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Controlled KK-theory and a Milnor exact sequence

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Abstract. We introduce controlled KK-theory groups associated to a pair (A, B) of separable C^*-algebras. Roughly, these consist of elements of the usual K-theory group $K_0(B)$ that approximately commute with elements of A. Our main results show that these groups are related to Kasparov's KK-groups by a Milnor exact sequence, in such a way that Rørdam's KL-group is identified with an inverse limit of our controlled KK-groups.

In the case that the C^*-algebras involved satisfy the UCT, our Milnor exact sequence agrees with the Milnor sequence associated to a KK-filtration in the sense of Schochet, although our results are independent of the UCT. Applications to the UCT will be pursued in subsequent work.

1. Introduction

Given two C^*-algebras A and B, Kasparov associated an abelian group $KK(A, B)$ of generalized morphisms between A and B. The Kasparov KK-groups were designed to have applications to index theory and the Novikov conjecture [16], but now play a fundamental role in many aspects of C^*-algebra theory (and elsewhere). This is particularly true in the Elliott program [10] to classify C^*-algebras by K-theoretic invariants.

Our immediate goal in this paper is to introduce *controlled KK-theory groups* and relate them to Kasparov's KK-theory groups. The idea – which we will pursue in subsequent work – is that the controlled groups allow more flexibility in computations. Our groups are analogues of the controlled K-theory groups introduced by the second author as part of his work on the Novikov conjecture [36], and later developed by him in collaboration with Oyono-Oyono [18]. Having said that, our approach in this paper is independent of, and in some sense dual to, these earlier developments: controlled K-theory abstracts the approach to the Novikov conjecture through operators of controlled propagation as in [36], while the controlled KK-theory we introduce here abstracts the "dual" approach to the Novikov conjecture through almost flat bundles as in [3] and [33, Chapter 11].

Our larger goal is to establish a new sufficient condition for a nuclear C^*-algebra to satisfy the UCT of Rosenberg and Schochet [22], analogously to recent results on the Künneth formula using controlled K-theory ideas [19, 31]. These applications appear in the companion paper [34].

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Our goal in this paper is to establish the basic theory, which we hope will be useful in other settings: indeed, since the initial submission of this paper, we also found quite different applications to representation stability in group theory [32].

1.1. Controlled KK-theory and the Milnor sequence

We now discuss a version of our controlled KK-theory groups in more detail.

Let B be a separable C^*-algebra, let $B \otimes \mathcal{K}$ be its stabilization, and let $M(B \otimes \mathcal{K})$ be its stable multiplier algebra. Define $\mathcal{P}(B)$ to consist of all projections in $p \in M_2(M(B \otimes \mathcal{K}))$ such that $p - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is in the ideal $M_2(B \otimes \mathcal{K})$. Then the formula

$$\pi_0\big(\mathcal{P}(B)\big) \longrightarrow K_0(B), \quad [p] \longmapsto [p] - \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] \tag{1.1}$$

gives a bijection from the set of path components of $\mathcal{P}(B)$ to the usual K_0-group of B.

Now, assume for simplicity that A is a separable, unital, and nuclear[1] C^*-algebra. Let $\pi\colon A \to \mathcal{B}(\ell^2)$ be an infinite amplification of a faithful unital representation,[2] and use the composition

$$A \longrightarrow \mathcal{B}(\ell^2) = M(\mathcal{K}) \subseteq M(B \otimes \mathcal{K})$$

of π and the canonical inclusion of $M(\mathcal{K})$ into $M(B \otimes \mathcal{K})$ to consider A as a C^*-subalgebra of $M(B \otimes \mathcal{K})$. We also identify A with a C^*-subalgebra of $M_2(M(B \otimes \mathcal{K}))$ via the diagonal action. For a finite subset X of A and $\varepsilon > 0$, define

$$\mathcal{P}_\varepsilon(X, B) := \big\{p \in \mathcal{P}(B) \mid \big\|[p, a]\big\| < \varepsilon \text{ for all } a \in X\big\}.$$

Define the *controlled KK-theory group*[3] *associated to X and ε* to be

$$KK_\varepsilon(X, B) := \pi_0\big(\mathcal{P}_\varepsilon(X, B)\big).$$

Thanks to the isomorphism of line (1.1), we think of $KK_\varepsilon(X, B)$ as "the part[4] of $K_0(B)$ that commutes with X up to ε". This idea – of considering elements of K-theory that asymptotically commute with some representation – is partly inspired by the E-theory of Connes and Higson [4].

Now, let (X_n) be a nested sequence of finite subsets of A with dense union, and let (ε_n) be a decreasing sequence of positive numbers that tend to zero. As it is easier to commute with X_n up to ε_n that it is to commute with X_{n+1} up to ε_{n+1}, we get a sequence

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[1] The assumptions of unitality and nuclearity are not necessary, but simplify the definitions - see the body of the paper for the general versions. The fact that the version we give here is equivalent to the general definition is a consequence of Proposition A.19 and Remark A.20.

[2] I.e., take a faithful unital representation on a separable Hilbert space, and add it to itself countably many times. It turns out the choices involved here do not matter in any serious way.

[3] It is a group in a natural way, in a way that is compatible with the group structure on $K_0(B)$ via the isomorphism in line (1.1).

[4] The word "part" is potentially misleading: there is a map $KK_\varepsilon(X, B) \to K_0(B)$ defined by forgetting the commutation condition, but it need not be injective in general.

of "forget control" homomorphisms

$$\cdots \longrightarrow KK_{\varepsilon_n}(X_n, B) \longrightarrow KK_{\varepsilon_{n-1}}(X_{n-1}, B) \longrightarrow \cdots \longrightarrow KK_{\varepsilon_1}(X_1, B).$$

Thus we may build the inverse limit $\varprojlim KK_{\varepsilon_n}(X_n, B)$ of abelian group theory associated to this sequence; moreover, we may consider this inverse limit as a topological abelian group by giving each $KK_\varepsilon(X, B)$ the discrete topology and taking the inverse limit in the category of topological abelian groups. Replacing B with its suspension SB, we may also build the $\lim^1$-group[5] $\varprojlim^1 KK_{\varepsilon_n}(X_n, SB)$ associated to the corresponding sequence. We are now ready to state a special case of our main theorem.

Theorem 1.1. *For any separable C^*-algebras A and B with A unital and nuclear,*[6] *there is a short exact sequence*

$$0 \longrightarrow \varprojlim{}^1 KK_{\varepsilon_n}(X_n, SB) \longrightarrow KK(A, B) \longrightarrow \varprojlim KK_{\varepsilon_n}(X_n, B) \longrightarrow 0.$$

We will explain the idea of the proof below, but first give a more precise version involving Rørdam's KL-groups [20, Section 5], and some comparisons of our results to the previous literature.

1.2. The topology on KK and Schochet's Milnor sequence

Recall that $KK(A, B)$ is equipped with a canonical topology, which makes it a (possibly non-Hausdorff) topological group. This topology can be described in several different ways that turn out to be equivalent, as established by Dadarlat in [6] (see also [24]). Define[7] $KL(A, B)$ to be to be the associated maximal Hausdorff quotient, i.e., the quotient $KK(A, B)/\overline{\{0\}}$ of $KK(A, B)$ by the closure of the zero element.

The following theorem relating our controlled KK-theory groups to the topology on KK is a more precise version of Theorem 1.1, and is what we actually establish in the main body of the paper.

Theorem 1.2. *For any separable C^*-algebras A and B with A unital and nuclear,*[8] *there are canonical isomorphisms*

$$\varprojlim{}^1 KK_{\varepsilon_n}(X_n, SB) \cong \overline{\{0\}} \quad \text{and} \quad \varprojlim KK_{\varepsilon_n}(X_n, B) \cong KL(A, B).$$

Moreover, the second isomorphism above is an isomorphism of topological groups.

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[5]The $\lim^1$ functor is the first derived functor of the inverse limit functor. See for example [30, Section 3.5] or [26, Chapter 3] for concrete definitions of the inverse limit and $\lim^1$ groups, and for some examples of computations. See [13] for the general case.

[6]There is also a very similar version for general separable A: see the main body of the paper.

[7]The original definition of $KL(A, B)$ is due to Rørdam [20, Section 5], and only makes sense if the pair (A, B) satisfies the UCT. The definition we are using was suggested by Dadarlat [6, Section 5], and is equivalent to Rørdam's when A satisfies the UCT by [6, Theorem 4.1].

[8]Again, the unitality and nuclearity assumptions on A are not necessary, with appropriate changes to the definitions.

The short exact sequence in Theorem 1.1 is an analogue of Schochet's *Milnor exact sequence* [23] associated to a *KK-filtration*. A KK-filtration consists of a KK-equivalence of A with the direct limit of an increasing sequence (A_n) of C^*-algebras where each A_n has unitization the continuous functions on some finite CW complex. Schochet [23, Theorem 1.5] shows that such a filtration exists if and only if A satisfies the UCT. Schochet [23, Theorem 1.5] then shows that there is an exact sequence

$$0 \longrightarrow \varprojlim{}^1 KK(A_n, SB) \longrightarrow KK(A, B) \longrightarrow \varprojlim KK(A_n, B) \longrightarrow 0.$$

It follows from Theorem 1.2 and [25, Proposition 4.1] that our Milnor sequence from Theorem 1.1 agrees with Schochet's when A satisfies the UCT. Our Milnor sequence can thus be thought of as a generalization of Schochet's sequence that works in the absence of the UCT.

1.3. Discussion of proofs

Continuing to assume for simplicity that A is unital and nuclear, let us identify A with a C^*-subalgebra of $M(B \otimes \mathcal{K})$ as in the statement of Theorem 1.1. Then we define $\mathcal{P}(A, B)$ to be the collection of all continuous, bounded, projection-valued functions $p\colon [1, \infty) \to M_2(M(B \otimes \mathcal{K}))$ such that $[p_t, a] \to 0$ as $t \to \infty$ for all $a \in A$, and so that $p_t - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is in $M_2(B \otimes \mathcal{K})$ for all t. Define $KK_{\mathcal{P}}(A, B)$ to be the quotient of $\mathcal{P}(A, B)$ by "homotopy", i.e., by stipulating that p and q are equivalent if they are restrictions to the endpoints of an element of $\mathcal{P}(A, C[0, 1] \otimes B)$.

One can then show[9] that $KK(A, B)$ is naturally isomorphic to $KK_{\mathcal{P}}(A, B)$. The first important ingredient in this is the description of $KK(A, B)$ as the K-theory of an appropriate *localization algebra*, which was done by Dadarlat, Wu and the first author in [9, Theorem 4.4] (inspired by ideas of the second author in the case of commutative C^*-algebras [35]). The other important (albeit implicit) ingredient we use for the isomorphism $KK(A, B) \cong KK_{\mathcal{P}}(A, B)$ is the fundamental theorem of Kasparov that the equivalence relations on Kasparov cycles induced by operator homotopy, and by homotopy, give rise to the same KK-groups: see [15, Section 6, Theorem 1], and see also [27, Theorem 19] and [1, Section 18.5].

Having described $KK(A, B)$ using continuous paths of projections, we can now also describe its topology in this language: roughly, a sequence $(p^n)_{n=1}^{\infty}$ converges to p in $\mathcal{P}(A, B)$ if for all $\varepsilon > 0$ and finite $X \subseteq A$ there is t_0 such that for all $t \geq t_0$, p^n_t can be connected to p_t via a homotopy passing through $\mathcal{P}_\varepsilon(X, B)$. This topology on $\mathcal{P}(A, B)$ induces a topology on $KK_{\mathcal{P}}(A, B)$, and we show that the latter topology agrees with the usual one on $KK(A, B)$ using an abstract characterization due to Dadarlat [6, Section 3] (and based on ideas of Pimsner).

[9] We do not actually show this, only the more general version where A is non-unital and not-necessarily nuclear; nonetheless, this result follows directly from the same methods.

Having got this far, it is not too difficult to see that there is a well-defined map

$$KK_{\mathcal{P}}(A,B) \longrightarrow \varprojlim KK_{\varepsilon_n}(X_n,B) \tag{1.2}$$

defined by evaluating a path $(p_t)_{t\in[1,\infty)}$ in $\mathcal{P}(A,B)$ at larger and larger values of t, and that there is a well-defined map

$$\varprojlim{}^1 KK_{\varepsilon_n}(X_n,SB) \longrightarrow KK_{\mathcal{P}}(A,B) \tag{1.3}$$

defined by treating an element of $\mathcal{P}_\varepsilon(X,SB)$ as a projection-valued function from $[0,1]$ to $\mathcal{P}_\varepsilon(X,B)$ that agrees with $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ at its endpoints, and stringing a countable sequence of these together to get an element $(p_t)_{t\in[1,\infty)}$ in $\mathcal{P}(A,B)$. It is then essentially true by definition that the map in line (1.2) contains the closure of $\{0\}$ in its kernel so descends to a map

$$KK_{\mathcal{P}}(A,B)/\overline{\{0\}} \longrightarrow \varprojlim KK_{\varepsilon_n}(X_n,B),$$

and moreover that the map in line (1.3) takes image if the closure of $\{0\}$ so corestricts to a map

$$\varprojlim{}^1 KK_{\varepsilon_n}(X_n,SB) \longrightarrow \overline{\{0\}}.$$

To establish Theorem 1.2, we show that the maps in the two previous displayed lines are isomorphisms.

1.4. Notation and conventions

We write ℓ^2 for $\ell^2(\mathbb{N})$.

Throughout, the symbols A and B are reserved for separable C^*-algebras; the letters C, D and others may sometimes refer to non-separable C^*-algebras. The unit ball of a C^*-algebra C is denoted by C_1, its unitization is C^+, and its multiplier algebra is $M(C)$.

Our conventions on Hilbert modules follow those of Lance [17]. We will always assume that Hilbert modules are over separable C^*-algebras, and are countably generated as discussed on [17, p. 60]. If it is not explicitly specified otherwise, all Hilbert modules will be over the C^*-algebra called B. For Hilbert B-modules E and F, we write $\mathcal{L}(E,F)$ (respectively $\mathcal{K}(E,F)$) for the spaces of adjointable (respectively compact) operators from E to F in the usual sense of Hilbert module theory [17, pp. 8–10]. We use the standard shorthands $\mathcal{L}(E) := \mathcal{L}(E,E)$ and $\mathcal{K}(E) := \mathcal{K}(E,E)$.

In this paper, unless stated otherwise, a *representation of A* will refer to a representation of A on a Hilbert *module*, i.e., a $*$-homomorphism $\pi\colon A\to\mathcal{L}(E)$ for some Hilbert module E (almost always over B, as above). For $a\in A$, we generally just write $a\in\mathcal{L}(E)$ for $\pi(a)$ to avoid notational clutter (even if π is not assumed injective). We write E^∞ for the (completed) infinite direct sum Hilbert module $\bigoplus_{n=1}^\infty E$ [17, p. 6], or equivalently the external tensor product $\ell^2\otimes E$ [17, p. 35]. If $\pi\colon A\to\mathcal{L}(E)$ is a representation, we write $\pi^\infty\colon A\to\mathcal{L}(E^\infty)$ for the amplified representation; in tensor product notation we have $\pi^\infty = 1_{\ell^2}\otimes\pi\colon A\to\mathcal{L}(\ell^2\otimes E)$. We say a representation (π,E) has *infinite multiplicity* if it isomorphic to (σ^∞,F^∞) for some other representation (σ,F).

The symbol "$\otimes$" always denotes a completed tensor product: either the (external or internal) tensor product of Hilbert modules [17, Chapter 4], or the minimal tensor product of C^*-algebras.

