



Mathematical Analysis. – *Hidden trace regularity for Riemann–Liouville fractional equations*, by PAOLA LORETI and DANIELA SFORZA, accepted on 14 May 2025.

ABSTRACT. – We begin with a brief overview of the most commonly used fractional derivatives, namely, the Caputo and Riemann–Liouville derivatives. We then focus on the study of the fractional time wave equation with the Riemann–Liouville derivative, addressing key questions such as well-posedness, regularity, and a trace result in appropriate interpolation spaces. Additionally, we explore the duality relationship with the Caputo fractional time derivative. The analysis is based on expanding the solution in terms of Mittag-Leffler functions.

KEYWORDS. – integro-differential equations, fractional derivatives, regularity.

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1. INTRODUCTION

Although theoretical results concerning fractional differential equations are still being studied by mathematicians – including well-posedness in Sobolev spaces, regularity properties, asymptotic behavior of solutions, and so on – the modeling and consistency with real phenomena drive the applications of fractional calculus. This is particularly true in engineering and science, especially in fields like rheology, viscoelasticity, control theory, bioengineering, and many others, see e.g. [6, 27].

One of the fascinating aspects of this topic is that the definition of fractional derivatives relies on non-local operators such as Riemann–Liouville integrals of the type

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds,$$

where Γ is the Gamma function, see [14, 15, 28]. The integral is well defined provided that $\alpha > 0$ and u is a locally integrable function. The operator defined by the Riemann–Liouville integral is non-local in nature. This means that the value of the fractional integral at a point t depends not only on the value of the function u at t , but also on values of the function at all points $s \in (0, t)$. This non-local behavior is a direct consequence of the kernel $s^{\alpha-1}$, which gives the integral a memory-like property. This is in contrast to classical integer-order calculus, where the derivative or integral at a point depends only on local values. The structure of the kernel $s^{\alpha-1}$ plays a crucial role

in defining the behavior of the fractional integral. For example, in the case $\alpha \in (0, 1)$, the kernel causes the fractional integral to be singular at $s = t$.

It is noteworthy to mention the paper [3], where G. Da Prato et al. address singularities at the origin, which are just characterized by the singular kernel $t^{-\beta}$, where $0 < \beta < 1$.

The existing literature on fractional calculus primarily focuses on the Riemann–Liouville derivative $D^\alpha u$ and Caputo derivative ${}^C D^\alpha u$ [2, 5], defined for $\alpha \in (1, 2)$ as

$$D^\alpha u = (I^{2-\alpha} u)_{tt}, \quad {}^C D^\alpha u = I^{2-\alpha} u_{tt}.$$

One of the key advantages of the Caputo fractional derivative is the more straightforward formulation of initial conditions. In traditional fractional calculus, initial conditions often involve fractional derivatives, which can be complex to interpret. However, the Caputo derivative is defined in such a way that the initial conditions can be specified in a more conventional manner, similar to integer-order derivatives, making it easier to apply in practical problems.

The analysis of fractional PDEs mainly focuses on the fractional Caputo equation due to its more regularity and simplicity in the formulation of the initial conditions. Although a connection between the two problems can be established (see Remark 3.2), we opt for a direct analysis due to the complications arising from the substitution.

We initiated our study of trace regularity for integro-differential problems in the context of regular kernels, as discussed in [16, 17]. In these works, we establish a hidden regularity result for wave equations that include a convolution-type integral term.

We introduce the concept of the trace of the normal derivative of the solution, thereby extending well-known results from linear wave equations without memory to integro-differential equations, even when nonlinear terms are present. Specifically, for the case without memory, we refer to [10], where the authors found out a trace theory interpretation for solutions of hyperbolic systems, and [11], where the term “hidden regularity” was first introduced in the context of semilinear wave equations.

In [18], we prove a hidden regularity result for weak solutions of time fractional diffusion-wave equations with the Caputo fractional derivative of order $\alpha \in (1, 2)$. Moreover, in [19], we introduce a notion of weak solution for abstract fractional differential equations with Caputo derivative, also providing two examples of concrete equations: time-fractional wave equations and time-fractional Petrovsky systems.

Important tools for representing the solutions of fractional differential equations, already in the scalar case, are the Mittag-Leffler functions.

In [22, 23], the foremost mathematician G. M. Mittag-Leffler introduced for $\alpha > 0$ the entire function

$$(1.1) \quad E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z \in \mathbb{C},$$

called after him as the Mittag-Leffler function. Subsequently, in the paper [24], he extended the definition (1.1) to the case $\alpha \in \mathbb{C}$ with $\Re \alpha > 0$. The Mittag-Leffler function can be considered a direct generalization of the exponential function, retaining some of its properties.

In [32], A. Wiman also noted that analogous asymptotic results hold for the two-parametric generalization $E_{\alpha,\beta}(z)$ of the Mittag-Leffler function defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}.$$

In the 20th century, this function was virtually unknown to most scientists, as it was ignored in most books on special functions. The recent growing interest in this function is mainly due to its close relationship with fractional calculus and especially with fractional problems arising from applications. For further reading about the Mittag-Leffler functions, we refer to the book [4].

In this paper, we are interested in the fractional Riemann–Liouville problem:

$$(1.2) \quad \begin{aligned} D_{0+}^{\alpha} u &= \Delta u && \text{in } (0, T) \times \Omega, \\ u &= 0 && \text{on } (0, T) \times \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded open set with C^2 boundary $\partial\Omega$ and the order $\alpha \in (\frac{3}{2}, 2)$. We introduce a notion of weak solution and establish an existence result. For $u_1 \in L^2(\Omega)$ and $u_2 \in H_0^1(\Omega)$, the function

$$u(t, x) = \sum_{n=1}^{\infty} [\langle u_1, e_n \rangle t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^{\alpha}) + \langle u_2, e_n \rangle t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n t^{\alpha})] e_n(x)$$

is the unique weak solution of (1.2) written as series, belonging to $L^2(0, T; H_0^1(\Omega))$ and satisfying

$$D_{0+}^{\alpha-1} u(0, \cdot) = u_1, \quad D_{0+}^{\alpha-2} u(0, \cdot) = u_2,$$

see Theorem 3.3. Assuming the condition $\alpha \in (\frac{3}{2}, 2)$, our main results are the following. First, we prove a regularity result when the initial data belong to interpolation spaces connected to the Laplace operator: for $u_1, \nabla u_2 \in D((-\Delta)^{\mu_{\alpha}})$, $\mu_{\alpha} \geq 0$, we have the following:

$$(i) \quad \text{for } \theta \in (\mu_{\alpha}, \frac{2\alpha-3}{2\alpha} + \mu_{\alpha}),$$

$$\|\nabla u\|_{L^2(0,T;D((-\Delta)^{\theta}))} \leq C(\|u_1\|_{D((-\Delta)^{\mu_{\alpha}})} + \|\nabla u_2\|_{D((-\Delta)^{\mu_{\alpha}})}),$$

$$(ii) \quad \text{for } \theta \in (\frac{3-\alpha}{2\alpha} - \mu_{\alpha}, \frac{1}{2} - \mu_{\alpha}),$$

$$\|D_{0+}^{\alpha} u\|_{L^2(0,T;D(A^{-\theta}))} \leq C(\|u_1\|_{D((-\Delta)^{\mu_{\alpha}})} + \|\nabla u_2\|_{D((-\Delta)^{\mu_{\alpha}})}),$$

for some constant $C > 0$, see Theorem 4.1.

A key step in proving the trace regularity result (see Theorem 5.4) is to ensure that the above estimates hold for the same value of θ . To guarantee that the intersection of the intervals $(\mu_\alpha, \frac{2\alpha-3}{2\alpha} + \mu_\alpha)$ and $(\frac{3-\alpha}{2\alpha} - \mu_\alpha, \frac{1}{2} - \mu_\alpha)$ is non-empty, we must choose $\frac{3(2-\alpha)}{4\alpha} < \mu_\alpha < \frac{1}{4}$, see Remark 4.2.

Finally, we prove the trace regularity result: for any $u_1, \nabla u_2 \in D((-\Delta)^{\mu_\alpha})$,

$$\frac{3(2-\alpha)}{4\alpha} < \mu_\alpha < \frac{1}{4},$$

we can define the normal derivative $\partial_\nu u$ of u such that for any $T > 0$, we have

$$\int_0^T \int_{\partial\Omega} |\partial_\nu u|^2 d\sigma dt \leq C (\|u_1\|_{D((-\Delta)^{\mu_\alpha})} + \|\nabla u_2\|_{D((-\Delta)^{\mu_\alpha})}),$$

see Theorem 5.4. It is worth noting that the trace result is established under weaker conditions on the initial data compared to the classical assumptions, see Remark 5.5. The plan of the paper is the following. In Section 2, we list some preliminaries. Section 3 focuses on weak solutions. Sections 4 and 5 are devoted to our main results: regularity in the interpolation spaces and trace regularity, respectively. Finally, in Section 6, we study the decay rate of so-called energy for the Riemann–Liouville problem and the duality existing between the Caputo problem and the Riemann–Liouville problem.

2. PRELIMINARIES

This section collects some known notations, definitions, and results that we will need in the following.

