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Lax additivity

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Abstract. We introduce notions of lax semiadditive and lax additive $(\infty, 2)$-categories, categorifying the classical notions of semiadditive and additive 1-categories. To establish a well-behaved axiomatic framework, we develop a calculus of lax matrices and use it to prove that in locally cocomplete $(\infty, 2)$-categories lax limits and lax colimits agree and are absolute. In the lax additive setting, we categorify fundamental constructions from homological algebra such as mapping complexes and mapping cones and establish their basic properties.

1. Introduction

In this article, we propose an axiomatic framework of *lax additive* $(\infty, 2)$-categories, intended as a natural context to develop foundational aspects of categorified homological algebra (analogously to the familiar development of classical homological algebra building on additive categories).

Our motivation stems from several recent developments, some of the most directly relevant ones being:

- Categorified analogs of classical homological techniques have been very successful in the study of Fukaya-type categories in homological mirror symmetry. The categorical Picard–Lefschetz theory developed in [18] is a particularly well-proven example.
- Kapranov and Schechtman have proposed to study categorified analogs of perverse sheaves, termed perverse schobers [13]. While this beautiful circle of ideas has already created substantial impact, the theory is still somewhat experimental and as of now there does not seem to exist a satisfying rigorous definition of perverse schobers in some natural generality.
- Various foundational results from classical homological algebra have been shown to admit categorified variants replacing abelian groups by stable ∞-categories. An illustrative example is the categorified Dold–Kan correspondence (cf. [4, 12]) which can be regarded as a "proof of concept" for the feasibility of categorifying some of the foundations of homological algebra.
- Several examples of stable ∞-categories of algebraic or geometric origin have been shown to admit natural upgrades to complexes of stable ∞-categories (cf. [3]).

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We see this work as a first step towards capturing the basic 2-categorical principles underlying these perspectives, with the final goal of creating a unified picture of mutual benefit. That being said, we recommend to read this paper as a companion to [3], where the abstract axiomatic theory developed here appears in a very hands-on way, illustrated by many examples and explicit constructions.

Beyond these concrete applications, we feel that the 2-categorical theory of lax additivity developed in this work does have some intrinsic category theoretic appeal, justifying its documentation in a standalone contribution. For example, we systematically introduce various types of lax matrices along with categorified matrix multiplication rules. Based on this calculus, we prove natural categorified variants of classical foundational results on (semi-)additive categories such as:

Theorem (Corollary 4.15 and Theorem 4.19). *In locally cocomplete*[1] $(\infty, 2)$*-categories*

- *lax limits and lax colimits coincide (when they exist) and*
- *all lax limits and lax colimits are absolute, i.e., preserved by locally cocontinuous*[2] *functors.*

As a further illustration of the theory, we categorify basic additive constructions from homological algebra such as mapping complexes and mapping cones and establish their basic properties.

1.1. Rules of categorification

We begin with an overview of our preferred type of "categorification": It arises from the insight that in some important respects stable ∞-categories behave like categorified abelian groups, leading to the "categorification rules" in Table 1.

Classical	Categorified
(1) abelian group A	stable ∞-category $\mathcal{A}$
(2) element $x \in A$	object $X \in \mathcal{A}$
(3) $y - x$	$\mathrm{cone}(X \xrightarrow{f} Y)$
(4) $\sum (-1)^i x_i$	$\mathrm{tot}(X_0 \xrightarrow{d} X_1 \xrightarrow{d} X_2 \xrightarrow{d} \cdots \xrightarrow{d} X_n)$
(5) direct sum decomposition $C \cong A \oplus B$	semiorthogonal decomposition $\mathcal{C} \simeq \langle \mathcal{A}, \mathcal{B} \rangle$
(6) external direct sum $A \oplus B$	gluing along a functor/lax sum $\mathcal{A} \underset{F}{\overset{\leftrightarrow}{\oplus}} \mathcal{B}$

Table 1. Categorification rules.

[1] An $(\infty, 2)$-category is locally cocomplete if all its hom-categories have colimits and if composition of 1-arrows preserves colimits in each variable.

[2] A functor of $(\infty, 2)$-categories is locally cocontinuous if it induces a colimit-preserving functor on hom-categories.

While the first two rules should be apparent, we start commenting on rule (3). This is a first crucial difference between the classical and the categorified context: In order to take a "difference" between objects X, Y of a stable ∞-category $\mathcal{A}$, we need to be given the additional datum of a morphism $f: X \to Y$—the difference will then be the cone of f. Compliance with this rule will force us to include certain 2-categorical data which becomes invisible upon passing to the Grothendieck group K_0. This typically results in rather natural *lax variants* of 1-categorical constructions.

Rule (4) is a natural generalization of Rule (3): An alternating sum over n elements will be categorified by the totalization of an n-term complex in $\mathcal{A}$. Here we do not only need to specify the differentials of this complex, but also a coherent system of null homotopies—this is necessary to make sense of the totalization in the ∞-categorical context.

Rule (5) is almost evident after having accepted Rule (3): While in a direct sum $A \oplus B$, every element is uniquely the sum of elements from the components A and B, respectively, in a semiorthogonal decomposition $\langle \mathcal{A}, \mathcal{B} \rangle$, every object is uniquely an *extension* of an object $A \in \mathcal{A}$ by an object $B \in \mathcal{B}$. Put differently, by shifting the exact triangle of the extension, every object is uniquely the cone of a morphism $A[-1] \to B$, thus connecting back to Rule (3).

Conceptually distinct to a direct sum *decomposition* of a given abelian group are the universal properties satisfied by the *external* direct sum of a pair of abelian groups. For its categorificaiton, it is not sufficient to just provide a pair of stable ∞-categories. As an additional datum, we need to specify a functor $F: \mathcal{A} \to \mathcal{B}$ (similar to the additional choice of a morphism $f: X \to Y$ needed in Rule (3)). The categorified "direct sum" is then the *lax sum*

$$\mathcal{A} \overset{\leftrightarrow}{\underset{F}{\oplus}} \mathcal{B}$$

of the diagram of stable ∞-categories described by F (see Rule (6)), which is given by the commonly known construction of gluing along a functor. The fact that this sum categorifies both the product and coproduct will be explained below in the context of lax additivity. The lax sum admits a semiorthogonal decomposition with components $\mathcal{A}$ and $\mathcal{B}$, and vice versa, any stable ∞-category with a semiorthogonal decomposition can be described as a lax sum if and only if the semiorthogonal decomposition admits a *gluing functor*.

1.2. Lax additivity

Of course abelian groups are rarely studied in isolation; rather we consider the category Ab of abelian groups. This category has many important features, but most importantly for us it is *additive*. Continuing our train of thought, Table 1 has a natural continuation in Table 2 which explains what it means to say that the $(\infty, 2)$-category of stable ∞-category, or more generally any $(\infty, 2)$-category $\mathbb{A}$, is *lax additive*.

Accepting our basic premise that abelian groups are to be categorified by stable ∞-categories, Rule (9) requires no further comment. Rule (8) is a convenient intermediate step, categorifying the situation where the addition on hom-sets does not necessarily have

	Classical	Categorified
(7)	additive (∞-)category $\mathcal{A}$	lax additive $(\infty,2)$-category $\mathbb{A}$
(8)	hom-sets $\mathcal{A}(X,Y)$ have addition	hom-categories $\mathbb{A}(\mathcal{X},\mathcal{Y})$ have colimits
(9)	hom-sets $\mathcal{A}(X,Y)$ are abelian groups	hom-categories $\mathbb{A}(\mathcal{X},\mathcal{Y})$ are stable
(10)	finite direct sums $\bigoplus_{s=1}^{k} x_s = \prod_{s=1}^{k} x_s = \coprod_{s=1}^{k} x_s$	general lax bilimits $\bigoplus^{\text{lax}}_{s:S} \mathcal{X}_s = \operatorname{laxlim}_{s:S} \mathcal{X}_s = \operatorname{laxcolim}_{s:S} \mathcal{X}_s$
(11)	binary direct sums $X \oplus Y = X \times Y = X \amalg Y$	lax/oplax Δ^1-bilimits $\mathcal{X} \underset{F}{\overrightarrow{\times}} \mathcal{Y} = \mathcal{X} \underset{F}{\overleftarrow{\amalg}} \mathcal{Y} = \mathcal{X} \underset{F}{\overleftarrow{\times}} \mathcal{Y} = \mathcal{X} \underset{F}{\overrightarrow{\amalg}} \mathcal{Y}$
(12)	matrices $\left(\begin{smallmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{smallmatrix}\right)$	lax matrices $\begin{pmatrix} M_{11} & \leftarrow & M_{12} \\ \downarrow & & \downarrow \\ M_{21} & \leftarrow & M_{22} \end{pmatrix}$
(13)	matrix multiplication $(nm)_{us} = \sum_{t=t'} (n_{ut'} \circ m_{ts})$	lax matrix multiplication $(NM)_{us} = \operatorname{colim}_{\gamma: t \to t'} (N_{ut'} \circ \gamma \circ M_{ts})$
(14)	matrix multiplication, reparameterized $(nm)_{us} = \sum_t (-1)^t (n_{ut} \circ m_{ts})$	lax-oplax matrix multiplication $(NM)_{us} = \operatorname{tot}_t (N_{ut} \circ M_{ts})$

Table 2. Categorification rules (cont.).

inverses; just like the uncategorified case, many basic lemmas are most naturally expressed in this generality leading to the notion of lax *semi*-additive $(\infty,2)$-category.

The direct sum of abelian groups is both a categorical product and a categorical coproduct, a universal property that is taken as the definition in general additive categories. Rule (10) states that the same definition can be categorified, by replacing finite products and coproducts, i.e., limits and colimits indexed by finite discrete categories $S = \{1,\dots,k\}$, with *lax* limits and colimits indexed by arbitrary ∞-categories S. Apart from this change, the theory is exactly analogous: if the hom-categories have colimits (categorifying addition) then such lax limits and colimits always agree if they exist, yielding the notion of lax bilimits. Thus we obtain the main concept of this article:

Definition (Definition 5.5). An $(\infty,2)$-category $\mathbb{A}$ is lax additive if

- it is enriched in stable ∞-categories with colimits and
- it has all lax limits and lax colimits, which then automatically agree.

In a lax additive $(\infty,2)$-category, we can say even more for the special choice of $S = \Delta^1$, in which case the constructions categorify the binary direct sum. For this choice of S, all four possible universal 2-categorical constructions (lax/oplax, limit/colimit) associated to an S-diagram $\mathcal{X} \xrightarrow{F} \mathcal{Y}$ agree with each other. This is the content of Rule (11). Note that this further explains the statement of Rule (6) and this (op)lax bilimit is also called the lax sum $\mathcal{X} \underset{F}{\overset{\leftrightarrow}{\oplus}} \mathcal{Y}$.

A convenient feature of additive categories is that maps $m: x_1 \oplus x_2 \to y_1 \oplus y_2$ between direct sums can be represented as matrices of the form

$$m = \begin{pmatrix} m_{11}: x_1 \to y_1, & m_{12}: x_2 \to y_1 \\ m_{21}: x_1 \to y_2, & m_{22}: x_2 \to y_2 \end{pmatrix}.$$

Composing such maps then just amounts to the usual matrix multiplication. Rule (13) shows how the usual matrix multiplication formula can be categorified, yielding an analogous theory of matrices indexed in each coordinate not by a finite set but by arbitrary ∞-categories. These matrices are just a dependent version of bimodules, which by Morita theory encode functors between module categories.

In Section 1.1 we have already seen how it is conceptually easier to categorify subtraction rather than addition and more generally alternating rather than ordinary sums. In the lax additive setting we see a similar feature, expressed in Rule (14), where a suitable "coordinate change" yields a more convenient formula for the categorified matrix multiplication when we reparameterize it to use alternating sums rather than ordinary sums. Categorically speaking this reparameterization involves the identification of lax and oplax limits and is therefore only available for certain special indexing categories such as Δ^1.

1.3. Mapping cones and mapping complexes

Building on the notions introduced above, we explain how to categorify two fundamental constructions within the lax additive framework: the mapping complex between two chain complexes (see Construction 7.32) and the mapping cone of a chain map (see Construction 8.9).

We summarize the categorified formulas for their respective differentials in Table 3. We note that upon passing to K_0, the signs in the right-hand side and left-hand side differ only in a non-essential way, the chain complexes are isomorphic.

Classical	Categorified
(15) $\delta g_\bullet := \big(dg + (-1)^\bullet gd\big)$	$\delta^{\mathrm{lax}} g_\bullet := \mathrm{fib}(dg \to gd)[\bullet]$
(16) $d_{\mathrm{Cone}(f)} := \begin{pmatrix} -d & 0 \\ -f & d \end{pmatrix}$	$d_{\mathrm{Cone}(f)} := \begin{pmatrix} d & \to & 0 \\ \downarrow & & \downarrow \\ f & \to & d \end{pmatrix}$

Table 3. Categorification rules (cont.).

We further note that unlike in the classical setting, we do not require any signs in the differential of $\mathrm{Cone}(f)$; the appropriate signs guaranteeing $\delta^2 = 0$ are inbuilt into the alternating sums defining the lax-oplax matrix multiplication.

The categorified mapping cone yields a natural notion of null-homotopy H of a chain map f, given by $\delta^{\mathrm{lax}}(H) = f$. The categorified mapping cone interacts with this notion as expected.

Theorem (Corollary 9.3). *For every chain map $f\colon \mathcal{A} \to \mathcal{B}$ for which sufficient adjoints exist and for each chain complex $\mathcal{C}$, there is an equivalence between the (stable) ∞-categories of*

- *chain maps* $\mathrm{Cone}(f) \to \mathcal{C}$ *and*
- *chain maps $g\colon \mathcal{B} \to \mathcal{C}$ together with a lax null-homotopy H of gf.*

The above mapping cone construction is only possible if certain adjoints of some of the involved functors exist. However, one can further analyze which data is precisely necessary to construct mapping cones, and give a more general construction of the categorified mapping cone, which takes as input extra data containing a degree 1 map $\mathcal{A} \to \mathcal{B}$. We also prove a more general version of the above theorem, classifying maps out of this generalized mapping cone, see Theorem 9.2. In the case that sufficient adjoints exist, there exists a canonical choice for this extra data and Theorem 9.2 specializes to the above theorem.

2. Additive 1-categories

To explain our philosophy, let us first remind the reader of the classical story for ordinary additive categories.

We start by recalling the definition.

Definition 2.1. A category $\mathcal{A}$ is called *additive* if:

(1) The category $\mathcal{A}$ is enriched in abelian monoids; i.e., each hom-set $\mathcal{A}(x, y)$ has an associative, commutative addition $+$ with neutral element 0 such that composition

$$\mathcal{A}(x, y) \times \mathcal{A}(y, z) \to \mathcal{A}(x, z)$$

preserves $+$ and 0 in each argument.

(2) Each commutative monoid $(\mathcal{A}(x, y), +, 0)$ admits negatives, hence is an abelian group.

(3) The category $\mathcal{A}$ admits finite products and coproducts (including empty ones).

(4) For each finite set of objects $x_1, \dots, x_n \in \mathcal{A}$, the natural map

$$\coprod_{s=1}^{n} x_s \to \prod_{t=1}^{n} x_t, \tag{2.2}$$

whose components $x_s \to x_t$ are

$$\begin{cases} 1\colon x_s \to x_s, & \text{if } s = t \\ 0 \in \mathcal{A}(x_s, x_t), & \text{otherwise,} \end{cases} \tag{2.3}$$

is an isomorphism.

A category only satisfying (1), (3) and (4) is called semiadditive.

One typically identifies finite products and coproducts via the canonical map (2.2) and uses the symbol $\oplus$ (called direct sum or biproduct) for both.

The use of the phrase "is called additive if" implies that being additive is a property of the category $\mathcal{A}$ rather than extra structure. This is justified by the fact that the addition on the hom-sets of an additive category is uniquely determined. Explicitly, it is given by the following formula: Given two maps $f, g\colon x \to y$, their sum is the composite

$$x \to x \oplus x \xrightarrow{f \oplus g} y \oplus y \to y,$$

where the first map is the diagonal $x \to x \times x$ and the last map is the codiagonal $y \amalg y \to y$.

In this sense, the biproduct structure $\oplus$ determines the addition structure $+$ on the hom-sets. The converse is also true, as explained by the following lemma.

Lemma 2.4. *Let $\mathcal{A}$ be a category enriched in abelian monoids. Let $x_1, \dots, x_n$ be a finite set of objects in $\mathcal{A}$.*

(1) *Let x be an object of $\mathcal{A}$ equipped with a cone $P = (p_s\colon x \to x_s)_{s=1}^n$ and a cocone $I = (i_s\colon x_s \to x)_{s=1}^n$ satisfying the two equations*

(a) $\sum_{s=1}^n i_s \circ p_s = 1 \in \mathcal{A}(x, x)$,

(b) $p_t \circ i_s = \begin{cases} 1 \in \mathcal{A}(x_s, x_t), & \text{if } s = t, \\ 0, & \text{otherwise}. \end{cases}$

Then P and I exhibit x as the product $\prod_{s=1}^n x_n$ and as the coproduct $\coprod_{s=1}^n x_n$, respectively. Moreover, the canonical comparison map (2.2) is the identity $1\colon x \to x$.

(2) *Assume the product $x = \prod_{s=1}^n$ exists and let $P = (p_s\colon x \to x_s)_{s=1}^n$ be the product cone. Then there exists a unique cocone $I = (i_s\colon x_s \to x)_{s=1}^n$ satisfying conditions (1a) and (1b) above.*

(3) *Dually, for every coproduct cocone $I = (i_s\colon x_s \to x)_{s=1}^n$ there exists a unique cone $P = (p_s\colon x \to x_s)_{s=1}^n$ satisfying (1a) and (1b).*

Since $\oplus$ and $+$ determine each other, we have the following corollary:

Corollary 2.5.

(1) *Let $\mathcal{A}$ be a category enriched in abelian monoids. If $\mathcal{A}$ admits finite products (equivalently, finite coproducts) then it is semiadditive.*

(2) *Let $F\colon \mathcal{A} \to \mathcal{A}'$ be a functor between additive categories. The following are equivalent:*

(a) *the functor F preserves finite products;*

(b) *the functor F preserves finite coproducts;*

(c) *the functor F preserves the addition on the hom-sets.*

Lemma 2.4 is well known. However, its proof will serve as a guide for its categorified counterpart, so we shall explain it here.

Proof of Lemma 2.4. To prove (1), we assume (1a) and (1b) and show that P is a product cone; the statement about I is dual. We need to show that for each $t \in \mathcal{A}$ the natural map

$$P_*\colon \mathcal{A}(t,x) \to \prod_{s=1}^{n} \mathcal{A}(t,x_s); \quad f \mapsto (p_s \circ f)_{s=1}^{n}$$

is a bijection. Using I we can produce an explicit inverse via the formula

$$I_*\colon (f_s)_{s=1}^{n} \mapsto \sum_{s=1}^{n} i_s f_s.$$

It satisfies

$$\begin{aligned}(P_* \circ I_*)(f_s)_{s=1}^{n} &= P_*\left(\sum_{s=1}^{n} i_s f_s\right)\\ &= \left(p_u \circ \sum_{s=1}^{n} i_s f_s\right)_{u=1}^{n}\\ &= \left(\sum_{s=1}^{n} p_u i_s f_s\right)_{u=1}^{n}\\ &= (f_u)_{u=1}^{n}\end{aligned}$$

(using equation (1b) in the last step) and

$$\begin{aligned}(I_* \circ P_*)(f) &= I_*(p_s \circ f)_{s=1}^{n}\\ &= \sum_{s=1}(i_s p_s f)\\ &= \left(\sum_{s=1} i_s p_s\right) \circ f\\ &= 1 \circ f = f\end{aligned}$$

(using equation (1a) in the last step), as desired. Moreover, equation (1b) says precisely that the identity $1\colon x \to x$ satisfies the defining equation to be the map (2.2).

Next we prove (2); the statement (3) is dual. By the universal property of the product cone P, there are unique maps $i_s\colon x_s \to x$ satisfying equation (1b). These maps then assemble into the desired cocone I. To verify equation (1a) it suffices to postcompose with all the product projections p_u and compute

$$\begin{aligned}p_u \circ \sum_{s=1}^{n} i_s p_s &= \sum_{s=1}^{n} p_u i_s p_s\\ &= p_u = 1 \circ p_u\end{aligned}$$

(using equation (1b) in the second step). ■

There is one further aspect of additive categories whose categorification will be discussed here: matrix calculus. This is based on the observation that in any category $\mathcal{A}$ any map

$$f\colon \coprod_{s=1}^{n} x_s \to \prod_{t=1}^{m} y_t$$

from a coproduct to a product can be encoded through the bijection

$$\mathcal{A}\left(\coprod_{s=1}^{n} x_s, \prod_{t=1}^{m} y_t\right) \cong \prod_{t=1}^{m}\prod_{s=1}^{n} \mathcal{A}(x_s, y_t)$$

as an $m \times n$-matrix $(f_{ts})_{t=1,s=1}^{m,n}$ whose entry f_{ts} is a map $x_s \to y_t$.

The special feature of semiadditive categories is that it makes sense to consider the composite

$$h\colon \coprod_{s=1}^{n} x_s \xrightarrow{f} \prod_{t=1}^{m} y_t \cong \coprod_{t=1}^{n} y_t \xrightarrow{g} \prod_{u=1}^{l} z_u$$

of two such maps, using the identification (2.2). This composite corresponds to a matrix

$$(h_{us})_{u=1,s=1}^{l,n} \in \prod_{u=1}^{l}\prod_{s=1}^{n} \mathcal{A}(x_s, z_u).$$

It is not hard to verify that the matrix corresponding to the composite h arises from the matrices of f and g by the usual rule for matrix multiplication:

$$h_{us} = \sum_{t=1}^{m} g_{ut} f_{ts}. \tag{2.6}$$

From this perspective, the identification (2.2) is just the identity matrix which has identities on the diagonal an zeroes everywhere else.

3. Preliminaries

Throughout this paper we use the notation "$x : A$" borrowed from homotopy type theory to say that x is a term/inhabitant/element/object of the (∞-)groupoid, (∞-)category, or even $(\infty, 2)$-category A. When we construct an object "$F(x) : B$ for each $x : A$", it is understood that $F(x)$ is supposed to be functorial in x in the appropriate sense. This allows us to unambiguously write formulas such as $\operatorname{colim}_{x:A} F(x)$ or $(F(x))_{x:A}$, which of course only make sense with the additional functoriality in mind.

We reserve the notation $x \in A$ for the case when A is discrete, i.e., (equivalent to) a set; in this case, the question of functoriality is vacuous.

3.1. $(\infty, 2)$-categories

In this paper, we think of $(\infty, 2)$-categories as categories enriched in the ∞-category Cat_∞ of ∞-categories. For a general treatment of enriched ∞-categories, we refer to the work of Gepner and Haugseng [9]. For different approaches to $(\infty, 2)$-categories we refer to [8, 16].

Our goal is not to develop any $(\infty, 2)$-categorical foundations but rather to develop the theory of lax additivity while assuming that such foundations are already laid. In practice, this means that none of our arguments and constructions are performed explicitly in a model, but only using the general high-level features which any theory of $(\infty, 2)$-categories is expected to share. We treat these ingredients axiomatically:

Let $\mathbb{C}$ be an $(\infty, 2)$-category.

- It has an underlying ∞-category $\mathbb{C}_1$, and an underlying ∞-groupoid $\mathbb{C}^{\simeq} = (\mathbb{C}_1)^{\simeq}$.
- It has a hom-functor
$$\mathbb{C}(-,-)\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbb{C}\mathrm{at}_\infty,$$
which takes values in the $(\infty, 2)$-category of ∞-categories. Occasionally, it is convenient to consider the hom-functor
$$\mathbb{C}(-,-)\colon \mathbb{C}_1^{\mathrm{op}} \times \mathbb{C}_1 \to \mathrm{Cat}_\infty$$
as a functor of the underlying ∞-categories, and its associated Cartesian fibration
$$\mathrm{Tw}^*(\mathbb{C}) = \int^* \mathbb{C}(-,-) \to \mathbb{C}_1 \times \mathbb{C}_1^{\mathrm{op}}.$$
- There are composition functors
$$\mathbb{C}(X, Y) \times \mathbb{C}(Y, Z) \to \mathbb{C}(X, Z), \tag{3.1}$$
functorial in $X, Y, Z : \mathbb{C}^{\simeq}$. Composition is coherently associative; this is formalized in [9] by encoding the $(\infty, 2)$-category $\mathbb{C}$ as an algebra in the monoidal ∞-category $(\mathrm{Cat}_\infty, \times)$ of a certain generalized nonsymmetric operad $\Delta^{\mathrm{op}}_{\mathbb{C}^{\simeq}} \to \Delta^{\mathrm{op}}$.
- More generally, the composition map (3.1) is also natural in $X : \mathbb{C}_1^{\mathrm{op}}$, $Z : \mathbb{C}_1$ and dinatural in $Y : \mathbb{C}_1$ (and not just in their groupoid cores). Thinking in terms of fibrations, this means that composition can be written as the dashed functor
$$\begin{tikzcd} \int_Y^X \mathbb{C}(X,Y) \times_{(Y:\mathbb{C}_1)} \int_Z^Y \mathbb{C}(Y,Z) \arrow[r, dashed] \arrow[d] & \int_Z^X \mathbb{C}(X,Z) \arrow[d] \\ (X : \mathbb{C}_1) \times (Y : \mathbb{C}_1) \times (Z : \mathbb{C}_1) \arrow[r] & (X : \mathbb{C}_1) \times (Z : \mathbb{C}_1) \end{tikzcd}$$
of mixed (Cartesian, coCartesian) fibrations.
- It makes sense to talk about adjunctions $f \dashv f^{\mathrm{R}}\colon X \to Y$ in $\mathbb{C}$. These are characterized by the fact that
$$(f\circ) \dashv (f^{\mathrm{R}}\circ)\colon \mathbb{C}(T, X) \to \mathbb{C}(T, Y) \quad \text{and} \quad (\circ f^{\mathrm{R}}) \dashv (\circ f)\colon \mathbb{C}(X, T) \to \mathbb{C}(Y, T)$$
are adjunctions of ∞-categories for all $T : \mathbb{C}$.

For the purpose of developing the theory of lax additivity we do not need the full coherent associativity of the composition law, but only its incoherent shadow. More precisely, it suffices to postcompose the enrichment with the symmetric monoidal functor $(\mathrm{Cat}_\infty, \times) \to (\mathrm{hoCat}_\infty, \times)$, and think of $\mathbb{C}$ as enriched in the homotopy category of ∞-categories up to equivalence.

3.2. Lax limits and colimits

We start by recalling the definition of a lax limits and colimits in a $(\infty, 2)$-category. Let S be an ∞-category.

First, let $\mathcal{X}\colon S \to \mathbb{C}\mathrm{at}_\infty$ be a diagram of ∞-categories. Let $\operatorname{laxlim} \mathcal{X}$ be the ∞-category of sections of the (covariant) Grothendieck construction $\int_* \mathcal{X} \to S$ associated to the functor $\mathcal{X}$. Informally, objects of $\operatorname{laxlim} \mathcal{X}$ consist of

(1) for each object s of S, an object x_s in $\mathcal{X}_s$,

(2) for each edge $f\colon s \to t$ in S, a morphism $x_f\colon \mathcal{X}_f(x_s) \to x_t$ in $\mathcal{X}_t$,

(3) for each 2-simplex $s \xrightarrow{f} t \xrightarrow{g} u$ (with the composite gf implicit) in S a 2-simplex

$$\begin{tikzcd} & \mathcal{X}_g(x_t) \arrow[dr, "x_g"] & \\ \mathcal{X}_{gf}(x_s) \arrow[ur, "\mathcal{X}_g(x_f)"] \arrow[rr, "x_{gf}"] & & x_u \end{tikzcd}$$

in $\mathcal{X}_u$,

(4) and so on for higher simplices of S.

We will denote an object of $\operatorname{laxlim}_{s:S} \mathcal{X}$ as a tuple $(x_s)_{s:S}$.

Now, let $\mathbb{C}$ be an arbitrary $(\infty, 2)$-category and $\mathcal{X}\colon S \to \mathbb{C}$ a diagram. A lax cone over the diagram X with vertex L is an object $(\phi_s)_{s:S} : \operatorname{laxlim}_{s:S} \mathbb{C}(L, \mathcal{X}_s)$, where $s \mapsto \mathbb{C}(L, \mathcal{X}_s)$ is the S-shaped diagram in Cat_∞ obtained as the composite

$$S \xrightarrow{\mathcal{X}} \mathbb{C} \xrightarrow{\mathbb{C}(L,-)} \mathrm{Cat}_\infty .$$

Unpacking the above, one sees that such a cone consists of

(1) for each object $s : S$, a structure map $\phi_s\colon L \to \mathcal{X}_s$,

(2) for each arrow $f\colon s \to t$ in S, a lax cone

$$\begin{tikzcd} & L \arrow[dl, "\phi_s"'] \arrow[dr, "\phi_t"] \arrow[d, phantom, "\Rightarrow"] & \\ \mathcal{X}_s \arrow[rr, "\mathcal{X}_f"'] & {} & \mathcal{X}_t, \end{tikzcd}$$

i.e., a map $\phi_f\colon \mathcal{X}_f \phi_s \to \phi_t$ in $\mathbb{C}(L, \mathcal{X}_t)$,

(3) together with coherent pasting identifications, $\phi_g \circ \mathcal{X}_g \phi_f \simeq \phi_{gf}$ for composable arrows $s \xrightarrow{f} t \xrightarrow{g} u$ in S.

For each other object $L' : \mathbb{C}$ we have a canonical composition map

$$\begin{tikzcd}
\mathbb{C}(L', L) \times \operatorname{laxlim}_{s:S} \mathbb{C}(L, \mathcal{X}_s) \arrow[r, "- \circ -"] \arrow[d, "\Delta \times \mathrm{id}"] & \operatorname{laxlim}_{s:S} \mathbb{C}(L', \mathcal{X}_s) \\
\operatorname{laxlim}_{s:S} \mathbb{C}(L', L) \times \operatorname{laxlim}_{s:S} \mathbb{C}(L, \mathcal{X}_s) \arrow[d, "\simeq"] & \\
\operatorname{laxlim}_{s:S} \mathbb{C}(L', L) \times \mathbb{C}(L, \mathcal{X}_s) \arrow[uur, bend right] &
\end{tikzcd}$$

which explicitly sends a cone $\Phi = (\phi_s)_{s:S}$ with vertex L and a morphism $l \colon L' \to L$ to the cone $\Phi \circ l = (\phi_s \circ l)_{s:S}$.