If E is a Banach space and X a locally compact Hausdorff space, we let $C_b(X, E)$ (respectively, $C_{ub}(X, E)$, $C_0(X, E)$) denote the Banach space of continuous and bounded (respectively uniformly continuous and bounded, continuous and vanishing at infinity) functions from X to E. We write elements of these spaces as e or $(e_x)_{x\in X}$, with $e_x \in E$ denoting the value of e at a point $x \in X$. We will sometimes say that e is a "..." if it is a pointwise a "...": for example, "$u \in C_b([1,\infty), \mathcal{L}(F_1, F_2))$ is unitary" means "u_t is unitary in $\mathcal{L}(F_1, F_2)$ for all $t \in [1,\infty)$"; if E is a C^*-algebra so $C_b(X, E)$ is also a C^*-algebra, then this is consistent with the standard use of "unitary" and so on. With $u \in C_b([1,\infty), \mathcal{L}(F_1, F_2))$ as above, if b is an element of $\mathcal{L}(F_1)$ we write ub for the function $t \mapsto u_t b$ in $C_{ub}([1,\infty), \mathcal{L}(F_1, F_2))$ and similarly for cu with $c \in \mathcal{L}(F_2)$ and so on.

For K-theory,

$$K_*(A) := K_0(A) \oplus K_1(A)$$

denotes the graded K-theory group of a C^*-algebra, and $KK_*(A, B) := KK_0(A, B) \oplus KK_1(A, B)$ the graded KK-theory group. We will typically just write $KK(A, B)$ instead of $KK_0(A, B)$.

1.5. Outline of the paper

Sections 2 and 3 are background. In Section 2 we recall some facts about "absorbing" representations. Most of the material in Section 2 is essentially contained in papers of Kasparov [14], Thomsen [29], Dadarlat–Eilers [7, 8], and Dadarlat [6]. In Section 3 we recall the localization algebra of Dadarlat, Wu and the first author [9] (inspired by earlier ideas of the second author [35]), and establish some technical results about this.

Sections 4 and 5 introduce a group $KK_{\mathcal{P}}(A, B)$ that consists of homotopy classes of paths of projections that asymptotically commute with A and relate it to KK-theory: the culminating results show that $KK(A, B)$ and $KK_{\mathcal{P}}(A, B)$ are isomorphic as topological groups. In Section 4 we introduce $KK_{\mathcal{P}}(A, B)$, show that it is a commutative monoid, and then that it is isomorphic to $KK(A, B)$ (whence a group). In Section 5 we introduce a topology on $KK_{\mathcal{P}}(A, B)$. We then use a characterization of Dadarlat [6, Section 3] to identify this with the canonical topology on $KK(A, B)$ that was introduced and studied by Brown, Salinas, Schochet, Pimsner, and Dadarlat in various guises.

Sections 6 and 7 establish Theorem 1.2 (and therefore Theorem 1.1). Section 6 identifies the quotient $KK_{\mathcal{P}}(A, B)/\overline{\{0\}}$ with $\varprojlim KK_{\varepsilon}(X, B)$ (and therefore identifies $KL(A, B)$ with this inverse limit). Section 7 identifies the closure of zero in $KK_{\mathcal{P}}(A, B)$ with the appropriate $\lim^1$ group, completing the proof of the main results.

Finally, Appendix A gives some alternative pictures of our controlled KK-groups that will be useful for our subsequent work. In particular, we give a slightly simplified picture in the case that A is unital.

2. Strongly absorbing representations

Throughout this section, A and B refer to separable C^*-algebras. All Hilbert modules are countably generated, and all are over B unless explicitly stated otherwise. All representations of A are on Hilbert B-modules unless explicitly stated otherwise.

In this section we establish conventions and terminology regarding representations on Hilbert modules. The ideas in this section are not original: they come from papers of Kasparov [14], Thomsen [29], Dadarlat–Eilers [7, 8], and Dadarlat [6]. Nonetheless, we need some variants of the material appearing in the literature, so record what we need here for the reader's convenience; we provide proofs where we could not find the precise statement we need in the literature.

The definition of absorbing representation below is essentially[10] due to Thomsen [29, Definition 2.6].

Definition 2.1. A representation $\pi\colon A \to \mathcal{L}(F)$ is *absorbing* (for the pair (A, B)) if for any Hilbert B-module E and ccp map $\sigma\colon A \to \mathcal{L}(E)$, there is a sequence (v_n) of isometries in $\mathcal{L}(E, F)$ such that:

(i) $\sigma(a) - v_n^*\pi(a)v_n \in \mathcal{K}(E)$ for all $a \in A$ and $n \in \mathbb{N}$;

(ii) $\|\sigma(a) - v_n^*\pi(a)v_n\| \to 0$ as $n \to \infty$ for all $a \in A$.

We want something slightly stronger.

Definition 2.2. A representation $\pi\colon A \to \mathcal{L}(F)$ is *strongly absorbing* (for the pair (A, B)) if (π, F) is the infinite amplification $(\sigma^\infty, E^\infty)$ of an absorbing representation (σ, E).

Remark 2.3. If (π, F) is an infinite multiplicity (for example, strongly absorbing) representation then we can write it as an infinite direct sum of copies of itself. It follows that there is a sequence $(s_n)_{n=1}^\infty$ of isometries in $\mathcal{L}(F)$ with mutually orthogonal ranges, that commute with the image of the representation, and are such that the sum $\sum_{n=1}^\infty s_n s_n^*$ converges strictly[11] to the identity.

In [29, Theorem 2.7], Thomsen shows that an absorbing representation of A on $\ell^2 \otimes B$ always exists. The following is therefore immediate from the fact that $(\ell^2 \otimes B)^\infty \cong \ell^2 \otimes B$.

Proposition 2.4. *There is a strongly absorbing representation of A on $\ell^2 \otimes B$.* ∎

The point of using strongly absorbing representations rather than just absorbing[12] ones is to get the following lemma.

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[10]Thomsen's definition is a little more restrictive: he insists that B be stable, and that the B-modules used all be copies $B \otimes \mathcal{K}(\ell^2)$. Thanks to a combination of Kasparov's stabilization theorem [14, Theorem 2] and Remark A.15 below, our extra generality makes no real difference.

[11]For the strict topology coming from the identification $\mathcal{L}(F) = M(\mathcal{K}(F))$. As the partial sums are uniformly bounded, we may equivalently use the topology of pointwise convergence as operators on F.

[12]We do not know that the lemma fails for absorbing representations, but cannot prove it either.

Lemma 2.5. *Let $\pi\colon A \to \mathcal{L}(F)$ be a strongly absorbing representation, and let $\sigma\colon A \to \mathcal{L}(E)$ be a ccp map. Then there is a sequence (v_n) of isometries in $\mathcal{L}(E, F)$ such that:*

(i) $\sigma(a) - v_n^*\pi(a)v_n \in \mathcal{K}(E)$ *for all* $a \in A$ *and* $n \in \mathbb{N}$*;*

(ii) $\|\sigma(a) - v_n^*\pi(a)v_n\| \to 0$ *as* $n \to \infty$ *for all* $a \in A$*;*

(iii) $v_n^* v_m = 0$ *for all* $n \neq m$.

Proof. Let $(\pi, F) = (\theta^\infty, G^\infty)$ for some absorbing representation (θ, G). Let (w_n) be a sequence of isometries in $\mathcal{L}(E, G)$ with the properties as in the definition of an absorbing representation for σ. For each n, let $s_n \in \mathcal{L}(G, F)$ be the inclusion of G in F as the nth summand, and set $v_n := s_n w_n \in \mathcal{L}(E, F)$. Direct checks show that (v_n) has the right properties. ∎

We will need the following result, which is implicit[13] in [7].

Proposition 2.6. *Let $\pi\colon A \to \mathcal{L}(E)$ be a strongly absorbing representation. Then for any ccp map $\sigma\colon A \to \mathcal{L}(F)$ there is an isometry $v \in C_{ub}([1, \infty), \mathcal{L}(F, E))$ such that $v^*\pi(a)v - \sigma(a) \in C_0([1, \infty), \mathcal{K}(F))$.*

Moreover, if $\sigma\colon A \to \mathcal{L}(F)$ is also a strongly absorbing representation, then there is a unitary $u \in C_{ub}([1, \infty), \mathcal{L}(F, E))$ such that $u^*\pi(a)u - \sigma(a) \in C_0([1, \infty), \mathcal{K}(F))$.

For the proof of Proposition 2.6 will need two lemmas. The first is a well-known algebraic trick.

Lemma 2.7. *Let $\pi\colon A \to \mathcal{L}(E)$ and $\sigma\colon A \to \mathcal{L}(F)$ be representations, and let $v \in \mathcal{L}(E, F)$ be an isometry. If $v \in C_{ub}([1, \infty), \mathcal{L}(F, E))$ is such that*

$$v^*\pi(a)v - \sigma(a) \in C_0\big([1, \infty), \mathcal{K}(F)\big) \quad \textit{for all } a \in A,$$

then $\pi(a)v - v\sigma(a)$ is an element of $C_0([1, \infty), \mathcal{K}(F, E))$ for all $a \in A$.

Proof. This follows from the fact that

$$\big(\pi(a)v - v\sigma(a)\big)^*\big(\pi(a)v - v\sigma(a)\big)$$

equals

$$v^*\pi(a^*a)v - \sigma(a^*a) - \big(v^*\pi(a^*)v - \sigma(a^*)\big)\sigma(a) - \sigma(a^*)\big(v^*\pi(a)v - \sigma(a)\big)$$

for all $a \in A$. ∎

The second lemma we need is [8, Lemma 2.16]; we recall the statement for the reader's convenience but refer to the reference for a proof.

[13] It is also explicit in [9, Theorem 2.6], but with π only assumed absorbing, not strongly absorbing. However, there seems to be a gap in the proof of [9, Theorem 2.6], so it seems to be necessary to assume all absorbing modules used in [9] are actually strongly absorbing. None of the results of [9] are further affected if one does this.

Lemma 2.8. *Let $\pi\colon A \to \mathcal{L}(E)$ and $\sigma\colon A \to \mathcal{L}(F)$ be representations. Let $\sigma^\infty\colon A \to \mathcal{L}(F^\infty)$ be the infinite amplification of σ, and let $w \in \mathcal{L}(F^\infty, F \oplus F^\infty)$ be defined by $(\xi_1, \xi_2, \xi_3, \ldots) \mapsto \xi_1 \oplus (\xi_2, \xi_3, \ldots)$. Then for any isometry $v \in \mathcal{L}(F^\infty, E)$, the operator*

$$u := (1_F \oplus v)wv^* + 1_E - vv^* \in \mathcal{L}(E, F \oplus E)$$

is unitary and satisfies

$$\|\sigma(a) \oplus \pi(a) - u\pi(a)u^*\| \leq 6\|v\sigma^\infty(a) - \pi(a)v\| + 4\|v\sigma^\infty(a^*) - \pi(a^*)v\|.$$

Moreover, if $v\sigma^\infty(a) - \pi(a)v \in \mathcal{K}(F^\infty, E)$ for all $a \in A$, then $\sigma(a) \oplus \pi(a) - u\pi(a)u^* \in \mathcal{K}(F \oplus E)$ for all $a \in A$. ∎

Proof of Proposition 2.6. Assume first that $\sigma\colon A \to \mathcal{L}(F)$ is ccp. Let (v_n) be a sequence of isometries in $\mathcal{L}(F, E)$ as in Lemma 2.5. For each $n \geq 1$ and each $t \in [n, n+1]$, define

$$v_t := (n + 1 - t)^{1/2} v_n + (t - n)^{1/2} v_{n+1}.$$

Then the resulting family $v := (v_t)_{t \in [1,\infty)}$ is an isometry in $C_{ub}([1,\infty), \mathcal{L}(F, E))$ such that $v^*\pi(a)v - \sigma(a) \in C_0([1,\infty), \mathcal{K}(F))$ for all $a \in A$; we leave the direct checks involved to the reader.

Assume now that $\sigma\colon A \to \mathcal{L}(F)$ is also a strongly absorbing representation. Using the first part of the proof applied to the infinite amplification $\sigma^\infty\colon A \to \mathcal{L}(F^\infty)$, we get $v \in C_{ub}([1,\infty), \mathcal{L}(F^\infty, E))$ such that $v^*\pi(a)v - \sigma^\infty(a) \in C_0([1,\infty), \mathcal{K}(F^\infty))$ for all $a \in A$. Lemma 2.7 implies that $\pi(a)v - v\sigma^\infty(a)$ is an element of $C_0([1,\infty), \mathcal{K}(F^\infty, E))$ for all $a \in A$. Building a unitary out of each v_t using the formula in Lemma 2.8 gives now a unitary $u_E \in C_{ub}([1,\infty), \mathcal{L}(E, F \oplus E))$ such that $\sigma(a) \oplus \pi(a) - u_E\pi(a)u_E^* \in C_0([1,\infty), \mathcal{K}(F \oplus E))$ for all $a \in A$. The situation is symmetric, so there is also a unitary $u_F \in C_{ub}([1,\infty), \mathcal{L}(F, F \oplus E))$ such that

$$\sigma(a) \oplus \pi(a) - u_F\sigma(a)u_F^* \in C_0\big([1,\infty), \mathcal{K}(F \oplus E)\big) \quad \text{for all } a \in A.$$

Defining $u = u_E^* u_F$, we are done. ∎

We need one more technical result about strongly absorbing representations. The statement and proof are essentially[14] the same as a result of Dadarlat [6, Proposition 3.2]. We give a proof for the reader's convenience.

Proposition 2.9. *Let $\pi\colon A \to \mathcal{L}(B \otimes \ell^2)$ be a strongly absorbing representation of A on the standard Hilbert B-module. Let C be a separable nuclear C^*-algebra, and let $C \otimes B \otimes \ell^2$ denote the $C \otimes B$-Hilbert module given by the exterior tensor product. Then the amplification $1_C \otimes \pi\colon A \to \mathcal{L}(C \otimes B \otimes \ell^2)$[15] is strongly absorbing for the pair $(A, C \otimes B)$.*

[14] The main difference is that we drop a unitality assumption on C. This is important for our applications later in the paper, as we will want to apply the result with $C = C_0(0,1)$ in order to treat suspensions of C^*-algebras.

[15] Here and throughout, by "1_C" we mean either the unit of C if C is unital, or the unit of the unitization C^+ of C, acting as a multiplier on C.

Proof. As $1_C \otimes \pi$ is isomorphic to the infinite amplification of itself, it suffices to show that $1_C \otimes \pi$ is absorbing. Let $(1_C \otimes \pi)^+ \colon A^+ \to \mathcal{L}(C \otimes B \otimes \ell^2)$ be the canonical unital extension of $1_C \otimes \pi$ to the unitization A^+ of A (even if A is already unital). Using Kasparov's stabilization theorem [14, Theorem 2], the equivalence of (1) and (2) from [29, Theorem 2.5], [29, Theorem 2.1], and the canonical identifications $C \otimes B \otimes \mathcal{K}(\ell^2) = \mathcal{K}(C \otimes B \otimes \ell^2)$ and $\mathcal{L}(C \otimes B \otimes \ell^2) = M(\mathcal{K}(C \otimes B \otimes \ell^2))$, it suffices to show that if $\sigma \colon A^+ \to C \otimes B \otimes \mathcal{K}(\ell^2)$ is any ccp map then there is a sequence (w_n) in $\mathcal{L}(C \otimes B \otimes \mathcal{K}(\ell^2))$ such that

$$\lim_{n\to\infty} \left\| \sigma(a) - w_n^*(1_C \otimes \pi)^+(a) w_n \right\| = 0 \quad \text{for all } a \in A^+$$

and such that

$$\lim_{n\to\infty} \|w_n^* b\| = 0 \quad \text{for all } b \in C \otimes B \otimes \mathcal{K}(\ell^2).$$

Let $\delta \colon C^+ \to \mathcal{B}(\ell^2)$ be a unital representation of the unitization of C such that

$$\delta^{-1}\big(\mathcal{K}(\ell^2)\big) = \{0\}.$$

Let $\iota \colon C^+ \to \mathcal{L}(C)$ be the canonical multiplication representation. Kasparov's version of Voiculescu's theorem [14, Theorem 5] combined with nuclearity of C^+ imply that there is a sequence $(v_{n,(0)})_{n=1}^{\infty}$ of isometries in $\mathcal{L}(C, C^+ \otimes \ell^2)$ such that

$$\left\| \iota(c) - v_{n,(0)}^* \big(1_{C^+} \otimes \delta(c)\big) v_{n,(0)} \right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty$$

for all $c \in C^+$. Perturbing $v_{n,(0)}$ slightly, we may assume that actually $v_{n,(0)}$ has image in $C^+ \otimes \ell^2\{1, \dots, k(n)\}$ for some $k(n)$.

