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Fluid Mechanics. – *On approximations of stochastic optimal control problems with an application to climate equations*, by Franco Flandoli, Giuseppina Guatteri, Umberto Pappalettera and Gianmario Tessitore, accepted on 7 May 2025.

Abstract. – The paper is devoted to the optimal control of a system with two time-scales, in a regime when the limit equation is not of averaging type but, in the spirit of Wong–Zakai principle, it is a stochastic differential equation for the slow variable, with noise emerging from the fast one. It proves that it is possible to control the slow variable by acting only on the fast scales. The concrete problem, of interest for climate research, is embedded into an abstract framework in Hilbert spaces, with a stochastic process driven by an approximation of a given noise. The principle established here is that the convergence of the uncontrolled problem is sufficient for the convergence of both the optimal costs and the optimal controls. This target is reached using Girsanov transform and the representation of the optimal cost and the optimal controls using a Forward-Backward System. A challenge in this program is represented by the generality considered here of unbounded control actions.

Keywords. – two-scale system, climate model, optimal stochastic control, backward stochastic differential equation.

Mathematics Subject Classification 2020. – 76M35.

1. Introduction

In this paper, we are concerned with optimal control problems associated with stochastic equations in abstract Hilbert spaces [6–8], and their convergence. More precisely, we introduce a family of stochastic equations indexed by a parameter $\varepsilon \in (0,1)$, driven by a stochastic process obtained as the *approximation* of some given noise W via some general approximation map Γ^ε. Then, we solve a control problem for every $\varepsilon \in (0,1)$ and we are interested in understanding if the convergence of both the optimal costs and the optimal controls holds true as $\varepsilon \to 0$.

In order to answer this question, we develop a general framework for studying the approximations of stochastic optimal control problems.

Apart from very natural technical assumptions, the only hypothesis on the approximation maps Γ^ε is the validity of some Wong–Zakai type of convergence. That is, we assume that the solution of the uncontrolled equation driven by the approximation of the noise $\Gamma^\varepsilon(W)$ converges in probability, as $\varepsilon \to 0$, towards the solution of the uncontrolled equation driven by W (cf. Assumption 4.1). The key result of this work is that the convergence of the uncontrolled problems is sufficient for the convergence of both the optimal costs and the optimal controls (cf. Theorem 4.2)

In this paper, the above goal is achieved through the representation of the optimal cost and the optimal controls using a Forward-Backward System of Stochastic Differential Equations (FBSDEs) (see e.g. the system (3.9) here). This technique has been widely used in the last twenty-five years both in finite and in infinite-dimensional frameworks (see for instance [28] or [12, Chapter 6] and references within). It has the advantage to characterize not only the optimal state of a stochastic control problem but also the optimal feedback law, without requiring the regularity of the value function. This is, in extreme synthesis, the reason why we are able to obtain our main abstract convergence result, see Theorem 4.2 and, in particular, the convergence of optimal controls stated in it.

On the other hand, it should be stated from the beginning that the techniques we employ rely heavily on non-trivial Girsanov transform methods, which in turn require a specific structure of the controlled state equation. Roughly speaking, we must assume that the control acts only in the directions of the state space influenced by the noise. Moreover, the approximating operator Γ^{ε} must act in the same way on both the noise and the control. Nevertheless, several natural examples exhibit the required structure—among them, the case in which both the noise and the control act on the boundary of a domain (and on the same parts of the boundary), see [9], or the case of a state equation with a delay in the state, see [17].

In this work, we consider the case in which the running cost is quadratic and coercive with respect to the control variable, while exhibiting bounded behavior in the state variable (cf. Assumption 2.5). This choice of allowing "unbounded" control actions introduces significant technical challenges in the development of the FBSDE approach to optimal control problems. First of all, it interacts with the necessity of adopting a weak formulation for the control problem. Indeed, the final convergence argument (cf. Section 2.4) works if uncontrolled state equations refer to the same stochastic framework. Consequently, we are led to express the control problem in a weak form. This implies that a rigorous formulation of the problem, along with the characterization of the class of admissible controls, involves a change of probability that necessitates the introduction of a localization argument (cf. Definition 2.7 and Proposition 2.11).

In addition, the Hamiltonian non-linearity ψ, introduced in Section 3.1, which drives the backward equation in the FBSDE system (cf. equation (3.9)), is non-Lipschitz with respect to its second variable, denoted by Z. To address this point, we adapt the techniques developed in [3, 23], taking profit, in particular, of the specific properties of BMO martingales (cf. Section 3.2 here and [22]). The appropriate use of this class of martingales, along with the corresponding estimates, constitutes a crucial element in the proof of our main general result, cf. Theorem 4.2. In synthesis, the combination of a coercive quadratic cost function and a non-Lipschitz Hamiltonian introduces considerable complexities. These necessitate the use of advanced stochastic analysis techniques, notably those concerning BMO martingales, to ensure the well-posedness of the optimal control problems and their convergence, within the weak formulation framework.

Our main motivation for studying general approximations of stochastic optimal control problems comes from the desire of understanding the behavior of controlled slow-fast systems of stochastic equations $(X^\varepsilon, Q^\varepsilon)$, depending on a small parameter $0 < \varepsilon \ll 1$. Indeed, in certain prototypical situations, the slow component X^ε of the system converges as $\varepsilon \to 0$ towards a limiting closed equation. Here, the term "closed" refers to the fact that the equation for the limit $\widehat{X}$ no longer depends on the fast variable. In this case, we intend to study a control problem for the systems $(X^\varepsilon, Q^\varepsilon)$ and for the limit equation $\widehat{X}$. A natural question is whether the control problem for $(X^\varepsilon, Q^\varepsilon)$ can be solved at every $\varepsilon > 0$, and whether or not the solutions of the control problems converge as $\varepsilon \to 0$ to a solution of the control problem for $\widehat{X}$. The relevance of this problem becomes clear in view of the interpretation of slow-fast systems as general models of climate-weather interaction (see next subsection for additional details). With the lens of this interpretation, the convergence of control problems translates into the following question: Is it possible to "control" the evolution of the climate by acting only at meteorologic scales?

We believe this setting is robust enough to be amenable to further generalizations of the control problems (1.3) and (1.4), cf. the discussion in Section 5.

1.1. A motivating example: Climatic model

Let us start with a motivating example. We consider a slow-fast system having the following form:

$$\begin{cases} dX_t^\varepsilon = AX_t^\varepsilon dt + b(X_t^\varepsilon)dt + \sigma(X_t^\varepsilon)Q_t^\varepsilon dt, & t \in [0,T], \\ X^\varepsilon(0) = x_0, \\ dQ_t^\varepsilon = -\dfrac{1}{\varepsilon}Q_t^\varepsilon dt + \dfrac{1}{\varepsilon}G\,dW_t, & t \in [0,T], \\ Q^\varepsilon(0) = 0. \end{cases} \tag{1.1}$$

Solutions of (1.1) are pairs of stochastic processes $(X^\varepsilon, Q^\varepsilon)$, where the "slow" component (X^ε) takes values in a Hilbert space K and the "fast" component (Q^ε) takes values in a Hilbert space H. We denote by $|\cdot|_K$ and $|\cdot|_H$ the norms on these spaces, and by $\langle\cdot,\cdot\rangle_K$ and $\langle\cdot,\cdot\rangle_K$ the inner products. For simplicity we assume $x_0 \in K$ to be given and deterministic. In the lines above, (W_t) is a cylindrical Wiener process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with complete and right-continuous filtration $(\mathcal{F}_t^W)$, and G is a Hilbert–Schmidt operator on the Hilbert space H with $Gv = \sum_{i=1}^{\infty} \lambda_i \langle e_i, v\rangle_H e_i$ for every $v \in H$, where $\sum_{i=1}^{\infty} \lambda_i^2 < \infty$ and $(e_i)_{i\in\mathbb{N}}$ is an orthonormal basis in H. Finally, $A : K \to K$ is linear continuous, $b : K \to K$ is Lipschitz and $\sigma : K \to \mathcal{L}(H; K)$ is of class C_b^2. The maps b and σ can be expressed in terms of their coordinates $b(x)_j := \langle b(x), f_j\rangle_K$ and $(\sigma(x)e_m)_j := \sigma^{j,m}(x) := \langle \sigma(x)e_m, f_j\rangle_K$, where $(f_j)_{j\in\mathbb{N}}$ is an orthonormal basis in K and $j, m \in \mathbb{N}$.

This kind of slow-fast systems has been extensively studied in pure and applied mathematics. Among other important question, one is naturally led to ask what the behavior of this system is in the limit of infinite separation of scales $\varepsilon \to 0$. Heuristically, the fast oscillations of the process Q^ε prevent it from converging as a genuine function, and the convergence of Q^ε usually holds in a space of distributions with respect to time. On the other hand, the slow component X^ε can converge as a function but its limiting dynamics should retain information about the statistics of Q^ε. When the limit $\widehat{X} := \lim_{\varepsilon\to 0} X^\varepsilon$ solves a closed equation, we say that the limiting equation for $\widehat{X}$ is a *stochastic model reduction* of (1.1).

The first rigorous examples of stochastic model reduction of finite-dimensional equations are due to Kurtz [24] and Majda, Timofeyev, and Vanden Eijnden [26]. In particular, the latter successfully gave a stochastic model reduction of the truncated Barotropic Equations, identifying the slow variable X^ε as a quantity evolving on *climatic* time-scale and the slow variable Q^ε as a quantity evolving on *meteorologic* time-scale. The small constant $\varepsilon > 0$ represents the ratio between the speed of the evolution at these different time-scales. A similar interpretation was given in [2].

Under the previous assumptions on (1.1) and assuming K finite dimensional, in [2] it is proved that, as ε goes to zero, the sequence (X^ε) converges in probability in the $C([0, T], K)$ norm towards the solution $(\widehat{X})$ of the “reduced” equation

$$\begin{cases} d\widehat{X}_t = A\widehat{X}_t dt + \hat{b}(\widehat{X}_t)dt + \sigma(\widehat{X}_t)G dW_t, \quad t \in [0, T], \\ \widehat{X}(0) = x_0, \end{cases} \tag{1.2}$$

where

$$\big(\hat{b}(x)\big)_i := \big(b(x)\big)_i + \frac{1}{2}\sum_{m=1}^{\infty} \lambda_m^2 \sum_{j=1}^{d} D_j \sigma^{i,m}(x)\sigma^{j,m}(x), \quad x \in K,\ i \in \mathbb{N}.$$

Notice that under the present assumptions, $\hat{b} : K \to K$ is Lipschitz.

It is wort noticing that, differently to “standard” two-scale stochastic models where the fast evolution equation is obtained by a simple change of the time-scale with ratio ε (for the controlled version of such systems see, e.g. [1, 18, 19, 21, 27]), here, the oscillations induced by the noise in the fast equation are magnified by a factor $1/\sqrt{\varepsilon}$.

