{\mathrm{op}}$$
where the target category is $(\lim_{\mathfrak{T}} \mathrm{QCoh}(\underline{\mathfrak{T}}_{\mathbb{Q}_p}))^{h\varphi}$ *as $\mathfrak{T}$ varies among log-frames over $\mathfrak{S}_k$. Locally on $\mathfrak{S}$, dualizable coefficients can be described as* $\mathrm{Perf}(\underline{\mathcal{W}}_{\mathbb{Q}_p}/\mathcal{G})^{h\varphi}$ *for some weakly-terminal semi-perfect log-frame $\mathcal{W}$.*
3. *There is a natural functor* HKCoeff $\to$ logRigCoeff *which restricts to an equivalence between the dualizable objects on each stratum of the natural stratification.*

Example 10. In the absolute case $\mathfrak{S} = \mathcal{O}_K^{\times}$ we have $\mathfrak{S}_k = k^0$ (the log-point) and $\mathcal{W} = \mathrm{Spf} W(k)[[T^{1/p^\infty}]]$ whose p-adic generic fiber is the "open preperfectoid disc" $\widehat{\mathbb{B}}^{\circ}$ over K_0. The equivalence of point (3) translates into $\mathrm{Perf}(\widehat{\mathbb{B}}^{\circ}/\mathcal{G})^{h\varphi} \simeq \mathrm{Perf}_{\varphi,N}(K_0)$.

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Non-Archimedean methods in complex geometry

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(joint work with Sébastien Boucksom)

Non-Archimedean geometry is often used to study problems of arithmetic nature, in which case one typically works over a non-Archimedean field of positive residue characteristic. As explained in my lecture, non-Archimedean geometry (in the sense of Berkovich) over well chosen fields of residue characteristic zero can be used to study degenerations in complex geometry.

1. Analytifications

To any scheme X of finite type over $\mathbb{C}$ is associated a complex analytic space X^h. For example, when X is smooth and projective, X^h is a compact Kähler manifold. The construction of course uses the standard absolute value $|\cdot|_\infty$ on $\mathbb{C}$.

Berkovich [1] gave an analogous construction of a space X^{an} when X is instead is defined over a non-Archimedean field. When X is proper, X^{an} is a compact Hausdorff space. This construction can be used to study certain degenerations. For example, suppose $X \to \mathbb{D}^*$ is a projective, holomorphic family over the punctured unit disc $\mathbb{D}^* \subset \mathbb{C}$ that is meromorphic at $t = 0 \in \mathbb{D}$. We can then view X as a (possibly) degenerating family X_t, $t \in \mathbb{D}^*$, of compact Kähler manifolds, and the asymptotic geometry of X_t as $t \to 0$ can be studied through the analytification $X^{\mathrm{an}}_{\mathbb{C}((t))}$ as an approach to deep conjectures in Mirror Symmetry.

However, in my lecture, the focus was on the case when the ground field is $\mathbb{C}$ equipped with the *trivial* absolute value $|\cdot|_0$. In this case we write $X^{\mathrm{na}} := X^{\mathrm{an}}$ for the analytification of a (say) smooth projective complex variety X. Note that X^{na} is an object fundamentally of algebro-geometric nature as any field can be equipped with the trivial absolute value. The trivial equality

$$\lim_{\rho\to 0+} |\cdot|^{\infty} = |\cdot|_0$$

pointwise on $\mathbb{C}$ can be viewed as a degeneration and in fact plays a key (albeit hidden) role in what follows.

2. Calabi's question

Consider a smooth complex projective variety X of dimension $n \geq 1$, together with an ample line bundle L. Let $\mathcal{H}$ be the set of smooth positive metrics on the holomorphic line bundle L^h on X^h. To any such metric ϕ we can associate a curvature (Kähler) form $dd^c\phi$ on X^h in the first Chern class $c_1(L^h)$ of L; in fact, the curvature operator induces a bijection between $\mathcal{H}/\mathbb{R}$ and the set of Kähler forms/metrics in $c_1(L^h)$.

In the 1950's, Calabi asked whether one can associate a *canonical* Kähler form $\omega \in c_1(L^h)$ to any polarized smooth complex variety (X, L) (or more generally, any compact Kähler manifold with a fixed Kähler class). For example, ω is a *constant scalar curvature Kähler* (cscK) metric if there exists a constant $c \in \mathbb{R}$ such that

$$\frac{\mathrm{vol}(B_\omega(x,r))}{\mathrm{vol}(B_{\mathbb{C}^n}(0,r))} = 1 + cr^2 + O(r^3)$$

as $r \to 0+$ for any $x \in X^h$. For brevity we say that (X, L) is cscK if there exists a cscK metric in $c_1(L^h)$.

When $n = 1$, X^h is a compact Riemann surface, and cscK metrics correspond to the spherical, flat, or hyperbolic metric, depending on the genus. In particular, (X, L) is cscK for any ample line bundle L. This is no longer true in dimension $n \geq 2$: Matsushima and Lichnerowicz proved in the late 1950's that (X, L) being cscK implies that the automorphism group $\mathrm{Aut}(X, L)$ is reductive. In particular, if X the blowup of $\mathbb{P}^2$ at a point, then (X, L) is not cscK for any L.

3. The Yau–Tian–Donaldson conjecture

With increasing precision, it was successively conjectured by Yau [11], Tian [9], and Donaldson [7] that (X, L) being cscK (which is a transcendental condition) should be equivalent to a stability condition on (X, L) of algebro-geometric nature.

There has been a tremendous amount of work on the Yau–Tian–Donaldson conjecture, especially in the case when X is a smooth Fano variety, and $L = -K_X$ is the anticanonical bundle; a cscK metric is then the same thing as a *Kähler–Einstein metric*. Suffice it to say that it was finally proved by Chen–Donaldson–Sun [5] and Tian [10] that (X, L) is cscK iff (X, L) is *K-polystable*, a condition on certain degenerations on (X, L) known as *test configurations* described below.

In [4] we prove the following version of the Yau–Tian–Donaldson conjecture for cscK metrics in the general (not necessarily Fano) case.

Theorem. A polarized smooth projective complex variety (X, L) is cscK iff (X, L) is $\hat{\mathrm{K}}$-polystable.

Below I explain the rough meaning of $\hat{\mathrm{K}}$-polystability, and give a very rough idea of the proof, but let me already now point out that $\hat{\mathrm{K}}$-polystability is formulated in terms of the analytification $(X^{\mathrm{na}}, L^{\mathrm{na}})$ of (X, L) with respect to the trivial absolute value on $\mathbb{C}$, and is therefore ultimately of algebro-geometric nature.

4. Ideas of proof

For simplicity, let me assume that the automorphism group $\mathrm{Aut}(X)$ is finite. We try to use a variational approach to construct a cscK metric. To this end, we use the *Mabuchi functional* $M\colon \mathcal{H} \to \mathbb{R}$; this has the property that $M'(\phi) = 0$ iff $dd^c\phi$ is a cscK metric. It turns out that (by convexity) that a critical point must be a minimum. Instead of looking directly for a minimum on $\mathcal{H}$, we extend M to a functional $M\colon \widehat{\mathcal{H}} \to \mathbb{R}\cup\{+\infty\}$ defined on the metric completion of $\mathcal{H}$ with respect to a natural *Darvas metric*; all of this is done using complex pluripotential theory. Due to deep work of many people, it is known that (X, L) being cscK is equivalent to M admitting a minimum on $\widehat{\mathcal{H}}$.

To explain the stability notions involved, let us first return to the notion of a test configuration for (X, L). Roughly speaking, this is a $\mathbb{C}^\times$-equivariant, flat, normal, projective family $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ which is trivial over $\mathbb{P}^1\backslash\{0\}$, and such that the fiber over $1 \in \mathbb{P}^1$ can be identified with (X, L). It serves as a trivially valued analogue of a model of (X, L) over the valuation ring, and induces a metric on L^{na}. Letting $\mathcal{H}^{\mathrm{na}}$ be the set of metrics associated to test configurations, there is a natural functional $M^{\mathrm{na}} \to \mathbb{Q}$ defined in terms of intersection numbers, and we say that (X, L) is *K-stable*[1] if $\mathrm{M}(\varphi) \geq 0$ for all $\varphi \in \mathcal{H}^{\mathrm{na}}$, with equality iff φ is associated to the trivial test configuration $(X, L) \times \mathbb{P}^1$.

