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Magnetofluid Dynamics. – *Unique continuation of static overdetermined magnetohydrodynamic equations*, by Irena Lasiecka, Buddhika Priyasad and Roberto Triggiani, accepted on 26 May 2025.

To Beppe: friend, colleague, collaborator.

Abstract. – This paper establishes the Unique Continuation Property (UCP) for a suitably overdetermined Magnetohydrodynamics (MHD) eigenvalue problem, which is equivalent to the Kalman, finite-rank, controllability condition for the finite-dimensional unstable projection of the linearized dynamic MHD problem. It is the "ignition key" to obtain uniform stabilization of the dynamic non-linear MHD system near an unstable equilibrium solution, by means of finitely many, interior, localized feedback controllers [Res. Math. Sci. 12 (2025), article no. 7]. The proof of the UCP result uses a pointwise Carleman-type estimate for the Laplacian following the approach that was introduced in [Nonlinear Anal. 71 (2009), 4967–4976] for the Navier–Stokes equations and further extended in [Appl. Math. Optim. 84 (2021), 2099–2146] for the Boussinesq system.

Keywords. – magnetohydrodynamics equations, unique continuation, uniform stabilization, Carleman estimates.

Mathematics Subject Classification 2020. – 76W05.

1. Introduction, the role of the unique continuation in uniform stabilization, main results, literature

1.1. *Unique continuation properties (UCP) of overdetermined static problems: the "ignition key" for uniform feedback stabilization*

The dynamic MHD equations: After the initial work carried out by the Nobel laureate Hannes Alfvén in 1970, magnetohydrodynamics (henceforth referred to as MHD) has culminated as an emerging discipline in plasma physics. MHD refers to phenomena arising in electrically conducting magnetic fluids. It is caused by the induction of current in a conductive fluid flow due to a magnetic field and moreover by polarization of the fluid and reciprocal changes in the magnetic field. MHD has been used extensively in plasma confinement, liquid metal cooling of nuclear reactors and electromagnetic casting (EMC). The system of MHD equations – below in (1.1) – consists of the Navier–

Stokes equations of a viscous incompressible fluid flow suitably coupled by high-order coupling with Maxwell-Ohm equations (of parabolic character) of an electromagnetic field [2, 25, 26, 33, 34, 36].

Uniform stabilization of fluids: Recent work by the authors has focused on the problem of feedback stabilization (asymptotic turbulent suppression) of fluids such as Navier–Stokes equations as well as Boussinesq systems, and MHD equations, on bounded domains Ω in $\mathbb{R}^d$, $d = 2, 3$, by means of finite-dimensional, static, feedback controls. These are either localized in the interior [14, 17, 20] or else localized at the boundary [16, 18]. More specifically, to illustrate, a 20-year-old open problem (introduced by A. Fursikov around 2000 [10, 11]) as to whether the 3d-Navier–Stokes equation could be stabilized by static, boundary-based localized feedback controllers which moreover are finite dimensional was positively proved in [16]. Moreover, this reference established minimal dimension. It required abandoning the usual Sobolev–Hilbert setting of the literature in favor of a new Besov space setting with tight indices ("close" to $L^3(\Omega)$ for $d = 3$).

Critical role of UCP: Following the strategy for feedback stabilization of parabolic dynamics introduced in [27], and extensively used since in the literature, a first step of the analysis consists in feedback stabilizing with an arbitrarily large decay rate, the finite-dimensional unstable component of the full dynamics with static, state feedback controls. The ability to do so requires the property of controllability of the finite-dimensional unstable component, which in fact is equivalent to the needed "spectrum allocation property" [35]. Showing such controllability property (Kalman's algebraic rank condition) for PDE problems requires a fundamental UCP. Section 3 explains how the present UCP of Theorem 1.2 is used to establish the needed Kaman algebraic rank condition arising in the problem of uniform stabilization of the MHD system by finite-dimensional interior, localized, static, feedback controllers. The corresponding Kalman algebraic condition (in fact, for the adjoint problem, as usual) is given by (3.19a). Establishing (3.19a) amounts to showing the following property. Let $\overline{\lambda}_i$ be an unstable eigenvalue of the linear adjoint operator $\widetilde{\mathbb{A}}_q^*$ in (3.16), and let $\mathbf{\Phi}_{i1}^*, \dots, \mathbf{\Phi}_{i\ell_i}^*$ be the corresponding eigenvectors, which are linearly independent on all of Ω. Then (3.19a) requires that $\mathbf{\Phi}_{i1}^*, \dots, \mathbf{\Phi}_{i\ell_i}^*$ remain linearly independent when restricted over the (arbitrary) small interior subdomain $\omega \subset \Omega$. The proof of Theorem 3.1 shows that this latter property amounts to proving the overdetermined eigenproblem (3.22) for the operator $\widetilde{\mathbb{A}}_q^*$; equivalently, in PDE form, the eigenvalue problem (3.23a–d) subject to the overdetermined condition (3.23e) implies UCP (3.25). Numerous illustrations from classical parabolic problems to fluid (Navier–Stokes, Boussinesq system) are given in the extensive article [31], which employs UCP for fluids [28–30, 32]. Thus, the mathematical focus of the present paper is to establish the required UCP of an eigenvalue problem for the MHD equations subject to an overdetermined condition. Such UCP is the primary subject of the present paper. The proof (based on pointwise Carleman-type estimate for the Laplacian) is a natural extension of those given in [29] for the Navier–Stokes equations and in [32] for the Boussinesq system. As mentioned, we refer to the paper [31] where the role of UCP in the case of parabolic dynamics is extensively treated. The authors' subsequent study of asymptotic turbulent suppression of the MHD system first with localized interior control in Besov spaces is given in [20]. This is to be next followed by the localized boundary-type control case in Besov spaces.

1.2. *Controlled dynamic MHD equations*

As already noted, we wish to introduce the present unique continuation theorem in the context of a uniform stabilization problem. Let, at first, Ω be an open connected bounded domain in $\mathbb{R}^d$, $d = 2, 3$, with sufficiently smooth boundary $\Gamma = \partial\Omega$. More specific requirements will be given below. Let ω be an arbitrarily small open smooth subset of the interior Ω, $\omega \subset \Omega$, of positive measure. Let m denote the characteristic function of ω: $m(\omega) \equiv 1$, $m(\Omega \setminus \omega) \equiv 0$. We consider the following MHD equations perturbed by forces f and g, and subject to the action of a pair u, v of interior localized controls, to be described below, where $Q = (0, \infty) \times \Omega$, $\Sigma = (0, \infty) \times \Gamma$:

(1.1a)
$$y_t - \nu\Delta y + (y \cdot \nabla)y + \nabla\pi + \frac{1}{2}\nabla(B \cdot B) - (B \cdot \nabla)B = m(x)u(t, x) + f(x) \text{ in } Q,$$

$$B_t + \eta \operatorname{curl}\operatorname{curl} B + (y \cdot \nabla)B - (B \cdot \nabla)y = m(x)v(t, x) + g(x) \quad \text{in } Q, \tag{1.1b}$$

$$\operatorname{div} y = 0, \quad \operatorname{div} B = 0 \quad \text{in } Q, \tag{1.1c}$$

$$y = 0, \quad B \cdot n = 0, \quad (\operatorname{curl} B) \times n = 0 \quad \text{on } \Sigma, \tag{1.1d}$$

$$y(0, x) = y_0, \quad B(0, x) = B_0 \quad \text{on } \Omega. \tag{1.1e}$$

We note the formula

$$\operatorname{curl}\operatorname{curl} B = -\Delta B + \nabla \operatorname{div} B$$

so that equation (1.1b) can be more conveniently rewritten as

$$B_t - \eta\Delta B + (y \cdot \nabla)B - (B \cdot \nabla)y = m(x)v(t, x) + g(x) \tag{1.1b$'$}$$

invoking $\operatorname{div} B \equiv 0$ in Q from (1.1c). Furthermore, we denote total pressure ϱ in the dynamic equation as $\varrho := \pi + \frac{1}{2}(B \cdot B)$ and in the static case $\varrho_e := \pi_e + \frac{1}{2}(B_e \cdot B_e)$.

1.3. *Stationary MHD equations*

The following result represents our basic starting point. See [3].

THEOREM 1.1. *Consider the following steady-state MHD equations in Ω:*

$$-\nu\Delta y_e + (y_e\cdot\nabla)y_e + \nabla\varrho_e - (B_e\cdot\nabla)B_e = f(x) \quad \textit{in } \Omega, \tag{1.2a}$$

$$-\eta\Delta B_e + (y_e\cdot\nabla)B_e - (B_e\cdot\nabla)y_e = g(x) \quad \textit{in } \Omega, \tag{1.2b}$$

$$\text{div } y_e = 0, \quad \text{div } B_e = 0 \quad \textit{in } \Omega, \tag{1.2c}$$

$$y_e = 0, \quad B_e\cdot n = 0, \quad (\text{curl } B_e)\times n = 0 \quad \textit{on } \Gamma. \tag{1.2d}$$

Let $1 < q < \infty$. For any $f, g \in L^q(\Omega)$, there exists a solution (not necessarily unique) $(y_e, B_e, \pi_e) \in (W^{2,q}(\Omega))^d \times (W^{2,q}(\Omega))^d \times W^{1,q}(\Omega)$, $q > d$.

1.4. Translated MHD system

We return to Theorem 1.1 and choose an equilibrium triplet $\{y_e, B_e, \pi_e\}$ to be kept fixed throughout the analysis. Then, we translate by $\{y_e, p_e\}$ the original N-S problem (1.1). Thus, we introduce new variables

$$z = y - y_e, \quad \mathbb{B} = B - B_e \quad p = \varrho - \varrho_e \tag{1.3a}$$

and obtain the translated problem given by

$$\begin{aligned} z_t - \nu\Delta z + (y_e\cdot\nabla)z + (z\cdot\nabla)y_e + (z\cdot\nabla)z - (B_e\cdot\nabla)\mathbb{B} - (\mathbb{B}\cdot\nabla)B_e \\ -(\mathbb{B}\cdot\nabla)\mathbb{B} + \nabla p = mu \quad \text{in } Q, \end{aligned} \tag{1.3b}$$

$$\begin{aligned} \mathbb{B}_t - \eta\Delta\mathbb{B} + (z\cdot\nabla)B_e + (y_e\cdot\nabla)\mathbb{B} - (\mathbb{B}\cdot\nabla)y_e - (B_e\cdot\nabla)z \\ +(z\cdot\nabla)\mathbb{B} - (\mathbb{B}\cdot\nabla)z = mv \quad \text{in } Q, \end{aligned} \tag{1.3c}$$

$$\text{div } z = 0, \quad \text{div } \mathbb{B} = 0 \quad \text{in } Q, \tag{1.3d}$$

$$z = 0, \quad \mathbb{B}\cdot n = 0, \quad (\text{curl } \mathbb{B})\times n = 0 \quad \text{on } \Sigma, \tag{1.3e}$$

$$z(0,x) = y_0(x) - y_e(x), \quad \mathbb{B}(0,x) = B_0(x) - B_e(x) \quad \text{on } \Omega. \tag{1.3f}$$

1.5. Translated linearized MHD system

The translated linearized problem is

$$w_t - \nu\Delta w + (y_e\cdot\nabla)w + (w\cdot\nabla)y_e - (\mathbb{W}\cdot\nabla)B_e - (B_e\cdot\nabla)\mathbb{W} + \nabla p = mu \ \text{ in } Q, \tag{1.4a}$$

$$\mathbb{W}_t - \eta\Delta\mathbb{W} + (w\cdot\nabla)B_e - (B_e\cdot\nabla)w + (y_e\cdot\nabla)\mathbb{W} - (\mathbb{W}\cdot\nabla)y_e = mv \ \text{ in } Q, \tag{1.4b}$$

$$\text{div } w = 0, \quad \text{div } \mathbb{W} = 0 \ \text{ in } Q, \tag{1.4c}$$

$$w = 0, \quad \mathbb{W}\cdot n = 0, \quad (\text{curl } \mathbb{W})\times n = 0 \ \text{ on } \Sigma, \tag{1.4d}$$

$$w(0,x) = y_0 - y_e, \quad \mathbb{W}(0,x) = B_0 - B_e \ \text{ on } \Omega. \tag{1.4e}$$

1.6. *The required unique continuation theorem to uniformly stabilize the linear problem (1.4) by localized finitely many static feedback controls in [20]*

As described in the introduction, the solution to the desired uniform stabilization problem of the original, non-linear problem (1.1a–e) in the vicinity of an unstable equilibrium solution $\{y_e, B_e\}$ is given in [20]. A critical preliminary step is the uniform stabilization of the linear problem (1.4a–e). To achieve it, the following unique continuation result is critical. Its implication on the sought-after uniform stabilization is shown in [20].

THEOREM 1.2 (UCP, direct problem). *Let ω be an arbitrary open, connected smooth subset of Ω, thus of positive measure, Figure 1. Let*

$$\{\phi, \xi, p\} \in \left(W^{2,q}(\Omega)\right)^d \times \left(W^{2,q}(\Omega)\right)^d \times W^{1,q}(\Omega), \quad q > d,$$

solve the original eigenvalue problem

$$-\nu\Delta\phi + (y_e\cdot\nabla)\phi + (\phi\cdot\nabla)y_e - (B_e\cdot\nabla)\xi - (\xi\cdot\nabla)B_e + \nabla p = \lambda\phi \quad \text{in } \Omega, \tag{1.5a}$$

$$-\eta\Delta\xi + (\phi\cdot\nabla)B_e - (B_e\cdot\nabla)\phi + (y_e\cdot\nabla)\xi - (\xi\cdot\nabla)y_e = \lambda\xi \quad \text{in } \Omega, \tag{1.5b}$$

$$\operatorname{div}\phi = 0, \quad \operatorname{div}\xi = 0 \quad \text{in } \Omega, \tag{1.5c}$$

$$\phi = 0, \quad \xi\cdot n = 0, \quad (\operatorname{curl}\xi)\times n = 0 \quad \text{on } \Gamma, \tag{1.5d}$$

along with the overdetermined condition

$$\phi \equiv 0, \quad \xi \equiv 0 \quad \text{in } \omega. \tag{1.6}$$

Then,

$$\phi \equiv 0, \quad \xi \equiv 0, \quad p \equiv \text{const} \quad \text{in } \Omega. \tag{1.7}$$

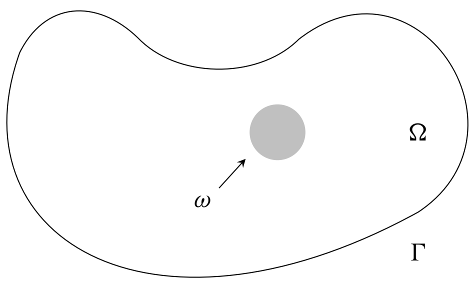

FIGURE 1. The localized interior set ω within Ω.