As a matter of fact, the two classes of slow-fast models show a very different behavior in the limit. Here, a new noise term and a correction drift appear in the reduced equation, while in the other case, the reduced equation is obtained by "averaging" the original coefficients with respect to a suitable "invariant measure".

It should also be pointed out that equations of the form (1.2) have already appeared in the study of climate since the seminal work of Hasselmann [20] on *stochastic climate models*. Indeed, Hasselmann proposes a general stochastic model to predict the evolution of quantity on *climatic* time-scales, without referring to any particular specification of the coefficients A, $\hat{b}$, σ of (1.2). The deep aspect of Hasselmann proposal is that a (small intensity) noise should be taken into account for a more correct description of the system.

In a second moment, the general theory of stochastic climate models has been specialized to particular systems, possibly adding *ad hoc* assumptions on the coefficients. To mention a few works in this direction, let us cite [16] on sea-surface temperature anomalies and thermocline variability, [11] on an energy balance model addressing temperature fluctuations due to rising carbon dioxide levels, [4] on magneto-hydrodynamics models, [15] on random attractors, and [25] on climatic tipping points.

1.2. *Controlled climatic model*

We wish to study a controlled version of this model, with control acting at the meteorologic scale. Namely, fixing a Hilbert space U and given a progressively measurable control process u taking values in U, we consider the system

$$\tag{1.3} \begin{cases} dX_t^{\varepsilon,u} = AX_t^{\varepsilon,u}dt + b(X_t^{\varepsilon,u})dt + \sigma(X_t^{\varepsilon,u})Q_t^{\varepsilon,u}dt & t \in [0,T], \\ X(0) = x_0, \\ dQ_t^{\varepsilon,u} = -\frac{1}{\varepsilon}Q_t^{\varepsilon,u}dt + \frac{1}{\varepsilon}Gr(u_t)dt + \frac{1}{\varepsilon}G\,dW_t, & t \in [0,T], \\ Q(0) = 0, \end{cases}$$

where $r : U \to H$ is a Lipschitz map. We also introduce a controlled reduced equation; namely,

$$\tag{1.4} \begin{cases} d\widehat{X}_t^u = A\widehat{X}_t dt + \hat{b}(\widehat{X}_t)dt + \sigma(\widehat{X}_t^u)Gr(u_t)dt + \sigma(\widehat{X}_t^u)G\,dW_t, & t \in [0,T], \\ \widehat{X}(0) = x_0. \end{cases}$$

The control problems above come with the two cost functionals: J^ε, related to system (1.3), and J, related to equation (1.4), which we assume both quadratic and coercive in u. In detail, the costs are given by

$$J^\varepsilon(x_0,u) := \mathbb{E}\left[\int_0^T l(X_s^{\varepsilon,u},u_s)ds + h(X_T^{\varepsilon,u})\right]$$

and

$$J(x_0, u) := \mathbb{E}\left[\int_0^T l(X_s^u, u_s)ds + h(X_T^u)\right]$$

so that the functionals above are well defined.

For the precise assumptions on l and h, as well as for the definition of the class of admissible controls, we refer to Assumption 2.5, Definition 2.7, and Theorem 3.6 below.

Our main result goes as follows, see also Theorem 4.2 for a precise statement. We prove the following:

(i) The control problems (1.3) and (1.4) admit an optimal control, denoted, respectively, as $\underline{u}^\varepsilon$ and $\hat{u}$.

(ii) The optimal controls are square integrable.

(iii) As $\varepsilon \to 0$, the optimal costs converge: $J^\varepsilon(x_0, \underline{u}^\varepsilon) \to J(x_0, \hat{u})$.

(iv) As $\varepsilon \to 0$, the optimal controls converge: $\mathbb{E}\int_0^T |\underline{u}_t^\varepsilon - \hat{u}_t|^2 \, dt \to 0$.

It is perhaps worth noticing that although we start by approximating problems with control acting at meteorological time-scale, we end up with a limit reduced problem with control acting at climatic time-scale.

We hope that this work, devoted to a simplified model, may serve as a useful starting point for examining the behavior of optimal controls in related, more realistic, contexts.

2. A GENERAL FRAMEWORK FOR THE APPROXIMATION OF STOCHASTIC OPTIMAL CONTROL PROBLEMS

We adopt a weak formulation of the problem, which is particularly well suited to our framework, as it enables the definition of all control problems on a common stochastic basis. This approach, combined with the representation of both the optimal cost and the optimal control through the solution of a backward stochastic differential equation (BSDE), allows us to establish the convergence, as $\varepsilon \to 0$, of the optimal costs, $J^\varepsilon(x_0, \underline{u}^\varepsilon) \to J(x_0, \hat{u})$, as well as the optimal controls, $\underline{u}^\varepsilon \to \hat{u}$. It is important to note that, due to the unbounded nature of the controls, the weak formulation cannot be obtained via a straightforward application of the Girsanov transform. Instead, we must employ a localization argument.

2.1. Settings

Let us reprise the notation of the previous motivating examples although precise working assumptions in the more general framework will be stated later. We fix H, K, and

U real separable Hilbert spaces (in our motivating model, H hosts the fast variables, K should host the slow variables of the system, and U will describe the actions of the control processes). Also recall that $(W_t)_{t\geq 0}$ is an H-valued Wiener process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with complete and right-continuous filtration $(\mathcal{F}_t^W)_{t\geq 0}$.

Given an arbitrary Banach space E and $p \in [1, \infty)$, let us denote by $L_W^{p,\mathrm{loc}}(\Omega \times [0, T], E)$ the space of $(\mathcal{F}_t^W)$-progressively measurable stochastic processes in

$$L_W^{p,\mathrm{loc}}(\Omega \times [0, T]; E) := \left\{\Phi : \Omega \times [0, T] \to E : \int_0^T |\Phi_t|_E^p \, dt < \infty \ \mathbb{P}\text{-a.s.}\right\}.$$

We will need also the spaces $L_W^2(\Omega \times [0, T]; E)$ of square integrable progressive measurable processes $\Phi : \Omega \times [0, T] \to E$ verifying

$$|\Phi|^2_{L^2_W(\Omega\times[0,T];E)} := \mathbb{E} \int_0^T |\Phi_t|_E^2 \, dt < \infty$$

and, for $p \in [1, \infty]$, the spaces $L_W^p(\Omega; C([0, T]; E))$ of progressive measurable processes Y with continuous paths in E, such that the norm

$$|Y|_{L^p_W(\Omega;C([0,T];E))} := \Big| \sup_{s\in[0,T]} |Y_s|_E \Big|_{L^p(\Omega)}$$

is finite, with the subspace of predictable processes Y with continuous paths in E.

We finally denote by $\mathcal{I}$ the space of H-valued continuous Itô-semimartingales of the form

$$I_t = \int_0^t \Phi_s \, ds + \int_0^t \Psi_s \, dW_s, \tag{2.1}$$

with $\Phi \in L_W^{1,\mathrm{loc}}(\Omega \times [0, T]; H)$ and $\Psi \in L_W^{2,\mathrm{loc}}(\Omega \times [0, T]; L_2(H))$ where $L_2(H)$ stands for the space of Hilbert–Schmidt operators from H to H.

We introduce a class of functionals $(\Gamma^\varepsilon)_{\varepsilon>0}$ from $\mathcal{I}$ to the class of càdlàg processes

$$\Omega \times [0, T] \to K$$

and we assume the following.

ASSUMPTION 2.1. *For every $I \in \mathcal{I}$, $\Gamma^\varepsilon[I]$ is an $(\mathcal{F}_t^W)_{t\geq 0}$-adapted process and its law only depends on the law of I.*

For the sake of the presentation, let us point out that one could think of the family of processes $(\Gamma^\varepsilon[I])_{\varepsilon>0}$ as some *adapted* approximation of the noise I, e.g., adapted piecewise linear interpolation, convolution, or colored-in-time approximation à la Ornstein–Uhlenbeck.

2.2. *State equations*

The first class of state equations represents evolution equations with *smoothed* noises; namely,

$$(S_\varepsilon) \qquad \begin{cases} dX_t^\varepsilon = AX_t^\varepsilon dt + b(X_t^\varepsilon)dt + \sigma(X_t^\varepsilon)\Gamma^\varepsilon[GW]_t dt, & t \in (0,T], \\ X^\varepsilon(0) = x_0. \end{cases}$$

The second class corresponds to stochastic equations with white-in-time noises:

$$(S) \qquad \begin{cases} d\widehat{X}_t = \big(A\widehat{X}_t + \hat{b}(\widehat{X}_t)\big)\,dt + \sigma(\widehat{X}_t)G\,dW_t, & t \in (0,T], \\ X(0) = x_0. \end{cases}$$

EXAMPLE 2.2. The motivating example of the introduction can be rephrased within this general framework by defining

$$\begin{aligned} \Gamma^\varepsilon[I]_t &:= \frac{1}{\varepsilon}\int_0^t e^{-(t-s)\varepsilon^{-1}}\,dI_s \\ &= \frac{1}{\varepsilon}\int_0^t e^{-(t-s)\varepsilon^{-1}}\Phi_s\,ds + \frac{1}{\varepsilon}\int_0^t e^{-(t-s)\varepsilon^{-1}}\Psi_s\,dW_s. \end{aligned} \tag{2.2}$$

Indeed, the cost functionals J^ε and J only involve the slow component of the solution to systems (1.1) and (1.2), so (1.1) and (1.2) can be replaced by (S_ε) and (S) without loss of useful information.

In both (S_ε) and (S), the coefficients satisfy the following.

ASSUMPTION 2.3. *We assume the following:*

(Hp 1-A) $A : D(A) \to K$ *is a (possibly unbounded) linear operator with domain* $D(A) \subseteq K$ *that generates a* C_0*-semigroup* $(e^{tA})_{t\geq 0}$.

(Hp 1-b) b *and* $\hat{b}$ *are Lipschitz maps from* K *to* K *and we fix a constant* $L_b > 0$ *such that*

$$\big|b(x) - b(y)\big|_K \leq L_b\,|x-y|_K, \quad \forall x, y \in K; \tag{2.3}$$

and the same holds for $\hat{b}$.

(Hp 1-σ) σ *is a Lipschitz map from* K *to* $L(H;K)$ *and we fix a constant* $L_\sigma > 0$ *such that*

$$\big|\sigma(x) - \sigma(y)\big|_{L(H;K)} \leq L_\sigma |x-y|_K, \quad \forall x, y \in K. \tag{2.4}$$

The following existence and uniqueness result is a consequence of straightforward fixed-point arguments.