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded open set with C^2 boundary $\partial\Omega$. We consider $L^2(\Omega)$ endowed with the inner product and norm defined by

$$\langle u, v \rangle = \int_{\Omega} u(x)v(x) dx, \quad \|u\|_{L^2(\Omega)} = \left(\int_{\Omega} |u(x)|^2 dx \right)^{1/2}, \quad u, v \in L^2(\Omega).$$

We denote the Riemann–Liouville fractional integral operators of order $\beta > 0$ by

$$(2.1) \quad I_{0+}^\beta u(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} u(s) ds,$$

$$(2.2) \quad I_{T-}^\beta u(t) = \frac{1}{\Gamma(\beta)} \int_t^T (s-t)^{\beta-1} u(s) ds,$$

where $u \in L^2(0, T)$ and $\Gamma(\beta) = \int_0^\infty t^{\beta-1} e^{-t} dt$ is the gamma function. The semigroup property of the fractional integral operators I_{0+}^β is given by

$$(2.3) \quad I_{0+}^\beta I_{0+}^\gamma = I_{0+}^{\beta+\gamma} \quad \beta, \gamma > 0.$$

The Riemann–Liouville fractional derivative $D_{0+}^\alpha u$ of order $\alpha \in (1, 2)$ is defined as follows:

$$D_{0+}^\alpha u(t) = \frac{d^2}{dt^2} I_{0+}^{2-\alpha} u(t) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dt^2} \int_0^t (t-s)^{1-\alpha} u(s) ds.$$

As $\alpha - 1 \in (0, 1)$, we get

$$(2.4) \quad D_{0+}^{\alpha-1} u = \frac{d}{dt} I_{0+}^{2-\alpha} u;$$

hence,

$$(2.5) \quad D_{0+}^\alpha u = \frac{d}{dt} D_{0+}^{\alpha-1} u.$$

Moreover, since $\alpha - 2 \in (-1, 0)$, we have

$$(2.6) \quad D_{0+}^{\alpha-2} u = I_{0+}^{2-\alpha} u.$$

A kind of integration by parts is the well-known property:

$$(2.7) \quad \int_0^T I_{0+}^\alpha f(t) g(t) dt = \int_0^T f(t) I_{T-}^\alpha g(t) dt$$

(see e.g. [7]). The symbol Δ denotes the Laplace operator as usual. We consider the operator $-\Delta$ in $L^2(\Omega)$ with domain $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$. The fractional powers $(-\Delta)^\theta$ are defined for any $\theta > 0$, see e.g. [25] and [20, Example 4.34]. We recall that the spectrum of $-\Delta$ consists of a sequence of positive eigenvalues tending to $+\infty$ and there exists an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ of $L^2(\Omega)$ consisting of eigenvectors of $-\Delta$. Moreover, we assume that the eigenvalues are distinct numbers and denote by λ_n the eigenvalue with eigenvector e_n , that is, $-\Delta e_n = \lambda_n e_n$. For $\theta > 0$, the domain $D((-\Delta)^\theta)$ of $(-\Delta)^\theta$ consists of those functions $u \in L^2(\Omega)$ such that

$$\sum_{n=1}^{\infty} \lambda_n^{2\theta} |\langle u, e_n \rangle|^2 < +\infty$$

and

$$(-\Delta)^\theta u = \sum_{n=1}^{\infty} \lambda_n^\theta \langle u, e_n \rangle e_n \quad u \in D((-\Delta)^\theta).$$

$D((-\Delta)^\theta)$ is a Hilbert space with the norm given by

$$(2.8) \quad \|u\|_{D((-\Delta)^\theta)} = \|(-\Delta)^\theta u\|_{L^2(\Omega)} = \left(\sum_{n=1}^{\infty} \lambda_n^{2\theta} |\langle u, e_n \rangle|^2 \right)^{1/2} \quad u \in D((-\Delta)^\theta).$$

For any $0 < \theta_1 < \theta_2$, we have $D((-\Delta)^{\theta_2}) \subset D((-\Delta)^{\theta_1})$.

We also have $D((-\Delta)^\theta) \subset H^{2\theta}(\Omega)$ and, in particular, $D((-\Delta)^{\frac{1}{2}}) = H_0^1(\Omega)$. Identifying the dual $(L^2(\Omega))'$ with $L^2(\Omega)$ itself, we have

$$D((-\Delta)^\theta) \subset L^2(\Omega) \subset (D((-\Delta)^\theta))'.$$

From now on we set

$$D((-\Delta)^{-\theta}) = (D((-\Delta)^\theta))';$$

hence, the elements of $D((-\Delta)^{-\theta})$ are bounded linear functionals on $D((-\Delta)^\theta)$. If $\varphi \in D((-\Delta)^{-\theta})$ and $u \in D((-\Delta)^\theta)$, the value $\varphi(u)$ is denoted by

$$(2.9) \quad \varphi(u) = \langle \varphi, u \rangle_{-\theta, \theta}.$$

In addition, $D((-\Delta)^{-\theta})$ is a Hilbert space with the norm given by

$$(2.10) \quad \|\varphi\|_{D((-\Delta)^{-\theta})} = \left(\sum_{n=1}^{\infty} \lambda_n^{-2\theta} |\langle \varphi, e_n \rangle_{-\theta, \theta}|^2 \right)^{1/2} \quad \varphi \in D((-\Delta)^{-\theta}).$$

We also recall that

$$(2.11) \quad \langle \varphi, u \rangle_{-\theta, \theta} = \langle \varphi, u \rangle \quad \text{for } \varphi \in L^2(\Omega), u \in D((-\Delta)^\theta),$$

see e.g. [1, Chapitre V].

The Mittag-Leffler function depending on arbitrary constants $\alpha, \beta > 0$ is defined as

$$(2.12) \quad E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad z \in \mathbb{C}.$$

The power series $E_{\alpha, \beta}(z)$ is an entire function of $z \in \mathbb{C}$. The symbol $E_\alpha(z)$ usually denotes $E_{\alpha, 1}(z)$.

LEMMA 2.1. *Let $\alpha \in (1, 2)$ and $\beta > 0$.*

(i) *There exists a constant $C = C(\alpha, \beta) > 0$ such that*

$$(2.13) \quad |E_{\alpha, \beta}(-\mu)| \leq \frac{C}{1 + \mu}, \quad \text{for any } \mu \geq 0.$$

(ii) *For any $\lambda \in \mathbb{R}$,*

$$(2.14) \quad I_{0+}^{2-\alpha}(\tau^{\beta-1} E_{\alpha, \beta}(\lambda \tau^\alpha))(t) = t^{1-\alpha+\beta} E_{\alpha, 2-\alpha+\beta}(\lambda t^\alpha).$$

The proof of (i) can be found in [26, Theorem 1.6]. For point (ii), see e.g. [7, formula (2.1.53)].

LEMMA 2.2. *If $\alpha, \lambda > 0$, then we have*

$$(2.15) \quad \frac{d}{dt} E_{\alpha}(-\lambda t^{\alpha}) = -\lambda t^{\alpha-1} E_{\alpha, \alpha}(-\lambda t^{\alpha}), \quad t > 0,$$

$$(2.16) \quad \frac{d}{dt} (t^k E_{\alpha, k+1}(-\lambda t^{\alpha})) = t^{k-1} E_{\alpha, k}(-\lambda t^{\alpha}), \quad k \in \mathbb{N}, \quad t \geq 0,$$

$$(2.17) \quad \frac{d}{dt} (t^{\alpha-1} E_{\alpha, \alpha}(-\lambda t^{\alpha})) = t^{\alpha-2} E_{\alpha, \alpha-1}(-\lambda t^{\alpha}), \quad t \geq 0.$$

We recall a result about the existence of solutions for fractional scalar differential equations; for the proof see e.g. [7, Theorem 4.1].

LEMMA 2.3. *For $\alpha \in (1, 2)$ and $\lambda, u_1, u_2 \in \mathbb{R}$, the solution of the Cauchy type problem*

$$(2.18) \quad \begin{cases} D_{0+}^{\alpha} u(t) + \lambda u(t) = 0, & t \geq 0, \\ D_{0+}^{\alpha-1} u(0) = u_1, \quad D_{0+}^{\alpha-2} u(0) = u_2, \end{cases}$$

is given by

$$u(t) = u_1 t^{\alpha-1} E_{\alpha, \alpha}(-\lambda t^{\alpha}) + u_2 t^{\alpha-2} E_{\alpha, \alpha-1}(-\lambda t^{\alpha}), \quad t \geq 0.$$

REMARK 2.4. We point out that the notation $D_{0+}^{\alpha-k} u(0)$, $k = 1, 2$, in (2.18) means that the limit is taken at almost all points of the right-sided neighborhood $(0, \varepsilon)$, $\varepsilon > 0$, of 0 as follows:

$$D_{0+}^{\alpha-1} u(0) = \lim_{t \rightarrow 0+} D_{0+}^{\alpha-1} u(t), \quad D_{0+}^{\alpha-2} u(0) = \lim_{t \rightarrow 0+} I_{0+}^{2-\alpha} u(t),$$

see [7, p. 136].

To conclude this section, we mention an elementary result useful for managing estimates.

LEMMA 2.5. *For any $\beta \in (0, 1)$, the function $x \rightarrow \frac{x^{\beta}}{1+x}$ gains its maximum on $[0, +\infty[$ at point $\frac{\beta}{1-\beta}$ and the maximum value is given by*

$$(2.19) \quad \max_{x \geq 0} \frac{x^{\beta}}{1+x} = \beta^{\beta} (1-\beta)^{1-\beta}, \quad \beta \in (0, 1).$$

The symbol $\sim$ between norms indicates two equivalent norms.

3. WEAK SOLUTIONS OF RIEMANN–LIOUVILLE FRACTIONAL EQUATIONS

We begin by introducing a general notion of a weak solution that does not depend on initial conditions (for an analogous definition in the case of classical differential

equations, see e.g. [1, p. 136]). In Theorem 3.3 below, we give an existence and uniqueness result in a particular case. Precisely, we prove the following: there exists a unique series expansion that is a weak solution satisfying given initial conditions.