Now, in [3] we developed (global) pluripotential theory over a trivially valued field, much in the same way as in the complex analytic case. This allows us to equip $\mathcal{H}^{\mathrm{na}}$ with a Darvas metric, and extend the Mabuchi functional as a functional $M^{\mathrm{na}}\colon \widehat{\mathcal{H}}^{\mathrm{na}} \to \mathbb{R}\cup\{+\infty\}$ on the metric completion. We say that (X, L) is *$\hat{K}$-stable* if $M(\varphi) \geq 0$ for all $\varphi \in \widehat{\mathcal{H}}^{\mathrm{na}}$, with equality iff φ is associated to the trivial test configuration.

To prove the Yau–Tian–Donaldson conjecture, we must prove that M admits a minimizer on $\widehat{\mathcal{H}}$ iff (X, L) is $\hat{\mathrm{K}}$-stable in the sense above. This is done in two steps, which using various oversimplifications can be described as follows.

First, the space $\widehat{\mathcal{H}}$ is uniquely geodesic and in fact enjoys a stronger property called *Busemann convexity.* Moreover, M is convex along geodesics. Using this, one proves that the existence of a minimum for M on $\widehat{\mathcal{H}}$ is equivalent to

[1] K-polystability is equivalent to K-stability when $\mathrm{Aut}(X)$ is finite.

$\lim_{t\to\infty} t^{-1}M(\phi_t) > 0$ for every non-trivial geodesic ray $\{\phi_t\}_{t\geq 0}$ emanating from some given point.

Second, every $\varphi \in \widehat{\mathcal{H}}^{\mathrm{na}}$ defines a geodesic ray $\{\phi_t\}_{t\geq 0}$ in $\widehat{\mathcal{H}}$, and we have

$$\lim_{t\to\infty} \frac{1}{t} M(\phi_t) = \begin{cases} M^{\mathrm{na}}(\varphi) & \text{for all such rays} \\ +\infty & \text{for all other rays} \end{cases}$$

The proof of the equality in the second case is due to Chi Li [8], who also proved an inequality in the first case. The proof of the equality in the first case lies deeper, and ultimately relies on a complex-analytic regularity theorem by Berman–Demailly [2] and Di Nezza–Trapani [6].

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Completed cohomology of the Drinfeld tower for $\mathbf{GL_2(Q_p)}$

Gabriel Dospinescu

(joint work with Pierre Colmez, Wiesława Nizioł and Juan Esteban Rodriguez Camargo)

Let p be an odd prime and let $\mathrm{Gal}_{\mathbf{Q}_p}$ be the absolute Galois group of $\mathbf{Q}_p$. Let $G = \mathrm{GL}_2(\mathbf{Q}_p)$, let D be a quaternion division algebra over $\mathbf{Q}_p$ and fix a large enough finite extension L of $\mathbf{Q}_p$, serving as coefficient field for our representations.

If H is a p-adic Lie group let $\mathrm{Ban}(H)$ be the category of admissible unitary L-Banach space representations of H. If Γ is a topological group let $\mathrm{Ban}(H)^{\Gamma}$ be the category of Γ-equivariant objects of $\mathrm{Ban}(H)$. Finally, let $\mathrm{Irr}(\mathrm{Gal}_{\mathbf{Q}_p})$ be the set of isomorphism classes of continuous 2-dimensional absolutely irreducible L-representations of $\mathrm{Gal}_{\mathbf{Q}_p}$.

The classical Jacquet-Langlands correspondence relates certain irreducible smooth representations of G over L (the discrete series) and the irreducible smooth representations of $D^{\times}$ over L. We are interested here in the Banach, locally analytic and modulo p versions of such a correspondence. Recall that the p-adic local Langlands correspondence establishes a bijection $V \mapsto \Pi(V), \Pi \mapsto V(\Pi)$ between $\mathrm{Irr}(\mathrm{Gal}_{\mathbf{Q}_p})$ and the set $\widehat{G}_{\mathrm{ss}}$ of isomorphism classes of absolutely irreducible objects of $\mathrm{Ban}(G)$ which are supersingular (i.e. not isomorphic to a subquotient of a unitary parabolic induction). There were several hints that this correspondence can be realised geometrically in the p-adic étale cohomology of various spaces, as we will briefly recall.

Let $X_{\infty} = X(p^{\infty})$ be the inverse limit (in the category of $\mathbf{Q}$-schemes) of the modular curves X_n of level $\Gamma(p^n)$ as $n \to \infty$. Then $H^1_{\mathrm{et}}(X_{\infty} \otimes \overline{\mathbf{Q}}, L)$ is the completed cohomology of the tower of modular curves $(X_n)_{n\geq 1}$ and it belongs to $\mathrm{Ban}(G)^{\mathrm{Gal}_{\mathbf{Q}}}$. Emerton proved in [2] that under mild hypothesis on the absolutely irreducible odd representation $\rho : \mathrm{Gal}_{\mathbf{Q}} \to \mathrm{GL}_2(L)$, unramified outside of p one has an isomorphism in $\mathrm{Ban}(G)$

$$\mathrm{Hom}_{\mathrm{Gal}_{\mathbf{Q}}}(\rho, H^1_{\mathrm{et}}(X_{\infty} \otimes \overline{\mathbf{Q}}, L)) \simeq \Pi(\rho|_{\mathrm{Gal}_{\mathbf{Q}_p}}).$$

The scheme $X_{\infty} \otimes_{\mathbf{Q}} \mathbf{C}_p$ has an "analytification" X_{∞}^{an}, which is a perfectoid space over $\mathbf{C}_p$ (as a proétale sheaf on perfectoid $\mathbf{C}_p$-algebras it is the inverse limit of the sheaves represented by the analytifications of $X_n \otimes_{\mathbf{Q}} \mathbf{C}_p$). Scholze constructed a G-equivariant Hodge-Tate period map $\pi_{\mathrm{HT}} : X_{\infty}^{\mathrm{an}} \to \mathbf{P}^1$ to the adic projective line over $\mathbf{C}_p$. Let $\Omega = \mathbf{P}^1 \setminus \mathbf{P}^1(\mathbf{Q}_p)$ be Drinfeld's upper half-plane, an open rigid subvariety of $\mathbf{P}^1$ stable under the action of G. The inverse image $\pi_{\mathrm{HT}}^{-1}(\Omega)$ is the disjoint union of finitely many copies of a perfectoid space LT_{∞} with a continuous G-action, the Lubin-Tate space at infinite level. The map $\pi_{\mathrm{HT}} : \mathrm{LT}_{\infty} \to \Omega$ is a proétale $D^{\times}$-torsor and we let $\mathrm{Dr}_n = \mathrm{LT}_{\infty}/(1 + p^n \mathcal{O}_D)$. As n increases we obtain a tower of $G \times D^{\times}$-equivariant finite étale coverings of $\coprod_{n\in\mathbf{Z}} \Omega$, defined over the completion of $\mathbf{Q}_p^{\mathrm{nr}}$, and endowed with a Weil descent datum. For simplicity, we will pretend that they are defined over $\mathbf{Q}_p$ (the spaces $\mathrm{Dr}_n/p^{\mathbf{Z}}$ are defined over $\mathbf{Q}_p$ and one can always reduce our problems to these spaces, by twisting by unramified characters). In our previous work [1] we proved that for any $V \in \mathrm{Irr}(\mathrm{Gal}_{\mathbf{Q}_p})$ which is nice (i.e. de Rham, with Hodge-Tate weights $0, 1$ and non trianguline) we have

$$\mathrm{Hom}_{\mathrm{Gal}_{\mathbf{Q}_p}}(V, \varinjlim_{n} H^1_{\mathrm{et}}(\mathrm{Dr}_n, L)) \simeq \mathrm{JL}^{\mathrm{cl}}(V) \otimes_L \Pi(V)^*,$$

for some (finite dimensional) irreducible smooth L-representation $\mathrm{JL}^{\mathrm{cl}}(V)$ depending only on $D_{\mathrm{pst}}(V)$ and constructed using the classical Jacquet-Langlands and local Langlands correspondence.

We would like to have a similar picture for $\Pi(V)$ for *any* $V \in \mathrm{Irr}(\mathrm{Gal}_{\mathbf{Q}_p})$. Let

$$H_\infty = H^1_{\mathrm{et}}(\mathrm{LT}_\infty, L).$$

Work of Scholze shows that there is a functor

$$S^1 : \mathrm{Ban}(G) \to \mathrm{Ban}(D^\times)^{\mathrm{Gal}_{\mathbf{Q}_p}},\ \Pi \mapsto S^1(\Pi) = \mathrm{Hom}_G(\Pi^*, H_\infty).$$

This functor has obvious integral and torsion versions that we will use implicitly.

