The Klein Project – elementary mathematics from a higher standpoint

ICMI column in this issue presented by Hans-Georg Weigand

The Klein Project[1] is an IMU and ICMI project with the aim of producing mathematics resources for secondary teachers on contemporary mathematics. It was inspired by Felix Klein's book "Elementary Mathematics from a Higher Standpoint," first published over 100 years ago. It is intended as a stimulus for mathematics teachers, to help them make connections between the mathematics they teach, or they can be asked to teach, and the field of academic mathematics, while taking into account the evolution of this field over the last century. In the following, the origin of this project, the aims and intentions, the present state and the upcoming aims are described.

Felix Klein and the *Elementary Mathematics from a Higher Standpoint*

2008 marked the 100th anniversary of the founding of the ICMI (International Commission on Mathematical Instruction). The first president of this commission was Felix Klein (1849–1925). On the occasion of this event, the IMU (International Mathematics Union) and ICMI initiated a project to revive Felix Klein's vision, for which he wrote "Elementary Mathematics from a Higher Standpoint" in 1908 [1]. The aim of this three-volume work was to show mathematics teachers the breadth of mathematical research at the time, to provide them with background knowledge beyond the usual curriculum, and to strengthen the relationship with mathematics teaching at the secondary-school level. This work has become a classic reference both nationally in Germany and internationally; its title alone has become its programme.

Felix Klein always emphasized the great importance of teaching at the university, and he vehemently demanded a modernization of teaching at high school through a stronger emphasis on mathematics and the natural sciences. This was ultimately reflected in the "Merano Resolutions" (1905). In 1908, the German Committee for Mathematics and Science Education (DAMNU) was founded. Felix Klein was the chair of this committee for teacher education.

In the same year, ICMI was founded, and Klein became the first chair of this commission until 1916.

The first volume of "Elementary Mathematics from a Higher Standpoint" was published in 1908 [1]. In the foreword, Felix Klein addressed the "mathematical public and especially the teachers of mathematics at our secondary schools" [2, p. IV]. After concentrating on the aims of learning and teaching – especially in the "Merano Resolutions" –, he now focused on the high school and the mathematics content. His aim was to "present the content and basis of the subjects to be covered in mathematics lessons, with reference to the actual teaching process, in the simplest and most stimulating way possible from the point of view of modern science." [2, Preface, p. IV].

For Klein, teacher education was an ongoing and constantly redefinable task, in which problems – similar to those in schools – arise again and again in the same or similar ways. For over 100 years, Klein highlighted a central problem in teacher education, expressed using the quite famous notion of "double discontinuity."

A mathematics freshman at the university is confronted with problems he had not been concerned with at school. After finishing university and becoming a teacher, he/she is expected to teach traditional elementary mathematics which he was not confronted with at university. Therefore, he teaches mathematics the way he was taught some years ago and his university studies remain only a more or less pleasant memory which do not influence his teaching.[2] [1, p. 1]

The first discontinuity concerns the well-known problem of transition from school to university. The second discontinuity concerns the transition from university back to school. Felix Klein aims to address this with his "Elementary Mathematics." The first volume covers the areas of arithmetic, algebra and analysis, the second is about geometry, and the third is about precision and approximation in mathematics. Klein relates the concepts of these areas, often used intuitively at school, to the new findings from mathematical

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[1] https://www.mathunion.org/icmi/projects/klein-project

[2] Cited from [3].

research on the fundamentals of mathematics at the beginning of the 20th century.

The original intention of the Klein Project

The "Klein Project," initiated by the IMU and the ICMI, aimed to publish a work that takes up the ideas of Felix Klein in a modern and up-to-date form and presents them in a way that teachers can understand. To this end, an international project team was formed with the task of presenting mathematical research and its applications in such a way that prospective and practising teachers are encouraged to convey an up-to-date and contemporary picture of mathematics to students in their lessons. The 2010 "Klein Design Team" consisted of the following members: the ICMI President Bill Barton (New Zealand), Michèle Artigue (France), Ferdinando Arzarello (Italy), Yuriko Baldin (Brazil), Graeme Cohen (Australia), Bill McCallum (USA), Tomás Recio (Spain), Christiane Rousseau (Canada), and Hans-Georg Weigand (Germany). Another idea of the project was to develop the contents and ideas of this new book concerning the mathematics of the 21st century in close cooperation with school teachers and in the context of the so-called "Klein Conferences." Following the first "Klein Conference" in Madeira (Portugal), 2009, there have been some more in different parts of the world: 2010 in Castro Urdiales (Spain), Oxford (UK), Belo Horizonte (Brazil) and Pittsburg (USA), 2011 in Palo Alto (USA), 2012 in Paris (France) and 2013 in Berlin (Germany). Some quite well-known mathematicians, experts in the areas of, for example, geometry, logic, discrete mathematics and statistics, and also in philosophy of mathematics, were asked to contribute to this book. However, immediately after the submission of the first drafts of articles, it became obvious that the big challenge around the transfer of the present state of these areas of mathematics to high school mathematics is to make it understandable for teachers. Moreover, what also became obvious, is the impossibility of writing the book at a consistent level if too many different authors contributed individual chapters.