DEFINITION 3.1. Let $\alpha \in (1, 2)$ and $T > 0$. We define u as a weak solution to the boundary value problem

$$(3.1) \quad \begin{aligned} D_{0+}^{\alpha} u(t, x) &= \Delta u(t, x) & (t, x) &\in (0, T) \times \Omega, \\ u(t, x) &= 0 & (t, x) &\in (0, T) \times \partial\Omega, \end{aligned}$$

if $u \in L^2(0, T; H_0^1(\Omega))$, $D_{0+}^{\alpha-1} u \in L^2(0, T; L^2(\Omega))$, and for any $\varphi \in H_0^1(\Omega)$, one has

$$\int_{\Omega} D_{0+}^{\alpha-1} u(t, x) \varphi(x) dx \in H^1(0, T)$$

and

$$(3.2) \quad \frac{d}{dt} \int_{\Omega} D_{0+}^{\alpha-1} u(t, x) \varphi(x) dx + \int_{\Omega} \nabla u(t, x) \cdot \nabla \varphi(x) dx = 0, \quad \text{a.e. } t \in (0, T).$$

REMARK 3.2. We observe that if u is the weak solution of (3.1), then $v = I_{0+}^{2-\alpha} u$ is the weak solution of the Caputo fractional problem

$$(3.3) \quad \begin{aligned} {}^C D_{0+}^{\alpha} v(t, x) &= \Delta v(t, x) & (t, x) &\in (0, T) \times \Omega, \\ v(t, x) &= 0 & (t, x) &\in (0, T) \times \partial\Omega, \end{aligned}$$

where ${}^C D_{0+}^{\alpha}$ denotes the Caputo derivative. Indeed, $v_{tt} = D_{0+}^{\alpha} u = \Delta u$ and hence

$${}^C D_{0+}^{\alpha} v(t, x) = I_{0+}^{2-\alpha} v_{tt} = \Delta I_{0+}^{2-\alpha} u = \Delta v.$$

Keeping in mind Remark 2.4, we establish an existence and regularity result in the case $\alpha > \frac{3}{2}$.

THEOREM 3.3. Assume $\alpha > \frac{3}{2}$.

(i) For $u_1 \in L^2(\Omega)$ and $u_2 \in H_0^1(\Omega)$, the function

$$(3.4) \quad u(t, x) = \sum_{n=1}^{\infty} \left[\langle u_1, e_n \rangle t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n t^{\alpha}) + \langle u_2, e_n \rangle t^{\alpha-2} E_{\alpha, \alpha-1}(-\lambda_n t^{\alpha}) \right] e_n(x)$$

is the unique series expansion that is a weak solution of (3.1) satisfying the initial data

$$(3.5) \quad D_{0+}^{\alpha-1} u(0, \cdot) = u_1, \quad D_{0+}^{\alpha-2} u(0, \cdot) = u_2.$$

Moreover,

$$(3.6) \quad D_{0+}^{\alpha-1} u(t, x) = \sum_{n=1}^{\infty} [\langle u_1, e_n \rangle E_{\alpha}(-\lambda_n t^{\alpha}) - \lambda_n \langle u_2, e_n \rangle t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n t^{\alpha})] e_n(x),$$

$$D_{0+}^{\alpha-1} u \in C([0, T]; D(A^{-\theta})), \quad \theta \in \left(\frac{2-\alpha}{2\alpha}, \frac{1}{2}\right).$$

(ii) If $u_1 \in H_0^1(\Omega)$ and $u_2 \in H^2(\Omega) \cap H_0^1(\Omega)$, then

$$u \in L^2(0, T; H^2(\Omega)), \quad D_{0+}^{\alpha} u(t, \cdot) \in L^2(0, T; L^2(\Omega))$$

is given by

$$(3.7) \quad D_{0+}^{\alpha} u(t, x) = - \sum_{n=1}^{\infty} \lambda_n [\langle u_1, e_n \rangle t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n t^{\alpha}) + \langle u_2, e_n \rangle t^{\alpha-2} E_{\alpha, \alpha-1}(-\lambda_n t^{\alpha})] e_n(x),$$

and

$$(3.8) \quad D_{0+}^{\alpha} u(t, x) = \Delta u(t, x) \quad \text{a.e. } (t, x) \in (0, T) \times \Omega.$$

PROOF. (i) First, we show that

$$\sum_{n=1}^{\infty} [\langle u_1, e_n \rangle t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n t^{\alpha}) + \langle u_2, e_n \rangle t^{\alpha-2} E_{\alpha, \alpha-1}(-\lambda_n t^{\alpha})] e_n(x)$$

is the unique series giving the weak solution of (3.1) with initial data (3.5). For this purpose, we seek the solution in the form $u(t) = \sum_{n=1}^{\infty} u_n(t) e_n$ where the functions $u_n(t) = \langle u(t), e_n \rangle$ are unknown. We take $\varphi = e_n$ in (3.2) to get

$$\frac{d}{dt} \int_{\Omega} D_{0+}^{\alpha-1} u(t, x) e_n(x) dx - \int_{\Omega} u(t, x) \Delta e_n(x) dx = 0, \quad t \in [0, T];$$

hence, by (2.5), we have

$$D_{0+}^{\alpha} u_n + \lambda_n u_n = \frac{d}{dt} D_{0+}^{\alpha-1} u_n + \lambda_n u_n = 0.$$

So $u_n(t)$ is the solution of problem (2.18) with $\lambda = \lambda_n$ where, thanks to (3.5), the initial conditions are given by

$$D_{0+}^{\alpha-1} u_n(0, \cdot) = \langle u_1, e_n \rangle, \quad D_{0+}^{\alpha-2} u_n(0, \cdot) = \langle u_2, e_n \rangle.$$

Therefore, by Lemma 2.3, we get

$$(3.9) \quad u_n(t) = \langle u_1, e_n \rangle t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n t^{\alpha}) + \langle u_2, e_n \rangle t^{\alpha-2} E_{\alpha, \alpha-1}(-\lambda_n t^{\alpha}), \quad t \geq 0;$$

hence, the uniqueness follows.

Now, we show that, taking $u_1 \in L^2(\Omega)$ and $u_2 \in H_0^1(\Omega)$, $u(t, \cdot) = \sum_{n=1}^{\infty} u_n(t) e_n$, with $u_n(t)$ given by (3.9), is a weak solution of (3.1) satisfying the initial conditions (3.5). First, we note that $u(t, \cdot) \in H_0^1(\Omega)$ for $t \in (0, T]$. Indeed,

$$\begin{aligned} \|u(t, \cdot)\|_{H_0^1(\Omega)}^2 &= \sum_{n=1}^{\infty} \lambda_n |u_n(t)|^2 \\ &\leq 2 \sum_{n=1}^{\infty} \lambda_n |\langle u_1, e_n \rangle t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n t^\alpha)|^2 \\ &\quad + 2 \sum_{n=1}^{\infty} \lambda_n |\langle u_2, e_n \rangle t^{\alpha-2} E_{\alpha, \alpha-1}(-\lambda_n t^\alpha)|^2 \end{aligned}$$

and thanks to (2.13), we have

$$\begin{aligned} \lambda_n |\langle u_1, e_n \rangle t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n t^\alpha)|^2 &\leq C t^{\alpha-2} |\langle u_1, e_n \rangle|^2 \frac{\lambda_n t^\alpha}{(1 + \lambda_n t^\alpha)^2} \leq C t^{\alpha-2} |\langle u_1, e_n \rangle|^2, \\ \lambda_n |\langle u_2, e_n \rangle t^{\alpha-2} E_{\alpha, \alpha-1}(-\lambda_n t^\alpha)|^2 &\leq C t^{2\alpha-4} \lambda_n |\langle u_2, e_n \rangle|^2; \end{aligned}$$

hence,

$$(3.10) \quad \|u(t, \cdot)\|_{H_0^1(\Omega)}^2 \leq C t^{\alpha-2} \|u_1\|_{L^2(\Omega)}^2 + C t^{2\alpha-4} \|u_2\|_{H_0^1(\Omega)}^2.$$

As $\alpha > \frac{3}{2}$, we obtain $u \in L^2(0, T; H_0^1(\Omega))$ and

$$\int_0^T \|u(t, \cdot)\|_{H_0^1(\Omega)}^2 dt \leq C T^{\alpha-1} \|u_1\|_{L^2(\Omega)}^2 + C T^{2\alpha-3} \|u_2\|_{H_0^1(\Omega)}^2.$$

Moreover, thanks to (2.14), we have

$$(3.11) \quad I_{0+}^{2-\alpha} u_n(t) = \langle u_1, e_n \rangle t E_{\alpha, 2}(-\lambda_n t^\alpha) + \langle u_2, e_n \rangle E_\alpha(-\lambda_n t^\alpha);$$

hence, being $u \in L^2(0, T; H_0^1(\Omega))$, by the dominated convergence theorem, we get

$$(3.12) \quad I_{0+}^{2-\alpha} u(t, \cdot) = \sum_{n=1}^{\infty} [\langle u_1, e_n \rangle t E_{\alpha, 2}(-\lambda_n t^\alpha) + \langle u_2, e_n \rangle E_\alpha(-\lambda_n t^\alpha)] e_n.$$

The series $\sum_{n=1}^{\infty} I_{0+}^{2-\alpha} u_n(t) e_n$ is convergent in $L^2(\Omega)$ uniformly in $t \in [0, T]$. As a consequence, we have $I_{0+}^{2-\alpha} u \in C([0, T]; L^2(\Omega))$ and

$$D_{0+}^{\alpha-2} u(0, \cdot) = \lim_{t \rightarrow 0+} I_{0+}^{2-\alpha} u(t, \cdot) = \sum_{n=1}^{\infty} \langle u_2, e_n \rangle e_n = u_2,$$

taking into account that $E_{\alpha, 2}(0) = \frac{1}{\Gamma(2)}$ and $E_\alpha(0) = \frac{1}{\Gamma(1)} = 1$.

