



**Differential Geometry.** – *The notions of skeleton and crack, and singularities of the oriented distance function*, by MICHEL C. DELFOUR, accepted on 12 December 2024.

**ABSTRACT.** – In problems where a geometric object is the variable, the object can be identified with the *oriented distance function* which can simultaneously deal with the smooth sets of classical Differential Geometry and sets with a lousy boundary.

This paper reviews some properties of the distance function  $d_A$  to a set  $A$ , the oriented distance function  $b_A = d_A - d_{\mathbb{C}A}$  ( $\mathbb{C}A$ , the complement of  $A$ ), and the associated notions of *skeletons*, *b-crack*, and *crack*. It gives the respective partitions of the boundaries  $\partial A$  and  $\partial(\partial A)$  and the partition of the singularities of  $\nabla b_A$ . It turns out that the notion of *b-crack* is possibly too broad since it also includes *corners* in the *core boundary*  $\partial \bar{A} \cap \partial \mathbb{C}A$  of the set  $A$ . On one hand, this analysis leads to the notions of *crack-free sets* and *strongly crack-free sets* and, on the other hand, to the notion of *cracked sets* in Image Segmentation and Mathematical Morphology.

**KEYWORDS.** – distance function, algebraic distance function, skeleton, cracks, singularities, imaging, mathematical morphology.

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## 1. INTRODUCTION

In problems where a geometric object is the variable, the object can be identified with a family of functions parametrized by sets, such as the *characteristic function* in Geometric Measure Theory, the *distance function* in Non-smooth Analysis, the *support function* in Convex Analysis, or the *oriented distance function* which can simultaneously deal with the sets of classical Differential Geometry with smooth boundary and sets with lousy boundaries [5, 8]. Such functions are each identifiable with an equivalence class of sets. They play the same role of a *state variable* in Control Theory. Natural metrics can be constructed on spaces or subspaces of such functions to measure the distance between two objects and to induce topologies from which existence and characterization of optimal objects can be obtained for design, identification, or control purposes. The choice of the function and the metric is problem dependent and corresponds to specific technological, physical, or geometric entities associated with the problem at hand.

The *distance function* from  $x \in X$  to a subset  $A$  of a Banach space  $X$  is defined as

$$d_A(x) \stackrel{\text{def}}{=} \begin{cases} \inf_{a \in A} \|a - x\|, & \text{if } A \neq \emptyset, \\ +\infty, & \text{if } A = \emptyset. \end{cases}$$

When  $A \neq \emptyset$ ,  $d_A$  is Lipschitz continuous of constant one. The function  $d_A$  is identified with the equivalence class

$$[A]_d \stackrel{\text{def}}{=} \{B \subset X : \bar{B} = \bar{A}\}$$

and  $\bar{A}$  is the nice closed representative in the class  $[A]_d$ .

The *oriented distance function* from  $x \in X$  to  $A$  is defined as

$$(1) \quad b_A(x) \stackrel{\text{def}}{=} d_A(x) - d_{\mathbb{C}A}(x) = \begin{cases} +\infty, & \text{if } A = \emptyset, \\ \in \mathbf{R}, & \text{if } \partial A \neq \emptyset, \\ -\infty, & \text{if } \mathbb{C}A = \emptyset, \end{cases}$$

where  $\mathbb{C}A = X \setminus A$ . The function  $b_A(x)$  is finite in  $X$  if and only if  $\partial A \neq \emptyset$ . In that case,  $b_A$  is Lipschitz continuous of constant one and  $b_A$  coincides with the following *algebraic distance function* to the boundary of  $A$ :

$$(2) \quad b_A(x) = \begin{cases} d_A(x) = d_{\partial A}(x), & x \in \text{int } \mathbb{C}A, \\ 0, & x \in \partial A, \\ -d_{\mathbb{C}A}(x) = -d_{\partial A}(x), & x \in \text{int } A. \end{cases}$$

The function  $b_A$  is identified with the finer equivalence class

$$[A]_b \stackrel{\text{def}}{=} \{B \subset X : \partial B = \partial A \text{ and } \bar{B} = \bar{A}\}.$$

In general, there is no nice open or closed representative of  $A$  in  $[A]_b$ . It provides a level set description of the set  $A$  whose boundary  $\partial A$  coincides with the zero level set of  $b_A$ . Noting that  $b_{\mathbb{C}A} = -b_A$ , it means that we have explicitly chosen the negative sign for the interior of  $A$  and the positive sign for the interior of its complement. So  $b_A$  is an increasing function from its interior to its exterior.

For subsets of the  $N$ -dimensional Euclidean vector space<sup>1</sup>  $\mathbf{R}^N$  with a  $C^2$  boundary, the restriction of the gradient  $\nabla b_A$  to  $\partial A$  coincides with the outward unit normal to  $\partial A$  and the Hessian matrix  $D^2 b_A(x)$  is the second fundamental form with eigenvalues 0 for the subspace spanned by the normal  $\nabla b_A(x)$  and the  $N - 1$  principal eigenvalues for the tangent space to  $\partial A$  at  $x$ .

(1)  $\mathbf{R}^N$  is endowed with the inner product  $(x, y) \stackrel{\text{def}}{=} \sum_{i=1}^N x_i y_i$  and norm  $\|x\| \stackrel{\text{def}}{=} \sqrt{(x, x)}$ .

Another big advantage of  $b_A$  over  $d_A$  is for the constraint  $x \in \bar{A}$ . The equality constraint  $d_A(x) = 0$  is equivalent to the inequality constraint  $b_A(x) \leq 0$  which is much less demanding (see [5, Eqs. (5.19) and (5.20) and Thms. 5.7–5.9]) in the context of constrained optimization since a theorem of the Kuhn–Tucker type for inequality constraints can be used over a Lagrange multiplier theorem for equality constraints.

In this paper, we focus on the oriented distance function which can provide an analytical characterization of the whole range of subsets of the  $N$ -dimensional Euclidean vector space  $\mathbf{R}^N$  from the smooth sets of classical Differential Geometry to the lousy sets with irregular boundaries encountered in Mechanics and Image Processing.

Sections 2 and 3 recall some properties of distance and oriented distance functions and the associated notions of *skeleton*, *crack*, and *b-crack* [8]. Section 4 gives the respective partitions of the boundaries  $\partial A$  and  $\partial(\partial A)$ . Section 5 gives the partition of the singularities of  $\nabla b_A$  in relation with the partition of  $\partial(\partial A)$ . Section 6 relates the oriented distance function to the broader partition of  $\partial A$ . Section 7 revisits the notion of *b-crack* which is probably too broad since it also includes corners in the *core boundary*  $\partial \bar{A} \cap \partial \bar{C}A$  of the set  $A$ . From this we introduce two notions: *crack-free sets* and *strongly crack-free sets*. In the later case, they are characterized by the property that  $b_{\text{int } A} = b_A = b_{\bar{A}}$ . They are a generalization to sets with a non-empty boundary of the notion of *Carathéodory set* for open sets introduced by Henrot [13] in 1994. For a *strongly crack-free set*  $A$ , there is both a nice open and a nice closed representative in the equivalence class  $[A]_b$ . At the opposite, we have families of *cracked sets* with applications in Image Segmentation [7, Def. 3.2, p. 37] and Mathematical Morphology (Rivière [19], Matheron [17], and Serra [20]), but this is beyond the scope of the present paper.

In the remainder of this paper,  $\mathbf{R}^N$  is the  $N$ -dimensional Euclidean vector space endowed with the Hilbertian inner product and norm as defined in Footnote 1.

## 2. DISTANCE FUNCTION

In this section, we recall definitions and some properties of the distance function.

### 2.1. Projection onto $\bar{A}$ , skeleton, and cracks

For a non-empty subset  $A$  of  $\mathbf{R}^N$ , the set of singularities of the gradient of  $d_A$  can be partitioned into the *skeleton*<sup>2</sup> and the *set of cracks* of  $A$ .

(2) Our definition of a skeleton does not coincide with the one used in *morphological mathematics* where it is defined as the closure  $\overline{\text{Sk}(A)}$  of our skeleton  $\text{Sk}(A)$  (cf. for instance, Matheron [17] or Rivière [19], and Serra [20] for the pioneering applications in mining engineering in 1968 and later in image processing).

DEFINITION 1 ([8, Def. 2.1, pp. 279–289], [5, Def. 4.6, p. 1935]). Let  $A \neq \emptyset$  be a subset of  $\mathbf{R}^N$ .

- (i) The *set of singularities of the gradient of  $d_A$*

$$\text{Sing}(\nabla d_A) \stackrel{\text{def}}{=} \{x \in \mathbf{R}^N : \nabla d_A(x) \nexists\}.$$

- (ii) Given  $x \in X$ , a point  $p \in \bar{A}$  such that  $|p - x| = d_A(x)$  is called a *projection onto  $\bar{A}$* . The *set of all projections onto  $\bar{A}$*  will be denoted by

$$\Pi_A(x) \stackrel{\text{def}}{=} \{p \in \bar{A} : |p - x| = d_A(x)\}.$$

When  $\Pi_A(x)$  is a singleton, its unique element is denoted by  $p_A(x)$ .

- (iii) The *skeleton of  $A$*  is the set of all points of  $X$  whose projection onto  $\bar{A}$  is not unique. It will be denoted by

$$\text{Sk}(A) \stackrel{\text{def}}{=} \{x \in \mathbf{R}^N : \Pi_A(x) \text{ is not a singleton}\}.$$

- (iv) The *set of cracks* is defined as the complement of  $\text{Sk}(A)$

$$\text{Ck}(A) \stackrel{\text{def}}{=} \text{Sing } \nabla d_A \setminus \text{Sk}(A).$$

Note that  $\text{Sk}(\bar{A}) = \text{Sk}(A)$ ,  $\text{Ck}(\bar{A}) = \text{Ck}(A)$ , and  $\text{Sing}(\bar{A}) = \text{Sing}(A)$  since  $d_A = d_{\bar{A}}$ .

REMARK 1. For all  $x \in \bar{A}$ ,  $\Pi_A(x) = \{x\}$  is a singleton and

$$\forall x \in \mathbf{R}^N \setminus \bar{A}, \quad \emptyset \neq \Pi_A(x) \subset \overline{\mathbb{C}\bar{A}} \setminus \mathbb{C}\bar{A} = \partial \bar{A}.$$

We shall see in equation (3) of Theorem 1 (iii) that

$$\text{Sk}(A) = \{x \in \mathbf{R}^N \setminus \bar{A} : \nabla d_A^2(x) \nexists\} = \{x \in \mathbf{R}^N \setminus \bar{A} : \nabla d_A(x) \nexists\};$$

that is,  $\text{Sk}(A)$  coincides with the singularities of  $\nabla d_A^2$  (resp.  $\nabla d_A$ ) in  $\mathbf{R}^N \setminus \bar{A}$ .