Many of the results in the next theorem (in joint work with Juan Esteban Rodriguez Camargo) were known under mild hypotheses on the reduction mod p of Π, by work of Hansen-Mann [3], Hu-Wang [4], [5] and Ludwig [6]. In the statement of the theorem "almost" means "up to finite dimensional representations" (i.e. we implicitly work with the composition of S^1 and the natural functor to the quotient of $\mathrm{Ban}(D^\times)^{\mathrm{Gal}_{\mathbf{Q}_p}}$ by the category of finite dimensional representations). Most results hold for $\mathrm{GL}_2(F)$, with $F/\mathbf{Q}_p$ finite. The proof heavily uses the work of Mann [7] on 6-functor formalisms for rigid analytic varieties.

Theorem 1. *(1) The functor S^1 is almost exact, in particular almost compatible with reduction modulo p.*

(2) There is a natural almost isomorphism $S^1(\Pi)^* \simeq \mathrm{Hom}_G(\Pi, H_\infty(1))$.

(3) Letting GK *be the Gelfand-Kirillov (or canonical) dimension, we have*

$$\mathrm{GK}(S^1(\Pi)) \leq \mathrm{GK}(\Pi),$$

with equality for Π Cohen-Macaulay, in particular $\mathrm{GK}(S^1(\Pi)) = 1$ for $\Pi \in \widehat{G}_{\mathrm{ss}}$ (and so $S^1(\Pi)$ is infinite-dimensional).

(4) There is a natural almost isomorphism

$$S^1(\Pi)^{\mathrm{la}} \simeq \mathrm{Hom}_G((\Pi^{\mathrm{la}})^*, H^1_{\mathrm{proet}}(\mathrm{LT}_\infty, L)).$$

For $\Pi \in \widehat{G}_{\mathrm{ss}}$ the representations $S^1(\Pi)^{\mathrm{la}}$ and Π^{la} have the same infinitesimal character.

The next result (joint with Colmez and Nizioł) is proved using results of Boston, Lenstra and Ribet, as well as an "analytic continuation" argument using patching. Again, the result was known in many cases by previous work of many people.

Theorem 2. *For any $\Pi \in \widehat{G}_{\mathrm{ss}}$ there is a $\mathrm{JL}(\Pi) \in \mathrm{Ban}(D^\times)$ such that*

$$S^1(\Pi) \simeq V(\Pi) \otimes_L \mathrm{JL}(\Pi),$$

in other words $S^1(\Pi)$ is $V(\Pi)$-isotypic as $\mathrm{Gal}_{\mathbf{Q}_p}$-representation.

Let

$$\tilde{H}_{\mathrm{Dr}} = (\varprojlim_k \varinjlim_n H^1_{\mathrm{et}}(\mathrm{Dr}_n, \mathcal{O}_L/p^k))[1/p]$$

be the completed cohomology of the tower $(\mathrm{Dr}_n)_{n\geq 1}$. Contrary to the completed cohomology of the modular curves this is much smaller than $H^1_{\mathrm{et}}(\mathrm{LT}_\infty, L)$ and does not have any reasonable finiteness property since each $H^1_{\mathrm{et}}(\mathrm{Dr}_n, \mathcal{O}_L/p^k)$ is huge (while $H^1_{\mathrm{et}}(X_n \otimes \mathbf{C}_p, \mathcal{O}_L/p^k)$ is finite!). Another key difficulty compared to modular curves is that we don't know whether $H^2_{\mathrm{et}}(\mathrm{Dr}_n, \mathcal{O}_L/p^k)$ is 0 or huge. One can still compare $\tilde{H}_{\mathrm{Dr}}$ and $Z := H^1_{\mathrm{et}}(\mathrm{LT}_\infty, \mathcal{O}_L)$ as follows:

$$\tilde{H}_{\mathrm{Dr}} = (\varprojlim_k (Z/p^k Z)^{D^\times-\mathrm{sm}})[1/p].$$

Our most delicate result (joint with Colmez and Niziol̀) is the following factorisation theorem, where we say that V is generic if its reduction modulo p (semi-simplified) is not the direct sum of two characters whose quotient is 1 or the mod p cyclotomic character (there is a more technical notion of being very generic which we don't recall here).

Theorem 3.

(1) For any generic $V \in \mathrm{Irr}(\mathrm{Gal}_{\mathbf{Q}_p})$ we have a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Gal}_{\mathbf{Q}_p}}(V, \tilde{H}_{\mathrm{Dr}}) \simeq \Pi(V)^* \widehat{\otimes}_L \mathrm{JL}(\Pi(V)),$$

the tensor product being p-adically completed.

(2) If V is very generic then $\mathrm{JL}(\Pi(V))$ is irreducible or has finite dimensional locally algebraic vectors (this happens if and only if V is nice), the quotient being irreducible. Moreover there is a natural isomorphism

$$\mathrm{Hom}_{D^\times \times \mathrm{Gal}_{\mathbf{Q}_p}}(S^1(\Pi(V)), H_\infty) \simeq \Pi(V)^*.$$

Towards the end of the talk we discussed variants modulo p of these results as well as some speculations on what happens for $\mathrm{GL}_2(F)$.

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Moduli of truncated local G-shtukas

Eva Viehmann
(joint work with Torsten Wedhorn)

In [1], Torsten Wedhorn and the speaker develop a general formalism of frames that allows to study moduli spaces of local G-shtukas and displays and of their truncations, as well as of prismatic displays and F-gauges. In this talk I explain as a motivation and introduction the easiest case, namely moduli of truncated local G-shtukas.

Let G be a reductive group over a finite field $\mathbb{F}_q$. As a first approach consider the prestacks assigning to any $\mathbb{F}_q$-algebra R the Hecke triples over R, i.e.

$$\{(\mathcal{E}_1, \mathcal{E}_2, \alpha) \mid \mathcal{E}_1, \mathcal{E}_2 \text{ are } G\text{-bundles on } \mathrm{Spec} R[\![z]\!], \alpha : \mathcal{E}_{1,R(\!(z)\!)} \xrightarrow{\cong} \mathcal{E}_{2,R(\!(z)\!)}\}$$

resp. local G-shtukas over R,

$$\{(\mathcal{E}, \alpha) \mid \mathcal{E} \text{ a } G\text{-bundle on } \mathrm{Spec} R[\![z]\!], \alpha : \varphi^*\mathcal{E}_{R(\!(z)\!)} \xrightarrow{\cong} \mathcal{E}_{R(\!(z)\!)}\}$$

where $\varphi : R(\!(z)\!) \to R(\!(z)\!)$ is the Frobenius with $\sum_i a_i z^i \mapsto \sum_i a_i^q z^i$.

Let R^{Hck} be the scheme obtained by gluing two copies of Spec $R[\![z]\!]$ along Spec $R(\!(z)\!)$ and let

$$R^{\mathrm{Sht}} := \mathrm{colim}(\ \mathrm{Spec} R[\![z]\!] \underset{\sigma}{\overset{\tau}{\rightrightarrows}} R^{\mathrm{Hck}}\ ).$$

Then we have an equivalence between G-bundles over R^{Hck} resp. R^{Sht} and Hecke triples resp. local G-shtukas over R. However, these descriptions do not have immediate generalizations describing corresponding truncations of any level N. We introduce as a variant of R^{Hck} the so-called ∞-truncated Hecke stack over R defined as

$$R^{\mathrm{Hck},\infty} := [\mathbb{G}_m \backslash \mathrm{Spec}(R[\![z]\!][t,u]/(tu - z))]$$

where $\mathbb{G}_m$ acts on t with degree -1 and on u with degree 1. Then the locus in this stack where t (or u) is invertible is isomorphic to Spec$((R[\![z]\!])$, yielding an open embedding $R^{\mathrm{Hck}} \to R^{\mathrm{Hck},\infty}$.

Proposition 1. *Restriction via this map induces a fully faithful functor*

$$Bun_G(R^{Hck,\infty}) \longrightarrow Bun_G(R^{Hck})$$

which is an equivalence if R is a field.

Its essential image can be understood by considering the complement of the image of R^{Hck}, which is isomorphic to $B\mathbb{G}_m$. Considering the fiber at this point assigns to G-bundles over $R^{\mathrm{Hck},\infty}$ a type map to $W\backslash X_*(T)$ that is locally constant on R. The above essential image then consist of G-bundles which are locally of some fixed type $\mu \in W\backslash X_*(T)$.

We can now define for any $N \in \mathbb{N}$ the N-truncated Hecke stack

$$R^{\mathrm{Hck},N} := \left[\mathbb{G}_m \backslash \mathrm{Spec}(R[\![z]\!][t,u]/(tu - z, z^N))\right]$$

and, using an analogous coequalizer construction as above, also the N-truncated local shtuka stack $R^{\mathrm{Sht},N}$.