THEOREM 2.4. *Under Assumption 2.3, for every $\varepsilon > 0$, there exists an adapted process X^ε with continuous trajectories solving equation (S_ε) in a mild-pathwise sense, that is, such that for $\mathbb{P}$ almost every $\omega \in \Omega$, it holds for all $t \in [0, T]$ that*

$$X_t^\varepsilon(\omega) = x_0 + \int_0^t e^{(t-s)A}\Big(b(X_s^\varepsilon(\omega)) + \sigma(X_s^\varepsilon(\omega))\Gamma^\varepsilon[GW]_s(\omega)\Big)ds.$$

Moreover, there exists a unique mild solution $\widehat{X}$ of equation (S) that belongs to $L_W^p(\Omega; C([0, T]; K))$ for all $p > 1$.

2.3. *Controlled equations*

Next, let us introduce the controlled equations we are going to study in this general framework. Let $X^{\varepsilon,u}$ solve

$$\text{(2.5)} \quad \begin{cases} dX_t^{\varepsilon,u} = AX_t^{\varepsilon,u}dt + b(X_t^{\varepsilon,u})dt \\ \qquad\qquad + \sigma(X_t^{\varepsilon,u})\Big(\Gamma^\varepsilon\Big[G\Big(W + \int_0^\cdot r(X_s^{\varepsilon,u}, u_s)ds\Big)\Big]\Big)_t dt, \quad t \in (0, T], \\ X^\varepsilon(0) = x_0, \end{cases}$$

and let X^u solve

$$\text{(2.6)} \quad \begin{cases} dX_t^u = \big(AX_t^u + \hat{b}(X_t^u)\big)\, dt + \sigma(X_t^u)G\, dW_t \\ \qquad\qquad + \sigma(X_t^u)G\, r(X_t^u, u_t)dt, \quad t \in (0, T], \\ X(0) = x_0. \end{cases}$$

Formally speaking, our purpose is to minimize the cost functionals (formally written) over all the admissible controls u

$$\text{(2.7)} \quad \begin{aligned} J^\varepsilon(x_0, u) &= \mathbb{E}\Big[\int_0^T l(X_s^{\varepsilon,u}, u_s)ds + h(X_T^{\varepsilon,u})\Big], \\ J(x_0, u) &= \mathbb{E}\Big[\int_0^T l(X_s^u, u_s)ds + h(X_T^u)\Big]. \end{aligned}$$

However, a precise formalization of the control problem will be given later. For the time being, let us state our main assumptions on the functions r, l, and h.

ASSUMPTION 2.5. *We assume the following:*

(Hp 2-r) *$r : K \times U \to H$ is measurable such that for some constants $M_r, L_r > 0$,*

$$\text{(2.8)} \quad \big|r(x, u)\big|_H \le M_r\big(1 + |u|_U\big), \qquad \forall x \in K,\ u \in U;$$

$$\text{(2.9)} \quad \big|r(x, u) - r(y, u)\big|_H \le L_r\big(|x - y|_K \wedge 1\big)\big(|u|_U + 1\big), \quad \forall x, y \in K,\ u \in U.$$

Moreover, we assume that there exists $u_\star \in U$ such that $r(x, u_\star) = 0$ for all $x \in H$.

(Hp 2-l) $l : K\times U \to \mathbb{R}$ *is a measurable map such that for some constants* $M_l, m_l, c_l > 0$,

$$m_l|u|_U^2 - c_l \le l(x,u) \le M_l\big(1+|u|_U^2\big), \quad \forall x\in K,\ u\in U; \tag{2.10}$$

$$\big|l(x,u)-l(y,u)\big| \le L_l|x-y|_K, \qquad \forall x,y\in K,\ u\in U. \tag{2.11}$$

Notice that the above implies that, for a suitable constant $C_l > 0$,

$$\big|l(x,u)\big| \le C_l\big(1+|u|_U^2\big) \quad \forall x\in K,\ u\in U, \tag{2.12}$$

and hence we deduce that there exists a constant $C > 0$ *such that*

$$\big|l(x,u)-l(y,u)\big| \le C\big(|x-y|_K\wedge 1\big)\big(1+|u|_U^2\big) \quad \forall x,y\in K,\ u\in U. \tag{2.13}$$

(Hp 2-h) $h : K \to \mathbb{R}$ *such that for some constant* $M_h > 0$,

$$\big|h(x)\big| \le M_h, \qquad \forall x\in K; \tag{2.14}$$

$$\big|h(x)-h(y)\big| \le L_h|x-y|_K, \quad \forall x,y\in K. \tag{2.15}$$

REMARK 2.6. Notice that if $r(x,u) = r_0(x)u$ with r_0 bounded and Lipschitz, then assumption (Hp 2-r) holds with $u_\star = 0$.

2.4. *Rigorous formalization of the control problem*

We start by considering, for $\varepsilon > 0$, the formal cost functional

$$J^\varepsilon(x_0,u) = \mathbb{E}\left[\int_0^T l(X_s^{\varepsilon,u},u_s)ds + h(X_T^{\varepsilon,u})\right].$$

If we assume the following boundedness condition on the controls

$$\int_0^T |u_t|_U^2 dt \le c < \infty, \quad \mathbb{P} \text{ almost surely},$$

then a straightforward application of Girsanov transform, together with the fact that the law of the solution to equations (S_ε) does not depend on the specific stochastic basis, yields

$$J^\varepsilon(x_0,u) = \mathbb{E}\left[\mathcal{E}_T\big(r(X^\varepsilon,u)\big)\left(\int_0^T l(X_s^\varepsilon,u_s)\,ds + h(X_T^\varepsilon)\right)\right],$$

where $\mathcal{E}_t(r(X^\varepsilon,u)) := \exp\{\int_0^t r(X_s^\varepsilon,u_s)\,dW_s - \frac{1}{2}\int_0^t |r(X_s^\varepsilon,u_s)|^2\,ds\}$ and, we recall, (X^ε) solves the uncontrolled evolution equation (S_ε).

However, while the condition $u \in L^2_W(\Omega \times [0,T]; U)$ is necessary due to the quadratic dependence of the running cost l on the control variable u (see Assumption (2.10)), the additional requirement

$$\int_0^T |u_t|_U^2 \, dt \le c$$

appears excessively restrictive in this setting. In fact, optimal controls within the space $L^2_W(\Omega \times [0,T]; U)$ are not necessarily expected to satisfy such a bound. This is illustrated, for instance, by the form of the optimal feedback control $\hat{u}$ in Example 3.3.

For this reason, we do not impose the condition of $\mathbb{P}$-essential boundedness on the $L^2([0,T]; U)$ norm of the control trajectories. This choice, along with our adoption of a weak probabilistic formulation of the control problem, introduces several technical challenges.

In particular, even the characterization of the class of admissible controls becomes non-trivial. We now introduce the following definition, which we consider to be a natural one.

Definition 2.7. For every $u \in L^{2,\mathrm{loc}}_W(\Omega \times [0,T]; U)$, we set the following.

(1) Let τ_n, $n \in \mathbb{N}$, be a sequence of stopping times defined by

$$\tau_n := \inf\left\{t \ge 0 : \int_0^t |u_s|_U^2 \, ds \ge n\right\}.$$

(2) Let $u_\star$ be such that $r(x, u_\star) = 0$ for every $x \in H$ (see (Hp 2-r) in Assumption 2.5) and let u^n denote the control

$$u^n_s := u_s I_{\{0 \le s \le \tau_n \wedge T\}} + u_\star I_{\{\tau_n \wedge T < s \le T\}}, \quad s \in [0,T].$$

(3) Let $\mathcal{E}_t(r(X^\varepsilon, u^n))$ be the exponential martingale

$$\mathcal{E}_t\big(r(X^\varepsilon, u^n)\big) := \exp\left\{\int_0^t r(X^\varepsilon_s, u^n_s) \, dW_s - \frac{1}{2}\int_0^t \big|r(X^\varepsilon_s, u^n_s)\big|^2 \, ds\right\}.$$

We define the space of admissible controls $\mathcal{U}^\varepsilon_{\mathrm{ad}}$ as the space:

$$\mathcal{U}^\varepsilon_{\mathrm{ad}} := \left\{u \in L^2_W(\Omega \times [0,T]; U) : \sup_{n \in \mathbb{N}} \mathbb{E}\left(\mathcal{E}_T\big(r(X^\varepsilon, u^n)\big) \int_0^{T \wedge \tau^n} |u_s|_U^2 \, ds\right) < \infty\right\}.$$

Notice that $\mathcal{U}^\varepsilon_{\mathrm{ad}}$, in the definition above, may in principle depend on x_0.

Remark 2.8. Notice that, in view of the fact that $r(x, u_\star) = 0$, we have

$$\mathcal{E}_T\big(r(X^\varepsilon, u^n)\big) = \mathcal{E}_{T \wedge \tau^n}\big(r(X^\varepsilon, u^n)\big) = \mathcal{E}_{T \wedge \tau^n}\big(r(X^\varepsilon, u)\big).$$

REMARK 2.9. The sequence $n \mapsto \mathbb{E}(\mathcal{E}_T(r(X^\varepsilon, u^n)) \int_0^{T\wedge\tau^n} |u_s|_U^2\, ds)$ is non-decreasing in n. Indeed, if $m < n$, we have $\tau_m \le \tau_n$ almost surely, and therefore

$$\begin{aligned}
&\mathbb{E}\bigg(\mathcal{E}_T\big(r(X^\varepsilon, u^n)\big) \int_0^{T\wedge\tau^n} |u_s|_U^2\, ds\bigg) \\
&\quad = \mathbb{E}\bigg(\mathcal{E}_{T\wedge\tau^n}\big(r(X^\varepsilon, u^n)\big) \int_0^{T\wedge\tau^n} |u_s|_U^2\, ds\bigg) \\
&\quad \ge \mathbb{E}\bigg(\mathcal{E}_{T\wedge\tau^n}\big(r(X^\varepsilon, u^n)\big) \int_0^{T\wedge\tau^m} |u_s|_U^2\, ds\bigg) \\
&\quad = \mathbb{E}\bigg(\mathbb{E}\bigg(\mathcal{E}_{T\wedge\tau^n}\big(r(X^\varepsilon, u^n)\big) \int_0^{T\wedge\tau^m} |u_s|_U^2\, ds \Big| \mathcal{F}^W_{\tau_m\wedge T}\bigg)\bigg) \\
&\quad = \mathbb{E}\bigg(\mathcal{E}_{T\wedge\tau^m}\big(r(X^\varepsilon, u^n)\big) \int_0^{T\wedge\tau^m} |u_s|_U^2\, ds\bigg) \\
&\quad = \mathbb{E}\bigg(\mathbb{E}\bigg(\mathcal{E}_{T\wedge\tau^m}\big(r(X^\varepsilon, u^m)\big) \int_0^{T\wedge\tau^m} |u_s|_U^2\, ds \Big| \mathcal{F}^W_{\tau_m\wedge T}\bigg)\bigg) \\
&\quad = \mathbb{E}\bigg(\mathcal{E}_T\big(r(X^\varepsilon, u^m)\big) \int_0^{T\wedge\tau^m} |u_s|_U^2\, ds\bigg).
\end{aligned}$$