EXAMPLE 1. Even for a set  $A$  with smooth boundary  $\partial A$ , the gradient  $\nabla d_A(x)$  may not exist far from the boundary as shown in Figure 1, where  $\nabla d_A(x)$  exists and  $\|\nabla d_A(x)\| = 1$  everywhere outside  $\bar{A}$  except on  $\text{Sk}(A)$ , a semi-infinite line. Here,  $\text{Ck}(A) = \partial A$ . In contrast for a square  $S$  in  $\mathbf{R}^2$ ,  $\text{Sk}(S) = \emptyset$  and  $\text{Ck}(S) = \partial S$ .

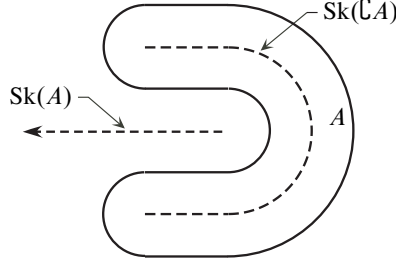
## 2.2. Singularities of $\nabla d_A$ , projections, and cracks

We first recall some results from [8, Thm. 3.3, Chap. 6, pp. 283–285].

THEOREM 1. Let  $\emptyset \neq A \subset \mathbf{R}^N$ .

- (i) If  $A \neq \emptyset$ , the set  $\Pi_A(x)$  is non-empty, compact,

$$\forall x \notin \bar{A}, \quad \Pi_A(x) \subset \partial \bar{A}, \quad \text{and} \quad \forall x \in \bar{A}, \quad \Pi_A(x) = \{x\}.$$

FIGURE 1. Skeletons  $\text{Sk}(A)$ ,  $\text{Sk}(\mathbb{C}A)$ , and  $\text{Sk}(\partial A) = \text{Sk}(A) \cup \text{Sk}(\mathbb{C}A)$ .

(ii) *The function*

$$x \mapsto f_A(x) \stackrel{\text{def}}{=} \frac{1}{2} [\|x\|^2 - d_A(x)^2] : \mathbf{R}^N \rightarrow \mathbf{R}$$

is convex, finite, and Hadamard<sup>3</sup> semidifferentiable and hence  $d_A^2$  is Hadamard semidifferentiable:

$$d_H d_A^2(x; v) = \inf_{p \in \Pi_A(x)} 2(x - p) \cdot v \quad \text{and} \quad d_H f_A(x; v) = \sup_{p \in \Pi_A(x)} p \cdot v.$$

$\nabla d_A^2(x)$  exists if and only if  $\Pi_A(x) = \{p_A(x)\}$  is a singleton. In that case,<sup>4</sup>

$$p_A(x) = \nabla f_A(x) = x - \frac{1}{2} \nabla d_A^2(x)$$

and

$$\text{Sk}(A) = \{x \in \mathbf{R}^N : \nabla d_A^2(x) \neq 0\}.$$

(iii) Since  $\text{Sk}(A) \subset \mathbf{R}^N \setminus \bar{A}$ ,

$$(3) \quad \text{Sk}(A) = \{x \in \mathbf{R}^N \setminus \bar{A} : \nabla d_A^2(x) \neq 0\} = \{x \in \mathbf{R}^N \setminus \bar{A} : \nabla d_A(x) \neq 0\}.$$

If  $\nabla d_A(x)$  exists at  $x \in \bar{A}$ , then  $\nabla d_A(x) = 0$ . In general,

$$(4) \quad \bar{A} = \{x \in \bar{A} : \nabla d_A(x) \neq 0\} \cup \{x \in \bar{A} : \nabla d_A(x) = 0\}$$

and  $\nabla d_A(x) = 0$  can occur at a boundary point  $x \in \partial \bar{A}$  (see Example 2 below).

(<sup>3</sup>) See the appendix for the definition of Hadamard semidifferentiability and differentiability and some properties.

(<sup>4</sup>) This is only true when  $\mathbf{R}^N$  is endowed with the Hilbertian inner product and norm as defined in Footnote 1 (see [6, Sec. 5, p. 1354] for a counterexample).

(iv) The functions  $d_A^2$  and  $d_A$  are differentiable almost everywhere and

$$m_N(\text{Sk}(A)) = m_N(\text{Ck}(A)) = m_N(\text{Sing}(\nabla d_A)) = 0,$$

where  $m_N$  is the  $N$ -dimensional Lebesgue measure. Moreover, the set  $\text{Sk}(A)$  is countably  $N - 1$  rectifiable.<sup>5</sup>

(v) For  $\partial A \neq \emptyset$ ,

$$\text{Sk}(\partial A) \subset \mathbf{R}^N \setminus \partial A, \quad \text{Ck}(\partial A) \subset \partial(\partial A).$$

If  $\partial A = \emptyset$ , then  $A = \mathbf{R}^N$ . The case  $A = \emptyset$  is ruled out by assumption.

(vi) Given  $A \subset \mathbf{R}^N$ ,  $\emptyset \neq A$  (resp.,  $\emptyset \neq \mathbb{C}A$ ),

$$\begin{aligned} \chi_{\bar{A}}(x) &= 1 - |\nabla d_A(x)|, & \chi_{\text{int } \mathbb{C}A}(x) &= |\nabla d_A(x)| \text{ in } \mathbf{R}^N \setminus \text{Sing}(\nabla d_A) \\ (\text{resp.}, \chi_{\overline{\mathbb{C}A}}(x) &= 1 - |\nabla d_{\mathbb{C}A}(x)|, & \chi_{\text{int } A}(x) &= |\nabla d_{\mathbb{C}A}(x)| \text{ in } \mathbf{R}^N \setminus \text{Sing}(\nabla d_{\mathbb{C}A})) \end{aligned}$$

and the above identities hold for almost all  $x$  in  $\mathbf{R}^N$ .

From property (4),  $\nabla d_A(x)$  does not exist or  $\nabla d_A(x) = 0$  at points on the boundary  $\partial \bar{A}$ . The next example shows that  $\nabla d_A(x) = 0$  can occur at a cusp  $x \in \partial \bar{A}$ .

EXAMPLE 2. Let  $B_+ = B_1(1, 0)$  and  $B_- = B_1(-1, 0)$  in  $\mathbf{R}^2$  and  $A = B_+ \cup B_-$ . Then,

$$\begin{aligned} \text{int } A &= A = B_+ \cup B_-, & \bar{A} &= \overline{B_+} \cup \overline{B_-}, & \text{int } \bar{A} &= B_+ \cup B_-, \\ \mathbb{C}A &= \mathbb{C}B_+ \cap \mathbb{C}B_- = \overline{\mathbb{C}A}, & \partial A &= \partial \bar{A} = \partial B_+ \cup \partial B_-. \end{aligned}$$

There is a cusp at the point  $(0, 0)$  where  $\nabla d_A((0, 0)) = 0$ . For directions such that  $\|v\| = 1$  and  $v \neq (0, \pm 1)$  and  $t > 0$  small,  $(0, 0) + tv \in A$ ,  $d_A((0, 0) + tv) = 0$ , and

$$\lim_{t \searrow 0} \frac{d_A((0, 0) + tv) - d_A((0, 0))}{t} = 0.$$

For  $v = (0, \pm 1)$  and  $t > 0$ ,  $(0, 0) + tv = (0, \pm t)$ ,  $d_A((0, \pm t)) = \sqrt{1^2 + (\pm t)^2} - 1$  and

$$\lim_{t \searrow 0} \frac{d_A((0, 0) + tv) - d_A((0, 0))}{t} = \lim_{t \searrow 0} \frac{\sqrt{1^2 + t^2} - 1}{t} = 0.$$

Since  $d_A$  is Lipschitzian and  $v \mapsto dd_A((0, 0); v)$  is linear,  $\nabla d_A((0, 0))$  exists [3, Def. 3.4 (iii), p. 1048, Thm. 3.10, p. 1952].  $\nabla d_A(x)$  does not exist at other points of  $\partial A$ .

(<sup>5</sup>) This is a consequence of the results in L. Zajíček [22]. The topological structure of the skeleton  $\text{Sk}(A)$  was studied in A. Lieutier [15]. His result was generalized (and improved) to the Riemannian setting in P. Albano, P. Cannarsa, K. T. Nguyen, and C. Sinestrari [1] and P. Cannarsa, W. Cheng, and A. Fathi [2].

Therefore,  $\text{Sk}(A)$ ,  $\text{Ck}(A)$ , and  $\text{Sing}(\nabla d_A)$  are not closed subsets of  $\mathbf{R}^2$

$$\begin{aligned}\text{Sk}(A) &= \{(x, 0) : x \neq 0\} \subset \mathbf{R}^N \setminus \bar{A}, \quad \text{Ck}(A) = \partial \bar{A} \setminus \{(0, 0)\} \subset \partial \bar{A}, \\ \text{Sing}(\nabla d_A) &= \{(x, 0) : x \neq 0\} \cup (\partial \bar{A} \setminus \{(0, 0)\})\end{aligned}$$

and we have a neat partition of the singularities of  $\nabla d_A$ .

If we change the definition of the skeleton  $\text{Sk}(A)$  to  $\overline{\text{Sk}(A)}$  as in Morphological Mathematics (see Footnote 2), we get

$$\overline{\text{Sing}(\nabla d_A)} = \overline{\text{Sk}(A)} \cup \text{Ck}(A), \quad \overline{\text{Sk}(A)} \cap \text{Ck}(A) = \emptyset.$$

We could also use the closure of  $\text{Ck}(A)$

$$\overline{\text{Sing}(\nabla d_A)} = \text{Sk}(A) \cup \overline{\text{Ck}(A)}, \quad \text{Sk}(A) \cap \overline{\text{Ck}(A)} = \emptyset.$$

Is a cluster point of  $\text{Sing}(\nabla d_A)$  also a cluster point of both  $\text{Sk}(A)$  and  $\text{Ck}(A)$  for an arbitrary subset  $A$ ?