We have to show that $D_{0+}^{\alpha-1} u$ is given by (3.6) and belongs to $C([0, T]; D(A^{-\theta}))$, for $\theta \in (\frac{2-\alpha}{2\alpha}, \frac{1}{2})$. First, we note that, thanks to (3.11), (2.16) for $k = 1$ and (2.15), we get

$$\frac{d}{dt} I_{0+}^{2-\alpha} u_n(t) = \langle u_1, e_n \rangle E_\alpha(-\lambda_n t^\alpha) - \langle u_2, e_n \rangle \lambda_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha).$$

Since

$$\left\| \sum_{n=1}^{\infty} \frac{d}{dt} I_{0+}^{2-\alpha} u_n(t) e_n \right\|_{D(A^{-\theta})}^2 = \sum_{n=1}^{\infty} \lambda_n^{-2\theta} \left| \frac{d}{dt} I_{0+}^{2-\alpha} u_n(t) \right|^2,$$

due to (2.13), (2.19) and $0 < \theta < \frac{1}{2}$, we obtain

$$\begin{aligned} & \lambda_n^{-2\theta} |\langle u_2, e_n \rangle \lambda_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)|^2 \\ & \leq C t^{2\alpha\theta+\alpha-2} \lambda_n |\langle u_2, e_n \rangle|^2 \left(\frac{(\lambda_n t^\alpha)^{\frac{1}{2}-\theta}}{1 + \lambda_n t^\alpha} \right)^2 \leq C t^{2\alpha\theta+\alpha-2} \lambda_n |\langle u_2, e_n \rangle|^2. \end{aligned}$$

Therefore, taking into account that $\theta > \frac{2-\alpha}{2\alpha}$, for any $t \in [0, T]$, we get

$$\left\| \sum_{n=1}^{\infty} \frac{d}{dt} I_{0+}^{2-\alpha} u_n(t) e_n \right\|_{D(A^{-\theta})}^2 \leq C \|u_1\|_{L^2(\Omega)}^2 + C T^{2\alpha\theta+\alpha-2} \|u_2\|_{H_0^1(\Omega)}^2.$$

Furthermore, the series $\sum_{n=1}^{\infty} \frac{d}{dt} I_{0+}^{2-\alpha} u_n(t) e_n$ is convergent in $D(A^{-\theta})$ uniformly in $t \in [0, T]$. For that reason, the function $I_{0+}^{2-\alpha} u$ is differentiable, so keeping in mind (2.4), we get

$$\begin{aligned} D_{0+}^{\alpha-1} u(t, \cdot) &= \frac{d}{dt} I_{0+}^{2-\alpha} u(t, \cdot) \\ &= \sum_{n=1}^{\infty} [\langle u_1, e_n \rangle E_\alpha(-\lambda_n t^\alpha) - \langle u_2, e_n \rangle \lambda_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)] e_n, \end{aligned}$$

that is, (3.6). Since $D_{0+}^{\alpha-1} u \in C([0, T]; D(A^{-\theta}))$, from (3.6), it follows that

$$D_{0+}^{\alpha-1} u(0, \cdot) = \lim_{t \rightarrow 0+} D_{0+}^{\alpha-1} u(t, \cdot) = \sum_{n=1}^{\infty} \langle u_1, e_n \rangle e_n = u_1.$$

Using again (2.13), we obtain

$$\|D_{0+}^{\alpha-1} u(t, \cdot)\|_{L^2(\Omega)}^2 \leq C \|u_1\|_{L^2(\Omega)}^2 + C t^{\alpha-2} \|u_2\|_{H_0^1(\Omega)}^2,$$

and hence,

$$\int_0^T \|D_{0+}^{\alpha-1} u(t, \cdot)\|_{L^2(\Omega)}^2 dt \leq C \|u_1\|_{L^2(\Omega)}^2 + C T^{\alpha-1} \|u_2\|_{H_0^1(\Omega)}^2,$$

that is, $D_{0+}^{\alpha-1} u \in L^2(0, T; L^2(\Omega))$.

Afterwards, if $\varphi = \sum_{n=1}^{\infty} \langle \varphi, e_n \rangle e_n$ belongs to $H_0^1(\Omega)$, then due to (3.6), we have

$$\begin{aligned} & \int_{\Omega} D_{0+}^{\alpha-1} u(t, x) \varphi(x) dx \\ &= \sum_{n=1}^{\infty} [\langle u_1, e_n \rangle E_{\alpha}(-\lambda_n t^{\alpha}) - \lambda_n \langle u_2, e_n \rangle t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n t^{\alpha})] \langle \varphi, e_n \rangle. \end{aligned}$$

We observe that (2.15) and (2.17) yield

$$\begin{aligned} (3.13) \quad & \frac{d}{dt} \langle D_{0+}^{\alpha-1} u(t, \cdot), e_n \rangle \\ &= -\lambda_n [\langle u_1, e_n \rangle t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n t^{\alpha}) + \langle u_2, e_n \rangle t^{\alpha-2} E_{\alpha, \alpha-1}(-\lambda_n t^{\alpha})], \end{aligned}$$

and hence

$$\begin{aligned} (3.14) \quad & \sum_{n=1}^{\infty} \frac{d}{dt} \langle D_{0+}^{\alpha-1} u(t, \cdot), e_n \rangle \langle \varphi, e_n \rangle \\ &= - \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{2}} [\langle u_1, e_n \rangle t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n t^{\alpha}) + \langle u_2, e_n \rangle t^{\alpha-2} E_{\alpha, \alpha-1}(-\lambda_n t^{\alpha})] \lambda_n^{\frac{1}{2}} \langle \varphi, e_n \rangle. \end{aligned}$$

Thanks to (2.13) and (2.19), we get

$$\begin{aligned} & \sum_{n=1}^{\infty} \lambda_n |\langle u_1, e_n \rangle t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n t^{\alpha}) + \langle u_2, e_n \rangle t^{\alpha-2} E_{\alpha, \alpha-1}(-\lambda_n t^{\alpha})|^2 \\ & \leq C t^{\alpha-2} \sum_{n=1}^{\infty} |\langle u_1, e_n \rangle|^2 \left(\frac{\lambda_n^{\frac{1}{2}} t^{\frac{\alpha}{2}}}{1 + \lambda_n t^{\alpha}} \right)^2 + C t^{2\alpha-4} \sum_{n=1}^{\infty} \lambda_n |\langle u_2, e_n \rangle|^2 \\ & \leq C t^{\alpha-2} \|u_1\|_{L^2(\Omega)}^2 + C t^{2\alpha-4} \|u_2\|_{H_0^1(\Omega)}^2. \end{aligned}$$

Since $\alpha > \frac{3}{2}$, from (3.14), we deduce

$$\begin{aligned} & \int_0^T \left| \sum_{n=1}^{\infty} \frac{d}{dt} \langle D_{0+}^{\alpha-1} u(t, \cdot), e_n \rangle \langle \varphi, e_n \rangle \right| dt \\ & \leq C T^{\alpha-1} \|u_1\|_{L^2(\Omega)}^2 + C T^{2\alpha-3} \|u_2\|_{H_0^1(\Omega)}^2 + C \|\varphi\|_{H_0^1(\Omega)}^2; \end{aligned}$$

that is, $\sum_{n=1}^{\infty} \frac{d}{dt} \langle D_{0+}^{\alpha-1} u(t, \cdot), e_n \rangle \langle \varphi, e_n \rangle$ belongs to $L^1(0, T)$. Therefore, for any $t \in [0, T]$, we have

$$\begin{aligned} \int_0^t \sum_{n=1}^{\infty} \frac{d}{ds} \langle D_{0+}^{\alpha-1} u(s, \cdot), e_n \rangle \langle \varphi, e_n \rangle ds &= \sum_{n=1}^{\infty} \int_0^t \frac{d}{ds} \langle D_{0+}^{\alpha-1} u(s, \cdot), e_n \rangle \langle \varphi, e_n \rangle ds \\ &= \int_{\Omega} D_{0+}^{\alpha-1} u(t, x) \varphi(x) dx - \int_{\Omega} u_1(x) \varphi(x) dx. \end{aligned}$$

As a consequence, the function $t \rightarrow \int_{\Omega} D_{0+}^{\alpha-1} u(t, x) \varphi(x) dx$ is absolutely continuous and, thanks to (3.14), its L^1 -derivative is given by

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} D_{0+}^{\alpha-1} u(t, x) \varphi(x) dx \\ &= - \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{2}} [\langle u_1, e_n \rangle t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n t^{\alpha}) + \langle u_2, e_n \rangle t^{\alpha-2} E_{\alpha, \alpha-1}(-\lambda_n t^{\alpha})] \lambda_n^{\frac{1}{2}} \langle \varphi, e_n \rangle. \end{aligned}$$

On the other hand, by (3.4), we get

$$\begin{aligned} & \int_{\Omega} \nabla u(t, x) \cdot \nabla \varphi(x) dx \\ &= \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{2}} [\langle u_1, e_n \rangle t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n t^{\alpha}) + \langle u_2, e_n \rangle t^{\alpha-2} E_{\alpha, \alpha-1}(-\lambda_n t^{\alpha})] \lambda_n^{\frac{1}{2}} \langle \varphi, e_n \rangle, \end{aligned}$$

and hence, for a.e. $t \in (0, T)$, we have

$$\frac{d}{dt} \int_{\Omega} D_{0+}^{\alpha-1} u(t, x) \varphi(x) dx + \int_{\Omega} \nabla u(t, x) \cdot \nabla \varphi(x) dx = 0;$$

that is, (3.2) holds and $\frac{d}{dt} \int_{\Omega} D_{0+}^{\alpha-1} u(t, x) \varphi(x) dx \in L^2(0, T)$.

In conclusion, u given by (3.4) is the weak solution of (3.1) satisfying the initial data (3.5).

(ii) We assume $u_1 \in H_0^1(\Omega)$ and $u_2 \in H^2(\Omega) \cap H_0^1(\Omega)$. We prove the regularity results about the weak solution u arguing as before and using again (2.13) and (2.19). First, we obtain

$$\int_0^T \|u(t, \cdot)\|_{H^2(\Omega)}^2 dt \leq C T^{\alpha-1} \|u_1\|_{H_0^1(\Omega)}^2 + C T^{2\alpha-3} \|u_2\|_{H^2(\Omega)}^2,$$

that is $u \in L^2(0, T; H^2(\Omega))$.