The idea of Klein Vignettes

Right from the start of the project, the idea was – of course – to develop an attractive website in addition to the new book and to the Klein Conferences. After the not very encouraging experiences with the book articles, the idea was to collect first some central contents, aspects, theorems, connections and explanations of current mathematics in the form of shorter articles on the website. These articles were called "Klein Vignettes." Vignettes are intended to give teachers a sense of connectedness between the mathematics of the teachers' world and contemporary research and applications in the mathematical sciences. All Klein Vignettes should have the same or at least a similar structure. Thus, a Klein Vignette will start with something with which the teacher is familiar and then move towards a greater understanding of the subject through a piece of interesting mathematics. It will ultimately illustrate a key principle of mathematics. A Klein Vignette is a short piece of writing about a single mathematical topic. It assumes the reader has some undergraduate mathematical knowledge, and at least a willingness to engage with mathematical writing and reading. Klein Vignettes are not just "about" mathematics, they contain significant mathematics which is likely to be new to most secondary teachers and give them a perspective of current mathematics. In the following years many Klein Vignettes were produced and listed in a blog.[3] Some titles are as follows:

- How Google works: Markov chains and eigenvalues
- Goodstein sequences – the power of a detour via infinity
- Matrices and digital images
- A regular heptadecagon is constructible
- The enormous theorem
- Art and mathematics: knots and links
- How do I solve this equation? Look at the symmetries! – The idea behind Galois theory
- Dimension
- Fair voting: the quest for gold
- Banach's microscope to find a fixed point
- Benford's law: learning to fraud or to detect frauds?
- Map colouring and Gröbner bases
- Recurrence and induction
- Trying to predict a floating leaf: chaos and predictions
- The revenge of the infinitesimals
- The hairy ball theorem
- Calculators, power series and Chebyshev polynomials
- The shocking behaviour of moving fluids
- Public-key cryptography
- A tale of two triangles: Heron triangles and elliptic curves

The Klein Vignettes are available in English, and many are translated into Arabic, French, German, Italian, Spanish, Portuguese, Khmer and Mandarin.

When discussing these vignettes with teachers, they appreciated the information about some pieces of present mathematics. However, they also asked for support and help to transfer some of these ideas into the classroom. This was the reason to launch "Klein Bridging Vignettes," which bridge the gap between the mathematics explained in classical Klein Vignettes and their use in the classroom. We wanted to motivate especially mathematics educators to write Klein Bridging Vignettes for the Klein Project. Some examples are as follows:

- From synthetic geometry to dynamic geometry and back: the case of circular inversion

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[3] https://blog.kleinproject.org

- What is the way of packing oranges? – Kepler's conjecture on the packing of spheres
- Higher dimensions
- Symmetry step by step

Besides the Klein Vignettes and the Bridging Vignettes, we also established a column "Site of the month" on the website. These sites present some short information about new published books, interesting projects and websites in relation to the aims and topics of the Klein Project. In this respect, some of the books are as follows:

- Étienne Ghys, La petite histoire des flocons de neige
- Richard Guy, Unsolved Problems in Number Theory
- Eli Maor, The Pythagorean Theorem
- Hans-Georg Weigand, William McCallum, Marta Menghini, Michael Neubrand and Gert Schubring (eds.), The Legacy of Felix Klein
- David Richeson, Euler's Gem: The Polyhedron Formula and the Birth of Topology
- Douglas Hofstadter, Metamagical Themas
- Renate Tobies, Felix Klein: Visions for Mathematics, Applications, and Education
- David E. Rowe, A Richer Picture of Mathematics: The Göttingen Tradition and Beyond
- Otto Neugebauer, The Exact Sciences in Antiquity
- David Hilbert and Stefan Cohn-Vossen, Geometry and the Imagination
- Maria G. Bartolini Bussi and Michela Maschietto, Macchine matematiche: Dalla storia alla scuola
  And these are some of the projects:
- The German Imaginary project[4]
- The Global Math Project[5]
- The Mathematics News Snapshots for High School (MNS) project[6]
- 3Blue1Brown[7]
- Wolfram MathWorld[8]
- Mathematical Etudes[9]
- Accromath[10]

The dissemination of the Klein Project

So far, the ideas of the Klein Project have been disseminated at many conferences, such as IMPA in Rio de Janeiro (Brazil) 2014,