We prove that the resulting notion of truncated local G-shtukas of level N is compatible with

- natural truncation maps from a higher to a lower level
- the description of truncations of level 1 via G-zips and
- the notion of truncations of local G-shtukas of any level N over an algebraically closed field k as being $G(\mathbb{F}_q[\![z]\!])$-φ-conjugacy classes of elements of $G(\mathbb{F}_q(\!(z)\!))/K_N$ where K_N is the kernel of the reduction map $G(\mathbb{F}_q[\![z]\!]) \to G(\mathbb{F}_q[\![z]\!]/(z^N))$.

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The tame fundamental group of a rigid space

Katharina Hübner
(joint work with Piotr Achinger, Marcin Lara, Jakob Stix)

We started our project (which is still in progress) with the aim of defining a well behaved fundamental group for a rigid space X over an algebraically closed field k. If X is projective, it comes from an algebraic variety over k and its étale fundamental group is the same as the fundamental group of this variety. In particular, the étale fundamental group is topologically finitely generated, which can be seen using Grothendieck's Lefschetz theorem (see [1], Corollary 7.2). If X is only proper over k, we expect the étale fundamental group to be topologically finitely generated but this conjecture is still open.

One reason why it is so difficult to study the étale fundamental group of a proper rigid space is that finite generation cannot be proved locally. Unless we work in characteristic 0 (i.e. over $\mathbb{C}$) the étale fundamental group of a non-proper rigid space tends to be rather huge. Moreover, there is no Chow's lemma that could help us relate to projective spaces.

We approached the problem from a different direction by replacing the étale fundamental group by a tame fundamental group. According to the general philosophy of tame fundamental groups we expect fundamental groups to become topologically finitely generated once we impose a tame ramification condition on étale morphisms. Moreover, we expect the Künneth formula to hold and and the tame fundamental group to be invariant under base change to a bigger algebraically closed field.

For an étale morphism $f : Y \to X$ of rigid spaces over (k, k^+) there is a natural notion of tameness. Namely, we require all residue field extensions $k(y)/k(x)$ for points $y \in Y$ and $x \in X$ with $f(y) = x$ to be tamely ramified. If $\mathcal{X}$ is a formal model of X this tameness condition can roughly be thought of as requiring tameness along the special fiber of $\mathcal{X}$. Accodingly we obtain a tame fundamental group $\pi_1^t(X, \bar{x})$.

The above notion of tameness turns out to be not strong enough. Any finite étale cover of the special fiber of $\mathcal{X}$ lifts to a finite étale cover of $\mathcal{X}$ and hence gives rise to a finite tame cover of X. If X is not proper, there are infinitely many covers of this kind and we would actually like to exclude those. We realize this idea by studying finite étale covers of X that induce tame covers over $\bar{X}$, Huber's universal compactification. Being tame over $\bar{X}$ is a stronger condition than being tame over X. In terms of formal models tameness over $\bar{X}$ comes down to tameness along the special fiber and also along a compactification of the special fiber.

The tame fundamental group we were seeking for can now be defined as the fundamental group of the Galois category of of finite tame covers of $\bar{X}$. We also speak of the tame fundamental group of X relative to k and use the notation

$$\pi_1^{t/k}(X, \bar{x}) := \pi_1^t(\bar{X}, \bar{x}).$$

Our main therem establishes finite generation of $\pi_1^{t/k}(X, \bar{x})$:

Theorem. *Let X be a connected, quasi-compact and quasi-separated rigid-analytic space over the algebraically closed affinoid field (k, k^+). Then $\pi_1^{t/k}(X, \bar{x})$ is topologically finitely generated.*

The proof strategy comes down to reducing the topologically finite generation of $\pi_1^{t/k}(X, \bar{x})$ to the topologically finite generation of the tame fundamental group of a variety over the residue field of k^+. As a first step we reduce to the case where X is smooth and has a semistable model $\mathcal{X}$. In this situation the category of finite tame morphisms $X \to Y$ is equivalent to finite Kummer étale morphisms $\mathcal{Y} \to \mathcal{X}$. We are actually interested in finite tame morphisms $\bar{Y} \to \bar{X}$ of the respective universal compactifications. These can be viewed as finite tame morphisms $X \to Y$ with an extra tameness condition for points in $\bar{Y} \setminus Y$. Under the equivalence with Kummer étale morphisms $\mathcal{Y} \to \mathcal{X}$ this is translated to the condition that the morphism of special fibers $\mathcal{Y}_s \to \mathcal{X}_s$ be tame over the residue field $k^{\succ}$ of k. In addition finite Kummer étale morphisms $\mathcal{Y} \to \mathcal{X}$ correspond to finite Kummer étale morphisms $\mathcal{Y}_s \to \mathcal{X}_s$ of the special fibers endowed with the pullback log structure from $\mathcal{X}$ and $\mathcal{Y}$.

As a result we are now studying finite Kummer étale morphisms of log schemes over $k^{\succ}$ such that the underlying morphism of schemes is tame over $k^{\succ}$. The log schemes we are confronted with are saturated but not necessarily fine. We developed the underlying foundations in [2]. In the end we use a version of van Kampen's theorem to chop the log scheme into pieces along its log stratification. In this way we reduce to the case of schemes where topologically finite generation of the the tame fundamental group is known.

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Sen Theory via moduli spaces – and stacks of equivariant vector bundles on the Fargues–Fontaine curve

Eugen Hellmann

(joint work with Ben Heuer)

1. Motivation

Given a reductive group G over $\mathbb{Q}_p$, the categorical approach to the p-adic local Langlands program is looking for a (fully) faithful functor

$$\mathrm{Rep}^{\mathrm{la}}_{\mathbb{Q}_p} G \to \mathrm{Coh}(\mathrm{LS}_{X_{\mathrm{FF}},\check{G}})$$

from the category of locally analytic G-representations on $\mathbb{Q}_p$-vector spaces to the category of coherent sheaves on the stack of equivariant $\check{G}$-bundles on the Fargues–Fontaine curve X_{FF}. This is only a very rough formulation ignoring many issues, in particular appropriate finiteness conditions and the derived nature of the correspondence.

On the representation theoretic side one can act on a locally analytic representation π by mapping it to $\pi \otimes_{\mathbb{Q}_p} V_\lambda$, equipped with diagonal G-action, where V_λ is an irreducible algebraic G-representations. The aim of this talk is to conjecturally realize this action as some kind of "Hecke"-action on the Galois side.

We begin by explaining the objects on the Galois side.

- given G we have a split dual group $\check{G}$ over $\mathbb{Q}_p$ together with an action $\rho : \mathrm{Gal}_{\mathbb{Q}_p} \to \mathrm{Aut}_{\mathbb{Q}_p}(G)$.
- the Fargues–Fontaine curve X_{FF} is an adic space over $\mathbb{Q}_p$ equipped with a (continuous) action of $\mathrm{Gal}_{\mathbb{Q}_p}$.
- the group object $\check{G}_{\mathrm{FF}} = \check{G} \times X_{\mathrm{FF}}$ over X_{FF} comes equipped with a lift of the $\mathrm{Gal}_{\mathbb{Q}_p}$-action by twisting the canonical lift (the structure morphism $X_{\mathrm{FF}} \to \mathrm{Spa}(\mathbb{Q}_p)$ is $\mathrm{Gal}_{\mathbb{Q}_p}$-equivariant) by ρ.
- we define the stack (on the category of rigid analytic spaces over $\mathbb{Q}_p$)

$$\mathrm{LS}_{X_{\mathrm{FF}},\check{G}} : S \mapsto \left\{ \begin{array}{c} \check{G}_{\mathrm{FF}}\text{-bundles on } S \times X_{\mathrm{FF}} \text{ together with} \\ \text{a continuous lift of the } \mathrm{Gal}_{\mathbb{Q}_p}\text{-action} \end{array} \right\}.$$

Example/Proposition 1: *Let $G = T$ be a torus. And let $\widehat{T}$ denote the rigid analytic space parametrizing continuous characters of $T(\mathbb{Q}_p)$. Then*

$$\mathrm{LS}_{X_{\mathrm{FF}},\check{T}} \cong \widehat{T}/\check{T}_0,$$

where $\check{T}_0$ is the multiplicative group with $X^*(\check{T}_0) = X_*(T)_{\mathrm{Gal}_{\mathbb{Q}_p}}$, and the stack quotient is for the trivial action.