In a dual way, we can define the ∞-category of lax cocones on $\mathcal{X}$ with vertex L as $\operatorname{laxlim}_{s:S^{\mathrm{op}}} \mathbb{C}(\mathcal{X}_s, L)$. Explicitly, such a cocone $(\psi_s)_{s:S}$ has structure maps $\psi_s \colon \mathcal{X}_s \to L$ and lax triangles

$$\begin{tikzcd}
& L & \\
\mathcal{X}_s \arrow[ur, "\psi_s"] \arrow[rr, "\mathcal{X}_f"'] \arrow[rr, phantom, "\Leftarrow \phi_f", shift left=3] & & \mathcal{X}_t \arrow[ul, "\psi_t"']
\end{tikzcd}$$

over each arrow $f : s \to t$ of S.

Definition 3.2. A cone $P = (p_s)_{s:S} : \operatorname{laxlim}_{s:S} \mathbb{C}(L, \mathcal{X}_s)$ is called a *lax limit cone* if for each object $L' : \mathbb{C}$ the functor

$$P \circ - \colon \mathbb{C}(L', L) \to \operatorname*{laxlim}_{s:S} \mathbb{C}(L', \mathcal{X}_s); \quad F \mapsto (p_s \circ F)_{s:S}$$

is an equivalence of ∞-categories; in this case we call the object L a lax limit of the diagram $\mathcal{X} \colon S \to \mathbb{C}$ and write $L = \operatorname{laxlim}_{s:S} \mathcal{X}_s$.

Dually, we say that a cocone $I = (i_s)_{s:S^{\mathrm{op}}} : \operatorname{laxlim}_{s:S^{\mathrm{op}}} \mathbb{C}(\mathcal{X}_s, L)$ is a *lax colimit cone* if for each $L' : \mathbb{C}$ the functor

$$- \circ I \colon \mathbb{C}(L, L') \to \operatorname*{laxlim}_{s:S^{\mathrm{op}}} \mathbb{C}(\mathcal{X}_s, L'); \quad F \mapsto (F \circ i_s)_{s:S^{\mathrm{op}}}$$

is an equivalence; in this case we call L a lax colimit of $\mathcal{X}$ and write $L = \operatorname{laxcolim}_{s:S} \mathcal{X}_s$.

Remark 3.3. Our definition starts by *defining* the lax limits of ∞-categories to be sections of the Grothendieck construction and then defining lax limits and lax colimits in arbitrary $(\infty, 2)$-categories by considering (co)representables. One can also define lax limits and colimits as a special case of *weighted colimits*, which can be defined directly in terms of ordinary limits/colimits. When using the latter definition, one can then *compute* that lax limits of ∞-categories as sections of the Grothendieck construction, see [10, Proposition 7.1 and Corollary 7.7].

Example 3.4. Let $\mathbb{C} = \mathbb{C}\mathrm{at}_\infty$ be the $(\infty,2)$-category of ∞-categories. Let $\mathcal{X}\colon S \to \mathbb{C}\mathrm{at}_\infty$ be a diagram.

(1) As the notation suggests, the lax limit of $\mathcal{X}$ is the ∞-category $L := \mathrm{laxlim}_{s:S}\,\mathcal{X}_s := \mathrm{Fun}_S(S, \int_{s:S}\mathcal{X}_s)$ of sections of the corresponding Grothendieck construction. Indeed, naturally in $L' : \mathbb{C}\mathrm{at}_\infty$ we have the equivalence

$$\begin{aligned}\mathbb{C}\mathrm{at}_\infty(L', L) &= \mathrm{Fun}\left(L', \mathrm{Fun}_S\left(S, \int_* \mathcal{X}\right)\right)\\ &\simeq \mathrm{Fun}_S\left(S \times L', \int_* \mathcal{X}\right)\\ &\simeq \mathrm{Fun}_S\left(S, \int_{s:S} \mathrm{Fun}(L', \mathcal{X}_s)\right)\\ &= \operatorname*{laxlim}_{s:S}\big(\mathbb{C}\mathrm{at}_\infty(L', \mathcal{X}_s)\big),\end{aligned}$$

which is induced by composition with the canonical cone

$$P = \big(p_s\colon L = \operatorname*{laxlim}_S \mathcal{X} \to \mathcal{X}_s\big)_{s:S}$$

given by evaluation of sections.

(2) The lax colimit of the diagram $\mathcal{X}$ is the contravariant Grothendieck construction

$$\operatorname*{laxcolim}_{s:S} \mathcal{X}_s = \int^{s:S} \mathcal{X}_s,$$

exhibited by the canonical cocone

$$I = \left(i_s\colon \mathcal{X}_s \to \int^* \mathcal{X}\right)_{s:S^{\mathrm{op}}}$$

that includes the individual fibers.

Assume now that the diagram $\mathcal{X}$ takes values in *stable* ∞-categories,

(3) The ∞-category $\mathrm{laxlim}_{s:S}\,\mathcal{X}_s = \mathrm{Fun}_S(S, \int_* \mathcal{X})$ is again stable because limits and colimits of sections are computed pointwise. For the same reason, every functor $F\colon L' \to \mathrm{laxlim}_{s:S}\,\mathcal{X}$ is exact if and only if each composite $p_s \circ F$ is exact. It follows that the cone P exhibits the ∞-category $\mathrm{laxlim}_{s:S}\,\mathcal{X}_s$ also as a lax limit in the $(\infty,2)$-category of stable ∞-categories and exact functors.

(4) The ∞-category $\int^* \mathcal{X}$, which is the lax colimit of $\mathcal{X}$ in $\mathbb{C}\mathrm{at}_\infty$, is typically *not* stable; to compute the lax colimit of $\mathcal{X}$ in the $(\infty,2)$-category of stable ∞-categories one therefore has to stabilize this ∞-category, which is a rather tricky operation. However, it will follow from the theory of lax matrices that—as long as the stable categories in question have enough colimits, for example because S is finite or because $\mathcal{X}$ takes vales in *presentable* stable ∞-categories—this lax colimit indeed just agrees with the lax *limit* which can be computed in $\mathbb{C}\mathrm{at}_\infty$; see Corollary 4.15 below.

Remark 3.5. Definition 3.2 can easily be modified to also define *partially lax* limits and colimits (sometimes also called *marked (co)limits*): If the indexing category S is equipped with some collection M of marked arrows, we can define the M-partially lax limit of a diagram $\mathcal{X}\colon S \to \mathbb{C}\mathrm{at}_\infty$ as the ∞-category of those sections of the Grothendieck construction $\int_* \mathcal{X} \to S$, whose value on the arrows in M is cocartesian. Then one defines partially lax limits and colimits in an arbitrary $(\infty, 2)$-category $\mathbb{C}$ analogously to Definition 3.2. See also [1, 2] for a general treatment of partially (op)lax (co)limits.

In this paper, we mostly deal with (fully) lax limits or colimits (i.e., M only consists of the equivalences of S).

Example 3.6. The only partially lax limits we need in this paper are the directed pullback and directed pushout, which we denote by $\mathcal{A} \overset{\curvearrowright}{\underset{\mathcal{B}}{\times}} \mathcal{C}$ and $\mathcal{A} \overset{\curvearrowright}{\underset{\mathcal{B}}{\amalg}} \mathcal{C}$. Abstractly, they are equipped with the universal squares

$$\begin{tikzcd} \mathcal{A} \overset{\curvearrowright}{\underset{\mathcal{B}}{\times}} \mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & \mathcal{A} \arrow[d, "f"] \\ \mathcal{C} \arrow[r, "g"'] & \mathcal{B} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{B} \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\swarrow"] & \mathcal{A} \arrow[d] \\ \mathcal{C} \arrow[r] & \mathcal{A} \overset{\curvearrowright}{\underset{\mathcal{B}}{\amalg}} \mathcal{C} \end{tikzcd}$$

inhabited by a (possibly noninvertible) 2-morphism (for given $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and f, g which we omit from the notation). Concretely, they can be defined as partially lax limits/colimits with the arrow indexing g being marked or, equivalently, as partially oplax limits/colimits with the arrow indexing f being marked.

4. Lax matrices

Throughout this section, let $\mathbb{C}$ be an $(\infty, 2)$-category enriched in ∞-categories with colimits, i.e.,

- each hom-category $\mathbb{C}(X, Y)$ has all colimits and
- and each composition functor $\mathbb{C}(X, Y) \times \mathbb{C}(Y, Z) \to \mathbb{C}(X, Z)$ preserves colimits in each variable separately.

Analogously to the case of ordinary coproducts and products (which corresponds to the case where the category S is just a set), we can interpret maps from a lax colimit to a lax limit as a sort of matrices: By the defining property we have

$$\begin{aligned} \mathbb{C}\big(\operatorname*{laxcolim}_{s:S} \mathcal{X}_s, \operatorname*{laxlim}_{t:T} \mathcal{Y}_t\big) &\simeq \operatorname*{laxlim}_{t:T} \mathbb{C}\big(\operatorname*{laxcolim}_{s:S} \mathcal{X}_s, \mathcal{Y}_t\big) \\ &\simeq \operatorname*{laxlim}_{t:T} \operatorname*{laxlim}_{s:S^{\mathrm{op}}} \mathbb{C}(\mathcal{X}_s, \mathcal{Y}_t) \\ &\simeq \operatorname*{laxlim}_{(t,s):T\times S^{\mathrm{op}}} \mathbb{C}(\mathcal{X}_s, \mathcal{Y}_t) \end{aligned}$$

so that we can interpret a map $\alpha\colon \mathrm{laxcolim}_{s:S}\,\mathcal{X}_s \to \mathrm{laxlim}_{t:T}\,\mathcal{Y}_t$ as a tuple $(\alpha_{t,s})_{(t,s):T\times S^{\mathrm{op}}}$ which we think of as a matrix whose rows are indexed by T and whose columns are indexed by S^{op}. We define

$$\mathrm{laxMat}_{\mathbb{C}}(\mathcal{X},\mathcal{Y}) := \underset{(t,s):T\times S^{\mathrm{op}}}{\mathrm{laxlim}}\ \mathbb{C}(\mathcal{X}_s,\mathcal{Y}_t).$$

Note that this is a well-defined ∞-category even when $\mathrm{laxcolim}\,\mathcal{X}$ and/or $\mathrm{laxlim}\,\mathcal{Y}$ does not exist. When $\mathcal{X}\colon\{*\}\to\mathbb{C}$ is just an object $X=\mathcal{X}_*$, we still use the notation $\mathrm{laxMat}_{\mathbb{C}}(X,\mathcal{Y})=\mathrm{laxMat}_{\mathbb{C}}(\mathcal{X},\mathcal{Y})$ and observe that it is precisely the ∞-category of lax cones on $\mathcal{Y}$ with vertex X; and analogously for lax cocones.

Example 4.1. Let $S=T=\Delta^1=\{0\xrightarrow{f}1\}$ be the walking arrow and consider two diagrams $\mathcal{X}\colon S\to\mathbb{C}$ and $\mathcal{Y}\colon T\to\mathbb{C}$. Then we can compactly describe objects of

$$\mathrm{laxMat}_{\mathbb{C}}(\mathcal{X},\mathcal{Y}) = \mathrm{laxlim}\begin{pmatrix}\mathbb{C}(\mathcal{X}_0,\mathcal{Y}_0) & \longleftarrow & \mathbb{C}(\mathcal{X}_1,\mathcal{Y}_0)\\ \downarrow & & \downarrow\\ \mathbb{C}(\mathcal{X}_0,\mathcal{Y}_1) & \longleftarrow & \mathbb{C}(\mathcal{X}_1,\mathcal{Y}_1)\end{pmatrix}$$

as $T\times S^{\mathrm{op}}$-indexed diagrams in the Grothendieck construction, which we depict as follows:

$$\begin{pmatrix}\alpha_{00} & \longleftarrow & \alpha_{01}\\ \downarrow & & \downarrow\\ \alpha_{10} & \longleftarrow & \alpha_{11}\end{pmatrix}.$$

Explicitly unpacking this notation, such a matrix consists of:

(1) four 1-morphisms

$$\alpha_{00}\colon\mathcal{X}_0\to\mathcal{Y}_0\quad \alpha_{01}\colon\mathcal{X}_1\to\mathcal{Y}_0$$
$$\alpha_{10}\colon\mathcal{X}_0\to\mathcal{Y}_1\quad \alpha_{11}\colon\mathcal{X}_1\to\mathcal{Y}_1$$

(2) four 2-morphisms

$$\begin{tikzcd}\mathcal{Y}_f\circ\alpha_{00} \arrow[d, "\alpha_{f0}"] \\ \alpha_{10}\end{tikzcd} \qquad \begin{tikzcd}\alpha_{00} & \alpha_{01}\circ\mathcal{X}_f \arrow[l, "\alpha_{0f}"']\end{tikzcd}$$

$$\begin{tikzcd}\alpha_{10} & \alpha_{11}\circ\mathcal{X}_f \arrow[l, "\alpha_{1f}"']\end{tikzcd} \qquad \begin{tikzcd}\mathcal{Y}_f\circ\alpha_{01} \arrow[d, "\alpha_{f1}"] \\ \alpha_{11}\end{tikzcd}$$

(3) assembling into a commutative square

$$\begin{tikzcd}\mathcal{Y}_f\circ\alpha_{00} \arrow[d, "\alpha_{f1}"] & \mathcal{Y}_f\circ\alpha_{01}\circ\mathcal{X}_f \arrow[l, "\mathcal{Y}_f\circ\alpha_{0f}"'] \arrow[d, "\alpha_{f1}\circ\mathcal{X}_f"] \\ \alpha_{10} & \alpha_{11}\circ\mathcal{X}_f. \arrow[l, "\alpha_{1f}"']\end{tikzcd}$$

We now introduce the lax matrix multiplication which categorifies the classical formula (2.6). The classical formula involves a finite sum of elements in some hom-set $\mathcal{A}(x_s, z_u)$ of the category $\mathcal{A}$. Our categorified analog of these sums will be categorical colimits.

Construction 4.2. Let S be an ∞-category and $\mathfrak{X}\colon S \to \mathbb{C}$ a diagram. Passing to the Cartesian fibrations classifying the respective hom-functors, we obtain a commutative square

$$\begin{tikzcd} \mathrm{Tw}^*(S) = \int^* S(-,-) \arrow[r, "\alpha"] \arrow[d, "p"] & \int^* \mathbb{C}(-,-) = \mathrm{Tw}^*(\mathbb{C}) \arrow[d, "q"] \\ S \times S^{\mathrm{op}} \arrow[r, "\mathfrak{X}\times\mathfrak{X}^{\mathrm{op}}"] & \mathbb{C}_1 \times \mathbb{C}_1^{\mathrm{op}} \end{tikzcd}$$

which amounts to the dashed section

$$\begin{tikzcd} & \int^{(s,t)} \mathbb{C}(\mathfrak{X}_s, \mathfrak{X}_t) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Tw}^*(\mathbb{C}) \arrow[d, "q"] \\ \mathrm{Tw}^*(S) \arrow[ur, dashed, bend left, "\alpha"] \arrow[r, "p"] & S \times S^{\mathrm{op}} \arrow[r, "\mathfrak{X}\times\mathfrak{X}^{\mathrm{op}}"] & \mathbb{C}_1 \times \mathbb{C}_1^{\mathrm{op}} \end{tikzcd}$$

of the pullback-fibration $(\mathfrak{X} \times \mathfrak{X}^{\mathrm{op}})^*(q)$ which informally sends an arrow $(f\colon s \to t) : \mathrm{Tw}^*(S)$ to $\mathfrak{X}_f : \mathbb{C}(\mathfrak{X}_s, \mathfrak{X}_t)$.

We can now construct the composite functor

$$\begin{aligned} &\operatorname*{laxlim}_{s:S} \mathbb{C}(L, \mathfrak{X}_s) \times \operatorname*{laxlim}_{t:S^{\mathrm{op}}} \mathbb{C}(\mathfrak{X}_t, L') \\ &\quad = \mathrm{Fun}_{S\times S^{\mathrm{op}}}\Big(S \times S^{\mathrm{op}}, \int_{(s,t)} \mathbb{C}(L, \mathfrak{X}_s) \times \mathbb{C}(\mathfrak{X}_t, L')\Big) \\ &\quad \to \mathrm{Fun}_{S\times S^{\mathrm{op}}}\Big(\mathrm{Tw}^*(S), \int_{(s,t)} \mathbb{C}(L, \mathfrak{X}_s) \times \mathbb{C}(\mathfrak{X}_t, L')\Big) \\ &\quad \to \mathrm{Fun}_{S\times S^{\mathrm{op}}}\Big(\mathrm{Tw}^*(S), \int^{(s,t)} \mathbb{C}(\mathfrak{X}_s, \mathfrak{X}_t) \times_{S\times S^{\mathrm{op}}} \int_{(s,t)} \mathbb{C}(L, \mathfrak{X}_s) \times \mathbb{C}(\mathfrak{X}_t, L')\Big) \\ &\quad = \mathrm{Fun}_{S\times S^{\mathrm{op}}}\Big(\mathrm{Tw}^*(S), \int_s \mathbb{C}(L, \mathfrak{X}_s) \times_S \int^{(s,t)} \mathbb{C}(\mathfrak{X}_s, \mathfrak{X}_t) \times_{S^{\mathrm{op}}} \int_t \mathbb{C}(\mathfrak{X}_t, L')\Big) \\ &\quad \to \mathrm{Fun}\big(\mathrm{Tw}^*(S), \mathbb{C}(L, L')\big) \xrightarrow{\mathrm{colim}} \mathbb{C}(L, L'), \end{aligned}$$

where

- the first arrow is pullback of sections along $p\colon \mathrm{Tw}^*(S) \to S \times S^{\mathrm{op}}$,
- the second arrow adds the section α in the first component of the fiber product,
- the third arrow is given by composition with the composition map

$$\int_s \mathbb{C}(L, \mathfrak{X}_s) \times_S \int^{(s,t)} \mathbb{C}(\mathfrak{X}_s, \mathfrak{X}_t) \times_{S^{\mathrm{op}}} \int_t \mathbb{C}(\mathfrak{X}_t, L') \to \mathbb{C}(L, L'),$$

- the last arrow is just the colimit functor in the ∞-category $\mathbb{C}(L, L')$.

On objects, this functor takes a lax cone and a lax cocone on $\mathcal{X}$,

$$\Phi : \operatorname*{laxlim}_{s:S} \mathbb{C}(L, \mathcal{X}_s) \quad \text{and} \quad \Psi : \operatorname*{laxlim}_{t:S^{\mathrm{op}}} \mathbb{C}(\mathcal{X}_t, L'),$$

and sends them to the map $\Psi \circ_S \Phi \colon L \to L'$ defined by the formula

$$\Psi \circ_S \Phi := \operatorname*{colim}_{(f:s\to t):\mathrm{Tw}^*(S)} (\Psi_t \circ \mathcal{X}_f \circ \Phi_s). \tag{4.3}$$

Remark 4.4. When $S = \{1, 2, \ldots, n\}$ is a finite set the (Cartesian) twisted arrow category $\mathrm{Tw}^*(S) \to S \times S$ can be canonically identified with the diagonal $\Delta\colon S \to S \times S$. Under this identification the formula (4.3) simplifies to

$$(\Psi \circ_S \Phi) = \coprod_{\substack{s,t\in S\\ s=t}} \Psi_t \circ \mathrm{id} \circ \Phi_s = \coprod_{s=1}^{n} \Psi_s \circ \Phi_s$$

which is just the usual multiplication (2.6) of the row vector Ψ with the column vector Φ.

When $S = \{*\}$ is even a singleton, this formula just returns the original composition in the $(\infty, 2)$-category $\mathbb{C}$. For this reason we drop the subscript S and just write $-\circ-$ instead of $-\circ_{\{*\}}-$.

We assemble our categorified analog of row-column multiplication to the lax version of matrix multiplication:

Construction 4.5. Let S, U be ∞-categories, and $\mathcal{X}\colon S \to \mathbb{C}$ and $\mathcal{Z}\colon U \to \mathbb{C}$ two diagrams. For each object $Y : \mathbb{C}$ we consider the functor

$$\begin{aligned}
&\operatorname*{laxlim}_{s:S^{\mathrm{op}}} \mathbb{C}(\mathcal{X}_s, Y) \times \operatorname*{laxlim}_{u:U} \mathbb{C}(Y, \mathcal{Z}_u)\\
&\quad = \operatorname*{laxlim}_{(u,s):U\times S^{\mathrm{op}}} \mathbb{C}(\mathcal{X}_s, Y) \times \mathbb{C}(Y, \mathcal{Z}_u)\\
&\quad \to \operatorname*{laxlim}_{(u,s):U\times S^{\mathrm{op}}} \mathbb{C}(\mathcal{X}_s, \mathcal{Z}_u),
\end{aligned}$$

induced by composition of $\mathbb{C}$. On objects it takes a lax cocone and a lax cone,

$$\Psi : \operatorname*{laxlim}_{s:S^{\mathrm{op}}} \mathbb{C}(\mathcal{X}_s, Y) \quad \text{and} \quad \Phi : \operatorname*{laxlim}_{u:S} \mathbb{C}(Y, \mathcal{Z}_u),$$

and sends them to the matrix $\Phi \circ \Psi : \mathrm{laxlim}_{(u,s):U\times S^{\mathrm{op}}}$ described by the formula

$$(\Phi \circ \Psi)_{us} = \Phi_u \circ \Phi_s.$$

More generally, we can replace the object $Y : \mathbb{C}$ by a diagram $\mathcal{Y}\colon T \to \mathbb{C}$ and consider the functor

$$\begin{aligned}
&\mathrm{laxMat}_{\mathbb{C}}(\mathcal{X}, \mathcal{Y}) \times \mathrm{laxMat}_{\mathbb{C}}(\mathcal{Y}, \mathcal{Z})\\
&\quad = \operatorname*{laxlim}_{(u,s):U\times S^{\mathrm{op}}} \Big(\operatorname*{laxlim}_{t:T} \mathbb{C}(\mathcal{X}_s, \mathcal{Y}_t) \times \operatorname*{laxlim}_{t':T^{\mathrm{op}}} \mathbb{C}(\mathcal{Y}_{t'}, \mathcal{Z}_u)\Big)\\
&\quad \xrightarrow{\mathrm{laxlim}_{u,s}(-\circ_T-)} \operatorname*{laxlim}_{(u,s):U\times S^{\mathrm{op}}} \mathbb{C}(\mathcal{X}_s, \mathcal{Z}_u) = \mathrm{laxMat}_{\mathbb{C}}(\mathcal{X}, \mathcal{Z})
\end{aligned}$$

which is given in componentwise in u, s by the composition functor from Construction 4.2 (applied to lax cones and cocones on $\mathcal{Y}$). Explicitly, this functor is given by the formula

$$\Phi \circ_T \Psi = \Big(\operatorname*{colim}_{(f:t\to t'):\mathrm{Tw}^*(T)} (\Phi_{ut'} \circ \mathcal{Y}_f \circ \Psi_{ts}) \Big)_{(u,s):U\times S^{\mathrm{op}}}.$$

This is what we call the lax matrix multiplication.

Finally, we can assemble all lax matrices of different shapes into a category $\mathrm{holaxMat}_{\mathbb{C}}$:

- Objects are equivalence classes of diagrams $\mathcal{X}\colon S \to \mathbb{C}$, where S is any small ∞-category.
- Morphisms from $\mathcal{X}\colon S \to \mathbb{C}$ to $\mathcal{Y}\colon T \to \mathbb{C}$ are equivalence classes of matrices Φ : $\mathrm{laxMat}_{\mathbb{C}}(\mathcal{X}, \mathcal{Y})$.
- Composition is given by lax matrix multiplication of Construction 4.5.

Remark 4.6. Similarly to Remark 4.4, we drop the subscript T in the case where $T = \{*\}$ is just a point.

Note that the lax matrix multiplication is functorial by construction, making it in particular well defined on equivalence classes. To prove that $\mathrm{holaxMat}_{\mathbb{C}}$ is indeed a category, we will thus only need to construct the identity matrix and prove that lax matrix multiplication is associative up to equivalence.

In fact, we shall prove a slightly stronger statement.

Lemma 4.7. *The lax matrix multiplication of Construction 4.5 is*

(1) *associative up to natural equivalence, i.e., for diagrams $\mathcal{W}\colon R \to \mathbb{C}$ and $\mathcal{X}\colon S \to \mathbb{C}$ and $\mathcal{Y}\colon T \to \mathbb{C}$ and $\mathcal{Z}\colon U \to \mathbb{C}$ we have*

$$(- \circ_T -) \circ_S - \simeq - \circ_T (- \circ_S -)$$

as functors

$$\mathrm{laxMat}_{\mathbb{C}}(\mathcal{W}, \mathcal{X}) \times \mathrm{laxMat}_{\mathbb{C}}(\mathcal{X}, \mathcal{Y}) \times \mathrm{laxMat}_{\mathbb{C}}(\mathcal{Y}, \mathcal{Z}) \to \mathrm{laxMat}_{\mathbb{C}}(\mathcal{W}, \mathcal{Z})$$

(2) *unital up to natural equivalence, i.e., for each diagram $\mathcal{X}\colon S \to \mathbb{C}$ there is a matrix $\mathcal{I}^{\mathcal{X}}$: $\mathrm{laxMat}_{\mathbb{C}}(\mathcal{X}, \mathcal{X})$ with components*

$$\mathcal{I}^{\mathcal{X}}_{ts} = \operatorname*{colim}_{f:S(s,t)} \mathcal{X}_f : \mathbb{C}(\mathcal{X}_s, \mathcal{X}_t) \tag{4.8}$$

such that

$$\mathcal{I}^{\mathcal{X}} \circ_S - \simeq \mathrm{id} \quad \textit{and} \quad - \circ_S \mathcal{I}^{\mathcal{X}} \simeq \mathrm{id}$$

as endofunctors of $\mathrm{laxMat}_{\mathbb{C}}(\mathcal{Y}, \mathcal{X})$ *and* $\mathrm{laxMat}_{\mathbb{C}}(\mathcal{X}, \mathcal{Y})$, *respectively (for each other diagram $\mathcal{Y}\colon U \to \mathbb{C}$).*

Remark 4.9. The category $\mathrm{holaxMat}_{\mathbb{C}}$ is of course only the truncation of an $(\infty, 2)$-category of lax matrices, which one could construct with more effort. Even Lemma 4.7 only shows that lax matrix multiplication is associative up to natural equivalence, but does not exhibit any sort of coherence such as the pentagon. We will not be needing this additional layer of coherence in this article so Lemma 4.7 will suffice.

The proof of associativity is relatively straightforward.

Proof of Lemma 4.7, part (1). For matrices

$$F : \operatorname*{laxlim}_{S\times R^{\mathrm{op}}}(\mathcal{W}_r, \mathcal{X}_s), \quad G : \operatorname*{laxlim}_{T\times S^{\mathrm{op}}}(\mathcal{X}_s, \mathcal{Y}_t), \quad H : \operatorname*{laxlim}_{U\times T^{\mathrm{op}}}(Y_t, Z_u)$$

we compute

$$\begin{aligned}
\big((H \circ_T G) \circ_S F\big)_{ur} &\simeq \operatorname*{colim}_{f:s\to s'}(H \circ_T G)_{us'}\mathcal{X}_f F_{sr} \\
&\simeq \operatorname*{colim}_{f:s\to s'}\big(\operatorname*{colim}_{g:t\to t'} H_{ut'}\mathcal{Y}_g G_{ts'}\big)\mathcal{X}_f F_{sr} \\
&= \operatorname*{colim}_{f:s\to s'}\operatorname*{colim}_{g:t\to t'}(H_{ut'}\mathcal{Y}_g G_{ts'}\mathcal{X}_f F_{sr}) \\
&= \operatorname*{colim}_{(f,g):\mathrm{Tw}^*(S)\times\mathrm{Tw}^*(T)}(H_{ut'}\mathcal{Y}_g G_{ts'}\mathcal{X}_f F_{sr}) \\
&\simeq \cdots \simeq \big(H \circ_T (G \circ_S F)\big)_{ur}
\end{aligned}$$

naturally in F, G, H and $u : U, r : R^{\mathrm{op}}$; where the third step uses that composition in $\mathbb{C}$ preserves colimits in each variable. ∎

Before we can prove part (2) of Lemma 4.7 we need to construct the unit matrices $\mathcal{I}^{\mathcal{X}} : \mathrm{laxMat}_{\mathbb{C}}(\mathcal{X}, \mathcal{X})$.

Construction 4.10. Consider the commutative square

$$\begin{tikzcd}
\mathrm{Tw}_*(S) = \int_* S(-,-) \arrow[r, "\alpha"] \arrow[d, "p"] & \int_* \mathbb{C}(-,-) = \mathrm{Tw}_*(\mathbb{C}) \arrow[d, "q"] \\
S^{\mathrm{op}} \times S \arrow[r, "\mathcal{X}^{\mathrm{op}}\times\mathcal{X}"] & \mathbb{C}_1^{\mathrm{op}} \times \mathbb{C}_1
\end{tikzcd}$$

induced by a diagram $\mathcal{X}\colon S \to \mathbb{C}$. Here the vertical maps are the coCartesian fibrations classifying the respective hom-functors of S and $\mathbb{C}$. The fact that the $(\infty, 2)$-category $\mathbb{C}$ is enriched in ∞-categories with colimits means that there exists an (essentially unique) left q-Kan extension of α along p, giving rise to the dashed lift

$$\begin{tikzcd}
\mathrm{Tw}_*(S) \arrow[r, "\alpha"] \arrow[d, "p"] & \mathrm{Tw}_*(\mathbb{C}) \arrow[d, "q"] \\
S^{\mathrm{op}} \times S \arrow[r, "\mathcal{X}^{\mathrm{op}}\times\mathcal{X}"] \arrow[ur, dashed, "\mathcal{I}'"] & \mathbb{C}_1^{\mathrm{op}} \times \mathbb{C}_1 .
\end{tikzcd}$$

Since the pullback of the coCartesian fibration q along $\mathcal{X}^{\mathrm{op}} \times \mathcal{X}$ is, by definition, the coCartesian fibration $\int_{(s,t):S^{\mathrm{op}}\times S} \mathbb{C}(\mathcal{X}_s, \mathcal{X}_t) \to S^{\mathrm{op}} \times S$, this lift $\mathcal{I}'$ corresponds to a section of this fibration, i.e., an object

$$\mathcal{I}^{\mathcal{X}} = \mathcal{I} : \operatorname*{laxlim}_{(s,t):S^{\mathrm{op}}\times S} \mathbb{C}(\mathcal{X}_s, \mathcal{X}_t).$$

By the pointwise formula, we can explicitly compute the value of $\mathcal{I}^{\mathcal{X}}$ at $(s,t) : S^{\mathrm{op}} \times S$ as the colimit of the composite

$$\mathrm{Tw}_*(S)/(s,t) \xrightarrow{\alpha} \mathrm{Tw}_*(\mathbb{C})/(\mathcal{X}_s, \mathcal{X}_t) \to \mathbb{C}(\mathcal{X}_s, \mathcal{X}_t),$$

which is the functor that informally maps

$$(f' : s' \to t', g : s \to s', h : t' \to t) \mapsto \mathcal{X}_h \circ \mathcal{X}_{f'} \circ \mathcal{X}_s.$$

Since the inclusion $S(s,t) \simeq \mathrm{Tw}_*(S)_{(s,t)} \hookrightarrow \mathrm{Tw}_*(S)/(s,t)$ has a left adjoint (because q is a coCartesian fibration), it is homotopy terminal; thus we can compute the components $\mathcal{I}^{\mathcal{X}}_{ts}$ via the desired explicit formula (4.8).