Thus, if $u \in \mathcal{U}^\varepsilon_{\mathrm{ad}}$, then there exists finitely the limit

$$\lim_{n\to+\infty} \mathbb{E}\bigg(\mathcal{E}_T\big(r(X^\varepsilon, u^n)\big) \int_0^{T\wedge\tau^n} |u_s|_U^2\, ds\bigg) \in \mathbb{R}.$$

REMARK 2.10. To further justify the choice of the class $\mathcal{U}^\varepsilon_{\mathrm{ad}}$ of admissible controls, recall that if we define $d\,\mathbb{P}^n := \mathcal{E}_{T\wedge\tau^n}(r(X^\varepsilon, u^n))d\,\mathbb{P}$ and $W_t^n := W_t - \int_0^t r(X_s^\varepsilon, u_s^n)ds$, then (X^ε) satisfies (2.5) with (W) replaced by the $\mathbb{P}^n$ Wiener process $(W_t^n)_{t\ge 0}$. Namely,

$$\begin{cases}
dX_t^\varepsilon = AX_t^\varepsilon dt + b(X_t^{\varepsilon,u})dt \\
\qquad\qquad + \sigma(X_t^\varepsilon)\bigg(\Gamma^\varepsilon\bigg[G\bigg(W^n + \int_0^\cdot r(X_s^\varepsilon, u_s)ds\bigg)\bigg]\bigg)_t dt, \quad t \in (0, T], \\
X^\varepsilon(0) = x_0,
\end{cases}$$

and $\mathbb{E}(\mathcal{E}_{T\wedge\tau^n}(r(X^\varepsilon, u^n)) \int_0^{T\wedge\tau^n} |u_s|_U^2\, ds)$ coincides with $\mathbb{E}^{\mathbb{P}^n}(\int_0^{T\wedge\tau^n} |u_s|_U^2\, ds)$ and it seems natural to ask that $\sup_n \mathbb{E}^{\mathbb{P}^n}(\int_0^{T\wedge\tau^n} |u_s|_U^2\, ds) < \infty$.

We are eventually ready to rigorously introduce our cost functional J^ε. Namely, we set for any $u \in \mathcal{U}^\varepsilon_{\mathrm{ad}}$,

$$J^\varepsilon(x_0, u) := \lim_{n\to\infty} \mathbb{E}\bigg[\mathcal{E}_T\big(r(X^\varepsilon, u^n)\big)\bigg(\int_0^T l(X_s^\varepsilon, u_s^n)\, ds + h(X_T^\varepsilon)\bigg)\bigg]. \tag{2.16}$$

Such functional is well defined; indeed, we can prove the following result.

PROPOSITION 2.11. *For any $u \in \mathcal{U}^{\varepsilon}_{\mathrm{ad}}$, the cost functional $J^{\varepsilon}(x_0, u)$ given in (2.16) is well defined.*

PROOF. We show that $\mathbb{E}(\mathcal{E}_T(r(X^{\varepsilon}, u^n)) \int_0^T l(X^{\varepsilon}_s, u^n_s)\, ds)$ is a (real-valued) Cauchy sequence. Let $m > n$. By the martingale property of $(\mathcal{E}_t(r(X^{\varepsilon}, u)))_t$ and (2.12), we have

$$
\begin{aligned}
(2.17)\quad & \left|\mathbb{E}\left(\mathcal{E}_T\big(r(X^{\varepsilon}, u^m)\big)\int_0^T l(X^{\varepsilon}_s, u^m_s)\, ds\right) - \mathbb{E}\left(\mathcal{E}_T\big(r(X^{\varepsilon}, u^n)\big)\int_0^T l(X^{\varepsilon}_s, u^n_s)\, ds\right)\right| \\
&\le \left|\mathbb{E}\left(\mathcal{E}_{T\wedge\tau_m}\big(r(X^{\varepsilon}, u)\big)\int_0^{T\wedge\tau_n} l(X^{\varepsilon}_s, u^m_s)\, ds\right)\right. \\
&\quad + \mathbb{E}\left(\mathcal{E}_{T\wedge\tau_m}\big(r(X^{\varepsilon}, u)\big)\int_{T\wedge\tau_n}^{T} l(X^{\varepsilon}_s, u^m_s)\, ds\right) \\
&\quad - \mathbb{E}\left(\mathcal{E}_{T\wedge\tau_n}\big(r(X^{\varepsilon}, u)\big)\int_0^{T\wedge\tau_n} l(X^{\varepsilon}_s, u_s)\, ds\right) \\
&\quad \left. - \mathbb{E}\left(\mathcal{E}_{T\wedge\tau_n}\big(r(X^{\varepsilon}, u)\big)\int_{T\wedge\tau_n}^{T} l(X^{\varepsilon}_s, u_\star)\, ds\right)\right| \\
&= \left|\mathbb{E}\left(\mathcal{E}_{T\wedge\tau_m}\big(r(X^{\varepsilon}, u)\big)\int_{T\wedge\tau_n}^{T} l(X^{\varepsilon}_s, u^m_s)\, ds\right)\right. \\
&\quad \left. - \mathbb{E}\left(\mathcal{E}_{T\wedge\tau_n}\big(r(X^{\varepsilon}, u)\big)\int_{T\wedge\tau_n}^{T} l(X^{\varepsilon}_s, u_\star)\, ds\right)\right| \\
&\le \left|\mathbb{E}\left(\mathcal{E}_{T\wedge\tau_m}\big(r(X^{\varepsilon}, u)\big)\int_{T\wedge\tau_n}^{T\wedge\tau_m} l(X^{\varepsilon}_s, u_s)\, ds\right)\right| \\
&\quad + TC_l\big(1 + |u_\star|^2_U\big)\big(\mathbb{E}\big(\mathcal{E}_{T\wedge\tau_n}\big(r(X^{\varepsilon}, u)\big)I_{\{\tau_n<T\}}\big) \\
&\quad + \mathbb{E}\big(\mathcal{E}_{T\wedge\tau_m}\big(r(X^{\varepsilon}, u)\big)I_{\{\tau_m<T\}}\big)\big).
\end{aligned}
$$

We start from the last two terms. It holds, by the Markov inequality, that

$$
\begin{aligned}
\mathbb{E}\big(\mathcal{E}_{T\wedge\tau_n}\big(r(X^{\varepsilon}, u)\big)I_{\{\tau_n<T\}}\big) &= \mathbb{E}\big(\mathcal{E}_{T\wedge\tau_n}\big(r(X^{\varepsilon}, u)\big)I_{\{\int_0^{T\wedge\tau_n}|u_s|^2_U\, ds \ge n\}}\big) \\
&\le \frac{1}{n}\,\mathbb{E}\left(\mathcal{E}_{T\wedge\tau_n}\big(r(X^{\varepsilon}, u)\big)\int_0^{T\wedge\tau_n}|u_s|^2_U\, ds\right),
\end{aligned}
$$

and the same holds for $\mathbb{E}(\mathcal{E}_{T\wedge\tau_m}(r(X^{\varepsilon}, u))I_{\{\tau_m<T\}})$. In particular, since $u \in \mathcal{U}_{\mathrm{ad}}$, both terms go to zero as $n, m \to \infty$.

Regarding the first term, we notice that it is smaller than

$$
\begin{aligned}
&\mathbb{E}\left(\mathcal{E}_{T\wedge\tau_m}\big(r(X^{\varepsilon}, u)\big)\int_{T\wedge\tau_n}^{T\wedge\tau_m}|u_s|^2_U\, ds\right) \\
&= \mathbb{E}\left(\mathcal{E}_{T\wedge\tau_m}\big(r(X^{\varepsilon}, u)\big)\int_0^{T\wedge\tau_m}|u_s|^2_U\, ds\right) - \mathbb{E}\left(\mathcal{E}_{T\wedge\tau_n}\big(r(X^{\varepsilon}, u)\big)\int_0^{T\wedge\tau_n}|u_s|^2_U\, ds\right).
\end{aligned}
$$

Since $u \in \mathcal{U}^{\varepsilon}_{\mathrm{ad}}$, the sequence $(\mathbb{E}(\mathcal{E}_{T\wedge\tau_n}(r(X^{\varepsilon},u))\int_0^{T\wedge\tau_n}|u_s|_U^2\,ds))_n$ is a Cauchy sequence, see also Remark 2.9. Therefore, the difference above as well converges to zero as $n,m\to 0$.

In a similar way, we show that $\mathbb{E}(\mathcal{E}_T(r(X^{\varepsilon},u^n))h(X^{\varepsilon}_T))$ is a Cauchy sequence. ■

We can define the admissible controls $\mathcal{U}^{\varepsilon}$ and the cost functional of the limit control problem (2.6) in a similar way.

3. BSDE representation of the value function and of the optimal control

3.1. *Hamiltonian function associated with the cost functional*

We introduce the Hamiltonian function ψ:

$$\psi : K\times H^* \to \mathbb{R},\quad \psi(x,z) := \inf_{u\in U}\left\{l(x,u) - \langle z, r(x,u)\rangle\right\},$$

where $\langle z, r(x,u)\rangle$ denotes the duality between H and H^*. Thanks to Assumptions 2.5, we have the following.