2. Stacks related to Sen Theory

Recall that the Fargues–Fontaine curve has a (unique) $\mathrm{Gal}_{\mathbb{Q}_p}$-stable point $i_\infty : \{\infty\} \hookrightarrow X_{\mathrm{FF}}$. This allows us to define the following variants:

$$\mathrm{LS}_{B_{\mathrm{dR}}^+,\check{G}} : S \mapsto \{\mathrm{Gal}_{\mathbb{Q}_p}\text{-equivariant } (\check{G}_{\mathrm{FF}})^{\widehat{}}_{\infty}\text{-bundles on } (S \times X_{\mathrm{FF}})^{\widehat{}}_{\infty}\}$$

$$\mathrm{LS}_{C,\check{G}} : S \mapsto \{\mathrm{Gal}_{\mathbb{Q}_p}\text{-equivariant } i_\infty^*\check{G}_{\mathrm{FF}}\text{-bundles on } (S \times \{\infty\})\}.$$

Note that in the case $G = \mathrm{GL}_n$ these stacks just parametrize families of semilinear $\mathrm{Gal}_{\mathbb{Q}_p}$-representations on projective B_{dR}^+-modules (respectively C-vector spaces) of rank n.

By definition we obtain canonical maps

$$\mathrm{LS}_{X_{\mathrm{FF}},\check{G}} \to \mathrm{LS}_{B_{\mathrm{dR}}^+,\check{G}} \to \mathrm{LS}_{C,\check{G}} \to (\mathrm{Lie}\,\check{G}_\infty)/\!\!/\check{G}_\infty,$$

where $\check{G}_\infty$ is the reductive group over $\mathbb{Q}_p$ such that $(\check{G}_\infty)_C \cong i_\infty^* G_{\mathrm{FF}}$, equivariant for the $\mathrm{Gal}_{\mathbb{Q}_p}$-action. Moreover, $(\mathrm{Lie}\,\check{G}_\infty)/\!\!/\check{G}_\infty$ denotes the GIT quotient for the adjoint action. The last map generalizes the morphism (in the GL_n-case) that maps a semilinear $\mathrm{Gal}_{\mathbb{Q}_p}$-representation on a C-vector space to the characteristic polynomial of its Sen operator.

Example/Proposition 2: *Let $G = T$ be a torus.*
(i) The composition

$$\widehat{T} \to \mathrm{LS}_{X_{\mathrm{FF}},\check{T}} \to \mathrm{Lie}\,\check{T}_\infty = (\mathrm{Lie}\,T)^*$$

is given by mapping a character δ to its derivative at 1.
(ii) The canonical map $\mathrm{LS}_{B_{\mathrm{dR}}^+,\check{T}} \to \mathrm{LS}_{C,\check{T}}$ is an isomorphism.

Theorem 3 (H.–Heuer):
(i) The stacks

$$\mathrm{LS}_{C,\check{G}} \to (\mathrm{Lie}\,\check{G}_\infty)/\!\!/\check{G}_\infty \leftarrow (\mathrm{Lie}\,\check{G}_\infty)/\check{G}_\infty$$

over $(\mathrm{Lie}\,\check{G}_\infty)/\!\!/\check{G}_\infty$ are isomorphic étale locally on $(\mathrm{Lie}\,\check{G}_\infty)/\!\!/\check{G}_\infty$.
(ii) Let $G = \mathbb{G}_m$ and $\mathrm{Lie}\,\mathbb{G}_m = \mathbb{A}^1$. The stack $\mathrm{LS}_{C,\mathbb{G}_m} \to \mathbb{A}^1$ is an étale $\mathbb{G}_m$-gerbe. Its cohomology class is given by the image of the coordinate function on $\mathbb{A}^1$ under a canonical map

$$H^0_{\text{ét}}(\mathbb{A}^1, \mathbb{G}_a) \to H^2_{\text{ét}}(\mathbb{A}^1, \mathbb{G}_m).$$

Remark: In particular the H^2-class of the gerbe $\mathrm{LS}_{C,\mathbb{G}}$ does not come by pullback from the Brauer group of $\mathbb{Q}_p$!

The above Theorem has a variant for B_{dR}^+-representations. For simplicity we only describe the result in the case $G = \mathrm{GL}_n$. We consider the stack of formal logarithmic derivations

$$\mathrm{FLD}_n : \mathrm{Spa}\,A \mapsto \left\{ \begin{array}{c} \text{projective } A[\![t]\!]\text{-modules } \Lambda \text{ of rank } n \\ \text{together with a derivation } \partial : \Lambda \to \Lambda \text{ above } t\frac{t}{dt} \end{array} \right\}.$$

Taking the quotient by t and taking the characteristic polynomial yields maps

$$\mathrm{FLD}_n \to (\mathrm{Lie}\,\mathrm{GL}_n)/\mathrm{GL}_n \to (\mathrm{Lie}\,\mathrm{GL}_n)/\!\!/\mathrm{GL}_n.$$

Theorem 4 (H.–Heuer): *The stacks* $\mathrm{LS}_{B_{\mathrm{dR}}^+,\mathrm{GL}_n}$ *and* FLD_n *are isomorphic étale locally on* $(\mathrm{Lie}\,\mathrm{GL}_n)/\!\!/\mathrm{GL}_n$.

Remark: Unlike $(\mathrm{Lie}\,\mathrm{GL}_n)/\mathrm{GL}_n$ the stack FLD_n is not an Artin stack. It is pro-Artin, has representable diagonal (though the diagonal is not of finite type) and the formal completions along points of $(\mathrm{Lie}\,\mathrm{GL}_n)/\!\!/\mathrm{GL}_n$ can explicitly be described as the formal completions of smooth Artin stacks.

3. Hecke stacks

For simplicity we continue to assume $G = \mathrm{GL}_n$.
The stack $\mathrm{LS}_{B_{\mathrm{dR}}^+,\mathrm{GL}_n}$ has the variant

$$\mathrm{Hck}_{B_{\mathrm{dR}+},n} : \mathrm{Spa}\,A \mapsto \left\{ \begin{array}{c} \text{projective } A\hat{\otimes}_{\mathbb{Q}_p} B_{\mathrm{dR}}^+\text{-modules } \Lambda_1, \Lambda_2 \text{ of rank } n \\ \text{with a semilinear continuous } \mathrm{Gal}_{\mathbb{Q}_p}\text{-action} \\ \text{together with an isomorphism } \Lambda_1[\tfrac{1}{t}] \cong \Lambda_2[\tfrac{1}{t}]. \end{array} \right\}.$$

Similarly, there is a variant $\mathrm{Hck}_{\partial,n}$ of the stack FLD_n. By abuse of notation we write $\mathrm{Coh}(\mathrm{Hck}_{\partial,n})$ for the subcategory of coherent sheaves that are supported on some closed substack $\mathrm{Hck}_{\partial,n,\le\mu} \subset \mathrm{Hck}_{\partial,n}$, where the relative position of Λ_2 with respect to Λ_1 is bounded by μ. Similarly we write $\mathbf{D}^b_{\mathrm{coh}}(\mathrm{Hck}_{\partial,n})$ for the corresponding derived category of coherent sheaves.

By general constructions we obtain a convolution $(\mathcal{F},\mathcal{G}) \mapsto \mathcal{F} * \mathcal{G}$ that makes $\mathbf{D}^b_{\mathrm{coh}}(\mathrm{Hck}_{\partial,n})$ and $\mathbf{D}^b_{\mathrm{coh}}(\mathrm{Hck}_{B_{\mathrm{dR}}^+,n})$ into monoidal categories. Moreover, modifying an equivariant vector bundle on X_{FF} by a $\mathrm{Gal}_{\mathbb{Q}_p}$-stable B_{dR}^+-lattice yields morphisms

$$\mathrm{LS}_{X_{\mathrm{FF}},\mathrm{GL}_n} \leftarrow \mathrm{LS}_{X_{\mathrm{FF}},\mathrm{GL}_n} \times_{\mathrm{LS}_{B_{\mathrm{dR}}^+,\mathrm{GL}_n}} \mathrm{Hck}_{B_{\mathrm{dR}}^+,n} \to \mathrm{LS}_{X_{\mathrm{FF}},\mathrm{GL}_n}.$$

Again, general constructions imply that there is a “Hecke“-action

$$\mathbf{D}^b_{\mathrm{coh}}(\mathrm{Hck}_{B_{\mathrm{dR}}^+,n}) \times \mathbf{D}^b_{\mathrm{coh}}(\mathrm{LS}_{X_{\mathrm{FF}},\mathrm{GL}_n}) \to \mathbf{D}^b_{\mathrm{coh}}(\mathrm{LS}_{X_{\mathrm{FF}},\mathrm{GL}_n}).$$

We expect that this action has a relation to the action of algebraic representations V_λ on $\mathrm{Rep}^{\mathrm{la}}_{\mathbb{Q}_p} G$. In fact the latter extends to an action of the monoidal category HC_G of Harish-Chandra bimodules. These are representations of $\mathrm{Lie}\,G \times \mathrm{Lie}\,G$ together with a lift of the restriction to the diagonal copy of $\mathrm{Lie}\,G$ to an algebraic representation of G. Algebraic representations V_λ are viewed as objects in HC_G by induction along the diagonal.