In line with the literature of Navier–Stokes equations, it will be convenient to introduce the following first-order operators:

$$\mathcal{L}_1\phi = \mathcal{L}^+_{y_e}\phi = (y_e \cdot \nabla)\phi + (\phi \cdot \nabla)y_e, \tag{1.8a}$$

$$\mathcal{L}_2\xi = \mathcal{L}^+_{B_e}\xi = (B_e \cdot \nabla)\xi + (\xi \cdot \nabla)B_e, \tag{1.8b}$$

$$\mathcal{M}_1\xi = \mathcal{L}^-_{y_e}\xi = (y_e \cdot \nabla)\xi - (\xi \cdot \nabla)y_e, \tag{1.9a}$$

$$\mathcal{M}_2\phi = \mathcal{L}^-_{B_e}\phi = (B_e \cdot \nabla)\phi - (\phi \cdot \nabla)B_e, \tag{1.9b}$$

$\mathcal{L}^+_{y_e}$ and $\mathcal{L}^+_{B_e}$ being the Oseen operators for y_e and B_e, respectively. For convenience in the analysis below, we re-write the ϕ-equation (1.5a) and the ξ-equation (1.5b), by use of (1.8), (1.9) as

$$-\nu\Delta\phi + \mathcal{L}_1\phi - \mathcal{L}_2\xi + \nabla p = \lambda\phi \quad \text{in } \Omega, \tag{1.5a$'$}$$

$$-\eta\Delta\xi + \mathcal{M}_1\xi - \mathcal{M}_2\phi = \lambda\xi \quad \text{in } \Omega. \tag{1.5b$'$}$$

1.7. Literature

A comparison between the results on uniform stabilization of the MHD problem (1.1a–e) in Besov spaces [20] and past results in the literature (all in Hilbert spaces) is given in [20, Section 1.5]. In particular, [20] requires a new maximal L^p-regularity [15] (see also [19]) in the Besov setting, while by contrast in the Hilbert setting of [24], only analyticity is needed (which is equivalent to maximal L^2-regularity [9]), a less challenging task. Paper [20] *constructs explicitly* the finite-dimensional, stabilizing feedback controllers, and of minimal dimension r. Instead [24] simply asserts the existence of a non-constructed feedback operator with a finite-dimensional range of unspecified dimension. The finite-dimensional decomposition approach introduced in [27], and followed in both [20, 24], requires a UCP result to assert Kalman algebraic, finite-rank condition of the finite-dimensional unstable component of the overall system. To achieve this end, [23, 24] establish a UCP for a *dynamic* coupled problem, by virtue of Carleman-type inequalities for *parabolic* equations, coupled with elliptic estimates. In contrast, we only need to establish a UCP for a *static eigenvalue problem* [(1.5), (1.6) of the present paper] still by Carleman-type estimates [20], a much more direct task. The original Carleman estimates, characterized by a suitable exponential weight function, were introduced in [8] in 1939 to establish the uniqueness of solutions for a PDE in two variables. The method of Carleman estimates was subsequently extended to the study of uniqueness in inverse problems, as first introduced in [6, 7]. For further developments and applications in this context, see also [13]. The subject has since grown substantially, with a vast and rich body of literature.

In the present paper, we shall make use of the pointwise Carleman estimates for the Laplacian operator [21, Corollary 4.3, p. 254] or [22, Corollary 4.2, equation (4.15), p. 73]. These, in turn, are obtained by specializing pointwise Carleman estimates for the second-order hyperbolic equations [21] or Schrödinger equations [22]. See Theorem 2.1 in Step 6 below.

The proof of Section 2 below is a further extension of the pointwise Carleman estimates-based proof originally introduced in [29] for the Navier–Stokes equations (one vectorial variable), further extended in [32] to the Boussinesq system (involving a vector–scalar variable coupling), and further extended for the MHD system (2.1a–b) involving two coupled vector-variables $\{\phi, \xi\}$. Now one needs to introduce a switching operator $\mathcal{S}$ and deal with higher-order coupling

$$-(\xi \cdot \nabla)\mathcal{S}\begin{bmatrix} y_e \\ B_e \end{bmatrix} - (B_e \cdot \nabla)\mathcal{S}\begin{bmatrix} \phi \\ \xi \end{bmatrix}$$

over the Boussinesq case. To this end, it is critical to invoke results from [28, equation (5.21)] or [30, equation (3.24)] to assert (in (2.25b–c) below) that $\operatorname{div} \mathcal{L}_1$ and $\operatorname{div} \mathcal{L}_2$ are first-order differential operators, one unit below than what appears at first sight. These results here, in turn, have a critical implication on the order of the commutator $T_\chi^{0,0,1}$ in (2.25c).

2. Proof of Theorem 1.2

Orientation

We first rewrite the problem in the variable $\left[\begin{smallmatrix} \phi \\ \xi \end{smallmatrix}\right]$ more conveniently by use of the switching operator $\mathcal{S}$ in order to fit into the strategy of [32], hence [29]. Next, in Step 2, we introduce a suitable cut-off function χ. Steps 3 and 4 consider the resulting $(\chi\phi)$-problem, $(\chi\xi)$-problem, hence the $\chi u = \left[\begin{smallmatrix} \chi\phi \\ \chi\xi \end{smallmatrix}\right]$-problem. In this effect, we need to extract two critical properties for the resulting commutators: (i) their order w.r.t. variables ϕ, ξ, p; (ii) the fact that their support falls inside Ω^* (see (2.7c), (2.9b), (2.25c)). In Step 6, we invoke the pointwise Carleman-type estimate for the Laplacian by specializing from [21] or [22]. This is then applied to the (χu)-problem in Step 7. The bound on the RHS of the (χu)-problem (2.17) in Step 8 uses critically also the two above mentioned properties of the commutators, leading to the final estimate for the (χu)-problem in (2.23). This, of course, involves the pressure term p. Hence, Step 10 considers the (χp)-problem given by (2.26a–b). Here, the critical property (already mentioned at the end of Section 1) that the operators $\operatorname{div} \mathcal{L}_1$ and $\operatorname{div} \mathcal{L}_2$ are of one degree less than "at first sight" is critically used to determine the order of the commutator $T_\chi^{0,0,1}$ in variables ϕ, ξ, p. The final estimate of the (χp)-problem is then achieved in Step 12, equation (2.31). Then, Step 13 combines the (χu)-problem (2.23) with the (χp)-estimate (2.31), leading to the final estimate of the original problem (2.1a–b) in Lemma 2.2, equation (2.35), Step 14. It is then in Step 15 that the strictly convex weight function $\psi(x)$ is selected, see Figure 5. This then yields to the final estimates, originally on Ω_1, for $u(x)$ and $p(x)$ in (2.40), (2.41), next in Ω in (2.44).

Step 0. Without loss of generality, we may normalize the constants $\nu = \eta \equiv 1$. We can rewrite equations (1.5a)–(1.5b) combined as in (2.1a) below, along with (1.5c) and (1.5d), and the over-determination (1.6)

$$\left.\begin{aligned}
&(2.1a) \quad (-\Delta)\begin{bmatrix}\phi\\ \xi\end{bmatrix} + (y_e\cdot\nabla)\begin{bmatrix}\phi\\ \xi\end{bmatrix} + (\phi\cdot\nabla)\begin{bmatrix}y_e\\ B_e\end{bmatrix}\\
&\qquad -(\xi\cdot\nabla)\begin{bmatrix}B_e\\ y_e\end{bmatrix} - (B_e\cdot\nabla)\begin{bmatrix}\xi\\ \phi\end{bmatrix} + \begin{bmatrix}\nabla p\\ 0\end{bmatrix} = \lambda\begin{bmatrix}\phi\\ \xi\end{bmatrix} \quad \text{in } \Omega,\\
&(2.1b) \quad \operatorname{div}\begin{bmatrix}\phi\\ \xi\end{bmatrix} \equiv \begin{bmatrix}0\\ 0\end{bmatrix} \quad \text{in } \Omega, \quad \text{and} \quad \begin{bmatrix}\phi\\ \xi\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} \quad \text{in } \omega.
\end{aligned}\right\}$$

Now we define a coordinate switching operator $\mathcal{S}$ such that $\left[\begin{smallmatrix}\xi\\ \phi\end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix}\phi\\ \xi\end{smallmatrix}\right]$ which is clearly bounded and continuous. Hence, rewrite the above equation as

$$\left.\begin{aligned}
&(2.2a) \quad (-\Delta)\begin{bmatrix}\phi\\ \xi\end{bmatrix} + (y_e\cdot\nabla)\begin{bmatrix}\phi\\ \xi\end{bmatrix} + (\phi\cdot\nabla)\begin{bmatrix}y_e\\ B_e\end{bmatrix}\\
&\qquad -(\xi\cdot\nabla)\mathcal{S}\begin{bmatrix}y_e\\ B_e\end{bmatrix} - (B_e\cdot\nabla)\mathcal{S}\begin{bmatrix}\phi\\ \xi\end{bmatrix} + \begin{bmatrix}\nabla p\\ 0\end{bmatrix} = \lambda\begin{bmatrix}\phi\\ \xi\end{bmatrix} \quad \text{in } \Omega,\\
&(2.2b) \quad \operatorname{div}\begin{bmatrix}\phi\\ \xi\end{bmatrix} \equiv \begin{bmatrix}0\\ 0\end{bmatrix} \quad \text{in } \Omega, \quad \text{and} \quad \begin{bmatrix}\phi\\ \xi\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} \quad \text{in } \omega.
\end{aligned}\right\}$$

Case 1. We write initially the proof for the case where ω is at a positive distance from $\partial\Omega$: dist $(\partial\Omega, \partial\omega) > 0$ (Figures 2 and 3).

Step 1. Henceforth, we introduce the state variable $u = \left[\begin{smallmatrix}\phi\\ \xi\end{smallmatrix}\right]$. Since $u = \{\phi, \xi\} \equiv 0$ in ω by (2.1b), then (2.1a) yields $\nabla p \equiv 0$ in ω; hence, $p = \text{const}$ in ω. We may, and will, take $p \equiv 0$ in ω, as p is only identified up to a constant. Then, we have

$$(2.3) \qquad u|_{\partial\omega} \equiv 0; \quad \left.\frac{\partial u}{\partial\nu}\right|_{\partial\omega} \equiv 0; \quad p|_{\partial\omega} \equiv 0; \quad \left.\frac{\partial p}{\partial\nu}\right|_{\partial\omega} \equiv 0,$$

where $\frac{\partial}{\partial\nu}$ denotes the normal derivative (ν = unit inward normal vector with respect to ω).

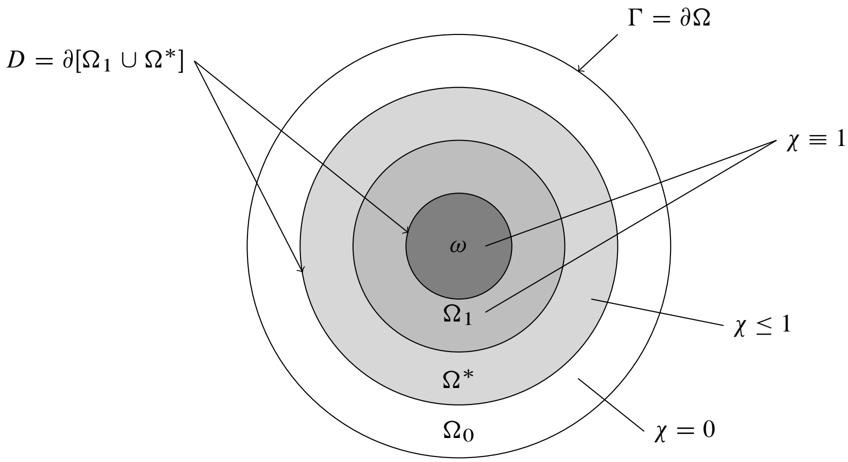

FIGURE 2. Case 1: $G = \Omega_1 \cup \Omega^*$.

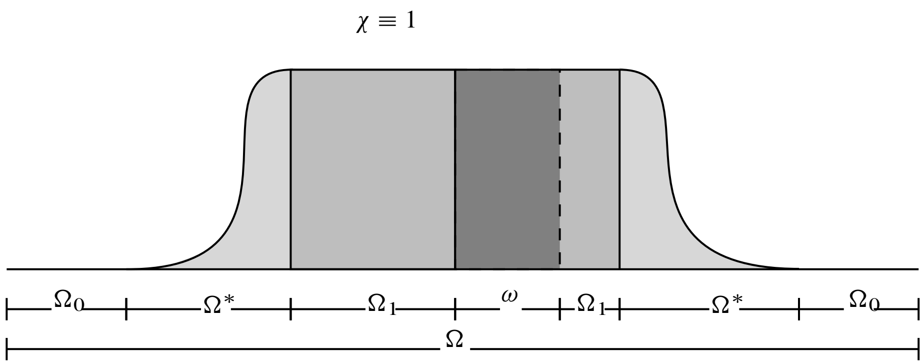

FIGURE 3. Case 1: the cut-off function χ.