Let (e_m) be an approximate unit for C. We may consider multiplication by e_m as defining an operator in $\mathcal{L}_C(C^+ \otimes H, C \otimes H)$ for any Hilbert space H, and therefore the product operators $e_m v_{n,(0)}$ make sense in $\mathcal{L}(C, C \otimes \ell^2\{1, \dots, k(n)\})$. For a suitable choice of $m(n)$ we have that if $v_{n,(1)} := e_{m(n)} v_{n,(0)}$ then

$$\left\| \iota(c) - v_{n,(1)}^* \big(1_C \otimes \delta(c)\big) v_{n,(1)} \right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty$$

for all $c \in C$. Let $\delta_n \colon C \to \mathcal{K}(\ell^2\{1, \dots, k(n)\})$ be the compression of δ to the first n basis vectors. Note that by choice of $k(n)$ we have $v_{n,(1)}^*(1_C \otimes \delta(c)) v_{n,(1)} = v_{n,(1)}^*(1_C \otimes \delta_n(c)) v_{n,(1)}$ for all n, and thus that

$$\left\| \iota(c) - v_{n,(1)}^* \big(1_C \otimes \delta_n(c)\big) v_{n,(1)} \right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty$$

for all $c \in C$.

Define

$$\Delta_n := \delta_n \otimes 1_{B \otimes \mathcal{K}(\ell^2)} \colon C \otimes B \otimes \mathcal{K}(\ell^2) \longrightarrow \mathcal{K}\big(\ell^2\{1, \dots, k(n)\}\big) \otimes B \otimes \mathcal{K}(\ell^2).$$

Define $v_{n,(2)} := v_{n,(1)} \otimes 1_{B \otimes \mathcal{K}(\ell^2)}$, so

$$v_{n,(2)} \in \mathcal{L}_{C \otimes B \otimes \mathcal{K}(\ell^2)}\big(C \otimes B \otimes \mathcal{K}(\ell^2),\ C \otimes \ell^2\{1, \dots, k(n)\} \otimes B \otimes \mathcal{K}(\ell^2)\big).$$

Note then that

$$\left\| c - v_{n,(2)}^{*}\big(1_C \otimes \Delta_n(c)\big)v_{n,(2)} \right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty$$

for all $c \in C \otimes B \otimes \mathcal{K}(\ell^2)$ and so in particular

$$\left\| \sigma(a) - v_{n,(2)}^{*}\big(1_C \otimes \Delta_n(\sigma(a))\big)v_{n,(2)} \right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty$$

for all $a \in A^{+}$.

To complete the proof, use an identification $\mathcal{K}(\ell^2\{1,\dots,k(n)\}) \otimes \mathcal{K}(\ell^2) \cong \mathcal{K}(\ell^2)$ to give an isomorphism $\phi : \mathcal{K}(\ell^2\{1,\dots,k(n)\}) \otimes B \otimes \mathcal{K}(\ell^2) \to B \otimes \mathcal{K}(\ell^2)$. Note that as π is absorbing there is a sequence $(v_{n,(3)})_{n=1}^{\infty}$ in $\mathcal{L}(B \otimes \mathcal{K}(\ell^2))$ such that

$$\left\| \phi\big(\Delta_n\big(\sigma(a)\big)\big) - v_{n,(3)}^{*}\pi(a)v_{n,(3)} \right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty$$

for all $a \in A^{+}$ (compare the equivalence of (1) and (2) from [29, Theorem 2.5] again). As π is strongly absorbing, we may moreover assume that the $v_{n,(3)}$ satisfy $v_{n,(3)}^{*}b \to 0$ for all $b \in B \otimes \mathcal{K}(\ell^2)$ by ensuring that for all m, $v_{n,(3)}v_{n,(3)}^{*}$ is orthogonal to any element of $B \otimes \mathcal{K}(\ell^2\{1,\dots,m\})$ for all large n. It is then not too difficult to check that we can choose $l(n)$ such that if we set

$$w_n := (1_C \otimes v_{l(n),(3)})v_{n,(2)} \in \mathcal{L}\big(C \otimes B \otimes \mathcal{K}(\ell^2)\big),$$

then (w_n) has the right properties. ∎

3. Localization algebras

Throughout this section, A and B refer to separable C^{*}-algebras. All Hilbert modules are countably generated, and all are over B unless explicitly stated otherwise. All representations of A are on Hilbert B-modules unless explicitly stated otherwise.

In this section, we define localization algebras following [9], and show that uniform continuity can be replaced with continuity in the definition without changing the K-theory. This result was first observed by Jianchao Wu (with a different proof), and we thank him for permission to include it here.

The following definition comes from [9, Section 3]. We use slightly different notation to that reference to differentiate between the continuous and uniformly continuous versions.

Definition 3.1. Let $\pi\colon A \to \mathcal{L}(E)$ be a representation of A on a Hilbert B-module. Define $C_{L,u}(\pi)$ to be the C^{*}-algebra of all bounded, uniformly continuous functions $b\colon [1,\infty) \to \mathcal{L}(E)$ such that $[b_t, a] \to 0$ for all $a \in A$, and such that ab_t is in $\mathcal{K}(E)$ for all $a \in A$ and all $t \in [1,\infty)$. We call $C_{L,u}(\pi)$ the *localization algebra* of π.

The following is[16] [9, Theorem 4.4].

[16] As explained in Footnote 13, the cited result should be stated with the assumption that the representation is strongly absorbing, not just absorbing.

Theorem 3.2. *Let $\pi: A \to \mathcal{L}(E)$ be a strongly absorbing representation. Then there is a canonical isomorphism $KK_*(A, B) \to K_*(C_{L,u}(\pi))$.* ∎

Definition 3.3. Let $\pi: A \to \mathcal{L}(E)$ be a representation of A on a Hilbert B-module. Define $C_{L,c}(\pi)$ to be the C^*-algebra of all bounded, continuous functions $b: [1,\infty) \to \mathcal{L}(E)$ such that $[b_t, a] \to 0$ for all $a \in A$, and such that ab_t is in $\mathcal{K}(E)$ for all $a \in A$ and all $t \in [1,\infty)$.

Clearly there is a canonical inclusion $C_{L,u}(\pi) \to C_{L,c}(\pi)$. Our main goal in this section is to establish the following result.

Theorem 3.4. *Let $\pi: A \to \mathcal{L}(F)$ be an infinite multiplicity (in particular, π could be strongly absorbing) representation of A on a Hilbert B-module. Then the canonical inclusion $C_{L,u}(\pi) \to C_{L,c}(\pi)$ induces an isomorphism on K-theory.*

We will need several preliminary lemmas. The first of these is no doubt well known: compare for example [12, Proposition 4.1.7].

Lemma 3.5. *There is a function $\omega: [0,\infty) \to [0,\infty)$ such that $\omega(0) = \lim_{t\to 0} \omega(t) = 0$, and with the following property. Assume that D is a C^*-algebra, and that $p_0, p_1 \in D$ are projections with the property that $\|p_0 - p_1\| < 1/2$. Then there is a homotopy $(p_t)_{t\in[0,1]}$ connecting p_0 and p_1 in D, and with the property that $\|p_s - p_t\| \leq \omega(|s-t|)$ for all $s, t \in [0,1]$.*[17]

Proof. Define $x := p_1p_0 + (1-p_1)(1-p_0)$ in the unitization D^+ of D. Then $x - 1 = (2p_1 - 1)(p_0 - p_1)$; as $\|2p_1 - 1\| = 1$, this implies that $\|1 - x\| < 1/2$. For $t \in [0,1]$, define $x_t := t1 + (1-t)x$, which also satisfies $\|1 - x_t\| < 1/2$ for all t. Hence each x_t is invertible, and the norm of its inverse is at most 2 by the usual Neumann series representation $x_t^{-1} = \sum_{n=0}^{\infty}(1-x_t)^n$. It follows that for each t, $\|(x_t^*x_t)^{-1}\| \leq 4$, whence the spectrum of $x_t^*x_t$ is contained in $[1/4,\infty)$. Define moreover $u_t := x_t(x_t^*x_t)^{-1/2}$, which is unitary.

Now, as the function $t \mapsto x_t$ is Lipschitz, it follows from the functional calculus and the fact that the spectrum of $x_t^*x_t$ is uniformly bounded away from zero that there is a function $\eta: [0,\infty) \to [0,\infty)$ such that $\eta(0) = \lim_{t\to 0}\eta(t) = 0$ and such that $\|u_s - u_t\| \leq \eta(|s-t|)$ for all $s, t \in [0,1]$: compare for example the proof of [21, Lemma 1.2.5].

One computes moreover that $u_1p_0u_1^* = p_1$: compare the proof of [12, Proposition 4.1.7]. Define finally $p_t := u_tp_0u_t^*$, which has the right properties by the discussion above. ∎

For the statement of the next lemma, recall that if C and D are C^*-algebras equipped with surjections $\pi_C: C \to Q$ and $\pi_D: D \to Q$ to a third C^*-algebra Q, then the *pullback* is the C^*-algebra

$$P := \{(c,d) \in C \oplus D \mid \pi_C(c) = \pi_D(d)\}.$$

[17] The lemma is not optimal: a slightly more elaborate argument with the holomorphic functional calculus would show that the function $t \mapsto p_t$ is Lipschitz with a uniform Lipschitz constant.

Such a pullback gives rise to a canonical diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & C \arrow[d, "\pi_C"] \\ D \arrow[r, "\pi_D"'] & Q \end{tikzcd} \tag{3.1}$$

where the arrows out of P are the natural (surjective) coordinate projections. We call any square isomorphic to one of this form a *pullback square*.

See for example [33, Proposition 2.7.15] for a proof of the next result.

Lemma 3.6. *Given a pullback diagram as in line* (3.1) *above, there is a six-term exact sequence*

$$\begin{tikzcd} K_0(P) \arrow[r] & K_0(C) \oplus K_0(D) \arrow[r] & K_0(Q) \arrow[d] \\ K_1(Q) \arrow[u] & K_1(C) \oplus K_1(D) \arrow[l] & K_1(P) \arrow[l] \end{tikzcd}$$

of K-theory groups. The sequence is natural for maps between pullback squares. ∎

We record two more well-known K-theoretic lemmas. See for example [33, Proposition 2.7.5 and Lemma 2.7.6] for proofs.[18]

Lemma 3.7. *If $\alpha, \beta\colon C \to D$ are $*$-homomorphisms with orthogonal images, then $\alpha + \beta\colon C \to D$ is also a $*$-homomorphism, and $(\alpha+\beta)_* = \alpha_* + \beta_*$ as maps on K-theory.* ∎

Lemma 3.8. *Let $\alpha, \beta\colon C \to D$ be $*$-homomorphisms, and assume that there is a partial isometry v in the multiplier algebra of D such that $\alpha(c)v^*v = \alpha(c)$ for all $c \in C$, and so that $v\alpha(c)v^* = \beta(c)$ for all $c \in C$. Then α and β induce the same maps on K-theory.* ∎

Proof of Theorem 3.4. Let $E := \bigsqcup_{n\geq 1}[2n, 2n+1]$ and $O := \bigsqcup_{n\geq 1}[2n-1, 2n]$, equipped with the restriction of the metric from $[1,\infty)$. Let $C_{L,u}(\pi; E)$ denote the collection of all bounded, uniformly continuous functions $b\colon E \to \mathcal{L}(F)$ such that $ab_t \in \mathcal{K}(F)$ for all $a \in A$, and such that $[a, b_t] \to 0$ as $t \to \infty$. Define $C_{L,u}(\pi; O)$ and $C_{L,u}(\pi; E \cap O)$ similarly, and define $C_{L,c}(\pi; E)$, $C_{L,c}(\pi; O)$, and $C_{L,c}(\pi; O \cap E)$ analogously, but with "uniformly continuous" replaced by "continuous". Then we have a commutative diagram of pullback squares

$$\begin{tikzcd} C_{L,u}(\pi) \arrow[rr] \arrow[dd] \arrow[dr] & & C_{L,u}(\pi; E) \arrow[dd] \arrow[dr] & \\ & C_{L,c}(\pi) \arrow[rr] \arrow[dd] & & C_{L,c}(\pi; E) \arrow[dd] \\ C_{L,u}(\pi; O) \arrow[rr] \arrow[dr] & & C_{L,u}(\pi; E\cap O) \arrow[dr] & \\ & C_{L,c}(\pi; O) \arrow[rr] & & C_{L,c}(\pi; E\cap O) \end{tikzcd}$$

[18] The statement of [33, Proposition 2.7.5] has a typo: vv^* should be v^*v where it appears there.

where the diagonal arrows are the canonical inclusions, and all the other arrows are the obvious restriction maps. Using Lemma 3.6 and the five lemma, it thus suffices to show that the maps $C_{L,u}(\pi; G) \to C_{L,c}(\pi; G)$ induce isomorphisms on K-theory for $G \in \{E, O, E \cap O\}$. For $E \cap O$, which just equals $\mathbb{N} \cap [1, \infty)$, this is clear: the map is the identity on the level of C^*-algebras as there is no difference between continuity and uniform continuity in this case. The cases of E and O are essentially the same, so we just focus on E.

Let now $E_{\mathbb{N}} := E \cap 2\mathbb{N} = \{2, 4, 6, \dots\}$ be the set of positive even numbers. Restriction defines a surjective $*$-homomorphism $C_{L,u}(\pi; E) \to C_{L,u}(\pi; E_{\mathbb{N}})$; we write $C^0_{L,u}(\pi; E)$ for the kernel. The same holds for the $C_{L,c}$ algebras, giving a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & C^0_{L,u}(\pi; E) \arrow[r] \arrow[d] & C_{L,u}(\pi; E) \arrow[r] \arrow[d] & C_{L,u}(\pi; E_{\mathbb{N}}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C^0_{L,c}(\pi; E) \arrow[r] & C_{L,c}(\pi; E) \arrow[r] & C_{L,c}(\pi; E_{\mathbb{N}}) \arrow[r] & 0 \end{tikzcd}$$

of short exact sequences where the vertical maps are the canonical inclusions. The right-hand vertical map is the identity as there is no difference between continuity and uniform continuity for maps out of $E_{\mathbb{N}}$. Hence by the five lemma and the usual long exact sequence in K-theory, it suffices to show that the left-hand vertical map induces an isomorphism on K-theory. For $r \in [0, 1]$ let us define a $*$-homomorphism $h_r \colon C^0_{L,u}(\pi; E) \to C^0_{L,u}(\pi; E)$ by the following prescription. For $b \in C^0_{L,u}(\pi; E)$ and $t \in [2n, 2n + 1]$, we set

$$(h_r b)_t := b_{2n+r(t-2n)}$$

(in words, h_r contracts $[2n, 2n+1]$ to $\{2n\}$ as r varies from 1 to 0). Using uniform continuity, $(h_r)_{r\in[0,1]}$ is a null-homotopy of $C^0_{L,u}(\pi; E)$, and therefore $K_*(C^0_{L,u}(\pi; E)) = 0$. It thus suffices to show that $K_*(C^0_{L,c}(\pi; E)) = 0$, which we spend the rest of the proof doing.