Expected Theorem (Bezrukavnikov–H.): *There is an exact and fully faithful embedding of monoidal categories*

$$\mathrm{HC}_{\mathrm{GL}_n} \to \mathrm{Coh}(\mathrm{Hck}_{\partial,n}).$$

Remark: (i) We expect to have a version of this theorem for arbitrary reductive groups.
(ii) Moreover, we expect that there is a "twisted" version of the theorem in which $\mathrm{Hck}_{\partial,n}$ is replaced by $\mathrm{Hck}_{B^+_{\mathrm{dR}},n}$.

A mod p Two Towers Isomorphism

Lucas Mann
(joint work with T. Barthel, R. Ray, T. Schlank, P. Srinivan, J. Weinstein, Y. Xu, Z. Yang, X. Zhou)

1. Motivation

The main motivation for this project lies in chromatic homotopy theory, a branch of topology. More precisely, we have the following variant of Hopkin's chromatic splitting conjecture [1]:

Conjecture 1 (Bichromatic Splitting)**.** *Fix integers $0 \leq t < n$ and a prime $p \gg n$. Then there is an isomorphism*

$$L_{K(p,t)} L_{K(p,n)} \mathbb{S} \cong L_{K(p,t)} \otimes \Lambda(x_1, \ldots, x_{n-t}), \qquad |x_i| = 2i - 1.$$

Here $\mathbb{S}$ denotes the *sphere spectrum*, which is the initial "derived" ring in topology and plays a similar role as $\mathbb{Z}$ in algebra. Its homology groups are the stable homotopy groups of spheres, a fundamental object of study in topology. The symbol $K(p,n)$ denotes the prime of $\mathbb{S}$ associated to p and n (or more precisely, the associated prime field). All primes of $\mathbb{S}$ are of this form for some p and some $0 \leq n \leq \infty$, and we have $K(p,0) = \mathbb{Q}$ and $K(p,\infty) = \mathbb{F}_p$. The functor $L_{K(p,n)}$ denotes the "completion" at the prime $K(p,n)$. In order to approach the above conjecture, one uses the following closely related algebraic version:

Conjecture 2 (Algebraic BCSC)**.** *Let n, t, p as above. Let Γ_n be a formal group of dimension 1 and height n over $\overline{\mathbb{F}}_p$, $A_n := W(\overline{\mathbb{F}}_p)[[u_1, \ldots, u_{n-1}]]$ the Lubin-Tate ring and $\mathbb{G}_n$ the automorphism group of Γ_n. Then*

$$H^*_{\mathrm{cts}}(\mathbb{G}_n, A_n/(p, u_1, \ldots, u_{t-1})[u_t^{-1}]) \cong H^*_{\mathrm{cts}}(\mathbb{G}_t \times \mathrm{GL}_{n-t}(\mathbb{Z}_p), \overline{\mathbb{F}}_p).$$

The case $t = 0$ of the above conjectures was recently solved by [2] using non-archimedean geometry. The core idea is to realize the left-hand side in Conjecture 2 via the $\mathcal{O}^+$-cohomology on the v-stack $\mathrm{LT}_{n,0}/\mathbb{G}_n$, where $\mathrm{LT}_{n,0} := (\mathrm{Spf}\, A_n)^\diamond_\eta$. Then one uses the following diagram:

$$\begin{array}{ccc} & \mathrm{LT}^\infty_{n,0} & \\ {}^{\mathrm{GL}_n(\mathbb{Z}_p)}\swarrow & & \searrow^{\mathbb{G}_n} \\ \mathrm{LT}_{n,0} & & \mathcal{H}^{n-1} \end{array}$$

Here $\mathrm{LT}^\infty_{n,0}$ is "infinite level Lubin–Tate space" or equivalently "infinite level Drinfeld space". Moreover $\mathcal{H}^{n-1}$ denotes Drinfeld's upper half space of dimension

$n-1$; it is the open subspace of $\mathbb{P}^{n-1}$ obtained by removing all $\mathbb{Q}_p$-rational hyperplanes. The maps in the above diagram are torsors under the respective groups. Hence altogether we obtain a canonical isomorphism of v-stacks $\mathrm{LT}_{n,0}/\mathbb{G}_n = \mathcal{H}^{n-1}/\mathrm{GL}_n(\mathbb{Z}_p)$. The cohomologies of these two v-stacks correspond to the two sides in Conjecture 2, proving the conjecture.

The goal of our project is to apply similar ideas in the case $t > 0$, where previously no analogous "two towers" picture was available. The project is active work in progress and we can so far prove the following:

Theorem 3. *Let $t = n-1$. Then there is a canonical isomorphism*

$$H^*_{\mathrm{cts}}(\mathbb{G}_n, \overline{\mathbb{F}}_p((u_{n-1}^{1/p^\infty}))) = H^*_{\mathrm{cts}}(\mathbb{G}_{n-1} \times \mathbb{Z}_p^\times, \overline{\mathbb{F}}_p).$$

Compared to Conjecture 2 the only difference is the completed perfection appearing on the left-hand side, which we currently do not know how to remove. The restriction to $t = n-1$ is mainly because it is the easiest case, but we expect our methods to apply to general t as well. In the following I will sketch the idea of the proof, which consists of two parts: First construct a mod p version of the two towers picture above, then compute the cohomology of the right-hand side.

2. The Two Towers

We fix a prime p and integers $0 < t < n$. We will make use of Scholze's *(small) v-stacks*, i.e. sheaves of groupoids on the v-site of perfectoid spaces over $\overline{\mathbb{F}}_p$.

Definition 4. We define the small v-stack $\mathrm{LT}_{n,t}$ by

$$\mathrm{LT}_{n,t}(\mathrm{Spa}(R,R^+)) := \mathrm{Hom}(A_n/(p,u_1,\dots,u_{t-1})[u_t^{-1}], A_n/(p,u_1,\dots,u_{t-1}), (R,R^+))$$

for any affinoid perfectoid space $\mathrm{Spa}(R,R^+)$ over $\overline{\mathbb{F}}_p$. One checks that $\mathrm{LT}_{n,t}$ is an $(n-t-1)$-dimensional perfectoid open disk over $\overline{\mathbb{F}}_p((u_t^{1/p^\infty}))$. In particular, if $t = n-1$ then $\mathrm{LT}_{n,t}$ is represented by $\overline{\mathbb{F}}_p((u_t^{1/p^\infty}))$.

There is a moduli description of $\mathrm{LT}_{n,t}$. Namely, its value on $\mathrm{Spa}(R,R^+)$ parametrizes pairs (H,α), where H is a p-divisible group over R^+ such that $H \otimes_{R^+} R$ has étale height $n-t$ and connected height t, and α is an isomorphism $\alpha\colon H \otimes_{R^+} R^+/\pi \to \Gamma_n \otimes_{\overline{\mathbb{F}}_p} R^+/\pi$. By Dieudonné theory, this can be equivalently described by pairs (M,α), where M is a Dieudonné module over $W(R^+)$ such that there is an isomorphism

$$M \otimes_{W(R^+)} W(R) \cong M_0 \oplus M_{1/t},$$

where M_0 and $M_{1/t}$ are semistable Dieudonné modules of slopes 0 and $1/t$ respectively, and α is an isomorphism of the base-change of M to R^+/π with the Dieudonné module of Γ_n. Based on this description, we can now introduce the "infinite level" version of $\mathrm{LT}_{n,t}$ and the analog of $\mathcal{H}^{n-1}$:

Definition 5.

(a) We define the v-stack $\mathrm{LT}_{n,t}^{\infty}$ as follows. For a given affinoid perfectoid $\mathrm{Spa}(R, R^+)$, $\mathrm{LT}_{n,t}^{\infty}(\mathrm{Spa}(R, R^+))$ consists of pairs (M, α) as above together with isomorphisms $M_0 \cong \mathbf{D}(0)^{n-t}$ and $M_{1/t} \cong \mathbf{D}(1/t)$, where $\mathbf{D}(\lambda)$ denotes the standard stable Dieudonné module of slope λ.

(b) We denote by $\Omega_{n,t}$ the small v-stack which to an affinoid perfectoid space $S = \mathrm{Spa}(R, R^+)$ associates the open subset of $\mathrm{Ext}^1(\mathcal{O}(1/t), \mathcal{O}^{n-t})$ where the extension is locally isomorphic to $\mathcal{O}(1/h)$. Here $\mathcal{O}(\lambda)$ denotes the standard stable vector bundle of slope λ on the relative Fargues–Fontaine curve over S.