STEP 2 (The cut-off function χ). Let χ be a smooth, non-negative, cut-off function defined as follows:

$$\chi \equiv \begin{cases} 1 & \text{in } \Omega_1 \cup \omega, \\ 0 & \text{in } \Omega_0; \end{cases} \qquad \operatorname{supp} \chi \subset [\Omega_1 \cup \omega \cup \Omega^*], \tag{2.4}$$

while monotonically decreasing from 1 to 0 in Ω^*, with $\chi \equiv 0$ also in a small layer of Ω^* bordering Ω_0 (Figures 2 and 3). Here,

(i) Ω_1 is a smooth subdomain of Ω which surrounds and borders ω (Figure 2);

(ii) in turn, Ω^* is a smooth subdomain of Ω which surrounds and borders Ω_1 (Figure 2);

(iii) in turn, Ω_0 is a smooth subdomain of Ω : $\Omega_0 \equiv \Omega \setminus \{\omega \cup \Omega_1 \cup \Omega^*\}$.

Step 3 (The $(\chi\phi)$-problem). Multiply the ϕ-equation re-written as in (1.5a′) by χ and obtain

$$(2.5)\quad (-\Delta)(\chi\phi) + \mathcal{L}_1(\chi\phi) - \mathcal{L}_2(\chi\xi) + \nabla(\chi p) = \lambda(\chi\phi) + F_\chi^{1,0,0}(\phi,\xi,p) \quad \text{in } \Omega,$$

or recalling (1.8a–b)

$$(2.6)\quad \begin{aligned} &(-\Delta)(\chi\phi) + (y_e\cdot\nabla)(\chi\phi) + \big((\chi\phi)\cdot\nabla\big)y_e \\ &\quad - (B_e\cdot\nabla)(\chi\xi) - \big((\chi\xi)\cdot\nabla\big)B_e + \nabla(\chi p) = \lambda(\chi\phi) + F_\chi^{1,0,0}(\phi,\xi,p) \text{ in } \Omega, \end{aligned}$$

with forcing term expressed in terms of the resulting commutators

$$(2.7a)\quad F_\chi \equiv F_\chi^{1,0,0}(\phi,\xi,p) \equiv [\chi,\Delta]\phi + [\mathcal{L}_1,\chi]\phi - [\mathcal{L}_2,\chi]\xi + [\nabla,\chi]p$$

$$(2.7b)\quad = \text{first order in } \phi;\ \text{zero order in } \xi \text{ and } p;$$

$$(2.7c)\quad \operatorname{supp} F_\chi^{1,0,0} \subset \Omega^*.$$

Notice that (2.7c) holds true since $\chi \equiv 1$ on Ω_1, on ω, and on a small layer within Ω^*, so that on the union taken over these three sets we have that $F_\chi \equiv 0$. We recall that the commutator $[\chi,\Delta]$ is defined by $[\chi,\Delta]\phi = \chi\Delta\phi - \Delta(\chi\phi)$. Thanks to the Leibniz formula, $[\chi,\Delta]$ is a linear combination of derivatives of order ≥ 1 of χ multiplied by derivatives of order ≤ 1 of ϕ. Consequently, the commutator $[\chi,\Delta]$ is of order $0+2-1=1$; the commutator $[\chi,\mathcal{L}_i]$ is of order $0+1-1=0$, $i=1,2$; the commutator $[\nabla,\chi]$ is of order $1+0-1=0$.

Step 4 (The $(\chi\xi)$-problem). Next, we multiply the ξ-equation re-written as in (1.5b′) by χ and obtain

$$(-\Delta)(\chi\xi) + \mathcal{M}_1(\chi\xi) - \mathcal{M}_2(\chi\phi) = \lambda(\chi\xi) + G_\chi^{1,0}(\xi,\phi) \quad \text{in } \Omega,$$

or

$$(2.8)\quad \begin{aligned} &(-\Delta)(\chi\xi) + (y_e\cdot\nabla)(\chi\xi) - \big((\chi\xi)\cdot\nabla\big)y_e \\ &\quad - (B_e\cdot\nabla)(\chi\phi) + \big((\chi\phi)\cdot\nabla\big)B_e = \lambda(\chi\xi) + G_\chi^{1,0}(\xi,\phi) \quad \text{in } \Omega, \end{aligned}$$

with forcing term expressed in terms of the resulting commutators

$$(2.9a)\quad G_\chi = G_\chi^{1,0}(\xi,\phi) \equiv [\chi,\Delta]\xi + [\mathcal{M}_1,\chi]\xi + [\mathcal{M}_2,\chi]\phi,$$

$$(2.9b)\quad = \text{first order in } \xi;\ \text{zero order in } \phi;\ \operatorname{supp} G_\chi^{1,0} \subset \Omega^*.$$

Notice that (2.9b) holds true since as in the case for (2.7c) we have that $G_\chi \equiv 0$ on $\omega \cup \Omega_1 \cup [\text{a small layer within } \Omega^*]$ since $\chi \equiv 1$ on such union.

STEP 5 (The (χu)-problem $\chi u = \{\chi\phi, \chi\xi\}$). We combine Steps 3 and 4 and obtain

$$(-\Delta)\left(\chi\begin{bmatrix}\phi\\ \xi\end{bmatrix}\right)+(y_e\cdot\nabla)\left(\chi\begin{bmatrix}\phi\\ \xi\end{bmatrix}\right)+((\chi\phi)\cdot\nabla)\begin{bmatrix}y_e\\ B_e\end{bmatrix}-((\chi\xi)\cdot\nabla)\begin{bmatrix}B_e\\ y_e\end{bmatrix}$$
$$-(B_e\cdot\nabla)\left(\chi\begin{bmatrix}\xi\\ \phi\end{bmatrix}\right)+\begin{bmatrix}\nabla(\chi p)\\ 0\end{bmatrix}=\lambda\left(\chi\begin{bmatrix}\phi\\ \xi\end{bmatrix}\right)+\begin{bmatrix}F_\chi^{1,0,0}\\ G_\chi^{1,0}\end{bmatrix}\quad \text{in } \Omega. \tag{2.10}$$

Moreover, let (Figure 2)

$$D = \partial\omega \cup \{\text{external boundary of } \Omega^*\} = \partial[\Omega_1 \cup \Omega^*]. \tag{2.11}$$

Since $\chi \equiv 0$ on Ω_0 and in a small layer of Ω^* bordering Ω_0, then $(\chi u) = \{(\chi\phi), (\chi\xi)\}$ and (χp) have zero Cauchy data on the [external boundary of Ω^*] = [interior boundary of Ω_0]. Moreover, since $u \equiv 0$ in ω and $p \equiv 0$ in ω by Step 1, then (χu) and (χp) have zero Cauchy data on $\partial\omega$. Thus,

$$(\chi u)|_D \equiv 0; \quad \left.\frac{\partial(\chi u)}{\partial\nu}\right|_D \equiv 0; \quad (\chi p)|_{\partial\omega} \equiv 0; \quad \left.\frac{\partial(\chi p)}{\partial\nu}\right|_{\partial\omega} \equiv 0, \tag{2.12}$$

where ν denotes a unit normal vector outward with respect to $[\Omega^* \cup \Omega_1]$ (Figure 2).

STEP 6 (A pointwise Carleman estimate). We shall invoke the following pointwise Carleman estimate for the Laplacian from [22, Corollary 4.2, equation (4.15), p. 73].

THEOREM 2.1. *The following pointwise estimate holds true at each point x of a bounded domain G in $\mathbb{R}^d$ for an H^2-function w, where $\varepsilon > 0$ and $0 < \delta_0 < 1$ are arbitrary:*

$$\delta_0\left[2\rho\tau-\frac{\varepsilon}{2}\right]e^{2\tau\psi(x)}\left|\nabla w(x)\right|^2+\left[4\rho k^2\tau^3(1-\delta_0)+\mathcal{O}(\tau^2)\right]e^{2\tau\psi(x)}\left|w(x)\right|^2$$
$$\le\left(1+\frac{1}{\varepsilon}\right)e^{2\tau\psi(x)}\left|\Delta w(x)\right|^2+\operatorname{div}V_w(x),\quad x\in G. \tag{2.13}$$

Here, $\psi(x)$ is any strictly convex function over G, with no critical points in $\bar{G}$, to be chosen below in Step 11 when $G = \Omega_1 \cup \Omega^*$; $\rho > 0$ is a constant, defined by $\mathcal{H}_\psi(x) \ge \rho I$, $x \in \bar{G}$, where $\mathcal{H}_\psi$ denotes the (symmetric) Hessian matrix of $\psi(x)$ [22, equation (1.1.6), p. 45]*; $k > 0$ is a constant, defined by* $\inf |\nabla\psi(x)| = k > 0$, *where inf is taken over G* [22, equation (1.1.7), p. 45]*; and τ is a free positive parameter, to be chosen sufficiently large. For what follows, it is not critical to recall what* $\operatorname{div} V_w(x)$ *is, only that, via the divergence theorem, we have*

$$\int_G \operatorname{div} V_w(x)dx = \int_{\partial G} V_w(x)\cdot\nu\, d\sigma = 0, \tag{2.14}$$

whenever the Cauchy data *of w vanish on its boundary ∂G: $w|_{\partial G} \equiv 0$; $\nabla w|_{\partial G} \equiv 0$. In* (2.14)*, ν is a unit normal vector outward with respect to G.*

STEP 7 (Pointwise Carleman estimates for (χu)). Next, we apply estimate (2.13) with $w = (\chi u)$ solution of (2.10). For definiteness, we select $\delta_0 = \frac{1}{2}$, $\varepsilon = \frac{1}{2}$. We obtain

$$\left[\rho\tau - \frac{1}{8}\right] e^{2\tau\psi(x)} \left|\nabla(\chi u)(x)\right|^2 + \left[2\rho k^2 \tau^3 + \mathcal{O}(\tau^2)\right] e^{2\tau\psi(x)} \left|(\chi u)(x)\right|^2 \\ \leq 3e^{2\tau\psi(x)} \left|\Delta(\chi u)(x)\right|^2 + \operatorname{div} V_{(\chi u)}(x), \quad x \in G. \tag{2.15}$$

Next, we integrate (2.15) over the domain $G \equiv [\Omega_1 \cup \Omega^*]$ (Figure 2), thus obtaining

$$\left[\rho\tau - \frac{1}{8}\right] \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} \left|\nabla(\chi u)(x)\right|^2 dx \\ + \left[2\rho k^2 \tau^3 + \mathcal{O}(\tau^2)\right] \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} \left|(\chi u)(x)\right|^2 dx \\ \leq 3 \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} \left|\Delta(\chi u)(x)\right|^2 dx + \int_{\partial[\Omega_1 \cup \Omega^*]} \cancel{V_{(\chi u)}(x)} \cdot \nu \, dD, \tag{2.16}$$

where, on the RHS of (2.16), the boundary integral over $D \equiv \partial[\Omega_1 \cup \Omega^*]$ = the boundary of $\Omega_1 \cup \Omega^*$, see (2.11) and Figure 2, vanishes in view of (2.14) with $w = (\chi u)$ having null Cauchy data on D, by virtue of (the LHS of) (2.12).

STEP 8 (Bound on the RHS of (2.16)). Here, we estimate the RHS of (2.16). Returning to the (χu)-problem (2.10), we rewrite it over $G \equiv [\Omega_1 \cup \Omega^*]$ as

$$\Delta\left(\chi \begin{bmatrix} \phi \\ \xi \end{bmatrix}\right) = (y_e \cdot \nabla)\left(\chi \begin{bmatrix} \phi \\ \xi \end{bmatrix}\right) + \left((\chi\phi) \cdot \nabla\right) \begin{bmatrix} y_e \\ B_e \end{bmatrix} \\ - \left((\chi\xi) \cdot \nabla\right) \begin{bmatrix} B_e \\ y_e \end{bmatrix} - (B_e \cdot \nabla)\left(\chi \begin{bmatrix} \xi \\ \phi \end{bmatrix}\right) \\ + \begin{bmatrix} \nabla(\chi p) \\ 0 \end{bmatrix} - \lambda \left(\chi \begin{bmatrix} \phi \\ \xi \end{bmatrix}\right) - \begin{bmatrix} F_\chi^{1,0,0} \\ G_\chi^{1,0} \end{bmatrix} \tag{2.17}$$

and multiply across by $e^{\tau\psi(x)}$ to get

$$e^{\tau\psi(x)} \Delta\left(\chi \begin{bmatrix} \phi \\ \xi \end{bmatrix}\right) = (e^{\tau\psi(x)} y_e \cdot \nabla)\left(\chi \begin{bmatrix} \phi \\ \xi \end{bmatrix}\right) + \left(e^{\tau\psi(x)}(\chi\phi) \cdot \nabla\right) \begin{bmatrix} y_e \\ B_e \end{bmatrix} \\ - \left(e^{\tau\psi(x)}(\chi\xi) \cdot \nabla\right) \begin{bmatrix} B_e \\ y_e \end{bmatrix} - (e^{\tau\psi(x)} B_e \cdot \nabla)\left(\chi \begin{bmatrix} \xi \\ \phi \end{bmatrix}\right) \\ + e^{\tau\psi(x)} \begin{bmatrix} \nabla(\chi p) \\ 0 \end{bmatrix} - \lambda e^{\tau\psi(x)} \left(\chi \begin{bmatrix} \phi \\ \xi \end{bmatrix}\right) - e^{\tau\psi(x)} \begin{bmatrix} F_\chi^{1,0,0} \\ G_\chi^{1,0} \end{bmatrix}. \tag{2.18}$$

Recalling $(y_e, B_e) \in (W^{2,q}(\Omega))^d \times (W^{2,q}(\Omega))^d$ by Theorem 1.1, as well as the embedding $W^{1,q}(\Omega) \hookrightarrow C(\bar{\Omega})$ for $q > d$, [1, p. 97, for Ω having cone property] [12, p. 79, requiring C^1-boundary], we have $|\nabla y_e(x)| + |\nabla B_e(x)| \le C_{y_e,B_e}$, $x \in \Omega$, for $q > d$, as assumed. In view of this, we return to (2.18) and obtain

$$
\begin{aligned}
(2.19)\qquad & e^{2\tau\psi(x)} \left| \Delta \left(\chi \begin{bmatrix} \phi \\ \xi \end{bmatrix} \right)(x) \right|^2 \\
&\le c_e e^{2\tau\psi(x)} \left\{ \left| \nabla \left(\chi \begin{bmatrix} \phi \\ \xi \end{bmatrix} \right)(x) \right|^2 + \left|(\chi\phi)(x)^2\right| + \left|(\chi\xi)(x)^2\right| \right\} \\
&\quad + c_\lambda e^{2\tau\psi(x)} \left| \left(\chi \begin{bmatrix} \phi \\ \xi \end{bmatrix} \right)(x) \right|^2 + e^{2\tau\psi(x)} \left|\nabla(\chi p)^2\right| \\
&\quad + e^{2\tau\psi(x)} \left| \begin{bmatrix} F_\chi^{1,0,0}(\phi,\xi,p)(x) \\ G_\chi^{1,0}(\xi,\phi)(x) \end{bmatrix} \right|^2, \quad x \in G,
\end{aligned}
$$

c_e = a constant depending on y_e and B_e, $c_\lambda = |\lambda|^2 + 1$. Thus, integrating (2.19) over $G \equiv [\Omega_1 \cup \Omega^*]$ as required by (2.16) yields with $u = [\begin{smallmatrix} \phi \\ \xi \end{smallmatrix}]$

$$
\begin{aligned}
(2.20)\qquad & \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} \left|\Delta(\chi u)(x)\right|^2 dx \\
&\le C_{\lambda,e} \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} \left[\left|\nabla(\chi u)(x)^2\right| + \left|(\chi u)(x)^2\right|\right] dx \\
&\quad + \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} \left|\nabla(\chi p)(x)^2\right| dx \\
&\quad + \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} \left| \begin{bmatrix} F_\chi^{1,0,0}(\phi,\xi,p)(x) \\ G_\chi^{1,0}(\xi,\phi)(x) \end{bmatrix} \right|^2 dx,
\end{aligned}
$$

$C_{\lambda,e}$ = a constant depending on λ, y_e, and B_e.