We will focus on the case of K_0 (which is in any case all we use in this paper); the case of K_1 is similar. Take then an arbitrary element $x \in K_0(C^0_{L,c}(\pi; E))$, which we may represent by a formal difference $x = [p] - [1_k]$ where p is a projection in the $m \times m$ matrices $M_m(C^0_{L,c}(\pi; E)^+)$ over the unitization $C^0_{L,c}(\pi; E)^+$ of $C_{L,c}(\pi; E)$ for some m, and $1_k \in M_m(\mathbb{C}) \subseteq M_m(C^0_{L,c}(\pi; E)^+)$ is the scalar matrix with 1s in the first k diagonal entries and 0s elsewhere for some $k \leq m$. Without loss of generality, we may think of p as a continuous projection-valued function

$$p \colon E \to M_m\big(\mathcal{L}(F)\big)$$

such that $a(p - 1_k) \in M_m(\mathcal{K}(F))$ for all $a \in A$ (here we use the amplification of the representation of A to a representation on $M_m(\mathcal{L}(F))$ to make sense of this), such that $[a, p_t] \to 0$ for all $a \in A$, and such that $p_{2n} = 1_k$ for all $n \in \mathbb{N}$.

Now, for each n, the restriction $p|_{[2n,2n+1]}$ is uniformly continuous, whence there is some $r_n \in (0, 1)$ such that if $t, s \in [2n, 2n+1]$ satisfy $|t-s| \leq 1-r_n$, then $\|p_t - p_s\| < 1/2$.

For each $l \in \mathbb{N} \cup \{0\}$, define $p^{(l)} \colon E \to M_m(\mathcal{L}(F))$ to be the function whose restriction to $[2n, 2n+1]$ is defined by

$$p_t^{(l)} := p_{2n+(t-2n)(r_n)^l}.$$

Fix a sequence $(s_l)_{l=0}^{\infty}$ of isometries as in Remark 2.3 and consider the formal difference

$$x_\infty := \left[\sum_{l=0}^{\infty} s_l p^{(l)} s_l^*\right] - \left[\sum_{l=0}^{\infty} s_l 1_k s_l^*\right]$$

where the sum converges strictly in $M_m(\mathcal{L}(F)) \cong \mathcal{L}(F^{\oplus m})$, pointwise in t (we are abusing notation slightly: we should really have replaced s_l by $1_{M_m(\mathbb{C})} \otimes s_l$). As $r_n < 1$ and as $p_{2n} = 1_k$ for each n, we see that for any t, $p_t^{(l)} - 1_k \to 0$ as $l \to \infty$; it follows from this and the fact that each s_l commutes with the representation of A that x_∞ gives a well-defined element of $K_0(C_{L,c}^0(\pi; E))$.

Now, let us consider the element $x_\infty + x$ of $K_0(C_{L,c}^0(\pi; E))$. We claim this equals x_∞. As K_0 is a group, this forces $x = 0$, and thus $K_0(C_{L,c}^0(\pi; E)) = 0$ as required. Indeed, first note that conjugating by the isometry

$$s := \sum_{l=0}^{\infty} s_{l+1} s_l^*$$

in the multiplier algebra of $C_{L,c}^0(\pi; E)$ and applying Lemma 3.8 shows that

$$x_\infty = \left[\sum_{l=1}^{\infty} s_l p^{(l-1)} s_l^*\right] - \left[\sum_{l=1}^{\infty} s_l 1_k s_l^*\right].$$

The choice of the sequence (r_n) and Lemma 3.5[19] guarantees the existence of a homotopy between $p^{(l-1)}$ and $p^{(l)}$ for each $l \geq 1$, and moreover that these homotopies can be assumed equicontinuous as l varies (this includes the fact that the restrictions to the various intervals $[2n, 2n+1]$ are equicontinuous as n varies). It follows that

$$x_\infty = \left[\sum_{l=1}^{\infty} s_l p^{(l)} s_l^*\right] - \left[\sum_{l=1}^{\infty} s_l 1_k s_l^*\right] \tag{3.2}$$

On the other hand, applying Lemma 3.8 again, we have that

$$x = [s_0 p s_0^*] - [s_0 1_k s_0^*].$$

[19]For the case of K_1 one needs an analogue of Lemma 3.5 that works for unitaries: this follows from an analogous functional calculus argument using that if u, v are unitaries with $\|u - v\| < 1/2$ then there is self-adjoint h with $\|h\| \leq \pi$, and $uv^* = e^{ih}$.

Combining this with line (3.2) above and also Lemma 3.7

$$\begin{aligned} x + x_\infty &= [s_0 p s_0^*] - [s_0 1_k s_0^*] + \left[\sum_{l=1}^{\infty} s_l p^{(l)} s_l^*\right] - \left[\sum_{l=1}^{\infty} s_l 1_k s_l^*\right] \\ &= \left[\sum_{l=0}^{\infty} s_l p^{(l)} s_l^*\right] - \left[\sum_{l=0}^{\infty} s_l 1_k s_l^*\right] \\ &= x_\infty \end{aligned}$$

and we are done. ■

We finish this section with some technical results that we will need later. The first goal is to show that $K_*(C_{L,c}(\pi))$ only really depends on information "at $t = \infty$" in some sense. This is made precise in Corollary 3.11 below, but we need some more notation first.

Definition 3.9. Let $\pi\colon A \to \mathcal{L}(E)$ be a representation of A on a Hilbert B-module. Define $I_{L,c}(\pi)$ to be the ideal in $C_{L,c}(\pi)$ consisting of all functions b such that $ab \in C_0([1,\infty), \mathcal{K}(E))$ for all $a \in A$. Define $Q_L(\pi) := C_{L,c}(\pi)/I_{L,c}(\pi)$ to be the corresponding quotient.

Lemma 3.10. *Let $\pi\colon A \to \mathcal{L}(E)$ be an infinite multiplicity representation of A. Then $I_{L,c}(\pi)$ has trivial K-theory.*

Proof. Set $I_{L,u}(\pi) := C_{L,u}(\pi) \cap I_{L,c}(\pi)$. The same argument in the proof of Theorem 3.4 shows that the inclusion $I_{L,u}(\pi) \to I_{L,c}(\pi)$ induces an isomorphism on K-theory. It thus suffices to prove that $K_*(I_{L,u}(\pi)) = 0$, which we now do.

Let $(s_n)_{n=0}^\infty$ be a sequence of isometries in $\mathcal{L}(E)$ that commute with A, and that have orthogonal ranges as in Remark 2.3. We regard each s_n as an isometry in the multiplier algebra of $I_{L,u}(\pi)$ by having it act pointwise in t. Define

$$\iota\colon I_{L,u}(\pi) \longrightarrow I_{L,u}(\pi), \quad b \longmapsto s_0 b s_0^*,$$

which is a $*$-homomorphism that induces the identity map on K-theory by Lemma 3.8. On the other hand, for each $s \geq 0$, define a $*$-endomorphism α_s of $I_{L,u}(\pi)$ by the formula $\alpha_s(b)_t := b_{t+s}$. Note that for any norm-bounded sequence (b_n) in $\mathcal{L}(E)$, the sum

$$\sum_{n=1}^{\infty} s_n b_n s_n^*$$

converges in the strict topology of $\mathcal{L}(E) = M(\mathcal{K}(E))$. Therefore we get a $*$-homomorphism

$$\alpha\colon I_{L,u}(\pi) \longrightarrow I_{L,u}(\pi), \quad \alpha(b) := \sum_{n=1}^{\infty} s_n \alpha_n(b) s_n^*$$

(the image is in $I_{L,u}(\pi)$ as $ab_t \to 0$ as $t \to \infty$ for all $a \in A$, which implies that for each fixed t and any $a \in A$, $a\alpha_n(b_t) \to 0$ as $n \to \infty$). Now, the maps α and ι have orthogonal ranges, whence by Lemma 3.7, $\alpha + \iota$ is also a $*$-homomorphism, and we have that as maps on K-theory, $\alpha_* + \iota_* = (\alpha + \iota)_*$. Define $s := \sum_{n=0}^{\infty} s_{n+1} s_n^*$ (convergence in the strict topology), which we think of as a multiplier of $I_{L,u}(\pi)$. Applying Lemma 3.8 again, we see that $\iota + \alpha$ induces the same map on K-theory as the map $b \mapsto s(\iota(b) + \alpha(b))s^*$, which is the map

$$I_{L,u}(\pi) \longrightarrow I_{L,u}(\pi), \quad b \longmapsto \sum_{n=1}^{\infty} s_n \alpha_{n-1}(b) s_n^*.$$

On the other hand, using that elements of $I_{L,u}(\pi)$ are uniformly continuous, we get a homotopy

$$b \longmapsto \sum_{n=1}^{\infty} s_n \alpha_{n-1+r}(b) s_n^*, \quad r \in [0,1]$$

between this map and α. In other words, we now have that $\alpha_* + \iota_* = \alpha_*$ as maps on K-theory. This forces ι_* to be the zero map on $K_*(I_{L,u}(\pi))$. However, we also observed already that ι_* is the identity map, so $K_*(I_{L,u}(\pi))$ is indeed zero. ∎

The following corollary is immediate from the six-term exact sequence in K-theory.

Corollary 3.11. *Let $\pi: A \to \mathcal{L}(E)$ be an infinite multiplicity representation of A on a Hilbert B-module. Then the canonical quotient map $C_{L,c}(\pi) \to Q_L(\pi)$ induces an isomorphism on K-theory.* ∎

We will need one more definition and lemma about the structure of $C_{L,c}(\pi)$.

Definition 3.12. Let $\pi: A \to \mathcal{L}(E)$ be a representation of A on a Hilbert B-module. Define

$$C_{L,c}(\pi; \mathcal{K}) := C_b\big([1,\infty), \mathcal{K}(E)\big) \cap C_{L,c}(\pi),$$

which is an ideal in $C_{L,c}(\pi)$.

Lemma 3.13. *Let $\pi: A \to \mathcal{L}(E)$ be a representation of A on a Hilbert B-module. With notation as in Definitions 3.9 and 3.12, we have*

$$C_{L,c}(\pi) = C_{L,c}(\pi; \mathcal{K}) + I_{L,c}(\pi).$$

In particular, the restriction of the quotient map $C_{L,c}(\pi) \to Q_L(\pi)$ to $C_{L,c}(\pi; \mathcal{K})$ is surjective.

Proof. Let (h_n) be a sequential approximate unit for A, and define $h \in C_{ub}([1,\infty), A)$ by setting $h_t := (n+1-t)h_n + (t-n)h_{n+1}$ for $t \in [n, n+1]$. Then a direct check using that $[a, h] \in C_0([1,\infty), A)$ for any $a \in A$ shows that h defines a multiplier of $C_{L,c}(\pi)$. Moreover, for any $b \in C_{L,c}(\pi)$, $b = (1-h)b + hb$, and one checks directly that $(1-h)b$ is in $I_{L,c}(\pi)$ and that hb is in $C_{L,c}(\pi; \mathcal{K})$. This gives the result on the sum, and the result on the quotient follows immediately. ∎

Our final goal in this section is to check that the isomorphisms from Theorem 3.2 and Theorem 3.4 are compatible with a special case of functoriality for KK-theory.

For this, let C be a separable C^*-algebra, and let $\phi\colon B \to C$ be a $*$-homomorphism. Let E be a Hilbert B-module, and let $E \otimes_\phi C$ be the internal tensor product defined using ϕ, which is a Hilbert C-module. As discussed on [17, p. 42] there is a canonical $*$-homomorphism

$$\Phi\colon \mathcal{L}(E) \longrightarrow \mathcal{L}(E \otimes_\phi C), \quad a \longmapsto a \otimes 1_C.$$

Abusing notation slightly, we also write Φ for the $*$-homomorphism

$$C_b\big([1,\infty), \mathcal{L}(E)\big) \longrightarrow C_b\big([1,\infty), \mathcal{L}(E \otimes_\phi C)\big)$$

defined by applying Φ pointwise. Let $\pi_B\colon A \to \mathcal{L}(E)$ and $\pi_C\colon A \to \mathcal{L}(F)$ be representations of A on Hilbert B- and C-modules, respectively.

Definition 3.14. With notation as above, a *covering isometry* for ϕ (with respect to π_B and π_C) is any isometry $v \in C_b([1,\infty), \mathcal{L}(E \otimes_\phi C, F))$ such that

$$v^*\pi_C(a)v - (\Phi \circ \pi_B)(a) \in C_0\big([1,\infty), \mathcal{K}(E \otimes_\phi C)\big)$$

for all $a \in A$.

Lemma 3.15. *With notation as above, if v is a covering isometry for ϕ, then the formula*

$$\phi^v\colon C_{L,c}(\pi_B) \to C_{L,c}(\pi_C), \quad \phi^v(b) := v\Phi(b)v^*$$

gives a well-defined $*$-homomorphism. Moreover, the induced map

$$\phi^v_*\colon K_*\big(C_{L,c}(\pi_B)\big) \longrightarrow K_*\big(C_{L,c}(\pi_C)\big)$$

on K-theory does not depend on the choice of v. Finally, if π_C is strongly absorbing, then a covering isometry for ϕ always exists, and can be taken to belong to $C_{ub}([1,\infty),$ $\mathcal{L}(E \otimes_\phi C, F))$ (i.e., to be uniformly continuous, not just continuous).

Proof. Let v be a covering isometry for ϕ. For notational simplicity, write $\sigma := \Phi \circ \pi_B$. Using Lemma 2.7 we have that

$$\pi_C(a)v - v\sigma(a) \in C_0\big([1,\infty), \mathcal{K}(E \otimes_\phi C, F)\big)$$

for all $a \in A$. Note that for $a \in A$ and $b \in C_{L,c}(\pi_B)$

$$\pi_C(a)\phi^v(b) = \big(\pi_C(a)v - v\sigma(a)\big)\Phi(b)v^* + v\Phi\big(\pi_B(a)b\big)v^*;$$

using that Φ takes $\mathcal{K}(E)$ to $\mathcal{K}(E \otimes_\phi C)$ (see [17, Proposition 4.7]), this shows that $\pi_C(a)\phi^v(b) \in C_b([1,\infty), \mathcal{K}(F))$. Similarly,

$$\begin{aligned}\big[\pi_C(a), \phi^v(b)\big] = {} & \big(\pi_C(a)v - v\sigma(a)\big)\Phi(b)v^* \\ & + v\Phi\big([\pi(a), b]\big)v^* + v\Phi(b)\big(v^*\pi_C(a) - \sigma(a)v^*\big),\end{aligned}$$

whence $[\pi_C(a), \phi^v(b)] \in C_0([1,\infty), \mathcal{K}(F))$. It follows that ϕ^v is indeed a well-defined $*$-homomorphism $C_{L,c}(\pi_B) \to C_{L,c}(\pi_C)$.

Let now v, w be possibly different covering isometries for ϕ. Using similar computations to the above, one checks that wv^* is an element of the multiplier algebra of $C_{L,c}(\pi_C)$ that conjugates the $*$-homomorphisms ϕ^v and ϕ^w to each other. The fact that $\phi^v_* = \phi^w_*$ as maps $K_*(C_{L,c}(\pi_B)) \to K_*(C_{L,c}(\pi_C))$ follows from this and Lemma 3.8.

Finally, if π_C is strongly absorbing, then covering isometries exist, and can be assumed uniformly continuous, by Proposition 2.6. ∎

Definition 3.16. Let $\pi_B\colon A \to \mathcal{L}(E)$ and $\pi_C\colon A \to \mathcal{L}(F)$ be representations of A on a Hilbert B-module and Hilbert C-module respectively, with π_C strongly absorbing. Let $\phi\colon B \to C$ be any $*$-homomorphism. Then Lemma 3.15 gives a well-defined homomorphism $K_*(C_{L,c}(\pi_B)) \to K_*(C_{L,c}(\pi_C))$, which we denote ϕ_*.