### 3. ORIENTED DISTANCE FUNCTION

#### 3.1. Properties

**THEOREM 2** ([8, Thm. 2.1, Chap. 7, pp. 337–338]). *Let  $A$  and  $B$  be subsets of  $\mathbf{R}^N$ . Then, the following properties hold:*

- (i)  $A \neq \emptyset$  and  $\mathbb{C}A \neq \emptyset \Leftrightarrow \partial A \neq \emptyset$ .
- (ii) Given  $A$  and  $B$ ,

$$\begin{aligned}A \supset B &\Leftrightarrow b_A \leq b_B \Leftrightarrow \bar{A} \supset \bar{B} \text{ and } \mathbb{C}\bar{A} \subset \mathbb{C}\bar{B}, \\ b_A = b_B &\Leftrightarrow \bar{B} = \bar{A} \text{ and } \mathbb{C}\bar{A} = \mathbb{C}\bar{B} \Leftrightarrow \bar{B} = \bar{A} \text{ and } \partial A = \partial B.\end{aligned}$$

In particular,  $b_{\bar{A}} \leq b_A \leq b_{\text{int } A}$  and

$$b_{\bar{A}} = b_A \Leftrightarrow \partial \bar{A} = \partial A \text{ and } b_{\text{int } A} = b_A \Leftrightarrow \partial \text{int } A = \partial A.$$

- (iii)  $|b_A| = d_A + d_{\mathbb{C}A} = \max\{d_A, d_{\mathbb{C}A}\} = d_{\partial A}$  and  $\partial A = \{x \in \mathbf{R}^N : b_A(x) = 0\}$ .
- (iv)  $b_A \geq 0 \Leftrightarrow \mathbb{C}\bar{A} \supset \partial A \supset \bar{A} \Leftrightarrow \partial A = \bar{A}$ .
- (v)  $b_A = 0 \Leftrightarrow \mathbb{C}\bar{A} = \partial A = \bar{A} \Leftrightarrow \partial A = \mathbf{R}^N$ .

From Theorem 2 (i), the function  $b_A$  is finite at each point if and only if  $\partial A \neq \emptyset$ . This excludes  $A = \emptyset$  and  $A = \mathbf{R}^N$ . The zero function  $b_A(x) = 0$  for all  $x$  in  $\mathbf{R}^N$  corresponds to the equivalence class of sets  $A$  such that  $\partial A = \mathbf{R}^N$ :

$$b_A = 0 \Leftrightarrow d_A = d_{\mathbb{C}A} \Leftrightarrow \bar{A} = \mathbb{C}\bar{A} \Leftrightarrow \bar{A} = \partial A = \mathbb{C}\bar{A} \Leftrightarrow \partial A = \mathbf{R}^N.$$

This class of sets is not empty. For instance, choose the subset  $A = \mathbb{Q}^N$  of points of  $\mathbf{R}^N$  with rational coordinates or the set of all lines parallel to one of the coordinate axes with rational coordinates.

### 3.2. Singularities of $\nabla b_A$ , projections onto $\partial A$ , and $b$ -cracks

We also have a connection between the gradient of  $b_A$  and the projection onto  $\partial A$  and the characteristic functions associated with  $\partial A$ , singularities of the gradients, and the notions of skeleton and  $b$ -cracks.

DEFINITION 2 ([8, Chap. 7, Sec. 3, p. 344]). Let  $A \subset \mathbf{R}^N$ ,  $\emptyset \neq \partial A$ , and the notation

$$\text{Sing}(\nabla b_A) \stackrel{\text{def}}{=} \{x \in \mathbf{R}^N : \nabla b_A(x) \nexists\}$$

for the *set of singularities of the gradient of  $b_A$* ,

$$\Pi_{\partial A}(x) \stackrel{\text{def}}{=} \{z \in \partial A : |z - x| = d_{\partial A}(x)\} = \{z \in \partial A : |z - x| = |b_A(x)|\}$$

for the *set of projections of  $x$  onto  $\partial A$*  (when  $\Pi_{\partial A}(x)$  is a singleton, the unique element is denoted by  $p_{\partial A}(x)$ ), and for the *skeleton* of  $\partial A$ ,

$$\text{Sk}(\partial A) \stackrel{\text{def}}{=} \{x \in \mathbf{R}^N : \Pi_{\partial A}(x) \text{ is not a singleton}\}.$$

Since  $d_{\partial A}(x) = |b_A(x)|$ , from Theorem 1 (ii),  $\nabla b_A^2(x)$  exists if and only if  $\Pi_{\partial A}(x) = \{p_{\partial A}(x)\}$  is a singleton. In that case,

$$p_{\partial A}(x) = x - \frac{1}{2} \nabla b_A^2(x).$$

In general, the functions  $b_A$  and  $d_{\partial A}$  are different and  $\text{Sing}(\nabla b_A)$  is smaller than  $\text{Sing}(\nabla d_{\partial A}) = \text{Sk}(\partial A) \cup \text{Ck}(\partial A)$ . As a result, using the oriented distance function  $b_A$  rather than  $d_{\partial A}$  requires a new definition of the set of cracks with respect to  $b_A$ .

DEFINITION 3 ([8, Def. 3.1, Chap. 7, p. 344]). Given  $A \subset \mathbf{R}^N$ ,  $\emptyset \neq \partial A$ , the *set of  $b$ -cracks*<sup>6</sup> of  $A$  is defined as

$$\text{Ck}_b(A) \stackrel{\text{def}}{=} \text{Sing}(\nabla b_A) \setminus \text{Sk}(\partial A)$$

and the *core boundary* of  $\partial A$  as  $\partial_c A \stackrel{\text{def}}{=} \partial \bar{A} \cap \overline{\partial \mathcal{C}A}$ .

(<sup>6</sup>) It may be interesting to get a fine upper bound of the Hausdorff dimension of  $\text{Ck}_b(A)$  and some information on its topological structure. For instance, the topological structure of the skeleton was studied in A. Lieutier [15]. His result was generalized (and improved) to the Riemannian setting in P. Albano, P. Cannarsa, K. T. Nguyen, and C. Sinestrari [1] and P. Cannarsa, W. Cheng, and A. Fathi [2].

Intuitively, the boundary  $\partial A$  of a set  $A$  is made up of *interior cracks* and *exterior cracks* where  $\nabla b_A(x)$  does not exist and the *core boundary* where  $\nabla b_A(x)$  may or may not exist as illustrated in the next example.

EXAMPLE 3. The set  $A$  in Figure 2 consists of the disk minus the eyes, nose, and mouth (interior cracks  $\text{int } \bar{A} \cap \partial A$ ) plus the rays outside the disk (exterior cracks  $\text{int } \bar{\mathbb{C}A} \cap \partial A$ ). The boundary  $\partial A$  consists of the interior boundary  $\text{int } \bar{A} \cap \partial A$ , the exterior boundary  $\text{int } \bar{\mathbb{C}A} \cap \partial A$ , and the core boundary  $\partial \bar{A} \cap \partial \bar{\mathbb{C}A}$  where  $\nabla b_A(x)$  exists and  $\|\nabla b_A(x)\| = 1$ . Note that  $\text{int } \partial A = \emptyset$  and hence  $\partial A = \partial(\partial A) = \partial \bar{A} \cup \partial \bar{\mathbb{C}A}$ .

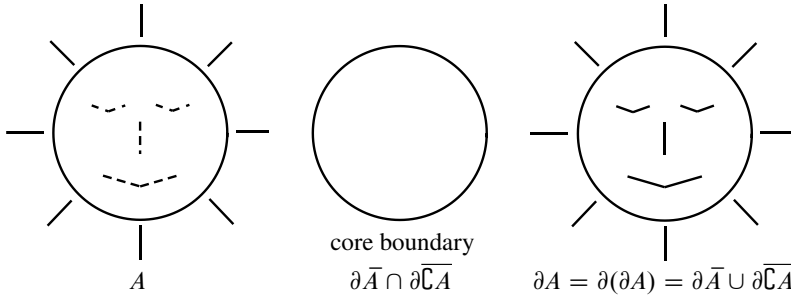


FIGURE 2. Smiling sun  $A$ , internal and external boundaries, *core boundary*, and boundary  $\partial A$ .

REMARK 2. (1) The set of *b-cracks* is larger than what would intuitively be perceived as a set of *cracks*. For instance, for an open square in  $\mathbf{R}^2$ , the four corners are *b-cracks*.

(2) In general,  $\text{Ck}_b(A) \subset \text{Ck}(\partial A)$ , but  $\text{Ck}_b(A)$  can be strictly smaller than  $\text{Ck}(\partial A)$ . Consider the open ball of unit radius  $A = B_1(0)$  in  $\mathbf{R}^N$ :

$$b_A(x) = \|x\| - 1, \quad \nabla b_A(x) = \frac{x}{\|x\|}, \quad D^2 b_A(x)_{ij} = \frac{1}{\|x\|} \left( \delta_{ij} - \frac{x_i}{\|x\|} \frac{x_j}{\|x\|} \right),$$

$$\text{Sing}(\nabla b_A) = \{0\}, \quad \text{Sk}(\partial A) = \{0\}, \quad \text{and} \quad \text{Ck}_b(A) = \emptyset.$$

For all  $x \in \partial A$ ,  $\nabla b_A(x)$  is the outward unit normal at  $x$ , and  $D^2 b_A(x)$  is the second fundamental form with eigenvalue 0 for the subspace spanned by the normal  $\nabla b_A(x)$  and repeated eigenvalues 1 for the tangent space to  $\partial A$  at  $x$  which is orthogonal to  $\nabla b_A(x)$ . But  $d_{\partial A}(x) = \|\nabla b_A(x)\| - 1$ , and

$$\text{Sing}(\nabla d_{\partial A}) = \{0\} \cup \partial B_1(0), \quad \text{Sk}(\partial A) = \{0\}, \quad \text{and} \quad \text{Ck}(\partial A) = \partial B_1(0)$$

as compared to  $\text{Ck}_b(A) = \emptyset$ . Using  $b_A$  rather than  $d_{\partial A}$  removes *artificial singularities*.

(3) In general,  $\text{Sk}(\partial A) \subset \mathbf{R}^N \setminus \partial A$ . If  $\partial A$  has positive reach  $h > 0$  in the sense of Federer [11], there exists  $h > 0$  such that  $b_A^2 \in C^{1,1}(U_h(\partial A))$  and the skeleton will remain at least at a distance  $h$  from  $\partial A$ , where

$$U_h(\partial A) \stackrel{\text{def}}{=} \{x \in \mathbf{R}^N : |b_A(x)| < h\}.$$

In the sequel, the following technical lemma will be useful.