We now recall from (2.16) and (2.9) that $F_\chi^{1,0,0}(\phi, \xi, p)$ is an operator which is first order in ϕ and zero order in p and ξ, while $G_\chi^{1,0}(\xi, \phi)$ is first order in ξ and zero order in ϕ; moreover, their support is in Ω^* : supp $F_\chi \subset \Omega^*$, supp $G_\chi \subset \Omega^*$. Thus, (2.20) becomes explicitly still with $u = [\begin{smallmatrix} \phi \\ \xi \end{smallmatrix}]$:

$$
\begin{aligned}
(2.21)\qquad & \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} \left|\Delta(\chi u)(x)\right|^2 dx \\
&\le C_{\lambda,e} \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} \left[\left|\nabla(\chi u)(x)\right|^2 + \left|(\chi u)(x)\right|^2\right] dx \\
&\quad + \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} \left|\nabla(\chi p)(x)\right|^2 dx \\
&\quad + c_\chi \int_{\Omega^*} e^{2\tau\psi(x)} \left[\left|\nabla u(x)\right|^2 + \left|u(x)\right|^2 + \left|p(x)\right|^2\right] dx,
\end{aligned}
$$

which is the sought-after bound on the last term of the RHS of (2.16). In (2.21), c_χ is a constant depending on χ.

STEP 9 (Final estimate for (χu)-problem (2.10), $u = [\begin{smallmatrix}\phi\\ \xi\end{smallmatrix}]$). We substitute (2.21) into the RHS of inequality (2.16) and obtain

$$\begin{aligned}
&\left[\rho\tau - \frac{1}{8}\right] \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} \left|\nabla(\chi u)(x)\right|^2 dx \\
&\qquad + \left[2\rho k^2 \tau^3 + \mathcal{O}(\tau^2)\right] \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} \left|(\chi u)(x)\right|^2 dx \\
&\quad \le C_{\lambda,e} \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} \left[\left|\nabla(\chi u)(x)\right|^2 + \left|(\chi u)(x)\right|^2\right] dx \\
&\qquad + 3 \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} \left|\nabla(\chi p)(x)\right|^2 dx \\
&\qquad + c_\chi \int_{\Omega^*} e^{2\tau\psi(x)} \left[\left|\nabla u(x)\right|^2 + \left|u(x)\right|^2 + \left|p(x)\right|^2\right] dx.
\end{aligned} \tag{2.22}$$

Moving the first integral term on the RHS of inequality (2.22) to the LHS of such inequality then yields for τ sufficiently large

$$\begin{aligned}
&\left\{\left[\rho\tau - \frac{1}{8}\right] - C_{\lambda,e}\right\} \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} \left|\nabla(\chi u)(x)\right|^2 dx \\
&\qquad + \left[2\rho k^2 \tau^3 + \mathcal{O}(\tau^2) - C_{\lambda,e}\right] \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} \left|(\chi u)(x)\right|^2 dx \\
&\quad \le 3 \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} \left|\nabla(\chi p)(x)\right|^2 dx \\
&\qquad + c_\chi \int_{\Omega^*} e^{2\tau\psi(x)} \left[\left|\nabla u(x)\right|^2 + \left|u(x)\right|^2 + \left|p(x)\right|^2\right] dx.
\end{aligned} \tag{2.23}$$

Inequality (2.23) is our final estimate for the (χu)-problem in (2.10), (2.7), (2.9).

STEP 10 (The (χp)-problem). We need to estimate the first integral term on the RHS of inequality (2.23). This will be accomplished in (2.31) below. To this end, we need to obtain preliminarily the PDE-problem satisfied by (χp) on $G \equiv \Omega_1 \cup \Omega^*$. This task will be accomplished in this step. Accordingly, we return to the ϕ-equation (1.5a′), take here the operation of "div" across, use $\operatorname{div}\phi \equiv 0$ and $\operatorname{div}\xi \equiv 0$ from (2.1b) = (1.5c), and obtain, recalling $\mathcal{L}_1(\phi)$ in (1.8a) and $\mathcal{L}_2(\xi)$ in (1.8b),

$$\Delta p = -\operatorname{div}\mathcal{L}_1(\phi) + \operatorname{div}\mathcal{L}_2(\xi) \quad \text{in } \Omega, \tag{2.24a}$$

where, actually [28, equation (5.21)], [30, equation (3.24)],

$$\operatorname{div}\mathcal{L}_1(\phi) = 2\left\{(\partial_x y_e \cdot \nabla)\phi\right\} = 2\left\{(\partial_x \phi \cdot \nabla) y_e\right\}, \tag{2.24b}$$

$$\operatorname{div}\mathcal{L}_2(\xi) = 2\left\{(\partial_x B_e \cdot \nabla)\xi\right\} = 2\left\{(\partial_x \xi \cdot \nabla) B_e\right\}, \tag{2.24c}$$

are first-order differential operators in ϕ and ξ, respectively. The proof of (2.24b) uses $\operatorname{div} \phi \equiv 0$ and $\operatorname{div} y_e \equiv 0$ in Ω from (2.1b) = (1.2c) and (1.1c). Next, multiply (2.24a) by χ. We obtain

$$\left.\begin{aligned} &\text{(2.25a)} && \Delta(\chi p) = -\operatorname{div} \mathcal{L}_1(\chi\phi) + \operatorname{div} \mathcal{L}_2(\chi\xi) + T_\chi^{0,0,1}(\phi, \xi, p) \quad \text{in } \Omega; \\ &\text{(2.25b)} && \left.\frac{\partial(\chi p)}{\partial\nu}\right|_D = 0, \quad (\chi p)|_D = 0, \quad D = \partial[\Omega_1 \cup \Omega^*]. \end{aligned}\right\}$$

(2.25c) $T_\chi^{0,0,1}(\phi, \xi, p) \equiv [\Delta, \chi]p + [\operatorname{div} \mathcal{L}_1, \chi]\phi - [\operatorname{div} \mathcal{L}_2, \chi]\xi =$ zero order in ϕ and ξ by (2.24b) and by (2.24c); first order in p; $\operatorname{supp} T_\chi^{0,0,1} \subset \Omega^*$,

while the B.C.s (2.25b) on the boundary D defined by (2.11) follow for two reasons:

(i) the RHS of (2.12) on (χp) on $\partial\omega$; actually, the RHS of (2.3) since $\chi \equiv 1$ on ω;

(ii) $\chi \equiv 0$ up to the external boundary of Ω^* and a small layer of Ω^* bordering Ω_0, so that $(\chi p) = 0$, $\frac{\partial(\chi p)}{\partial\nu} = 0$, on such external boundary of Ω^*.

Thus, (2.25b) is justified. In (2.25b), the reason for $\operatorname{supp} T_\chi^{0,0,1} \subset \Omega^*$ is the same as in (2.25c) and (2.9b).

Next, we apply the pointwise Carleman estimate (2.13) to problem (2.25a)–(2.25b), that is, for $w = (\chi p)$. We obtain with $G = \Omega_1 \cup \Omega^*$

(2.26)

$$\begin{aligned} &\delta_0\left[2\rho\tau - \frac{\varepsilon}{2}\right] e^{2\tau\psi(x)} \left|\nabla(\chi p)(x)\right|^2 + \left[4\rho k^2\tau^3(1-\delta_0) + \mathcal{O}(\tau^2)\right] e^{2\tau\psi(x)} \left|(\chi p)(x)\right|^2 \\ &\qquad \le \left(1 + \frac{1}{\varepsilon}\right) e^{2\tau\psi(x)} \left|\Delta(\chi p)(x)\right|^2 + \operatorname{div} V_{(\chi p)}(x), \quad x \in G. \end{aligned}$$

Again, it is not critical to recall what $\operatorname{div} V_{(\chi p)}(x)$ is, only the vanishing relationship (2.14) (for $w = (\chi p)$) on an appropriate bounded domain G. Indeed, we shall take again $G = \Omega_1 \cup \Omega^*$, integrate inequality (2.26) over G (after selecting again $\delta_0 = \frac{1}{2}$, $\varepsilon = \frac{1}{2}$), and obtain

$$\begin{aligned} \text{(2.27)} \quad &\left[\rho\tau - \frac{1}{8}\right] \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} \left|\nabla(\chi p)(x)\right|^2 dx \\ &\qquad + \left[2\rho k^2\tau^3 + \mathcal{O}(\tau^2)\right] \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} \left|(\chi p)(x)\right|^2 dx \\ &\le 3 \int_{\Omega_1 \cup \Omega^*} e^{2\tau\psi(x)} \left|\Delta(\chi p)(x)\right|^2 dx + \int_{\partial[\Omega_1 \cup \Omega^*]} \cancel{V_{(\chi p)}(x)} \cdot \nu \, dD, \end{aligned}$$

where, on the RHS of (2.27), the boundary integral over

$$D \equiv \partial[\Omega_1 \cup \Omega^*] = [\partial\omega \cup \text{external boundary of } \Omega^*],$$

see (2.11) and Figure 2, again vanishes in view of (2.25b) for $w = (\chi p)$. Thus, the vanishing of the last integral term of (2.27) is justified.

STEP 11. Here, we now estimate the last integral term on the RHS of (2.27). We multiply equation (2.25a) by $e^{\tau\psi(x)}$, thus obtaining

$$e^{\tau\psi(x)}\Delta(\chi p) = -e^{\tau\psi(x)} \operatorname{div} \mathscr{L}_1(\chi\phi) + e^{\tau\psi(x)} \operatorname{div} \mathscr{L}_2(\chi\xi) + e^{\tau\psi(x)} T_\chi^{0,0,1}(\phi, \xi, p), \tag{2.28}$$

$$e^{2\tau\psi(x)}\left|\Delta(\chi p)(x)\right|^2 \leq ce^{2\tau\psi(x)}\left\{\left| \operatorname{div} \mathscr{L}_1(\chi\phi)(x)\right|^2 + \left| \operatorname{div} \mathscr{L}_2(\chi\xi)(x)\right|^2 + \left|T_\chi^{0,0,1}(\phi, \xi, p)(x)\right|^2\right\}, \quad x \in G. \tag{2.29}$$

We now integrate (2.29) over $G \equiv [\Omega_1 \cup \Omega^*]$. In doing so, we recall from (2.24b), (2.24c) that $[\operatorname{div} \mathscr{L}_1]$ and $[\operatorname{div} \mathscr{L}_2]$ are first-order operators and accordingly, from (2.25c), that $T_\chi^{0,0,1}(\phi, \xi, p)$ is an operator which is zero order in ϕ and ξ, and first order in p, and that $T_\chi^{0,0,1}(\phi, \xi, p)$ has support in Ω^*. We thus obtain from (2.29)

$$\begin{aligned} &\int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)}\left|\Delta(\chi p)(x)\right|^2 dx \\ &\leq C_{y_e,B_e} \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)}\left[\left|\nabla(\chi\phi)(x)\right|^2 + \left|(\chi\phi)(x)\right|^2 + \left|\nabla(\chi\xi)(x)\right|^2 + \left|(\chi\xi)(x)\right|^2\right] dx \\ &\quad + C_\chi \int_{\Omega^*} e^{2\tau\psi(x)}\left[\left|\nabla p(x)\right|^2 + \left|p(x)\right|^2 + \left|\phi(x)\right|^2 + \left|\xi(x)\right|^2\right] dx \end{aligned} \tag{2.30}$$

with constant C_χ depending on χ.

STEP 12 (Final estimate of the (χp)-problem). We now substitute (2.30) into the RHS of (2.27), divide across by $[\rho\tau - \frac{1}{8}] > 0$ for τ large, and obtain

$$\begin{aligned} &\int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)}\left|\nabla(\chi p)(x)\right|^2 dx + \frac{\left[2\rho k^2\tau^3 + \mathcal{O}(\tau^2)\right]}{\left[\rho\tau - \frac{1}{8}\right]} \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)}\left|(\chi p)(x)\right|^2 dx \\ &\leq \frac{C_{y_e,B_e}}{(\rho\tau - \frac{1}{8})} \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)}\left[\left|\nabla(\chi\phi)(x)\right|^2 + \left|(\chi\phi)(x)\right|^2 + \left|\nabla(\chi\xi)(x)\right|^2 + \left|(\chi\xi)(x)\right|^2\right] dx \\ &\quad + \frac{C_\chi}{(\rho\tau - \frac{1}{8})} \int_{\Omega^*} e^{2\tau\psi(x)}\left[\left|\nabla p(x)\right|^2 + \left|p(x)\right|^2 + \left|\phi(x)\right|^2 + \left|\xi(x)\right|^2\right] dx. \end{aligned} \tag{2.31}$$

Inequality (2.31) is our final estimate on the (χp)-problem (2.25a).