Corollary 3.1. *The function ψ has the following properties (for suitable constants M_ψ and L_ψ):*

$$\left|\psi(x,z)\right| \le M_\psi\left(1+|z|^2_{H^*}\right) \qquad \forall x\in K,\ \forall z\in H^*, \tag{3.1}$$

$$\left|\psi(x,z)-\psi(x,z')\right| \le L_\psi\left(1+|z|_{H^*}+|z'|_{H^*}\right)|z-z'|_{H^*} \quad \forall x\in K,\ \forall z,z'\in H^*, \tag{3.2}$$

$$\left|\psi(x',z)-\psi(x,z)\right| \le L_\psi\left(1+|z|^2_{H^*}\right)\left(|x-x'|_K\wedge 1\right) \quad \forall x,x'\in K,\ \forall z\in H^*. \tag{3.3}$$

Proof. By (2.12) and (2.8), we easily get that

$$\psi(x,z) \le l(x,u_\star) - \langle z, r(x,u_\star)\rangle \le C_l\left(1+|u_\star|^2_U\right).$$

On the other hand, there exists a finite constant c such that, for every u satisfying $|u|_U \ge c(1+|z|_{H^*})$, it holds that, recalling that $m_l>0$,

$$l(x,u) - \langle z, r(x,u)\rangle \ge -c_l + m_l|u|^2_U - M_r|z|_{H^*}\left(1+|u|_U\right) \ge 0,$$

while for u satisfying $|u|_U \le c(1+|z|_{H^*})$, we have

$$\begin{aligned} l(x,u) - \langle z, r(x,u)\rangle &\ge -c_l + m_l|u|^2_U - M_r|z|_{H^*}\left(1+|u|_U\right)\\ &\ge -(c_l + cM_r + M_r)\left(1+|z|^2_{H^*}\right). \end{aligned}$$

Hence, we deduce that there exists a constant M_ψ such that

$$\left|\psi(x,z)\right| \le M_\psi\left(1+|z|^2_{H^*}\right).$$

Proceeding as above, we notice that

$$l(x,u)-\big\langle z,r(x,u)\big\rangle \geq -c_l+m_l|u|_U^2-M_r|z|_{H^*}\big(1+|u|_U\big)\geq l(x,u_\star)-\langle z,r(x,u_\star)\rangle$$

whenever $|u|\geq c(1+|z|)$ for a suitable constant c. Thus,

$$\psi(x,z)=\inf_{u\in U,\,|u|\leq c(1+|z|)}\big\{l(x,u)-\big\langle z,r(x,u)\big\rangle\big\}.$$

Next, the difference $|\psi(x,z)-\psi(x,z')|$ is controlled from above with

$$\begin{aligned}
&\Bigg|\inf_{|u|_U\leq c(1+|z|_{H^*}+|z'|_{H^*})}\big(l(x,u)-\big\langle z,r(x,u)\big\rangle\big)\\
&\qquad-\inf_{|u|_U\leq c(1+|z|_{H^*}+|z'|_{H^*})}\big(l(x,u)-\big\langle z',r(x,u)\big\rangle\big)\Bigg|\\
&\quad\leq \sup_{|u|_U\leq c(1+|z|_{H^*}+|z'|_{H^*})}|z-z'|_{H^*}\big|r(x,u)\big|\\
&\quad\leq (cM_r+M_r)\big(1+|z|_{H^*}+|z'|_{H^*}\big)|z-z'|_{H^*}\\
&\quad\leq L_\psi\big(1+|z|_{H^*}+|z'|_{H^*}\big)|z-z'|_{H^*}.
\end{aligned}$$

Finally, in view of (2.9) and (2.13), the difference $|\psi(x,z)-\psi(x',z)|$ is controlled from above with

$$\begin{aligned}
&\Bigg|\inf_{|u|_U\leq c(1+|z|_{H^*})}\big(l(x,u)-\big\langle z,r(x,u)\big\rangle\big)-\inf_{|u|_U\leq c(1+|z|_{H^*})}\big(l(x',u)-\big\langle z,r(x',u)\big\rangle\big)\Bigg|\\
&\quad\leq \sup_{|u|_U\leq c(1+|z|_{H^*})}\big|l(x,u)-l(x',u)\big|+\sup_{|u|_U\leq c(1+|z|_{H^*})}|z|_{H^*}\big|r(x,u)-r(x',u)\big|\\
&\quad\leq 2cC_l\big(|x-x'|_K\wedge 1\big)\big(1+|z|_{H^*}\big)^2+cL_r|z|_{H^*}\big(1+|z|_{H^*}\big)\big(|x-x'|_K\wedge 1\big)\\
&\quad\leq L_\psi\big(1+|z|_{H^*}^2\big)\big(|x-x'|_K\wedge 1\big).
\end{aligned}$$

■

In the following, we assume that the infimum of the definition of ψ is indeed achieved.

ASSUMPTION 3.2. *There exists a measurable function $\underline{u}(x,z):K\times H^*\to U$ such that*

(1) $\psi(x,z)=\inf_{u\in U}\{l(x,u)-\langle z,r(x,u)\rangle\}=l(x,\underline{u}(x,z))-\langle z,r(x,\underline{u}(x,z))\rangle$;

(2) *there exists a constant $L_{\underline{u}}>0$, such that*

$$\big|\underline{u}(x,z)-\underline{u}(x',z')\big|_U\leq L_{\underline{u}}\big[|z-z'|_{H^*}+\big(1+|z|_{H^*}\big)\big(1\wedge|x-x'|_K\big)\big]\quad\forall x,x'\in K,\ \forall z,z'\in H^*. \tag{3.4}$$

EXAMPLE 3.3. Assume that $U=H$ and let $l(x,u):=l_0(x)+|u|_K^2$ and $r(x,u):=r_0(x)u$ with l_0 and r_0 bounded continuous functions $K\to\mathbb{R}$ and $K\to\mathcal{L}(H)$, respectively. In this case, if one identifies H^* with H by the canonical Riesz isomorphism, one gets

$$\psi(x,z) = l_0(x) - \frac{1}{4}\left|r_0(x)^* z\right|^2_{H^*} \quad \text{and} \quad \underline{u}(x,z) = \frac{1}{2} r_0(x)^* z.$$

Thus, Assumptions 3.2 are verified.

3.2. *BMO martingales*

For the reader's convenience and in order to fix the notation, we report here a few basic facts on BMO martingales, following [3, 22].

Let $T \in (0,\infty)$ be given. A continuous $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ local martingale is a BMO_2 martingale on the time interval $[0,T]$ if

$$\begin{aligned}\|M\|_{\mathrm{BMO}_2} &:= \sup_{\tau\in\mathcal{T}} \left\| \mathbb{E}\Big((M_T - M_\tau)^2 \big| \mathcal{F}_\tau\Big)^{1/2} \right\|_{L^\infty(\Omega)} \\ &= \sup_{\tau\in\mathcal{T}} \left\| \mathbb{E}\Big(\langle M\rangle_T - \langle M\rangle_\tau \big| \mathcal{F}_\tau\Big)^{1/2} \right\|_{L^\infty(\Omega)} < \infty,\end{aligned}$$

where τ in the supremum varies in the class $\mathcal{T}$ of all stopping times satisfying $\tau \le T$ almost surely. If (Ψ) is a process in $L^{2,\mathrm{loc}}_W(\Omega \times [0,T]; H^*)$ and $M_t = \int_0^t \Psi_s dW_s$, then

$$\|M\|^2_{\mathrm{BMO}_2} = \sup_{\tau\in\mathcal{T}} \left\| \mathbb{E}\Big(\int_\tau^T |\Psi_s|^2_{H^*} ds \big| \mathcal{F}_\tau \Big) \right\|_{L^\infty(\Omega)}, \tag{3.5}$$

whenever the right-hand side is finite.

Moreover, again in the particular case $M_t = \int_0^t \Psi_s dW_s$, by [22, p. 26] (see also [3, formula (13), p. 831]), one has that for all $p \ge 1$, there exists a finite constant $c(p)$ such that

$$\mathbb{E}\Big(\int_0^T |\Psi_s|^2_{H^*} ds \Big)^p \le c(p) \|M\|^{2p}_{\mathrm{BMO}_2}. \tag{3.6}$$

Finally, the exponential martingale

$$\mathcal{E}(\Psi)_t := \exp\left(\int_0^t \Psi_s dW_s - \frac{1}{2}\int_0^t |\Psi_s|^2_{H^*} ds \right)$$

is uniformly integrable and, by [3, formula (6), p. 824], there exists $q^* > 1$, depending only on $\|M\|_{\mathrm{BMO}_2}$, such that for all $q \in (1, q^*)$, there is a suitable finite constant $C(q, \|M\|_{\mathrm{BMO}_2})$ such that for every stopping time $\tau \le T$, it holds that

$$\mathbb{E}\big(\mathcal{E}(M)^q_T | \mathcal{F}_\tau\big) \le C\big(q, \|M\|_{\mathrm{BMO}_2}\big) \mathcal{E}(M)^q_\tau. \tag{3.7}$$

In particular, taking $\tau = 0$, one gets

$$\mathbb{E}\big(\mathcal{E}(M)^q_T\big) \le C\big(q, \|M\|_{\mathrm{BMO}_2}\big). \tag{3.8}$$

3.3. *BSDE representation*

We are in the position to prove the following.

THEOREM 3.4. *Under Assumptions 2.3 and 2.5, there exists a unique triple of stochastic processes* $(X^\varepsilon, Y^\varepsilon, Z^\varepsilon)$ *adapted to the filtration* $(\mathcal{F}_t^W)_{t\in[0,T]}$ *such that* X^ε *has continuous trajectories,* $Y^\varepsilon \in L_W^\infty(\Omega; C([0,T];\mathbb{R}))$*,* $Z^\varepsilon \in L_W^2(\Omega \times [0,T]; H^*)$*, and* $(X^\varepsilon, Y^\varepsilon, Z^\varepsilon)$ *is a solution to the following system:*

$$\begin{cases} \dfrac{d}{dt}X_t^\varepsilon = \big(AX_t^\varepsilon + b(X_t^\varepsilon)\big) + \sigma(X_t^\varepsilon)\big(\Gamma^\varepsilon[GW]\big)_t & t \in (0,T], \\ -dY_t^\varepsilon = \psi(X_t^\varepsilon, Z_t^\varepsilon)\,dt - Z_t^\varepsilon\,dW_t \\ X^\varepsilon(0) = x_0, \quad Y_T^\varepsilon = h(X_T^\varepsilon). \end{cases} \tag{3.9}$$

Moreover,

$$\sup_{t\in[0,T]} |Y_t^\varepsilon|_{L_W^\infty(\Omega;C([0,T];\mathbb{R}))} + \left\| \int_0^\cdot Z_s^\varepsilon d W_s \right\|_{\mathrm{BMO}_2} \le \kappa, \tag{3.10}$$

where $\kappa > 0$ *is independent of* ε*.*

PROOF. By Theorem 2.4, the forward equation has a unique solution

$$X^\varepsilon \in L_W^2\big(\Omega; C\big([0,T]; K\big)\big).$$

Following [3, Proposition 11], see also [23, Proposition 2.1], there exists a unique $(Y^\varepsilon, Z^\varepsilon)$, such that

$$\sup_{t\in[0,T]} |Y_t^\varepsilon|_{L_W^\infty(\Omega;C([0,T];\mathbb{R}))} \le M_h + M_\psi T, \tag{3.11}$$

$$\mathbb{E}\int_0^T |Z^\varepsilon|_{H^*}^2\,dt \le C \tag{3.12}$$

for a constant $C > 0$ depends only on M_h, M_ψ, T.

Let us check the uniform bound for the BMO_2 norm of the martingale term in (3.10). We follow again [3].