LEMMA 3. *Let  $A$  be a subset of  $\mathbf{R}^N$ .*

- (i)  $\partial(\partial A) = \partial \bar{A} \cup \partial \overline{\mathbb{C}A}$ ,  $\text{int } \partial A = \text{int } \bar{A} \cap \text{int } \overline{\mathbb{C}A}$ , and  $\partial A = [\text{int } \bar{A} \cap \text{int } \overline{\mathbb{C}A}] \cup [\partial \bar{A} \cup \partial \overline{\mathbb{C}A}]$ .
- (ii)  $\partial \bar{A} \subset \partial A$ ,  $\text{int}(\partial \bar{A}) = \emptyset$ , and  $\partial A = \partial \bar{A} \cup (\text{int } \bar{A} \cap \partial A)$ .
- (iii)  $\partial \overline{\mathbb{C}A} \subset \partial A$ ,  $\text{int}(\partial \overline{\mathbb{C}A}) = \emptyset$ , and  $\partial A = \partial \overline{\mathbb{C}A} \cup (\text{int } \overline{\mathbb{C}A} \cap \partial A)$ .
- (iv)  $\text{int } \partial(\partial A) \subset \text{int } \partial \bar{A} \cup \text{int } \partial \overline{\mathbb{C}A} = \emptyset$ .
- (v) *In general,  $\overline{\text{int } A} = \overline{\text{int}(\text{int } A)}$ . For an open subset  $\Omega$ ,  $\Omega = \text{int } \Omega$  and  $\bar{\Omega} = \overline{\text{int}(\bar{\Omega})}$ .*

PROOF. (i) By definition,  $\partial A = \text{int } \partial A \cup \partial(\partial A)$  and

$$\begin{aligned}
 (5) \quad \partial(\partial A) &= \partial A \cap \overline{\mathbb{C}\partial A} = \bar{A} \cap \overline{\mathbb{C}A} \cap \overline{\mathbb{C}(\bar{A} \cap \overline{\mathbb{C}A})} \\
 &= \bar{A} \cap \overline{\mathbb{C}A} \cap \overline{(\mathbb{C}\bar{A} \cup \overline{\mathbb{C}\mathbb{C}A})} = \bar{A} \cap \overline{\mathbb{C}A} \cap (\overline{\mathbb{C}\bar{A}} \cup \overline{\mathbb{C}\mathbb{C}A}) \\
 &= (\bar{A} \cap \overline{\mathbb{C}\bar{A}}) \cup (\overline{\mathbb{C}A} \cap \overline{\mathbb{C}\mathbb{C}A}) = \partial \bar{A} \cup \partial \overline{\mathbb{C}A}.
 \end{aligned}$$

As for the interior of  $\partial A$ ,  $\text{int } \partial A = \overline{\mathbb{C}\mathbb{C}\partial A} = \overline{\mathbb{C}\mathbb{C}[\bar{A} \cap \overline{\mathbb{C}A}]}$  and

$$\text{int } \partial A = \mathbb{C}(\overline{\mathbb{C}\bar{A}} \cup \overline{\mathbb{C}[\overline{\mathbb{C}A}]}) = \mathbb{C}(\overline{\mathbb{C}\bar{A}} \cup \overline{\mathbb{C}[\overline{\mathbb{C}A}]})) = \overline{\mathbb{C}\mathbb{C}\bar{A}} \cap \overline{\mathbb{C}\mathbb{C}[\overline{\mathbb{C}A}]} = \text{int } \bar{A} \cap \text{int } \overline{\mathbb{C}A}.$$

(iii) Firstly,  $\partial \bar{A} = \bar{A} \cap \overline{\mathbb{C}\bar{A}} \subset \bar{A} \cap \overline{\mathbb{C}A} = \partial A$ . If  $\text{int}(\partial \bar{A}) \neq \emptyset$ , there exists  $x \in \partial \bar{A}$  and  $\varepsilon > 0$  such that  $B_\varepsilon(x) \subset \partial \bar{A} = \bar{A} \cap \overline{\mathbb{C}\bar{A}}$  and this yields the following contradiction:

$$B_\varepsilon(x) \subset \text{int } \bar{A} = \overline{\mathbb{C}\mathbb{C}\bar{A}} \quad \text{and} \quad B_\varepsilon(x) \subset \overline{\mathbb{C}\bar{A}}.$$

Finally,  $\partial A = \partial A \cap \bar{A} = \partial A \cap (\partial \bar{A} \cup \text{int } \bar{A}) = \partial \bar{A} \cup (\text{int } \bar{A} \cap \partial A)$ .

(iv) From (ii) with  $\mathbb{C}A$  in place of  $A$  and the fact that  $\partial A = \partial \mathbb{C}A$ .

(v) From the definition  $\text{int } \partial(\partial A) = \overline{\mathbb{C}\mathbb{C}\partial(\partial A)} = \overline{\mathbb{C}\mathbb{C}[\partial \bar{A} \cup \partial \overline{\mathbb{C}A}]}$ . Then,

$$\begin{aligned}
 \text{int } \partial(\partial A) &= \overline{\mathbb{C}[\overline{\mathbb{C}\partial \bar{A}}] \cap [\overline{\mathbb{C}\partial \overline{\mathbb{C}A}}]} \subset \overline{\mathbb{C}[\overline{\mathbb{C}\partial \bar{A}}] \cap [\overline{\mathbb{C}\partial \overline{\mathbb{C}A}}]} \\
 &= \overline{\mathbb{C}[\overline{\mathbb{C}\partial \bar{A}}]} \cup \overline{\mathbb{C}[\overline{\mathbb{C}\partial \overline{\mathbb{C}A}}]} = \text{int } \partial \bar{A} \cup \text{int } \partial \overline{\mathbb{C}A} = \emptyset
 \end{aligned}$$

from parts (ii) and (iii).

(vi) It is sufficient to prove it for an open set  $\Omega$  and apply it to  $\text{int } A$

$$\Omega \subset \bar{\Omega} \implies \left\{ \begin{array}{l} \mathbb{C}\Omega \supset \overline{\mathbb{C}\bar{\Omega}} \implies \mathbb{C}\Omega \supset \overline{\mathbb{C}\bar{\Omega}} \implies \Omega \subset \overline{\mathbb{C}\mathbb{C}\bar{\Omega}} \\ \mathbb{C}\bar{\Omega} \subset \overline{\mathbb{C}\bar{\Omega}} \implies \bar{\Omega} \supset \overline{\mathbb{C}\mathbb{C}\bar{\Omega}} \end{array} \right\} \implies \Omega \subset \underbrace{\overline{\mathbb{C}\mathbb{C}\bar{\Omega}}}_{=\text{int } \bar{\Omega}} \subset \bar{\Omega}$$

and, after taking the closures, we get the equality of the closures. In particular, for an open subset  $\Omega$  of  $\mathbf{R}^N$ ,  $\Omega = \text{int } \Omega$  and  $\bar{\Omega} = \overline{\text{int}(\bar{\Omega})}$ . ■

4. PARTITION OF THE BOUNDARIES  $\partial(\partial A)$  AND  $\partial A$ 

We now introduce partitions of the boundaries  $\partial(\partial A)$  and  $\partial A$ , which reveals their structure and the way the  $b$ -cracks are distributed. Recall from Lemma 3 (iv) that  $\partial(\partial A)$ ,  $\partial \bar{A}$ , and  $\text{int } \partial \bar{\mathbb{C}}A$  as nowhere dense sets have an empty interior [10, p. 250].

THEOREM 4. *Let  $A \subset \mathbf{R}^N$  such that  $\partial A \neq \emptyset$ .*

(i) *The set  $\partial(\partial A)$  is partitioned as follows:*

$$(6) \quad \begin{aligned} \partial(\partial A) &= [\partial(\partial A) \cap \text{int } \bar{A}] \cup [\partial(\partial A) \cap \text{int } \bar{\mathbb{C}}A] \cup [\partial \bar{A} \cap \partial \bar{\mathbb{C}}A] \\ &= [(\text{int } \bar{A} \cup \text{int } \bar{\mathbb{C}}A) \cap \partial(\partial A)] \cup [\partial \bar{A} \cap \partial \bar{\mathbb{C}}A]. \end{aligned}$$

(ii) *The set  $\partial A$  is partitioned as follows:*

$$(7) \quad \begin{aligned} \partial A &= [(\text{int } \bar{A} \cup \text{int } \bar{\mathbb{C}}A) \cap \partial A] \cup [\partial \bar{A} \cap \partial \bar{\mathbb{C}}A] \\ &= \underbrace{\text{int } \bar{A} \cap \text{int } \bar{\mathbb{C}}A}_{\text{int } \partial A} \cup \underbrace{[(\text{int } \bar{A} \cup \text{int } \bar{\mathbb{C}}A) \cap \partial(\partial A)] \cup [\partial \bar{A} \cap \partial \bar{\mathbb{C}}A]}_{\partial(\partial A) = \partial \bar{A} \cup \partial \bar{\mathbb{C}}A}. \end{aligned}$$

(iii) *For an open subset  $\Omega$  of  $\mathbf{R}^N$ ,  $\text{int } \partial \Omega = \emptyset$ ,  $\partial(\partial \Omega) = \partial \Omega$ , and*

$$(8) \quad \begin{aligned} \partial \Omega &= [\partial \Omega \cap \text{int } \bar{\Omega}] \cup \underbrace{[\partial \Omega \cap \text{int}(\bar{\mathbb{C}}\Omega)]}_{=\emptyset} \cup \underbrace{[\partial \bar{\Omega} \cap \partial(\bar{\mathbb{C}}\Omega)]}_{=\partial \bar{\Omega}} \\ &= [\partial \Omega \cap \text{int } \bar{\Omega}] \cup \partial \bar{\Omega}. \end{aligned}$$

(iv)  *$\partial A = \partial \bar{A}$  if and only if  $\text{int } \partial A = \emptyset$  and  $\partial \bar{A} \supset \partial(\partial A)$ ;  $\partial A = \partial \text{int } A$  if and only if  $\text{int } \partial A = \emptyset$  and  $\partial \bar{\mathbb{C}}A \supset \partial(\partial A)$ ; in both cases,  $\partial A = \partial(\partial A)$ .*

PROOF OF THEOREM 4. (i) Consider the following identity:

$$\partial(\partial A) = [\partial(\partial A) \cap \text{int } \bar{A}] \cup [\partial(\partial A) \cap \text{int } \bar{\mathbb{C}}A] \cup \underbrace{[\partial(\partial A) \setminus (\text{int } \bar{A} \cup \text{int } \bar{\mathbb{C}}A)]}_{\stackrel{\text{def}}{=} X}.$$