STEP 13 (Combining the (χu)-estimate (2.23) with the (χp)-estimate (2.31)). We return to estimate (2.23) and add to each side the term

$$\frac{\left[2\rho k^2\tau^3 + \mathcal{O}(\tau^2)\right]}{\left[\rho\tau - \frac{1}{8}\right]} \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)} \left|(\chi p)(x)\right|^2 dx$$

to get

$$\begin{aligned}
&\left\{\left[\rho\tau - \frac{1}{8}\right] - C_{\lambda,e}\right\} \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)} \left|\nabla(\chi u)(x)\right|^2 dx \\
&\quad + \left[2\rho k^2\tau^3 + \mathcal{O}(\tau^2) - C_{\lambda,e}\right] \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)} \left|(\chi u)(x)\right|^2 dx \\
&\quad + \frac{\left[2\rho k^2\tau^3 + \mathcal{O}(\tau^2)\right]}{\left[\rho\tau - \frac{1}{8}\right]} \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)} \left|(\chi p)(x)\right|^2 dx \\
&\le 3 \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)} \left|\nabla(\chi p)(x)\right|^2 dx \\
&\quad + \frac{\left[2\rho k^2\tau^3 + \mathcal{O}(\tau^2)\right]}{\left[\rho\tau - \frac{1}{8}\right]} \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)} \left|(\chi p)(x)\right|^2 dx \\
&\quad + c_\chi \int_{\Omega^*} e^{2\tau\psi(x)} \left[\left|\nabla u(x)\right|^2 + \left|u(x)\right|^2 + \left|p(x)\right|^2\right] dx.
\end{aligned} \tag{2.32}$$

Next, we substitute inequality (2.31) for the first two integral terms on the RHS of (2.32) and obtain

(2.33)

$$\begin{aligned}
&\left\{\left[\rho\tau - \frac{1}{8}\right] - C_{\lambda,e}\right\} \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)} \left|\nabla(\chi u)(x)\right|^2 dx \\
&\quad + \left\{\left[2\rho k^2\tau^3 + \mathcal{O}(\tau^2)\right] - C_{\lambda,e}\right\} \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)} \left|(\chi u)(x)\right|^2 dx \\
&\quad + \frac{\left[2\rho k^2\tau^3 + \mathcal{O}(\tau^2)\right]}{\left[\rho\tau - \frac{1}{8}\right]} \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)} \left|(\chi p)(x)\right|^2 dx \\
&\le \frac{C_{y_e,B_e}}{\left(\rho\tau - \frac{1}{8}\right)} \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)} \left[\left|\nabla(\chi\phi)(x)\right|^2 + \left|(\chi\phi)(x)\right|^2 + \left|\nabla(\chi\xi)(x)\right|^2 + \left|(\chi\xi)(x)\right|^2\right] dx \\
&\quad + \frac{C_\chi}{\left(\rho\tau - \frac{1}{8}\right)} \int_{\Omega^*} e^{2\tau\psi(x)} \left[\left|\nabla p(x)\right|^2 + \left|p(x)\right|^2 + \left|\phi(x)\right|^2 + \left|\xi(x)\right|^2\right] dx \\
&\quad + c_\chi \int_{\Omega^*} e^{2\tau\psi(x)} \left[\left|\nabla u(x)\right|^2 + \left|u(x)\right|^2 + \left|p(x)\right|^2\right] dx.
\end{aligned}$$

Recalling that $u = [\begin{smallmatrix}\phi\\ \xi\end{smallmatrix}]$, we re-write (2.33) explicitly as

(2.34)
$$
\begin{aligned}
&\left\{\left[\rho\tau - \frac{1}{8}\right] - C_{\lambda,e}\right\} \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)}\left[\left|\nabla(\chi\phi)(x)\right|^2 + \left|\nabla(\chi\xi)(x)\right|^2\right]dx \\
&\quad + \left\{\left[2\rho k^2\tau^3 + \mathcal{O}(\tau^2)\right] - C_{\lambda,e}\right\} \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)}\left[\left|(\chi\phi)(x)\right|^2 + \left|(\chi\xi)(x)\right|^2\right]dx \\
&\quad + \frac{\left[2\rho k^2\tau^3 + \mathcal{O}(\tau^2)\right]}{\left[\rho\tau - \frac{1}{8}\right]} \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)}\left|(\chi p)(x)\right|^2 dx \\
&\leq \frac{C_{y_e,B_e}}{(\rho\tau - \frac{1}{8})} \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)}\left[\left|\nabla(\chi\phi)(x)\right|^2 + \left|(\chi\phi)(x)\right|^2 + \left|\nabla(\chi\xi)(x)\right|^2 + \left|(\chi\xi)(x)\right|^2\right]dx \\
&\quad + \frac{C_\chi}{(\rho\tau - \frac{1}{8})} \int_{\Omega^*} e^{2\tau\psi(x)}\left[\left|\nabla p(x)\right|^2 + \left|p(x)\right|^2 + \left|\phi(x)\right|^2 + \left|\xi(x)\right|^2\right]dx \\
&\quad + c_\chi \int_{\Omega^*} e^{2\tau\psi(x)}\left[\left|\nabla\phi(x)\right|^2 + \left|\nabla\xi(x)\right|^2 + \left|\phi(x)\right|^2 + \left|\xi(x)\right|^2 + \left|p(x)\right|^2\right]dx.
\end{aligned}
$$

STEP 14 (Final estimate of problem (2.1a)–(2.1b)). Finally, we combine the integral terms with the same integrand on the LHS of (2.34) and obtain the final sought-after estimate which we formalize as a lemma.

LEMMA 2.2. *The following inequality holds true for all τ sufficiently large:*

(2.35)
$$
\begin{aligned}
&\left\{\left[\rho\tau - \frac{1}{8}\right] - C_{\lambda,e} - \frac{C_{y_e,B_e}}{(\rho\tau - \frac{1}{8})}\right\} \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)}\left[\left|\nabla(\chi\phi)(x)\right|^2 + \left|\nabla(\chi\xi)(x)\right|^2\right]dx \\
&+ \left\{\left[2\rho k^2\tau^3 + \mathcal{O}(\tau^2)\right] - C_{\lambda,e} - \frac{C_{y_e,B_e}}{(\rho\tau - \frac{1}{8})}\right\} \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)}\left[\left|(\chi\phi)(x)\right|^2 + \left|(\chi\xi)(x)\right|^2\right]dx \\
&+ \frac{\left[2\rho k^2\tau^3 + \mathcal{O}(\tau^2)\right]}{\left[\rho\tau - \frac{1}{8}\right]} \int_{\Omega_1\cup\Omega^*} e^{2\tau\psi(x)}\left|(\chi p)(x)\right|^2 dx \\
&\leq \frac{C_\chi}{(\rho\tau - \frac{1}{8})} \int_{\Omega^*} e^{2\tau\psi(x)}\left[\left|\nabla p(x)\right|^2 + \left|p(x)\right|^2 + \left|\phi(x)\right|^2 + \left|\xi(x)\right|^2\right]dx \\
&+ c_\chi \int_{\Omega^*} e^{2\tau\psi(x)}\left[\left|\nabla\phi(x)\right|^2 + \left|\nabla\xi(x)\right|^2 + \left|\phi(x)\right|^2 + \left|\xi(x)\right|^2 + \left|p(x)\right|^2\right]dx.
\end{aligned}
$$

We note explicitly two critical features of estimate (2.35): the integral terms on its LHS are over $[\Omega_1 \cup \Omega^*]$, while the integral terms on its RHS are over Ω^*. As already noted, (2.35) is the ultimate estimate regarding the original problem (2.1a)–(2.1b).

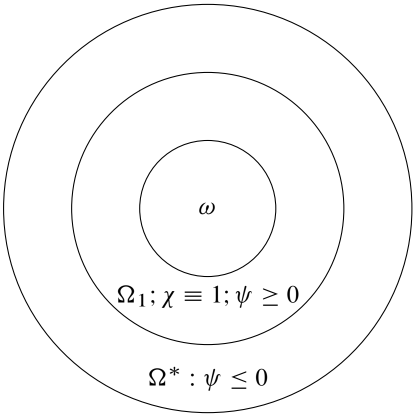

FIGURE 4. Construction of the domains Ω_1 and Ω^* in Case 1.

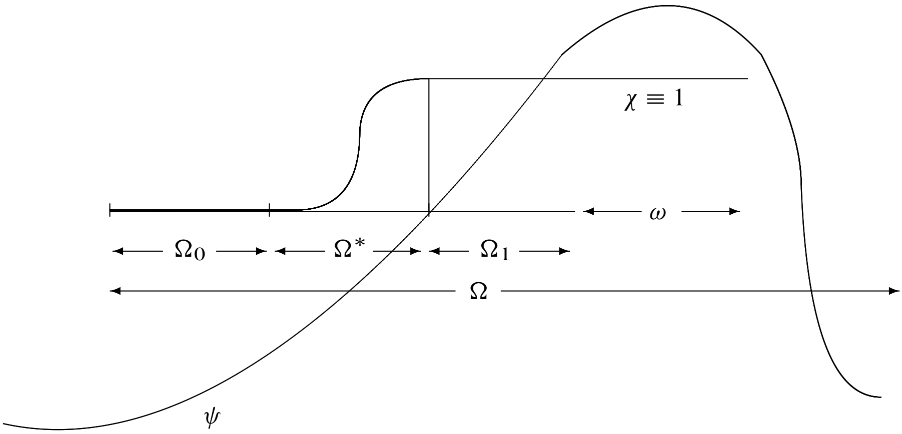

FIGURE 5. Choice of ψ in Case 1.

STEP 15 (The choice of weight function $\psi(x)$). We now choose the strictly convex function $\psi(x)$ as follows (Figures 4 and 5, as well as Figure 3):

$$\psi(x) \geq 0 \text{ on } \Omega_1, \text{ where } \chi \equiv 1 \text{ by (2.4), so that } e^{2\tau\psi(x)} \geq 1 \text{ on } \Omega_1, \tag{2.36}$$

$$\psi(x) \leq 0 \text{ on } \Omega_0 \cup \Omega^*, \text{ where } \chi < 1, \text{ so that } e^{2\tau\psi(x)} \leq 1 \text{ on } \Omega^*, \tag{2.37}$$

in such a way that $\psi(x)$ has no critical point in $\Omega \setminus \omega$, as required by Theorem 2.1 (ψ has no critical points on $G = \Omega_1 \cup \Omega^*$): that is, the critical point(s) of ψ will fall on ω, outside the region $G = \Omega_1 \cup \Omega^*$ where we have integrated.

Having chosen $\psi(x)$ as in (2.36), (2.37) with no critical points in $\Omega \setminus \omega$ – i.e., no critical points on $G = \Omega_1 \cup \Omega^*$ – we return to the basic estimate (2.35), with τ

sufficiently large (Figure 5). On the LHS of (2.35), we retain only integration over Ω_1, where $\psi \geq 0$; hence, $e^{2\tau\psi} \geq 1$ and $\chi \equiv 1$ by (2.4), so that $(\chi u) \equiv u$ on Ω_1; that is, $(\chi\phi) \equiv \phi$ on Ω_1 and $(\chi\xi) \equiv \xi$ on Ω_1. On the RHS of (2.35), we have $\psi \leq 0$ on Ω^*; hence, $e^{2\tau\psi} \leq 1$ on Ω^*. We thus obtain from (2.35) for τ sufficiently large

$$
\begin{aligned}
(2.38)\quad & \left\{\left[\rho\tau - \frac{1}{8}\right] - C_{\lambda,e} - \frac{C_{B_e,y_e}}{(\rho\tau - \frac{1}{8})}\right\} \int_{\Omega_1} \left[|\nabla\phi(x)|^2 + |\nabla\xi(x)|^2\right] dx \\
& + \left\{\left[2\rho k^2\tau^3 + \mathcal{O}(\tau^2)\right] - C_{\lambda,e} - \frac{6C_{B_e,y_e}}{(\rho\tau - \frac{1}{8})}\right\} \int_{\Omega_1} \left[|\phi(x)|^2 + |\xi(x)|^2\right] dx \\
& + \frac{\left[2\rho k^2\tau^3 + \mathcal{O}(\tau^2)\right]}{\left[\rho\tau - \frac{1}{8}\right]} \int_{\Omega_1} |p(x)|^2 dx \\
& \leq \frac{C_\chi}{(\rho\tau - \frac{1}{8})} \int_{\Omega^*} \left[|\nabla p(x)|^2 + |p(x)|^2 + |\phi(x)|^2 + |\xi(x)|^2\right] dx \\
& + c_\chi \int_{\Omega^*} \left[|\nabla\phi(x)|^2 + |\nabla\xi(x)|^2 + |\phi(x)|^2 + |\xi(x)|^2 + |p(x)|^2\right] dx.
\end{aligned}
$$