Theorem 6. *The above spaces fit into the following picture, where the maps are torsors under the respective groups:*

$$\begin{array}{ccc} & \mathrm{LT}_{n,t}^{\infty} & \\ {\scriptstyle \mathbb{G}_t \times \mathrm{GL}_{n-t}(\mathbb{Z}_p)} \swarrow & & \searrow {\scriptstyle \mathbb{G}_n} \\ \mathrm{LT}_{n,t} & & \Omega_{n,t} \end{array}$$

The proof of Theorem 6 uses the theory of meromorphic vector bundles on the Fargues–Fontaine curve, as developed in [3]. The spaces $\Omega_{n,t}$ appear in the geometric Langlands program and are not so difficult to describe.

3. Cohomology Computation

Let us now consider the case $t = n - 1$ and indicate how to prove Theorem 3. Using the two-towers isomorphism from Theorem 6 we compute

$$\begin{aligned} H^*_{\mathrm{cts}}(\mathbb{G}_n, \overline{\mathbb{F}}_p((u_{n-1}^{1/p^{\infty}}))) &= H^*(\mathrm{LT}_{n,t}/\mathbb{G}_h, \mathcal{O}^\flat) \\ &= H^*(\Omega_{n,t}/(\mathbb{G}_t \times \mathrm{GL}_{n-t}(\mathbb{Z}_p)), \mathcal{O}^\flat) \\ &= H^*_{\mathrm{cts}}(\mathbb{G}_t \times \mathrm{GL}_{n-t}(\mathbb{Z}_p)), H^*(\Omega_{n,t}, \mathcal{O}^\flat)). \end{aligned}$$

Here $\mathcal{O}^\flat$ denotes the v-sheaf of abelian groups sending $\mathrm{Spa}(R, R^+)$ to R. Using the above computation, we have reduced ourselves to proving the following.

Theorem 7. *We have $H^*(\Omega_{n,t}, \mathcal{O}^\flat) = \overline{\mathbb{F}}_p$.*

In order to prove this, we first show a version of the Drinfeld lemma:

$$H^*(\Omega_{n,t}, \mathcal{O}^\flat) = H^*(\Omega_{n,t} \times \mathrm{Spa}(\mathbb{C}_p)/\varphi, \mathcal{O}^\flat) = H^*(\Omega_{n,t} \times \mathrm{Spa}(\mathbb{C}_p), \mathcal{O}^\flat).$$

Now $\Omega_{n,t} \times \mathrm{Spa}(\mathbb{C}_p) \cong \mathcal{H}^1/\mathbb{Q}_p$, so we need to compute $H^*(\mathcal{H}^1, \mathcal{O}^\flat)$. We emphasize that $\mathcal{H}^1$ appears here for a different reason than in the $t = 0$ case!

We compute $H^*(\mathcal{H}^1, \mathcal{O}^\flat)$ using the 6-functor formalism for $\mathcal{O}^\flat$-cohomology introduced by Anschütz–Le Bras–Mann [4]. Namely, $\mathcal{H}^1$ is an open subset of $\mathbb{P}^1$ and its closed complement is given by the profinite set $\mathbb{P}^1(\mathbb{Q}_p)$. The 6-functor formalism allows us to use an excision sequence for this decomposition.

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Applications of p-adic character varieties to anabelian geometry

Ben Heuer

(joint work with Magnus Carlson, Ruth Wild)

1. Introduction

Initiated by Grothendieck in a letter to Faltings, anabelian geometry studies which arithmetic varieties are determined by their étale fundamental group. A famous instance of such "anabelian" varieties are hyperbolic curves over $\mathbb{Q}_p$:

Theorem 1 (Mochizuki, [5])**.** *Let X_0 and Y_0 be hyperbolic curves over a finite extension $K/\mathbb{Q}_p$. Then any open continuous homomorphism $\varphi : \pi_1(X_0) \to \pi_1(Y_0)$ over G_K comes from a dominant morphism $\phi : X_0 \to Y_0$ over K.*

In joint work with Magnus Carlson and Ruth Wild, very much in progress, we aim to give a new proof of this Theorem using p-adic non-abelian Hodge theory, specifically, the moduli-theoretic version of the p-adic Simpson correspondence. In this talk, we give an outline of our strategy.

Let C be the completion of an algebraic closure of K and let $X := X_{0,C}$ and $Y := Y_{0,C}$ be the base-changes to C. We start by following Mochizuki, whose proof begins with the following beautiful idea: By comparing the Hodge–Tate morphism

$$\mathrm{HT} : \mathrm{Hom}_{\mathrm{cts}}(\pi_1(X), C) \to H^0(X, \Omega_X),$$

and its canonical splitting induced by the Galois action, to the analogous morphism for Y, he associates to φ a G_K-equivariant homomorphism

$$h : H^0(Y, \Omega_Y) \to H^0(X, \Omega_X).$$

The idea is now to show that this extends to a homomorphism of graded rings

$$(1) \qquad \bigoplus_{i=0}^{\infty} H^0(Y, \Omega_Y^{\otimes i}) \to \bigoplus_{i=0}^{\infty} H^0(X, \Omega_X^{\otimes i}).$$

Once this is obtained, one can construct ϕ by simply applying $\mathrm{Proj}(-)$. But upgrading h to such a homomorphism of graded rings (1) is hard: This is what a large part of Mochizuki's article [5] is about.

Our aim is to find a new construction of the homomorphism (1) of graded rings.

2. Arithmetic properties of p-adic character varieties

Our idea is to construct the morphism (1) using p-adic character varieties:

The starting point is Faltings' p-adic Simpson correspondence, which can be interpreted as an equivalence of categories [1][2]

$$\{\text{v-vector bundles on } X\} \leftrightarrow \{\text{Higgs bundles on } X\}.$$

In joint work with Daxin Xu [4], we recently showed that this can be upgraded to a comparison of moduli spaces: The moduli space of stable v-vector bundles is a twist of the moduli space of stable Higgs bundles over the Hitchin base.

Second, we proved that the p-adic character variety $\mathrm{Rep}_{n,X}$, which parametrises irreducible continuous n-dimensional C-representations of $\pi_1(X)$, admits a natural open embedding into the moduli space of v-vector bundles of rank n.

The key object is now the resulting Hitchin fibration of the p-adic character variety, introduced in [3]:

$$\widetilde{\mathcal{H}}_n : \mathrm{Rep}_{n,X} \to \mathcal{A}_{n,X},$$

where $\mathcal{A}_n = \mathcal{A}_{n,X}$ denotes the Hitchin base

$$\mathcal{A}_{n,X} := \bigoplus_{i=1}^{n} H^0(X, \Omega_X^{\otimes i}(-i)) \otimes_C \mathbb{A}^1$$

with its natural Tate-twisted semi-linear G_K-action. Since $\widetilde{\mathcal{H}}$ is a twist of the map that sends a Higgs bundle to the characteristic polynomial of its Higgs field, we think of elements of $\mathcal{A}_n$ as polynomials

$$f(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_0$$

with coefficients $a_{n-i} \in H^0(X, \Omega^{\otimes i}(-i)) \otimes_C \mathbb{A}^1$.

Proposition 2. *The map $\widetilde{\mathcal{H}}_n$ has the following properties:*

(1) It is G_K-equivariant.
(2) It is open (and conjecturally surjective).
(3) It is compatible with $\oplus$: For any representations $V \in \mathrm{Rep}_n$, $W \in \mathrm{Rep}_m$, we have

$$\widetilde{\mathcal{H}}_{n+m}(V \oplus W) = \widetilde{\mathcal{H}}_n(V) \cdot \widetilde{\mathcal{H}}_m(W)$$

where the right hand side is interpreted as the product of polynomials.

(4) It is compatible with $\otimes$: For example, Rep_1 acts on Rep_n via a twisting action $(L, V) \mapsto L \otimes W$, while $\mathcal{A}_1 = H^0(X, \Omega_X(-1)) \otimes \mathbb{A}^1$ acts on $\mathcal{A}_n$ via $(\tau, f(T)) \mapsto \tau * f := f(T - \tau)$. Then for any $L \in \mathrm{Rep}_1$ and $V \in \mathrm{Rep}_n$,

$$\widetilde{\mathcal{H}}_n(L \otimes V) = \widetilde{\mathcal{H}}_1(L) * \widetilde{\mathcal{H}}_n(V).$$

This means that $\widetilde{\mathcal{H}}_n$ can be thought of as a generalisation of the map HT used by Mochizuki. For example, in Proposition 2:

(1) is the analogue of HT being G_K-equivariant,
(2) is the analogue of HT being surjective, and
(4) is the analogue of HT being linear.