It is a partition of  $\partial(\partial A)$  since  $\text{int } \bar{\mathbb{C}}A \cap \text{int } \bar{A} = \emptyset$ ,  $X \cap \text{int } \bar{A} = \emptyset$ , and  $X \cap \text{int } \bar{\mathbb{C}}A = \emptyset$ . By direct computation,

$$X = \partial(\partial A) \setminus (\text{int } \bar{A} \cup \text{int } \bar{\mathbb{C}}A) = \partial(\partial A) \cap [\bar{\mathbb{C}}A \cap \overline{\bar{\mathbb{C}}A}].$$

Note that  $\partial \bar{A} = \bar{A} \cap \bar{\mathbb{C}}A$  and  $\partial \bar{\mathbb{C}}A = \bar{\mathbb{C}}A \cap \overline{\bar{\mathbb{C}}A}$  yields

$$\partial \bar{A} \cap \partial \bar{\mathbb{C}}A = \partial A \cap \bar{\mathbb{C}}A \cap \overline{\bar{\mathbb{C}}A}$$

and

$$\partial A \cap \partial(\partial A) = \bar{A} \cap \bar{\mathbb{C}}A \cap \partial A \cap \bar{\mathbb{C}}\partial A = \partial A \cap \bar{\mathbb{C}}\partial A = \partial(\partial A).$$

Finally, from Lemma 3 (i),  $\partial(\partial A) = (\partial \bar{A} \cup \partial \overline{\mathbb{C}A})$ ,

$$\begin{aligned} \partial(\partial A) \cap \partial \bar{A} \cap \partial \overline{\mathbb{C}A} &= \partial A \cap \overline{\mathbb{C}A} \cap \overline{\mathbb{C}\overline{\mathbb{C}A}} \\ \implies X &= \partial(\partial A) \cap \partial \bar{A} \cap \partial \overline{\mathbb{C}A} = (\partial \bar{A} \cup \partial \overline{\mathbb{C}A}) \cap \partial \bar{A} \cap \partial \overline{\mathbb{C}A} = \partial \bar{A} \cap \partial \overline{\mathbb{C}A}. \end{aligned}$$

(ii) The second equation is a direct consequence of part (i) and Lemma 3 (i). For the first equation,

$$\begin{aligned} (\text{int } \bar{A} \cup \text{int } \overline{\mathbb{C}A}) \cap \partial A &= (\text{int } \bar{A} \cup \text{int } \overline{\mathbb{C}A}) \cap [\partial(\partial A) \cup \text{int } \partial A] \\ &= [(\text{int } \bar{A} \cup \text{int } \overline{\mathbb{C}A}) \cap \partial(\partial A)] \cup [(\text{int } \bar{A} \cup \text{int } \overline{\mathbb{C}A}) \cap \text{int } \partial A]. \end{aligned}$$

But since  $\text{int } \partial A = \text{int } \bar{A} \cap \text{int } \overline{\mathbb{C}A}$ ,

$$\begin{aligned} (\text{int } \bar{A} \cup \text{int } \overline{\mathbb{C}A}) \cap \partial A &= [(\text{int } \bar{A} \cup \text{int } \overline{\mathbb{C}A}) \cap \partial(\partial A)] \\ &\quad \cup [(\text{int } \bar{A} \cap \text{int } \overline{\mathbb{C}A})] \cap [\text{int } \bar{A} \cup \text{int } \overline{\mathbb{C}A}] \\ &= [(\text{int } \bar{A} \cup \text{int } \overline{\mathbb{C}A}) \cap \partial(\partial A)] \cup \underbrace{(\text{int } \bar{A} \cap \text{int } \overline{\mathbb{C}A})}_{=\text{int } \partial A}. \end{aligned}$$

(iii) From (ii) since  $\text{int } \partial \Omega = \text{int } \bar{\Omega} \cap \text{int } (\mathbb{C}\Omega) = \text{int } \bar{\Omega} \cap \mathbb{C}\bar{\Omega} = \emptyset$ .

(iv) For the first assertion. ( $\Leftarrow$ ) From Lemma 3 (i) and (ii),

$$\partial \bar{A} \subset \partial A = \underbrace{\text{int } \partial A}_{=\emptyset} \cup \partial \bar{A} \cup \partial \overline{\mathbb{C}A} = \partial \bar{A} \cup \partial \overline{\mathbb{C}A} = \underbrace{\partial(\partial A)}_{\text{by assumption}} \subset \partial \bar{A}$$

and, as a result,  $\partial \bar{A} = \partial A = \partial \bar{A} \cup \partial \overline{\mathbb{C}A} = \partial(\partial A)$ . ( $\Rightarrow$ ) From Lemma 3 (ii),  $\text{int}(\partial \bar{A}) = \emptyset$ ,

$$\begin{aligned} \partial(\partial A) &= \underbrace{\partial(\partial \bar{A})}_{\partial \bar{A} = \partial A \text{ by assumption}} \cup \underbrace{\text{int } \partial \bar{A}}_{=\emptyset} = \underbrace{\partial \bar{A}}_{\text{by assumption}} = \partial A = \text{int } \partial A \cup \partial(\partial A). \end{aligned}$$

This implies that  $\text{int } \partial A = \emptyset$  and  $\partial(\partial A) \supset \partial \bar{A}$ . The same proof holds for the other part since  $\partial \text{int } A = \partial \overline{\mathbb{C}A}$  and  $\partial A = \partial \mathbb{C}A$ . ■

## 5. PARTITION OF THE SINGULARITIES OF $\nabla b_A$

**THEOREM 5.** *Let  $A \subset \mathbf{R}^N$  be such that  $\partial A \neq \emptyset$ .*

(i) *The skeleton and the b-cracks can be characterized as follows:*

$$\begin{aligned} \text{Sk}(\partial A) &= \{x \in \mathbf{R}^N : \nabla b_A^2(x) \nexists\} = \{x \in \mathbf{R}^N \setminus \partial A : \nabla b_A(x) \nexists \text{ and } \nabla b_A^2(x) \exists\}, \\ \text{Sing}(\nabla b_A) &= \text{Sk}(\partial A) \cup \{x \in \partial(\partial A) : \nabla b_A(x) \nexists\}, \\ \text{Ck}_b(A) &= \text{Sing}(\nabla b_A) \setminus \text{Sk}(\partial A) = \{x \in \partial(\partial A) : \nabla b_A(x) \exists\}. \end{aligned}$$

(ii) The set of singularities of  $\nabla b_A$  can be decomposed as follows:

$$\begin{aligned} \text{Sing}(\nabla b_A) &= \text{Sk}(\partial A) \cup \{x \in (\text{int } \bar{A} \cup \text{int } \bar{\mathbb{C}A}) \cap \partial(\partial A) : \nabla b_A(x) \nexists\} \\ &\quad \cup \{x \in \partial \bar{A} \cap \partial \bar{\mathbb{C}A} : \nabla b_A(x) \nexists\} \end{aligned}$$

and

$$\begin{aligned} (\text{int } \bar{\mathbb{C}A} \cup \text{int } \bar{A}) \cap \partial(\partial A) &= \{x \in (\text{int } \bar{\mathbb{C}A} \cup \text{int } \bar{A}) \cap \partial(\partial A) : \nabla b_A(x) \nexists\} \\ &\quad \cup \{x \in (\text{int } \bar{\mathbb{C}A} \cup \text{int } \bar{A}) \cap \partial(\partial A) : \nabla b_A(x) = 0\}, \end{aligned}$$

where the second set corresponds to cusps such that  $\nabla d_A(x) = \nabla d_{\mathbb{C}A}(x) = 0$ .

PROOF. (i) By (3) in Theorem 1 (ii) with  $\partial A$  instead of  $A$  and the fact that  $d_{\partial A}^2 = b_A^2$ ,

$$\text{Sk}(\partial A) = \{x \in \mathbf{R}^N : \nabla b_A^2(x) \nexists\}.$$

But, since if  $\nabla b_A(x)$  exists, then  $\nabla b_A^2(x) = 2b_A(x)\nabla b_A(x)$  exists,

$$\text{Sk}(\partial A) = \{x \in \mathbf{R}^N : \nabla b_A^2(x) \nexists\} = \{x \in \mathbf{R}^N : \nabla b_A(x) \nexists \text{ and } \nabla b_A^2(x) \nexists\}.$$

The set of singularities of  $\nabla b_A$  can be decomposed as

$$\begin{aligned} \text{Sing}(\nabla b_A) &= \underbrace{\{x \in \mathbf{R}^N : \nabla b_A(x) \nexists \text{ and } \nabla b_A^2(x) \nexists\}}_{=\text{Sk}(\partial A)} \\ &\quad \cup \{x \in \mathbf{R}^N : \nabla b_A(x) \nexists \text{ and } \nabla b_A^2(x) \exists\}. \end{aligned}$$

In the second term, if  $\nabla b_A^2(x) \neq 0$ , then  $b_A(x) \neq 0$  and  $\nabla b_A(x)$  exists. From this contradiction,  $\nabla d_{\partial A}^2(x) = \nabla b_A^2(x) = 0$  and  $x \in \partial A$ . Therefore,

$$\text{Sing}(\nabla b_A) = \text{Sk}(\partial A) \cup \{x \in \partial A : \nabla b_A(x) \nexists\}.$$

Furthermore,  $\partial A = \partial(\partial A) \cup \text{int } \partial A$  and  $\nabla b_A(x) = 0$  for  $x \in \text{int } \partial A$ . Hence,

$$\begin{aligned} \text{Sing}(\nabla b_A) &= \text{Sk}(\partial A) \cup \{x \in \partial(\partial A) : \nabla b_A(x) \nexists\}, \\ \text{Ck}_b(A) &= \{x \in \partial(\partial A) : \nabla b_A(x) \nexists\}. \end{aligned}$$

(ii) Recall from Theorem 4 (i) the partition of  $\partial(\partial A)$ :

$$\partial(\partial A) = [(\text{int } \bar{A} \cup \text{int } \bar{\mathbb{C}A}) \cap \partial(\partial A)] \cup [\partial \bar{A} \cap \partial \bar{\mathbb{C}A}],$$

which implies that

$$\begin{aligned} &\{x \in \partial(\partial A) : \nabla b_A(x) \nexists\} \\ &= \{x \in (\text{int } \bar{A} \cup \text{int } \bar{\mathbb{C}A}) \cap \partial(\partial A) : \nabla b_A(x) \nexists\} \cup \{x \in \partial \bar{A} \cap \partial \bar{\mathbb{C}A} : \nabla b_A(x) \nexists\}. \end{aligned}$$