For τ sufficiently large, inequality (2.38) is of the type

$$
\begin{aligned}
(2.39a)\quad & \left(\tau - \text{const} - \frac{1}{\tau}\right) \int_{\Omega_1} \left[|\nabla\phi(x)|^2 + |\nabla\xi(x)|^2\right] dx \\
& + \left(\tau^3 - \text{const} - \frac{1}{\tau}\right) \int_{\Omega_1} \left[|\phi(x)|^2 + |\xi(x)|^2\right] dx + (\tau^2) \int_{\Omega_1} |p(x)|^2 dx \\
& \leq \frac{c}{\tau} \int_{\Omega^*} \left[|\nabla p(x)|^2 + |p(x)|^2 + |\phi(x)|^2 + |\xi(x)|^2\right] dx \\
& + \text{const} \int_{\Omega^*} \left[|\nabla\phi(x)|^2 + |\nabla\xi(x)|^2 + |\phi(x)|^2 + |\xi(x)|^2 + |p(x)|^2\right] dx
\end{aligned}
$$

or setting as usual $u = \{\phi, h\}$, we re-write (2.39a) equivalently as

$$
\begin{aligned}
(2.39b)\quad & \left(\tau - \text{const} - \frac{1}{\tau}\right) \int_{\Omega_1} |\nabla u(x)|^2 dx + \left(\tau^3 - \text{const} - \frac{1}{\tau}\right) \int_{\Omega_1} |u(x)|^2 dx \\
& + (\tau^2) \int_{\Omega_1} |p(x)|^2 dx \\
& \leq \frac{c}{\tau} \int_{\Omega^*} \left[|\nabla p(x)|^2 + |p(x)|^2 + |u(x)|^2\right] dx \\
& + \text{const} \int_{\Omega^*} \left[|\nabla u(x)|^2 + |u(x)|^2 + |p(x)|^2\right] dx \\
(2.39c)\quad & \leq \frac{c}{\tau} C_1(p, u; \Omega^*) + \text{const}\, C_2(p, u; \Omega^*).
\end{aligned}
$$

In going from (2.39b) to (2.39c), we have emphasized in the notation that we are working with a fixed solution $\{u, p\}$ of problem (2.1a)–(2.1b), so that the integrals on the RHS of (2.39b) are fixed numbers $C_1(p, u; \Omega^*)$ and $C_2(p, u; \Omega^*)$, depending on such fixed solution $\{u, p\}$ as well as Ω^*, $u = \{\phi, \xi\}$. Inequality (2.39) is more than we need. On its LHS, we may drop the ∇u-term over Ω_1; and alternatively either keep only the u-term over Ω_1, and divide the remaining inequality across by $(\tau^3 - \text{const} - \frac{1}{\tau})$ for τ large; or else keep only the p-term over Ω_1 and divide the corresponding inequality across by τ^2. We obtain, respectively,

$$\int_{\Omega_1} |u(x)|^2 dx \leq \left(\frac{C}{\tau^3} \frac{1}{\tau}\right) C_1(p, u; \Omega^*) + \frac{\text{const}}{\tau^3} C_2(p, u; \Omega^*) \to 0, \tag{2.40}$$

$$\int_{\Omega_1} |p(x)|^2 dx \leq \left(\frac{C}{\tau^2} \frac{1}{\tau}\right) C_1(p, u; \Omega^*) + \frac{\text{const}}{\tau^2} C_2(p, u; \Omega^*) \to 0 \tag{2.41}$$

as $\tau \to +\infty$. We thus obtain

$$u(x) \equiv \{\phi(x), \xi(x)\} \equiv 0 \text{ in } \Omega_1; \quad p(x) \equiv 0 \text{ in } \Omega_1, \tag{2.42}$$

and recalling (2.1b) and Step 1,

$$u(x) \equiv \{\phi(x), \xi(x)\} \equiv 0, \quad p(x) \equiv 0 \text{ in } \omega \cup \Omega_1. \tag{2.43}$$

The implication: Step 1⇒(2.43) is illustrated by Figure 6, with $u = \left[\begin{smallmatrix}\phi \\ \xi\end{smallmatrix}\right]$.

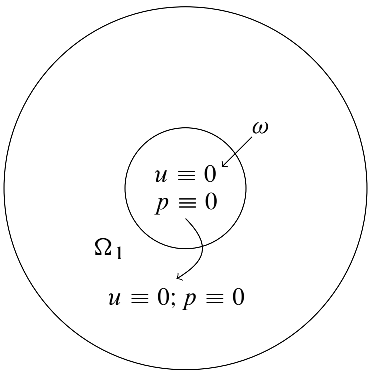

FIGURE 6. Case 1: from $\{u, p\} = 0$ on ω to $\{u, p\} = 0$ on Ω_1.

Finally, we can now push the external boundary of Ω_1 as close as we please to the boundary $\partial\Omega$ of Ω, and thus we finally obtain

$$u(x) \equiv \{\phi(x), \xi(x)\} \equiv 0 \text{ in } \Omega, \quad p(x) \equiv 0 \text{ in } \Omega. \tag{2.44}$$

Indeed, we have $u \equiv \{\phi, \xi\} \in (W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega))^d \times (W^{2,q}(\Omega))^d$ and $p \in W^{1,q}(\Omega)$. Moreover, $W^{1,q}(\Omega) \hookrightarrow C(\overline{\Omega})$ for $q > d$ [12, p. 78] as assumed, and more generally $W^{m,q}(\Omega) \hookrightarrow C^k(\overline{\Omega})$ for $qm > d$, $k = m - \frac{d}{q}$ [12, p. 79]. A fortiori, $u \in (C(\overline{\Omega}))^d$, $p \in C(\overline{\Omega})$, $q > d$, as assumed. Thus, if it should happen that $u(x_1) \neq 0$ at a point $x_1 \in \Omega$ near $\partial\Omega$, hence $u(x) \not\equiv 0$ in a suitable neighborhood N of x_1, then it would suffice to take Ω_1 as to intersect such N to obtain a contradiction. Theorem 1.2 is proved at least in Case 1 (Figures 1, 2, and 3).

CASE 2. Let ω be a full collar of boundary $\Gamma = \partial\Omega$ (Figures 7 and 8). Then, the above proof of Case 1 can be carried out with sets Ω_1, Ω^*, and Ω_0, as indicated in Figure 7.

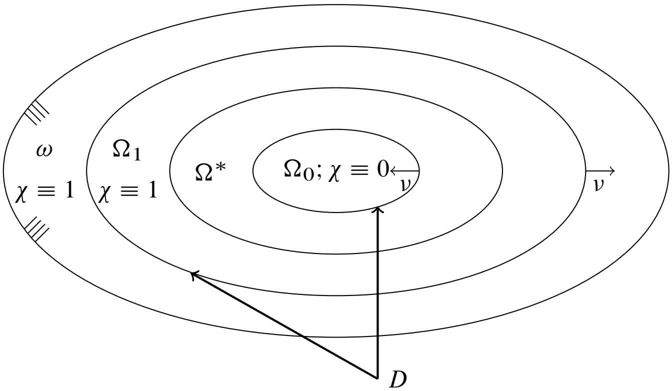

FIGURE 7. Case 2: ω is a collar of Γ; $G = \Omega_1 \cup \Omega^*$; $\partial G = D =$ [internal boundary of ω] $\cup$ [internal boundary of Ω^*].

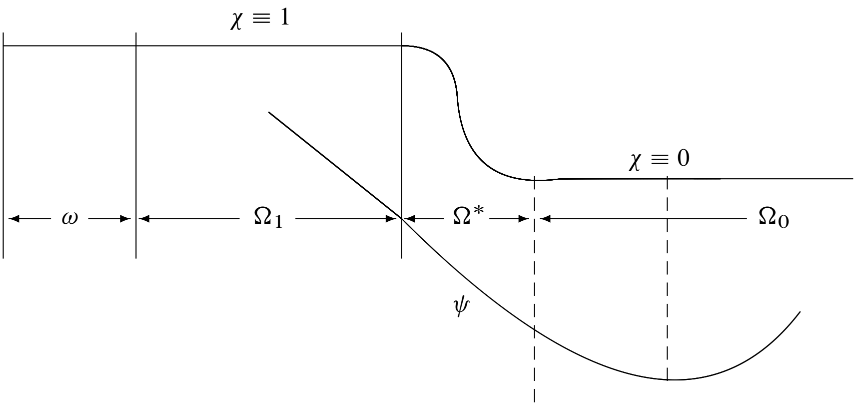

FIGURE 8. Case 2: the choice of ψ for Figure 7.

Let now ω be a partial collar of the boundary $\Gamma = \partial\Omega$. Then, the above proof of Case 1 can be carried out with sets Ω_1, Ω^*, and Ω_0, as indicated in Figure 9.

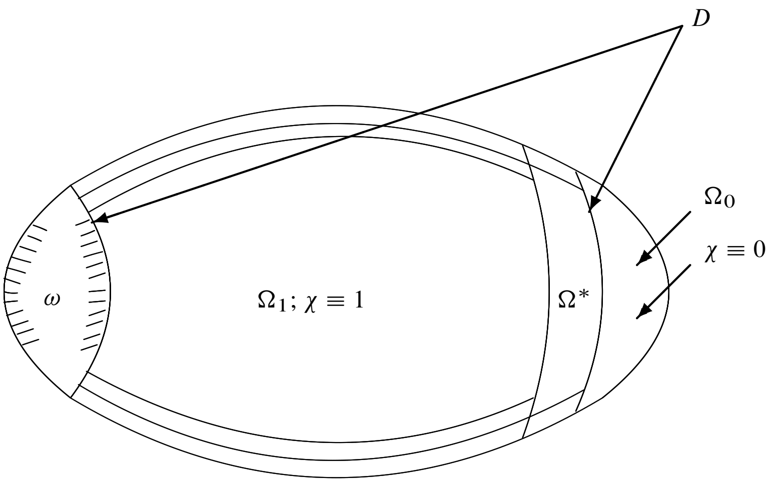

FIGURE 9. Case 2: ω is a collar of a portion of the boundary.

3. IMPLICATION OF THEOREM 1.2 ON THE SOLUTION OF THE CORRESPONDING UNIFORM STABILIZATION PROBLEM OF THE MHD SYSTEM BY FINITE-DIMENSIONAL INTERIOR LOCALIZED STATIC FEEDBACK CONTROLLERS

As is by now well known [31], a result such as Theorem 1.2 is the "ignition key" to solve a corresponding stabilization problem. Because of space constraints, we can only report here, in a very concise form, the direct implication of Theorem 1.2 in establishing (in the adjoint version) the Kalman algebraic condition for the finite-dimensional unstable component of the linearized dynamics (1.4a–e). The solution of the full local uniform stabilization problem of the original non-linear problem (1.1a–e) in the Besov space setting, by means of two finite-dimensional localized static controllers $\{u, v\}$ in feedback form and of minimal dimension, is given in [20, $d = 2, 3$].

3.1. Preliminaries

We preliminarily assume that the space $\mathbf{L}^p(\Omega)$ can be decomposed into the direct (non-orthonormal sum for $q \neq 2$)

$$\text{(3.1a)} \qquad \mathbf{L}^q(\Omega) = \mathbf{L}^q_\sigma(\Omega) \oplus \mathbf{G}^q(\Omega),$$

$$\text{(3.1b)} \qquad \mathbf{L}^q_\sigma(\Omega) = \overline{\left\{\mathbf{y} \in \mathbf{C}^\infty_c(\Omega) : \operatorname{div} \mathbf{y} = 0 \text{ in } \Omega\right\}}^{\|\cdot\|_q}$$
$$= \left\{\mathbf{g} \in \mathbf{L}^q(\Omega) : \operatorname{div} \mathbf{g} = 0;\ \mathbf{g} \cdot \nu = 0 \text{ on } \partial\Omega\right\},$$
$$\text{for any locally Lipschitz domain } \Omega \subset \mathbb{R}^d,\ d \geq 2,$$

$$\text{(3.1c)} \qquad \mathbf{G}^q(\Omega) = \left\{\mathbf{y} \in \mathbf{L}^q(\Omega) : \mathbf{y} = \nabla p,\ p \in W^{1,q}_{\text{loc}}(\Omega)\right\} \quad \text{where } 1 \leq q < \infty,$$

of the solenoidal vector space $\mathbf{L}^q_\sigma(\Omega)$ and the space of gradient fields (Helmholtz decomposition). Both of these are closed subspaces of $\mathbf{L}^q$. This is a mild assumption in the bounded domain Ω in $\mathbb{R}^d$, $d = 2, 3$. Let P_q be the Helmholtz projection, the unique linear, bounded, idempotent ($P_q^2 = P_q$) projection operator $P_q : \mathbf{L}^q(\Omega)$ onto $\mathbf{L}^q_\sigma(\Omega)$, having $\mathbf{G}^q(\Omega)$ as its null space. Introduce the operators

$$A_{1,q} w = -P_q \Delta w, \qquad \mathcal{D}(A_{1,q}) = \mathbf{W}^{2,q}(\Omega) \cap \mathbf{W}^{1,q}_0(\Omega) \cap \mathbf{L}^q_\sigma(\Omega), \tag{3.2}$$

$$A_{2,q} \mathbb{W} = -\Delta \mathbb{W}, \qquad \mathcal{D}(A_{2,q}) = \big\{\mathbb{W} \in \mathbf{W}^{2,q}(\Omega) \cap \mathbf{L}^q_\sigma(\Omega), (\text{curl } \mathbb{W}) \times n \equiv 0 \text{ on } \Gamma\big\}, \tag{3.3}$$

$$A_{o,y_e,q} w = P_q \mathcal{L}^+_{y_e} w = P_q\big[(y_e \cdot \nabla) w + (w \cdot \nabla) y_e\big], \qquad \mathcal{D}(A_{o,y_e,q}) = \mathcal{D}(A^{1/2}_{1,q}) \subset \mathbf{L}^q_\sigma(\Omega), \tag{3.4}$$

$$A_{o,B_e,q} \mathbb{W} = P_q \mathcal{L}^+_{B_e} \mathbb{W} = P_q\big[(B_e \cdot \nabla)\mathbb{W} + (\mathbb{W} \cdot \nabla) B_e\big], \qquad \mathcal{D}(A_{o,B_e,q}) = \mathcal{D}(A^{1/2}_{2,q}) \subset \mathbf{L}^q_\sigma(\Omega), \tag{3.5}$$

$$L^-_{B_e} w = \mathcal{L}^-_{B_e} w = \big[(B_e \cdot \nabla) w - (w \cdot \nabla) B_e\big], \qquad \mathcal{D}(L^-_{B_e}) = \mathcal{D}(A^{1/2}_{2,q}) \subset \mathbf{L}^q_\sigma(\Omega), \tag{3.6}$$

$$L^-_{y_e} \mathbb{W} = \mathcal{L}^-_{y_e} \mathbb{W} = (y_e \cdot \nabla)\mathbb{W} - (\mathbb{W} \cdot \nabla) y_e, \qquad \mathcal{D}(L^-_{y_e}) = \mathcal{D}(A^{1/2}_{2,q}) \subset \mathbf{L}^q_\sigma(\Omega). \tag{3.7}$$