Remark 4.11. Since all pointwise colimits (4.8) are taken over *spaces* $S(s,t)$ (as opposed to arbitrary ∞-categories), we see that for the construction of the unit matrix we could have relaxed our assumption on $\mathbb{C}$ and only required it to be enriched in ∞-categories with groupoidal colimits.

Remark 4.12. When S is a set this formula simplifies to

$$\mathcal{I}_{ts} = \operatorname*{colim}_{f : S(s,t)} (\mathcal{X}_f) = \begin{cases} \mathrm{id}_{\mathcal{X}_s}, & \text{if } s = t, \\ \emptyset, & \text{if } s \neq t \end{cases}$$

which is the direct analog of formula (2.3), with the initial object $\emptyset$ of $\mathbb{C}(x_s, x_t)$ taking the role of the zero object of a commutative monoid.

Example 4.13. Continuing Example 4.1, we consider a diagram $\mathcal{X} : \Delta^1 = \{0 \xrightarrow{10} 1\} \to \mathbb{C}$. The unit $\Delta^1 \times (\Delta^1)^{\mathrm{op}}$-matrix then looks as follows:

$$\begin{pmatrix} \mathrm{id}_{\mathcal{X}_0} & \longleftarrow & \emptyset \\ \downarrow & & \downarrow \\ \mathcal{X}_{10} & \longleftarrow & \mathrm{id}_{\mathcal{X}_1} \end{pmatrix} : \mathrm{laxMat}_{\mathbb{C}}(\mathcal{X}, \mathcal{X})$$

since the indexing space of the colimit $\operatorname{colim}_{f : S(s,t)} \mathcal{X}_f$ is either empty in the case $s = 1$, $t = 0$ or a singleton otherwise.

We now finish the proof of Lemma 4.7.

Proof of Lemma 4.7, part (2). We only treat the case of postcomposition with $\mathcal{I}^{\mathcal{X}}$; the other statement is dual. We need to show that for every diagram $\mathcal{Y} : R \to \mathbb{C}$, the functor

$$\mathcal{I}^{\mathcal{X}} \circ_S - : \mathrm{laxMat}_{\mathbb{C}}(\mathcal{Y}, \mathcal{X}) \to \mathrm{laxMat}_{\mathbb{C}}(\mathcal{Y}, \mathcal{X})$$

is naturally equivalent to the identity. Naturally in $u : S$, $r : R^{\mathrm{op}}$ and $F : \mathrm{laxMat}_{\mathbb{C}}(\mathcal{Y}, \mathcal{X})$, we compute (in the ∞-category $\mathbb{C}(\mathcal{Y}_r, \mathcal{X}_s)$):

$$\begin{aligned} (\mathcal{I}^{\mathcal{X}} \circ_S F)_{ur} &\simeq \operatorname*{colim}_{(f : s \to t) : \mathrm{Tw}^*(S)} \big(\operatorname*{colim}_{g : S(t,u)} \mathcal{X}_g \big) \mathcal{X}_f F_{sr} \\ &= \operatorname*{colim}_{\substack{(f : s \to t) : \mathrm{Tw}^*(S) \\ g : S(t,u)}} \mathcal{X}_{gf} F_{sr}, \end{aligned}$$

where the shape of the second colimit is the category

$$\int_{(f:s\to t):\mathrm{Tw}^*(S)} S(t,u) = \mathrm{Tw}^*(S) \times_{S^{\mathrm{op}}} (S/u)^{\mathrm{op}}.$$

Note that the diagram $(f,g) \mapsto \mathcal{X}_{gf} F_{sr}$ over which we are taking the colimit arises as the pullback of the diagram

$$S/u \to \mathbb{C}(\mathcal{Y}_r, \mathcal{X}_u), \quad (h: s \to u) \mapsto \mathcal{X}_h F_{sr}$$

along the functor

$$\gamma: \mathrm{Tw}^*(S) \times_{S^{\mathrm{op}}} (S/u)^{\mathrm{op}} \to S/u, \quad (f: s \to t,\ g: t \to u) \mapsto (gf: s \to u).$$

This functor γ has a left adjoint

$$S/u \to \mathrm{Tw}^*(S) \times_{S^{\mathrm{op}}} (S/u)^{\mathrm{op}}, \quad (f: s \to u) \mapsto (f: s \to u,\ \mathrm{id}_u: u \to u),$$

and is therefore homotopy terminal. This allows us to finish the computation:

$$\begin{aligned}(\mathcal{I}^{\mathcal{X}} \circ_S F)_{ur} &\simeq \operatorname*{colim}_{\substack{(f:s\to t):\mathrm{Tw}^*(S)\\ g:S(t,u)}} \mathcal{X}_{gf} F_{sr} \\ &\simeq \operatorname*{colim}_{(h:s\to u):S/u} \mathcal{X}_h F_{sr} \\ &\simeq \mathcal{X}_{\mathrm{id}_u} F_{ur} = F_{ur},\end{aligned}$$

using in the last step that $\mathrm{id}_u: u \to u$ is a terminal object of the comma category S/u. ■

We can characterize lax limit and colimits purely in terms of the matrix calculus encoded in the category $\mathrm{hoLaxMat}_{\mathbb{C}}$.

Lemma 4.14. *Let $\mathcal{X}: S \to \mathbb{C}$ be a diagram.*

(1) *A lax cone $P : \mathrm{laxMat}_{\mathbb{C}}(L, \mathcal{X})$ is a lax limit cone if and only if it is an isomorphism in the category* $\mathrm{hoLaxMat}_{\mathbb{C}}$.

(2) *A lax cocone $I : \mathrm{laxMat}_{\mathbb{C}}(\mathcal{X}, L)$ is a lax colimit cone if and only if it is an isomorphism in the category* $\mathrm{hoLaxMat}_{\mathbb{C}}$.

Proof. We prove the statement about lax cones; the other one is dual.

First assume that P is a lax limit cone. Let $\mathcal{Y}: T \to \mathbb{C}$ be a diagram in $\mathbb{C}$. Using the defining universal property of Definition 3.2 on $L' := \mathcal{Y}_t$ for each $t : T^{\mathrm{op}}$, we see that the functor

$$P \circ -: \mathrm{laxMat}_{\mathbb{C}}(\mathcal{Y}, L) = \operatorname*{laxlim}_{t:T^{\mathrm{op}}} \mathbb{C}(\mathcal{Y}_t, L) \xrightarrow{\simeq} \operatorname*{laxlim}_{(s,t):S\times T^{\mathrm{op}}} \mathbb{C}(\mathcal{Y}_t, \mathcal{X}_s) = \mathrm{laxMat}_{\mathbb{C}}(\mathcal{Y}, \mathcal{X})$$

is an equivalence of ∞-categories. In particular, after passing to equivalence classes of matrices, the map

$$P \circ -: \mathrm{hoLaxMat}_{\mathbb{C}}(\mathcal{Y}, L) \xrightarrow{\cong} \mathrm{hoLaxMat}_{\mathbb{C}}(\mathcal{Y}, \mathcal{X})$$

is a bijection. Since $\mathcal{Y}$ was arbitrary, it follows that $P\colon L \to \mathcal{X}$ is an isomorphism in the category $\mathrm{hoLaxMat}_{\mathbb{C}}$.

Conversely, assume that P has an inverse in $\mathrm{hoLaxMat}_{\mathbb{C}}$, i.e., that we have a lax cocone $I : \mathrm{laxMat}_{\mathbb{C}}(\mathcal{X}, L)$ satisfying $P \circ I \simeq \mathcal{I}^{\mathcal{X}}$ and $I \circ_S P \simeq \mathrm{id}_L$. Then for each $L' : \mathbb{C}$ we have equivalences

$$(P \circ -) \circ (I \circ_S -) = \big(P \circ (I \circ_S -)\big) \simeq (P \circ I) \circ_S - \simeq \mathcal{I}^{\mathcal{X}} \circ_S - \simeq \mathrm{id}$$

and

$$(I \circ_S -) \circ (P \circ -) = I \circ_S (P \circ -) \simeq (I \circ_S P) \circ - \simeq \mathrm{id}_L \circ - = \mathrm{id}$$

as endofunctors of

$$\mathrm{laxMat}_{\mathbb{C}}(L', \mathcal{X}) \quad \text{and} \quad \mathbb{C}(L', L),$$

respectively (using Lemma 4.7), showing that $P \circ -$ is an equivalence of ∞-categories, as required. ■

Recall the standing assumption of this section, that $\mathbb{C}$ is enriched in ∞-categories with colimits.

Corollary 4.15. *A diagram $\mathcal{X}\colon S \to \mathbb{C}$ admits a lax limit if and only if it admits a lax colimit. When they exist, the unit matrix $\mathcal{I}^{\mathcal{X}}$: $\mathrm{laxMat}_{\mathbb{C}}(\mathcal{X}, \mathcal{X})$ corresponds to an equivalence*

$$\underline{\mathcal{I}}^{\mathcal{X}}\colon \operatorname*{laxcolim}_S \mathcal{X} \xrightarrow{\simeq} \operatorname*{laxlim}_S \mathcal{X}. \tag{4.16}$$

Proof. The diagram $\mathcal{X}$ admits a lax (co)limit if and only if it is isomorphic in $\mathrm{hoLaxMat}_{\mathbb{C}}$ to an object $L\colon \{*\} \to \mathbb{C}$. In this case L is both the lax limit and the lax colimit, exhibited by mutually inverse lax (co)cones $I\colon \mathcal{X} \to L$ and $P\colon L \to \mathcal{X}$. By definition, the map $\underline{\mathcal{I}}^{\mathcal{X}}\colon L \to L$ corresponding to the matrix $\mathcal{I}^{\mathcal{X}}$ is determined (up to equivalence) by the property that $P \circ \underline{\mathcal{I}}^{\mathcal{X}} \circ I \simeq \mathcal{I}^{\mathcal{X}}$. Since the identity id_L satisfies this property, we conclude that $\underline{\mathcal{I}}^{\mathcal{X}} \simeq \mathrm{id}_L$; in particular this map is an equivalence. ■

Remark 4.17. The comparison map (4.16) does not just depend on the objects

$$L = \operatorname{laxlim} \mathcal{X} \quad \text{and} \quad L' = \operatorname{laxcolim} \mathcal{X}$$

but on the implicit lax (co)limit cones $P\colon \mathrm{laxMat}_{\mathbb{C}}(L, \mathcal{X})$ and $I\colon \mathrm{laxMat}_{\mathbb{C}}(\mathcal{X}, L')$. Specifically, the map $\underline{\mathcal{I}}^{\mathcal{X}}$ is characterized up to equivalence by the relation

$$P \circ \underline{\mathcal{I}}^{\mathcal{X}} \circ I \simeq \mathcal{I}^{\mathcal{X}}$$

or equivalently

$$\underline{\mathcal{I}}^{\mathcal{X}} \simeq (I \circ_S P)^{-1} \tag{4.18}$$

(since P and I are isomorphisms in $\mathrm{hoLaxMat}_{\mathbb{C}}$ and $\mathcal{I}^{X}$ is the identity on $\mathcal{X} : \mathrm{hoLaxMat}_{\mathbb{C}}$).

Having described lax (co)limits via lax matrix formulas in $\mathrm{hoLaxMat}_{\mathbb{C}}$, we can immediately deduce that all lax (co)limits are absolute with respect to the colimit-enrichment.

Theorem 4.19. *Let $\mathbb{C}, \mathbb{C}'$ be $(\infty, 2)$-categories enriched in ∞-categories with colimits. Let $\mathbb{F}: \mathbb{C} \to \mathbb{C}'$ be a functor which preserves colimits on hom-categories. Then $\mathbb{F}$ preserves all lax colimits and lax limits.*

Proof. Since $\mathbb{F}$ preserves colimits on hom-categories, it induces a well defined functor

$$\mathrm{hoLaxMat}_{\mathbb{F}}: \mathrm{hoLaxMat}_{\mathbb{C}} \to \mathrm{hoLaxMat}_{\mathbb{C}'},$$

given by applying $\mathbb{F}$ pointwise to diagram and matrices. Since this functor necessarily sends isomorphisms to isomorphisms, Lemma 4.14 implies that $\mathbb{F}$ sends lax (co)limit cones to lax (co)limit cones. ∎

Finally, we deduce that lax matrix multiplication corresponds to composition of maps between lax colimits/limits in the case where those lax (co)limits exist.

Proposition 4.20. *Let $\mathcal{X}: S \to \mathbb{C}$, $\mathcal{Y}: T \to \mathbb{C}$, $\mathcal{Z}: U \to \mathbb{C}$ be diagrams indexed by ∞-categories and admitting lax limits/colimits. Then there is a commutative square of ∞-categories*

$$\begin{tikzcd}
\mathbb{C}(\mathrm{laxcolim}_S\, \mathcal{X}, \mathrm{laxlim}_T\, \mathcal{Y}) \times \mathbb{C}(\mathrm{laxcolim}_T\, \mathcal{Y}, \mathrm{laxlim}_U\, \mathcal{Z}) \arrow[r, "\simeq"] \arrow[d, "- \circ (\underline{\mathcal{I}}^{\mathcal{Y}})^{-1} \circ -"] & \mathrm{laxMat}_{\mathbb{C}}(\mathcal{X}, \mathcal{Y}) \times \mathrm{laxMat}_{\mathbb{C}}(\mathcal{Y}, \mathcal{Z}) \arrow[d, "- \circ_T -"] \\
\mathbb{C}(\mathrm{laxcolim}_S\, \mathcal{X}, \mathrm{laxlim}_U\, \mathcal{Z}) \arrow[r, "\simeq"] & \mathrm{laxMat}_{\mathbb{C}}(\mathcal{X}, \mathcal{Z}).
\end{tikzcd}$$

In other words, after identifying lax colimits and lax limits via the canonical unit matrix, lax matrix multiplication corresponds precisely to function composition.

Proof. Denote by $I_{\mathcal{X}}: \mathrm{laxMat}_{\mathbb{C}}(\mathcal{X}, \mathrm{laxcolim}\, \mathcal{X})$ and $P_{\mathcal{X}}: \mathrm{laxMat}_{\mathbb{C}}(\mathrm{laxlim}\, \mathcal{X}, \mathcal{X})$ two lax (co)limits cones for the diagram $\mathcal{X}$ (and similarly for $\mathcal{Y}$ and $\mathcal{Z}$). The implicit identification

$$\mathbb{C}(\mathrm{laxcolim}\, \mathcal{X}, \mathrm{laxlim}\, \mathcal{Y}) \xrightarrow{\simeq} \mathrm{laxMat}_{\mathbb{C}}(\mathcal{X}, \mathcal{Y})$$

is given explicitly as $P_{\mathcal{Y}} \circ - \circ I_{\mathcal{X}}$, and similarly for the other horizontal maps. Therefore the desired commutative square is just the natural equivalence

$$\begin{aligned}
(P_{\mathcal{Z}} \circ - \circ I_{\mathcal{Y}}) \circ_T (P_{\mathcal{Y}} \circ - \circ I_{\mathcal{X}}) &\simeq P_{\mathcal{Z}} \circ - \circ (I_{\mathcal{Y}} \circ_T P_{\mathcal{Y}}) \circ - \circ I_{\mathcal{X}} \\
&\simeq P_{\mathcal{Z}} \circ \big(- \circ (\underline{\mathcal{I}}^{\mathcal{Y}})^{-1} \circ - \big) \circ I_{\mathcal{X}}.
\end{aligned}$$

using the equivalence (4.18) in the last step. ∎

Example 4.21. Continuing Example 4.13, we consider a Δ^1-diagram $\mathcal{Y} = (\mathcal{Y}_0 \xrightarrow{\mathcal{Y}_{10}} \mathcal{Y}_1)$ in $\mathbb{C}$ and two lax matrices

$$F = \begin{pmatrix} F_0 \\ \downarrow \\ F_1 \end{pmatrix} : \mathrm{laxMat}_{\mathbb{C}}(X, \mathcal{Y}) \quad \text{and} \quad G = (G_0 \leftarrow G_1) : \mathrm{laxMat}_{\mathbb{C}}(\mathcal{Y}, Z)$$

(just a lax cone and a lax cocone, really). The Cartesian twisted arrow category $\mathrm{Tw}^*(\Delta^1)$ is the poset

$$\begin{tikzcd} \mathrm{id}_0 & (0 \xrightarrow{10} 1) \arrow[l] \arrow[d] \\ & \mathrm{id}_1 \end{tikzcd}$$

The matrix product $G \circ_{\Delta^1} F : \mathrm{laxMat}_{\mathbb{C}}(X, Z) = \mathbb{C}(X, Z)$ is therefore the pushout of the diagram

$$\begin{tikzcd} G_0F_0 & G_1\mathcal{Y}_{10}F_0 \arrow[l, "gF_0"'] \arrow[d, "G_1 f"] \\ & G_1F_1 \end{tikzcd}$$

computed in the ∞-category $\mathbb{C}(X, Y)$, where $g\colon G_1\mathcal{Y}_{10} \to G_0$ and $f\colon \mathcal{Y}_{10}F_0 \to F_1$ are the 2-cells encoded in G and F, respectively. More general $\Delta^1 \times (\Delta^1)^{\mathrm{op}}$ matrices can then be multiplied in the usual row-by-column fashion since each entry $(GF)_{us}$ only depends on row u of G and column s of F. For example, we can compute (with $\mathcal{X} = \mathcal{Y}$ and $F = \mathcal{I}^{\mathcal{Y}}$)

$$\begin{aligned} G \circ_{\Delta^1} \mathcal{I} &= \begin{pmatrix} G_{00} \leftarrow G_{01} \\ \downarrow \qquad \downarrow \\ G_{10} \leftarrow G_{11} \end{pmatrix} \begin{pmatrix} \mathrm{id}_{\mathcal{Y}_0} \longleftarrow \emptyset \\ \downarrow \qquad \downarrow \\ \mathcal{Y}_{10} \longleftarrow \mathrm{id}_{\mathcal{Y}_1} \end{pmatrix} \\ &\simeq \begin{pmatrix} \operatorname{colim}\big(G_{00} \leftarrow G_{01}\mathcal{Y}_{10} \xrightarrow{=} G_{01}\mathcal{Y}_{10}\big) \longleftarrow \operatorname{colim}\big(\emptyset \xleftarrow{=} \emptyset \to G_{01}\big) \\ \downarrow \qquad\qquad\qquad\qquad\qquad \downarrow \\ \operatorname{colim}\big(G_{10} \leftarrow G_{11}\mathcal{Y}_{10} \xrightarrow{=} G_{11}\mathcal{Y}_{10}\big) \longleftarrow \operatorname{colim}\big(\emptyset \xleftarrow{=} \emptyset \to G_{11}\big) \end{pmatrix} \simeq G. \end{aligned}$$

5. Lax additivity

Classical semi-additivity of a category $\mathcal{A}$ manifests itself on two levels:

(1) Each hom-set of A has a commutative monoid structure which allows to take sums $\sum_{s \in S} f_s$ indexed by arbitary finite sets S.

(2) The category A allows direct sums $\bigoplus_{s \in S} x_s$ indexed by finite sets S which are both products and coproducts.

We categorify these notions by replacing (discrete) addition $\Sigma_{s \in S}$ on the hom-sets by colimits $\operatorname{colim}_{s:S}$ on the hom-categories and (discrete) coproducts/products $\coprod_{s \in S} \simeq \prod_{s \in S}$ by lax bilimits $\operatorname{laxcolim}_{s:S} \simeq \operatorname{laxlim}_{s:S}$, which are now indexed by arbitary small ∞-categories S rather than finite sets.

Definition 5.1. Let $\mathbb{C}$ be an $(\infty, 2)$-category enriched in ∞-categories with groupoidal colimits. (This means that each hom-category $\mathbb{C}(X, Y)$ has all colimits indexed by ∞-groupoids and that composition preserves such colimits in each variable.) Let $\mathcal{X}\colon S \to \mathbb{C}$

a diagram indexed by an ∞-category S. A *lax bilimit* of $\mathcal{X}$ consists of a lax colimit cone I : $\mathrm{laxMat}_{\mathbb{C}}(\mathcal{X}, L')$ and a lax limit cone P : $\mathrm{laxMat}_{\mathbb{C}}(L, \mathcal{X})$ such that the canonical map

$$\underline{\mathcal{I}}^{\mathcal{X}}\colon L' \to L$$

corresponding to the unit matrix $\mathcal{I}^{\mathcal{X}}$: $\mathrm{laxMat}_{\mathbb{C}}(\mathcal{X}, \mathcal{X})$ is an equivalence. We identify L and L' via $\underline{\mathcal{I}}^{\mathcal{X}}$ and write

$$\bigoplus_{S}^{\mathrm{lax}} \mathcal{X} \quad \text{or} \quad \bigoplus_{s:S}^{\mathrm{lax}} \mathcal{X}_s$$

for both/either of them.

When $\mathcal{X}\colon S \to \{*\} \xrightarrow{X} \mathbb{C}$ is a constant diagram, we write $S \otimes X$ or X^S for the constant lax bilimit $\bigoplus_{s:S}^{\mathrm{lax}} X = \bigoplus_{S}^{\mathrm{lax}} \mathcal{X}$. In the special case where $S = \Delta^1$ so that $\mathcal{X} = (\mathcal{X}_0 \xrightarrow{H} \mathcal{X}_1)$ is just an arrow in $\mathbb{C}$, we also write $\mathcal{X}_0 \overset{\mathrm{lax}}{\oplus}_H \mathcal{X}_1$ instead of $\bigoplus_{\Delta^1}^{\mathrm{lax}} \mathcal{X}$.

Remark 5.2. When convenient we drop the typographical distinction between a matrix F : $\mathrm{laxMat}_{\mathbb{C}}(\mathcal{X}, \mathcal{Y})$ and the associated map $\underline{F}$: $\bigoplus^{\mathrm{lax}} \mathcal{X} \to \bigoplus^{\mathrm{lax}} \mathcal{Y}$. More generally, Proposition 4.20 justifies dropping the typographical distinction between matrix multiplication $G \circ_T F$ and composition $\underline{G} \circ \underline{F}$ of the associated maps

$$\bigoplus_{S}^{\mathrm{lax}} \mathcal{X} \xrightarrow{\underline{F}} \bigoplus_{T}^{\mathrm{lax}} \mathcal{Y} \xrightarrow{\underline{G}} \bigoplus_{U}^{\mathrm{lax}} \mathcal{Z}$$

between lax bilimits in $\mathbb{C}$.

We can now finally define the notion of lax semiadditivity.

Definition 5.3. An $(\infty, 2)$-category $\mathbb{A}$ is called *(finitely) lax semiadditive* if

(1) it is enriched in ∞-categories with (finite) colimits (with functors preserving them),

(2) each diagram $S \to \mathbb{A}$ indexed by a (finite) small ∞-category S admits a lax bilimit.

Remark 5.4. We have seen in Corollary 4.15 that in the presence of sufficently many colimits in the hom-categories, *every* lax limit or colimit is automatically a lax bilimit. Thus the second condition could be weakened to just require the existence of lax limits *or* lax colimits.

The final step is to categorify the passage from semiadditve to additive categories which amounts to requiring the hom-monoids to be abelian groups. Following our philosophy of Section 1.1, abelian groups should be replaced by stable ∞-categories leading to the following easy definition.

Definition 5.5. A (finitely) lax semiadditive $(\infty, 2)$-category $\mathbb{A}$ is called *(finitely) lax additive* if every hom-∞-category $\mathbb{A}(X, Y)$ is stable.

Remark 5.6. Denote by $\mathbb{C}^{\mathrm{op}}$ the $(\infty,2)$-category obtained from $\mathbb{C}$ by reversing the directions of the 1-morphisms, i.e., $\mathbb{C}^{\mathrm{op}}(X,Y) = \mathbb{C}(Y,X)$. If $\mathbb{C}$ is enriched in (stable) ∞-categories with colimits, then so is $\mathbb{C}^{\mathrm{op}}$. Moreover, lax limits/colimits/bilimits in $\mathbb{C}^{\mathrm{op}}$ correspond to lax colimits/limits/bilimits in $\mathbb{C}$. Thus an $(\infty,2)$-category $\mathbb{A}$ is (finitely) lax (semi)additive if and only if $\mathbb{A}^{\mathrm{op}}$ is (finitely) lax (semi)additive.

Example 5.7. Lax limits in the $(\infty,2)$-category $\mathcal{P}r^L$ of presentable ∞-categories exist and are computed as underlying ∞-categories, i.e., as sections of the Grothendieck construction as in Example 3.4; indeed, for any small diagram $\mathcal{X}\colon S \to \mathcal{P}r^L$ the ∞-category $L := \mathrm{Fun}_S(S, \int_* \mathcal{X})$ is again presentable (see [15, Propositions 5.5.3.17 and 5.5.3.3]) and a functor into it preserves colimits if and only if it does so after postcomposing with each pointwise evaluation map $L \to \mathcal{X}_s$. Since $\mathcal{P}r^L$ is enriched in ∞-categories with colimits it follows that it is a lax semiadditive $(\infty,2)$-category. The full $(\infty,2)$-category $\mathcal{S}t^L$ of presentable *stable* ∞-categories is closed under lax colimits and enriched in stable ∞-categories, thus it is lax additive. The $(\infty,2)$-category $\mathcal{S}t^R$ of presentable stable ∞-categories and right adjoint functors is only finitely lax additive, since composition with a right adjoint functor is exact but does not preserve arbitrary colimits. The $(\infty,2)$-category $\mathcal{S}t$ of stable ∞-categories and exact functors is finitely lax additive.

Note that one can replace all presentability assumptions by just requiring the relevant ∞-categories to have colimits (or limits, in the case of $\mathcal{S}t^R$) and for the functors between them to preserve them.

Remark 5.8. In the $(\infty,2)$-category $\mathcal{S}t^L$, the lax bilimit of a diagram $F\colon \mathcal{A} \to \mathcal{B}$ comes with a semiorthogonal decompositions with components $\mathcal{A}$ and $\mathcal{B}$ and gluing functor F in the sense of [7], see also Appendix B. The idea to use matrices to describe coordinate change for such semiorthogonal decompositions already appears in [6]. Enlarging $\mathcal{S}t^L$ by a suitably defined $(\infty,2)$-category of exact profunctors, we may even describe general semiorthogonal decompositions via lax bilimits – in the context of dg categories, this corresponds to the gluing constructions for bimodules, as established in [14].

Construction 5.9. Let T, S be small ∞-categories and let

$$F\colon T \times S^{\mathrm{op}} \to \mathrm{Spaces} \simeq \mathcal{P}r^L(\mathrm{Spaces}, \mathrm{Spaces})$$

be a matrix of spaces. Let $\mathbb{A}$ be a lax semiadditive $(\infty,2)$-category. For each $X : \mathbb{A}$, denote by $F^X := F \otimes \mathrm{id}_X$ the matrix

$$F^X\colon T \times S^{\mathrm{op}} \xrightarrow{F} \mathrm{Spaces} \xrightarrow{-\otimes \mathrm{id}_X} \mathbb{A}(X,X),$$

where the second functor arises from the tensoring by Spaces on the ∞-category $\mathbb{A}(X,X)$ with colimits. In this way, the space-valued matrix F gives rise to a map $F^X\colon X^S \to X^T$, which we call the action of F on X. Similarly, when the hom-categories are pointed/stable, we can interpret in $\mathbb{A}$ every matrix of pointed spaces/spectra, by using the corresponding tensoring.

Lemma 5.10. *Action of matrices is compatible with matrix multiplication, i.e., we have equivalences $F^X \circ G^X \simeq (F \circ G)^X$ whenever F, G are composable matrices of spaces/pointed spaces/spectra and X is an object in a correspondingly enriched $(\infty, 2)$-category.*

Proof. We have, naturally in $u : U$, $s : S^{\mathrm{op}}$:

$$\begin{aligned}(F \circ G)^X_{us} &\simeq \big(\operatorname*{colim}_{f: t \to t'} F_{ut'} \otimes G_{ts}\big) \otimes \mathrm{id}_X \\ &\simeq \big(\operatorname*{colim}_{f: t \to t'} (F_{ut'} \otimes \mathrm{id}_X) \circ (G_{ts} \otimes \mathrm{id}_X)\big) \\ &\simeq (F^X \circ G^X)_{us}\end{aligned}$$

using that the tensoring preserves colimits and that $\mathrm{id}_X \circ \mathrm{id}_X \simeq \mathrm{id}_X$. ∎

Example 5.11. The universal identity $\Delta^1 \times (\Delta^1)^{\mathrm{op}}$-matrix is

$$\mathcal{I} = \begin{pmatrix} \{*\} \longleftarrow \emptyset \\ \downarrow \qquad \downarrow \\ \{*\} \longleftarrow \{*\} \end{pmatrix} : \Delta^1 \times (\Delta^1)^{\mathrm{op}} \to \text{Spaces}.$$

Indeed, every $(\infty, 2)$-category $\mathbb{C}$ whose hom-categories have initial objects, the unit matrix $\mathcal{I}^{\mathcal{X}}$ for the constant diagram $\mathcal{X} \colon \Delta^1 \to \{*\} \xrightarrow{X} \mathbb{C}$ is given by $\mathcal{I}^{\mathcal{X}} = \mathcal{I}^X := \mathcal{I} \otimes \mathrm{id}_X$.