To prove the regularity of $D_{0+}^{\alpha} u$, taking into account (2.5), we have to differentiate formula (3.6). Indeed, from (3.13), we deduce

$$\begin{aligned} (3.15) \quad & \int_0^T \left\| \sum_{n=1}^{\infty} \frac{d}{dt} \langle D_{0+}^{\alpha-1} u(t, \cdot), e_n \rangle e_n \right\|_{L^2(\Omega)}^2 dt \\ & \leq C T^{\alpha-1} \|u_1\|_{H_0^1(\Omega)}^2 + C T^{2\alpha-3} \|u_2\|_{H^2(\Omega)}^2. \end{aligned}$$

Therefore, the function $D_{0+}^{\alpha-1} u(t, \cdot)$ is absolutely continuous and, also because of (3.13), its derivative is given by

$$\begin{aligned} & \frac{d}{dt} D_{0+}^{\alpha-1} u(t, \cdot) \\ &= - \sum_{n=1}^{\infty} \lambda_n [\langle u_1, e_n \rangle t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n t^{\alpha}) + \langle u_2, e_n \rangle t^{\alpha-2} E_{\alpha, \alpha-1}(-\lambda_n t^{\alpha})] e_n. \end{aligned}$$

Since $D_{0+}^\alpha u = \frac{d}{dt} D_{0+}^{\alpha-1} u$, the formula (3.7) is true. Finally, the inequality (3.15) guarantees $D_{0+}^\alpha u \in L^2(0, T; L^2(\Omega))$ and (3.8) follows from (3.2) and the regularity of the solution u . ■

4. REGULARITY IN THE INTERPOLATION SPACES WHEN $\alpha > \frac{3}{2}$

We establish a result on the regularity of weak solutions by assuming that the data u_1 and u_2 belong to interpolation spaces connected to the Laplace operator.

THEOREM 4.1. *Assume $\alpha > \frac{3}{2}$. For $u_1, \nabla u_2 \in D((-\Delta)^{\mu_\alpha})$, $\mu_\alpha \geq 0$, the unique weak solution u to (3.1)–(3.5) given by (3.4) satisfies the following:*

(i) for $\theta \in (\mu_\alpha, \frac{2\alpha-3}{2\alpha} + \mu_\alpha)$,

$$(4.1) \quad \|\nabla u\|_{L^2(0,T;D((-\Delta)^\theta))} \leq C (\|u_1\|_{D((-\Delta)^{\mu_\alpha})} + \|\nabla u_2\|_{D((-\Delta)^{\mu_\alpha})}),$$

(ii) for $\theta \in (\frac{3-\alpha}{2\alpha} - \mu_\alpha, \frac{1}{2} - \mu_\alpha)$,

$$(4.2) \quad \|D_{0+}^\alpha u\|_{L^2(0,T;D((-\Delta)^{-\theta}))} \leq C (\|u_1\|_{D((-\Delta)^{\mu_\alpha})} + \|\nabla u_2\|_{D((-\Delta)^{\mu_\alpha})}),$$

for some constant $C > 0$.

PROOF. (i) In virtue of the expression (3.4) for the solution u , we have

$$\begin{aligned} (4.3) \quad & \|\nabla u(t, \cdot)\|_{D((-\Delta)^\theta)}^2 \\ &= \sum_{n=1}^{\infty} \lambda_n^{1+2\theta} |\langle u_1, e_n \rangle t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) + \langle u_2, e_n \rangle t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n t^\alpha)|^2 \\ &\leq 2t^{2\alpha-2} \sum_{n=1}^{\infty} \lambda_n^{1+2\theta} |\langle u_1, e_n \rangle E_{\alpha,\alpha}(-\lambda_n t^\alpha)|^2 \\ &\quad + 2t^{2\alpha-4} \sum_{n=1}^{\infty} \lambda_n^{1+2\theta} |\langle u_2, e_n \rangle E_{\alpha,\alpha-1}(-\lambda_n t^\alpha)|^2. \end{aligned}$$

To estimate the first sum, we use (2.13) and keep in mind that $u_1 \in D((-\Delta)^{\mu_\alpha})$, so we get

$$\sum_{n=1}^{\infty} \lambda_n^{1+2\theta} |\langle u_1, e_n \rangle E_{\alpha,\alpha}(-\lambda_n t^\alpha)|^2 \leq C \sum_{n=1}^{\infty} \lambda_n^{2\mu_\alpha} |\langle u_1, e_n \rangle|^2 \frac{\lambda_n^{1+2(\theta-\mu_\alpha)}}{(1 + \lambda_n t^\alpha)^2}.$$

Since

$$\frac{\lambda_n^{1+2(\theta-\mu_\alpha)}}{(1 + \lambda_n t^\alpha)^2} = \left(\frac{(\lambda_n t^\alpha)^{\frac{1}{2} + \theta - \mu_\alpha}}{1 + \lambda_n t^\alpha} \right)^2 t^{-\alpha - 2\alpha(\theta - \mu_\alpha)},$$

assuming $\mu_\alpha < \theta < \frac{1}{2} + \mu_\alpha$, we can apply (2.19) to have

$$(4.4) \quad \sum_{n=1}^{\infty} \lambda_n^{1+2\theta} |\langle u_1, e_n \rangle E_{\alpha,\alpha}(-\lambda_n t^\alpha)|^2 \\ \leq C t^{-\alpha-2\alpha(\theta-\mu_\alpha)} \sum_{n=1}^{\infty} \lambda_n^{2\mu_\alpha} |\langle u_1, e_n \rangle|^2 = C t^{-\alpha-2\alpha(\theta-\mu_\alpha)} \|u_1\|_{D((-\Delta)^{\mu_\alpha})}^2.$$

To evaluate the second sum in (4.3), we again use (2.13). Bearing in mind that $\nabla u_2 \in D((-\Delta)^{\mu_\alpha})$, we have

$$\sum_{n=1}^{\infty} \lambda_n^{1+2\theta} |\langle u_2, e_n \rangle E_{\alpha,\alpha-1}(-\lambda_n t^\alpha)|^2 \leq C \sum_{n=1}^{\infty} \lambda_n^{1+2\mu_\alpha} |\langle u_2, e_n \rangle|^2 \frac{\lambda_n^{2(\theta-\mu_\alpha)}}{(1 + \lambda_n t^\alpha)^2}.$$

We note that

$$\frac{\lambda_n^{2(\theta-\mu_\alpha)}}{(1 + \lambda_n t^\alpha)^2} = \left(\frac{(\lambda_n t^\alpha)^{\theta-\mu_\alpha}}{1 + \lambda_n t^\alpha} \right)^2 t^{-2\alpha(\theta-\mu_\alpha)}.$$

Thanks to $\mu_\alpha < \theta < \frac{1}{2} + \mu_\alpha$ and (2.19), we obtain

$$\sum_{n=1}^{\infty} \lambda_n^{1+2\theta} |\langle u_2, e_n \rangle E_{\alpha,\alpha-1}(-\lambda_n t^\alpha)|^2 \leq C t^{-2\alpha(\theta-\mu_\alpha)} \|\nabla u_2\|_{D((-\Delta)^{\mu_\alpha})}^2.$$

Plugging the previous estimate and (4.4) into (4.3) yields

$$\|\nabla u(t, \cdot)\|_{D((-\Delta)^\theta)}^2 \\ \leq C t^{\alpha-2-2\alpha(\theta-\mu_\alpha)} \|u_1\|_{D((-\Delta)^{\mu_\alpha})}^2 + C t^{2\alpha-4-2\alpha(\theta-\mu_\alpha)} \|\nabla u_2\|_{D((-\Delta)^{\mu_\alpha})}^2.$$

For $\alpha - 1 - 2\alpha(\theta - \mu_\alpha) > 0$ and $2\alpha - 3 - 2\alpha(\theta - \mu_\alpha) > 0$, we obtain that ∇u belongs to $L^2(0, T; D((-\Delta)^\theta))$. Since $\alpha < 2$, we have $\frac{2\alpha-3}{2\alpha} < \frac{\alpha-1}{2\alpha}$, and hence we must take $\theta < \frac{2\alpha-3}{2\alpha} + \mu_\alpha$, which is consistent with the previous condition $\theta < \frac{1}{2} + \mu_\alpha$. In conclusion, (4.1) follows.

(ii) In proving the estimate, we follow the same lines of reasoning used above.

Thanks to (3.7), one has

$$(4.5) \quad \|D_{0+}^\alpha u(t, \cdot)\|_{D((-\Delta)^{-\theta})}^2 \\ = \sum_{n=1}^{\infty} \lambda_n^{2-2\theta} |\langle u_1, e_n \rangle t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) + \langle u_2, e_n \rangle t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n t^\alpha)|^2 \\ \leq 2t^{2\alpha-2} \sum_{n=1}^{\infty} \lambda_n^{2-2\theta} |\langle u_1, e_n \rangle E_{\alpha,\alpha}(-\lambda_n t^\alpha)|^2 \\ + 2t^{2\alpha-4} \sum_{n=1}^{\infty} \lambda_n^{2-2\theta} |\langle u_2, e_n \rangle E_{\alpha,\alpha-1}(-\lambda_n t^\alpha)|^2.$$

The first sum can be evaluated as follows:

$$\sum_{n=1}^{\infty} \lambda_n^{2-2\theta} |\langle u_1, e_n \rangle E_{\alpha, \alpha}(-\lambda_n t^\alpha)|^2 \leq C \sum_{n=1}^{\infty} \lambda_n^{2\mu_\alpha} |\langle u_1, e_n \rangle|^2 \frac{\lambda_n^{2-2(\theta+\mu_\alpha)}}{(1 + \lambda_n t^\alpha)^2}.$$

Since

$$\frac{\lambda_n^{2-2(\theta+\mu_\alpha)}}{(1 + \lambda_n t^\alpha)^2} = \left(\frac{(\lambda_n t^\alpha)^{1-(\theta+\mu_\alpha)}}{1 + \lambda_n t^\alpha} \right)^2 t^{2\alpha(\theta+\mu_\alpha)-2\alpha},$$

we have

$$(4.6) \quad \sum_{n=1}^{\infty} \lambda_n^{2-2\theta} |\langle u_1, e_n \rangle E_{\alpha, \alpha}(-\lambda_n t^\alpha)|^2 \leq C t^{2\alpha(\theta+\mu_\alpha)-2\alpha} \|u_1\|_{D((-\Delta)^{\mu_\alpha})}^2.$$