EARCOME 7 in Cebu City (Philippines) 2015, ICME13 in Hamburg (Germany) 2016, EARCOME 8 in Taipei (Taiwan) 2018, PME 42 in Umeå (Sweden) 2018, PME 43 in Pretoria (South Africa) 2019, and ICME14 in Shanghai (China) 2020. At ICME13 in Hamburg, there was a "Thematic Afternoon" with the topic "What is and what might be the Legacy of Felix Klein?", organized by Hans-Georg Weigand, William McCallum, Marta Menghini, Michael Neubrand, Gert Schubring and Renate Tobies. The starting point of this afternoon was the point of view that, at present, the problems Felix Klein saw at university and high school are similar or even the same 100 years later. Speaking about Felix Klein's legacy meant discussing the solutions to problems that he suggested and hoping to get answers to some of the problems we are struggling with today. The results of the meeting are available in "The legacy of Felix Klein," a 2019 book by H.-G. Weigand and colleagues [4]. At a Felix Klein workshop at ICME14 in Shanghai, 2020, "Vignettes in Practice" were presented. Moreover, we are in contact with some other projects, such as:

- French project "Images des mathématiques"[11]
- "Liceo Matematico" project in Italy[12]
- Mathematics News Snapshots for High School (MNS) project in Israel[13]
- Cornerstone Maths project in England[14]

The transfer of the website to the IMU server

Since some years ago, the website of the Klein Project exists on the IMU server.[15] However, for many years the collected Klein Vignettes, the Klein Bridging Vignettes and the Site of the month were hosted by Bill McCallum on the server of the University of Arizona. During the years of the pandemics, the Klein Project did not progress. There have been only a few Klein Conferences for professional development organized, for example, by Yuriko Baldin in Brazil, Michèle Artigue in France or Ferdinando Arzarello in Italy. Moreover, it was obvious that the Klein Project needed some restructuring and revival. In December 2024, all the files of the Klein blog were transferred to the IMU server in Berlin. At the moment, the website is not yet well-structured and still remains under construction. The reconstruction of the Klein Project homepage will be used to start the project in a refreshed way.

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[4] https://www.imaginary.org
[5] https://globalmathproject.org
[6] https://mns.org.il
[7] https://www.3blue1brown.com
[8] https://mathworld.wolfram.com
[9] https://en.etudes.ru
[10] https://accromath.uqam.ca

[11] http://images.math.cnrs.fr
[12] https://www.liceomatematico.it
[13] https://mns.org.il
[14] https://www.ucl.ac.uk/cornerstone-maths
[15] https://www.mathunion.org/icmi/projects/klein-project

About the ongoing project

The Klein Design Team nowadays comprises Ferdinando Arzarello (Italy), Michèle Artigue (France), Yuriko Baldin (Brazil), Samuel Bengmark (Sweden), William McCallum (USA), and Hans-Georg Weigand (Germany). At the CERME14 conference in Bolzano in February 2025, a Klein information meeting was organized to recruit new people who will contribute to the project development. In particular, we were looking for mathematicians, mathematics educators and teachers who are interested in the following activities:

- writing new Klein vignettes and bridging vignettes
- translating existing vignettes into any language
- writing a page of the month
- providing technical support for the homepage and blog
- making connections with other projects
- encouraging collaboration with teachers in designing concrete classroom activities based on Klein Vignettes
- organizing a Klein conference
- establishing contacts with organizations that support the Klein Project financially

We were surprised by the great interest in the Klein Project. We subsequently received positive feedback from interested parties who wanted to get involved in the project. We are currently in the process of integrating these people into the project, but we are also grateful for anyone who would like to work on it – in whichever form.
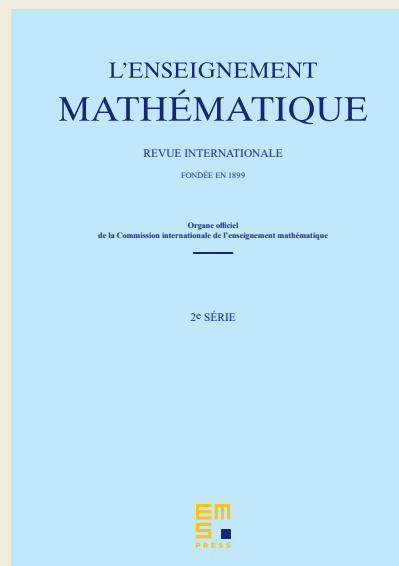
References

[1] F. Klein, *Klein Elementarmathematik vom höheren Standpunkte aus. Teil I. Arithmetik, Algebra, Analysis*. B. G. Teubner, Leipzig (1908)

[2] F. Klein, *Elementary mathematics from an advanced standpoint. Arithmetic–algebra–analysis*. Macmillan, London (1932)

[3] F. Klein, *Elementary mathematics from a higher standpoint. Vol. I–III*. Springer, Berlin, Heidelberg (2016)

[4] H.-G. Weigand, W. McCallum, M. Menghini, M. Neubrand and G. Schubring (eds.), *The legacy of Felix Klein*. ICME-13 Monographs, Springer, Cham (2019)

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