More generally, for any ∞-category S the universal identity $S \times S^{\mathrm{op}}$-matrix is the just the transpose of the hom-functor

$$\mathcal{I}_{\mathsf{T}} = S(-,-) \colon S^{\mathrm{op}} \times S \to \text{Spaces}.$$

Example 5.12. Consider the matrices

$$\mathrm{Cof} := \begin{pmatrix} \mathbb{S}^0 \longleftarrow \mathbb{S}^0 \\ \downarrow \qquad \downarrow \\ 0 \longleftarrow \mathbb{S}^0 \end{pmatrix} : \Delta^1 \times (\Delta^1)^{\mathrm{op}} \to \text{Spaces}_*$$

and

$$\mathrm{Fib} := \begin{pmatrix} 0 \longleftarrow \mathbb{S}[-1] \\ \downarrow \quad \square \quad \downarrow \\ \mathbb{S} \longleftarrow 0 \end{pmatrix} : \Delta^1 \times (\Delta^1)^{\mathrm{op}} \to Sp,$$

defined in pointed spaces and spectra, respectively. If $\mathcal{A}$ is a pointed ∞-category with colimits, then the matrix Cof acts as the cofiber map

$$\mathrm{Cof}_{\mathcal{A}} = \begin{pmatrix} \mathrm{id}_{\mathcal{A}} \longleftarrow \mathrm{id}_{\mathcal{A}} \\ \downarrow \qquad \downarrow \\ 0 \longleftarrow \mathrm{id}_{\mathcal{A}} \end{pmatrix} : \mathcal{A}^{\Delta^1} \to \mathcal{A}^{\Delta^1}.$$

Indeed, for every

$$\begin{pmatrix} a \\ \downarrow \\ b \end{pmatrix} : \mathcal{B} \to \mathcal{A}^{\Delta^1}$$

we can compute the matrix product

$$\begin{pmatrix} \mathrm{id}_A \leftarrow \mathrm{id}_A \\ \downarrow \qquad \downarrow \\ 0 \longleftarrow \mathrm{id}_A \end{pmatrix} \circ \begin{pmatrix} a \\ \downarrow \\ b \end{pmatrix} \simeq \begin{pmatrix} \mathrm{colim}(a \leftarrow a \to b) \\ \downarrow \\ \mathrm{colim}(0 \leftarrow a \to b) \end{pmatrix} \simeq \begin{pmatrix} b \\ \downarrow \\ \mathrm{cof}(a \to b) \end{pmatrix} = \mathrm{Cof}(a \to b).$$

If $\mathcal{A}$ is also stable, then a similar calculation shows that the matrix Fib is inverse to Cof and acts as the fiber map.

For every matrix $F\colon T \times S^{\mathrm{op}} \to \mathcal{A}(X, X)$ corresponding to an arrow $X^S \to X^T$ in $\mathbb{A}$, we denote by F_{T} the "transposed" matrix

$$F_{\mathsf{T}}\colon S^{\mathrm{op}} \times (T^{\mathrm{op}})^{\mathrm{op}} \simeq T \times S^{\mathrm{op}} \xrightarrow{F} \mathcal{A}(X, X),$$

describing the dual map $X^{T^{\mathrm{op}}} \to X^{S^{\mathrm{op}}}$.

Lemma 5.13. *In the setting of Construction 5.9 there are commutative diagrams*

$$\begin{array}{ccc} \mathbb{A}(X, Y^S) & \xrightarrow{\simeq} & \mathbb{A}(X, Y)^S \\ {\scriptstyle \mathbb{A}(X, F^Y)}\big\downarrow & & \big\downarrow{\scriptstyle F^{\mathbb{A}(X,Y)}} \\ \mathbb{A}(X, Y^T) & \xrightarrow{\simeq} & \mathbb{A}(X, Y)^T \end{array} \quad \textit{and} \quad \begin{array}{ccc} \mathbb{A}(S \otimes X, Y) & \xrightarrow{\simeq} & \mathbb{A}(X, Y)^{S^{\mathrm{op}}} \\ {\scriptstyle \mathbb{A}(F^X, Y)}\big\uparrow & & \big\uparrow{\scriptstyle F_{\mathsf{T}}^{\mathbb{A}(X,Y)}} \\ \mathbb{A}(T \otimes X, Y) & \xrightarrow{\simeq} & \mathbb{A}(X, Y)^{T^{\mathrm{op}}} \end{array}$$

Proof. We do the second computation; the first one is similar. Functorially in

$$M : \mathbb{A}(X, Y)^{T^{\mathrm{op}}} \simeq \mathbb{A}(T \otimes X, Y)$$

and $s : S^{\mathrm{op}}$ we compute

$$\begin{aligned} \mathbb{A}(F^X, Y)(M)_s = (M \circ F^X)_s &= \operatorname*{colim}_{(f:t \to t'):\mathrm{Tw}^*(T)} M_{t'} \circ F^X_{ts} \\ &= \operatorname*{colim}_{f:\mathrm{Tw}^*(T)} M_{t'} \circ (F_{ts} \otimes \mathrm{id}_X) \\ &\simeq \operatorname*{colim}_{(f:t \to t'):\mathrm{Tw}^*(T)} F_{ts} \otimes M_{t'} \\ &\simeq \operatorname*{colim}_{(g:t' \to t):\mathrm{Tw}^*(T^{\mathrm{op}})} F_{ts} \otimes M_{t'} \\ &\simeq \operatorname*{colim}_{(g:t' \to t):\mathrm{Tw}^*(T^{\mathrm{op}})} (F_{ts} \otimes \mathrm{id}_{\mathbb{A}(X,Y)})(M_{t'}) \\ &= \operatorname*{colim}_{(g:t' \to t):\mathrm{Tw}^*(T^{\mathrm{op}})} \big((F_{\mathsf{T}})_{st} \otimes \mathrm{id}_{\mathbb{A}(X,Y)}\big)(M_{t'}) \\ &= F_{\mathsf{T}}^{\mathbb{A}(X,Y)}(M)_s. \end{aligned}$$

Apart from expanding the various definitions, we have used

- that $M_{t'} \circ -\colon \mathbb{A}(X, X) \to \mathbb{A}(X, Y)$ preserves colimits and hence the tensoring $F_{ts} \otimes -$,
- the canonical identification $\mathrm{Tw}^*(T) \simeq \mathrm{Tw}^*(T^{\mathrm{op}})$ which reverses source and target,
- that colimits of functors $\mathbb{A}(X, Y) \to \mathbb{A}(X, Y)$ are computed pointwise, hence also the tensoring $F_{ts} \otimes -$. ■

Lemma 5.14. *Let $\mathbb{A}$ be a lax semiadditive $(\infty, 2)$-category. Then $\mathbb{A}$ is lax additive if and only if each hom-category $\mathbb{A}(X, Y)$ is pointed and the matrix* Cof *from Example 5.12 acts invertibly on each $X : \mathbb{A}$. If this is the case, then the inverse is given by the action of* Fib.

Proof. Assuming that all hom-categories $\mathbb{A}(X, Y)$ of the lax semiadditive $(\infty, 2)$-category $\mathbb{A}$ are pointed, they are stable if and only if the cofiber functor

$$\mathrm{Cof}^{\mathbb{A}(X,Y)} \colon \mathbb{A}(X, Y)^{\Delta^1} \to \mathbb{A}(X, Y)^{\Delta^1}$$

is invertible. Using Lemma 5.13, we can identify this functor with $\mathbb{A}(X, \mathrm{Cof}^Y)$. Thus $\mathbb{A}$ is lax additive if and only if all $\mathbb{A}(X, Y)$ are stable, if and only if all $\mathbb{A}(X, \mathrm{Cof}^Y)$ are invertible, if and only if all Cof^Y are invertible, as claimed. ■

5.1. Oplax additivity

So far we have focused our discussion exclusively on *lax* limits and colimits, as opposed to *oplax* ones. We could have of course passed to the 2-morphism dual everywhere (obtained from an $(\infty, 2)$-category $\mathbb{C}$ by replacing each hom-category $\mathbb{C}(X, Y)$ by its opposite) and told an analogous story using *oplax* colimits/limits/bilimits. This would lead to what we might call oplax (semi)additive $(\infty, 2)$-categories $\mathbb{A}$, which are enriched in (stable) ∞-categories with limits and allow the formation of oplax bilimits

$$\bigoplus_S^{\mathrm{oplax}} \mathcal{X} := \operatorname*{oplaxcolim}_S \mathcal{X} \simeq \operatorname*{oplaxlim}_S \mathcal{X}$$

of any diagram $\mathcal{X} \colon S \to \mathbb{A}$ indexed by a small ∞-category.

For the convenience of the reader, we summarize the main formulas of this dual theory; all the constructions and proofs are dual to the ones we saw earlier.

(1) For two diagrams $\mathcal{X} \colon S \to \mathbb{A}$ and $\mathcal{Y} \colon T \to \mathbb{C}$, we define

$$\mathrm{oplaxMat}_{\mathbb{C}}(\mathcal{X}, \mathcal{Y}) := \operatorname*{oplaxlim}_{(t,s):T \times S^{\mathrm{op}}} \mathbb{C}(\mathcal{X}_s, \mathcal{Y}_t)$$

as the category of oplax matrices from $\mathcal{X}$ to $\mathcal{Y}$. Explicitly, such matrices are sections of the contravariant Grothendieck construction

$$\int^{(t,s):T \times S^{\mathrm{op}}} \mathbb{A}(\mathcal{X}_s, \mathcal{Y}_t) \to T^{\mathrm{op}} \times S.$$

(2) The oplax matrix multiplication

$$\mathrm{oplaxMat}_{\mathbb{C}}(\mathcal{X}, \mathcal{Y}) \times \mathrm{oplaxMat}_{\mathbb{C}}(\mathcal{Y}, \mathcal{Z}) \to \mathrm{oplaxMat}_{\mathbb{C}}(\mathcal{X}, \mathcal{Z})$$

is given by the formula

$$(\Phi \circ \Psi)_{us} := \lim_{(f:t \to t'):\mathrm{Tw}_*(T)} \Phi_{ut'} \mathcal{Y}_f \Psi_{ts}.$$

(3) The oplax unit matrix for a diagram $\mathcal{X}\colon S \to \mathbb{C}$ is

$$\mathcal{I}^{\mathcal{X}} = \big((t,s) \mapsto \lim_{f:S(s,t)} \mathcal{X}_f\big) : \mathrm{oplaxMat}_{\mathbb{C}}(\mathcal{X},\mathcal{X}).$$

Example 5.15. The $(\infty,2)$-category $\mathcal{S}t^L$ is enriched in stable ∞-categories and admits oplax limits; thus it is finitely oplax additive. It is *not* oplax additive because composition of functors does not preserve arbitrary limits.

Example 5.16. As in Example 4.21, consider a Δ^1-diagram $\mathcal{Y} = (\mathcal{Y}_0 \xrightarrow{\mathcal{Y}_{10}} \mathcal{Y}_1)$ in $\mathbb{C}$. The oplax matrix product over $\mathcal{Y}$ of an oplax cocone (= Δ^1-row vector) G with an oplax cone (= $(\Delta^1)^{\mathrm{op}}$-column vector) F is given by the dual formula

$$(G_0 \to G_1) \circ_{\Delta^1} \begin{pmatrix} F_0 \\ \uparrow \\ F_1 \end{pmatrix} = \lim \begin{pmatrix} G_0F_0 & \longrightarrow & G_1\mathcal{Y}_{10}F_0 \\ & & \uparrow \\ & & G_1F_1 \end{pmatrix}.$$

General $(\Delta^1)^{\mathrm{op}} \times \Delta$-matrices can then be multiplied in the usual row-by-column way. The oplax unit matrix on $\mathcal{Y}$ is

$$\mathcal{I}^{\mathcal{Y}} = \begin{pmatrix} \mathrm{id}_{\mathcal{Y}_0} & \longrightarrow & \{*\} \\ \uparrow & & \uparrow \\ \mathcal{Y}_{10} & \longrightarrow & \mathrm{id}_{\mathcal{Y}_1} \end{pmatrix},$$

where $\{*\}$ is the terminal object of $\mathbb{C}(\mathcal{Y}_1,\mathcal{Y}_0)$.

6. Coordinate change for Δ^1-matrices

We have seen that any lax semiadditive $(\infty,2)$-category admits a nicely behaved calculus of lax matrices. However, if we apply K_0 componentwise to the lax matrix multiplication for lax Δ^1-bilimits (see Examples 4.13 and 4.21) we obtain the very unusual formula

$$\begin{pmatrix} a_0 & a_1 \end{pmatrix} \circ \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = a_0b_0 - a_1b_0 + a_1b_1$$

or, more generally $A \circ B = AI^{-1}B$, where $I = K_0(\mathcal{I}^{\Delta^1}) = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ is the new unit matrix.

The goal of this section is to introduce a convenient "coordinate change" in the lax additive (as opposed to merely lax *semi*additive) setting, which up to a sign recovers the usual matrix multiplication on K_0.

The key ingredient is the cofiber-fiber-equivalence

$$\mathrm{Cof}\colon \mathrm{Fun}(\Delta^1,\mathcal{A}) \overset{\simeq}{\leftrightarrows} \mathrm{Fun}(\Delta^1,\mathcal{A}) : \mathrm{Fib},$$

$$\big(\mathrm{fib}(u') = b \xrightarrow{u} a\big) \leftrightarrow \big(a \xrightarrow{u'} b' = \mathrm{cof}(u)\big)$$

for every stable ∞-category $\mathcal{A}$.

More precisely, we make use of the following dependent version of the cofiber-fiber-equivalence which identifies the oplax limit over an arrow with the lax limit.

Lemma 6.1 ([5, Lemma 1.3]). *Let $f:\mathcal{A} \to \mathcal{B}$ be a diagram of stable ∞-categories. Then there is a natural equivalence*

$$\mathrm{Cof}\colon \underset{\Delta^1}{\mathrm{oplaxlim}}(\mathcal{B} \xleftarrow{f} \mathcal{A}) \overset{\simeq}{\longleftrightarrow} \underset{\Delta^1}{\mathrm{laxlim}}(\mathcal{A} \xrightarrow{f} \mathcal{B}) : \mathrm{Fib} \tag{6.2}$$

described by the formula

$$\big(b = \mathrm{fib}(u'), a, b \xrightarrow{u} fa\big) \leftrightarrow \big(a, b' = \mathrm{cof}(u), fa \xrightarrow{v} b'\big).$$

While not strictly necessary, it is convenient to implement this equivalence by explicit matrices using a combination of the lax and oplax matrix calculus.

For the remainder of the section, let $\mathbb{A}$ be a lax additive $(\infty, 2)$-category. Then $\mathbb{A}$ is in particular enriched in ∞-categories with finite limits, so that we have available the finite oplax matrix calculus (dual to the one in Section 4) as long as we restrict to diagrams indexed by *finite* ∞-categories S.

Construction 6.3. Let $\mathcal{X}\colon \Delta^1 \to \mathbb{A}$ be a diagram, $\mathcal{X} = (\mathcal{X}_0 \xrightarrow{F} \mathcal{X}_1)$ and let $P = (p_s)$ and $I = (i_s)$ (indexed by $s : \Delta^1$) be the lax limit/colimit cone exhibiting the lax bilimit $\mathcal{X}_0 \overset{\mathrm{lax}}{\oplus}_F \mathcal{X}_1$. We construct the oplax cone and cocone

$$\mathrm{Fib} := \begin{tikzcd} & \mathcal{X}_0 \overset{\mathrm{lax}}{\oplus}_F \mathcal{X}_1 \arrow[dl, "p_0"'] \arrow[dr, "\mathrm{fib}"] \arrow[d, phantom, "\Leftarrow"] & \\ \mathcal{X}_0 \arrow[rr, "F"'] & {} & \mathcal{X}_1 \end{tikzcd} \quad \text{and} \quad \mathrm{Fib}^\vee := \begin{tikzcd} & \mathcal{X}_0 \overset{\mathrm{lax}}{\oplus}_F \mathcal{X}_1 \arrow[d, phantom, "\Rightarrow"] & \\ \mathcal{X}_0 \arrow[ur, "\mathrm{fib}^\vee"] \arrow[rr, "F"'] & {} & \mathcal{X}_1 \arrow[ul, "i_1"'] \end{tikzcd}$$

as follows:

- Recall that

$$p_0 = (\mathrm{id}_{\mathcal{X}_0} \leftarrow 0) \quad \text{and} \quad i_1 = \begin{pmatrix} 0 \\ \downarrow \\ \mathrm{id}_{\mathcal{X}_1} \end{pmatrix}$$

are just the top row and right column of the unit matrix $\mathcal{I}^{\mathcal{X}}$, viewed as a map out of or into the lax limit of $\mathcal{X}$, respectively.

- Additionally we define the row and column vectors

$$\mathrm{fib} := \big(0 \leftarrow [-1]_{\mathcal{X}_1}\big) \quad \text{and} \quad \mathrm{fib}^\vee := \begin{pmatrix} [-1]_{\mathcal{X}_0} \\ \downarrow \\ 0 \end{pmatrix}$$

obtained by passing to the vertical and horizontal fibers of the unit matrix $\mathcal{I}^{\mathcal{X}}$.

- By construction, (p_0, fib) and $(\mathrm{fib}^\vee, i_1)$ fit into a $(\Delta^1)^{\mathrm{op}} \times (\Delta^1)^{\mathrm{op}}$-matrix and a $\Delta^1 \times \Delta^1$-matrix,

$$\mathrm{Fib} := \begin{pmatrix} p_0 \\ \uparrow \\ \mathrm{fib} \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{\mathcal{X}_0} \longleftarrow 0 \\ \uparrow \quad \square \quad \uparrow \\ 0 \longleftarrow [-1]_{\mathcal{X}_1} \end{pmatrix} \text{ and } \mathrm{Fib}^\vee := (\mathrm{fib}^\vee \to i_1) = \begin{pmatrix} [-1]_{\mathcal{X}_0} \longrightarrow 0 \\ \downarrow \quad \square \quad \downarrow \\ 0 \longrightarrow \mathrm{id}_{\mathcal{X}_1} \end{pmatrix},$$

respectively. As indicated, we can view these matrices as a column vector of row vectors, or a row vector of column vectors, respectively, thus yielding the desired oplax cone and cocone.

The following lemma explains the name of the cones Fib and $\mathrm{Fib}^\vee$ in terms of the maps they represent/corepresent.

Lemma 6.4. *Let $Y : \mathbb{A}$.*

(1) *The induced map*

$$\operatorname*{laxlim}_{s:\Delta^1} \mathbb{A}(Y, \mathcal{X}_s) \xleftarrow{\simeq} \mathbb{A}\big(Y, \mathcal{X}_0 \overset{\mathrm{lax}}{\oplus}_F \mathcal{X}_1\big) \xrightarrow{\mathrm{Fib}\circ -} \operatorname*{oplaxlim}_{s:\Delta^1} \mathbb{A}(Y, \mathcal{X}_s)$$

is precisely the dependent fiber functor of Lemma 6.1 for the Δ^1-diagram $\mathbb{A}(Y, \mathcal{X}_0) \to \mathbb{A}(Y, \mathcal{X}_1)$.

(2) *The induced map*

$$\operatorname*{laxlim}_{s:(\Delta^1)^{\mathrm{op}}} \mathbb{A}(\mathcal{X}_s, Y) \xleftarrow{\simeq} \mathbb{A}\big(\mathcal{X}_0 \overset{\mathrm{lax}}{\oplus}_F \mathcal{X}_1, Y\big) \xrightarrow{-\circ \mathrm{Fib}^\vee} \operatorname*{oplaxlim}_{s:(\Delta^1)^{\mathrm{op}}} \mathbb{A}(\mathcal{X}_s, Y)$$

is precisely the dependent fiber functor of Lemma 6.1 for the $(\Delta^1)^{\mathrm{op}} = \Delta^1$-diagram $\mathbb{A}(\mathcal{X}_0, Y) \leftarrow \mathbb{A}(\mathcal{X}_1, Y)$.

Proof. A quick matrix computation for each $x = \begin{pmatrix} x_0 \\ \downarrow \\ x_1 \end{pmatrix} : \mathrm{laxlim}_s\, \mathbb{A}(Y, \mathcal{X}_s)$ shows

$$\mathrm{Fib} \circ x = \begin{pmatrix} p_0 \circ_{\Delta^1} x \\ \uparrow \\ \mathrm{fib} \circ_{\Delta^1} x \end{pmatrix} = \begin{pmatrix} x_0 \\ \uparrow \\ \mathrm{fib}(F x_0 \to x_1) \end{pmatrix} : \operatorname*{oplaxlim}_s \mathbb{A}(\mathcal{X}_s, Y),$$

as required. Similarly, for each $x^\vee = (x_0^\vee \leftarrow x_1^\vee) : \mathrm{laxlim}_{s:(\Delta^1)^{\mathrm{op}}} \mathbb{A}(\mathcal{X}_s, Y)$ we have

$$x^\vee \circ \mathrm{Fib}^\vee = \big(\mathrm{fib}(x_1^\vee F \to x_0^\vee) \to x_1^\vee\big) : \operatorname*{oplaxlim}_s \mathbb{A}(\mathcal{X}_s, Y),$$

as desired. ∎

As an immediate application of Lemma 6.1 we therefore get that the oplax cone/cocones Fib and $\mathrm{Fib}^\vee$ exhibit the lax bilimit $\mathcal{X}_0 \overset{\mathrm{lax}}{\oplus}_F \mathcal{X}_1$ also as an oplax limit and colimit. The following lemma makes a more precise statement, showing that Fib and $\mathrm{Fib}^\vee$ are inverse up to a shift.

Lemma 6.5. *The oplax cone* Fib *and cocone* $\mathrm{Fib}^\vee$ *from Construction 6.3 are, up to negative shift* $[-1]$*, mutually inverse with respect to the oplax matrix multiplication. In particular,* Fib *and* $\mathrm{Fib}^\vee[1]$ *(or* $\mathrm{Fib}[1]$ *and* $\mathrm{Fib}^\vee$*) exhibit the lax bilimit* $\mathcal{X}_0 \overset{\mathrm{lax}}{\oplus}_F \mathcal{X}_1$ *also as an oplax bilimit of the diagram* $\mathcal{X}\colon \Delta^1 \to \mathbb{A}$.

Proof. An explicit computation with the oplax matrix multiplication over Δ^1 (recall Example 5.16) shows:

$$\mathrm{Fib} \circ \mathrm{Fib}^\vee = \begin{pmatrix} p_0 \mathrm{fib}^\vee \longrightarrow p_0 i_1 \\ \uparrow \qquad\qquad \uparrow \\ \mathrm{fib}\mathrm{fib}^\vee \longrightarrow \mathrm{fib} i_1 \end{pmatrix} \simeq \begin{pmatrix} \mathrm{id}_{\mathcal{X}_0}[-1] \longrightarrow 0 \\ \uparrow \qquad\qquad\qquad \uparrow \\ F[-1] \longrightarrow \mathrm{id}_{\mathcal{X}_1}[-1] \end{pmatrix} = \mathcal{J}^{\mathcal{X}}[-1]$$

and

$$\begin{aligned} \mathrm{Fib}^\vee \circ_{\Delta^1} \mathrm{Fib} &\simeq \lim(\mathrm{fib}^\vee p_0 \to i_1 F p_0 \leftarrow i_1 \mathrm{fib}) \\ &\simeq \lim\left(\begin{pmatrix} [-1] \leftarrow 0 \\ \downarrow \qquad \downarrow \\ 0 \longleftarrow 0 \end{pmatrix} \to \begin{pmatrix} 0 \leftarrow 0 \\ \downarrow \quad \downarrow \\ F \leftarrow 0 \end{pmatrix} \leftarrow \begin{pmatrix} 0 \longleftarrow 0 \\ \downarrow \qquad \downarrow \\ 0 \leftarrow [-1] \end{pmatrix} \right) \\ &\simeq \begin{pmatrix} [-1] \longleftarrow 0 \\ \downarrow \qquad\qquad \downarrow \\ F[-1] \leftarrow [-1] \end{pmatrix} \simeq \mathcal{I}^{\mathcal{X}}[-1], \end{aligned}$$

where in the second computation we omit the straightforward verification that the unnamed arrows appearing in the last matrix are indeed those of $\mathcal{I}^{\mathcal{X}}[-1]$. ∎

Remark 6.6. While there is a distinguished choice for the cofiber-fiber equivalence (6.2), Lemma 6.5 provides two (but equally distinguished) ways to identify the lax bilimit $\mathcal{X}_0 \overset{\mathrm{lax}}{\oplus}_F \mathcal{X}_1$ and the oplax bilimit $\mathcal{X}_0 \overset{\mathrm{oplax}}{\oplus}_F \mathcal{X}_1$, depending on whether we look at the represented map (using Fib and treating them as (op)lax limits) or the corepresented map (using $\mathrm{Fib}^\vee$ and treating them as (op)lax colimits). These two ways are not equivalent: they differ precisely by a suspension.

Remark 6.7. We now have several different ways to represent maps $\mathcal{X}_0 \overset{\mathrm{lax}}{\oplus}_F \mathcal{X}_1 \xrightarrow{\alpha} \mathcal{Y}_0 \overset{\mathrm{lax}}{\oplus}_G \mathcal{Y}_1$, with the passage between them implemented by applying the cofiber-fiber equivalence to rows and/or columns of a matrix.

$$\begin{array}{ccc} \begin{pmatrix} \alpha_{00} \leftarrow \alpha_{01} \\ \uparrow \qquad \uparrow \\ \alpha^v_{10} \leftarrow \alpha^v_{11} \end{pmatrix} & \xrightarrow{-\circ \mathrm{Fib}^\vee} & \begin{pmatrix} \alpha^h_{00} \rightarrow \alpha_{01} \\ \uparrow \qquad \uparrow \\ \alpha^{hv}_{10} \rightarrow \alpha^v_{11} \end{pmatrix} \\ {\scriptstyle \mathrm{Fib}\circ -}\big\uparrow & & {\scriptstyle \mathrm{Fib}\circ -}\big\uparrow \\ \begin{pmatrix} \alpha_{00} \leftarrow \alpha_{01} \\ \downarrow \qquad \downarrow \\ \alpha_{10} \leftarrow \alpha_{11} \end{pmatrix} & \xrightarrow{-\circ \mathrm{Fib}^\vee} & \begin{pmatrix} \alpha^h_{00} \rightarrow \alpha_{01} \\ \downarrow \qquad \downarrow \\ \alpha^h_{01} \rightarrow \alpha_{11} \end{pmatrix} \end{array}.$$

Here, the notation is chosen as follows:

- Subscripts indicate the source and target of each entry, reading right to left. For example, $\alpha_{10}, \alpha^h_{10}, \alpha^v_{10}, \alpha^{hv}_{10}$ all live in $\mathbb{A}(\mathcal{X}_0, \mathcal{Y}_1)$, etc.
- Superscripts record in which direction (horizontal and/or vertical) one has to take fibers to obtain the new object from the original lax-lax matrix (lower left). For example, α^h_{10} is the fiber of the horizontal map $\alpha_{11}F \to \alpha_{10}$ while α^{hv}_{10} can be computed either as the fiber of the vertical map $G\alpha^h_{00} \to \alpha^h_{10}$ or equivalently as the fiber of the horizontal map $\alpha^v_{11}F \to \alpha^v_{10}$.

Remark 6.8. Consider two composable maps

$$\mathcal{X} \xrightarrow{\beta} \mathcal{Y}_0 \overset{\text{lax}}{\oplus}_G \mathcal{Y}_1 \xrightarrow{\alpha} \mathcal{Z}.$$

Each of the two maps β and α can be represented by a matrix in two ways, depending on whether we treat the middle term as a lax or oplax bilimit. The following table shows the four corresponding possible row-column-multiplications with the standard lax multiplication in the lower left. General 2×2 matrices describing maps between (op)lax bilimits over Δ^1 can then be multiplied in the usual row-by-column fashion.

$$\begin{array}{c|cc}
\circ & (\alpha_0 \leftarrow \alpha_1) & (\alpha^h_0 \to \alpha_1) \\
\hline
\begin{pmatrix} \beta_0 \\ \uparrow \\ \beta^v_1 \end{pmatrix} & \operatorname{cof}(\alpha_1\beta^v_1 \to \alpha_1 G\beta_0 \to \alpha_0\beta_0) & \lim \begin{pmatrix} \alpha^h_0\beta_0 & \to & \alpha_1 G\beta_0 \\ & & \uparrow \\ & & \alpha_1\beta^v_1 \end{pmatrix} [1] \\
\begin{pmatrix} \beta_0 \\ \downarrow \\ \beta_1 \end{pmatrix} & \operatorname{colim} \begin{pmatrix} \alpha_0\beta_0 & \leftarrow & \alpha_1 G\beta_0 \\ & & \downarrow \\ & & \alpha_1\beta_1 \end{pmatrix} & \operatorname{cof}(\alpha^h_0\beta_0 \to \alpha_1 G\beta_0 \to \alpha_1\beta_1)
\end{array} \tag{6.9}$$

Observe how the entry in top right differs from the standard oplax multiplication (see Example 5.16) by a shift [1]. The reason for this is that we used the canonical cofiber-fiber-equivalence (6.2) both horizontally and vertically, which amounts to using the identification $\mathrm{Fib}\colon \bigoplus^{\text{lax}} \mathcal{Y} \to \bigoplus^{\text{oplax}} \mathcal{Y}$ when discussing maps *into* the (op)lax bilimit but the identification $\mathrm{Fib}^\vee\colon \bigoplus^{\text{oplax}} \mathcal{Y} \to \bigoplus^{\text{lax}} \mathcal{Y}$ when discussing maps *from* the (op)lax bilimit; we have seen in Lemma 6.5, that these two identifications are only inverse up to shift.

The following table depicts the unit matrix with respect to each of the four multiplications; they are just obtained from the standard lax unit matrix (lower left) by passing to horizontal and/or vertical fibers.

$$\begin{array}{c|cc}
 & \text{lax} & \text{oplax} \\
\hline
\text{oplax} & \begin{pmatrix} \mathrm{id} & \longleftarrow & 0 \\ \uparrow & \square & \uparrow \\ 0 & \leftarrow & [-1] \end{pmatrix} & \begin{pmatrix} [-1] & \longrightarrow & 0 \\ \uparrow & & \uparrow \\ [-1] & \to & [-1] \end{pmatrix} \\
\text{lax} & \begin{pmatrix} \mathrm{id} & \longleftarrow & 0 \\ \downarrow & & \downarrow \\ \mathrm{id} & \leftarrow & \mathrm{id} \end{pmatrix} & \begin{pmatrix} [-1] & \to & 0 \\ \downarrow & \square & \downarrow \\ 0 & \longrightarrow & \mathrm{id} \end{pmatrix}
\end{array}$$

From the matrix multiplication formulas of (6.9) we can immediately see the advantage of this change of coordinates. By working with lax-oplax or oplax-lax matrices, we obtain, on K_0 the formulas

$$\begin{pmatrix} a_0 & a_1 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = \pm(a_0 b_0 - a_1 b_1).$$

The fact that matrix multiplication now involves an alternating sum rather than an ordinary sum is a feature, rather than a bug. In the next section we will see, for example, how we can express the differential of the mapping cone of a chain map $f\colon (A_\bullet, \alpha) \to (B_\bullet, \beta)$ by directly categorifying the canonical matrix $\delta = \left(\begin{smallmatrix} \alpha & 0 \\ f & \beta \end{smallmatrix}\right)$ without having to introduce any signs; the signs are already part of the matrix multiplication.