Regarding the second sum in (4.5), we note that

$$\sum_{n=1}^{\infty} \lambda_n^{2-2\theta} |\langle u_2, e_n \rangle E_{\alpha, \alpha-1}(-\lambda_n t^\alpha)|^2 \leq C \sum_{n=1}^{\infty} \lambda_n^{1+2\mu_\alpha} |\langle u_2, e_n \rangle|^2 \frac{\lambda_n^{1-2(\theta+\mu_\alpha)}}{(1 + \lambda_n t^\alpha)^2}$$

and

$$\frac{\lambda_n^{1-2(\theta+\mu_\alpha)}}{(1 + \lambda_n t^\alpha)^2} = \left(\frac{(\lambda_n t^\alpha)^{\frac{1}{2}-(\theta+\mu_\alpha)}}{1 + \lambda_n t^\alpha} \right)^2 t^{2\alpha(\theta+\mu_\alpha)-\alpha}.$$

Taking $\theta < \frac{1}{2} - \mu_\alpha$, we get

$$\sum_{n=1}^{\infty} \lambda_n^{2-2\theta} |\langle u_2, e_n \rangle E_{\alpha, \alpha-1}(-\lambda_n t^\alpha)|^2 \leq C t^{2\alpha(\theta+\mu_\alpha)-\alpha} \|\nabla u_2\|_{D((-\Delta)^{\mu_\alpha})}^2.$$

By inserting the previous estimate and (4.6) into (4.5), we obtain

$$\begin{aligned} & \|D_{0+}^\alpha u(t, \cdot)\|_{D((-\Delta)^{-\theta})}^2 \\ & \leq C t^{2\alpha(\theta+\mu_\alpha)-2} \|u_1\|_{D((-\Delta)^{\mu_\alpha})}^2 + C t^{2\alpha(\theta+\mu_\alpha)+\alpha-4} \|\nabla u_2\|_{D((-\Delta)^{\mu_\alpha})}^2. \end{aligned}$$

So, $D_{0+}^\alpha u(t, \cdot) \in L^2(0, T; D((-\Delta)^{-\theta}))$ if $2\alpha(\theta + \mu_\alpha) - 1 > 0$ and $2\alpha(\theta + \mu_\alpha) + \alpha - 3 > 0$, that is, $\theta > \frac{3-\alpha}{2\alpha} - \mu_\alpha$. In conclusion, (4.2) is proved. ■

REMARK 4.2. In Theorem 4.1, we can take $\mu_\alpha = 0$. In that case, we have $u_1, \nabla u_2 \in L^2(\Omega)$, (4.1) holds for $\theta \in (0, \frac{2\alpha-3}{2\alpha})$, while (4.2) holds for $\theta \in (\frac{3-\alpha}{2\alpha}, \frac{1}{2})$. We note that the intervals $(0, \frac{2\alpha-3}{2\alpha})$ and $(\frac{3-\alpha}{2\alpha}, \frac{1}{2})$ have no common points because $\frac{2\alpha-3}{2\alpha} < \frac{3-\alpha}{2\alpha}$ thanks to $\alpha < 2$.

To prove the trace regularity result, it is essential that the estimates (4.1) and (4.2) hold for the same value of θ . For this reason, we must choose a suitable value of $\mu_\alpha > 0$ such that the intersection of the intervals $(\mu_\alpha, \frac{2\alpha-3}{2\alpha} + \mu_\alpha)$ and $(\frac{3-\alpha}{2\alpha} - \mu_\alpha, \frac{1}{2} - \mu_\alpha)$ is

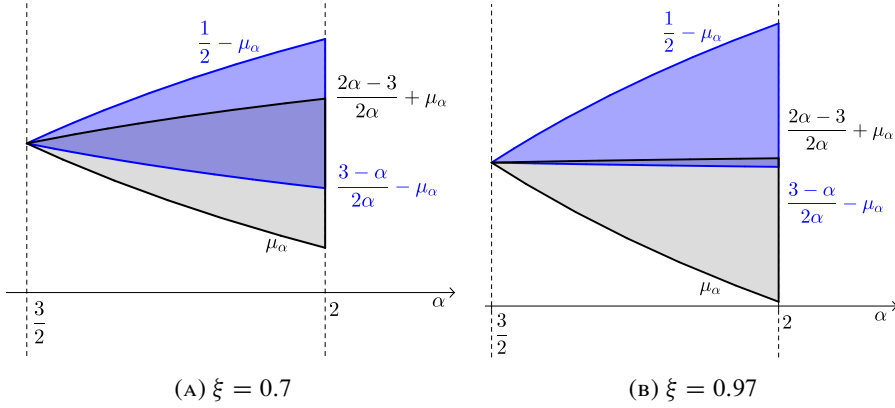


FIGURE 1. Non-empty intersections.

not empty. Consequently, we have to impose $\frac{3-\alpha}{2\alpha} - \mu_\alpha < \frac{2\alpha-3}{2\alpha} + \mu_\alpha$ and $\mu_\alpha < \frac{1}{2} - \mu_\alpha$; hence, we obtain the following condition $\frac{3(2-\alpha)}{4\alpha} < \mu_\alpha < \frac{1}{4}$.

To provide further clarification, we consider the expression

$$\mu_\alpha = \frac{3(2-\alpha)\xi}{4\alpha} + \frac{1-\xi}{4} \quad \text{with } \xi \in (0, 1).$$

As shown in Figure 1, we can observe the non-empty intersection of the intervals for two distinct values of ξ .

5. TRACE REGULARITY RESULTS

In this section, we follow arguments similar to those implemented in [8] for wave equations. First, we single out some technical results that we will use later.

LEMMA 5.1. *For $u \in H^2(\Omega)$ and a vector field $h : \bar{\Omega} \rightarrow \mathbb{R}^N$ of class C^1 , one has*

$$(5.1) \quad \int_{\Omega} \Delta u \cdot h \cdot \nabla u \, dx = \int_{\partial\Omega} \left[\partial_\nu u \cdot h \cdot \nabla u - \frac{1}{2} h \cdot \nu |\nabla u|^2 \right] d\sigma \\ - \sum_{i,j=1}^N \int_{\Omega} \partial_i h_j \partial_i u \partial_j u \, dx + \frac{1}{2} \int_{\Omega} \sum_{j=1}^N \partial_j h_j |\nabla u|^2 \, dx.$$

PROOF. We integrate by parts to get

$$(5.2) \quad \int_{\Omega} \Delta u \cdot h \cdot \nabla u \, dx = \int_{\partial\Omega} \partial_\nu u \cdot h \cdot \nabla u \, d\sigma - \int_{\Omega} \nabla u \cdot \nabla (h \cdot \nabla u) \, dx.$$

Since

$$\begin{aligned} \int_{\Omega} \nabla u \cdot \nabla (h \cdot \nabla u) dx &= \sum_{i,j=1}^N \int_{\Omega} \partial_i u \partial_i (h_j \partial_j u) dx \\ &= \sum_{i,j=1}^N \int_{\Omega} \partial_i u \partial_i h_j \partial_j u dx + \sum_{i,j=1}^N \int_{\Omega} h_j \partial_i u \partial_j (\partial_i u) dx, \end{aligned}$$

we evaluate the last term on the right-hand side again by an integration by parts, so we obtain

$$\begin{aligned} \sum_{i,j=1}^N \int_{\Omega} h_j \partial_i u \partial_j (\partial_i u) dx &= \frac{1}{2} \sum_{j=1}^N \int_{\Omega} h_j \partial_j \left(\sum_{i=1}^N (\partial_i u)^2 \right) dx \\ &= \frac{1}{2} \int_{\partial\Omega} h \cdot \nu |\nabla u|^2 d\sigma - \frac{1}{2} \int_{\Omega} \sum_{j=1}^N \partial_j h_j |\nabla u|^2 dx. \end{aligned}$$

Finally, combining the previous two identities with (5.2) yields (5.1). $\blacksquare$

Let us keep in mind the symbol $\langle \cdot, \cdot \rangle_{-\theta, \theta}$, $\theta \in (0, 1)$, denoting the duality defined in (2.9).

LEMMA 5.2. Assume $\alpha \in (1, 2)$ and the weak solution u of

$$(5.3) \quad D_{0+}^{\alpha} u(t, x) = \Delta u(t, x) \quad \text{in } (0, T) \times \Omega$$

belonging to $L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega))$ with $D_{0+}^{\alpha} u \in L^2(0, T; L^2(\Omega))$. Then, for a vector field $h : \bar{\Omega} \rightarrow \mathbb{R}^N$ of class C^1 and $\theta \in (0, 1)$, the following identity holds true:

$$\begin{aligned} (5.4) \quad & \int_{\partial\Omega} \left[\partial_{\nu} u(t, \sigma) h \cdot \nabla u(t, \sigma) - \frac{1}{2} h \cdot \nu |\nabla u(t, \sigma)|^2 \right] d\sigma \\ &= \langle D_{0+}^{\alpha} u(t, \cdot), h \cdot \nabla u(t, \cdot) \rangle_{-\theta, \theta} + \sum_{i,j=1}^N \int_{\Omega} \partial_i h_j(x) \partial_i u(t, x) \partial_j u(t, x) dx \\ &\quad - \frac{1}{2} \int_{\Omega} \sum_{j=1}^N \partial_j h_j |\nabla u(t, x)|^2 dx, \quad \text{a.e. } t \in (0, T). \end{aligned}$$