We apply the Itô formula to $\phi(Y_t + m)$, where m is chosen so that $Y_t + m \ge 0$ and $\phi(x) = (e^{2Cx} - 2Cx - 1)/(2C^2)$, so that for all $x \ge 0$, $\phi'(x) \ge 0$ and $\frac{1}{2}\phi''(x) - C\phi'(x) = 1$ are satisfied, for some $C > M_\psi$, given in Corollary 3.1. Then, taking the conditional expectation with respect to $\mathcal{F}_\tau$, for any stopping time $\tau \le T$, we have the following (we avoid the subscript in the norms for simplicity):

$$\phi(Y_\tau + m) + \frac{1}{2}\mathbb{E}^{\mathcal{F}_\tau}\left(\int_\tau^T \phi''(Y_s + m)|Z_s|^2\,ds\right) = \mathbb{E}^{\mathcal{F}_\tau}\phi(Y_T + m) + \mathbb{E}^{\mathcal{F}_\tau}\left(\int_\tau^T \phi'(Y_s + m)\psi(Z_s)\,ds\right).$$

Thus, for every $\tau \le T$,

$$\begin{aligned}
&\phi(Y_\tau + m) + \frac{1}{2}\mathbb{E}^{\mathcal{F}_t}\left(\int_\tau^T |Z_s|^2\, ds\right) \\
&\quad = \mathbb{E}^{\mathcal{F}_t}\phi(Y_T + m) + \mathbb{E}^{\mathcal{F}_t}\left(\int_\tau^T \phi'(Y_s + m)[\psi(Z_s) - C|Z_s|^2]\, ds\right) \\
&\quad \le \mathbb{E}^{\mathcal{F}_t}\phi(Y_T + m) + M_\psi \mathbb{E}^{\mathcal{F}_t}\left(\int_\tau^T \phi'(Y_s + m)\, ds\right). \qquad \blacksquare
\end{aligned}$$

THEOREM 3.5. *Under Assumptions 2.3, 2.5, and 3.2, we have that*

$$Y_0^\varepsilon = \inf_{u \in \mathcal{U}_{\mathrm{ad}}^\varepsilon} J^\varepsilon(u) = J^\varepsilon\big(\underline{u}(X^\varepsilon, Z^\varepsilon)\big), \tag{3.13}$$

where $(X^\varepsilon, Y^\varepsilon, Z^\varepsilon)$ is given in Theorem 3.4 and $\underline{u}$ is defined in Assumption 3.2.

PROOF. We need to get a fundamental relation for a generic $u \in \mathcal{U}_{\mathrm{ad}}^\varepsilon$, or at least for its approximations.

Proceeding as in Remark 2.10, we apply Girsanov transformation to ensure that

$$W_t^n := -\int_0^t r(X_s^\varepsilon, u_s^n)\, ds + W_t$$

is a cylindrical Wiener process under the probability $d\mathbb{P}^n := \mathcal{E}_T(r(X^\varepsilon, u^n))d\mathbb{P}$. Thus,

$$-dY_t^\varepsilon = \psi(X_t^\varepsilon, Z_t^\varepsilon)\, dt - Z_t^\varepsilon\, dW_t = \big[\psi(X_t^\varepsilon, Z_t^\varepsilon) - Z_s^\varepsilon r(X_t^\varepsilon, u_t^n)\big]\, dt - Z_t^\varepsilon\, dW_t^n.$$

Adding and subtracting the current cost and integrating between 0 and T, we have

$$Y_0^\varepsilon = \mathbb{E}\left(\mathcal{E}_T\big(r(X^\varepsilon, u^n)\big)\int_0^T \big[\psi(X_t^\varepsilon, Z_t^\varepsilon) - Z_s^\varepsilon r(X_t^\varepsilon, u_t^n) - l(X_t^\varepsilon, u_t^n)\big]\, dt\right) + J^{\varepsilon,n}(u),$$

where $J^{\varepsilon,n}(u) := \mathbb{E}[\mathcal{E}_T(r(X^\varepsilon, u^n))(\int_0^T (l(X_s^\varepsilon, u_s^n))\, ds + h(X_T^\varepsilon))]$.

The definition of ψ yields $J^{n,\varepsilon}(u) \ge Y_0^\varepsilon$ for every n and consequently, by definition (2.16),

$$J^\varepsilon(u) = \lim_{n\to\infty} J^{\varepsilon,n}(u) \ge Y_0^\varepsilon \quad \forall u \in \mathcal{U}_{\mathrm{ad}}^\varepsilon. \tag{3.14}$$

Now we define $\underline{u}^\varepsilon(s) = \underline{u}(X_s^\varepsilon, Z_s^\varepsilon)$, where $\underline{u}$ is given in Assumption 3.2 and $(X^\varepsilon, Z^\varepsilon)$ is the solution of (3.9). From Assumption 3.2, we have that

$$\big|\underline{u}^\varepsilon(s)\big| \le C_{\underline{u}}\big(1 + |Z_s^\varepsilon|\big) \tag{3.15}$$

for some constant $C_{\bar{u}}$.

We have to show that $\underline{u}^\varepsilon \in \mathcal{U}^\varepsilon_{\mathrm{ad}}$ and $Y_0^\varepsilon = J^e(\underline{u}^\varepsilon)$. To this purpose, we define (see Definition 2.7) the following:

(1) $\overline{\tau}_n = \inf\{0 \le t \le T : \int_0^t |\underline{u}^\varepsilon(s)|^2\, ds \ge n\}$,

(2) $\bar{u}^n(s) = \underline{u}^\varepsilon(s) I_{[0,\tau_n]}(s) + u^\star I_{(\tau_n,T]}(s)$,

(3) $\overline{W}_t^n : -\int_0^t r(X_s^\varepsilon, \bar{u}^n(s))ds + W_t$.

The backward component in (3.9) can be rewritten as

$$\begin{aligned} -dY_t^\varepsilon &= \big[\psi(X_t^\varepsilon, Z_t^\varepsilon) - Z_t^\varepsilon r\big(X_t^\varepsilon, \bar{u}^n(t)\big)\big]dt - Z_t^\varepsilon d\, W_t^n, \quad t \in [0,T],\\ Y_T^\varepsilon &= h(X_T^\varepsilon). \end{aligned}$$

Thus, recalling point (1) in Assumption 3.2,

$$\begin{aligned} Y_0^\varepsilon &= \mathbb{E}\big(\mathcal{E}_T\big(r(X^\varepsilon, \bar{u}^n)h(X_T^\varepsilon)\big)\big)\\ &\quad + \mathbb{E}\bigg(\mathcal{E}_T\big(r(X^\varepsilon, \bar{u}^n)\big)\int_0^T l\big(X_s^\varepsilon, \bar{u}^n(s)\big)\, ds\bigg). \end{aligned} \tag{3.16}$$

By (2.8) and (3.15), we have, $\mathbb{P}$-a.s.,

$$\big|r\big(X_s^\varepsilon, \bar{u}^n(s)\big)\big| \le M_r(1 + C_{\underline{u}}) + M_r C_{\underline{u}}|Z_s^\varepsilon|.$$

Thus, by (3.10),

$$\sup_{\varepsilon>0, n\in\mathbb{N}} \bigg\| \int_0^\cdot r^*\big(X_s^\varepsilon, \bar{u}^n(s)\big)d\, W_s \bigg\|_{\mathrm{BMO}_2} < +\infty$$

and finally the above estimate together with (3.8) yields that there exists $q > 1$ such that

$$\sup_{\varepsilon>0, n\in\mathbb{N}} \mathbb{E}\big(\mathcal{E}_T\big(r(X^\varepsilon, \bar{u}^n)\big)^q\big) < \infty. \tag{3.17}$$

Again, by (3.10) and (3.6), we obtain that, for every $p \ge 1$,

$$\sup_{\varepsilon>0} \mathbb{E}\bigg(\int_0^T |Z_s^\varepsilon|^2\, ds\bigg)^{p/2} < \infty. \tag{3.18}$$

Summing up, we have that $\lim_{n\to+\infty} \bar{u}_s^n = \underline{u}_s^\varepsilon = \underline{u}(X_s^\varepsilon, Z_s^\varepsilon)$ for all $s \in [0,T]$, $\mathbb{P}$-a.s.

Moreover, for all $p \ge 1$,

$$\begin{aligned} \mathbb{E}\bigg(\int_0^T |\bar{u}_s^n|^2\, ds\bigg)^{p/2} &\le c(p)T^{p/2}|u^\star|^p + c(p)\mathbb{E}\bigg(\int_0^T \big|\underline{u}(X_s^\varepsilon, Z_s^\varepsilon)\big|^2\, ds\bigg)^{p/2}\\ &\le \widetilde{C} + \widetilde{C}\mathbb{E}\bigg(\int_0^T |Z_s^\varepsilon|^2\, ds\bigg)^{p/2} < +\infty. \end{aligned}$$

In view of (3.17) and of the above estimate, the sequence

$$\left(\mathcal{E}_T\big(r(X^\varepsilon,\bar u^n)\big)\int_0^T|\bar u^n_s|^2\,ds\right)_n$$

turns out to be uniformly integrable. Moreover, $\int_0^T|\bar u^n_s|^2\,ds\to\int_0^T|\underline{u}^\varepsilon_s|^2\,ds$ $\mathbb{P}$-a.s. and

$$\mathcal{E}_T\big(r(X^\varepsilon,\bar u^n)\big)\to\exp\left\{\int_0^T r(X^\varepsilon_s,\underline{u}^\varepsilon_s)dW_s-\frac{1}{2}\int_0^T\left|r(X^\varepsilon_s,\underline{u}^\varepsilon_s)\right|^2ds\right\}\ \mathbb{P}\text{-a.s.}$$

Thus, the limit $\lim_{n\to\infty}\mathbb{E}(\mathcal{E}_T(r(X^\varepsilon,\bar u^n))\int_0^T|\bar u^\varepsilon_s|^2\,ds)$ exists in $\mathbb{R}$ and we can conclude that $\underline{u}^\varepsilon_s\in\mathcal{U}^\varepsilon_{\mathrm{ad}}$.

In a similar way, taking into account (2.12) and (2.14), we get that both

$$\left(\mathcal{E}_T\big(r(X^\varepsilon,\bar u^n)h(X^\varepsilon_T)\big)\right)_n\ \text{and}\ \left(\mathcal{E}_T\big(r(X^\varepsilon,\bar u^n)\big)\int_0^T l\big(X^\varepsilon_s,\bar u^n(s)\big)\,ds\right)_n$$

are uniformly integrable and $\mathbb{P}$-a.s. converging sequences of random variables. Letting $n\to\infty$ in (3.16), we have that

$$Y^\varepsilon_0=\lim_{n\to\infty}\mathbb{E}\left[\mathcal{E}_T\big(r(X^\varepsilon,\bar u^n)\big)\left(\int_0^T l(X^\varepsilon_s,\bar u^n_s)\,ds+h(X^\varepsilon_T)\right)\right]=J^\varepsilon(\underline{u}^\varepsilon)$$

and the claim follows. ■

Following exactly the same argument as in the proof of Theorem 3.5, we have the following result, whose proof is omitted for the sake of brevity.