However, $\widetilde{\mathcal{H}}_n$ also sees "non-abelian information" of $\pi_1(X)$ since it deals with representations instead of the additive characters that form the source of HT. The heuristic is that, for this reason, via the Hitchin base $\mathcal{A}_n$, the morphism $\widetilde{\mathcal{H}}_n$ will allow us to "reconstruct $H^0(X, \Omega_X^{\otimes i}(-i))$ for $i \geq 2$ from the datum of $\pi_1(X)$".

Furthermore, an important point is that either of property (3) or (4) in Proposition 2 means that we can use $\widetilde{\mathcal{H}}_n$ to detect the multiplicative structure on $\mathcal{A}_n$: For example, the infinitesimal action of $\mathcal{A}_1$ on $\mathcal{A}_n$ given by deriving $*$ is given by

$$\mathcal{A}_1 \times \mathcal{A}_n \to \mathcal{A}_n, \quad (w, (a_1, \ldots, a_n)) \mapsto (nw, (n-1)wa_1, \ldots, wa_{n-1})$$

where the products wa_i on the right hand side are to be interpreted in terms of the multiplication map $H^0(X, \Omega_X) \otimes H^0(X, \Omega_X^{\otimes(n-1)}) \to H^0(X, \Omega_X^{\otimes n})$.

3. Application to anabelian geometry

The starting point of our application is the observation that any open continuous homomorphism $\varphi : \pi_1(X_0) \to \pi_1(Y_0)$ over G_K induces a G_K-equivariant pullback morphism between the character varieties of the geometric fundamental groups:

$$\begin{array}{ccc} \mathrm{Rep}_{n,Y} & \xrightarrow{\varphi^*} & \mathrm{Rep}_{n,X} \\ \downarrow{\scriptstyle \widetilde{\mathcal{H}}_Y} & & \downarrow{\scriptstyle \widetilde{\mathcal{H}}_X} \\ \mathcal{A}_{n,Y} & \dashrightarrow & \mathcal{A}_{n,X} \end{array}$$

We wish to use this to compare $\widetilde{\mathcal{H}}_Y$ to $\widetilde{\mathcal{H}}_X$.

If we knew that there was a dotted morphism making the diagram commute, this would necessarily be unique by Proposition 2.(2). It would then follow from Proposition 2.(3) that for varying n, these morphisms are compatible and induce on C-points in the colimit over n the desired homomorphism of graded rings (1)

$$\bigoplus_{i=0}^{\infty} H^0(Y, \Omega_Y^{\otimes i}) \to \bigoplus_{i=0}^{\infty} H^0(X, \Omega_X^{\otimes i}).$$

Currently, we do not a priori know the existence of such a dotted arrow. However, we can instead modify the argument slightly:

As explained to us by Daxin Xu, a result in progress by Xu–Zhu (presented during this same Oberwolfach workshop) can be used to see that there is a local section of the Hitchin morphism $\widetilde{\mathcal{H}}_n$

$$\mathcal{A}_{n,Y,\epsilon} \to \mathrm{Rep}_{n,Y}$$

where $\mathcal{A}_{n,Y,\epsilon}$ is a small open neighbourhood of $0 \in \mathcal{A}_{n,Y}$. In our work in progress, we aim to use this to construct directly a Rep_1- and G_K-equivariant morphism

$$\mathcal{A}_{n,Y} \to \mathcal{A}_{n,X}.$$

The idea is then to use Proposition 2.(4) to show that this induces the desired homomorphism (1) of graded rings by considering the tangent spaces at 0.

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Valuative compactifications of complex analytic spaces

Jérôme Poineau

Let X be a complex algebraic variety, *i.e.* a separated scheme of finite type over $\mathbf{C}$. The aim of this talk is to construct a locally ringed space $X^{\urcorner}$ that is a compactification of the complex analytification $X(\mathbf{C})$ of X, and preserves its properties.

1. A non-Archimedean boundary

We denote by $\mathbf{C}_0$ the field $\mathbf{C}$ endowed with the trivial absolute value $|\cdot|_0$. We will work in the category of analytic spaces over $\mathbf{C}_0$ in the sense of V. Berkovich (see [2, 3]). Recall that we have an analytification functor $X \mapsto X_0^{\mathrm{an}}$ from algebraic varieties over $\mathbf{C}$ to analytic spaces over $\mathbf{C}_0$.

A non-Archimedean space X_∞ that plays a role similar to that of the boundary of X has already appeared in work by V. Berkovich (in a letter to V. Drinfeld), O. Ben–Bassat and M. Temkin [1] and A. Thuillier [6].

Let us describe it in the affine case $X = \mathrm{Spec}(A)$, where A is an algebra of finite type over $\mathbf{C}$. Then X_0^{an} may be defined as the set of multiplicative seminorms on A that induce the trivial absolute value $|\cdot|_0$ on $\mathbf{C}$. We may consider the subset $X^{\beth}$ of seminorms that admit a center of X, that it to say

$$X^{\beth} = \{x \in X_0^{\mathrm{an}} : |a(x)| \leq 1, a \in A\}.$$

The non-Archimedean boundary of X is then defined as

$$X_\infty := X_0^{\mathrm{an}} - X^{\beth}.$$

2. Hybrid spaces

The complex analytic space $X(\mathbf{C})$ and the Berkovich space X_∞ may be glued together by working in the category of Berkovich hybrid spaces.

Recall that we denote by $\mathbf{C}_{\mathrm{hyb}}$ the field $\mathbf{C}$ endowed with the norm $\|\cdot\|_{\mathrm{hyb}} := \max(|\cdot|_0, |\cdot|_\infty)$, where $|\cdot|_\infty$ is the usual absolute value on $\mathbf{C}$. It is a Banach ring. As a consequence, the theory defined in [2] and further developed in [4] provides us with a definition of analytic space over $\mathbf{C}_{\mathrm{hyb}}$ and an analytification functor $X \mapsto X^{\mathrm{hyb}}$.

For each complex variety X, the space X^{hyb} comes with a structure map $\mathrm{pr}\colon X^{\mathrm{hyb}} \to \mathcal{M}(\mathbf{C}_{\mathrm{hyb}})$, where $\mathcal{M}(\mathbf{C}_{\mathrm{hyb}})$ is the Berkovich spectrum of $\mathbf{C}_{\mathrm{hyb}}$. The latter may be described explicitly as

$$\mathcal{M}(\mathbf{C}_{\mathrm{hyb}}) = \{|\cdot|_0\} \cup \{|\cdot|_\infty^\varepsilon, 0 < \varepsilon \leq 1\}.$$

In particular, we have $\mathrm{pr}^{-1}(|\cdot|_0) = X_0^{\mathrm{an}}$ and, for each $\varepsilon \in (0,1]$, $\mathrm{pr}^{-1}(|\cdot|_\infty^\varepsilon)$ identifies to $X(\mathbf{C})$ (with a distances possibly rescaled).

3. The compactification

Let X be a complex algebraic variety. We set

$$X^+ := X^{\mathrm{hyb}} - X^{\sqsupset}.$$

Since $X^{\sqsupset}$ is a closed subset of X_0^{an}, which is itself closed in X^{hyb}, X^+ is an open subset of X^{hyb}. In particular, it inherits a structure of locally ringed space.

As X^{hyb}, the space X^+ contains several copies of $X(\mathbf{C})$. To remedy this problem, we mod out by the equivalence of seminorms : $|\cdot|_x \sim |\cdot|_x^\varepsilon$ for $\varepsilon > 0$. Denote by $X^{\urcorner}$ the quotient space and endowing it with the push-forward of the structure sheaf on X^+. Note that, on the non-Archimedean part, this procedure already appears in work by L. Fantini [5].

Theorem 1.

(1) The space $X^{\urcorner}$ is Hausdorff and compact.

(2) The morphism $X(\mathbf{C}) \to X^{\urcorner}$ induced by the identification of $X(\mathbf{C})$ with $\mathrm{pr}^{-1}(|\cdot|_\infty)$ in X^{hyb} is an open immersion with dense image.

Moreover, the association $X \mapsto X^{\urcorner}$ is functorial with respect to proper morphisms, and preserves many properties: normality, regularity, etc.

A partial GAGA theorem for coherent sheaves exists in this setting.

Theorem 2. *To each coherent sheaf F on X, we may functorially associate a coherent sheaf $F^{\urcorner}$ on $X^{\urcorner}$.*

(1) The functor

$$F \in \mathrm{Coh}(X) \longmapsto F^{\urcorner} \in \mathrm{Coh}(X^{\urcorner})$$

is an equivalence of categories.

(2) For each coherent sheaf F on X, we have a natural isomorphism

$$H^0(X, F) \xrightarrow{\sim} H^0(X^{\urcorner}, F^{\urcorner}).$$

The full GAGA theorem that one may expect does not hold, and one may check that, for each $n \geq 2$, the compactification of $\mathbf{C}^n$ has a non-vanishing H^n.

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