PROOF. Fix $\theta \in (0, 1)$; by means of the duality $\langle \cdot, \cdot \rangle_{-\theta, \theta}$, see (2.9), for a.e. $t \in (0, T)$, we evaluate each term of the equation (5.3) in $h \cdot \nabla u(t, \cdot)$ to get

$$\langle D_{0+}^{\alpha} u(t, \cdot), h \cdot \nabla u(t, \cdot) \rangle_{-\theta, \theta} = \langle \Delta u(t, \cdot), h \cdot \nabla u(t, \cdot) \rangle_{-\theta, \theta}.$$

Thanks to the regularity of the weak solution u and (2.11), the term on the right-hand side of the previous equation can be written as a scalar product in $L^2(\Omega)$, so we have

$$(5.5) \quad \langle D_{0+}^{\alpha} u(t, \cdot), h \cdot \nabla u(t, \cdot) \rangle_{-\theta, \theta} = \int_{\Omega} \Delta u(t, x) h \cdot \nabla u(t, x) dx.$$

By evaluating the term $\int_{\Omega} \Delta u(t, x) h \cdot \nabla u(t, x) dx$ with Lemma 5.1, from (5.1), we deduce

$$\begin{aligned} & \int_{\partial\Omega} \left[\partial_{\nu} u(t, \sigma) h \cdot \nabla u(t, \sigma) - \frac{1}{2} h \cdot \nu |\nabla u(t, \sigma)|^2 \right] d\sigma \\ &= \int_{\Omega} \Delta u(t, x) h \cdot \nabla u(t, x) dx + \sum_{i,j=1}^N \int_{\Omega} \partial_i h_j(x) \partial_i u(t, x) \partial_j u(t, x) dx \\ & \quad - \frac{1}{2} \int_{\Omega} \sum_{j=1}^N \partial_j h_j |\nabla u(t, x)|^2 dx. \end{aligned}$$

We conclude from (5.5) that (5.4) is proved. $\blacksquare$

THEOREM 5.3. *Let $u_1 \in H_0^1(\Omega)$, $u_2 \in H^2(\Omega) \cap H_0^1(\Omega)$, and u the weak solution of*

$$(5.6) \quad \begin{aligned} D_{0+}^{\alpha} u(t, x) &= \Delta u(t, x) & (t, x) &\in (0, T) \times \Omega, \\ u(t, x) &= 0 & (t, x) &\in (0, T) \times \partial\Omega, \end{aligned}$$

satisfying the initial data

$$(5.7) \quad D_{0+}^{\alpha-1} u(0, \cdot) = u_1, \quad D_{0+}^{\alpha-2} u(0, \cdot) = u_2.$$

Then, for any $T > 0$, there exists a constant $C = C(T)$ such that for any $\mu_{\alpha} \in (\frac{3(2-\alpha)}{4\alpha}, \frac{1}{4})$, u satisfies the inequality

$$(5.8) \quad \int_0^T \int_{\partial\Omega} |\partial_{\nu} u|^2 d\sigma dt \leq C (\|u_1\|_{D((-\Delta)^{\mu_{\alpha}})} + \|\nabla u_2\|_{D((-\Delta)^{\mu_{\alpha}})}).$$

PROOF. First, we note that by Theorem 3.3 (ii), the weak solution u to (3.1) given by (3.4) belongs to $L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega))$ and $D_{0+}^{\alpha} u(t, \cdot) \in L^2(0, T; L^2(\Omega))$. In particular, the normal derivative $\partial_{\nu} u$ is well defined.

We employ the identity in Lemma 5.2 with a suitable choice of the vector field h . Indeed, we take a vector field $h \in C^1(\bar{\Omega}; \mathbb{R}^N)$ satisfying the condition

$$h = \nu \quad \text{on } \partial\Omega$$

(see e.g. [8] for the existence of such vector field h). First, we consider the identity (5.4). Since

$$\nabla u = (\partial_{\nu} u) \nu \quad \text{on } (0, T) \times \partial\Omega$$

(see e.g. [21, Lemma 2.1] for a detailed proof), the left-hand side of (5.4) becomes

$$\frac{1}{2} \int_{\partial\Omega} |\partial_{\nu} u(t, \sigma)|^2 d\sigma.$$

If we integrate (5.4) over $[0, T]$, then we obtain

$$\begin{aligned} \int_0^T \int_{\partial\Omega} |\partial_\nu u(t, \sigma)|^2 d\sigma dt &= 2 \int_0^T \langle D_{0+}^\alpha u(t, \cdot), h \cdot \nabla u(t, \cdot) \rangle_{-\theta, \theta} dt \\ &\quad + 2 \sum_{i,j=1}^N \int_0^T \int_{\Omega} \partial_i h_j(x) \partial_i u(t, x) \partial_j u(t, x) dx dt \\ &\quad - \int_0^T \int_{\Omega} \sum_{j=1}^N \partial_j h_j |\nabla u(t, x)|^2 dx dt. \end{aligned}$$

Since $h \in C^1(\bar{\Omega}; \mathbb{R}^N)$ from the above inequality, we get

$$(5.9) \quad \|\partial_\nu u\|_{L^2(0,T;L^2(\partial\Omega))} \leq C (\|D_{0+}^\alpha u\|_{L^2(0,T;D(A^{-\theta}))} + \|\nabla u\|_{L^2(0,T;D(A^\theta))}),$$

for some constant $C > 0$.

To apply Theorem 4.1, since $\mu_\alpha \in (\frac{3(2-\alpha)}{4\alpha}, \frac{1}{4})$, we can choose

$$\theta \in \left(\mu_\alpha, \frac{2\alpha-3}{2\alpha} + \mu_\alpha \right) \cap \left(\frac{3-\alpha}{2\alpha} - \mu_\alpha, \frac{1}{2} - \mu_\alpha \right)$$

see Remark 4.2. So, we get $D_{0+}^\alpha u \in L^2(0, T; D(A^{-\theta}))$ and $\nabla u \in L^2(0, T; D(A^\theta))$; hence, from (5.9), (4.2), and (4.1), we deduce (5.8). ■

THEOREM 5.4. *Let $u_1, \nabla u_2 \in D((-\Delta)^{\mu_\alpha})$ with $\frac{3(2-\alpha)}{4\alpha} < \mu_\alpha < \frac{1}{4}$. If u is the weak solution of (5.6)–(5.7), then we define the normal derivative $\partial_\nu u$ of u such that for any $T > 0$, we have*

$$(5.10) \quad \int_0^T \int_{\partial\Omega} |\partial_\nu u|^2 d\sigma dt \leq C (\|u_1\|_{D((-\Delta)^{\mu_\alpha})} + \|\nabla u_2\|_{D((-\Delta)^{\mu_\alpha})}),$$

for some constant $C = C(T)$ independent of the initial data.

PROOF. For $u_1 \in H_0^1(\Omega)$ and $u_2 \in H^2(\Omega) \cap H_0^1(\Omega)$, denoted by u the weak solution of problem (5.6)–(5.7), thanks to Theorem 5.3, the inequality (5.10) holds for any $T > 0$. By density, there exists a unique continuous linear map

$$\mathcal{L} : D((-\Delta)^{\mu_\alpha}) \times D((-\Delta)^{\frac{1}{2}+\mu_\alpha}) \rightarrow L_{\text{loc}}^2((0, \infty); L^2(\partial\Omega))$$

such that

$$\mathcal{L}(u_1, u_2) = \partial_\nu u \quad \forall (u_1, u_2) \in H_0^1(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega))$$

and

$$\int_0^T \int_{\partial\Omega} |\mathcal{L}(u_1, u_2)|^2 d\sigma dt \leq C (\|u_1\|_{D((-\Delta)^{\mu_\alpha})} + \|\nabla u_2\|_{D((-\Delta)^{\mu_\alpha})}),$$

for any $u_1, \nabla u_2 \in D((-\Delta)^{\mu_\alpha})$. Consequently, we adopt the standard notation $\partial_\nu u$ in place of $\mathcal{L}(u_1, u_2)$, which leads to (5.10) holding. ■

REMARK 5.5. We observe that the assumption $u_1, \nabla u_2 \in D((-\Delta)^{\mu_\alpha})$ with $\frac{3(2-\alpha)}{4\alpha} < \mu_\alpha < \frac{1}{4}$ is weaker than assuming $u_1 \in H_0^1(\Omega)$ and $u_2 \in H^2(\Omega) \cap H_0^1(\Omega)$, due to

$$H_0^1(\Omega) = D((-\Delta)^{\frac{1}{2}}) \subset D((-\Delta)^{\frac{1}{4}}) \subset D((-\Delta)^{\mu_\alpha}).$$

6. FURTHER PROPERTIES

In this section, we establish the decay rate of so-called energy for the Riemann–Liouville problem.

Next, we investigate the duality existing between the Caputo problem and the Riemann–Liouville problem following an approach similar to that of J.-L. Lions [12, 13] for the case of the wave equation.

6.1. Decay rate of energy

The analysis of the decay rate in L^2 -norm started from the case $\alpha \in (0, 1)$ for the problem involving the Caputo fractional derivative instead of the Riemann–Liouville fractional derivative, see [31] and also [9]. In [30], the authors established the decay rate in L^2 -norm when $\alpha \in (1, 2)$. This result was improved in [33], in the case $\alpha \in (0, 1) \cup (1, 2)$.