THEOREM 3.6. *Under Assumptions 2.3, 2.5, and 3.2, there exists a unique solution* $(\widehat X,\widehat Y,\widehat Z)$ *with* $\widehat X\in L^2_W(\Omega;C([0,T];K))$, $\widehat Y\in L^2_W(\Omega;C([0,T];\mathbb{R}))$, $\widehat Z\in L^2_W(\Omega\times[0,T];H^*)$ *to*

$$\begin{cases} d\widehat X_t=\big(A\widehat X_t+\hat b(\widehat X_t)\big)\,dt+\sigma(\widehat X_t)G\,dW_t, & t\in(0,T],\\ -d\widehat Y_t=\psi(\widehat X_t,\widehat Z_t)\,dt-\widehat Z_t\,dW_t, & \\ X(0)=x_0,\quad \widehat Y_T=h(\widehat X_T). & \end{cases}\tag{3.19}$$

Moreover,

$$\sup_{t\in[0,T]}|\widehat Y_t|_{L^\infty_W(\Omega;C([0,T];\mathbb{R}))}+\left\|\int_0^\cdot\widehat Z_s\,dW_s\right\|_{\mathrm{BMO}_2}\le\kappa.\tag{3.20}$$

Finally, if

$$\hat u_t:=\underline{u}(\widehat X_t,\widehat Z_t),\tag{3.21}$$

where $\underline{u}$ *is given in Assumption 3.2, then,* $\hat u$ *is admissible (that is, belongs to* $\mathcal{U}_{\mathrm{ad}}$*) and*

$$\widehat Y_0=\inf_{u\in\mathcal{U}_{\mathrm{ad}}}\widehat J(u)=\widehat J(\hat u),$$

where

$$\widehat{J}(u) := \lim_{n\to\infty} \mathbb{E}\Big[\mathcal{E}_T\big(r(\widehat{X}, \hat{u}^n)\big)\Big(h(\widehat{X}_T) + \int_0^T \big(l(\widehat{X}_s, \hat{u}^n_s)\big)\, ds\Big)\Big]$$

and $\hat{\tau}_n = \inf\{0 \le t \le T : \int_0^t |\hat{u}(s)|^2\, ds \ge n\}$, $\hat{u}^n(s) = \hat{u}(s) I_{[0,\hat{\tau}_n]}(s) + u^\star I_{(\hat{\tau}_n,T]}(s)$.

4. Limit problem and convergence

We have now developed all the necessary tools to address the convergence of the control problems. Naturally, such convergence can only be expected when the approximating maps (Γ^ε) provide sufficiently accurate representations of the noise. Remarkably, it turns out that it is sufficient to assume the convergence of the forward equation alone—no additional assumptions on the control problems themselves are required. More precisely, let us assume the following natural condition.

Assumption 4.1. *Let X^ε and X be the solutions to (3.9) and (3.19), respectively. We assume that for every $t \in [0, T]$, $X^\varepsilon_t \to \widehat{X}_t$ in probability as $\varepsilon \to 0$.*

We are now in a position to prove our main convergence result.

Theorem 4.2. *Under Assumptions 2.3, 2.5, 3.2, and 4.1, we have that*

$$\lim_{\varepsilon\to 0} \inf_{u\in\mathcal{U}^\varepsilon_{\mathrm{ad}}} J^\varepsilon(u) = \inf_{u\in\mathcal{U}_{\mathrm{ad}}} \widehat{J}(u). \tag{4.1}$$

Moreover, if $\underline{u}^\varepsilon$ and $\hat{u}$ are the optimal admissible controls introduced in the previous section, then as we know $J^\varepsilon(\underline{u}^\varepsilon) = \inf_{u\in\mathcal{U}_{\mathrm{ad}}} J^\varepsilon(u)$; $\widehat{J}(\hat{u}) = \inf_{u\in\mathcal{U}_{\mathrm{ad}}} \widehat{J}(u)$; moreover,

$$\lim_{\varepsilon\to 0} \mathbb{E}\int_0^T |\underline{u}^\varepsilon_s - \hat{u}_s|^2\, ds = 0. \tag{4.2}$$

Proof. Let us consider the equation for the difference $Y^\varepsilon_t - \widehat{Y}_t := \widetilde{Y}^\varepsilon_t$. It solves

$$\widetilde{Y}^\varepsilon_t = h(X^\varepsilon_T) - h(\widehat{X}_T) + \int_t^T \big(\psi(X^\varepsilon_s, Z^\varepsilon_s) - \psi(\widehat{X}_s, \widehat{Z}_s)\big)\, ds + \int_t^T \widetilde{Z}^\varepsilon_s dW_s, \tag{4.3}$$

where $\widetilde{Z}^\varepsilon_t = Z^\varepsilon_t - Z_t$. Equation (4.3) can be rewritten as

$$\begin{aligned}\widetilde{Y}^\varepsilon_t = h(X^\varepsilon_T) - h(\widehat{X}_T) &+ \int_t^T \big(\psi(X^\varepsilon_s, \widehat{Z}_s) - \psi(\widehat{X}_s, \widehat{Z}_s)\big)\, ds \\ &+ \int_t^T K^\varepsilon_s \widetilde{Z}^\varepsilon_s\, ds + \int_t^T \widetilde{Z}^\varepsilon_s\, dW_s,\end{aligned}$$

where

$$K^\varepsilon_s =: v(X^\varepsilon_s, Z^\varepsilon_s, \widehat{Z}_s) \text{ and } v(x,z,z') = \begin{cases} \frac{\psi(x,z)-\psi(x,z')}{|z-z'|^2}(z-z') & \text{if } |z-z'| \neq 0, \\ 0 & \text{if } |z-z'| = 0. \end{cases}$$

Notice that, by (3.2), $K^\varepsilon_s \leq L_\psi(1+|Z^\varepsilon_s|+|\widehat{Z}_s|)$; thus, in view of by (3.10) and (3.20),

$$\sup_{\varepsilon>0} \left\| \int_0^\cdot K^\varepsilon_s dW_s \right\|_{\mathrm{BMO}_2} < +\infty.$$

Moreover, if $f^\varepsilon_s := \psi(X^\varepsilon_s, \widehat{Z}_s) - \psi(\widehat{X}_s, \widehat{Z}_s)$, then in view of Corollary 3.1,

$$|f^\varepsilon_s| \leq L_\psi\big(1+|\widehat{Z}_s|^2\big)\big(1 \wedge |X^\varepsilon_s - \widehat{X}_s|\big). \tag{4.4}$$

By (3.6) and (3.20) applied to $\int_0^\cdot \widehat{Z}_s\, dW_s$, we have that

$$\sup_{\varepsilon>0} \mathbb{E}\left(\int_0^T |f^\varepsilon_s| ds\right)^q < +\infty, \quad \text{for all } q \geq 1. \tag{4.5}$$

Thus, Assumption A3 in [3] is verified for any $p > 1$. Consequently, we can apply estimate (7) in [3] with $p^* = 2p$

$$\begin{aligned} &\big(\mathbb{E} \sup_{t\in[0,T]} |\widetilde{Y}^\varepsilon_t|^p\big)^{1/p} + \left(\mathbb{E}\left(\int_0^T |\widetilde{Z}^\varepsilon_t|^2\, dt\right)^{p/2}\right)^{1/p} \\ &\leq C\left\{\big[\mathbb{E}\big(\big|h(X^\varepsilon_T) - h(\widehat{X}_T)\big|\big)^{2p}\big]^{1/2p} + \left[\mathbb{E}\left(\int_0^T |f^\varepsilon_s|\, ds\right)^{2p}\right]^{1/2p}\right\} \\ &\quad \times \left(1 + \left[\mathbb{E}\left(\int_0^T |K^\varepsilon_s|^2\, ds\right)^{3p/2}\right]^{1/3p}\right) \\ &\leq \widetilde{C}\left\{\big[\mathbb{E}\big(\big|h(X^\varepsilon_T) - h(\widehat{X}_T)\big|\big)^{2p}\big]^{1/2p} + \left[\mathbb{E}\left(\int_0^T |f^\varepsilon_s|\, ds\right)^{2p}\right]^{1/2p}\right\}, \end{aligned} \tag{4.6}$$

where $\widetilde{C}$ depends on $\| \int_0^\cdot K^\varepsilon_s dW_s\|_{\mathrm{BMO}_2}$, see again [3].

Moreover, recalling that $\int_0^T |\widehat{Z}_s|^2 ds \in L^q$ for all $q \geq 1$, we readily deduce that the sequence $(\int_0^T |f^\varepsilon_s|\, ds)^{2p}$ is uniformly integrable. To prove that $\mathbb{E}(\int_0^T |f^\varepsilon_s| ds)^{2p} \to 0$, it is therefore enough to prove that $\int_0^T |f^\varepsilon_s| ds$ converges to 0 in probability.

We start by showing that, for almost every $s \in [0,T]$, $\mathbb{E}|f^\varepsilon_s| \to 0$. By (4.4), it is enough to show that $\mathbb{E}(1+|\widehat{Z}_s|^2)(1 \wedge |X^\varepsilon_s - \widehat{X}_s|) \to 0$. Indeed, in view of Assumption 4.1, $(1+|\widehat{Z}_s|^2)(1 \wedge |X^\varepsilon_s - \widehat{X}_s|)$ converges to 0 in probability and is dominated by $(1+|\widehat{Z}_s|^2)$.

Then, again by dominated convergence,

$$\mathbb{E}\int_0^T |f^\varepsilon_s| ds \to 0$$

and consequently, $\int_0^T |f^\varepsilon_s| ds \to 0$ in probability.

In the same way, taking into account assumption (Hp 2-h), we get that

$$\mathbb{E}\big(\big|h(X^\varepsilon_T) - h(\widehat{X}_T)\big|\big)^{2p} \to 0.$$

Thus, by (4.6),

$$\lim_{\varepsilon\to 0} \mathbb{E} \sup_{t\in[0,T]} |\widetilde{Y}^\varepsilon_t|^p = 0 \qquad \lim_{\varepsilon\to 0} \mathbb{E}\bigg(\int_0^T |\widetilde{Z}^\varepsilon_t|^2\, dt\bigg)^{p/2} = 0 \tag{4.7}$$

and we deduce (4.1).