On the other hand, for a $*$-homomorphism $\phi\colon B \to C$, let us write $\phi_*\colon KK(A,B) \to KK(A,C)$ for the usual functorially induced map on KK-theory. The following lemma gives compatibility between these two maps.

Lemma 3.17. *With notation as above, assume that both $\pi_B\colon A \to \mathcal{L}(E)$ and $\pi_C\colon A \to \mathcal{L}(F)$ are strongly absorbing, and let $KK(A,B) \to K_0(C_{L,c}(\pi_B))$ and $KK(A,C) \to K_0(C_{L,c}(\pi_C))$ be the isomorphisms defined as the composition of the isomorphisms from Theorems 3.2 and 3.4. Then the diagram*

$$\begin{tikzcd} KK(A,B) \arrow[r] \arrow[d, "\phi_*"] & K_0\big(C_{L,c}(\pi_B)\big) \arrow[d, "\phi_*"] \\ KK(A,C) \arrow[r] & K_0\big(C_{L,c}(\pi_C)\big) \end{tikzcd}$$

commutes.

Proof. The proof is unfortunately long as there is a lot to check, but the checks are fairly routine. We recall first the precise form of the isomorphism $KK_*(A,B) \to K_*(C_{L,u}(\pi_B))$ of Theorems 3.2 and 3.4. It is a composition of the following maps (see also [9, Definition 3.1] for the various algebras involved).

(i) The Paschke duality isomorphism $P\colon KK(A,B) \to K_1(\mathcal{D}(\pi_B)/\mathcal{C}(\pi_B))$ of [29, Theorem 3.2], where $\mathcal{D}(\pi_B) := \{b \in \mathcal{L}(E) \mid [b,a] \in \mathcal{K}(E) \text{ for all } a \in A\}$, and $\mathcal{C}(\pi_B) := \{b \in \mathcal{D}(\pi_B) \mid ab \in \mathcal{K}(E) \text{ for all } a \in A\}$.

(ii) The map on K-theory

$$\iota_*\colon K_1\big(\mathcal{D}(\pi_B)/\mathcal{C}(\pi_B)\big) \to K_1\big(\mathcal{D}_T(\pi_B)/\mathcal{C}_T(\pi_B)\big)$$

induced by the constant inclusion $\iota\colon \mathcal{D}(\pi_B) \to \mathcal{D}_T(\pi_B)$, where $\mathcal{D}_T(\pi_B) := C_{ub}([1,\infty), \mathcal{D}(\pi_B))$ and $\mathcal{C}_T(\pi_B) := C_{ub}([1,\infty), \mathcal{C}(\pi_B))$.

(iii) The map on K-theory

$$\eta_*^{-1}\colon K_1\big(\mathcal{D}_T(\pi_B)/\mathcal{C}_T(\pi_B)\big) \longrightarrow K_1\big(\mathcal{D}_L(\pi_B)/C_{L,u}(\pi_B)\big)$$

which is induced by the inverse (it turns out to be an isomorphism of C^*-algebras) of the map $\eta\colon \mathcal{D}_T(\pi_B)/\mathcal{C}_T(\pi_B) \to \mathcal{D}_L(\pi_B)/C_{L,u}(\pi_B)$ induced by the inclusion $\mathcal{D}_L(\pi_B) \to \mathcal{D}_T(\pi_B)$, where

$$\mathcal{D}_L(\pi_B) := \big\{b \in \mathcal{D}_T(\pi_B) \mid [a, b_t] \to 0 \text{ as } t \to \infty \text{ for all } a \in A\big\}.$$

(iv) The usual K-theory boundary map

$$\partial\colon K_1\big(\mathcal{D}_L(\pi_B)/C_{L,u}(\pi_B)\big) \longrightarrow K_0\big(C_{L,u}(\pi_B)\big).$$

(v) The isomorphism $\kappa_*\colon K_0(C_{L,u}(\pi_B)) \to K_0(C_{L,c}(\pi_B))$ of Theorem 3.4 induced by the canonical inclusion.

Now, if v is a uniformly continuous covering isometry for ϕ, then one sees from analogous arguments to those given in the proof of Lemma 3.15 that the formula

$$\phi^v(b)_t := v_t \Phi(b_t) v_t^*$$

from Lemma 3.15 also defines $*$-homomorphisms

$$\phi^v\colon \begin{Bmatrix} \mathcal{D}_L(\pi_B)/C_{L,u}(\pi_B) \to \mathcal{D}_L(\pi_C)/C_{L,u}(\pi_C) \\ \mathcal{D}_T(\pi_B)/\mathcal{C}_T(\pi_B) \to \mathcal{D}_T(\pi_C)/\mathcal{C}_T(\pi_C) \end{Bmatrix}.$$

Moreover, the formula

$$\phi^{v_1}(b) := v_1 \Phi(b) v_1^*$$

defines a $*$-homomorphism $\mathcal{D}(\pi_B)/\mathcal{C}(\pi_B) \to \mathcal{D}(\pi_C)/\mathcal{C}(\pi_B)$. Putting all this together, we get a diagram

$$\begin{array}{ccc}
K_1\big(\mathcal{D}(\pi_B)/\mathcal{C}(\pi_B)\big) & \xrightarrow{\phi_*^{v_1}} & K_1\big(\mathcal{D}(\pi_C)/\mathcal{C}(\pi_C)\big) \\
\downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} \\
K_1\big(\mathcal{D}_T(\pi_B)/\mathcal{C}_T(\pi_B)\big) & \xrightarrow{\phi_*^{v}} & K_1\big(\mathcal{D}_T(\pi_C)/\mathcal{C}_T(\pi_C)\big) \\
\downarrow{\scriptstyle \eta_*^{-1}} & & \downarrow{\scriptstyle \eta_*^{-1}} \\
K_1\big(\mathcal{D}_L(\pi_B)/C_{L,u}(\pi_B)\big) & \xrightarrow{\phi_*^{v}} & K_1\big(\mathcal{D}_L(\pi_C)/C_{L,u}(\pi_C)\big) \\
\downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} \\
K_0\big(C_{L,c}(\pi_B)\big) & \xrightarrow{\phi_*^{v}} & K_0\big(C_{L,c}(\pi_C)\big) \\
\downarrow{\scriptstyle \kappa_*} & & \downarrow{\scriptstyle \kappa_*} \\
K_0\big(C_{L,c}(\pi_B)\big) & \xrightarrow{\phi_*^{v}} & K_0\big(C_{L,c}(\pi_C)\big).
\end{array} \tag{3.3}$$

We claim that this commutes. Indeed, the first square commutes as ι_* is an isomorphism on K-theory [9, Proposition 4.3 (b)], whence its inverse on the level of K-theory is the map induced by the evaluation-at-one homomorphism $e : \mathcal{D}_T(\pi_B) \to \mathcal{D}(\pi_B)$, and the diagram

$$\begin{tikzcd} K_1\big(\mathcal{D}(\pi_B)/\mathcal{C}(\pi_B)\big) \arrow[r, "\phi_*^{v_1}"] & K_1\big(\mathcal{D}(\pi_C)/\mathcal{C}(\pi_C)\big) \\ K_1\big(\mathcal{D}_T(\pi_B)/\mathcal{C}_T(\pi_B)\big) \arrow[r, "\phi_*^{v}"] \arrow[u, "e_*"] & K_1\big(\mathcal{D}_T(\pi_C)/\mathcal{C}_T(\pi_C)\big) \arrow[u, "e_*"'] \end{tikzcd}$$

commutes on the level of $*$-homomorphisms. The second square in line (3.3) commutes as the diagram

$$\begin{tikzcd} K_1\big(\mathcal{D}_T(\pi_B)/\mathcal{C}_T(\pi_B)\big) \arrow[r, "\phi_*^{v}"] & K_1\big(\mathcal{D}_T(\pi_C)/\mathcal{C}_T(\pi_C)\big) \\ K_1\big(\mathcal{D}_L(\pi_B)/C_{L,u}(\pi_B)\big) \arrow[r, "\phi_*^{v}"] \arrow[u, "\eta_*"] & K_1\big(\mathcal{D}_L(\pi_C)/C_{L,u}(\pi_C)\big) \arrow[u, "\eta_*"'] \end{tikzcd}$$

commutes on the level of $*$-homomorphisms. The third square commutes by naturality of the boundary map in K-theory. Finally, the fourth square commutes as it commutes on the level of $*$-homomorphisms.

Now, the diagram in the statement of the lemma "factors" as the rectangle from line (3.3), "augmented" on the top with the diagram below

$$\begin{tikzcd} KK(A, B) \arrow[r, "\phi_*"] \arrow[d, "P"] & KK(A, C) \arrow[d, "P"] \\ K_1\big(\mathcal{D}(\pi_B)/\mathcal{C}(\pi_B)\big) \arrow[r, "\phi_*^{v_1}"] & K_1\big(\mathcal{D}(\pi_C)/\mathcal{C}(\pi_C)\big) \end{tikzcd}$$

involving the Paschke duality isomorphism P. To complete the proof, it suffices to show that this commutes. We will actually work with the diagram

$$\begin{tikzcd} KK(A, B) \arrow[r, "\phi_*"] & KK(A, C) \\ K_1\big(\mathcal{D}(\pi_B)/\mathcal{C}(\pi_B)\big) \arrow[r, "\phi_*^{v_1}"] \arrow[u, "P^{-1}"] & K_1\big(\mathcal{D}(\pi_C)/\mathcal{C}(\pi_C)\big) \arrow[u, "P^{-1}"'] \end{tikzcd} \tag{3.4}$$

involving the inverse Paschke duality isomorphism, as this is a little simpler.

Let then $[u] \in K_1(\mathcal{D}(\pi_B)/\mathcal{C}(\pi_B))$ be a class. The existence of a sequence of isometries as in Remark 2.3 implies that we may assume that u is in $\mathcal{D}(\pi_B)/\mathcal{C}(\pi_B)$ (not "just" in some matrix algebra over this C^*-algebra). Abusing notation slightly, we also write $u \in \mathcal{D}(\pi_B)$ for some choice of lift of this unitary. Then the triple (π_B, E, u) is an even

Kasparov cycle for (A, B) (in the ungraded picture of the even KK-group) and the inverse Paschke duality map is defined by

$$P^{-1}\colon K_1\big(\mathcal{D}(\pi_B)/\mathcal{C}(\pi_B)\big) \longrightarrow KK(A, B), \quad [u] \longmapsto [\pi_B, E, u]$$

(see [29, Section 3] or [28, Remarque 2.8]).

Now, the "up-right" composition

$$\begin{tikzcd} KK(A,B) \arrow[r, "\phi_*"] & KK(A,C) \\ K_1\big(\mathcal{D}(\pi_B)/\mathcal{C}(\pi_B)\big) \arrow[u, "P^{-1}"] & \end{tikzcd}$$

from line (3.4) takes $[u]$ first to $[\pi_B, E, u]$, and then to $[\pi_B \otimes 1_C, E \otimes_\phi C, u \otimes 1_C]$. On the other hand, the "right-up" composition

$$\begin{tikzcd} & KK(A,C) \\ K_1\big(\mathcal{D}(\pi_B)/\mathcal{C}(\pi_B)\big) \arrow[r, "\phi_*^{v_1}"] & K_1\big(\mathcal{D}(\pi_C)/\mathcal{C}(\pi_C)\big) \arrow[u, "P^{-1}"] \end{tikzcd}$$

from line (3.4) takes $[u]$ first to $[v_1(u\otimes 1_C)v_1^* + (1 - v_1v_1^*)]$ (compare [21, Exercise 8.5] to see where "$1 - v_1v_1^*$" is coming from), and then to $[\pi_C, F, v_1(u \otimes 1_C)v_1^* + (1 - v_1v_1^*)]$. Our task is therefore to show that

$$[\pi_B \otimes 1_C, E \otimes_\phi C, u \otimes 1_C] = \big[\pi_C, F, v_1(u \otimes 1_C)v_1^* + (1 - v_1v_1^*)\big] \tag{3.5}$$

in $KK(A, C)$.

For notational simplicity, write $p = v_1v_1^*$, and note that p commutes with $\pi_C(A)$ as the compression of π_C by p agrees with $v_1(\pi_B \otimes 1_C)v_1^*$, and the latter is a $*$-homomorphism. Note also that $(\pi_B \otimes 1_C, E \otimes_\phi C, u \otimes 1_C)$ is unitarily equivalent to $(p\pi_C p, pF, v_1(u \otimes 1_C)v_1^*)$ via the unitary isomorphism

$$v_1 : E \otimes_\phi C \longrightarrow pF.$$

On the other hand, the Kasparov module $((1-p)\pi_C(1-p), (1-p)F, 1-p)$ is degenerate, so represents zero in $KK(A, C)$. We thus have that

$$\begin{aligned}[\pi_B \otimes 1_C, E \otimes_\phi C, u \otimes 1_C] &= \big[p\pi_C p, pF, v_1(u \otimes 1_C)v_1^*\big] \\ &= \big[p\pi_C p, pF, v_1(u \otimes 1_C)v_1^*\big] \\ &\quad \oplus \big[(1-p)\pi_C(1-p), (1-p)F, 1-p\big].\end{aligned}$$

As $p = v_1v_1^*$ commutes with π_C, the last line above agrees with the right-hand side of line (3.5) above, and we are done. ∎

4. Paths of projections

Throughout this section, A and B refer to separable C^*-algebras. All Hilbert modules are countably generated, and all are over B unless explicitly stated otherwise. All representations of A are on Hilbert B-modules unless explicitly stated otherwise.

Our goal in this section is to introduce a new model of KK-theory based on paths of projections. We will need some more terminology about representations.

Definition 4.1. Let $\pi: A \to \mathcal{L}(E)$ be a representation of A on a Hilbert B-module.

(i) (π, E) is *graded* if it comes with a fixed decomposition $(\pi, E) = (\pi_0 \oplus \pi_1, E_0 \oplus E_1)$ as a direct sum of two subrepresentations.

(ii) If $\pi: A \to \mathcal{L}(E)$ is a graded representation, the *neutral projection* is the projection $e \in \mathcal{L}(E)$ onto the first summand in $E = E_0 \oplus E_1$.

(iii) A graded representation (π, E) is *balanced*[20] if it is graded and if $(\pi_0, E_0) = (\pi_1, E_1)$ in the given decomposition.

(iv) A graded representation (π, E) is *infinite multiplicity* (respectively, *strongly absorbing*) if (π_0, E_0) has infinite multiplicity in the sense of Section 1.4 (respectively, is strongly absorbing in the sense of Definition 2.2).

Note that a graded representation (π, E) is balanced and infinite multiplicity if and only if

$$(\pi, E) = (1_{\mathbb{C}^2 \otimes \ell^2 \otimes \ell^2} \otimes \sigma, \mathbb{C}^2 \otimes \ell^2 \otimes \ell^2 \otimes F) \tag{4.1}$$

for some representation $\sigma: A \to \mathcal{L}(F)$: here, a tensor factor of ℓ^2 comes from the infinite multiplicity assumption, and we use an identification $\ell^2 = \ell^2 \otimes \ell^2$ to split off an extra tensorial factor of ℓ^2. We record some useful observations arising from this as a lemma.

Lemma 4.2. *Let $\pi: A \to \mathcal{L}(E)$ be a graded, balanced, infinite multiplicity representation of A on a Hilbert B-module. Arising from a decomposition as in line (4.1), there are canonical unital inclusions*

$$M_2(\mathbb{C}) \subseteq \mathcal{L}(E) \quad \textit{and} \quad \mathcal{B}(\ell^2) \subseteq \mathcal{L}(E) \tag{4.2}$$

as the C^*-subalgebras

$$M_2(\mathbb{C}) \otimes 1_{\ell^2 \otimes \ell^2 \otimes F} \quad \textit{and} \quad 1_{\mathbb{C}^2} \otimes \mathcal{B}(\ell^2) \otimes 1_{\ell^2 \otimes F}$$

respectively. These inclusions have the following properties:

(i) *The neutral projection corresponds to the element $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{C})$.*

(ii) *The subalgebras $\mathcal{B}(\ell^2)$ and $M_2(\mathbb{C})$ of $\mathcal{L}(E)$ commute with each other, and with A.*

---

[20] Compare [12, Definition 8.3.10].