Acting similarly to [33], we prove the decay rate for the weak solution of the problem

$$\begin{aligned} (6.1) \quad & D_{0+}^\alpha u(t, x) = \Delta u(t, x) \quad (t, x) \in (0, T) \times \Omega, \\ & u(t, x) = 0 \quad (t, x) \in (0, T) \times \partial\Omega, \\ & D_{0+}^{\alpha-1} u(0, \cdot) = u_1, \quad D_{0+}^{\alpha-2} u(0, \cdot) = u_2. \end{aligned}$$

THEOREM 6.1. *The weak solution*

$$(6.2) \quad u(t, x) = \sum_{n=1}^{\infty} [\langle u_1, e_n \rangle t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n t^\alpha) + \langle u_2, e_n \rangle t^{\alpha-2} E_{\alpha, \alpha-1}(-\lambda_n t^\alpha)] e_n(x)$$

of (6.1) satisfies

$$(6.3) \quad \|u(t, \cdot)\|_{H^2(\Omega)} \leq \frac{C}{t} \|u_1\| + \frac{C}{t^2} \|u_2\|, \quad \text{for any } t > 0.$$

PROOF. Since

$$\|u(t, \cdot)\|_{H^2(\Omega)}^2 = \sum_{n=1}^{\infty} \lambda_n^2 |\langle u_1, e_n \rangle t^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n t^\alpha) + \langle u_2, e_n \rangle t^{\alpha-2} E_{\alpha, \alpha-1}(-\lambda_n t^\alpha)|^2,$$

thanks to (2.13), we have for $t > 0$,

$$\|u(t, \cdot)\|_{H^2(\Omega)}^2 \leq C t^{-2} \sum_{n=1}^{\infty} |\langle u_1, e_n \rangle|^2 \frac{\lambda_n^2 t^{2\alpha}}{1 + \lambda_n^2 t^{2\alpha}} + C t^{-4} \sum_{n=1}^{\infty} |\langle u_2, e_n \rangle|^2 \frac{\lambda_n^2 t^{2\alpha}}{1 + \lambda_n^2 t^{2\alpha}}.$$

Therefore,

$$\|u(t, \cdot)\|_{H^2(\Omega)}^2 \leq \frac{C}{t^2} \|u_1\|^2 + \frac{C}{t^4} \|u_2\|^2, \quad \text{for any } t > 0,$$

that is, (6.3). ■

We observe that the estimate (6.3) does not depend on α .

6.2. Duality

Let $T > 0$ and Ω be a bounded domain of class C^2 in $\mathbb{R}^N$, $N \geq 1$, with boundary $\partial\Omega$. We consider the Caputo fractional diffusion-wave equation with $\alpha \in (1, 2)$:

$$(6.4) \quad {}^C D_{0+}^{\alpha} u(t, x) = \Delta u(t, x) \quad (t, x) \in (0, T) \times \Omega,$$

with null initial conditions

$$(6.5) \quad u(0, x) = u_t(0, x) = 0 \quad x \in \Omega,$$

and boundary conditions

$$(6.6) \quad u(t, x) = g(t, x) \quad (t, x) \in (0, T) \times \partial\Omega.$$

We are interested in determining the dual system of the fractional differential problem (6.4)–(6.6) also due to the uniqueness properties of the dual system.

For the meaning and study of duality in the context of control theory, we refer to the seminal works [12, 13, 29] and references therein.

In relation to the fractional differential case, one of the main difficulties is determining the function spaces in which to set the problem and an appropriate meaning for the solution.

We introduce the *adjoint* system of (6.4)–(6.6) as

$$(6.7) \quad \begin{cases} D_{T-}^{\alpha} w(t, x) = \Delta w(t, x) & (t, x) \in (0, T) \times \Omega, \\ w(t, x) = 0 & (t, x) \in (0, T) \times \partial\Omega, \end{cases}$$

with final data

$$(6.8) \quad D_{T-}^{\alpha-1} w(T, \cdot) = w_1, \quad D_{T-}^{\alpha-2} w(T, \cdot) = w_2.$$

Set $v(t, x) = w(T - t, x)$, for any $\beta > 0$, we have

$$\begin{aligned} & \int_t^T (s - t)^{\beta-1} w(s, x) ds \\ &= \int_t^T (s - t)^{\beta-1} v(T - s, x) ds = \int_0^{T-t} (T - t - \sigma)^{\beta-1} v(\sigma, x) d\sigma. \end{aligned}$$

Therefore,

$$I_{T-}^{\beta} w(t, x) = I_{0+}^{\beta} v(T - t, x) \quad \beta > 0,$$

and hence,

$$\begin{aligned} D_{T-}^{\alpha} w(t, x) &= D_{0+}^{\alpha} v(T - t, x), \\ D_{T-}^{\alpha-1} w(T, \cdot) &= D_{0+}^{\alpha-1} v(0, \cdot), \\ D_{T-}^{\alpha-2} w(T, \cdot) &= D_{0+}^{\alpha-2} v(0, \cdot). \end{aligned}$$

The above considerations ensure that the backward problem (6.7)–(6.8) is equivalent to a forward problem of the type (5.6)–(5.7). Thanks to Theorem 5.4, by appropriately choosing the regularity of the final data w_1 and w_2 , the solution w of (6.7)–(6.8) will be sufficiently smooth to guarantee that $\partial_v w$ is well defined, allowing us to take $g = \partial_v w$ in (6.6).

The nonhomogeneous problem (6.4)–(6.6) can be written in the form

$$(6.9) \quad \begin{cases} {}^C D_{0+}^{\alpha} u(t, x) = \Delta u(t, x) & (t, x) \in (0, T) \times \Omega, \\ u(0, x) = u_t(0, x) = 0 & x \in \Omega, \\ u(t, x) = \partial_v w(t, x) & (t, x) \in (0, T) \times \partial\Omega. \end{cases}$$

In a similar way to [12, 13], we show duality by introducing a suitable operator.

THEOREM 6.2. *Let Ψ be a linear operator defined as*

$$(6.10) \quad \Psi(w_1, w_2) = (u(T, \cdot), -u_t(T, \cdot)),$$

for $w_1, \nabla w_2 \in D((-\Delta)^{\mu_{\alpha}})$ with $\frac{3(2-\alpha)}{4\alpha} < \mu_{\alpha} < \frac{1}{4}$, where u and w are the solution of (6.9) and (6.7)–(6.8), respectively.

Then,

$$(6.11) \quad \langle \Psi(w_1, w_2), (w_1, w_2) \rangle = \int_0^T \int_{\partial\Omega} |\partial_v w(t, x)|^2 dx dt.$$

PROOF. We multiply the equation in (6.9) by the solution $w(t, x)$ of the adjoint system (6.7)–(6.8) and integrate on $(0, T) \times \Omega$, that is,

$$(6.12) \quad \int_0^T \int_{\Omega} {}^C D_{0+}^{\alpha} u(t, x) w(t, x) dx dt = \int_0^T \int_{\Omega} \Delta u(t, x) w(t, x) dx dt.$$

Applying (2.7), we obtain for any $x \in \Omega$,

$$(6.13) \quad \int_0^T {}^C D_{0+}^\alpha u(t, x) w(t, x) dt \\ = \int_0^T I_{0+}^{2-\alpha} u_{tt}(t, x) w(t, x) dt = \int_0^T u_{tt}(t, x) I_{T-}^{2-\alpha} w(t, x) dt.$$

Integrating twice by parts and taking into account that $u(0, \cdot) = u_t(0, \cdot) = 0$, we have

$$(6.14) \quad \int_0^T u_{tt}(t, x) I_{T-}^{2-\alpha} w(t, x) dt \\ = u_t(T, x) I_{T-}^{2-\alpha} w(T, x) - \int_0^T u_t(t, x) \frac{\partial}{\partial t} I_{T-}^{2-\alpha} w(t, x) dt \\ = u_t(T, x) I_{T-}^{2-\alpha} w(T, x) - u(T, x) \frac{\partial}{\partial t} I_{T-}^{2-\alpha} w(T, x) + \int_0^T u(t, x) D_{T-}^\alpha w(t, x) dt.$$

Since $-1 < \alpha - 2 < 0$, we have $I_{T-}^{2-\alpha} w(T, x) = D_{T-}^{\alpha-2} w(T, x)$, see (2.6). Moreover, $0 < \alpha - 1 < 1$, so by (2.4) we have

$$\frac{\partial}{\partial t} I_{T-}^{2-\alpha} w(T, x) = D_{T-}^{\alpha-1} w(T, x).$$

Combining (6.13) with (6.14) yields

$$(6.15) \quad \int_0^T {}^C D_{0+}^\alpha u(t, x) w(t, x) dt \\ = u_t(T, x) D_{T-}^{\alpha-2} w(T, x) - u(T, x) D_{T-}^{\alpha-1} w(T, x) + \int_0^T u(t, x) D_{T-}^\alpha w(t, x) dt.$$

On the other hand, integrating twice by parts with respect to the variable x , we have

$$(6.16) \quad \int_\Omega \Delta u(t, x) w(t, x) dx = \int_\Omega u(t, x) \Delta w(t, x) dx - \int_{\partial\Omega} u(t, x) \partial_\nu w(t, x) dx.$$

Putting (6.15) and (6.16) into (6.12), we get

$$\int_0^T \int_\Omega u(t, x) D_{T-}^\alpha w(t, x) dx dt \\ + \int_\Omega u_t(T, x) D_{T-}^{\alpha-2} w(T, x) dx - \int_\Omega u(T, x) D_{T-}^{\alpha-1} w(T, x) dx \\ = \int_0^T \int_\Omega u(t, x) \Delta w(t, x) dx dt - \int_0^T \int_{\partial\Omega} u(t, x) \partial_\nu w(t, x) dx dt.$$

We recall that w is the solution of the adjoint problem (6.7): keeping $D_{T-}^{\alpha} w = \Delta w$ in mind, from the above equation, we have

$$\begin{aligned} \int_{\Omega} u_t(T, x) D_{T-}^{\alpha-2} w(T, x) dx - \int_{\Omega} u(T, x) D_{T-}^{\alpha-1} w(T, x) dx \\ = - \int_0^T \int_{\partial\Omega} u(t, x) \partial_{\nu} w(t, x) dx dt. \end{aligned}$$

By the final data (6.8) of w and the boundary conditions satisfied by u , see (6.9), we deduce that

$$\begin{aligned} \langle \Psi(w_1, w_2), (w_1, w_2) \rangle \\ = \int_{\Omega} u(T, x) w_1(x) dx - \int_{\Omega} u_t(T, x) w_2(x) dx = \int_0^T \int_{\partial\Omega} |\partial_{\nu} w(t, x)|^2 dx dt; \end{aligned}$$

hence, (6.11) is proved. $\blacksquare$

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