For the first term,

$$(\text{int } \bar{A} \cup \text{int } \bar{\mathcal{C}A}) \cap \partial(\partial A) = (\text{int } \bar{A} \cap \partial \bar{\mathcal{C}A}) \cup (\text{int } \bar{\mathcal{C}A} \cap \partial \bar{A}).$$

If  $x \in \text{int } \bar{A} \cap \partial \bar{\mathcal{C}A}$  and  $\nabla b_A(x)$  exists, then  $\nabla d_A(x) = 0$  and  $\nabla d_{\mathcal{C}A}(x)$  exists. But from the characterization of  $\partial \bar{\mathcal{C}A}$  in (4),

$$\partial \bar{\mathcal{C}A} = \{x \in \partial \bar{\mathcal{C}A} : \nabla d_{\mathcal{C}A}(x) \neq 0\} \cup \{x \in \partial \bar{\mathcal{C}A} : \nabla d_{\mathcal{C}A}(x) = 0\}.$$

Then,  $\nabla d_{\mathcal{C}A}(x) = 0$  and

$$\{x \in \text{int } \bar{A} \cap \partial \bar{\mathcal{C}A} : \nabla b_A(x) \neq 0\} = \{x \in \text{int } \bar{A} \cap \partial \bar{\mathcal{C}A} : \nabla d_{\mathcal{C}A}(x) = 0 \text{ and } \nabla d_A(x) = 0\}.$$

Similarly, for the other term  $\text{int } \bar{\mathcal{C}A} \cap \partial \bar{A}$ ,

$$\{x \in \text{int } \bar{\mathcal{C}A} \cap \partial \bar{A} : \nabla b_A(x) \neq 0\} = \{x \in \text{int } \bar{\mathcal{C}A} \cap \partial \bar{A} : \nabla d_A(x) = 0 \text{ and } \nabla d_{\mathcal{C}A}(x) = 0\}$$

and combining the two identities,

$$\begin{aligned} & (\text{int } \bar{\mathcal{C}A} \cup \text{int } \bar{A}) \cap \partial(\partial A) \\ &= \{x \in (\text{int } \bar{\mathcal{C}A} \cup \text{int } \bar{A}) \cap \partial(\partial A) : \nabla b_A(x) \neq 0\} \\ & \quad \cup \{x \in (\text{int } \bar{\mathcal{C}A} \cup \text{int } \bar{A}) \cap \partial(\partial A) : \nabla d_A(x) = \nabla d_{\mathcal{C}A}(x) = 0\} \\ &= \{x \in (\text{int } \bar{\mathcal{C}A} \cup \text{int } \bar{A}) \cap \partial(\partial A) : \nabla b_A(x) \neq 0\} \\ & \quad \cup \{x \in (\text{int } \bar{\mathcal{C}A} \cup \text{int } \bar{A}) \cap \partial(\partial A) : \nabla b_A(x) = 0\}. \end{aligned}$$

Finally, the set of singularities of  $\nabla b_A$  can be decomposed as follows:

$$\begin{aligned} \text{Sing}(\nabla b_A) &= \text{Sk}(\partial A) \cup \{x \in \partial(\partial A) : \nabla b_A(x) \neq 0\} \\ &= \text{Sk}(\partial A) \cup \{x \in (\text{int } \bar{A} \cup \text{int } \bar{\mathcal{C}A}) \cap \partial(\partial A) : \nabla b_A(x) \neq 0\} \\ & \quad \cup \{x \in \partial \bar{A} \cap \partial \bar{\mathcal{C}A} : \nabla b_A(x) \neq 0\}. \end{aligned} \quad \blacksquare$$

To illustrate Theorem 5, we conclude with the following example.

**EXAMPLE 4.** Let  $B_+ = B_1(1, 0)$  and  $B_- = B_1(-1, 0)$  and  $A = \mathbb{Q}^2 \cap (B_+ \cup B_-)$  as shown in Figure 3. Then, we have the following properties:

$$\begin{aligned} \text{int } A &= \emptyset, \quad \bar{A} = \bar{B}_+ \cup \bar{B}_-, \quad \bar{\mathcal{C}A} = \mathbf{R}^2, \quad \partial A = \bar{A}, \\ \partial(\partial A) &= \partial \bar{A} = \partial B_+ \cup \partial B_-, \quad \text{int } \bar{A} = \text{int } B_+ \cup \text{int } B_-, \quad \text{int } \bar{\mathcal{C}A} = \mathbf{R}^2 \\ &\implies \partial(\partial A) \cap (\text{int } \bar{A} \cup \text{int } \bar{\mathcal{C}A}) = \partial \bar{A} = \partial B_+ \cup \partial B_- \text{ and } \partial \bar{A} \cap \partial \bar{\mathcal{C}A} = \emptyset. \end{aligned}$$

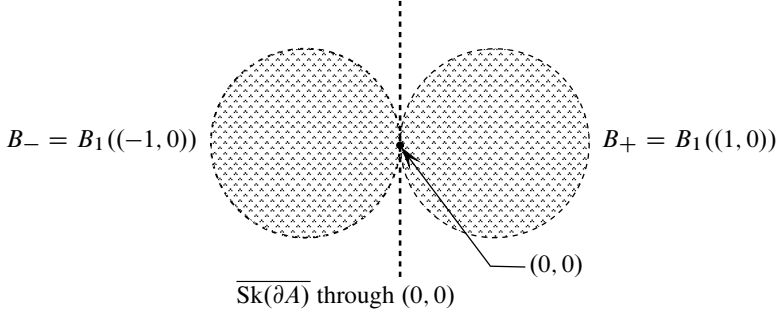


FIGURE 3. Tangent sets  $B_1(1, 0) \cap \mathbb{Q}^2$  and  $B_1(-1, 0) \cap \mathbb{Q}^2$  at  $(0, 0)$  showing part of  $\text{Sk}(\partial A) = \{(0, x_2) : x_2 \neq 0\} \cup \{(-1, 0), (1, 0)\}$ .

Moreover,  $d_{\mathbb{C}A} = 0$ ,  $b_A = d_A = d_{B_+ \cup B_-}$ ,  $\nabla d_A(0, 0) = \nabla d_{\mathbb{C}A}(0, 0) = 0$ , and

$$\begin{aligned} \text{Sk}(\partial A) &= \{(0, x_2) : x_2 \neq 0\} \cup \{(-1, 0), (1, 0)\}, \quad \overline{\text{Sk}(\partial A)} = \text{Sk}(\partial A) \cup \{(0, 0)\}, \\ \text{Ck}_b(A) &= \{x \in \partial(\partial A) : \nabla b_A(x) \nexists\} = (\partial B_+ \cup \partial B_-) \setminus \{(0, 0)\}, \\ \text{Sing}(\nabla b_A) \setminus \overline{\text{Sk}(\partial A)} &= (\partial B_+ \cup \partial B_-) \setminus \{(0, 0)\}. \end{aligned}$$

To be compared with,

$$\begin{aligned} \partial(\partial A) \cap (\text{int } \bar{A} \cup \text{int } \overline{\mathbb{C}A}) &= \partial \bar{A} = \partial B_+ \cup \partial B_-, \\ \{x \in \partial(\partial A) \cap (\text{int } \bar{A} \cup \text{int } \overline{\mathbb{C}A}) : \nabla b_A(x) \exists\} &= \{(0, 0)\}, \\ \{x \in \partial(\partial A) \cap (\text{int } \bar{A} \cup \text{int } \overline{\mathbb{C}A}) : \nabla b_A(x) \nexists\} &= (\partial B_+ \cup \partial B_-) \setminus \{(0, 0)\} = \text{Ck}_b(A). \end{aligned}$$

## 6. RELATIONS TO THE ORIENTED DISTANCE FUNCTION

The partitions of  $\partial A$  in Theorem 4 indicate that the two potential candidates for the definition of the *set of cracks* of a subset  $A$  of  $\mathbf{R}^N$  could be

$$(9) \quad (\text{int } \bar{A} \cup \text{int } \overline{\mathbb{C}A}) \cap \partial A \quad \text{or} \quad (\text{int } \bar{A} \cup \text{int } \overline{\mathbb{C}A}) \cap \partial(\partial A).$$

In general, the first one is strictly larger than the second which will not be a good candidate.

EXAMPLE 5. Let  $B$  be the open unit ball of center 0 in  $\mathbf{R}^2$  and  $A = B \cap \mathbb{Q}^2$ . Then,

$$\begin{aligned} \bar{A} &= \bar{B}, \quad \overline{\mathbb{C}A} = \mathbf{R}^2, \quad \partial A = \bar{B}, \quad \partial(\partial A) = \partial B, \quad \text{int } \bar{A} = B, \quad \text{int } \overline{\mathbb{C}A} = \mathbb{C}\bar{B} \\ \implies \begin{cases} (\text{int } \bar{A} \cup \text{int } \overline{\mathbb{C}A}) \cap \partial A &= (B \cup \mathbb{C}\bar{B}) \cap \bar{B} = B \neq \emptyset, \\ (\text{int } \bar{A} \cup \text{int } \overline{\mathbb{C}A}) \cap \partial(\partial A) &= (B \cup \mathbb{C}\bar{B}) \cap \partial B = \emptyset. \end{cases} \end{aligned}$$

This is a pathological case where  $\text{int } \partial A = B \neq \emptyset$ .

It will turn out that the candidates (9) are both too large and that the right choice for the *set of cracks* is

$$\{x \in (\text{int } \bar{A} \cup \text{int } \overline{\mathbb{C}A}) \cap \partial(\partial A) : \nabla b_A(x) \nexists\}.$$

Yet for a closed or an open set, the three coincide.

Nevertheless, what is specially interesting is that when the first one is empty, the set can be completely characterized in terms of the oriented distance function.

LEMMA 6. *Let  $A \subset \mathbf{R}^N$  such that  $\partial A \neq \emptyset$ . Then,*

- (i)  $\text{int } A \subset \text{int } \bar{A}$  and  $\partial \bar{A} \subset \partial A$ ,
- (ii)  $\partial A = \partial \mathbb{C}A$  and  $\partial \bar{A} = \partial \mathbb{C}\bar{A}$ .