Invoking (3.5), (3.6), we rewrite (1.4a–b) more conveniently as

$$\left.\begin{aligned} &w_t - \nu\Delta w + \mathcal{L}^+_{y_e}(w) - \mathcal{L}^+_{B_e}(\mathbb{W}) + \nabla p = mu \quad \text{in } Q, && (3.8)\\ &\mathbb{W}_t - \eta\Delta\mathbb{W} + \mathcal{L}^-_{y_e}(\mathbb{W}) + \mathcal{L}^-_{B_e}(w) = mv \quad \text{in } Q, && (3.9)\end{aligned}\right\}$$

to be accompanied by the divergence free contribution (1.4c) and the B.C. (1.4d). We next invoke the Helmholtz projection $P_q : \mathbf{L}^p(\Omega)$ onto $\mathbf{L}^q_\sigma(\Omega)$ to eliminate the pressure term ∇p in (1.9a), taking advantage of the divergence free conditions div $w \equiv 0$, div $\mathbb{W} \equiv 0$ in Q, and the conditions $w \equiv 0$ and $\mathbb{W} \cdot n \equiv 0$ on Σ, which are intrinsic conditions in the $\mathbf{L}^q_\sigma(\Omega)$-space. We obtain

$$w_t - \nu(P_q\Delta)w + (P_q\mathcal{L}^+_{y_e})(w) + (P_q\mathcal{L}^+_{B_e})(\mathbb{W}) = mP_q u \quad \text{in } Q \tag{3.10}$$

as $P_q\nabla p \equiv 0$, which along with equation (3.9) for $\mathbb{W}$ yields the following first-order PDE system

$$\frac{d}{dt}\begin{bmatrix} w \\ \mathbb{W}\end{bmatrix} = \begin{bmatrix} \nu P_q\Delta & 0 \\ 0 & \eta\Delta\end{bmatrix}\begin{bmatrix} w \\ \mathbb{W}\end{bmatrix} + \begin{bmatrix} -P_q\mathcal{L}^+_{y_e} & P_q\mathcal{L}^+_{B_e} \\ -\mathcal{L}^-_{B_e} & -\mathcal{L}^-_{y_e}\end{bmatrix}\begin{bmatrix} w \\ \mathbb{W}\end{bmatrix} + \begin{bmatrix} mP_q u \\ mP_q v\end{bmatrix}, \tag{3.11}$$

along with (1.4c–d). Thus, in view of (3.2)–(3.5), the abstract version of the PDE-coupled problem (1.14a–d) is

$$\frac{d}{dt}\begin{bmatrix} w \\ \mathbb{W} \end{bmatrix} = \begin{bmatrix} -\nu A_{1,q} & 0 \\ 0 & -\eta A_{2,q} \end{bmatrix}\begin{bmatrix} w \\ \mathbb{W} \end{bmatrix} + \begin{bmatrix} -A_{o,y_e,q} & A_{o,B_e,q} \\ L_{\overline{B_e}} & -L_{\overline{y_e}} \end{bmatrix}\begin{bmatrix} w \\ \mathbb{W} \end{bmatrix} + \begin{bmatrix} mP_q u \\ mP_q v \end{bmatrix}; \tag{3.12}$$

finally,

$$\frac{d}{dt}\begin{bmatrix} w \\ \mathbb{W} \end{bmatrix} = \widetilde{\mathbb{A}}_q \begin{bmatrix} w \\ \mathbb{W} \end{bmatrix} + \begin{bmatrix} mP_q u \\ mP_q v \end{bmatrix}, \ \boldsymbol{\eta} = \begin{bmatrix} w \\ \mathbb{W} \end{bmatrix}, \text{ on } \mathbf{Y}^q_\sigma(\Omega) = \mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega), \tag{3.13}$$

$$\begin{aligned} \widetilde{\mathbb{A}}_q &= \mathbb{A}_{o,q} + \Pi = \begin{bmatrix} -\nu A_{1,q} & 0 \\ 0 & -\eta A_{2,q} \end{bmatrix} + \begin{bmatrix} -A_{o,y_e,q} & A_{o,B_e,q} \\ L_{\overline{B_e}} & -L_{\overline{y_e}} \end{bmatrix} \\ \mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega) &\supset \mathcal{D}(\widetilde{\mathbb{A}}_q) = \mathcal{D}(A_{1,q}) \times \mathcal{D}(A_{2,q}) \to \mathbf{Y}^q_\sigma(\Omega). \end{aligned} \tag{3.14}$$

The operator $\widetilde{\mathbb{A}}_q$ is the generator of a strongly continuous (s.c.) analytic semigroup $e^{\widetilde{\mathbb{A}}_q t}$ on $\mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega) \equiv \mathbf{Y}^q_\sigma(\Omega)$ [20].

3.2. *Introduction to the stabilization problem*

For the problem of stabilization to be relevant, the assumption is that: the generator $\widetilde{\mathbb{A}}_q$ of a s.c. analytic compact semigroup is *unstable* on $\mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega) \equiv \mathbf{Y}^q_\sigma(\Omega)$, in the sense that there are N unstable eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_N$ of $\widetilde{\mathbb{A}}_q$ (see Figure 10),

$$\cdots \le \operatorname{Re}\lambda_{N+2} \le \operatorname{Re}\lambda_{N+1} < 0 \le \operatorname{Re}\lambda_N \le \cdots \le \operatorname{Re}\lambda_2 \le \operatorname{Re}\lambda_1,$$

where the eigenvalues of $\widetilde{\mathbb{A}}_q$ are numbered in order of decreasing real parts. Let M be the number of distinct unstable eigenvalues of $\widetilde{\mathbb{A}}_q$ (or $\widetilde{\mathbb{A}}^*_q$). For each $i = 1, \dots, M$, we denote by

$$\{\mathbf{\Phi}_{ij}\}_{j=1}^{\ell_i} = \left\{ \begin{bmatrix} \varphi_{ij} \\ \psi_{ij} \end{bmatrix} \right\}_{j=1}^{\ell_i}, \quad \{\mathbf{\Phi}^*_{ij}\}_{j=1}^{\ell_i} = \left\{ \begin{bmatrix} \varphi^*_{ij} \\ \psi^*_{ij} \end{bmatrix} \right\}_{j=1}^{\ell_i}$$

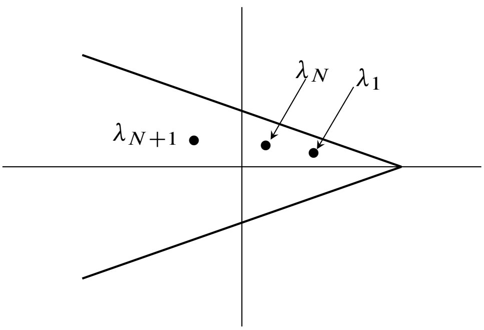

FIGURE 10. The eigenvalues of $\widetilde{\mathbb{A}}_q$.

the normalized, linearly independent eigenfunctions of $\widetilde{\mathbb{A}}_q$, respectively $\widetilde{\mathbb{A}}_q^*$, say, on $\mathbf{Y}^q_\sigma(\Omega) \equiv \mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega)$ and

$$(3.15) \quad \left(\mathbf{Y}^q_\sigma(\Omega)\right)^* \equiv \left(\mathbf{L}^q_\sigma(\Omega)\right)' \times \left(\mathbf{L}^q_\sigma(\Omega)\right)' = \mathbf{L}^{q'}_\sigma(\Omega) \times \mathbf{L}^{q'}_\sigma(\Omega), \quad \frac{1}{q} + \frac{1}{q'} = 1,$$

corresponding to the M distinct unstable eigenvalues $\lambda_1, \dots, \lambda_M$ of $\widetilde{\mathbb{A}}_q$ and $\overline{\lambda}_1, \dots, \overline{\lambda}_M$ of $\widetilde{\mathbb{A}}_q^*$, respectively,

$$\begin{aligned}
(3.16) \qquad \widetilde{\mathbb{A}}_q \mathbf{\Phi}_{ij} &= \lambda_i \mathbf{\Phi}_{ij} \in \mathcal{D}(\widetilde{\mathbb{A}}_q) = \mathcal{D}(A_{1,q}) \times \mathcal{D}(A_{2,q}) \\
&= \left[\mathbf{W}^{2,q}(\Omega) \cap \mathbf{W}^{1,q}_0(\Omega) \cap \mathbf{L}^q_\sigma(\Omega)\right] \times \left[\mathbf{W}^{2,q}(\Omega) \cap \mathbf{L}^q_\sigma(\Omega)\right], \\
(3.17) \qquad \widetilde{\mathbb{A}}_q^* \mathbf{\Phi}^*_{ij} &= \overline{\lambda}_i \mathbf{\Phi}^*_{ij} \in \mathcal{D}(\widetilde{\mathbb{A}}_q^*) \\
&= \left[\mathbf{W}^{2,q'}(\Omega) \cap \mathbf{W}^{1,q'}_0(\Omega) \cap \mathbf{L}^{q'}_\sigma(\Omega)\right] \times \left[\mathbf{W}^{2,q'}(\Omega) \cap \mathbf{L}^{q'}_\sigma(\Omega)\right].
\end{aligned}$$

$$\begin{array}{c} \overline{\lambda}_i \\ \swarrow \quad \swarrow \qquad \searrow \\ \mathbf{\Phi}^*_{i1}, \quad \mathbf{\Phi}^*_{i2}, \quad \dots \quad \mathbf{\Phi}^*_{i\ell_i} \end{array}$$

Linearly independent in $(\mathbf{Y}^q_\sigma(\Omega))^*$,
and ℓ_i is the geometric multiplicity of λ_i.

The critical consequence of Theorem 1.2 (actually, its adjoint version whose proof is essentially the same) is the following theorem.

THEOREM 3.1. *Consider the above presentation in Section 3.1.*

(i) *With reference to (3.17), we have, for each i,*

(3.18) *the vectors $\mathbf{\Phi}^*_{i1}, \dots, \mathbf{\Phi}^*_{i\ell_i}$ remain linearly independent in*

$$\mathbf{L}^{q'}_\sigma(\omega) \times \mathbf{L}^{q'}_\sigma(\omega) \equiv \left(\mathbf{Y}^q_\sigma(\Omega)\right)^* = \mathbf{Y}^{q'}_\sigma(\Omega), \ \frac{1}{q} + \frac{1}{q'} = 1.$$

(ii) *Consequently, it is possible to select vectors $\mathbf{u}_1, \dots, \mathbf{u}_K \in \mathbf{L}^q_\sigma(\omega) \times \mathbf{L}^q_\sigma(\omega)$, $\mathbf{u}_i = [u_i^1, u_i^2]$, $q > 1$, $K = \sup\{\ell_i : i = 1, \dots, M\}$, such that*

(3.19a)

$$\operatorname{rank} \begin{bmatrix} (\mathbf{u}_1, \mathbf{\Phi}^*_{i1})_\omega & \cdots & (\mathbf{u}_K, \mathbf{\Phi}^*_{i1})_\omega \\ (\mathbf{u}_1, \mathbf{\Phi}^*_{i2})_\omega & \cdots & (\mathbf{u}_K, \mathbf{\Phi}^*_{i2})_\omega \\ \vdots & & \vdots \\ (\mathbf{u}_1, \mathbf{\Phi}^*_{i\ell_i})_\omega & \cdots & (\mathbf{u}_K, \mathbf{\Phi}^*_{i\ell_i})_\omega \end{bmatrix} = \ell_i; \ \ell_i \times K \textit{ for each } i = 1, \dots, M,$$

(3.19b)

$$(\mathbf{u}_j, \mathbf{\Phi}^*_{i1})_\omega = \left(\begin{bmatrix} u_j^1 \\ u_j^2 \end{bmatrix}, \begin{bmatrix} \varphi^*_{ij} \\ \psi^*_{ij} \end{bmatrix} \right)_{\mathbf{L}^q_\sigma(\omega) \times \mathbf{L}^q_\sigma(\omega)}, \ \ell_i = \textit{geometric multiplicity of } \lambda_i.$$

Proof. (i) By contradiction, let us assume that the vectors $\{\mathbf{\Phi}^*_{i1}, \dots, \mathbf{\Phi}^*_{i\ell_i}\}$ are instead linearly dependent on $\mathbf{L}^{q'}_\sigma(\omega) \times \mathbf{L}^{q'}_\sigma(\omega)$, so that

$$\mathbf{\Phi}^*_{i\ell_i} = \sum_{j=1}^{\ell_i - 1} \alpha_j \mathbf{\Phi}^*_{i\ell_j} \text{ in } \mathbf{L}^{q'}_\sigma(\omega) \times \mathbf{L}^{q'}_\sigma(\omega). \tag{3.20}$$