Another convenient feature is that the identification between lax-oplax and oplax-lax matrices is compatible with the passage to adjoints in the following sense.

Construction 6.10. Assume that $G\colon \mathcal{Y}_0 \to \mathcal{Y}_1$ has a right adjoint $G \dashv G^{\mathrm{R}}$. Then Corollary A.3, applied to the adjunctions $(G\circ) \dashv (G^{\mathrm{R}}\circ)$ and $(\circ G^{\mathrm{R}}) \dashv (\circ G)$ yields equivalences

$$\mathbb{A}(-, \mathcal{Y}_0 \overset{\text{lax}}{\oplus}_G \mathcal{Y}_1) \simeq \mathbb{A}(-, \mathcal{Y}_1 \overset{\text{oplax}}{\oplus}_{G^{\mathrm{R}}} \mathcal{Y}_0) \quad \text{and} \quad \mathbb{A}(\mathcal{Y}_1 \overset{\text{lax}}{\oplus}_{G^{\mathrm{R}}} \mathcal{Y}_0, -) \simeq \mathbb{A}(\mathcal{Y}_0 \overset{\text{oplax}}{\oplus}_G \mathcal{Y}_1, -)$$

given explicitly by passing to vertical and horizontal transposes

$$\begin{pmatrix} G y_0 \\ \downarrow u \\ y_1 \end{pmatrix} \leftrightarrow \begin{pmatrix} G^{\mathrm{R}} y_1 \\ \bar{u}\uparrow \\ y_0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} y_1^\vee \xleftarrow{\bar{v}} y_0^\vee G^{\mathrm{R}} \end{pmatrix} \leftrightarrow \begin{pmatrix} y_0^\vee \xrightarrow{v} y_1^\vee G \end{pmatrix},$$

where we have added the application of the gluing functor in the matrix to make the effect of the transposition more apparent (usually we would just write something like $(y_0^\vee \to y_1^\vee)$).

Lemma 6.11. *For each $\mathcal{X}, \mathcal{Z} : \mathbb{A}$, we have a commutative square*

$$\begin{tikzcd}
\mathbb{A}(\mathcal{X}, \mathcal{Y}_0 \overset{\text{lax}}{\oplus}_G \mathcal{Y}_1) \times \mathbb{A}(\mathcal{Y}_0 \overset{\text{oplax}}{\oplus}_G \mathcal{Y}_1, \mathcal{Z}) \arrow[r] \arrow[d, "\simeq"] & \mathbb{A}(\mathcal{X}, \mathcal{Z}) \arrow[d, equal] \\
\mathbb{A}(\mathcal{X}, \mathcal{Y}_1 \overset{\text{oplax}}{\oplus}_{G^{\mathrm{R}}} \mathcal{Y}_0) \times \mathbb{A}(\mathcal{Y}_1 \overset{\text{lax}}{\oplus}_{G^{\mathrm{R}}} \mathcal{Y}_0, \mathcal{Z}) \arrow[r] & \mathbb{A}(\mathcal{X}, \mathcal{Z}),
\end{tikzcd}$$

where the horizontal maps are the oplax-lax and lax-oplax matrix multiplication, respectively, and the left vertical map is the equivalence of Construction 6.10.

Proof. For each

$$\begin{pmatrix} y_0^\vee \xrightarrow{v} y_1^\vee \end{pmatrix} : \mathcal{Y}_0 \overset{\text{oplax}}{\oplus}_G \mathcal{Y}_1 \to \mathcal{Z} \quad \text{and} \quad \begin{pmatrix} y_0 \\ \downarrow u \\ y_1 \end{pmatrix} : \mathcal{X} \to \mathcal{Y}_0 \overset{\text{lax}}{\oplus}_G \mathcal{Y}_1,$$

the two different row-column products are the cofiber in $\mathbb{A}(\mathcal{X},\mathcal{Z})$ of the two composite maps

$$y_0^\vee y_0 \xrightarrow{v y_0} y_1^\vee G y_0 \xrightarrow{y_1^\vee u} y_1^\vee y_1 \quad \text{and} \quad y_0^\vee y_0 \xrightarrow{y_0^\vee \bar{u}} y_0^\vee G^{\mathrm{R}} y_1 \xrightarrow{\bar{v} y_1} y_1^\vee y_1.$$

A straightforward computation using the triangle identities for $G \dashv G^{\mathrm{R}}$ shows that these two maps are in canonically identified; hence so are their cofibers. ∎

7. Chain complexes and chain maps

Throughout this section, let $\mathbb{A}$ be a finitely lax additive $(\infty, 2)$-category.

Let $\mathbb{Z} = (\mathbb{Z}, \leq)$ be the standard poset of integers. A chain complex in $\mathbb{A}$ is a functor $\mathbb{Z}^{\mathrm{op}} \to \mathbb{A}$, depicted as

$$\cdots \xrightarrow{\alpha} \mathcal{A}_2 \xrightarrow{\alpha} \mathcal{A}_1 \xrightarrow{\alpha} \mathcal{A}_0 \xrightarrow{\alpha} \cdots,$$

with the conditions that each α^k is a zero object in $\mathbb{A}(\mathcal{A}_n, \mathcal{A}_{n-1-k})$ for $k \geq 2$.

There are various notions of chain maps, corresponding to different notions of natural transformations of diagrams $\mathbb{Z}^{\mathrm{op}} \to \mathbb{A}$ in the 2-categorical context (see also Appendix A.2).

- A *chain map* (without further qualifier) is a commutative diagram of the form

$$\begin{tikzcd} \cdots \arrow[r, "\alpha"] & \mathcal{A}_2 \arrow[r, "\alpha"] \arrow[d, "f_2"] \arrow[dr, phantom, "\simeq"] & \mathcal{A}_1 \arrow[r, "\alpha"] \arrow[d, "f_1"] \arrow[dr, phantom, "\simeq"] & \mathcal{A}_0 \arrow[r, "\alpha"] \arrow[d, "f_0"] & \cdots \\ \cdots \arrow[r, "\beta"'] & \mathcal{B}_2 \arrow[r, "\beta"'] & \mathcal{B}_1 \arrow[r, "\beta"'] & \mathcal{B}_0 \arrow[r, "\beta"'] & \cdots. \end{tikzcd} \tag{7.1}$$

  Chain complexes and chain maps in $\mathbb{A}$ assemble into an $(\infty, 2)$-category $\mathbb{C}\mathrm{h}(\mathbb{A})$, defined as a full sub-2-category of $\mathrm{FUN}(\mathbb{Z}^{\mathrm{op}}, \mathbb{A})$.

- A *lax chain map* is a diagram of the form, commuting only up to possibly noninvertible 2-cells.

$$\begin{tikzcd} \cdots \arrow[r, "\alpha"] & \mathcal{A}_2 \arrow[r, "\alpha"] \arrow[d, "f_2"] \arrow[dr, phantom, "\Nearrow"] & \mathcal{A}_1 \arrow[r, "\alpha"] \arrow[d, "f_1"] \arrow[dr, phantom, "\Nearrow"] & \mathcal{A}_0 \arrow[r, "\alpha"] \arrow[d, "f_0"] & \cdots \\ \cdots \arrow[r, "\beta"'] & \mathcal{B}_2 \arrow[r, "\beta"'] & \mathcal{B}_1 \arrow[r, "\beta"'] & \mathcal{B}_0 \arrow[r, "\beta"'] & \cdots. \end{tikzcd}$$

  Chain complexes and chain maps in $\mathbb{A}$ assemble into an $(\infty, 2)$-category $\mathbb{C}\mathrm{h}^{\mathrm{lax}}(\mathbb{A})$, defined as a full sub-2-category of $\mathrm{FUN}_{\mathrm{lax}}(\mathbb{Z}^{\mathrm{op}}, \mathbb{A})$.

- Dually, we define the full sub-2-category $\mathbb{C}\mathrm{h}^{\mathrm{oplax}}(\mathbb{A}) \subset \mathrm{FUN}_{\mathrm{oplax}}(\mathbb{Z}^{\mathrm{op}}, \mathbb{A})$ of chain complexes and *oplax chain maps*, which explicitly look as follows:

$$\begin{tikzcd} \cdots \arrow[r, "\alpha"] & \mathcal{A}_2 \arrow[r, "\alpha"] \arrow[d, "f_2"] \arrow[dr, phantom, "\Swarrow"] & \mathcal{A}_1 \arrow[r, "\alpha"] \arrow[d, "f_1"] \arrow[dr, phantom, "\Swarrow"] & \mathcal{A}_0 \arrow[r, "\alpha"] \arrow[d, "f_0"] & \cdots \\ \cdots \arrow[r, "\beta"'] & \mathcal{B}_2 \arrow[r, "\beta"'] & \mathcal{B}_1 \arrow[r, "\beta"'] & \mathcal{B}_0 \arrow[r, "\beta"'] & \cdots. \end{tikzcd}$$

  Given two chain complexes $(\mathcal{A}_\bullet, \alpha)$ and $(\mathcal{B}_\bullet, \beta)$, we write

$$\mathrm{Map}_0^{\mathrm{lax}}(\mathcal{A}, \mathcal{B}) \hookleftarrow \mathrm{Map}(\mathcal{A}, \mathcal{B}) \hookrightarrow \mathrm{Map}_0^{\mathrm{oplax}}(\mathcal{A}, \mathcal{B})$$

for the three corresponding ∞-categories of lax chain maps, chain maps and oplax chain maps $\mathcal{A} \to \mathcal{B}$. More generally, we write

$$\mathrm{Map}_k^{\mathrm{lax}}(\mathcal{A}_\bullet, \mathcal{B}_\bullet) := \mathrm{Map}_0^{\mathrm{lax}}(\mathcal{A}_\bullet, \mathcal{B}_{k+\bullet}) \quad \text{and} \quad \mathrm{Map}_k^{\mathrm{oplax}}(\mathcal{A}_\bullet, \mathcal{B}_\bullet) := \mathrm{Map}_0^{\mathrm{oplax}}(\mathcal{A}_\bullet, \mathcal{B}_{k+\bullet})$$

for the (stable) ∞-category of lax/oplax degree-k-maps from $\mathcal{A}$ to $\mathcal{B}$. Abstractly, these various ∞-categories are just the hom-categories in the $(\infty, 2)$-categories $\mathrm{FUN}(\mathbb{Z}^{\mathrm{op}}, \mathbb{A})$, $\mathrm{FUN}_{\mathrm{lax}}(\mathbb{Z}^{\mathrm{op}}, \mathbb{A})$ and $\mathrm{FUN}_{\mathrm{oplax}}(\mathbb{Z}^{\mathrm{op}}, \mathbb{A})$. For us, a more useful description/definition will be as certain sections of certain tautological fibrations.

For the rest of this section, fix two chain complexes $(\mathcal{A}_\bullet, \alpha)$ and $(\mathcal{B}_\bullet, \beta)$ and an integer $k \in \mathbb{Z}$.

Construction 7.2. Consider the functor

$$\mathbb{Z}^{\mathrm{op}} \times \mathbb{Z} \xrightarrow{\mathcal{B} \times \mathcal{A}^{\mathrm{op}}} \mathbb{A} \times \mathbb{A}^{\mathrm{op}} \xrightarrow{\mathbb{A}^{\mathrm{op}}(-,-)} \mathcal{S}t; \quad (m, n) \mapsto \mathbb{A}(\mathcal{A}_n, \mathcal{B}_m)$$

and its two mixed Grothendieck constructions

$$q : \int_{n:\mathbb{Z}}^{m:\mathbb{Z}^{\mathrm{op}}} \mathbb{A}(\mathcal{A}_n, \mathcal{B}_m) \to \mathbb{Z} \times \mathbb{Z} \quad \text{and} \quad q' : \int_{m:\mathbb{Z}^{\mathrm{op}}}^{n:\mathbb{Z}} \mathbb{A}(\mathcal{A}_n, \mathcal{B}_m) \to \mathbb{Z}^{\mathrm{op}} \times \mathbb{Z}^{\mathrm{op}}, \tag{7.3}$$

(contravariant, covariant) and (covariant, contravariant), respectively. We can identify oplax and lax chain maps $\mathcal{A} \to \mathcal{B}$ with sections of q and q' on the diagonal. More precisely, we define

$$\mathrm{Map}_k^{\mathrm{oplax}}(\mathcal{A}, \mathcal{B}) := \mathrm{Fun}_{\mathbb{Z}\times\mathbb{Z}}\left(\mathbb{Z}(k), \int_{n:\mathbb{Z}}^{m:\mathbb{Z}^{\mathrm{op}}} \mathbb{A}(\mathcal{A}_n, \mathcal{B}_m)\right)$$

and

$$\mathrm{Map}_k^{\mathrm{lax}}(\mathcal{A}, \mathcal{B}) := \mathrm{Fun}_{\mathbb{Z}^{\mathrm{op}}\times\mathbb{Z}^{\mathrm{op}}}\left(\mathbb{Z}^{\mathrm{op}}(k), \int_{m:\mathbb{Z}^{\mathrm{op}}}^{n:\mathbb{Z}} \mathbb{A}(\mathcal{A}_n, \mathcal{B}_m)\right),$$

where

$$\mathbb{Z}(k) := \big\{(n+k, n) \mid m = n+k\big\} \subset \mathbb{Z} \times \mathbb{Z}$$

is the k-shifted diagonal. Concretely, such a section consists of objects $f_n : \mathbb{A}(\mathcal{A}_n, \mathcal{B}_{n+k})$ and morphisms

$$f_n \to f_{n+1} \quad \text{or} \quad f_{n+1} \to f_n$$

in the corresponding Grothendieck construction, amounting to morphisms

$$f_{n-1}\alpha \to \beta f_n \quad \text{or} \quad \beta f_n \to f_{n-1}\alpha, \tag{7.4}$$

in $\mathbb{A}(\mathcal{A}_n, \mathcal{B}_{n+k-1})$ respectively. We say that $f_\bullet$ is an *oplax* or *lax chain map of degree k*.

The full subcategories of $\mathrm{Map}_0^{\mathrm{oplax}}(\mathcal{A}, \mathcal{B})$ and $\mathrm{Map}_0^{\mathrm{lax}}(\mathcal{A}, \mathcal{B})$ spanned by those sections where the corresponding maps (7.4) are equivalences are canonically equivalent to each other by passing to inverses; we define this common full subcategory to be $\mathrm{Map}(\mathcal{A}, \mathcal{B})$; it consists of the *degree-k-chain maps*.

Remark 7.5. By the standing assumption that $\mathbb{A}$ is finitely lax additive, the diagram $\mathbb{A}(\mathcal{A}_\bullet, \mathcal{B}_\bullet)\colon \mathbb{Z}^{\mathrm{op}} \times \mathbb{Z} \to \mathcal{S}t$ takes values in stable ∞-categories and exact functors, hence any of its associated categories of sections will again be stable and restriction functors between them will be exact (see also Example 3.4 (3)).

This includes the ∞-categories $\mathrm{Map}_k^{\mathrm{oplax}}(\mathcal{A},\mathcal{B})$ and $\mathrm{Map}_k^{\mathrm{lax}}(\mathcal{A},\mathcal{B})$ of shifted diagonal sections as well as other variants defined below, such as the shifted upper triangular sections of Construction 7.7. Moreover, the full subcategory $\mathrm{Map}(\mathcal{A},\mathcal{B}) \hookrightarrow \mathrm{Map}_0^{\mathrm{lax}}(\mathcal{A},\mathcal{B})$ (defined by the condition that each of the maps $\beta f_{n+1} \to f_n\alpha$ is an equivalence) manifestly contains the zero section and is closed under fibers and cofibers; hence it is a stable subcategory.

Remark 7.6. We shall not unravel the definition of $\mathrm{FUN}_{\mathrm{lax}}$ and $\mathrm{FUN}_{\mathrm{oplax}}$ and show that the mapping categories therein do indeed agree with the ∞-categories constructed in Construction 7.2. For the purpose of this paper, the reader may take this construction as the definition.

It will be useful to study more general sections of the fibrations (7.3).

Construction 7.7. Denote by $\mathbb{Z}(\geq k) \hookrightarrow \mathbb{Z}\times\mathbb{Z}$ the full shifted triangular subposet of those (m,n) satisfying $m \geq n+k$. We write

$$\mathrm{Map}_{\geq k}^{\mathrm{oplax}}(\mathcal{A},\mathcal{B}) := \mathrm{Fun}_{\mathbb{Z}\times\mathbb{Z}}\left(\mathbb{Z}(\geq k), \int_{n:\mathbb{Z}}^{m:\mathbb{Z}^{\mathrm{op}}} \mathbb{A}(\mathcal{A}_n,\mathcal{B}_m)\right).$$

for the (stable) ∞-category of shifted upper triangular sections; see Remark 7.8 for a depiction of such sections. For each k we have the obvious (exact) restriction functors

$$\mathrm{Map}_k^{\mathrm{oplax}}(\mathcal{A},\mathcal{B}) \xleftarrow{|_k} \mathrm{Map}_{\geq k}^{\mathrm{oplax}}(\mathcal{A},\mathcal{B}) \xrightarrow{|_{\geq k+1}} \mathrm{Map}_{\geq k+1}^{\mathrm{oplax}}(\mathcal{A},\mathcal{B}).$$

Remark 7.8. A section $F = (F_{m,n})_{m\geq k+n}$ as in Construction 7.7 with $f_n^r = F_{r+n,n}$ can be depicted as follows:

$$\cdots \xrightarrow{\alpha} \mathcal{A}_2 \xrightarrow{\alpha} \mathcal{A}_1 \xrightarrow{\alpha} \mathcal{A}_0 \xrightarrow{\alpha} \mathcal{A}_{-1} \xrightarrow{\alpha} \cdots$$

$$\begin{tikzcd}
\vdots \arrow[d, "\beta"] & \ddots & & & & & \\
\mathcal{B}_{k+2} \arrow[d, "\beta"] & & f_2^k \arrow[ul] & f_1^{k+1} \arrow[l] & f_0^{k+2} \arrow[l] & f_{-1}^{k+3} \arrow[l] & \\
\mathcal{B}_{k+1} \arrow[d, "\beta"] & & & f_1^k \arrow[ul] \arrow[u] & f_0^{k+1} \arrow[l] \arrow[u] & f_{-1}^{k+2} \arrow[l] \arrow[u] & \\
\mathcal{B}_k \arrow[d, "\beta"] & & & & f_0^k \arrow[ul] \arrow[u] & f_{-1}^{k+1} \arrow[l] \arrow[u] & \\
\mathcal{B}_{k-1} \arrow[d, "\beta"] & & & & & f_{-1}^k \arrow[ul] \arrow[u] & \\
\vdots & & & & & & \ddots \arrow[ul]
\end{tikzcd} \tag{7.9}$$

The complexes $\mathcal{A}_\bullet$ and $\mathcal{B}_\bullet$ are drawn for reference to indicate how the section F spreads across the fibers of the fibration.

Note that the arrows and squares in the diagram (7.9) take place in a Grothendieck construction, so that one needs to suitably postcompose with β or precompose with α to obtain genuine arrows or squares in the hom-categories $\mathbb{A}(\mathcal{A}_n, \mathcal{B}_{n+r})$. Explicitly unpacking the data of such a section F, we see that for each $r \geq k$ and $n \in \mathbb{Z}$ it contains

- an object f_n^r in $\mathbb{A}(\mathcal{A}_n, \mathcal{B}_{n+r})$,
- arrows $\eta_n^r\colon f_n^r \to \beta f_n^{r+1}$ (vertical) and $\varepsilon_n^r\colon f_{n-1}^{r+1}\alpha \to f_n^r$ (horizontal) in $\mathbb{A}(\mathcal{A}_n, \mathcal{B}_{n+r})$,
- an arrow $f_n^r\alpha \to \beta f_{n+1}^r$ (diagonal) in $\mathbb{A}(\mathcal{A}_{n+1}, \mathcal{B}_{n+r})$,
- a commutative square

$$\begin{tikzcd} \beta f_n^r & & \beta f_{n-1}^{r+1}\alpha \arrow[ll, "\beta\varepsilon_n^r"'] \\ f_n^{r-1} \arrow[u, "\eta_n^{r-1}"] & & f_{n-1}^r\alpha \arrow[ll, "\varepsilon_n^{r-1}"] \arrow[u, "\eta_{n-1}^r\alpha"'] \arrow[ull] \end{tikzcd}$$

in $\mathbb{A}(\mathcal{A}_n, \mathcal{B}_{n+r-1})$ when $r > k$, and just the upper right triangle when $r = k$ (because f_n^{k-1} is not defined).

Observe how the diagonal part $f_\bullet^k = F|_k : \mathrm{Map}_k^{\mathrm{oplax}}(\mathcal{A}, \mathcal{B})$ precisely encodes the datum of an oplax degree-k chain map as in Construction 7.2.

The following lemma states that we can "crop" redundant zeroes in a section f : $\mathrm{Map}_{\geq k}^{\mathrm{oplax}}(\mathcal{A}, \mathcal{B})$.

Lemma 7.10. *Denote by*

$$U_k^r := \big\{(m, n) \mid k \leq m - n \leq k + r\big\} \subset \mathbb{Z}(\geq k)$$

the k-shifted diagonal strip of width r. The canonical restriction functor along $U_k^r \hookrightarrow \mathbb{Z}(\geq k)$ induces an equivalence

$$\mathrm{Map}_{\geq k}^{\mathrm{oplax}}(\mathcal{A}, \mathcal{B})_{|_{\geq k+r}=0} \xrightarrow{\simeq} \mathrm{Fun}_{\mathbb{Z}\times\mathbb{Z}}\left(U_k^r, \int_{n:\mathbb{Z}}^{m:\mathbb{Z}^{\mathrm{op}}} \mathbb{A}(\mathcal{A}_n, \mathcal{B}_m)\right)_{|_{\geq k+r}=0}, \tag{7.11}$$

where on both sides we are only considering those sections which are zero on the r-th off diagonal and beyond.

Proof. First of all, we claim that the restriction functor (7.11) admits a fully faithful left adjoint given by left q-Kan extension (q is the fibration (7.3)). The pointwise q-Kan extension formula trivializes, since for each $m \geq n + k + r$ the overcategory $U_k^r/(m, n)$ has a terminal object given by the vertical edge $(n + k + r, n) \to (m, n)$. We thus only have to argue that there are sufficiently many coCartesian edges over these vertical edges $(n + k + r, n) \to (m, n)$. Since we are, by definition, only considering sections whose value at $(n+k+r, n)$ is zero, this is automatic; the resulting Kan extended diagram is zero on $\mathbb{Z}(\geq k+r)$. The result follows since by construction the essential image of this left q-Kan extension is precisely $\mathrm{Map}^{\mathrm{oplax}}_{\geq k}(\mathcal{A}, \mathcal{B})_{|_{\geq k+r}=0}$. ∎

Remark 7.12. Even when the r-th off-diagonal is zero, we cannot crop the diagram any further without losing information. In other words, the restriction functor

$$\mathrm{Map}^{\mathrm{oplax}}_{\geq k}(\mathcal{A}, \mathcal{B})_{|_{\geq k+r}=0} \xrightarrow{\simeq} \mathrm{Fun}_{\mathbb{Z}\times\mathbb{Z}}\left(U_k^{r-1}, \int_{n:\mathbb{Z}}^{m:\mathbb{Z}^{\mathrm{op}}} \mathbb{A}(\mathcal{A}_n, \mathcal{B}_m)\right),$$

is *not* typically an equivalence because the commutative squares

$$\begin{tikzcd} \beta f_n^{r-1} & 0 \arrow[l] \\ f_n^{r-2} \arrow[u] & f_{n-1}^{r-1}\alpha \arrow[l] \arrow[u] \end{tikzcd}$$

at the edge of the strip U_k^r carry more data than just the composable arrows

$$f_{n-1}^{r-1}\alpha \to f_n^{r-2} \to \beta f_n^{r-1},$$

namely a trivialization of their composite.

Remark 7.13. In the special case $r=2$, Lemma 7.10 says that a section $f : \mathrm{Map}^{\mathrm{oplax}}_{\geq k}(\mathcal{A}, \mathcal{B})$ satisfying $f|_{\geq k+2} = 0$ amounts to the following data:

- objects $f_n := f_n^k : \mathbb{A}(\mathcal{A}_n, \mathcal{B}_{k+n})$,
- objects $h_n := f_n^{k+1} : \mathbb{A}(\mathcal{A}_n, \mathcal{B}_{k+n+1})$,
- commutative squares

$$\begin{tikzcd} \beta h_n & 0 \arrow[l] \\ f_n \arrow[u] & h_{n-1}\alpha \arrow[l] \arrow[u] \end{tikzcd} \tag{7.14}$$

  in $\mathbb{A}(\mathcal{A}_n, \mathcal{B}_{k+n})$.

Lemma 7.15. *The canonical evaluation maps at the individual k-shifted diagonal entries assemble into an equivalence*

$$\mathrm{Map}^{\mathrm{oplax}}_{\geq k}(\mathcal{A}, \mathcal{B})_{|_{\geq k+1}=0} \xrightarrow{\simeq} \prod_{n\in\mathbb{Z}} \mathbb{A}(\mathcal{A}_n, \mathcal{B}_{k+n}), \qquad f_\bullet^\bullet \mapsto (f_n^k)_{n\in\mathbb{Z}}$$

of (stable) ∞-categories. Here the left-hand side denotes the kernel of the restriction functor

$$\mathrm{Map}^{\mathrm{oplax}}_{\geq k}(\mathcal{A}, \mathcal{B}) \xrightarrow{|_{\geq k+1}} \mathrm{Map}^{\mathrm{oplax}}_{\geq k+1}(\mathcal{A}, \mathcal{B}),$$

i.e., the full subcategory of those shifted upper triangular sections that vanish on the first off-diagonal and beyond.

Proof. By Lemma 7.10 (with $r = 1$), we may restrict our sections to the strip $U_k^1 \subset \mathbb{Z}(\geq k)$, which, as a poset, is simply isomorphic to $\mathbb{Z}$ via $(m, n) \mapsto m + n$. Therefore a diagram of shape U_k^1 just amounts to a sequence of objects and arrows. If such a diagram is zero on odd-indexed objects $2n + 1 \triangleq (n + 1, n)$, then all arrows are uniquely determined and the only relevant data are the values at the even-indexed objects $2n \triangleq (n, n)$. ■

Lemma 7.16. *The restriction functor*

$$\mathrm{Map}^{\mathrm{oplax}}_{\geq k}(\mathcal{A}, \mathcal{B}) \to \mathrm{Map}^{\mathrm{oplax}}_{\geq k+1}(\mathcal{A}, \mathcal{B})$$

admits a left adjoint j and a right adjoint j', both fully faithful, given by left and right q-Kan extension, respectively. A section (7.9) lies in the essential image of j / j' if and only if each leftmost horizontal/bottommost vertical edge is coCartesian/Cartesian, i.e., induces an equivalence

$$f^{k+1}_{n-1}\alpha \xrightarrow{\simeq} f^k_n \;/\; f^k_n \xrightarrow{\simeq} \beta f^{k+1}_n.$$

Proof. The pointwise left q-Kan extension formula at $(n + k, n)$ along the inclusion $\mathbb{Z}(\geq k{+}1) \hookrightarrow \mathbb{Z}(\geq k)$ trivializes, since the overcategory $\mathbb{Z}(\geq n{+}k{+}1)/(n + k, n)$ has a terminal object $(n + k, n - 1)$. Therefore the desired left q-Kan extension exists if and only if each horizontal edge $(n + k, n) \leftarrow (n + k, n - 1)$ admits a coCartesian lift. Since the fibration

$$q\colon \int_{n:\mathbb{Z}}^{m:\mathbb{Z}^{\mathrm{op}}} \mathbb{A}(\mathcal{A}_n, \mathcal{B}_m) \to \mathbb{Z} \times \mathbb{Z}$$

is (by construction) coCartesian in the second variable, this is always the case.

The argument for the right adjoint is dual. ■

Going forward, it will be convenient to reformulate such statements using recollements of stable ∞-categories. See Appendix B for a brief summary of the theory as we will use it without further explicit mention.

Corollary 7.17. *The restriction functor $|_{\geq k+1}$ is part of a recollement*

$$\begin{tikzcd}[column sep=large] \prod_{n\in\mathbb{Z}} \mathbb{A}(\mathcal{A}_n, \mathcal{B}_{k+n}) \arrow[r, hook] & \mathrm{Map}^{\mathrm{oplax}}_{\geq k}(\mathcal{A}, \mathcal{B}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "|_{\geq k+1}"] & \mathrm{Map}^{\mathrm{oplax}}_{\geq k+1}(\mathcal{A}, \mathcal{B}) \arrow[l, bend right, hook', "j"'] \arrow[l, bend left, hook, "j'"] \end{tikzcd} \tag{7.18}$$

of stable ∞-categories with gluing functor

$$f^\bullet_\bullet \mapsto \left(\mathrm{fib}\left(f^{k+1}_{n-1}\alpha \to \beta f^{k+1}_n\right)\right)_n.$$

Proof. Lemma 7.16 provides the two fully faithful adjoints j and j' of the functor $|_{\geq k+1}$ as left and right q-Kan extension, respectively. Since the kernel of this functor is identified with $\prod_{n\in\mathbb{Z}} \mathbb{A}(\mathcal{A}_n, \mathcal{B}_{k+n})$ by Lemma 7.15, this determines the desired recollement.

It remains to compute the gluing functor. From the pointwise formulas for the relative Kan extensions we see that for each $f_\bullet^\bullet : \mathrm{Map}^{\mathrm{oplax}}_{\geq k+1}(\mathcal{A}, \mathcal{B})$ the canonical transformation $j(f) \to j'(f)$ is given on the main diagonal by the structure map

$$j(f)^k_n = f^{k+1}_{n-1}\alpha \to \beta f^{k+1}_n = j'(f)^k_n$$

(for $n \in \mathbb{Z}$); passing to fibers yields the desired formula for the gluing functor. ∎

Remark 7.19. Note that neither of the two adjoints in the left half of the recollement (7.18) are the tautological functor $f_\bullet^\bullet \mapsto (f^k_n)_{n\in\mathbb{Z}}$ that evaluates a section at the individual entries of the k-shifted diagonal.