(iii) *The compositions*

$$\mathcal{B}(\ell^2) \longrightarrow \mathcal{L}(E) \longrightarrow \mathcal{L}(E)/\mathcal{K}(E) \text{ and } M_2(\mathbb{C}) \longrightarrow \mathcal{L}(E) \longrightarrow \mathcal{L}(E)/\mathcal{K}(E)$$

of the inclusions in line (4.2) with the quotient map to the Calkin algebra are injective. ∎

The following is the key definition of this section.

Definition 4.3. Let $\pi: A \to \mathcal{L}(E)$ be a graded representation of A on a Hilbert B-module, and define $\mathcal{P}^\pi(A, B)$ to be the set of self-adjoint contractions

$$p \in C_b\big([1,\infty), \mathcal{L}(E)\big)$$

such that:

(i) $p - e \in C_b([1,\infty), \mathcal{K}(E))$;[21]

(ii) for all $a \in A$, $[a, p] \in C_0([1,\infty), \mathcal{L}(E))$;

(iii) for all $a \in A$, $a(p^2 - p) \in C_0([1,\infty), \mathcal{K}(E))$.

Our next goal is to define an equivalence relation on $\mathcal{P}^\pi(A, B)$ such that the equivalence classes give a realization of $KK(A, B)$. For this (and other purposes later), it will be convenient to introduce a parameter space Y. Let then $C = C_0(Y)$ be a separable commutative C^*-algebra: for our applications, Y will be one of the intervals $[0, 1]$ or $(0, 1)$, or the one-point compactification $\overline{\mathbb{N}}$ of the natural numbers. Let $\pi: A \to \mathcal{L}(E)$ be a representation of A on a Hilbert B-module, and let $C \otimes E$ denote the tensor product Hilbert $C \otimes B$-module. Let $1 \otimes \pi: A \to \mathcal{L}(C \otimes E)$ be the amplification of π. If π is graded then $1 \otimes \pi$ inherits a grading in a natural way, and so if we are in the graded case we may consider $\mathcal{P}^{1\otimes\pi}(A, C \otimes B)$.

The following lemma characterizes elements of $\mathcal{P}^{1\otimes\pi}(A, C \otimes B)$ in terms of doubly parametrized families $(p_t^y)_{t\in[1,\infty), y\in Y}$.

Lemma 4.4. *Let $\pi: A \to \mathcal{L}(E)$ be a graded representation of A on a Hilbert B-module. With notation as above, there is a canonical identification between elements p of $\mathcal{P}^{1\otimes\pi}(A, C \otimes B)$ and doubly parametrized families of self-adjoint contractions $(p_t^y)_{t\in[1,\infty), y\in Y}$ that define a function*

$$p: [1,\infty) \longrightarrow C_b\big(Y, \mathcal{L}(E)\big), \quad t \longmapsto (y \mapsto p_t^y)$$

with the following properties:

(i) *the function $p - e$ is in $C_b([1,\infty), C_0(Y, \mathcal{K}(E)))$;*

(ii) *$[p, a] \in C_0([1,\infty), C_b(Y, \mathcal{L}(E)))$ for all $a \in A$;*

(iii) *$a(p^2 - p) \in C_0([1,\infty), C_0(Y, \mathcal{K}(E)))$ for all $a \in A$.*

[21] To make sense of this, we follow our usual conventions and identify e with a constant function in $C_b([1,\infty), \mathcal{L}(E))$, and similarly for elements of A in the subsequent parts of the definition.

Proof. An element of $\mathcal{P}^{1\otimes\pi}(A, C \otimes B)$ is a function $p\colon [1,\infty) \to \mathcal{L}(C \otimes E)$ satisfying the conditions of Definition 4.3. Using the canonical identifications

$$\mathcal{K}(C \otimes E) = C \otimes \mathcal{K}(E) = C_0\big(Y, \mathcal{K}(E)\big)$$

(for the first, see for example [17, pp. 37 and 10]) and the fact that

$$p - e \in C_b\big([1,\infty), \mathcal{K}(C \otimes E)\big),$$

we identify p with a function

$$p\colon [1,\infty) \longrightarrow C_0\big(Y, \mathcal{K}(E)\big) + \{e\} \subseteq C_b\big(Y, \mathcal{L}(E)\big).$$

The remaining checks are direct. ∎

Definition 4.5. Let $\pi\colon A \to \mathcal{L}(E)$ be a graded representation of A on a Hilbert B-module. Elements p^0 and p^1 of $\mathcal{P}^\pi(A, B)$ are *homotopic* if (with notation as in Lemma 4.4) there is an element

$$p = (p_t^s)_{t\in[1,\infty), s\in[0,1]} \in \mathcal{P}^{1\otimes\pi}\big(A, C[0,1] \otimes B\big)$$

that agrees with p^0 and p^1 at the endpoints $s \in \{0,1\}$. We write $p^0 \sim p^1$ if p^0 and p^1 are homotopic, and write $KK^\pi_{\mathcal{P}}(A, B)$ for the quotient set $\mathcal{P}^\pi(A, B)/\sim$.

We will need the following elementary fact a few times, so record it here for ease of reference.

Lemma 4.6. *Assume that p and q are elements of $\mathcal{P}^\pi(A, B)$ such that $p_t - q_t \to 0$ as $t \to \infty$. Then $p \sim q$.*

Proof. A straight line homotopy $(sp + (1-s)q)_{s\in[0,1]}$ works: we leave the direct checks involved to the reader. ∎

In order to define a semigroup structure on $KK^\pi_{\mathcal{P}}(A, B)$, we assume π is graded, balanced, and infinite multiplicity as in Definition 4.1, and fix a tensorial decomposition as in line (4.1) (which will remain fixed for the rest of the section). Fix also two isometries s_1 and s_2 in $\mathcal{B}(\ell^2)$ that satisfy the Cuntz relation $s_1s_1^* + s_2s_2^* = 1$. Using the canonical (unital) inclusion $\mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ from line (4.2) in Lemma 4.2, we think of these isometries as adjointable operators on E that commute with $A \subseteq \mathcal{L}(E)$ and with the neutral projection $e \in \mathcal{L}(E)$.

Lemma 4.7. *Let $\pi\colon A \to \mathcal{L}(E)$ be a graded, balanced, and infinite multiplicity representation. Then with notation as above, the operation defined by*

$$[p] + [q] := [s_1ps_1^* + s_2qs_2^*]$$

makes $KK^\pi_{\mathcal{P}}(A, B)$ into an abelian semigroup. The operation does not depend on the choice of s_1, s_2 within $\mathcal{B}(\ell^2)$.

Proof. The Borel functional calculus shows that the unitary group of $\mathcal{B}(\ell^2)$ is connected in the norm topology. Hence conjugation by a unitary in $\mathcal{B}(\ell^2)$ induces the trivial map on $KK^{\pi}_{\mathcal{P}}(A,B)$, and so conjugating by the unitaries $s_1s_2^*+s_2s_1^*$ and $s_1s_1s_1^*+s_1s_2s_1^*s_2^*+s_2s_2^*s_2^*$ show that the operation is commutative and associative. On the other hand, if $t_1,t_2\in\mathcal{B}(\ell^2)$ also satisfy the Cuntz relation, then conjugating by the unitary $s_1t_1^*+s_2t_2^*$ shows that the pairs (s_1,s_2) and (t_1,t_2) induce the same operation on $KK^{\pi}_{\mathcal{P}}(A,B)$. ∎

Our next goal is to show that the semigroup $KK^{\pi}_{\mathcal{P}}(A,B)$ is a monoid. We first state a well-known lemma about paths of projections in a C^*-algebra. It follows from the arguments of [12, Proposition 4.1.7 and Corollary 4.1.8], for example.

Lemma 4.8. *Let I be either $[a,b]$ or $[a,\infty)$ for some $a,b\in\mathbb{R}$, and let $(p_t)_{t\in I}$ be a continuous path of projections in a unital C^*-algebra D. Then there is a continuous path of unitaries $(u_t)_{t\in I}$ in D such that $u_a=1$, and such that $p_t=u_tp_au_t^*$ for all t.* ∎

We are now ready for the key technical lemma we will use to show that $KK^{\pi}_{\mathcal{P}}(A,B)$ is a monoid.

Lemma 4.9. *Let $\pi\colon A\to\mathcal{L}(E)$ be a graded, balanced, and infinite multiplicity representation of A on a Hilbert B-module. Let p be an element of $\mathcal{P}^{\pi}(A,B)$, and let v be an isometry in the canonical copy of $\mathcal{B}(\ell^2)\subseteq\mathcal{L}(E)$ from line (4.2) from Lemma 4.2. Then the element*

$$q:=vpv^*+(1-vv^*)e\in C_b\big([1,\infty),\mathcal{L}(E)\big)$$

is in $\mathcal{P}^{\pi}(A,B)$ and satisfies $p\sim q$.

Proof. For each $n\geq 1$, a compactness argument gives a finite rank projection

$$e_n\in\mathcal{K}(\ell^2)\subseteq\mathcal{B}(\ell^2)\subseteq\mathcal{L}(E)$$

(where the last inclusion is that from line (4.2) from Lemma 4.2) such that

$$\big\|(1-e_n)(p_t-e)\big\|<\frac{1}{n}$$

for all $t\in[1,n+1]$. Choose now a projection $r_1\geq e_1$ such that r_1-e_1 and $1-r_1$ both have infinite rank. Given r_n, define r_{n+1} to be the max of r_n and e_{n+1}. In this way we get an increasing sequence $r_1\leq r_2\leq\cdots$ of projections in $\mathcal{B}(\ell^2)$ such that $r_n\geq e_m$ for all n and all $m\leq n$, and such that r_n-e_m and $1-r_n$ both have infinite rank for all n and all $m\leq n$. For each n, $(1-e_n)r_n$ and $(1-e_n)r_{n+1}$ are projections with infinite-dimensional kernel and image as operators on $(1-e_n)\ell^2$, and are thus connected by a continuous path of projections $(r_t^0)_{t\in[n,n+1]}$ in $\mathcal{B}((1-e_n)\ell^2)$. Set $r_t:=e_n+r_t^0$ for $t\in[n,n+1]$. In this way we get a continuous path of projections $r=(r_t)_{t\in[1,\infty)}$ in $\mathcal{B}(\ell^2)\subseteq\mathcal{L}(E)$ such that if $\lfloor t\rfloor$ is the floor function of t then

$$\big\|(1-r_t)(p_t-e)\big\|\leq\big\|(1-e_{\lfloor t\rfloor})(p_t-e)\big\|<\frac{1}{\lfloor t\rfloor},\tag{4.3}$$

and such that r_t and $1-r_t$ have infinite rank as operators on ℓ^2 for each t.

Note now that as r_t commutes with e, line (4.3) implies in particular that $\|[r_t, p_t]\| < 2/\lfloor t \rfloor$. Define $p' \in C_b([1,\infty), \mathcal{L}(E))$ by

$$p'_t := r_t p_t r_t + (1 - r_t)e.$$

As $r_t p_t r_t - r_t e$ is in $\mathcal{K}(E)$ for all t, we see that $p'_t - e$ is in $\mathcal{K}(E)$ for all t. Moreover,

$$\|p'_t - p_t\| \leq \big\|[r_t, p_t]\big\| + \big\|(1 - r_t)(p_t - e)\big\| \longrightarrow 0 \quad \text{as } t \longrightarrow \infty,$$

and so $p' := (p'_t)$ defines an element of $\mathcal{P}^\pi(A, B)$ such that $p' \sim p$ by Lemma 4.6.

Now, let $v \in \mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ be an isometry as in the statement of the lemma. Lemma 4.8 gives a continuous path $(u^r_t)_{t\in[1,\infty)}$ of unitaries in $\mathcal{B}(\ell^2)$ such that $r_t = u^r_t r_1 (u^r_t)^*$ for all t. Similarly, we get a continuous path of unitaries $(u^v_t)_{t\in[1,\infty)}$ such that

$$u^v_t\big(1 - vv^* + v(1 - r_1)v^*\big)(u^v_t)^* = 1 - vv^* + v(1 - r_t)v^* \quad \text{for all } t.$$

Choose any partial isometry $w \in \mathcal{B}(\ell^2)$ such that $ww^* = r_1$ and $w^*w = 1 - vv^* + v(1 - r_1)v^*$ (such exists as r_1 and $1 - vv^* + v(1 - r_1)v^*$ are both infinite rank), and define $w_t := u^r_t w (u^v_t)^*$. Then $(w_t)_{t\in[1,\infty)}$ is a continuous path of partial isometries in $\mathcal{B}(\ell^2)$ such that $w_t w^*_t = 1 - r_t$ and $w^*_t w_t = 1 - vv^* + v(1 - r_t)v^*$. Define

$$u_t := vr_t + w^*_t \in \mathcal{B}(\ell^2) \subseteq \mathcal{L}(E).$$

Then $u = (u_t)_{t\in[1,\infty)}$ is a continuous path of unitaries such that $up'u^* = vp'v^* + (1 - vv^*)e$. Let $(h^s : \mathcal{U}(\ell^2) \to \mathcal{U}(\ell^2))_{s\in[0,1]}$ be a norm-continuous contraction of the unitary group of ℓ^2 to the identity element (such exists by Kuiper's theorem: see for example [5, Theorem on p. 433]) and note that the path $(h^s(u)p'h^s(u^*))_{s\in[0,1]}$ shows that $p' \sim vp'v^* + (1 - vv^*)e$. In conclusion, we have that

$$p \sim p' \sim vp'v^* + (1 - vv^*)e \sim vpv^* + (1 - vv^*)e$$

and are done. ∎

Corollary 4.10. *Let $\pi : A \to \mathcal{L}(E)$ be a graded, balanced, and infinite multiplicity representation of A on a Hilbert B-module. Then for any $p \in \mathcal{P}^\pi(A, B)$, we have $s_1 p s^*_1 + s_2 e s^*_2 \sim p$.*

In particular, the semigroup $KK^\pi_{\mathcal{P}}(A, B)$ is a commutative monoid with identity given by the class $[e]$ of the neutral projection.

Proof. Apply Lemma 4.9 with $v = s_1$, whence $1 - vv^* = s_2 s^*_2$, and use that s_2 commutes with e. ∎

Our next goal, which is the main point of this section, is to show that if $\pi : A \to \mathcal{L}(E)$ is a graded, balanced, and strongly absorbing representation of A on a Hilbert B-module, then there is a canonical semigroup isomorphism $KK^\pi_{\mathcal{P}}(A, B) \cong KK(A, B)$ (and therefore in particular, $KK^\pi_{\mathcal{P}}(A, B)$ is a group). We need some preliminaries.

Let $\pi: A \to \mathcal{L}(E)$ be a graded, balanced, and infinite multiplicity representation of A on a Hilbert B-module, and fix the decomposition of line (4.1) and the Cuntz isometries of Lemma 4.7. Let $Q_L(\pi)$ and $C_{L,c}(\pi;\mathcal{K})$ be as in Definitions 3.9 and 3.12 respectively, so Lemma 3.13 gives us a surjection $\rho: C_{L,c}(\pi;\mathcal{K}) \to Q_L(\pi)$. This induces a $*$-homomorphism $\overline{\rho}: M(C_{L,c}(\pi;\mathcal{K})) \to M(Q_L(\pi))$ on multiplier algebras, which is uniquely determined by the condition that $\overline{\rho}(m)\cdot\rho(b) = \rho(mb)$ for all $m \in M(C_{L,c}(\pi;\mathcal{K}))$ and $b \in C_{L,c}(\pi;\mathcal{K})$ (see [17, Chapter 2] for this). We define

$$M := \overline{\rho}\big(M\big(C_{L,c}(\pi;\mathcal{K})\big)\big), \tag{4.4}$$

which is a unital C^*-subalgebra[22] of $M(Q_L(\pi))$ containing $Q_L(\pi)$ as an ideal.