PROOF. (i)  $\text{int } \bar{A} = \text{int } \bar{A} \cup \text{int } \bar{A} = \bar{A} = \text{int } A \cup \partial A$ ,  $\text{int } A \subset \text{int } \bar{A}$ , and  $\partial \bar{A} \subset \partial A$ .

(ii) By definition,

$$\partial A = \bar{A} \cap \overline{\mathbb{C}A} = \overline{\mathbb{C}(\mathbb{C}A)} \cap \overline{\mathbb{C}A} = \partial \mathbb{C}A.$$

Similarly,  $\partial \bar{A} = \partial \mathbb{C}\bar{A}$ . ■

THEOREM 7. *Let  $A \subset \mathbf{R}^N$  such that  $\partial A \neq \emptyset$ .*

- (i) *The following statements are equivalent:*
  - (a)  $\text{int } \bar{A} \cap \partial A = \emptyset$ .
  - (b)  $\partial A = \partial \bar{A}$  (which is equivalent to  $\text{int } A = \text{int } \bar{A}$ ).
  - (c)  $d_{\mathbb{C}A} = d_{\mathbb{C}\bar{A}}$ .
  - (d)  $b_A = b_{\bar{A}}$ .
- (ii) *The following statements are equivalent:*
  - (a)  $\text{int } \overline{\mathbb{C}A} \cap \partial A = \emptyset$ .
  - (b)  $\partial A = \partial \text{int } A$  (which is equivalent to  $\partial \mathbb{C}A = \partial \overline{\mathbb{C}A}$ ).
  - (c)  $d_A = d_{\text{int } A}$  (which is equivalent to  $d_{\mathbb{C}A} = d_{\overline{\mathbb{C}A}}$ ).
  - (d)  $b_A = b_{\text{int } A}$ .
- (iii) *The following statements are equivalent:*
  - (a)  $(\text{int } \bar{A} \cup \text{int } \overline{\mathbb{C}A}) \cap \partial A = \emptyset$ .
  - (b)  $\partial A = \partial \bar{A} = \partial \text{int } A (= \partial \overline{\mathbb{C}A})$ .
  - (c)  $d_{\mathbb{C}\bar{A}} = d_{\mathbb{C}A}$  and  $d_A = d_{\text{int } A}$ .
  - (d)  $b_{\text{int } A} = b_A = b_{\bar{A}}$ .

REMARK 3. From (iii) (b),  $\partial A = \partial \bar{A} = \partial \overline{\mathbb{C}A} = \partial \bar{A} \cap \partial \overline{\mathbb{C}A} = \partial(\partial A)$ . Open or closed subsets of  $\mathbf{R}^N$  with a smooth boundary satisfy the conditions of part (iii).

PROOF. (i) (a) $\Rightarrow$ (b)  $\partial A = \partial A \cap \bar{A} = \partial A \cap (\partial \bar{A} \cup \text{int } \bar{A}) = \partial A \cap (\partial \bar{A}) = \partial \bar{A}$ .

(b) $\Rightarrow$ (c)  $\partial \bar{A} = \partial A$  implies  $d_{\partial \bar{A}} = d_{\partial A}$  and

$$d_{\bar{A}} + d_{\mathbb{C}\bar{A}} = d_{\partial \bar{A}} = d_{\partial A} = d_A + d_{\mathbb{C}A} \implies d_{\mathbb{C}\bar{A}} = d_{\mathbb{C}A}.$$

(c) $\Rightarrow$ (d)  $b_{\bar{A}} = d_{\bar{A}} - d_{\mathbb{C}\bar{A}} = d_A - d_{\mathbb{C}A} = b_A \Rightarrow b_{\bar{A}} = b_A$ .

(d) $\Rightarrow$ (a)  $b_A = b_{\bar{A}}$  implies  $d_{\partial A} = |b_A| = |b_{\bar{A}}| = d_{\partial \bar{A}}$  and  $\partial A = \partial \bar{A}$ . Then,

$$\text{int } \bar{A} \cap \partial A = \text{int } \bar{A} \cap \partial \bar{A} = \emptyset.$$

(ii) The same proof follows as in part (i) with  $A$  replaced by  $\mathbb{C}A$  using the fact that  $\partial A = \partial \mathbb{C}A$ .

(iii) is a consequence of the equivalences in (i) and (ii).  $\blacksquare$

THEOREM 8. Let  $A \subset \mathbf{R}^N$  such that  $\partial A \neq \emptyset$ .

(i) The following statements are equivalent:

(a)  $(\text{int } \bar{A} \cup \text{int } \overline{\mathbb{C}A}) \cap \partial A = \emptyset$ .

(b)  $(\text{int } \bar{A} \cup \text{int } \overline{\mathbb{C}A}) \cap \partial(\partial A) = \emptyset$  and  $\text{int } \bar{A} \cap \text{int } \overline{\mathbb{C}A} = \emptyset$ .

(ii) If  $A$  is open or closed, then  $\text{int } \bar{A} \cap \text{int } \overline{\mathbb{C}A} = \emptyset$  and the following statements are equivalent:

(a)  $(\text{int } \bar{A} \cup \text{int } \overline{\mathbb{C}A}) \cap \partial A = \emptyset$ .

(b)  $(\text{int } \bar{A} \cup \text{int } \overline{\mathbb{C}A}) \cap \partial(\partial A) = \emptyset$ .

PROOF. (i) Clearly, (b) $\Rightarrow$ (a). In the other direction, from Lemma 3(i),  $\text{int } \partial A = \text{int } \bar{A} \cap \text{int } \overline{\mathbb{C}A}$  and

$$\begin{aligned} (\text{int } \bar{A} \cup \text{int } \overline{\mathbb{C}A}) \cap \partial A &= (\text{int } \bar{A} \cup \text{int } \overline{\mathbb{C}A}) \cap \underbrace{\text{int } \partial A}_{\text{int } \bar{A} \cap \text{int } \overline{\mathbb{C}A}} \cup (\text{int } \bar{A} \cup \text{int } \overline{\mathbb{C}A}) \cap \partial(\partial A) \\ &= (\text{int } \bar{A} \cap \text{int } \overline{\mathbb{C}A}) \cup (\text{int } \bar{A} \cup \text{int } \overline{\mathbb{C}A}) \cap \partial(\partial A). \end{aligned}$$

(ii) If  $A = \Omega$  is open,  $\overline{\mathbb{C}\Omega} = \mathbb{C}\Omega$  and

$$\begin{aligned} \text{int } \overline{\mathbb{C}\Omega} &= \text{int } \mathbb{C}\Omega = \overline{\mathbb{C}\mathbb{C}\Omega} = \overline{\mathbb{C}\mathbb{C}\mathbb{C}\Omega} = \overline{\mathbb{C}\Omega} \\ \text{int } \bar{\Omega} \cap \text{int } \overline{\mathbb{C}\Omega} &= \text{int } \bar{\Omega} \cap \mathbb{C}\Omega \subset \bar{\Omega} \cap \mathbb{C}\Omega = \emptyset. \end{aligned}$$

If  $A$  is closed,  $\mathbb{C}A$  is open and from the previous case applied to  $\mathbb{C}A$ :

$$\emptyset = \text{int } \overline{\mathbb{C}A} \cap \text{int } \overline{\mathbb{C}\mathbb{C}A} = \text{int } \overline{\mathbb{C}A} \cap \text{int } \bar{A}.$$

Therefore, (a) is equivalent to (b).  $\blacksquare$

## 7. THE SET OF CRACKS AND CRACK-FREE SETS

The terminology *crack-free* can be intuitively associated with a set  $A$  that does not have pieces of its boundary that can be *erased* by the closure of the set and the closure of its complement. For instance, an open set  $\Omega$  in  $\mathbf{R}^N$  whose boundary is a  $C^0$  submanifold of dimension  $N - 1$  is crack-free. That intuitive property rules out pathological sets whose boundary has a non-empty interior and, a priori, sets whose boundary has a non-zero *volume*.

From Theorem 5 (i) and (ii),

$$\begin{aligned} \text{Ck}_b(A) &= \text{Sing}(\nabla b_A) \setminus \text{Sk}(\partial A) \\ &= \{x \in \partial \bar{A} \cap \partial \overline{\mathcal{C}A} : \nabla b_A(x) \neq 0\} \cup \{x \in (\text{int } \bar{A} \cup \text{int } \overline{\mathcal{C}A}) \cap \partial(\partial A) : \nabla b_A(x) \neq 0\}. \end{aligned}$$

The set  $\partial \bar{A} \cap \partial \overline{\mathcal{C}A}$  is the *core boundary* of  $\partial(\partial A)$  which can contain singularities at *corners*. This means that  $\text{Ck}_b(A)$  is too large as a candidate for the notion of crack.

The other potential candidate from Theorem 7 (iii)

$$(\text{int } \bar{A} \cup \text{int } \overline{\mathcal{C}A}) \cap \partial A$$

is even larger but, when it is empty, it can be characterized by oriented distance functions and  $\partial A$  reduces to the core boundary

$$\partial A = \partial(\partial A) = \partial \bar{A} = \partial \overline{\mathcal{C}A} = \partial \bar{A} \cap \partial \overline{\mathcal{C}A}.$$

This brings up the following notions.

DEFINITION 4. Let  $A \subset \mathbf{R}^N$  such that  $\partial A \neq \emptyset$ .

(i) The *set of cracks* of  $A$  which splits into the *interior cracks* and the *exterior cracks*:

$$\begin{aligned} &\{x \in (\text{int } \bar{A} \cup \text{int } \overline{\mathcal{C}A}) \cap \partial(\partial A) : \nabla b_A(x) \neq 0\} \\ &= \{x \in \text{int } \bar{A} \cap \partial \overline{\mathcal{C}A} : \nabla b_A(x) \neq 0\} \cup \{x \in \text{int } \overline{\mathcal{C}A} \cap \partial \bar{A} : \nabla b_A(x) \neq 0\}. \end{aligned}$$

(ii) The set  $A$  is *crack-free* if

$$\{x \in (\text{int } \bar{A} \cup \text{int } \overline{\mathcal{C}A}) \cap \partial(\partial A) : \nabla b_A(x) \neq 0\} = \emptyset.$$

(iii) The set  $A$  is *strongly crack-free* if

$$(\text{int } \bar{A} \cup \text{int } \overline{\mathcal{C}A}) \cap \partial A = \emptyset.$$

Necessary and sufficient conditions for a strongly crack-free set are now given by any of the equivalent conditions of Theorem 7 (iii) which simplifies for open or closed sets by Theorem 8 (ii).