Remark 7.20. We can think of $\mathrm{Map}^{\mathrm{oplax}}_{\geq k}(\mathcal{A}, \mathcal{B})_{|\geq k+1 = 0}$ as the ∞-category of degree-k chain maps $f : \mathcal{A}_\bullet \to \mathcal{B}_{k+\bullet}$ with trivialized structure map $f\alpha \to \beta f$. Note that this is not a full subcategory of $\mathrm{Map}^{\mathrm{oplax}}_k(\mathcal{A}, \mathcal{B})$. Indeed, the restriction functor

$$\mathrm{Map}^{\mathrm{oplax}}_{\geq k}(\mathcal{A}, \mathcal{B})_{|\geq k+1 = 0} \to \mathrm{Map}^{\mathrm{oplax}}_k(\mathcal{A}, \mathcal{B})$$

which forgets the trivialization is neither full nor faithful.

The restriction functor to the diagonal does not, in general, have analogous adjoints. This does happen in the special case where the differentials of the chain complexes $(\mathcal{A}_\bullet, \alpha)$ and/or $(\mathcal{B}_\bullet, \beta)$ have left adjoints.

Lemma 7.21. *Consider the restriction functor*

$$|_k : \mathrm{Map}^{\mathrm{oplax}}_{\geq k}(\mathcal{A}, \mathcal{B}) \to \mathrm{Map}^{\mathrm{oplax}}_k(\mathcal{A}, \mathcal{B})$$

(1) *Assume that each differential β has a left adjoint. Then this restriction functor has a fully faithful left adjoint given by relative left Kan extension. Explicitly it is given by*
$$f^{k+1}_n := \beta^{\mathrm{L}} f^k_n \quad \textit{and} \quad f^r_n = 0 \quad \textit{for } r \geq k+2$$
with the non-trivial vertical arrows amounting to the units $f^k_n \to \beta\beta^{\mathrm{L}} f^k_n$ of the adjunction.

(2) *Assume that each differential α has a left adjoint. Then this restriction functor has a fully faithful right adjoint given by relative right Kan extension. Explicitly it is given by*
$$f^{k+1}_n := f^k_{n+1}\alpha^{\mathrm{L}} \quad \textit{and} \quad f^r_n = 0 \quad \textit{for } r \geq k+2$$
with the non-trivial horizontal arrows amounting to the counits $f^k_{n+1}\alpha^{\mathrm{L}}\alpha \to f^k_{n+1}$ of the adjunction.

Proof. The two statements are dual; we focus on (2).

We observe that the relevant undercategories $(m,n)/\mathbb{Z}(k)$ (for $(m,n) : \mathbb{Z}(\geq k)$) have an initial object $(m, m-k)$. Therefore the desired pointwise right q-Kan extension exists if we can guarantee that each horizontal edge $(m, m-k) \to (m,n)$ has a Cartesian lift.

In general, the fibration

$$q\colon \int_{n:\mathbb{Z}}^{m:\mathbb{Z}^{\mathrm{op}}} \mathbb{A}(\mathcal{A}_n, \mathcal{B}_m) \to \mathbb{Z} \times \mathbb{Z}$$

is only Cartesian in the first variable, not in the second. Being Cartesian in the second variable amounts to each $\mathbb{A}(\alpha, \mathcal{B}_m)$ having a right adjoint which is guaranteed because each $\alpha\colon \mathcal{A}_n \to \mathcal{A}_{n-1}$ has a left adjoint by assumption. The explicit formulas are an immediate consequence of this pointwise construction using $\beta^{\mathrm{L}}\beta^{\mathrm{L}} = 0$ to obtain the vanishing beyond the first off-diagonal. ■

Lemma 7.22. *There is an equivalence, canonical up to shift, between*

- *the full subcategory*

$$\{f \mid \forall r \neq k+1 : f_\bullet^r = 0\} \subset \mathrm{Map}_{\geq k}^{\mathrm{oplax}}(\mathcal{A}, \mathcal{B})$$

  of those sections f which are non-zero only on the first off-diagonal and
- *the ∞-category $\mathrm{Map}_{k+1}^{\mathrm{oplax}}(\mathcal{A}, \mathcal{B})$ of oplax degree-$(k+1)$ chain maps.*

Explicitly it sends a section $f_\bullet^\bullet$ to a chain map with components $g_n := f_n^{k+1}[-n]$: $\mathbb{A}(\mathcal{A}_n, \mathcal{B}_{k+n+1})$.

Remark 7.23. Note that the equivalence of Lemma 7.22 is *not* induced by the obvious restriction functor

$$\mathrm{Map}_{\geq k}^{\mathrm{oplax}}(\mathcal{A}, \mathcal{B}) \xrightarrow{|_{k+1}} \mathrm{Map}_{k+1}^{\mathrm{oplax}}(\mathcal{A}, \mathcal{B})$$

which, when restricted to $\{f \mid \forall r \neq k+1 : f_\bullet^r = 0\}$ only hits oplax chain maps with trivial structure map.

Proof. According to Remark 7.13, the data of a section $f : \mathrm{Map}_k^{\mathrm{oplax}}(\mathcal{A}, \mathcal{B})$ with $f|_{\geq k+2} = 0$ and $f_\bullet^k = 0$ amounts to

- 1-morphisms $h_n := f_n^{k+1} : \mathbb{A}(\mathcal{A}_n, \mathcal{B}_{k+n+1})$
- and commutative squares

$$\begin{tikzcd} \beta h_n & 0 \arrow[l] \\ 0 \arrow[u] & h_{n-1}\alpha \arrow[l] \arrow[u] \end{tikzcd}$$

  in $\mathbb{A}(\mathcal{A}_n, \mathcal{B}_{k+n})$ which amount precisely to morphisms $\phi_n\colon h_{n-1}[-n+1]\alpha \to \beta h_n[-n]$.

Thus setting $g_n := h_n[-n]$, this is precisely the data of an oplax degree-$(k+1)$ map $g = (g_\bullet, \phi_\bullet) : \mathrm{Map}_{k+1}^{\mathrm{oplax}}(\mathcal{A}, \mathcal{B})$. ■

Proposition 7.24. *Assume that all differentials α and β have left adjoints. Then the restriction functor*

$$|_k\colon \mathrm{Map}_{\geq k}^{\mathrm{oplax}}(\mathcal{A}, \mathcal{B})_{|_{\geq k+2}=0} \to \mathrm{Map}_k^{\mathrm{oplax}}(\mathcal{A}, \mathcal{B}) \tag{7.25}$$

is part of a recollement

$$\begin{tikzcd}
\mathrm{Map}^{\mathrm{oplax}}_{k+1}(\mathcal{A},\mathcal{B}) \arrow[r, hook, "i"] & \mathrm{Map}^{\mathrm{oplax}}_{\geq k}(\mathcal{A},\mathcal{B})_{|_{\geq k+2}=0} \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "|_k"] & \mathrm{Map}^{\mathrm{oplax}}_{k}(\mathcal{A},\mathcal{B}); \arrow[l, bend right, hook, "j"'] \arrow[l, bend left, hook, "j'"]
\end{tikzcd} \tag{7.26}$$

with gluing functor

$$\rho\colon f_\bullet \mapsto \big(\mathrm{fib}(\beta^{\mathrm{L}} f_n \to f_{n+1}\alpha^{\mathrm{L}})[-n]\big)_n. \tag{7.27}$$

In particular, we have the dashed equivalence of (stable) ∞-categories.

$$\begin{tikzcd}
\mathrm{Map}^{\mathrm{oplax}}_{\geq k}(\mathcal{A},\mathcal{B})_{|_{\geq k+2}=0} \arrow[r, dashed, leftrightarrow, "\simeq"] \arrow[d, "|_k"] & \mathrm{Map}^{\mathrm{oplax}}_{k}(\mathcal{A},\mathcal{B}) \overset{\rightarrow}{\underset{\rho}{\times}} \mathrm{Map}^{\mathrm{oplax}}_{k+1}(\mathcal{A},\mathcal{B}) \arrow[dl, "p_0"] \\
\mathrm{Map}^{\mathrm{oplax}}_{k}(\mathcal{A},\mathcal{B}). &
\end{tikzcd} \tag{7.28}$$

Proof. By Lemma 7.21, the restriction functor (7.25) has adjoints j and j' given by relative left and right Kan extension. Using Lemma 7.22 to identify the kernel then yields the recollement (7.26) and the induced equivalence (7.28) by the general theory.

From the explicit construction in Lemma 7.22 it follows that the canonical transformation $j \to j'$ between the two adjoints is given explicitly at $f : \mathrm{Map}^{\mathrm{oplax}}_k(\mathcal{A},\mathcal{B})$ by the canonical mate

$$j(f)^{k+1}_n = \beta^{\mathrm{L}} f_n \to f_{n+1}\alpha^{\mathrm{L}} = j'(f)^{k+1}_n$$

on the first off-diagonal; it is an equivalence ($f_n \xrightarrow{=} f_n$ or $0 \xrightarrow{=} 0$) everywhere else. The gluing functor

$$\mathrm{Map}^{\mathrm{oplax}}_k(\mathcal{A},\mathcal{B}) \to \mathrm{Ker}(|_k)$$

is given by the fiber of this transformation, therefore yields the desired formula (7.27) under the identification of Lemma 7.22. ∎

Definition 7.29. We denote by

$$\mathrm{Map}^{\mathrm{oplax}}_{\mathrm{ex}\geq k}(\mathcal{A},\mathcal{B}) \subset \mathrm{Map}^{\mathrm{oplax}}_{\geq k}(\mathcal{A},\mathcal{B})$$

the full subcategory spanned by those sections (f^r_n) such that all the induced squares

$$\begin{tikzcd}
\beta f^{r+1}_n & \beta f^{r+2}_{n-1}\alpha \arrow[l] \\
f^r_n \arrow[u] \arrow[ur, phantom, "\square"] & f^{r+1}_{n-1}\alpha \arrow[l] \arrow[u]
\end{tikzcd}$$

in $\mathbb{A}(\mathcal{A}_n, \mathcal{B}_{r+n})$ are biCartesian (for all $r \geq k$) and call such sections *exact*.

Lemma 7.30. *There is an equivalence of ∞-categories between*

- *the full subcategory*

$$\mathrm{Map}^{\mathrm{oplax}}_{\mathrm{ex}\geq k}(\mathcal{A},\mathcal{B})_{|\geq k+2=0} \subset \mathrm{Map}^{\mathrm{oplax}}_{\geq k}(\mathcal{A},\mathcal{B})$$

  of those sections which are exact and vanish beyond the first off-diagonal and

- *the ∞-category $\mathrm{Map}^{\mathrm{lax}}_{k+1}(\mathcal{A},\mathcal{B})$, of* lax *degree-$(k+1)$ chain maps.*

Explicitly it sends a section (f^r_n) to a chain map with components $g_n := f^{k+1}_n[-n]$.

Proof. In Remark 7.13, if we restrict to squares (7.14) which are biCartesian, the data just amounts (by rotating the exact triangle forward and shifting by $[-n]$) to objects $h_n = f^{k+1}_n : \mathbb{A}(\mathcal{A}_n,\mathcal{B}_{k+n+1})$ and maps $\beta h_n[-n] \to h_{n-1}[-n+1]\alpha$ in $\mathbb{A}(\mathcal{A}_n,\mathcal{B}_{k+n})$. This is precisely the data of a lax degree-$k+1$ chain map g with components $g_n := h_n[-n]$, as desired. ∎

Remark 7.31. Lemmas 7.22 and 7.30 explain how the ∞-category $\mathrm{Map}^{\mathrm{oplax}}_{\geq k}(\mathcal{A},\mathcal{B})_{|\geq k+2=0}$ contains both the oplax and the lax degree-$(k+1)$ maps $\mathcal{A}\to\mathcal{B}$. From the explicit constructions it is immediate that these two inclusions are compatible, in the sense that there is a commutative square

$$\begin{tikzcd}
\mathrm{Map}(\mathcal{A}_\bullet,\mathcal{B}_{k+\bullet+1}) \arrow[r, hook] \arrow[d, hook] & \mathrm{Map}^{\mathrm{oplax}}_{k+1}(\mathcal{A},\mathcal{B}) \arrow[d, hook] \\
\mathrm{Map}^{\mathrm{lax}}_{k+1}(\mathcal{A},\mathcal{B}) \arrow[r, hook] & \mathrm{Map}^{\mathrm{oplax}}_{\geq k}(\mathcal{A},\mathcal{B})_{|\geq k+2=0}
\end{tikzcd}$$

and we have

$$\mathrm{Map}(\mathcal{A}_\bullet,\mathcal{B}_{k+\bullet+1}) = \mathrm{Map}^{\mathrm{oplax}}_{k+1}(\mathcal{A},\mathcal{B}) \cap \mathrm{Map}^{\mathrm{lax}}_{k+1}(\mathcal{A},\mathcal{B})$$

as full subcategories of $\mathrm{Map}^{\mathrm{oplax}}_{\geq k}(\mathcal{A},\mathcal{B})_{|\geq k+2=0}$.

Construction 7.32 (lax mapping complex). Let $(f^r_n) = (f,h) : \mathrm{Map}^{\mathrm{oplax}}_{\geq k}(\mathcal{A},\mathcal{B})$ be a section as in Remark 7.13. If each square (7.14) is biCartesian, then both of the maps

$$f_n\alpha \to \beta h_n\alpha \quad \text{and} \quad \beta h_n\alpha \to \beta f_{n+1}$$

are equivalences, since their fibers/cofibers are

$$h_{n-1}\alpha\alpha = 0 \quad \text{and} \quad \beta\beta h_{n+1} = 0,$$

respectively. Therefore the oplax degree-k chain map $f = f^k$ is an actual chain map $\mathcal{A}_\bullet \to \mathcal{B}_{k+\bullet}$. Therefore the canonical restriction functor

$$|_k : \mathrm{Map}^{\mathrm{oplax}}_{\geq k}(\mathcal{A},\mathcal{B}) \to \mathrm{Map}^{\mathrm{oplax}}_{k}(\mathcal{A},\mathcal{B})$$

restricts to a functor

$$\delta\colon \mathrm{Map}^{\mathrm{lax}}_{k+1}(\mathcal{A},\mathcal{B}) \simeq \mathrm{Map}^{\mathrm{oplax}}_{\mathrm{ex}\geq k}(\mathcal{A},\mathcal{B})_{|\geq k+2=0} \xrightarrow{|_k} \mathrm{Map}(\mathcal{A}_\bullet,\mathcal{B}_{k+\bullet})$$

whose kernel is precisely $\mathrm{Map}(\mathcal{A}_\bullet,\mathcal{B}_{k+\bullet+1})$. These differentials δ assemble to what we call the *lax mapping complex* $\mathrm{Map}^{\mathrm{lax}}_\bullet(\mathcal{A},\mathcal{B})$:

$$\begin{tikzcd}
\cdots \mathrm{Map}^{\mathrm{lax}}_2(\mathcal{A},\mathcal{B}) \arrow[rr, dashed, "\delta"] \arrow[dr, "\delta"'] & & \mathrm{Map}^{\mathrm{lax}}_1(\mathcal{A},\mathcal{B}) \arrow[rr, dashed, "\delta"] \arrow[dr, "\delta"'] & & \mathrm{Map}^{\mathrm{lax}}_0(\mathcal{A},\mathcal{B}) \cdots \\
& \mathrm{Map}(\mathcal{A}_\bullet,\mathcal{B}_{1+\bullet}) \arrow[ur, hook] & & \mathrm{Map}(\mathcal{A}_\bullet,\mathcal{B}_\bullet) \arrow[ur, hook] &
\end{tikzcd} \tag{7.33}$$

Unraveling, we get the explicit formula for the differential

$$\delta(g_\bullet)_n = \mathrm{fib}(\beta g_n \to g_{n-1}\alpha)[n].$$

Remark 7.34. Assume that the differentials α and β have right adjoints α^{R} and β^{R}, respectively. Denote by $\mathcal{A}^{\mathrm{R}}_\bullet := (\mathcal{A}_{-\bullet},\alpha^{\mathrm{R}})$ and $\mathcal{B}^{\mathrm{R}}_\bullet := (\mathcal{B}_{-\bullet},\beta^{\mathrm{R}})$ the chain complex obtained from $\mathcal{A}$ and $\mathcal{B}$ by passing to right adjoints of the differentials. Note that for each $n \in \mathbb{N}$ there is a tautological equivalence of ∞-categories

$$\mathrm{Map}^{\mathrm{oplax}}_k(\mathcal{A}^{\mathrm{R}},\mathcal{B}^{\mathrm{R}}) \xleftrightarrow{\simeq} \mathrm{Map}^{\mathrm{lax}}_{-k}(\mathcal{A},\mathcal{B}),$$
$$(f_\bullet, f\alpha^{\mathrm{R}} \to \beta^{\mathrm{R}} f) \leftrightarrow (f_{-\bullet}, \beta f \to f\alpha)$$

by noting that both sides are sections

$$\begin{tikzcd}
& \int_{n:\mathbb{Z}}^{m:\mathbb{Z}^{\mathrm{op}}} \mathbb{A}(\mathcal{A}^{\mathrm{R}}_n,\mathcal{B}^{\mathrm{R}}_m) \arrow[r, equal] \arrow[d, "q"] & \int_{m:\mathbb{Z}^{\mathrm{op}}}^{n:\mathbb{Z}} \mathbb{A}(\mathcal{A}_n,\mathcal{B}_m) \arrow[d, "q'"] \\
\mathbb{Z}(k) \arrow[r, hook] \arrow[ur, dashed] & \mathbb{Z}\times\mathbb{Z} \arrow[r, leftrightarrow, "\cong", "(-1)\cdot"'] & \mathbb{Z}^{\mathrm{op}}\times\mathbb{Z}^{\mathrm{op}}
\end{tikzcd}$$

of the same fibration. An explicit computation shows that under this equivalence, the differential

$$\delta\colon \mathrm{Map}^{\mathrm{lax}}_{k+1}(\mathcal{A},\mathcal{B}) \to \mathrm{Map}^{\mathrm{lax}}_k(\mathcal{A},\mathcal{B})$$

of the lax mapping complex (7.33) is identified with the gluing functor

$$\rho\colon \mathrm{Map}^{\mathrm{oplax}}_{-k-1}(\mathcal{A}^{\mathrm{R}},\mathcal{B}^{\mathrm{R}}) \to \mathrm{Map}^{\mathrm{lax}}_{-k}(\mathcal{A}^{\mathrm{R}},\mathcal{B}^{\mathrm{R}})$$

of Proposition 7.24 applied to the chain complexes $\mathcal{A}^{\mathrm{R}}$ and $\mathcal{B}^{\mathrm{R}}$.

Once we have constructed the mapping complex, we immediately get the corresponding notion of categorified chain homotopy.

Definition 7.35. Let $f\colon\mathcal{A}\to\mathcal{B}$ be a chain map. A *lax null-homotopy* of f is a lax degree-1 map $h\colon \mathrm{Map}^{\mathrm{lax}}_1(\mathcal{A},\mathcal{B})$ with $\delta(h) = f$.

Remark 7.36. Clearly, one could dualize Construction 7.32 and all the preceding lemmas to obtain the *oplax* mapping complex and the resulting notion of an *oplax* null-homotopy. This is the version which appears in [3, Section 4.6], where this oplax mapping complex was constructed (in the special case $\mathbb{C} = \mathcal{S}t_k^L$) via the product totalization of the canonical double complex $\mathbb{C}(\mathcal{A}_{-\bullet}, \mathcal{B}_\bullet)$. We shall not give a detailed proof that these two different constructions agree; this is relatively straightforward by inspection of the terms of the complex and the explicit formulas for the differential.

Definition 7.37. A commutative square

$$\begin{tikzcd} \mathcal{A} \arrow[r, "g"] \arrow[d, "f"'] & \mathcal{B} \arrow[d, "f'"] \\ \mathcal{C} \arrow[r, "g'"'] & \mathcal{D} \end{tikzcd}$$

in an $(\infty, 2)$-category $\mathbb{A}$ is called *vertically left/right adjointable* if both f and f' have a left/right adjoint and the corresponding canonical mate

$$f'^{\mathrm{L}} g' \to g f^{\mathrm{L}} \;/\; g f^{\mathrm{R}} \to f'^{\mathrm{R}} g'$$

is an equivalence. *Horizontally left/right adjointable* is defined analogously but with g and g' having adjoints.

For chain maps, we distinguish two types of adjointability conditions: in the direction of the differentials and in the direction of the chain map itself.

Definition 7.38. Let $f\colon (A_\bullet, \alpha) \to (\mathcal{B}_\bullet, \beta)$ be a chain map.

- We say that f is *left diff-adjointable/right diff-adjointable* if each square in the corresponding diagram (7.1) is horizontally left/right adjointable, i.e., all differentials α and β admit left/right adjoints and the canonical mate $\beta^{\mathrm{L}} f \to f\alpha^{\mathrm{L}} / f\alpha^{\mathrm{R}} \to \beta^{\mathrm{R}} f$ is an equivalence.
- We say that f is *left/right adjointable* if each square in the corresponding diagram (7.1) is vertically left/right adjointable, i.e., each component f_n has a left/right adjoint and the canonical mate $f^{\mathrm{L}}\beta \to \alpha f^{\mathrm{L}} / \alpha f^{\mathrm{R}} \to f^{\mathrm{R}}\beta$ is an equivalence.

8. The oplax mapping cone construction

Let $\mathbb{A}$ be a finitely lax additive $(\infty, 2)$-category. The goal of this section is to construct the oplax mapping cone $\mathrm{Cone}^{\leftarrow}(f)$ of a chain map (7.1) in $\mathbb{A}$ by categorifying the usual formula

$$\mathrm{Cone}(f)_{n+1} := A_n \oplus B_{n+1} \xrightarrow{\left(\begin{smallmatrix} -\alpha & 0 \\ -f & \beta \end{smallmatrix}\right)} A_{n-1} \oplus B_n =: \mathrm{Cone}(f)_n \tag{8.1}$$

for the differential. According to the philosophy outlined in Section 1.1, we need additional data to specify the mapping cone complex:

- To construct the terms of the mapping cone complex

$$\mathrm{Cone}^{\leftarrow}(f) := \mathcal{A}_{n-1} \overset{\mathrm{lax}}{\oplus} \mathcal{B}_n$$

  as a lax bilimit, we need to specify 1-morphisms $h\colon \mathcal{A}_{n-1} \to \mathcal{B}_n$ or $k\colon \mathcal{B}_n \to \mathcal{A}_{n-1}$.
- We need some suitable 2-categorical data to be able to write down the $\Delta^1 \times \Delta^1$-analog of the differential matrix (8.1).

We will also see that in the presence of sufficient compatible adjoints to the differentials α, β and/or f, one can canonically construct such data using the various units/counits and in this case we recover the fiber and cofiber of f as in [3, Section 4.3].

Definition 8.2. We denote by

$$\mathrm{Map}^{\mathrm{lh}}(\mathcal{A}, \mathcal{B}) := \mathrm{Map}(\mathcal{A}, \mathcal{B}) \times_{\mathrm{Map}_0^{\mathrm{oplax}}(\mathcal{A},\mathcal{B})} \mathrm{Map}_{\geq 0}^{\mathrm{oplax}}(\mathcal{A}, \mathcal{B})_{|\geq 2 = 0}$$

the ∞-category of those sections (7.9) which are zero beyond the first off-diagonal and restrict to an honest chain map (as opposed to an oplax one) on the diagonal. Such sections are called *lh-enhanced morphisms* of chain complexes and are written

$$F\colon (\mathcal{A}_\bullet, \alpha_\bullet) \overset{\mathrm{lh}}{\Rightarrow} (\mathcal{B}_\bullet, \beta_\bullet).$$

The mnemonic "lh" stands for "left-horizontal" and is explained by Lemma 8.10, where we construct canonical lh-enhancements in the presence of left adjoints in the horizontal (= differential) direction.

Remark 8.3. Remark 7.13 tells us that an lh-enhanced morphism $F\colon (\mathcal{A}_\bullet, \alpha_\bullet) \overset{\mathrm{lh}}{\Rightarrow} (\mathcal{B}_\bullet, \beta_\bullet)$ consists of 1-morphisms

$$f_n\colon \mathcal{A}_n \to \mathcal{B}_n \quad \text{and} \quad h_n\colon \mathcal{A}_n \to \mathcal{B}_{n+1}$$

together with (not necessarily biCartesian) commutative squares

$$\begin{tikzcd} h_{n-1}\alpha_n \arrow[r] \arrow[d, "\varepsilon_n"] & 0 \arrow[d] \\ f_n \arrow[r, "\eta_n"] & \beta_{n+1} h_n \end{tikzcd} \tag{8.4}$$

in $\mathbb{A}(\mathcal{A}_n, \mathcal{B}_n)$ such that each composite

$$f_{n-1}\alpha_n \xrightarrow{\eta_{n-1}\alpha_n} \beta_n h_{n-1}\alpha_n \xrightarrow{\beta_n \varepsilon_n} \beta_n f_n \tag{8.5}$$

is an equivalence (i.e., exhibits $f\colon \mathcal{A} \to \mathcal{B}$ as a chain map). We say that $F = (F, h, \varepsilon, \eta)$ is an *lh-enhancement* of the underlying chain map $f\colon \mathcal{A} \to \mathcal{B}$.

We can also depict such an lh-enhanced morphism of chain complexes as follows

$$\begin{tikzcd}
\cdots \arrow[r, "\alpha"] & \mathcal{A}_2 \arrow[r, "\alpha"] \arrow[d, "f"] & \mathcal{A}_1 \arrow[r, "\alpha"] \arrow[d, "f"] \arrow[dl, "h"'] \arrow[l, phantom, "\Leftarrow", near end, shift right=2] & \mathcal{A}_0 \arrow[r, "\alpha"] \arrow[d, "f"] \arrow[dl, "h"'] \arrow[l, phantom, "\Leftarrow", near end, shift right=2] & \cdots \\
\cdots \arrow[r, "\beta"'] & \mathcal{B}_2 \arrow[r, "\beta"'] \arrow[r, phantom, "\Leftarrow", near end, shift left=2] & \mathcal{B}_1 \arrow[r, "\beta"'] \arrow[r, phantom, "\Leftarrow", near end, shift left=2] & \mathcal{B}_0 \arrow[r, "\beta"'] & \cdots
\end{tikzcd} \tag{8.6}$$

but note that this picture is not complete, since it does not depict the trivialization $\eta \circ \varepsilon \simeq 0$ encoded in the square (8.4).

The forgetful functor

$$\mathrm{Map}^{\mathrm{lh}}(\mathcal{A}, \mathcal{B}) \to \mathrm{Map}(\mathcal{A}, \mathcal{B})$$

sends an lh-enhanced morphism $F = (f, h, \varepsilon, \eta)$ to its underlying chain map by forgetting h, ε and η and only remembering the maps f and the equivalences $f\alpha \simeq \beta f$; in the picture (8.6) this just amounts to pasting the triangular 2-cells to form (commutative) squares. For each chain map $f : \mathcal{A}_\bullet \to \mathcal{B}_\bullet$, we write $\mathrm{Map}^{\mathrm{lh}}_f(\mathcal{A}_\bullet, \mathcal{B}_\bullet)$ for the fiber of this forgetful functor over the object $f : \mathrm{Map}(\mathcal{A}_\bullet, \mathcal{B}_\bullet)$; it is the (typically not stable) ∞-category of lh-enhancements of the chain map f.

Remark 8.7. An lh-enhanced morphism is called exact if each square (8.4) is biCartesian. We denote by

$$\mathrm{Map}^{\mathrm{lh-ex}}(\mathcal{A}, \mathcal{B}) := \mathrm{Map}^{\mathrm{oplax}}_{\mathrm{ex}\geq 0}(\mathcal{A}, \mathcal{B})_{|\geq 2 = 0} \subset \mathrm{Map}^{\mathrm{lh}}(\mathcal{A}, \mathcal{B})$$

the full subcategory of exact lh-enhanced morphisms.

Remark 8.8. Note that under the identification of Lemma 7.30, an exact lh-enhancement of a chain map is precisely a lax null-homotopy in the sense of Definition 7.35.

The following construction of the oplax mapping cone is a tautological reformulation of what the data of an lh-enhanced morphism entails.

Construction 8.9 (Oplax mapping cone). Let $F = (f, h, \varepsilon, \eta) : (\mathcal{A}_\bullet, \alpha) \overset{\mathrm{lh}}{\Rightarrow} (\mathcal{B}_\bullet, \beta)$ be an lh-enhanced morphism of chain complexes. We define the *oplax mapping cone* of F to be the chain complex

$$\mathrm{Cone}^{\leftarrow}(F) : \cdots \to \mathcal{A}_n \overset{\mathrm{lax}}{\oplus_h} \mathcal{B}_{n+1} \xrightarrow{\delta_{n+1}} \mathcal{A}_{n-1} \overset{\mathrm{lax}}{\oplus_h} \mathcal{B}_n \xrightarrow{\delta_n} \mathcal{A}_{n-2} \overset{\mathrm{lax}}{\oplus_h} \mathcal{B}_{n-1} \to \cdots,$$

where the differential is the lax-oplax matrix

$$\delta_{n+1} := \begin{pmatrix} \alpha_n & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ f_n & \longrightarrow & \beta_{n+1} \end{pmatrix} : \mathcal{A}_n \overset{\mathrm{oplax}}{\oplus_h} \mathcal{B}_{n+1} \to \mathcal{A}_{n-1} \overset{\mathrm{lax}}{\oplus_h} \mathcal{B}_n$$

induced by the commutative square (8.4). Using the matrix multiplication formula from Remark 6.8 we compute the squared differential

$$\delta \circ \delta \simeq \begin{pmatrix} \mathrm{cof}(\alpha\alpha \to 0) & \longrightarrow & \mathrm{cof}(0 \to 0) \\ \downarrow & & \downarrow \\ \mathrm{cof}(f\alpha \to \beta h \alpha \to \beta f) & \longrightarrow & \mathrm{cof}(0 \to \beta\beta) \end{pmatrix}.$$

It is zero because $\alpha^2 \simeq 0$, $\beta^2 \simeq 0$ and the fact that the composite map (8.5) is an equivalence.

Having constructed the mapping cone $\mathrm{Cone}^{\leftarrow}(F)$ with respect to the choice of the auxiliary lh-enhancement of the underlying chain map f, it is natural to ask whether there are universal ways to produce such lh-enhancements. These exists as long as the differentials α and/or β admit left adjoints.