Lemma 4.11. *With notation as in line (4.4) above, M has trivial K-theory.*

Proof. The unital inclusion $\mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ of Lemma 4.2 induces a unital inclusion $\mathcal{B}(\ell^2) \subseteq M(C_{L,c}(\pi;\mathcal{K}))$ by having $\mathcal{B}(\ell^2)$ act pointwise in the variable t (this uses that $\mathcal{B}(\ell^2)$ commutes with A). This in turn descends to a unital inclusion $\mathcal{B}(\ell^2) \subseteq M$. Let $(s_n)_{n=0}^{\infty}$ be a sequence of isometries in $\mathcal{B}(\ell^2) \subseteq M$ with orthogonal ranges.

Consider the maps

$$\iota_0: M\big(C_{L,c}(\pi;\mathcal{K})\big) \longrightarrow M\big(C_{L,c}(\pi;\mathcal{K})\big), \quad b \longmapsto s_0 b s_0^*,$$

and

$$\alpha_0: M\big(C_{L,c}(\pi;\mathcal{K})\big) \longrightarrow M\big(C_{L,c}(\pi;\mathcal{K})\big), \quad b \longmapsto \sum_{n=1}^{\infty} s_n b s_n^*$$

(the sum converges in the strict topology of $\mathcal{L}(E)$, pointwise in t). With $I_{L,c}(\pi)$ as in Definition 3.9, the kernel of the map $\overline{\rho}: M(C_{L,c}(\pi;\mathcal{K})) \to M$ is

$$\big\{m \in M\big(C_{L,c}(\pi;\mathcal{K})\big) \mid mb \in I_{L,c}(\pi) \text{ for all } b \in C_{L,c}(\pi;\mathcal{K})\big\},$$

whence ι_0 and α_0 descend to well-defined $*$-homomorphisms $\iota, \alpha: M \to M$.

As α and ι have orthogonal ranges, Lemma 3.7 implies that $\alpha+\iota$ is a $*$-homomorphism and that as maps on K-theory, $\alpha_* + \iota_* = (\alpha+\iota)_*$. Moreover, conjugating by the isometry $s := \sum_{n=0}^{\infty} s_n s_{n+1}^* \in \mathcal{B}(\ell^2) \subseteq M$ (the sum converges in the strong topology of $\mathcal{B}(\ell^2)$) and applying Lemma 3.8 implies that $(\alpha+\iota)_* = \alpha_*$ as maps on K-theory. We thus have

$$\alpha_* + \iota_* = (\alpha+\iota)_* = \alpha_*,$$

whence $\iota_* = 0$. However, ι_* is an isomorphism by Lemma 3.8 again, whence $K_*(M)$ is zero as required. ∎

We need one more preliminary definition and lemma before we get to the isomorphism $KK^{\pi}_{\mathcal{P}}(A,B) \cong KK(A,B)$.

[22] It could be all of $M(Q_L(\pi))$, although this does not seem to be obvious: note that the noncommutative Tietze extension theorem [17, Proposition 6.8] is not available here as $C_{L,c}(\pi;\mathcal{K})$ is not σ-unital.

Definition 4.12. For an ideal I in a C^*-algebra N, the *double* of I along N is the C^*-algebra defined by

$$D_N(I) := \{(a,b) \in N \oplus N \mid a - b \in I\}.$$

Note that $D_N(I)$ fits into a short exact sequence

$$0 \longrightarrow I \longrightarrow D_N(I) \longrightarrow N \longrightarrow 0 \tag{4.5}$$

with the maps $I \to D_N(I)$ and $D_N(I) \to N$ given by $a \mapsto (a,0)$ and $(a,b) \mapsto b$, respectively.

Lemma 4.13. *Assume that I is an ideal in a unital C^*-algebra N, let $D_N(I)$ be the double from Definition 4.12, and assume that $K_*(N) = 0$. Then $D_N(I)$ has the following properties:*

(i) *the inclusion $I \to D_N(I)$ from line (4.5) induces an isomorphism on K-theory;*

(ii) *any class in $K_0(D_N(I))$ of the form $[p,p]$ for some projection $p \in M_n(N)$ is zero;*

(iii) *for any $[p,q] \in K_0(D_N(I))$, we have $-[p,q] = [q,p]$;*

(iv) *any element in $K_0(D_N(I))$ can be written as $[p,q]$ for a projection (p,q) in a matrix algebra $M_n(D_N(I))$.*

Proof. Part (i) follows from the six-term exact sequence in K-theory and the assumption that $K_*(N) = 0$. Part (ii) follows as any such class is in the image of the map induced on K-theory by the $*$-homomorphism

$$N \longrightarrow D_N(I), \quad a \longmapsto (a,a)$$

and is thus zero as $K_*(N) = 0$. For part (iii), say $[p,q] \in K_0(D_N(I))$ with $p,q \in M_n(N)$. Then $[p,q] + [q,p] = [p \oplus q, q \oplus p]$. As $p - q \in M_n(I)$, the formula

$$[0,\pi/2] \ni s \longmapsto \left(\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}, \begin{pmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{pmatrix} \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} \cos(s) & -\sin(s) \\ \sin(s) & \cos(s) \end{pmatrix} \right)$$

defines a homotopy between $(p \oplus q, q \oplus p)$ and $(p \oplus q, p \oplus q)$ passing through projections in $M_{2n}(D_N(I))$. The latter defines the zero class in K_0 by part (ii), which gives part (iii). Part (iv) follows directly from part (iii). ∎

Here is the main result of this section.

Theorem 4.14. *Let $\pi: A \to \mathcal{L}(E)$ be a graded, balanced, and strongly absorbing representation of A on a Hilbert B-module. Let M be as in line (4.4), $Q_L(\pi)$ as in Definition 3.9 and $D_M(Q_L(\pi))$ be as in Definition 4.12. Then the formula*

$$KK^{\pi}_{\mathcal{P}}(A,B) \longrightarrow K_0\big(D_M\big(Q_L(\pi)\big)\big), \quad [p] \longmapsto [p,e] \tag{4.6}$$

defines an isomorphism of commutative monoids. In particular, $KK^{\pi}_{\mathcal{P}}(A,B)$ is an abelian group.

Moreover, there is a canonical isomorphism $KK^{\pi}_{\mathcal{P}}(A,B) \cong KK(A,B)$.

Proof. We first have to show that the map in line (4.6) is well defined. It is not difficult to see that if $p \in \mathcal{P}^\pi(A, B)$, then (p, e) is a projection in $D_M(Q_L(\pi))$. For well-definedness, we need to show that if $p^0 \sim p^1$ in $\mathcal{P}^\pi(A, B)$, then the projections (p^0, e) and (p^1, e) in $D_M(Q_L(\pi))$ define the same K-theory class. Let then $(p^s)_{s\in[0,1]}$ be a homotopy implementing the equivalence between p^0 and p^1. Let

$$1 \otimes \pi \colon A \longrightarrow \mathcal{L}\big(C[0,1] \otimes E\big)$$

be the amplification of π to the $C[0,1] \otimes B$-module $C[0,1] \otimes E$, and let $C_{L,c}(1 \otimes \pi)$ be the associated localization algebra. Note that $p := (p^s)_{s\in[0,1]}$ defines an element of the multiplier algebra $M(C_{L,c}(1 \otimes \pi))$ such that $p - e$ is in $C_{L,c}(1 \otimes \pi)$, and so that $[p, e]$ is a well-defined class in $D_{M_C}(Q_L(1 \otimes \pi))$, where M_C is defined analogously to M, but starting with $1 \otimes \pi$.

As π is graded, balanced, and strongly absorbing we can write $E = E_0 \oplus E_1$ with $E_0 \cong \ell^2 \otimes B$ (compare Remark A.14 for the last point). Hence we may apply Proposition 2.9 to conclude that $1 \otimes \pi$ is (graded, balanced, and) strongly absorbing, and thus Theorems 3.2 and 3.4 give an isomorphism

$$KK\big(A, C[0,1] \otimes B\big) \xrightarrow{\cong} K_0\big(C_{L,c}(1 \otimes \pi)\big).$$

Let $\varepsilon^0, \varepsilon^1 \colon C[0,1] \otimes B \to B$ be given by evaluation at the endpoints. Lemma 3.17 then gives a commutative diagram

$$\begin{tikzcd} KK\big(A, C[0,1] \otimes B\big) \arrow[r, "\cong"] \arrow[d, "\varepsilon^i_*"] & K_0\big(C_{L,c}(\iota \otimes \pi)\big) \arrow[d, "\varepsilon^i_*"] \\ KK(A, B) \arrow[r, "\cong"] & K_0\big(C_{L,c}(\pi)\big) \end{tikzcd}$$

for $i \in \{0, 1\}$. Homotopy invariance of KK-theory gives that the maps

$$\varepsilon^0_*, \varepsilon^1_* \colon KK\big(A, C[0,1] \otimes B\big) \longrightarrow KK(A, B)$$

are the same, whence the maps $\varepsilon^0_*, \varepsilon^1_* \colon K_0(C_{L,c}(1 \otimes \pi)) \to K_0(C_{L,c}(\pi))$ are too. On the other hand each ε^i induces maps $\varepsilon^i \colon Q_L(1 \otimes \pi) \to Q_L(\pi)$ and $\varepsilon^i \colon C_{L,c}(1 \otimes \pi; \mathcal{K}) \to C_{L,c}(\pi; \mathcal{K})$, and therefore induces a map $D_{M_C}(Q_L(1 \otimes \pi)) \to D_M(Q_L(1 \otimes \pi))$. All this gives rise to a commutative diagram

$$\begin{tikzcd} C_{L,c}(1 \otimes \pi) \arrow[r] \arrow[d, "\varepsilon^i"] & Q_L(1 \otimes \pi) \arrow[r] \arrow[d, "\varepsilon^i"] & D_{M_C}\big(Q_L(1 \otimes \pi)\big) \arrow[d, "\varepsilon^i"] \\ C_{L,c}(\pi) \arrow[r] & Q_L(\pi) \arrow[r] & D_M\big(Q_L(\pi)\big) \end{tikzcd}$$

where the first pair of horizontal maps are the canonical quotients, and the second pair are the inclusions $a \mapsto (a, 0)$. The horizontal maps induce isomorphisms on K-theory by Corollary 3.11 (first pair), and Lemmas 4.11 and 4.13 (second pair). Hence the maps $\varepsilon^0_*, \varepsilon^1_*\colon K_0(D_{M_C}(Q_L(1\otimes\pi))) \to K_0(D_M(Q_L(\pi)))$ are the same. We thus see that

$$[p^0, e] = \varepsilon^0_*[p,e] = \varepsilon^1_*[p,e] = [p^1, e],$$

which is the statement needed for well-definedness of the map in line (4.6).

We now show that the map in line (4.6) is a homomorphism. Indeed, for $p, q \in \mathcal{P}^\pi(A,B)$, the element $[s_1ps_1^* + s_2qs_2^*]$ of $KK^\pi_{\mathcal{P}}(A,B)$ gets sent to

$$[s_1ps_1^* + s_2qs_2^*, e] = [s_1ps_1^* + s_2qs_2^*, s_1es_1^* + s_2es_2^*],$$

where we have used that $s_1s_1^* + s_2s_2^* = 1$ and that s_1, s_2 commute with e. As $s_1xs_1^*$ is orthogonal to $s_1ys_2^*$ for any x, y we have that

$$[s_1ps_1^* + s_2qs_2^*, s_1es_1^* + s_2es_2^*] = [s_1ps_1^*, s_1es_1^*] + [s_2qs_2^*, s_2es_2^*]$$

and as conjugation by s_1 and s_2 has no effect on K-theory by Lemma 3.8, this equals

$$[p,e] + [q,e],$$

i.e., the sum of the images of $[p]$ and $[q]$.

We now show that the map in line (4.6) is surjective. Using Lemma 4.13, an arbitrary element of $K_0(D_M(Q_L(\pi)))$ can be represented as a class $[p,q]$ with p, q projections in $M_n(M)$ for some n, and with $p - q \in M_n(Q_L(\pi))$. We have that $[1-q, 1-q] = 0$ by Lemma 4.13, and thus $[p,q] = [p \oplus 1-q, q\oplus 1-q]$. The matrix $u = \begin{pmatrix} q & 1-q \\ 1-q & q\end{pmatrix}$ is a unitary in $M_{2n}(M)$, whence conjugating by (u,u) we see that

$$[p,q] = [p\oplus 1-q, q\oplus 1-q] = \big[u(p\oplus q)u^*, u(q\oplus 1-q)u^*\big] = \big[u(p\oplus q)u^*, 1_n \oplus 0_n\big],$$

where 1_n and 0_n are the unit and zero in $M_n(M)$. Choose now $2n$ isometries $v_1, \ldots, v_{2n}$ in $\mathcal{B}(\mathbb{C}^2 \otimes \ell^2) \subseteq \mathcal{L}$ such that $\sum_{i=1}^n v_iv_i^* = e$ and $\sum_{i=1}^{2n} v_iv_i^* = 1_{2n}$. The matrix

$$v := \begin{pmatrix} v_1 & v_2 & \cdots & v_{2n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in M_{2n}(M)$$

is an isometry, whence conjugation by (v,v) induces the trivial map on $K_0(D_M(Q_L(\pi)))$ by Lemma 3.8. Hence

$$[p,q] = \big[vu(p\oplus q)u^*v^*, v(1_n \oplus 0)v^*\big] = [r \oplus 0_{2n-1}, e \oplus 0_{2n-1}],$$

where $r \in M$ is a projection such that $a := r - e$ is in $Q_L(\pi)$. We may lift a to a self-adjoint element $b \in C_{L,c}(\pi;\mathcal{K})$ by Lemma 3.13. Consider the self-adjoint element

$$(b+e, e) \in D_{M(C_{L,c}(\pi;\mathcal{K}))}\big(C_{L,c}(\pi;\mathcal{K})\big),$$

which maps to $(r, e) \in D_M(Q_L(\pi))$ under the $*$-homomorphism

$$D_{M(C_{L,c}(\pi;\mathcal{K}))}\big(C_{L,c}(\pi;\mathcal{K})\big) \longrightarrow D_M\big(Q_L(\pi)\big)$$

induced by the quotient map $C_{L,c}(\pi;\mathcal{K}) \to Q_L(\pi)$ of Lemma 3.13. Note that if $f\colon \mathbb{R} \to [-1,1]$ is the function defined by

$$f(t) := \begin{cases} 1 & t > 1, \\ t & -1 \leq t \leq 1, \\ -1 & t < -1, \end{cases}$$

then in $D_{M(C_{L,c}(\pi;\mathcal{K}))}(C_{L,c}(\pi;\mathcal{K}))$

$$f(b,e) = \big(f(b+e), f(e)\big) = \big(f(b+e), e\big),$$

and this element still maps to (r, e) by naturality of the functional calculus. Set $c := f(b+e)$. Then one checks that c is an element of $\mathcal{P}^\pi(A, B)$ such that $[c, e] = [r, e] = [p, q]$, so we are done with surjectivity.

To see injectivity of the map in line (4.6), assume that $[p] \in KK^\pi_{\mathcal{P}}(A, B)$ is such that $[p, e]$ is zero in $K_0(D_M(Q_L(\pi)))$. In particular, $[p, e] = [e, e]$ by Lemma 4.13, and therefore there is a projection $(q_1, q_2) \in M_n(D_M(Q_L(\pi)))$ and a homotopy $p_{(1)} = (p^s_{(1)})_{s\in[0,1]}$ between $(p \oplus q_1, e \oplus q_2)$ and $(e \oplus q_1, e \oplus q_2)$ in $M_{n+1}(D_M(Q_L(\pi)))$. We will manipulate this homotopy to build a homotopy between p and e in $\mathcal{P}^\pi(A, B)$.