**THEOREM 9.** *Let  $\Omega$  be an open subset of  $\mathbf{R}^N$  such that  $\partial\Omega \neq \emptyset$ . Then, the following conditions are equivalent:*

- (a)  $\Omega$  is strongly crack-free;
- (b)  $\partial\Omega = \partial\bar{\Omega}$ ;
- (c)  $\Omega = \text{int } \bar{\Omega}$ ;
- (d)  $b_\Omega = b_{\bar{\Omega}}$  in  $\mathbf{R}^N$ .

Moreover,  $\text{int } \partial\Omega = \emptyset$ ,  $\partial(\partial\Omega) = \partial\Omega$ , and the set of  $b$ -cracks corresponds to the set  $\{x \in \partial\Omega : \nabla b_\Omega(x) \neq 0\}$ .

**REMARK 4.** In 1994, Henrot [13] introduced the terminology *Carathéodory set* for open sets which are *strongly crack-free* (in our terminology).

It was later adopted by Tiba [21], Neittaanmaki, Sprekels, and Tiba [18], and Henrot and Pierre [14]. However, this terminology does not seem to be standard.

For instance, in the literature on polynomial approximations in the complex plane  $\mathbb{C}$ , a Carathéodory set is defined as follows.

**DEFINITION 5** (Dovgoshey [9] or Gaier [12]). A bounded subset  $A$  of  $\mathbb{C}$  is said to be a *Carathéodory set* if the boundary of  $A$  coincides with the boundary of the unbounded component of the complement of  $\bar{A}$ . A *Carathéodory domain* is a Carathéodory set if, in addition,  $A$  is simply connected.

In V. A. Martirosian and S. E. Mkrtchyan [16],  $A$  is further assumed to be measurable. This definition excludes not only interior cracks but also bounded holes inside the set  $A$  as can be seen from the example of the annulus  $A = \{x \in \mathbf{R}^2 : 1 < |x| < 2\}$  in  $\mathbf{R}^2$ . So, it is more restrictive than  $\partial A = \partial\bar{A}$ .

To avoid ambiguities, we chose the intuitive terminology *strongly crack-free*.

**THEOREM 10.** *Let  $A$  be a closed subset of  $\mathbf{R}^N$  such that  $\partial A \neq \emptyset$ . The following conditions are equivalent:*

- (a)  $A$  is strongly crack-free;
- (b)  $\partial A = \partial\bar{\mathcal{C}A} (= \partial(\text{int } A))$ ;
- (c)  $\text{int } \mathcal{C}A = \text{int } \bar{\mathcal{C}A}$ ;
- (d)  $b_{\text{int } A} = b_A : \mathbf{R}^N \rightarrow \mathbf{R}$ .

Moreover,  $\text{int } \partial A = \emptyset$ ,  $\partial(\partial A) = \partial A$ , and the set of  $b$ -cracks corresponds to the set  $\{x \in \partial A : \nabla b_A(x) \neq 0\}$ .

To complete the picture, we provide a last example which further illustrates some of the subtleties of the various notions introduced in this paper.

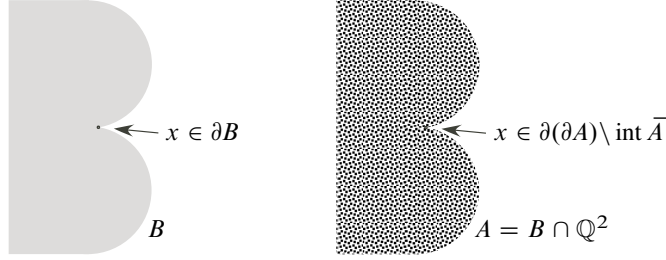


FIGURE 4. Open set  $B$  (left) and set  $A = B \cap \mathbb{Q}^2$  of points of  $B$  with rational coordinates (right).

EXAMPLE 6. Let  $B$  be the open subset to the left in Figure 4:

$$B \stackrel{\text{def}}{=} \left\{ (x, y) : \begin{array}{l} -3 < x \leq 0, -2 < y < 2 \\ 0 < x < 1, x^2 + (y-1)^2 < 1 \\ 0 < x < 1, x^2 + (y+1)^2 < 1 \end{array} \right\}.$$

$\nabla d_B(x)$  exists everywhere except at the corners  $(-3, \pm 2)$ . This is obvious except at the point  $x = (0, 0) \in \partial B$  where the boundary  $\partial B$  has a cusp. Indeed, for  $v \neq (1, 0)$  and  $t > 0$  small,

$$\frac{d_B((0, 0) + tv) - d_B(0, 0)}{t} = 0 \implies dd_B(0, 0; v) = 0;$$

for  $v = (1, 0)$ , and  $t > 0$  small,

$$\begin{aligned} \frac{d_B((0, 0) + t(1, 0)) - d_B(0, 0)}{t} &= \frac{d_B((t, 0))}{t} = \frac{\sqrt{1+t^2} - 1}{t} \rightarrow 0 \\ &\implies dd_B((0, 0); (1, 0)) = 0 \text{ and } \nabla d_B(0, 0) = 0 \end{aligned}$$

since  $d_B$  is Lipschitzian and  $v \mapsto dd_B((0, 0); v)$  is linear [3, Def. 3.4 (iii), p. 1048, Thm. 3.10, p. 1952]. But  $\nabla d_{\mathbb{C}B}(0, 0)$  does not exist since

$$dd_{\mathbb{C}B}((0, 0); \pm(0, 1)) = \lim_{t \searrow 0} \frac{d_{\mathbb{C}B}((0, 0) \pm t(0, 1)) - d_{\mathbb{C}B}((0, 0))}{t} = 1.$$

Define  $A$  as the set of points of  $B$  with rational coordinates, that is,  $A = B \cap \mathbb{Q}^2$ . It is readily seen that

$$\begin{aligned} \text{int } A &= \emptyset, \bar{A} = \bar{B}, \partial A = \bar{A} = \bar{B}, \partial(\partial A) = \partial \bar{A} = \partial B, \text{int } \bar{A} = \text{int } \bar{B}, \\ \text{int } \mathbb{C}A &= \mathbb{C}\bar{B}, \mathbb{C}A = \mathbf{R}^2, \partial \mathbb{C}A = \emptyset, \text{int } \mathbb{C}A = \mathbf{R}^2 \\ &\implies d_{\mathbb{C}A} = 0 \text{ and } d_A = d_B \implies b_A = d_B \\ &\implies \nabla d_A(0, 0) = \nabla d_B(0, 0) = 0 \implies \nabla b_A(0, 0) = 0. \end{aligned}$$

The boundary  $\partial(\partial A)$  has a cusp at  $(0, 0)$  where  $\nabla b_A(0, 0) = 0$ . Therefore,  $(0, 0) \notin \text{Ck}_b(A)$  but the two corners  $(-3, \pm 2) \in \text{Ck}_b(A)$ . Moreover,  $(0, 0) \in \partial(\partial A) \cap \text{int } \overline{\mathcal{L}A}$  where  $\nabla b_A(0, 0) = 0$  (since  $\overline{\mathcal{L}A} = \mathbf{R}^2$ ),  $\partial \overline{\mathcal{L}A} = \emptyset$ , and  $\partial \overline{A} \cap \partial \overline{\mathcal{L}A} = \emptyset$ . Finally,

$$\begin{aligned} \text{Sk}(\partial A) &= \{(-2, y) : 1 \leq y < 2 \text{ and } -2 < y \leq -1\} \cup \{(x, 0) : x > 0\}, \\ \text{Ck}_b(A) &= \text{Sing}(\nabla b_A) \setminus \text{Sk}(\partial A) = \{(-2, 2) \cup (-2, -2)\} \text{ since } \nabla b_A(0, 0) = 0, \\ \overline{\text{Sk}(\partial A)} &= \{(-2, y) : 1 \leq y \leq 2 \text{ and } -2 \leq y \leq -1\} \cup \{(x, 0) : x \geq 0\}, \\ \text{Sing}(\nabla b_A) \setminus \overline{\text{Sk}(\partial A)} &= \emptyset, \\ \partial \overline{A} \cap \partial \overline{\mathcal{L}A} &= \emptyset \implies \{x \in \partial \overline{A} \cap \partial \overline{\mathcal{L}A} : \nabla b_A(x) \nexists\} = \emptyset, \\ \partial(\partial A) \cap (\text{int } \overline{A} \cup \text{int } \overline{\mathcal{L}A}) &= \partial B \cap (B \cup \mathbf{R}^2) = \partial B, \end{aligned}$$

where at  $(0, 0) \in \partial B$ ,  $\nabla d_A(0, 0) = \nabla d_{\mathcal{L}A}(0, 0) = 0$ . Since  $b_A = d_B$ ,  $\nabla b_A(x) = \nabla d_B(x)$  does not exist at  $x \in \partial B \setminus (0, 0)$ .

#### APPENDIX: HADAMARD SEMIDIFFERENTIAL

Recall the following equivalent definition of the *Hadamard semidifferential*.

DEFINITION 6 ([3, Defs. 3.3 and 3.4, and Thm. 3.6(ii)]). Let  $X$  and  $Y$  be Banach spaces and  $f : X \rightarrow Y$  a function.

(i)  $f$  is *Hadamard semidifferentiable* at  $x \in X$  in the direction  $v \in X$  if

$$(10) \quad d_H f(x; v) \stackrel{\text{def}}{=} \lim_{\substack{w \rightarrow v \\ t \searrow 0}} \frac{f(x + tw) - f(x)}{t} \text{ exists in } Y.$$

(ii)  $f$  is *Hadamard semidifferentiable* at  $x \in X$  if it is Hadamard semidifferentiable at  $x \in X$  in all directions  $v \in X$ .

(iii)  $f$  is *Hadamard differentiable* at  $x \in X$  if  $f$  is *Hadamard semidifferentiable* at  $x \in X$  and the function  $v \mapsto Df(x)v \stackrel{\text{def}}{=} d_H f(x; v) : X \rightarrow Y$  is linear.

In finite dimension, the Hadamard differentiability coincides with the Fréchet differential (see [4, Thm. 3.2, p. 97]), but in infinite dimension, it is weaker and yet retains all the properties of the classical differential calculus.

A (finite) convex function  $f : \mathbf{R}^N \rightarrow \mathbf{R}$  is Hadamard semidifferentiable (see [4, Thm. 4.8, p. 136]).

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