Lemma 8.10. *Let $(A_\bullet, \alpha)$ and $(B_\bullet, \beta)$ be two chain complexes in $\mathbb{A}$. Consider the forgetful functor*

$$p \colon \mathrm{Map}^{\mathrm{lh}}(\mathcal{A}_\bullet, \mathcal{B}_\bullet) \to \mathrm{Map}(\mathcal{A}_\bullet, \mathcal{B}_\bullet)$$

(1) *If each differential β has a left adjoint, then p admits a fully faithful left adjoint $(-)_\beta$.*
(2) *If each differential α has a left adjoint, then p admits a fully faithful right adjoint $(-)_\alpha$.*
(3) *Assume that both differentials α and β admit left adjoints. The canonical transformation $(-)_\beta \to (-)_\alpha$ is an equivalence precisely on those chain maps f : $\mathrm{Map}(\mathcal{A}_\bullet, \mathcal{B}_\bullet)$ which are left diff-adjointable.*

Proof. The first two statements are a direct consequence of Lemma 7.21 (for $k = 0$) by observing that both adjoints (if they exists) take values in

$$\mathrm{Map}^{\mathrm{lh}}(\mathcal{A}, \mathcal{B}) \subset \mathrm{Map}^{\mathrm{oplax}}_{\geq 0}(\mathcal{A}, \mathcal{B})$$

when restricted to

$$\mathrm{Map}(\mathcal{A}, \mathcal{B}) \subset \mathrm{Map}^{\mathrm{oplax}}_{0}(\mathcal{A}, \mathcal{B}).$$

To prove (3) fix a chain map f : $\mathrm{Map}(\mathcal{A}_\bullet, \mathcal{B}_\bullet)$ and consider the component $f_\beta \to f_\alpha$. The only place where it can possibly not be an equivalence is on the first off-diagonal. Unraveling the pointwise formula, one observes that the value at these off-diagonal places is given by the mates $\beta^{\mathrm{L}} f \to f \alpha^{\mathrm{L}}$ of the equivalences $f\alpha \to \beta f$; by definition f is left diff-adjointable precisely if these mates are all equivalences. ■

Remark 8.11. Assume that all differentials α and β admit left adjoints. Then the recollement (7.26) (for $k = 0$) restricts to a recollement

$$\begin{tikzcd}
\mathrm{Map}_1^{\mathrm{oplax}}(\mathcal{A}, \mathcal{B}) \arrow[r, hook] & \mathrm{Map}^{\mathrm{lh}}(\mathcal{A}, \mathcal{B}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "p"] & \mathrm{Map}(\mathcal{A}, \mathcal{B}) \arrow[l, hook', bend right] \arrow[l, hook, bend left]
\end{tikzcd}$$

and therefore to an equivalence

$$\begin{tikzcd}
\mathrm{Map}^{\mathrm{lh}}(\mathcal{A}, \mathcal{B}) \arrow[r, dashed, leftrightarrow, "\simeq"] \arrow[d] & \mathrm{Map}(\mathcal{A}, \mathcal{B}) \overset{\rightarrow}{\underset{\rho}{\times}} \mathrm{Map}_1^{\mathrm{oplax}}(\mathcal{A}, \mathcal{B}) \arrow[d, "p_0"] \\
\mathrm{Map}(\mathcal{A}, \mathcal{B}) \arrow[r, equal] & \mathrm{Map}(\mathcal{A}, \mathcal{B}).
\end{tikzcd}$$

Pointwise over each $f : \mathrm{Map}(\mathcal{A}, \mathcal{B})$ we thus have an equivalence

$$\mathrm{Map}^{\mathrm{lh}}_f(\mathcal{A}, \mathcal{B}) \simeq \rho(f)/\operatorname{Map}^{\mathrm{oplax}}_1(\mathcal{A}, \mathcal{B}).$$

A glance at the explicit formula (7.27) shows that the gluing functor ρ is zero precisely on those chain maps which are left diff-adjointable; in this case the ∞-category of lh-enhancements is simply equivalent to $\mathrm{Map}^{\mathrm{oplax}}_1(\mathcal{A}, \mathcal{B})$, hence in particular stable.

The following corollary summarizes the situation over each chain map f: When the differentials of the chain complexes admit adjoints, each chain map f can be canonically enhanced in two ways yielding an initial or terminal object in the category $\mathrm{Map}^{\mathrm{lh}}_f(\mathcal{A}, \mathcal{B})$ of lh-enhancements of f. If the chain map is left diff-adjointable, this ∞-category is stable and these two canonical lh-enhancements agree.

Corollary 8.12. *Let $f\colon (\mathcal{A}_\bullet, \alpha) \to (\mathcal{B}_\bullet, \beta)$ be a chain map.*

(1) *If each differential β admits a left adjoint β^{L} then f admits an initial lh-enhancement f_β.*

(2) *Dually, if each differential α admits a left adjoint α^{L}, then f admits a terminal lh-enhancement f_α.*

(3) *If the chain map f is left diff-adjointable then the two lh-enhancements f_β and f_α coincide. In this case we denote this lh-enhancement by f_{lh}.*

Proof. Follows from the adjunctions of Lemma 8.10 viewed pointwise over

$$f : \mathrm{Map}(\mathcal{A}_\bullet, \mathcal{B}_\bullet). \qquad \blacksquare$$

We now identify the mapping cones constructed from the initial and terminal lh-enhancement with those constructed in [3, Construction 4.3.3] using the directed pushout and directed pullback.

Proposition 8.13. *Let $f\colon (\mathcal{A}_\bullet, \alpha) \to (\mathcal{B}_\bullet, \beta)$ be a chain map.*

(1) *Assume that each α admits a left adjoint and let f_α be its terminal lh-enhancement of Corollary 8.12. Consider the oplax square*

$$\begin{tikzcd}
\mathcal{A}_i \arrow[rr, "\alpha"] \arrow[d, "f_i"'] \arrow[drr, phantom, "\Leftarrow" near start] & & \mathcal{A}_{i-1} \arrow[d] \arrow[dll, "h_\alpha"'] \arrow[dll, phantom, "\Leftarrow" near end, shift right=3] \\
\mathcal{B}_i \arrow[rr] & & \mathcal{A}_{i-1} \overset{\mathrm{oplax}}{\oplus}_{h_\alpha} \mathcal{B}_i
\end{tikzcd}$$

obtained by pasting ε with the oplax colimit cone. This square is a directed pushout, thus yields an identification $\mathcal{A}_{i-1} \overset{\curvearrowright}{\coprod_{\mathcal{A}_i}} \mathcal{B}_i \xrightarrow{\simeq} \mathcal{A}_{i-1} \overset{\mathrm{oplax}}{\oplus}_h \mathcal{B}_i$. Under this identification the differential of $\mathrm{Cone}^{\leftarrow}(f_\alpha)$ *corepresents the map*

$$(a_{i-1}^\vee \alpha \to b_i^\vee f) \mapsto \big(\mathrm{cof}(a_{i-1}^\vee \alpha \to b_i^\vee f)\alpha \xrightarrow{\simeq} b_i^\vee \beta f\big). \tag{8.14}$$

(2) *Dually, if each β admits a left adjoint, then the terms of the cone* $\mathrm{Cone}^{\leftarrow}(f_\beta)$ *are canonically identified with* $\mathcal{A}_{i-1} \overset{\curvearrowright}{\underset{\mathcal{B}_{i-1}}{\times}} \mathcal{B}_i$ *and the differential represents the map*

$$(fa_i \to \beta b_{i+1}) \mapsto \big(f\alpha a_i \xrightarrow{\simeq} \beta\,\mathrm{fib}(fa_i \to \beta b_{i+1})\big)[1]. \tag{8.15}$$

Proof. Let $f_\alpha = (f, h_\alpha, \varepsilon_\alpha, \eta_\alpha)$ be the terminal lh-enhancement of f. We have to show that for each test object $\mathcal{C} : \mathbb{A}$, the functor

$$\begin{aligned}\underset{\Delta^1}{\mathrm{oplaxlim}}\,&\big(\mathbb{A}(\mathcal{A}_{i-1},\mathcal{C}) \xleftarrow{\circ h_\alpha} \mathbb{A}(\mathcal{B}_i,\mathcal{C})\big)\\ &= \mathbb{A}(\mathcal{A}_{i-1} \overset{\mathrm{oplax}}{\oplus}_{h_\alpha} \mathcal{B}_i, \mathcal{C}) \to \mathbb{A}(\mathcal{A}_{i-1},\mathcal{C}) \overset{\curvearrowright}{\underset{\mathbb{A}(\mathcal{A}_i,\mathcal{C})}{\times}} \mathbb{A}(\mathcal{B}_i,\mathcal{C})\end{aligned} \tag{8.16}$$

is an equivalence of (stable) ∞-categories. Explicitly, this functor sends a section $a'_{i-1} \xrightarrow{u} b_i^\vee h_\alpha$ to the composite

$$a'_{i-1}\alpha \xrightarrow{u\alpha} b_i^\vee h_\alpha \alpha = b_i^\vee f_i \alpha^{\mathrm{L}} \alpha \xrightarrow{b_i^\vee f_i \mathrm{cu}_\alpha} b_i^\vee f_i,$$

(where $\mathrm{cu}_\alpha\colon \alpha^{\mathrm{L}}\alpha \to \mathrm{id}$ is the counit of the adjunction $\alpha^{\mathrm{L}} \dashv \alpha$), which is precisely its transpose under the adjunction $(\circ\alpha) \dashv (\circ\alpha^{\mathrm{L}})$. Thus the functor (8.16) is an equivalence by Lemma A.41 applied to

$$\mathbb{A}(\mathcal{A}_{i-1},\mathcal{C}) \xrightarrow{\circ\alpha} \mathbb{A}(\mathcal{A}_i,\mathcal{C}) \xleftarrow{\circ f_i} \mathbb{A}(\mathcal{B}_i,\mathcal{C}).$$

To compute the map corepresented by the differential, we compute for each

$$(a_{i-1}^\vee \xrightarrow{u} b_i^\vee) : \mathbb{A}(\mathcal{A}_{i-1} \overset{\mathrm{oplax}}{\oplus}_{h_\alpha} \mathcal{B}_i, \mathcal{C})$$

the matrix product

$$(a_{i-1}^\vee \xrightarrow{u} b_i^\vee) \circ \begin{pmatrix} \alpha & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ f_i & \longrightarrow & \beta \end{pmatrix} = \big(\mathrm{cof}(a_{i-1}^\vee \alpha \to b_i^\vee f_i) \to \mathrm{cof}(0 \to b_i^\vee \beta)\big), \tag{8.17}$$

where in the first entry we are taking the cofiber of the map $\bar{u}\colon a_{i-1}^\vee \alpha \xrightarrow{u\alpha} b_i^\vee h_\alpha \alpha \xrightarrow{b^\vee \varepsilon_\alpha} b_i^\vee f_i$. Note that in the matrix representation (8.17) we are omitting the application of the gluing functor as is customary. If we put this implicit application back in, we obtain the map

$$\mathrm{cof}(a_{i-1}^\vee \alpha \to b_i^\vee f_i) \to b_i^\vee \beta h_\alpha = b_i^\vee \beta f_{i+1} \alpha^{\mathrm{L}} : \mathbb{A}(\mathcal{A}_i, \mathcal{C})$$

which yields the desired map

$$\mathrm{cof}(a_{i-1}^\vee \alpha \to b_i^\vee f_i)\alpha \to b_i^\vee \beta f_{i+1} : \mathbb{A}(\mathcal{A}_{i+1}, \mathcal{C})$$

after transposing; it is just the equivalence $b_i^\vee f_i \alpha \simeq b_i^\vee \beta f_{i+1}$ because $a_{i-1}^\vee \alpha\alpha = 0$.

The proof of the dual statement is analogous: We apply $\mathbb{A}(\mathcal{C}, -)$ to reduce to the case of ∞-categories, where we apply Lemma A.42. Then we only have to perform the dual matrix computation

$$\begin{pmatrix} \alpha \longrightarrow 0 \\ \downarrow \quad\quad \downarrow \\ f_i \longrightarrow \beta \end{pmatrix} \circ \begin{pmatrix} a_i \\ \downarrow \\ b_{i+1} \end{pmatrix} = \begin{pmatrix} \operatorname{cof}(\alpha a_i \to 0) \\ \downarrow \\ \operatorname{cof}(f_i a_i \to \beta b_{i+1}) \end{pmatrix} = \begin{pmatrix} \alpha a_i \\ \downarrow \\ \operatorname{fib}(f_i a_i \to \beta b_{i+1}) \end{pmatrix} [1]$$

to obtain the desired formula. ■

Definition 8.18. The external shift of a chain complex $(\mathcal{A}_\bullet, \alpha)$ is defined as

$$\mathcal{A}[n]_\bullet := \big(\mathcal{A}_{\bullet - n}, \alpha[n]\big),$$

where the terms are reindexed and the differentials are shifted internally in the stable ∞-categories $\mathbb{A}(\mathcal{A}_i, \mathcal{A}_{i-1})$.

Construction 8.19. Let $f\colon (\mathcal{A}_\bullet, \alpha) \to (\mathcal{B}_\bullet, \beta)$ be a chain map. We define

$$\operatorname{Cof}(f) := \operatorname{Cone}^{\leftarrow}(f_\alpha) \quad \text{and} \quad \operatorname{Fib}(f) := \operatorname{Cone}^{\leftarrow}(f_\beta)[-1],$$

whenever these are defined, i.e., whenever α or β has a left adjoint, respectively.

Remark 8.20. Proposition 8.13 essentially states that this definition of $\operatorname{Fib}(f)$ and $\operatorname{Cof}(f)$ agrees with the one from [3, Construction 4.3.3] in the case $\mathbb{A} = \mathcal{S}t_k^L$.

Corollary 8.21. *Let $f\colon (\mathcal{A}_\bullet, \alpha) \to (\mathcal{B}_\bullet, \beta)$ be a left diff-adjointable chain map. We have an equivalence*

$$\operatorname{Cof}(f) \simeq \operatorname{Fib}(f)[1].$$

Proof. Since we assume that the chain map f is left diff-adjointable, Corollary 8.12 states that f_α and f_β are canonically equivalent as lh-enhancements of the chain map f. Therefore the chain complexes

$$\operatorname{Cof}(f) = \operatorname{Cone}^{\leftarrow}(f_\alpha) \quad \text{and} \quad \operatorname{Fib}(f)[1] = \operatorname{Cone}^{\leftarrow}(f_\beta)$$

are also equivalent. ■

So far we have used that one can express the directed pullback $\mathcal{A}_{i-1} \overset{\curvearrowright}{\underset{\mathcal{B}_{i-1}}{\times}} \mathcal{B}_i$ and the directed pushout $\mathcal{A}_{i-1} \overset{\curvearrowright}{\underset{\mathcal{A}_i}{\amalg}} \mathcal{B}_i$ as a lax limit/colimit of a composite involving horizontal left adjoints. To prove [3, Proposition 4.3.12] we need an analogous discussion using vertical right adjoints. This change corresponds to changing the direction of the gluing map between $\mathcal{B}_n$ and $\mathcal{A}_{n-1}$. We start by defining the corresponding notion of enhancement.

Definition 8.22. An *rv-enhanced morphism* $F\colon (\mathcal{A}_\bullet, \alpha_\bullet) \overset{\mathrm{rv}}{\Longrightarrow} (\mathcal{B}_\bullet, \beta_\bullet)$ of chain complexes consists of 1-morphisms

$$f_n\colon \mathcal{A}_n \to \mathcal{B}_n \quad \text{and} \quad k_n\colon \mathcal{B}_n \to \mathcal{A}_{n-1}$$

together with an oplax-lax matrix of the form

$$\delta = \begin{pmatrix} \beta \leftarrow f \\ \uparrow \quad\;\; \uparrow \\ 0 \leftarrow \alpha \end{pmatrix} : \mathcal{B}_{n+1} \overset{\text{lax}}{\oplus_k} \mathcal{A}_n \to \mathcal{B}_n \overset{\text{oplax}}{\oplus_k} \mathcal{A}_{n-1} \tag{8.23}$$

such that the composite map $f\alpha \to fkf \to \beta f$ is an equivalence (yielding the underlying chain map f of F). The resulting chain complex $(\mathrm{Cone}^{\leftarrow}(F)_\bullet := \mathcal{B}_\bullet \overset{\text{lax}}{\oplus_k} \mathcal{A}_{\bullet-1}, \delta)$ is called the oplax mapping cone of F.

The mnemonic "rv" stands for "right-vertical" and reflects the fact that there are canonical rv-enhancements in the presence of right adjoints in the vertical (= chain map) direction.

We shall now explain how such rv-enhanced morphisms assemble into an ∞-category. For simplicity we will restrict to those, where each f_n admits a right adjoint $g_n := {f_n}^{\mathrm{R}}$.

Construction 8.24. Define

$$\mathrm{Map}^{\mathrm{lax}}_{\leq k}(\mathcal{B}, \mathcal{A}) := \mathrm{Fun}_{\mathbb{Z}^{\mathrm{op}} \times \mathbb{Z}^{\mathrm{op}}}\left(\mathbb{Z}(\leq k)^{\mathrm{op}}, \int_{m:\mathbb{Z}^{\mathrm{op}}}^{n:\mathbb{Z}} \mathbb{A}(\mathcal{B}_n, \mathcal{A}_m) \right)$$

to consist of sections defined on

$$\mathbb{Z}(\leq k)^{\mathrm{op}} := \big\{ (m,n) \mid m \leq n + k \big\} \subset \mathbb{Z}^{\mathrm{op}} \times \mathbb{Z}^{\mathrm{op}}.$$

Pictorially, such sections look as follows:

$$\begin{tikzcd}
 & \cdots \arrow[r, "\beta"] & \mathcal{B}_2 \arrow[r, "\beta"] & \mathcal{B}_1 \arrow[r, "\beta"] & \mathcal{B}_0 \arrow[r, "\beta"] & \cdots \\
\vdots \arrow[d, "\alpha"] & \ddots \arrow[dr] & & & & \\
\mathcal{A}_{k+2} \arrow[d, "\alpha"] & & g_2^k \arrow[d] \arrow[dr] & & & \\
\mathcal{A}_{k+1} \arrow[d, "\alpha"] & & g_2^{k-1} \arrow[r] \arrow[d] & g_1^k \arrow[d] \arrow[dr] & & \\
\mathcal{A}_k \arrow[d, "\alpha"] & & g_2^{k-2} \arrow[r] & g_1^{k-1} \arrow[r] & g_0^k \arrow[dr] & \\
\vdots & & & & & \ddots
\end{tikzcd}$$

Denote by

$$\mathrm{Map}^{\mathrm{rv-L}}(\mathcal{A}, \mathcal{B}) \subset \mathrm{Map}^{\mathrm{lax}}_{\leq 0}(\mathcal{B}, \mathcal{A})$$

the full subcategory of those sections (g_n^r) satisfying the following:

- The lax chain map $g_\bullet = g^0_\bullet\colon \mathcal{B}_\bullet \to \mathcal{A}_\bullet$ on the main diagonal is left adjointable, i.e. each $g_n\colon \mathcal{B}_n \to \mathcal{A}_n$ has a left adjoint ${g_n}^{\mathrm{L}}$ and the canonical mate ${g_{n-1}}^{\mathrm{L}}\alpha \to \beta {g_n}^{\mathrm{L}}$ is an equivalence.
- The section is zero beyond the first off-diagonal, i.e. $g^r_\bullet = 0$ for $r \leq -2$.

Using the dual of Remark 7.13 and by passing from an adjointable lax chain map $g_\bullet = g^0_\bullet\colon \mathcal{B}_\bullet \to \mathcal{A}_\bullet$ to its adjoint $f_\bullet := {g_\bullet}^{\mathrm{L}}\colon \mathcal{A}_\bullet \to \mathcal{B}_\bullet$ (which is an honest chain map), it is not hard to see that the data of such a section amounts precisely to that of an rv-enhanced morphism whose underlying chain map f admits pointwise adjoints; the 1-morphisms $k_n\colon \mathcal{B}_n \to \mathcal{A}_{n-1}$ are the term g_n^{-1} on the first off-diagonal and the matrices (8.23) amount precisely to the squares

$$\begin{tikzcd} k_{n+1} \arrow[r] \arrow[d] & g_n \arrow[d] \\ 0 \arrow[r] & k_n. \end{tikzcd}$$

Therefore we can view $\mathrm{Map}^{\mathrm{rv-L}}(\mathcal{A}, \mathcal{B})$ as the ∞-category of those rv-enhanced morphisms, whose underlying chain map f admits pointwise adjoints. We have the canonical forgetful functor

$$\begin{tikzcd} \mathrm{Map}^{\mathrm{lax}}_{\leq 0}(\mathcal{B}, \mathcal{A})_{|\leq -2 = 0} \arrow[r, "|_0"] & \mathrm{Map}^{\mathrm{lax}}_{0}(\mathcal{B}, \mathcal{A}) & \\ \mathrm{Map}^{\mathrm{rv-L}}(\mathcal{A}, \mathcal{B}) \arrow[u, hook] \arrow[r, "|_0"] \arrow[dr, dashed] & \{\text{left adjointable } g_\bullet\} \arrow[u, hook] \arrow[d, leftrightarrow, "\simeq"'] & \\ & \{\text{pointwise right adjointable } f_\bullet\} \arrow[r, hook] & \mathrm{Map}(\mathcal{A}, \mathcal{B}) \end{tikzcd} \tag{8.25}$$

sending such an rv-enhanced morphism to its underlying chain map. For each pointwise adjointable chain map $f\colon \mathcal{A} \to \mathcal{B}$ we write $\mathrm{Map}^{\mathrm{rv}}_f(\mathcal{A}, \mathcal{B}) = \mathrm{Map}^{\mathrm{rv-L}}_f(\mathcal{A}, \mathcal{B})$ for the fiber of this dashed functor; it is the ∞-category of rv-enhancements of f.

Lemma 8.26. *The forgetful functor* (8.25) *is part of a recollement*

$$\begin{tikzcd} \mathrm{Map}^{\mathrm{lax}}_{-1}(\mathcal{B}, \mathcal{A}) \arrow[r, hook] & \mathrm{Map}^{\mathrm{rv-L}}(\mathcal{A}, \mathcal{B}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \left\{\begin{smallmatrix} \textit{pointwise right adjointable} \\ f_\bullet\colon \mathcal{A} \to \mathcal{B} \end{smallmatrix}\right\}, \arrow[l, bend right, "j"'] \arrow[l, bend left, "j'"] \end{tikzcd}$$

whose gluing functor ρ computes the fiber of the canonical mate, i.e.,

$$\rho(f) = \big(\mathrm{fib}(\alpha {f_n}^{\mathrm{R}} \to {f_{n-1}}^{\mathrm{R}}\beta)[n]\big)_n.$$

Proof. Similarly to Lemma 7.16 and Corollary 7.17 we compute that the relative left and right Kan extension along the diagonal $\mathbb{Z}^{\mathrm{op}} \hookrightarrow \mathbb{Z}(\leq 0)^{\mathrm{op}}$ always exist, yielding fully faithful left and right adjoints j and j' to the restriction functors $|_0$. Explicitly we have

$$j(g)_n^{-1} = \alpha g_n \to g_{n-1}\beta = j'(g)_n^{-1}$$

(the structure map of $g_\bullet$) on the first off-diagonal and zero beyond it. Moreover, the kernel of the forgetful functor is the full subcategory

$$\{g \mid \forall r \neq -1 : g^r_\bullet = 0\} \subset \mathrm{Map}^{\mathrm{lax}}_{\leq 0}(\mathcal{B}, \mathcal{A}),$$

which, similarly to Lemma 7.22 we can identify with $\mathrm{Map}^{\mathrm{lax}}_{-1}(\mathcal{B}, \mathcal{A})$ via the assignment

$$g^\bullet_\bullet \mapsto \left(g_n^{-1}[n]\right)_n.$$

The desired result follows by passing to adjoints, i.e., $g_\bullet := f_\bullet{}^{\mathrm{R}}$. ■

Remark 8.27. The gluing functor for the recollement

$$\begin{tikzcd} \mathrm{Map}^{\mathrm{lax}}_{-1}(\mathcal{B}, \mathcal{A}) \arrow[r, hook] & \mathrm{Map}^{\mathrm{lax}}_{\leq 0}(\mathcal{B}, \mathcal{A})_{|\leq -2 = 0} \arrow[l, bend right=40] \arrow[l, bend left=40] \arrow[r, "|_0"] & \mathrm{Map}^{\mathrm{lax}}_{0}(\mathcal{B}, \mathcal{A}) \arrow[l, hook, bend right=40] \arrow[l, hook, bend left=40] \end{tikzcd}$$

(before restricting the cokernel to the subcategory of left adjointable maps $g: \mathcal{B} \to \mathcal{A}$) is nothing but the differential of the lax mapping complex $\mathrm{Map}^{\mathrm{lax}}_\bullet(\mathcal{B}, \mathcal{A})$.

As a direct consequence we get the following result, which provides the two canonical rv-enhancements of a pointwise right adjointable chain map.

Corollary 8.28. *Let $f: (\mathcal{A}_\bullet, \alpha) \to (\mathcal{B}_\bullet, \beta)$ be a pointwise right adjointable chain map. The ∞-category* $\mathrm{Map}^{\mathrm{rv}}_f(\mathcal{A}, \mathcal{B})$ *has*

(1) *an initial object $f^\alpha := j(f)$ with $k^\alpha = \alpha f^{\mathrm{R}}$ and where the vertical map $\alpha \to kf = \alpha f^{\mathrm{R}} f$ in the matrix (8.23) is the unit;*

(2) *a terminal object $f^\beta := j'(f)$ with $k^\beta = f^{\mathrm{R}}\beta$ and where the horizontal map $\beta \leftarrow fk = ff^{\mathrm{R}}\beta$ in the matrix (8.23) is the counit.*

(3) *These two rv-enhancements coincide if and only if the chain map f is right adjointable. In this case we denote this rv-enhancement by f^{rv}.*

Analogously to Proposition 8.13, we can exhibit the terms of the corresponding oplax mapping cones $\mathrm{Cone}^{\leftarrow}(f^\alpha)$ and $\mathrm{Cone}^{\leftarrow}(f^\beta)$ as a directed pushout or directed pullback, respectively.

Proposition 8.29. *Let $f: (\mathcal{A}_\bullet, \alpha) \to (\mathcal{B}_\bullet, \beta)$ be a chain map and assume that each f_i admits a right adjoint.*

(1) *The oplax square*

$$\begin{tikzcd} \mathcal{A}_i \arrow[r, "\alpha"] \arrow[d, "f"'] \arrow[dr, phantom, "\Swarrow", near start] & \mathcal{A}_{i-1} \arrow[d] \\ \mathcal{B}_i \arrow[r] \arrow[ur, "k^\alpha"] \arrow[ur, phantom, "\Swarrow", near end, shift right=3] & \mathcal{B}_i \overset{\mathrm{lax}}{\oplus}_{k^\alpha} \mathcal{A}_{i-1} \end{tikzcd}$$

yields an identification $\mathcal{A}_{i-1} \overset{\curvearrowright}{\coprod_{\mathcal{A}_i}} \mathcal{B}_i \xrightarrow{\simeq} \mathcal{B}_i \overset{\mathrm{lax}}{\oplus}_{k^\alpha} \mathcal{A}_{i-1}$. *Under this identification, the differential of* $\mathrm{Cone}^{\leftarrow}(f^\alpha)$ *again corepresents the map (8.14).*

(2) *Dually, the terms of the cone* $\mathrm{Cone}^{\leftarrow}(f^{\beta})$ *are canonically identified with* $\mathcal{A}_{i-1} \overset{\curvearrowright}{\underset{\mathcal{B}_{i-1}}{\times}} \mathcal{B}_i$ *and the differential again represents the map* (8.15).

Proof. Similar to Proposition 8.13; omitted. ∎

Corollary 8.30. *The chain complexes* $\mathrm{Cone}^{\leftarrow}(f^{\alpha})$ *and* $\mathrm{Cone}^{\leftarrow}(f^{\beta})$ *also yield a construction for* $\mathrm{Cof}(f)$ *and* $\mathrm{Fib}(f)[1]$*, respectively. In particular,* $\mathrm{Cof}(f)$ *and* $\mathrm{Fib}(f)[1]$ *agree when the chain map* f *is right adjointable.*

Corollary 8.31. *When* f *is both right adjointable and left diff-adjointable, the two canonical oplax mapping cones* $\mathrm{Cone}^{\leftarrow}(f_{\mathrm{lh}})$ *and* $\mathrm{Cone}^{\leftarrow}(f^{\mathrm{rv}})$ *agree.*

Remark 8.32. Throughout this section there was a bias in our discussion, since we implicitly treated chain maps as being *oplax*, i.e. having directed squares of the form

$$\begin{tikzcd}
\mathcal{A}_{i+1} \arrow[r, "\alpha"] \arrow[d, "f_{i+1}"'] & \mathcal{A}_i \arrow[d, "f_i"] \arrow[dl, phantom, "\Swarrow"] \\
\mathcal{B}_{i+1} \arrow[r, "\beta"'] & \mathcal{B}_i .
\end{tikzcd}$$

This was already apparent in the chosen direction for directed pushouts and directed pullbacks in Example 3.6 and accounts for the two possible choices we had when it came to adjointability conditions: having vertical right adjoints or horizontal left adjoints. We could rewrite this whole section with the opposite conventions and obtain the *lax mapping cone* $\mathrm{Cone}^{\rightarrow}(F)$ associated to a chain map f with suitable enhancements. In the case where f is left adjointable or right diff-adjointable we could again construct a canonical lax mapping cone $\mathrm{Cone}^{\rightarrow}(F)$ whose terms are identified both with $\mathcal{B}_i \overset{\curvearrowright}{\underset{\mathcal{A}_i}{\amalg}} \mathcal{A}_{i-1}$ and with $\mathcal{B}_i \overset{\curvearrowright}{\underset{\mathcal{B}_{i-1}}{\times}} \mathcal{A}_{i-1}$.

9. Universal property of the lax mapping cone

The main reason for introducing the mapping cone of a chain map $f\colon (A_\bullet, \alpha) \to (B_\bullet, \beta)$ between chain complexes in an additive category $\mathcal{A}$ is that it yields an explicit model for the cofiber of f in the stable ∞-category $\mathcal{K}(\mathcal{A})$ of chain complexes up to chain homotopy. In other words, it satisfies

$$\mathrm{Map}_{\mathcal{K}(\mathcal{A})}\big(\mathrm{Cone}(f), C\big) \simeq \mathrm{fib}\big(\mathrm{Map}(B, C) \to \mathrm{Map}(A, C)\big) = \big\{(g\colon B \to C,\ h\colon gf \simeq 0)\big\}$$

naturally in $C : \mathcal{K}(\mathcal{A})$.