It remains to prove (4.2). Recalling that $\underline{u}^\varepsilon_s = \underline{u}(X^\varepsilon_s, Z^\varepsilon_s)$ and $\hat{u}_s = \underline{u}(\widehat{X}_s, \widehat{Z}_s)$ by (3.4), we have

$$\begin{aligned}\mathbb{E}\int_0^T |\underline{u}^\varepsilon_s - \hat{u}_s|^2 ds &\le L^2_{\underline{u}}\mathbb{E}\int_0^T |Z^\varepsilon_s - \widehat{Z}_s|^2 ds \\ &\quad + L^2_{\underline{u}}\mathbb{E}\int_0^T \big(1 + |\widehat{Z}_s|^2\big)\big(1 \wedge |X^\varepsilon_s - \widehat{X}_s|\big)^2 ds.\end{aligned}$$

Then, (4.2) follows by Assumption (4.1) and relation (4.7) exactly as in the above detailed proof that $\mathbb{E}(\int_0^T |f^\varepsilon_s| ds)^{2p} \to 0$. ■

REMARK 4.3. By [2, Theorem 2.2], our motivating example (1.1)–(1.2) satisfies Assumptions 2.3 and 4.1 when K is a finite-dimensional Hilbert space. Therefore, our Theorem 4.2 applies as soon as the cost functional satisfies Assumptions 2.5 and 3.2, and we have the convergence of the optimal costs and the associated control problem (1.3) towards those of (1.4).

5. Examples and further developments

5.1. *Wong–Zakai type approximations*

Let us consider a simple finite-dimensional stochastic equation

$$\begin{cases} d_t X_t = \sigma(X^\varepsilon_t) dB_t, & t \in [0,T], \\ X(0) = x_0, \end{cases} \tag{5.1}$$

where $(B_t)_{t\in[0,T]}$ is an $\mathbb{R}^d$-valued standard Brownian motion, and $\sigma : \mathbb{R}^n \to L(\mathbb{R}^d, \mathbb{R}^n)$ is a regular map. We define the usual Itô–Stratonovich correction map $\sigma_2 : \mathbb{R}^n \to L(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^n)$ by

$$\sigma_2(x)(v,w) = \nabla_x\big(\sigma(x)v\big)\big(\sigma(x)w\big), \quad \text{for all } x \in \mathbb{R}^n,\ v, w \in \mathbb{R}^d.$$

Following [5], we assume that σ and σ_2 are of class C^3 with bounded first, second, and third derivative.

Concerning the regularization of noise, let $\rho : \mathbb{R} \to [0, +\infty)$ be a smooth function ρ with compact support satisfying $\rho(s) = 0$, for $s < 0$ and $\int_{-\infty}^{0} \rho(s)ds = 1$. For all $\varepsilon > 0$, let $\rho_\varepsilon(s) := \varepsilon^{-1}\rho(\varepsilon^{-1}s)$.

Given a continuous semimartingale I in the class $\mathcal{I}$ introduced in paragraph 2.1, let

$$I_t^\varepsilon = (\rho^\varepsilon \star I)_t \text{ and } \Gamma^\varepsilon[I]_t = \dot{I}_t^\varepsilon; \quad \text{in particular,} \quad B_t^\varepsilon = (\rho^\varepsilon \star I)_t \text{ and } \Gamma^\varepsilon[B]_t = \dot{B}_t^\varepsilon,$$

where I has been extended to 0 before 0 and after T. Notice that Assumption 2.1 is satisfied due to the asymmetry of the mollifier ρ.

Concerning the control problem, let $(u_t)_{t\in[0,T]} \in L_B^2([0,T];\mathbb{R}^m)$ and r, l, h satisfy Assumption 2.5 with $K = \mathbb{R}^n$ and $U = \mathbb{R}^m$. We consider the approximating controlled equations

$$\begin{cases} d_t X_t^{u,\varepsilon} = \sigma(X_t^{u,\varepsilon})\Gamma^\varepsilon[\int_0^\cdot r(X_s, u_s)ds + B]_t dt, & t \in [0,T], \\ X(0) = x_0, \end{cases} \tag{5.2}$$

which can be rewritten as

$$\begin{cases} d_t X_t^{u,\varepsilon} = \sigma(X_t^{u,\varepsilon})[\rho_e \star r(X,u)]_t dt + \sigma(X_t^{u,\varepsilon})\dot{B}_t^\varepsilon dt, & t \in [0,T], \\ X(0) = x_0, \end{cases}$$

and the limit controlled equation

$$\begin{cases} d_t X_t^u = \mathrm{Tr}[\sigma_2(X_t^u)] + \sigma(X_t^u) r(X_t^u, u_t)dt + \sigma(X_t^u)dB_t, & t \in [0,T], \\ X(0) = x_0, \end{cases}$$

together with the cost functionals J^ε and J defined as in (2.7). In [5], it is shown that, under the present assumptions, if X^ε solves

$$\begin{cases} d_t X_t^\varepsilon = \sigma(X_t^\varepsilon)\dot{B}_t^\varepsilon dt, & t \in [0,T], \\ X(0) = x_0, \end{cases} \tag{5.3}$$

and $\widehat{X}$ solves

$$\begin{cases} d_t \widehat{X}_t = \mathrm{Tr}\left[\sigma_2(X_t^u)\right]dt + \sigma(\widehat{X}_t)dB_t, & t \in [0,T], \\ X(0) = x_0, \end{cases} \tag{5.4}$$

then $X^\varepsilon \to \widehat{X}$, $\mathbb{P}$-a.s. in a suitable Holder norm and in particular $X_t^\varepsilon \to \widehat{X}_t$, $\mathbb{P}$-a.s. for all $t > 0$.

Thus, if Assumption 3.2 holds, we are in a condition to apply Theorem 4.2 and conclude that

$$\lim_{\varepsilon\to 0} \inf_{u\in\mathcal{U}_{\mathrm{ad}}^\varepsilon} J^\varepsilon(u) = \inf_{u\in\mathcal{U}_{\mathrm{ad}}} \widehat{J}(u),$$

where the definition of $\mathcal{U}_{\mathrm{ad}}^\varepsilon$ and of $\mathcal{U}_{\mathrm{ad}}$ is given in Section 2.4.

Moreover, there exist optimal controls $\underline{u}^\varepsilon$ in $\mathcal{U}^\varepsilon_{\mathrm{ad}}$ and $\hat{u}$ in $\mathcal{U}_{\mathrm{ad}}$ such that

$$J^\varepsilon(\underline{u}^\varepsilon) = \inf_{u\in\mathcal{U}^\varepsilon_{\mathrm{ad}}} J^\varepsilon(u) \quad \text{and} \quad \widehat{J}(\hat{u}) = \inf_{u\in\mathcal{U}_{\mathrm{ad}}} \widehat{J}(u).$$

Finally,

$$\lim_{\varepsilon\to 0} \mathbb{E} \int_0^T |\underline{u}^\varepsilon_s - \hat{u}_s|^2\, ds = 0.$$

5.2. *Quadratic fast-fast interaction*

A possible extension of our results could take into account climatic systems with fast-fast interaction at meteorologic scales, replacing (1.1) with

$$\begin{cases} dX^\varepsilon_t = AX^\varepsilon_t dt + b(X^\varepsilon_t)dt + \sigma(X^\varepsilon_t)Q^\varepsilon_t dt, & t\in[0,T], \\ X^\varepsilon(0) = x_0, \\ dQ^\varepsilon_t = q(Q^\varepsilon_t, Q^\varepsilon_t)dt - \dfrac{1}{\varepsilon}Q^\varepsilon_t dt + \dfrac{1}{\varepsilon}G\,dW_t, & t\in[0,T], \\ Q^\varepsilon(0) = 0. \end{cases} \tag{5.5}$$

In the equation above, $q : H \times H \to H$ is a continuous bilinear map. For simplicity, we suppose that K is finite dimensional. Hereafter, we shall implicitly assume conditions on q guaranteeing the existence and uniqueness of solutions to (5.5) for a sufficient class of noises W.

Technically speaking, the results in [2, 26] do not cover the case of quadratic self-interaction for the fast variable and therefore require $q = 0$. In view of their geophysical interpretation, assuming $q = 0$ is a restrictive modeling assumption (cf. [26, equation (2.4)]) since most equations of geophysical fluid dynamics do have quadratic non-linearities. These difficulties have been recently overcome in a series of papers [10, 13, 14].

A stochastic model reduction of (5.5) is performed in [10], where convergence in probability towards a reduced equation is proved. The reduced equation has the form

$$\begin{cases} d\widehat{X}_t = A\widehat{X}_t dt + \hat{b}(\widehat{X}_t)dt + \sigma(\widehat{X}_t)G\,dW_t + \sigma(\widehat{X}_t)\hat{q}\,dt, & t\in[0,T], \\ \widehat{X}(0) = x_0, \end{cases}$$

where $\hat{q}$ is the average of the fast-fast interaction with respect to the centered Gaussian measure with covariance $Q := \frac{1}{2}G^*G$; namely,

$$\hat{q} := \int_H q(w,w)\mathcal{N}(0,Q)(dw).$$

In view of the results of this paper, one can introduce the controlled fast-slow system with quadratic fast-fast interaction

$$
\begin{cases}
dX_t^{\varepsilon,u} = AX_t^{\varepsilon,u}dt + b(X_t^{\varepsilon,u})dt + \sigma(X_t^{\varepsilon,u})Q_t^{\varepsilon,u}dt, & t \in [0,T], \\
X(0) = x_0, \\
dQ_t^{\varepsilon,u} = q(Q_t^{\varepsilon,u}, Q_t^{\varepsilon,u})dt - \frac{1}{\varepsilon}Q_t^{\varepsilon,u}dt + \frac{1}{\varepsilon}Gr(u_t)dt + \frac{1}{\varepsilon}GdW_t, & t \in [0,T], \\
Q(0) = 0,
\end{cases}
\tag{5.6}
$$

and the controlled reduced equation

$$
\begin{cases}
d\widehat{X}_t^u = A\widehat{X}_t^u dt + \hat{b}(\widehat{X}_t)dt + \sigma(\widehat{X}_t^u)Gr(u_t)dt \\
\qquad\qquad + \sigma(\widehat{X}_t^u)GdW_t + \sigma(\widehat{X}_t^u)\hat{q}dt, \quad t \in [0,T], \\
\widehat{X}(0) = x_0.
\end{cases}
\tag{5.7}
$$

Convergence of the optimal control problems falls into our general theory by considering the maps

$$\Gamma^\varepsilon(I) := Q,$$

where Q is the unique solution of

$$dQ_t = q(Q_t, Q_t)dt - \frac{1}{\varepsilon}Q_t dt + \frac{1}{\varepsilon}dI_t.$$

The analogue of Theorem 4.2 follows.

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