Define the following function (depending on i) in $\mathbf{L}^{q'}_\sigma(\Omega) \times \mathbf{L}^{q'}_\sigma(\Omega)$

$$\mathbf{\Phi}^* = \left[\sum_{j=1}^{\ell_i - 1} \alpha_j \mathbf{\Phi}^*_{i\ell_j} - \mathbf{\Phi}^*_{i\ell_i} \right] \in \mathbf{L}^{q'}_\sigma(\Omega) \times \mathbf{L}^{q'}_\sigma(\Omega), \quad i = 1, \dots, M, \tag{3.21}$$

so that $\mathbf{\Phi}^* \equiv 0$ in ω by (3.20). As each $\mathbf{\Phi}^*_{ij}$ is an eigenvalue of $\widetilde{\mathbb{A}}^*_q$ (or $(\widetilde{\mathbb{A}}^u_{q,N})^*$) corresponding to the eigenvalue $\overline{\lambda}_i$, see (3.17), so is the linear combination $\mathbf{\Phi}^*$. This property along with $\mathbf{\Phi}^* \equiv 0$ in ω yields that $\mathbf{\Phi}^*$ satisfies the following overdetermined eigenvalue problem for the operator $\widetilde{\mathbb{A}}^*_q$ (or $(\widetilde{\mathbb{A}}^u_{q,N})^*$):

$$\widetilde{\mathbb{A}}^*_q \mathbf{\Phi}^* = \overline{\lambda} \mathbf{\Phi}^*, \quad \operatorname{div} \mathbf{\Phi}^* = 0 \text{ in } \Omega; \quad \mathbf{\Phi}^* = 0 \text{ in } \omega \quad \big(\text{by } (3.20)\big). \tag{3.22}$$

(ii) But the linear combination $\mathbf{\Phi}^*$ in (3.25) of the eigenfunctions $\mathbf{\Phi}^*_{ij} \in \mathcal{D}(\widetilde{\mathbb{A}}^*_q)$ satisfies itself the Dirichlet B.C $\mathbf{\Phi}^*|_{\partial\Omega} = 0$. Thus, the explicit PDE version of problem (3.22) with $\mathbf{\Phi}^* = \{\varphi^*, \xi^*\}$ is

$$\left.\begin{aligned}
&-\nu\Delta\varphi^* + \mathcal{L}^*_1\varphi^* - \mathcal{L}^*_2\xi^* + \nabla p = \overline{\lambda}\varphi^* && \text{in } \Omega, && (3.23a)\\
&-\eta\Delta\xi^* + \mathcal{M}^*_1\xi^* - \mathcal{M}^*_2\varphi^* = \overline{\lambda}\xi^* && \text{in } \Omega, && (3.23b)\\
&\operatorname{div}\varphi^* \equiv 0, \quad \operatorname{div}\xi^* \equiv 0 && \text{in } \Omega, && (3.23c)\\
&\varphi^* \equiv 0, \quad \xi^* \cdot n \equiv 0, \quad \operatorname{curl}\xi^* \times n \equiv 0 && \text{on } \Gamma, && (3.23d)\\
&\varphi^* \equiv 0, \quad \xi^* \equiv 0 && \text{in } \omega, && (3.23e)
\end{aligned}\right\}$$

$$\mathbf{\Phi}^* \in \mathcal{D}(\widetilde{\mathbb{A}}^*_q); \quad \mathcal{L}^*_i\varphi^* = (y_e \cdot \nabla)\varphi^* + (\varphi^* \cdot \nabla)^* y_e, \tag{3.24}$$

with overdetermined conditions (3.23e), where $(f.\nabla)^* y_e$ is a d-vector whose ith component is $\sum_{j=1}^d (D_i y_{e_j}) f_j$. The adjoint version of Theorem 1.2 (essentially with the same proof) implies

$$\varphi^* \equiv 0, \ \xi^* \equiv 0, \ p^* \equiv \text{const in } \Omega; \text{ or } \mathbf{\Phi}^* = 0 \text{ in } \mathbf{L}^q_\sigma(\Omega) \times \mathbf{L}^q_\sigma(\Omega); \tag{3.25}$$

that is, by (3.21),

$$\mathbf{\Phi}^*_{i\ell_i} = \alpha_1 \mathbf{\Phi}^*_{i1} + \alpha_2 \mathbf{\Phi}^*_{i2} + \dots + \alpha_{\ell_i - 1} \mathbf{\Phi}^*_{i\ell_i - 1} \text{ in } \mathbf{L}^{q'}_\sigma(\Omega) \times \mathbf{L}^{q'}_\sigma(\Omega), \tag{3.26}$$

i.e. the set $\{\mathbf{\Phi}^*_{i1}, \dots, \mathbf{\Phi}^*_{i\ell_i}\}$ in linearly dependent on $\mathbf{L}^{q'}_\sigma(\Omega) \times \mathbf{L}^{q'}_\sigma(\Omega)$. But this is false, by the very selection of such eigenvectors, see (3.26) and statement preceding it. Thus, the condition (3.20) cannot hold. ∎

REMARK 1. Condition (3.19) is the Kalman algebraic condition for asserting the controllability of the finite-dimensional unstable component η_N of the $\left[\begin{smallmatrix} w \\ \mathbb{W} \end{smallmatrix}\right] = \eta$-dynamics, $\eta = \eta_N + \zeta_N$. It is then equivalent to the arbitrary spectrum location property [5] and hence it implies that such originally unstable finite-dimensional dynamics can be stabilized with an arbitrary large decay rate by a (finite-dimensional) state feedback control.

Some relevant references are [4, 36].

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REFERENCES

[1] R. A. ADAMS, *Sobolev spaces*. Pure Appl. Math. 65, Academic Press, New York, 1975. Zbl 0314.46030 MR 0450957

[2] H. AMANN, *Linear and quasilinear parabolic problems. Vol. I. Abstract linear theory*. Monogr. Math. 89, Birkhäuser, Boston, MA, 1995. Zbl 0819.35001 MR 1345385

[3] C. AMROUCHE – S. BOUKASSA, Existence and regularity of solution for a model in magnetohydrodynamics. *Nonlinear Anal.* **190** (2020), article no. 111602. Zbl 1482.35166 MR 3998839

[4] C. AMROUCHE – E. WEHBE, MHD flow with Navier type boundary conditions. *Appl. Math. Lett.* **134** (2022), article no. 108361. Zbl 1504.35279 MR 4462955

[5] G. BASILE – G. MARRO, *Controlled and conditioned invariants in linear system theory*. Prentice Hall, Englewood Cliffs, NJ, 1992. Zbl 0758.93002 MR 1149379

[6] A. L. BUKHGEĬM, Carleman estimates for Volterra operators and the uniqueness of inverse problems. In *Nonclassical problems of mathematical physics*, pp. 56–64, Akad. Nauk SSSR Sibirsk. Otdel., Vychisl. Tsentr, Novosibirsk, 1981. Zbl 0581.45010 MR 0739163

[7] A. L. BUKHGEIM – M. V. KLIBANOV, Global uniqueness of a class of multidimensional inverse problems. *Soviet Math. Dokl.* **24** (1981), 244–247. Zbl 0497.35082

[8] T. Carleman, Sur un problème d'unicité pur les systèmes d'équations aux dérivées partielles à deux variables indépendantes. *Ark. Mat. Astr. Fys.* **26** (1939), no. 17, 9. Zbl 0022.34201 MR 0000334

[9] L. de Simon, Un'applicazione della teoria degli integrali singolari allo studio delle equazioni differenziali lineari astratte del primo ordine. *Rend. Sem. Mat. Univ. Padova* **34** (1964), 205–223. Zbl 0196.44803 MR 0176192

[10] A. Fursikov, Real process corresponding to the 3D Navier-Stokes system, and its feedback stabilization from the boundary. In *Partial differential equations*, pp. 95–123, Amer. Math. Soc. Transl. Ser. 2 206, American Mathematical Society, Providence, RI, 2002. MR 1939489

[11] A. V. Fursikov, Stabilizability of two-dimensional Navier-Stokes equations with help of a boundary feedback control. *J. Math. Fluid Mech.* **3** (2001), no. 3, 259–301. Zbl 0983.93021 MR 1860125

[12] S. Kesavan, *Topics in functional analysis and applications*. John Wiley & Sons, New York, 1989. Zbl 0666.46001 MR 0990018

[13] M. V. Klibanov, Uniqueness of solutions in "the large" of some multidimensional inverse problems. In *Nonclassical problems of mathematical physics*, pp. 101–114, Akad. Nauk SSSR Sibirsk. Otdel., Vychisl. Tsentr, Novosibirsk, 1981. Zbl 0536.35075 MR 0739168

[14] I. Lasiecka – B. Priyasad – R. Triggiani, Uniform stabilization of Boussinesq systems in critical L^q-based Sobolev and Besov spaces by finite dimensional interior localized feedback controls. *Discrete Contin. Dyn. Syst. Ser. B* **25** (2020), no. 10, 4071–4117. Zbl 1452.35229 MR 4147376

[15] I. Lasiecka – B. Priyasad – R. Triggiani, Maximal L^p-regularity for an abstract evolution equation with applications to closed-loop boundary feedback control problems. *J. Differential Equations* **294** (2021), 60–87. Zbl 1470.35214 MR 4265661

[16] I. Lasiecka – B. Priyasad – R. Triggiani, Uniform stabilization of 3D Navier-Stokes equations in low regularity Besov spaces with finite dimensional, tangential-like boundary, localized feedback controllers. *Arch. Ration. Mech. Anal.* **241** (2021), no. 3, 1575–1654. Zbl 1468.76022 MR 4284531

[17] I. Lasiecka – B. Priyasad – R. Triggiani, Uniform stabilization of Navier-Stokes equations in critical L^q-based Sobolev and Besov spaces by finite dimensional interior localized feedback controls. *Appl. Math. Optim.* **83** (2021), no. 3, 1765–1829. Zbl 1468.49039 MR 4261273

[18] I. Lasiecka – B. Priyasad – R. Triggiani, Finite-dimensional boundary uniform stabilization of the Boussinesq system in Besov spaces by critical use of Carleman estimate-based inverse theory. *J. Inverse Ill-Posed Probl.* **30** (2022), no. 1, 35–79. Zbl 1509.35186 MR 4373271

[19] I. Lasiecka – B. Priyasad – R. Triggiani, Maximal L^p-regularity of an abstract evolution equation: application to closed-loop feedback problems, with boundary controls and boundary sensors. *Nonlinear Anal. Real World Appl.* **84** (2025), article no. 104295. Zbl 1559.35356 MR 4842732

[20] I. Lasiecka – B. Priyasad – R. Triggiani, Uniform stabilization in Besov spaces with arbitrary decay rates of the magnetohydrodynamic system by finite-dimensional interior localized static feedback controllers. *Res. Math. Sci.* **12** (2025), no. 1, article no. 7. Zbl 1558.35212 MR 4841826

[21] I. Lasiecka – R. Triggiani – X. Zhang, Nonconservative wave equations with unobserved Neumann B.C.: global uniqueness and observability in one shot. In *Differential geometric methods in the control of partial differential equations (Boulder, CO, 1999)*, pp. 227–325, Contemp. Math. 268, American Mathematical Society, Providence, RI, 2000. Zbl 1096.93503 MR 1804797

[22] I. Lasiecka – R. Triggiani – X. Zhang, Global uniqueness, observability and stabilization of nonconservative Schrödinger equations via pointwise Carleman estimates. I. $H^1(\Omega)$-estimates. *J. Inverse Ill-Posed Probl.* **12** (2004), no. 1, 43–123. Zbl 1057.35042 MR 2040524

[23] C.-G. Lefter, On a unique continuation property related to the boundary stabilization of magnetohydrodynamic equations. *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)* **56** (2010), no. 1, 1–15. Zbl 1212.93256 MR 2605928

[24] C.-G. Lefter, Feedback stabilization of magnetohydrodynamic equations. *SIAM J. Control Optim.* **49** (2011), no. 3, 963–983. Zbl 1228.93100 MR 2806571

[25] M. Sermange – R. Temam, Some mathematical questions related to the MHD equations. *Comm. Pure Appl. Math.* **36** (1983), no. 5, 635–664. Zbl 0524.76099 MR 0716200

[26] V. A. Solonnikov, Some stationary boundary-value problems of magnetohydrodynamics. *Trudy Mat. Inst. Steklov.* **59** (1960), 174–187. MR 0170131

[27] R. Triggiani, On the stabilizability problem in Banach space. *J. Math. Anal. Appl.* **52** (1975), no. 3, 383–403. Zbl 0326.93023 MR 0445388

[28] R. Triggiani, Linear independence of boundary traces of eigenfunctions of elliptic and Stokes operators and applications. *Appl. Math. (Warsaw)* **35** (2008), no. 4, 481–512. Zbl 1152.35420 MR 2457273

[29] R. Triggiani, Unique continuation from an arbitrary interior subdomain of the variable-coefficient Oseen equation. *Nonlinear Anal.* **71** (2009), no. 10, 4967–4976. Zbl 1181.35034 MR 2548728

[30] R. Triggiani, Unique continuation of boundary over-determined Stokes and Oseen eigenproblems. *Discrete Contin. Dyn. Syst. Ser. S* **2** (2009), no. 3, 645–677. Zbl 1198.35200 MR 2525772

[31] R. Triggiani, Unique continuation properties of static over-determined eigenproblems: the ignition key for uniform stabilization of dynamic fluids by feedback controllers. In *Fluids under control*, pp. 63–155, Adv. Math. Fluid Mech., Birkhäuser/Springer, Cham, 2024. Zbl 1557.76034 MR 4738450

[32] R. Triggiani – X. Wan, Unique continuation properties of over-determined static Boussinesq problems with application to uniform stabilization of dynamic Boussinesq systems. *Appl. Math. Optim.* **84** (2021), no. 2, 2099–2146. Zbl 1487.35306 MR 4304899

[33] N. Yamaguchi, L^q–L^r estimates of solution to the parabolic Maxwell equations and their application to the magnetohydrodynamic equations. *Math. Anal. Fluid Gas Dyn.* **1353** (2004), 72–91.

[34] Z. Yoshida – Y. Giga, On the Ohm-Navier-Stokes system in magnetohydrodynamics. *J. Math. Phys.* **24** (1983), no. 12, 2860–2864. Zbl 0533.76125 MR 0727780

[35] J. Zabczyk, *Mathematical control theory: An introduction*. Systems & Control: Foundations & Applications, Birkhäuser, Boston, MA, 1992. Zbl 1071.93500 MR 1193920

[36] C. Zhao – K. Li, On existence, uniqueness and L^r-exponential stability for stationary solutions to the MHD equations in three-dimensional domains. *ANZIAM J.* **46** (2004), no. 1, 95–109. Zbl 1060.35114 MR 2075516

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