Already before passing to the stable ∞-category $\mathcal{K}(\mathcal{A})$, one can see a naive version of this universal property characterizing the mapping cone up to isomorphism in $\mathrm{Ch}(\mathcal{A})$ via

$$\mathrm{Ch}(\mathcal{A})\big(\mathrm{Cone}(f), C\big) \cong \big\{(g, h) \mid g\colon B \to C, h\colon gf \simeq 0\big\} \tag{9.1}$$

naturally in $C : \mathrm{Ch}(\mathcal{A})$. In other words: maps out of $\mathrm{Cone}(f)$ are chain maps $g\colon B \to C$ together with a null-homotopy of gf.

Ultimately, we are of course interested in understanding the categorified analog of the homotopically meaningful universal property. However, this is currently out of reach since we do not even know what the correct analog of the stable ∞-category $\mathcal{K}(\mathcal{A})$ should be and in what sense we are supposed to view the mapping cone as a cofiber. Therefore, we now instead describe the categorified analog of (9.1) in the hopes that it might lead to a better understanding of the theory of categorified chain complexes up to homotopy.

Theorem 9.2. *Let $F\colon \mathcal{A} \overset{\mathrm{lh}}{\Rightarrow} \mathcal{B}$ be an lh-enhanced morphism of chain complexes with underlying chain map f.*

(1) *For each chain complex $\mathcal{C} : \mathrm{Ch}(\mathbb{A})$ there is a natural equivalence of (stable) ∞-categories between*
  - *chain maps $\mathrm{Cone}^{\leftarrow}(F) \to \mathcal{C}$ and*
  - *chain maps $g\colon \mathcal{B} \to \mathcal{C}$ together with an exact lh-enhancement E of gf and a morphism $E \to gF$ of lh-enhancements of gf.*

(2) *For each chain complex $\mathcal{C} : \mathrm{Ch}(\mathbb{A})$ there is a natural equivalence of (stable) ∞-categories between*
  - *chain maps $\mathcal{C} \to \mathrm{Cone}^{\leftarrow}(F)[-1]$ and*
  - *chain maps $g\colon \mathcal{C} \to \mathcal{A}$ together with an exact lh-enhancement E of fg and a morphism $Fg \to E$ of lh-enhancements of fg.*

Before proving Theorem 9.2, we isolate the special case where F is the initial or terminal lh-enhancement of f.

Corollary 9.3. *Let $f\colon (\mathcal{A}_\bullet, \alpha) \to (\mathcal{B}_\bullet, \beta)$ be a chain map.*

(1) *Assume that all differentials α have left adjoints. Then for each chain complex $\mathcal{C} : \mathrm{Ch}(\mathbb{A})$ there is an equivalence of (stable) ∞-categories between*
  - *chain maps $\mathrm{Cof}(f) \to \mathcal{C}$ and*
  - *chain maps $g\colon \mathcal{B} \to \mathcal{C}$ together with a lax null-homotopy E of gf.*

(2) *Assume that all differentials β have left adjoints. Then for each chain complex $\mathcal{C} : \mathrm{Ch}(\mathbb{A})$ there is an equivalence of (stable) ∞-categories between*
  - *chain maps $\mathcal{C} \to \mathrm{Fib}(f)$ and*
  - *chain maps $g\colon \mathcal{C} \to \mathcal{A}$ together with a lax null-homotopy E of fg.*

Proof. We prove the first statement; the second is dual. Let f_α be the terminal lh-enhancement of f and recall that we have $\mathrm{Cof}(f) = \mathrm{Cone}^{\leftarrow}(f_\alpha)$. Observe further, that composition with g sends the lh-enhanced morphism f_α to $g(f_\alpha) \simeq (gf)_\alpha$, which is thus a terminal object of $\mathrm{Map}^{\mathrm{lh}}_{gf}(\mathcal{A}_\bullet, \mathcal{C}_\bullet)$. Therefore the claim follows from Theorem 9.2 after identifying exact lh-enhancements with lax null-homotopies (see Remark 8.8). ■

Proof of Theorem 9.2. Fix an lh-enhanced morphism

$$F = (f, h, \varepsilon, \eta)\colon (\mathcal{A}_\bullet, \alpha_\bullet) \overset{\mathrm{lh}}{\Rightarrow} (\mathcal{B}_\bullet, \beta_\bullet)$$

and a test chain complex $(\mathcal{C},\gamma)$ in $\mathbb{A}$. We unravel the data encoded in a chain map

$$(\mathrm{Cone}^{\leftarrow}(F)_{\bullet},\delta)\to(\mathcal{C}_{\bullet},\gamma),$$

using lax-oplax matrices. For each n, we have a map

$$G_n=(k_{n-1}\xrightarrow{\mu_{n-1}}g_n)\colon \mathrm{Cone}^{\leftarrow}(F)_n=\mathcal{A}_{n-1}\overset{\mathrm{lax}}{\oplus}\mathcal{B}_n\to\mathcal{C}_n$$

and an equivalence $G_n\delta_{n+1}\xrightarrow{\simeq}\gamma_{n+1}G_{n+1}$, which we can expand to

$$\begin{aligned}\big(\mathrm{cof}\,(k_{n-1}\alpha_n\to g_nf_n)\to g_n\beta_{n+1}\big)=(k_{n-1}\to g_n)&\begin{pmatrix}\alpha_n & \longrightarrow & 0\\ \downarrow & & \downarrow\\ f_n & \longrightarrow & \beta_{n+1}\end{pmatrix}\\ &\xrightarrow{\simeq}(\gamma_{n+1}k_n\to\gamma_{n+1}g_{n+1})\,.\end{aligned}$$

Therefore, the map $G_n\delta_{n+1}\to\gamma_{n+1}G_{n+1}$ amounts to a cube (read back to front)

$$\begin{tikzcd}
k_{n-1}\alpha_n \arrow[rr] \arrow[dr] \arrow[d, "\mu_{n-1}\alpha_n"'] \arrow[dd, bend right=60, "\zeta_n"'] & & 0 \arrow[dr] \arrow[dd] & \\
g_nh_{n-1}\alpha_n \arrow[urr] \arrow[d, "g_n\varepsilon_n"'] & 0 \arrow[rr] \arrow[dd] & & 0 \arrow[dd] \\
g_nf_n \arrow[rr, "g_n\eta_n"] \arrow[dr, "\nu_n"'] & & g_n\beta_{n+1} \arrow[dr, "\phi_{n+1}"] & \\
& \gamma_{n+1}k_n \arrow[rr, "\gamma_{n+1}\mu_n"] & & \gamma_{n+1}g_{n+1}
\end{tikzcd}\tag{9.4}$$

$$\mathbb{A}(\mathcal{A}_n,\mathcal{C}_n)\xleftarrow[-\circ h_n]{}\mathbb{A}(\mathcal{B}_{n+1},\mathcal{C}_n)$$

in the contravariant Grothendieck construction of $-\circ h_n$. The fact that this map is an equivalence amounts to saying that the left and right squares of the cube are biCartesian. In particular, we can focus on the right face and see an equivalence $\phi_{n+1}\colon g_n\beta_{n+1}\xrightarrow{\simeq}\gamma_{n+1}g_{n+1}$, exhibiting $g_{\bullet}\colon(\mathcal{B}_{\bullet},\beta)\to(\mathcal{C}_{\bullet},\gamma)$ as a chain map.

Consider the functor

$$\Xi\colon\mathbb{Z}^{\mathrm{op}}\times\mathbb{Z}\times\Delta^1\to\mathcal{S}t;\quad (m,n,-)\mapsto\big(\mathbb{A}(\mathcal{A}_n,\mathcal{B}_m)\xrightarrow{g_n\circ-}\mathbb{A}(\mathcal{A}_n,\mathcal{C}_m)\big),$$

which is well defined because $g\colon\mathcal{B}_{\bullet}\to\mathcal{C}_{\bullet}$ is a chain map.

By direct comparison with the diagram (9.5) below, one verifies that all of the data (9.4) can then be equivalently encoded as $\mathbb{Z}(\geq 0)\times\Delta^1$-sections of the mixed (contravariant, contravariant, covariant) Grothendieck construction of Ξ that satisfy

- the restriction to $\mathbb{Z}(\geq 0)\times 0$ is the original lh-enhanced morphism $F\colon\mathcal{A}\overset{\mathrm{lh}}{\Rightarrow}\mathcal{B}$,

- the value on each edge $(n,n,1) \to (n,n,0)$ is Cartesian,
- the restriction to $\mathbb{Z}(\geq 0) \times \{1\}$ is an exact lh-enhanced morphism $E\colon \mathcal{A} \overset{\mathrm{lh}}{\Rightarrow} \mathcal{C}$.

$$\mathcal{A}_{n+1} \xrightarrow{\alpha} \mathcal{A}_n \xrightarrow{\alpha} \mathcal{A}_{n-1}$$

$$\begin{tikzcd}
 & \mathcal{C}_{n+1} \arrow[dd, "\gamma"] & & gf \arrow[dl, "!"'] & & k \arrow[ll, "\zeta"'] \arrow[dl, "\mu"'] & & 0 \arrow[ll] \arrow[dl] \\
\mathcal{B}_{n+1} \arrow[ur, "g"] \arrow[dd, "\beta"] & & f & & h \arrow[ll, "\varepsilon"'] & & 0 \arrow[ll] & \\
 & \mathcal{C}_n \arrow[dd, "\gamma"] & & & & gf \arrow[uu, "\nu"] \arrow[dl, "!"'] & & k \arrow[ll, "\zeta"'] \arrow[uu] \arrow[dl, "\mu"'] \\
\mathcal{B}_n \arrow[ur, "g"] \arrow[dd, "\beta"] & & & & f \arrow[uu, "\eta"] & & h \arrow[ll, "\varepsilon"'] \arrow[uu] & \\
 & \mathcal{C}_{n-1} & & & & & & gf \arrow[uu, "\nu"] \arrow[dl, "!"'] \\
\mathcal{B}_{n-1} \arrow[ur, "g"] & & & & & & f \arrow[uu, "\eta"] &
\end{tikzcd} \tag{9.5}$$

In other words, we have exact lh-enhanced morphisms $E\colon \mathcal{A} \overset{\mathrm{lh}}{\Rightarrow} \mathcal{C}$ equipped with a map $E \to gF$ which induces an equivalence on the underlying chain maps. This completes the proof. ∎

A. Some lemmas from (2-)category theory

A.1. About (op)lax limits of ∞-categories

We collect here a few useful lemmas regarding various types of 2-categorical limits of ∞-categories or stable ∞-categories.

Construction A.1. Let S be an ∞-category and $\mathcal{X}\colon S \to \mathrm{Cat}_\infty$ an S-indexed diagram of ∞-categories. Let $p\colon \int_S \mathcal{X} \to S$ be its (covariant) Grothendieck construction. Assume that for every arrow $f\colon s \to t$ in S, the functor $\mathcal{X}_f$ admits a right adjoint. In this case, the cocartesian fibration p is also cartesian; it corresponds to the diagram $\mathcal{X}^{\mathrm{R}}\colon S^{\mathrm{op}} \to \mathrm{Cat}_\infty$ which is obtained form $\mathcal{X}$ by passing to right adjoints. Therefore we obtain a tautological identification

$$\operatorname*{laxlim}_S \mathcal{X} = \{\text{sections of p}\} = \operatorname*{oplaxlim}_{S^{\mathrm{op}}} \mathcal{X}^{\mathrm{R}}. \tag{A.2}$$

Corollary A.3. *Let $f\colon \mathcal{A} \to \mathcal{B}$ be a diagram of ∞-categories and assume that f has a right adjoint f^{R}. Then there is a natural identification*

$$\mathcal{A} \overset{\rightarrow}{\underset{f}{\times}} \mathcal{B} = \mathcal{B} \overset{\leftarrow}{\underset{f^{\mathrm{R}}}{\times}} \mathcal{A}$$

given by the formula

$$(a, b, fa \xrightarrow{u} b) \leftrightarrow (a, b, a \xrightarrow{\bar{u}} f^{\mathrm{R}} b).$$

Proof. This is just the special case $S = \Delta^1$ of the identification (A.2). ∎

Lemma A.4. *Let $\mathcal{A} \xrightarrow{f} \mathcal{C} \xleftarrow{g} \mathcal{B}$ be a diagram of ∞-categories.*

(1) *Assume that f has a right adjoint f^{R}. Then there is a natural equivalence*

$$\mathcal{A} \overset{\curvearrowright}{\underset{\mathcal{C}}{\times}} \mathcal{B} \simeq \mathcal{B} \overset{\leftarrow}{\underset{f^{\mathrm{R}} g}{\times}} \mathcal{A}$$

given by the formula

$$(a, b, fa \xrightarrow{u} gb) \leftrightarrow (a, b, a \xrightarrow{\bar{u}} f^{\mathrm{R}} gb).$$

(2) *Assume that g has a left adjoint g^{L}. Then there is a natural equivalence*

$$\mathcal{A} \overset{\curvearrowright}{\underset{\mathcal{C}}{\times}} \mathcal{B} \simeq \mathcal{A} \overset{\rightarrow}{\underset{g^{\mathrm{L}} f}{\times}} \mathcal{B}$$

given by the formula

$$(a, b, fa \xrightarrow{u} gb) \leftrightarrow (a, b, g^{\mathrm{L}} fa \xrightarrow{\bar{u}} b).$$

Proof. We compute

$$\begin{aligned} \mathcal{A} \overset{\curvearrowright}{\underset{\mathcal{C}}{\times}} \mathcal{B} &= \mathcal{A} \times_{\mathcal{C}^{\{0\}}} \mathcal{C}^{\{0\to 1\}} \times_{\mathcal{C}^{\{1\}}} \mathcal{B} \simeq \mathcal{A} \overset{\rightarrow}{\underset{f}{\times}} \mathcal{C} \times_{\mathcal{C}} \mathcal{B} \\ &\simeq \mathcal{C} \overset{\leftarrow}{\underset{f^{\mathrm{R}}}{\times}} \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \simeq \mathcal{A}^{\{0\to 1\}} \times_{\mathcal{A}^{\{1\}}} \mathcal{C} \times_{\mathcal{C}} \mathcal{B} \\ &\simeq \mathcal{A}^{\{0\to 1\}} \times_{\mathcal{A}^{\{1\}}} \mathcal{B} \simeq \mathcal{B} \overset{\leftarrow}{\underset{f^{\mathrm{R}} g}{\times}} \mathcal{A}, \end{aligned}$$

where we have used Corollary A.3 in the third step and the explicit construction of the lax/oplax limit in steps two, four and six. Chasing through the chain of identifications one immediately obtains the desired formula.

The second statement is analogous, this time using the description

$$\mathcal{A} \overset{\curvearrowright}{\underset{\mathcal{C}}{\times}} \mathcal{B} \simeq \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \overset{\leftarrow}{\underset{f}{\times}} \mathcal{C}$$

and applying Corollary A.3 in the other direction. ∎

Corollary A.5. *Let $\mathbb{C}$ be an $(\infty,2)$-category.*

(1) *For each arrow $f\colon \mathcal{A} \to \mathcal{B}$ in $\mathbb{C}$ with a right adjoint f^{R}, we have natural equivalences*

$$\mathcal{A} \underset{f}{\overset{\rightarrow}{\times}} \mathcal{B} \simeq \mathcal{B} \underset{f^{\mathrm{R}}}{\overset{\leftarrow}{\times}} \mathcal{A} \quad \textit{and} \quad \mathcal{B} \underset{f^{\mathrm{R}}}{\overset{\leftarrow}{\amalg}} \mathcal{A} \simeq \mathcal{A} \underset{f}{\overset{\rightarrow}{\amalg}} \mathcal{B}.$$

(2) *For each diagram $\mathcal{A} \xrightarrow{f} \mathcal{C} \xleftarrow{g} \mathcal{B}$ in $\mathbb{C}$, we have a natural equivalences*

$$\mathcal{A} \underset{g^{\mathrm{L}} f}{\overset{\rightarrow}{\times}} \mathcal{B} \simeq \mathcal{A} \underset{\mathcal{C}}{\overset{\curvearrowright}{\times}} \mathcal{B} \simeq \mathcal{B} \underset{f^{\mathrm{R}} g}{\overset{\leftarrow}{\times}} \mathcal{A}.$$

assuming that g has a left adjoint or f has a right adjoint.

(3) *For each diagram $\mathcal{A} \xleftarrow{f} \mathcal{C} \xrightarrow{g} \mathcal{B}$ in $\mathbb{C}$, we have a natural equivalences*

$$\mathcal{B} \underset{f g^{\mathrm{R}}}{\overset{\leftarrow}{\amalg}} \mathcal{A} \simeq \mathcal{A} \underset{\mathcal{C}}{\overset{\curvearrowright}{\amalg}} \mathcal{B} \simeq \mathcal{A} \underset{g f^{\mathrm{L}}}{\overset{\rightarrow}{\amalg}} \mathcal{B}$$

assuming that g has a right adjoint or f has a left adjoint.

Each of these equivalences represents (in the case "$\amalg$") or corepresents (in the case "$\times$") the corresponding equivalences of Corollary A.3 and Lemma A.4.

Proof. All the relevant objects are characterized either by their represented or corepresented functor, hence we may reduce to the case of lax limits and directed pullbacks in $\mathbb{C}\mathrm{at}_\infty$. This case is established in Corollary A.3 and Lemma A.4. ■

A.2. About adjoints in diagram 2-categories

Let $\mathbb{B}, \mathbb{C}$ be two $(\infty,2)$-categories.

- By $\mathrm{FUN}_{\mathrm{lax}}(\mathbb{B},\mathbb{C})$ and $\mathrm{FUN}_{\mathrm{oplax}}(\mathbb{B},\mathbb{C})$ we denote the $(\infty,2)$-category of functors $\mathbb{B} \to \mathbb{C}$ and *lax/oplax* natural transformations $\eta\colon F \to G$ between them, which assigns to each morphism $f\colon B \to B'$ in $\mathbb{B}$ a square

$$\begin{tikzcd} FB \arrow[r, "Ff"] \arrow[d, "\eta^B"'] \arrow[dr, phantom, "\nearrow"] & FB' \arrow[d, "\eta^{B'}"] \\ GB \arrow[r, "Gf"'] & GB' \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} FB \arrow[r, "Ff"] \arrow[d, "\eta^B"'] \arrow[dr, phantom, "\swarrow"] & FB' \arrow[d, "\eta^{B'}"] \\ GB \arrow[r, "Gf"'] & GB' \end{tikzcd} \tag{A.6}$$

  respectively. Formally, the functors $\mathrm{FUN}_{\mathrm{lax}}(\mathbb{B},-)$ and $\mathrm{FUN}_{\mathrm{oplax}}(\mathbb{B},-)$ can be defined as right adjoints to the lax and oplax Gray tensor products; for example, see [11, Section 3] and references therein.
- By $\mathrm{FUN}(\mathbb{B},\mathbb{C})$ we denote the standard internal hom in the $(\infty,2)$-category of $(\infty,2)$-categories; it can be identified with the wide, locally full subcategory of $\mathrm{FUN}_{\mathrm{lax}}(\mathbb{B},\mathbb{C})$ and $\mathrm{FUN}_{\mathrm{oplax}}(\mathbb{B},\mathbb{C})$ containing only those 1-morphisms η, where the squares (A.6) contain invertible 2-cells.

If each component η^B of a lax natural transformation $\eta\colon F \to G$ has a left adjoint ${\eta^B}^{\mathrm{L}}$, then these assemble to an oplax natural transformation $\eta^{\mathrm{L}}\colon F \to G$ whose oplax naturality squares

$$\begin{tikzcd} GB \arrow[r, "Gf"] \arrow[d, "{\eta^B}^{\mathrm{L}}"'] \arrow[dr, phantom, "\swarrow"] & GB' \arrow[d, "{\eta^{B'}}^{\mathrm{L}}"] \\ FB \arrow[r, "Ff"'] & FB' \end{tikzcd}$$

are the canonical mates of the squares (A.6). Dually, each oplax transformation η has a canonical mate η^{R} (which is a lax transformation), whenever its components have right adjoints. Finally, note that each natural transformation η can be viewed both as a lax and as an oplax transformation, thus has both mates η^{L} (oplax) and η^{R} (lax), provided that all the required componentwise adjoints exist.

The following result due to Haugseng characterizes the morphisms in $\mathrm{FUN}(\mathbb{B}, \mathbb{C})$ which have a adjoints.

Proposition A.7 ([11, Theorem 4.6]). *Let $\eta\colon F \to G\colon \mathbb{B} \to \mathbb{C}$ be a natural transformation.*

(1) *As a morphism in $\mathrm{FUN}_{\mathrm{lax}}(\mathbb{B}, \mathbb{C})$, the transformation η has a right adjoint if and only if each component η^B has a right adjoint in $\mathbb{C}$. The right adjoint η^{R} is its canonical mate, where η is viewed as an oplax transformation.*

(2) *As a morphism in $\mathrm{FUN}_{\mathrm{oplax}}(\mathbb{B}, \mathbb{C})$, the transformation η has a left adjoint if and only if each component η^B has a left adjoint in $\mathbb{C}$. The left adjoint η^{L} is its canonical mate, where η is viewed as a lax transformation.*

This result also explains our terminology from Definition 7.37.

Corollary A.8. *Let $\eta\colon F \to G\colon \mathbb{B} \to \mathbb{C}$ be a natural transformation. As a morphism in $\mathrm{FUN}(\mathbb{B}, \mathbb{C})$ it has*

(1) *a right adjoint if and only if the naturality square (A.6) is vertically right adjointable,*

(2) *a left adjoint if and only if the naturality square (A.6) is vertically left adjointable,*

In each case, the left/right adjoint is the corresponding canonical mate.

Proof. Beyond the existence of adjoints, the right/left vertical adjointability condition states precisely that the 2-cells in the canonical mates η^{R}, η^{L} are again invertible, thus providing a right/left adjoint in $\mathrm{FUN}(\mathbb{B}, \mathbb{C})$ and not just in $\mathrm{FUN}_{\mathrm{lax}}(\mathbb{B}, \mathbb{C})/\mathrm{FUN}_{\mathrm{oplax}}(\mathbb{B}, \mathbb{C})$. ∎

Remark A.9. Let S be an ∞-category and $\alpha\colon \mathcal{X} \to \mathcal{Y}\colon S \to \mathbb{C}$ a natural transformation of S-diagrams in $\mathbb{C}$. Assume that each component α_s has a left/right adjoint β_s and that all naturality squares of α are vertically left/right adjointable. Corollary A.8 tells us that in this case the components β_s assemble to a natural transformation $\beta\colon \mathcal{Y} \to \mathcal{X}$ which is in the diagram category $\mathrm{FUN}(S, \mathbb{C})$ a left/right adjoint to α. Assuming that $\mathbb{C}$ has lax limits or colimits of shape S, we can apply the 2-functors

$$\mathrm{laxcolim}\colon \mathrm{FUN}(S, \mathbb{C}) \to \mathbb{C} \quad \text{and} \quad \mathrm{laxlim}\colon \mathrm{FUN}(S, \mathbb{C}) \to \mathbb{C}$$

to get corresponding adjunctions

$$\operatorname*{laxcolim}_s \alpha_s \colon \operatorname{laxcolim} \mathcal{X} \leftrightarrow \operatorname{laxcolim} \mathcal{Y} : \operatorname*{laxcolim}_s \beta_s$$

and

$$\operatorname*{laxlim}_s \alpha_s \colon \operatorname{laxlim} \mathcal{X} \leftrightarrow \operatorname{laxlim} \mathcal{Y} : \operatorname*{laxlim}_s \beta_s$$

which in the lax semiadditive case are identified with each other.

B. Recollements of stable ∞-categories

For the convenience of the reader, we quickly summarize the basic theory of recollements of stable ∞-categories, or equivalently that of semiorthogonal decompositions with a gluing functor, as it is used in Sections 7 and 8 without further mention. For comprehensive treatments, see for instance [17, Appendix A.8] and [7, Section 2].

All ∞-categories in this section are stable, all functors exact.

Definition B.1. A recollement of stable ∞-categories is a diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "i"] & \mathcal{C} \arrow[l, bend right, "q"'] \arrow[l, bend left, "q'"] \arrow[r, "p"] & \mathcal{B} \arrow[l, bend right, "j"'] \arrow[l, bend left, "j'"] \end{tikzcd} \tag{B.2}$$

of adjunctions $q \dashv i \dashv q'$ and $j \dashv p \dashv j'$, such that

- the functor i is fully faithful and exhibits $\mathcal{A}$ as the kernel of p;
- the functors j and j' are fully faithful and exhibit $\mathcal{B}$ as the kernel of q and of q', respectively.

In such a recollement, one often views $\mathcal{A}$ as a full subcategory of $\mathcal{C}$ via i; then $\mathcal{B}$ is viewed either as its left or its right orthogonal complement

$${}^{\perp}\mathcal{A} := \{c : \mathcal{C} \mid \mathcal{C}(c, i(-)) = 0\} = \ker(q) \simeq \mathcal{B}$$

and

$$\mathcal{A}^{\perp} := \{c : \mathcal{C} \mid \mathcal{C}(i(-), c) = 0\} = \ker(q') \simeq \mathcal{B}$$

depending on whether we view $\mathcal{B}$ as embedded in $\mathcal{C}$ via j or via j'. Note that while these two complements are both identified with $\mathcal{B}$, they are *not* the same subcategory of $\mathcal{C}$, unless the recollement is trivial, i.e., $\mathcal{C} = \mathcal{A} \times \mathcal{B}$.

Remark B.3. We prefer the formulation of Definition B.1 because it presents the full datum of a recollement in a way which is nicely symmetric between left and right adjoints. In this way our definition seemingly differs from Lurie's (see [17, Definition A.8.1]), who in a more general left exact setting defines recollements asymmetrically only in terms of two full subcategories $\mathcal{C}_0 \triangleq \mathcal{A}$ and $\mathcal{C}_1 \triangleq \mathcal{A}^{\perp}$ of $\mathcal{C}$, whose inclusions i and j' admit left adjoints $L_0 \triangleq q$ and $L_1 \triangleq p$, respectively. In the stable setting one then automatically has the other two adjoints q' and j, see [17, Remark A.8.19] (this can also be deduced from [7, Proposition 2.3.2]), justifying our more redundant definition.

One has the following computation.

Lemma B.4. *Given a recollement* (B.2)*, the units and counits of the various adjunctions yield canonical identifications*

$$q'j \cong \operatorname{fib}(j \to j') \cong qj'[-1]$$

of functors $\mathcal{B} \to \mathcal{A}$*, where the canonical map* $j \to j'$ *can be obtained either by transposing the counit* $pj \xrightarrow{\cong} \mathrm{id}_{\mathcal{B}}$ *along the adjunction* $p \dashv j'$ *or, equivalently, by transposing the unit* $\mathrm{id}_{\mathcal{B}} \xrightarrow{\cong} pj'$ *along the adjunction* $j \dashv p$*. Note that in the middle there is the implicit claim that* $\operatorname{fib}(j \to j') \colon \mathcal{B} \to \mathcal{C}$ *factors through* i*, so that we can view it as a functor* $\mathcal{B} \to \mathcal{A}$*.*

Definition B.5. Any of the equivalent functors $F \colon \mathcal{B} \to \mathcal{A}$ of Lemma B.4 is called the *gluing functor* of the recollement (B.2).

Construction B.6. Conversely given a $F \colon \mathcal{B} \to \mathcal{A}$, one can construct a canonical recollement

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook, "(0\to a)" description] & \mathcal{B} \overset{\to}{\underset{F}{\times}} \mathcal{A} \arrow[l, bend right=50, "\mathrm{cof}"'] \arrow[l, bend left=50, "\mathrm{ev}_a"] \arrow[r, "\mathrm{ev}_b"] & \mathcal{B} \arrow[l, bend right=50, "(b \xrightarrow{!} Fb)"'] \arrow[l, bend left=50, "(b\to 0)"] \end{tikzcd} \tag{B.7}$$

where in the middle we have the lax limit

$$\mathcal{B} \overset{\to}{\underset{F}{\times}} \mathcal{A} = \big\{(b \to a) = (b : B, a : A, Fb \to a)\big\},$$

i.e., the category of sections of the Grothendieck construction for the functor $\Delta^1 \to \mathcal{S}t$ classifying F. Manifestly, the gluing functor of the recollement (B.7) is the original functor $F \colon \mathcal{A} \to \mathcal{B}$.

The main structural result of the theory is that one can construct recollements starting with only very minimal amount of data.

Theorem B.8. *A recollement* (B.2) *can be uniquely recovered/constructed from any one of the following pieces of data:*

(1) *A fully faithful functor* $i \colon \mathcal{A} \hookrightarrow \mathcal{C}$ *which admits a left and a right adjoint. In this case* $\mathcal{B}$ *is determined as the Verdier quotient* $\mathcal{C}/\mathcal{A}$*.*

(2) *A functor* $p \colon \mathcal{C} \to \mathcal{B}$ *which admits a left and a right adjoint, both fully faithful. In this case* $\mathcal{A}$ *is determined as the kernel of* p*.*

(3) *An arbitrary functor* $F \colon \mathcal{B} \to \mathcal{A}$*. In this case the recollement is determined by Construction B.6.*

Proof. Part (1) is the statement of [7, Proposition 2.3.3] or [17, Proposition A.8.20]. Part (3) follows from [7, Proposition 2.2.11] or [17, Remark A.8.18]. The only thing to note there is that Lurie's convention is dual to ours: instead of reconstructing the recollement from the composite $F = q'j : \mathcal{B} \to \mathcal{A}$, he uses the composite $F' = qj' : \mathcal{B} \to \mathcal{A}$ (which is written $L_0|\mathcal{C}_1$ in his notation); and instead of considering sections of the coCartesian Grothendieck construction, he uses the Cartesian one. Passing from one convention to the other is just a matter of replacing each stable ∞-category with its opposite.

Part (2) follows from [7, Proposition 2.3.2]: since the inclusion $j : \mathcal{B} \to \mathcal{C}$ admits a right adjoint, there is a semiorthogonal decomposition $(\mathcal{A}, \mathcal{B})$ of $\mathcal{C}$ with $\mathcal{A} = \ker(p)$, which in turn implies that $i : \mathcal{A} \to \mathcal{C}$ admits a left adjoint. A similar argument shows that i admits a right adjoint. ∎

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