- Replacing $p_{(1)}$ by $p_{(2)} := p_{(1)} \oplus (q_2, q_1)$, we get a homotopy between $(p \oplus q_1 \oplus q_2, e \oplus q_2 \oplus q_1)$ and $(e \oplus q_1 \oplus q_2, e \oplus q_2 \oplus q_1)$.
- As $q_1 - q_2 \in M_n(Q_L(\pi))$, we get a homotopy

$$s \longmapsto \left(p \oplus q_1 \oplus q_2, e \oplus \begin{pmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{pmatrix} \begin{pmatrix} q_2 & 0 \\ 0 & q_1 \end{pmatrix} \begin{pmatrix} \cos(s) & -\sin(s) \\ \sin(s) & \cos(s) \end{pmatrix}\right)$$

  between $(p \oplus q_1 \oplus q_2, e \oplus q_2 \oplus q_1)$ and $(p \oplus q_1 \oplus q_2, e \oplus q_1 \oplus q_2)$, and similarly between $(e \oplus q_1 \oplus q_2, e \oplus q_2 \oplus q_1)$ and $(e \oplus q_1 \oplus q_2, e \oplus q_1 \oplus q_2)$. Concatenating these with the homotopy $p_{(2)}$ gives a homotopy $(p^s_{(3)})_{s\in[0,1]}$ between $(p \oplus q_1 \oplus q_2, e \oplus q_1 \oplus q_2)$ and $(e \oplus q_1 \oplus q_2, e \oplus q_1 \oplus q_2)$.
- Setting $r = q_1 \oplus q_2$ and replacing $p_{(3)}$ with

$$p^s_{(4)} := p_{(3)} \oplus \big((1-r), (1-r)\big)$$

  gives a homotopy between $(p \oplus r \oplus (1-r), e \oplus r \oplus 1 - r \oplus 0_{4n})$ and $(e \oplus r \oplus (1-r), e \oplus r \oplus (1-r))$.
- Set $u = \left(\begin{smallmatrix} r & 1-r \\ 1-r & r \end{smallmatrix}\right)$, which is a unitary in $M_{4n}(M)$. Moreover, u is self-adjoint, so connected to the identity via some path $(u^s)_{s\in[0,1]}$ of unitaries. Then

$$(1 \oplus u^s, 1 \oplus u^s)\big(p \oplus r \oplus (1-r), e \oplus r \oplus (1-r)\big)(1 \oplus u^s, 1 \oplus u^s)^*$$

defines a homotopy between $(p \oplus r \oplus (1-r), e \oplus r \oplus (1-r))$ and $(p \oplus 1_{2n} \oplus 0_{2n}, e \oplus 1_{2n} \oplus 0_{2n})$. Similarly, we get a homotopy between $(e \oplus r \oplus (1-r), e \oplus r \oplus (1-r))$ and $(e \oplus 1_{2n} \oplus 0_{2n}, e \oplus 1_{2n} \oplus 0_{2n})$. Concatenating these with $p_{(4)}$ gives a homotopy $p_{(5)}$ between $(p \oplus 1_{2n} \oplus 0_{2n}, e \oplus 1_{2n} \oplus 0_{2n})$ and $(e \oplus 1_{2n} \oplus 0_{2n}, e \oplus 1_{2n} \oplus 0_{2n})$.

- Write $p^s_{(5)} = (p^s_0, p^s_1)$ for paths of projections $(p^s_0)_{s \in [0,1]}$ and $(p^s_1)_{s\in[0,1]}$ in $M_{4n+1}(M)$. Then Lemma 4.8 gives a continuous path of unitaries $(v^s)_{s\in[0,1]}$ in $M_{4n+1}(M)$ with $v^0 = 1$, and $p^s_1 = v_s(e \oplus 1_{2n} \oplus 0_{2n})v^*_s$ for all $s \in [0,1]$. Note in particular that

$$v_1(e \oplus 1_{2n} \oplus 0_{2n})v_1^* = (e \oplus 1_{2n} \oplus 0_{2n}),$$

even though we may not have that $v^1 = 1$. Define then

$$p^s_{(6)} := (v_s, v_s)^* p^s_{(5)}(v_s, v_s),$$

which gives a new homotopy between $(p \oplus 1_{2n} \oplus 0_{2n}, e \oplus 1_{2n} \oplus 0_{2n})$ and $(e \oplus 1_{2n} \oplus 0_{2n}, e \oplus 1_{2n} \oplus 0_{2n})$ with the additional property of being constant in the second variable.

- Let $M_{1\times 4n}(M)$ be the $1 \times 4n$ row matrices, and choose an isometry $w \in M_{1\times 4n}(\mathcal{B}(\ell^2)) \subseteq M$ be such that $w(1_{2n} \oplus 0_{2n})w^* = s_2 e s_2^*$. Define

$$t := \begin{pmatrix} s_1 & w \end{pmatrix} \in M_{1\times 4n+1}\big(\mathcal{B}(\ell^2)\big) \subseteq M_{1\times 4n+1}(M),$$

which is an isometry, and define $p^s_{(7)} := t p^s_{(6)} t^*$. Then this is a homotopy in $D_M(Q_L(\pi))$ between $(s_1 p s_1^* + s_2 e s_2^*, s_1 e s_1^* + s_2 e s_2^*)$ and $(s_1 e s_1^* + s_2 e s_2^*, s_1 e s_1^* + s_2 e s_2^*)$ that is constant in the second variable.

Now, restricting the homotopy $p_{(7)}$ to the first variable gives a homotopy of projections in M, say $(p^s)_{s\in[0,1]}$, between $s_1 p s_1^* + s_2 e s_2^*$ and e, and such that $p^s - e$ is in $Q_L(\pi)$ for all s. The function

$$[0,1] \longrightarrow D_M\big(Q_L(\pi)\big), \quad s \longmapsto (p^s, e)$$

defines an idempotent, say q, in $C[0,1] \otimes D_M(Q_L(\pi))$. As the natural $*$-homomorphism

$$C[0,1] \otimes D_{M(C_{L,c}(\pi;\mathcal{K}))}\big(C_{L,c}(\pi;\mathcal{K})\big) \longrightarrow C[0,1] \otimes D_M\big(Q_L(\pi)\big)$$

is surjective (this follows from Lemma 3.13), q lifts to a self-adjoint contraction of the form

$$(a, e) \in C[0,1] \otimes D_{M(C_{L,c}(\pi;\mathcal{K}))}\big(C_{L,c}(\pi;\mathcal{K})\big)$$

analogously to the argument at the end of the surjectivity part. The element a defines a homotopy in $\mathcal{P}^\pi(A, B)$ between $s_1 p s_1^* + s_2 e s_2^*$ and e. On the other hand, $s_1 p s_1^* + s_2 e s_2^* \sim p$ by Corollary 4.10, whence we have

$$p \sim s_1 p s_1^* + s_2 e s_2^* \sim e.$$

Corollary 4.10 then shows that $[p] = 0$, and so we have injectivity.

To complete the proof of Theorem 4.14, we need to show that

$$KK^{\pi}_{\mathcal{P}}(A,B) \cong KK(A,B).$$

This follows by combining: the isomorphism $KK^{\pi}_{\mathcal{P}}(A,B) \cong K_0(D_M(Q_L(\pi)))$ established above; the isomorphism $K_*(D_M(Q_L(\pi))) \cong K_*(Q_L(\pi))$ of Lemma 4.13; the isomorphism $K_*(Q_L(\pi)) \cong K_*(C_{L,c}(\pi))$ of Corollary 3.11; and the isomorphism $K_0(C_{L,c}(\pi)) \cong KK(A,B)$ of Theorems 3.2 and 3.4. ∎

Finally, in this section, we establish a technical lemma about functoriality that we will need later.

Lemma 4.15. *Let $\pi\colon A \to \mathcal{L}(E)$ be a graded, balanced, and strongly absorbing representation of A on a Hilbert B-module, and let $C = C_0(Y)$ be a separable and commutative C^*-algebra. For $y \in Y$, let $e^y\colon C_0(Y) \to \mathbb{C}$ be the $*$-homomorphism defined by evaluation at y. Let $\phi_B\colon KK(A,B) \to KK^{\pi}_{\mathcal{P}}(A,B)$ be the isomorphism of Theorem 4.14. Then if p is an element of $\mathcal{P}^{1\otimes\pi}(A, C\otimes B)$ with corresponding family $(p^y_t)_{t\in[1,\infty),y\in Y}$ as in Lemma 4.4, we have that*

$$e^y_*\big(\phi^{-1}_{C\otimes B}[p]\big) = \phi^{-1}_B[p^y].$$

Proof. The map

$$\mathcal{P}^{1\otimes\pi}(A, C\otimes B) \longrightarrow \mathcal{P}^{\pi}(A,B), \quad p \longmapsto p^y$$

induces a homomorphism

$$e^y_*\colon KK^{1\otimes\pi}_{\mathcal{P}}(A, C\otimes B) \longrightarrow KK^{\pi}_{\mathcal{P}}(A,B).$$

Moreover, with notation as in the first paragraph of the proof of Theorem 4.14, e^y induces $*$-homomorphisms

$$e^y\colon Q_L(1\otimes\pi) \longrightarrow Q_L(\pi) \quad \text{and} \quad e^y\colon D_{M_C}\big(Q_L(1\otimes\pi)\big) \longrightarrow D_M\big(Q_L(\pi)\big).$$

Consider now the diagram

$$\begin{array}{ccc}
KK(A, C\otimes B) & \xrightarrow{\ e^y_*\ } & KK(A,B) \\
\downarrow & & \downarrow \\
K_0\big(C_{L,c}(1\otimes\pi)\big) & \xrightarrow{\ e^y_*\ } & K_0\big(C_{L,c}(\pi)\big) \\
\downarrow & & \downarrow \\
K_0\big(Q_L(1\otimes\pi)\big) & \xrightarrow{\ e^y_*\ } & K_0\big(Q_L(\pi)\big) \\
\downarrow & & \downarrow \\
K_0\big(D_{M_C}\big(Q_L(1\otimes\pi)\big)\big) & \xrightarrow{\ e^y_*\ } & K_0\big(D_M\big(Q_L(\pi)\big)\big) \\
\downarrow & & \downarrow \\
KK^{1\otimes\pi}_{\mathcal{P}}(A, C\otimes B) & \xrightarrow{\ e^y_*\ } & KK^{\pi}_{\mathcal{P}}(A,B)
\end{array}$$

where: the first pairs of vertical arrows are the isomorphisms of Theorems 3.2 and 3.4; the second pair of vertical arrows are induced by the canonical quotient map; the third pair of vertical arrows are induced by the inclusion $a \mapsto (a, 0)$; and the last pair of vertical arrows are the isomorphisms of Theorem 4.14. The first square commutes by Lemma 3.17 (using also Proposition 2.9 to see that the representation $1 \otimes \pi$ is strongly absorbing). It is straightforward to see that the remaining squares commute: we leave this to the reader. As the isomorphisms $\phi_{C\otimes B}$ and ϕ_B are by definition the compositions of the all the vertical arrows on the left and right respectively, the result follows. ∎

5. The topology on KK

Throughout this section, A and B refer to separable C^*-algebras. All Hilbert modules are countably generated, and all are over B unless explicitly stated otherwise. All representations of A are on Hilbert B-modules unless explicitly stated otherwise.

Our goal in this section is to recall the canonical topology on $KK(A, B)$, and describe it in terms of the isomorphism $KK(A, B) \cong KK^{\pi}_{\mathcal{P}}(A, B)$ of Theorem 4.14.

We need a quantitative version of Definition 4.3; this will also be important to us later when we define our controlled KK-theory groups. See Definition 4.1 for graded representations and the neutral projection e used in the next two definitions.

Definition 5.1. Let $\pi: A \to \mathcal{L}(E)$ be a graded representation of A on a Hilbert B-module. Let X be a finite subset of the unit ball A_1 of A, and let $\varepsilon > 0$. Define $\mathcal{P}^{\pi}_{\varepsilon}(X, B)$ to be the set of self-adjoint contractions p in $\mathcal{L}(E)$ satisfying the following conditions:

(i) $p - e$ is in $\mathcal{K}(E)$;

(ii) $\|[p, a]\| < \varepsilon$ for all $a \in X$;

(iii) $\|a(p^2 - p)\| < \varepsilon$ for all $a \in X$.

For the next definition, see Definition 4.3 for the notation $\mathcal{P}^{\pi}(A, B)$.

Definition 5.2. Let A and B be separable C^*-algebras, and let $\pi: A \to \mathcal{L}(E)$ be a graded representation of A on a Hilbert B-module. For a finite subset X of A_1 and $\varepsilon > 0$, define a function $\tau_{X,\varepsilon}: \mathcal{P}^{\pi}(A, B) \to [1, \infty)$ by

$$\tau_{X,\varepsilon}(p) := \inf\{t_0 \in [1, \infty) \mid p_t \in \mathcal{P}^{\pi}_{\varepsilon}(X, B) \text{ for all } t \geq t_0\}.$$

For each $p \in \mathcal{P}^{\pi}(A, B)$, define $U(p; X, \varepsilon)$ to be the subset of $\mathcal{P}^{\pi}(A, B)$ consisting of all q such that there exists $t \geq \max\{\tau_{X,\varepsilon}(p), \tau_{X,\varepsilon}(q)\}$ and a norm continuous path $(p^s)_{s\in[0,1]}$ in $\mathcal{L}(E)$ such that each p^s is in $\mathcal{P}^{\pi}_{\varepsilon}(X, B)$, and with endpoints $p^0 = p_t$ and $p^1 = q_t$.

For the next lemma, recall the homotopy equivalence relation $\sim$ on $\mathcal{P}^{\pi}(A, B)$ from Definition 4.5.

Lemma 5.3. *Let $\pi: A \to \mathcal{L}(E)$ be a graded representation of A on a Hilbert B-module. Let $p \in \mathcal{P}^{\pi}(A, B)$, X be a finite subset of A_1, and $\varepsilon > 0$. Then:*

(i) *if $p' \sim p$, then $U(p; X, \varepsilon) = U(p'; X, \varepsilon)$;*

(ii) *if $q \in U(p; X, \varepsilon)$ and $q \sim q'$, then $q' \in U(p; X, \varepsilon)$.*

Proof. Part (ii) follows from part (i) on noting that q is in $U(p; X, \varepsilon)$ if and only if p is in $U(q; X, \varepsilon)$. It thus suffices to prove (i).

Assume then that $p \sim p'$, so there is a homotopy $(p^s)_{s\in[0,1]}$ in $\mathcal{P}^{1\otimes\pi}(A, C[0,1] \otimes B)$ between p and p'. The definition of a homotopy gives $t_p \geq \max\{\tau_{X,\varepsilon}(p), \tau_{X,\varepsilon}(p')\}$ such that $p^s_{t_p}$ is in $\mathcal{P}^\pi_\varepsilon(X, B)$ for all $s \in [0, 1]$. Let q be an element of $U(p; X, \varepsilon)$, and let $t_q \geq \{\tau_{X,\varepsilon}(q), \tau_{X,\varepsilon}(p)\}$ be such that there is a homotopy $(q^s)_{s\in[0,1]}$ connecting p_{t_q} and q_{t_q}. Write I for whichever of the intervals $[t_p, t_q]$ or $[t_q, t_p]$ makes sense. Then concatenating the homotopies $(p^s_{t_p})_{s\in[0,1]}$, $(p_t)_{t\in I}$ and $(q^s)_{s\in[0,1]}$ shows that q is in $U(p'; X, \varepsilon)$. Hence $U(p; X, \varepsilon) \subseteq U(p'; X, \varepsilon)$. The opposite inclusion follows by symmetry. ∎

Definition 5.4. Let $\pi: A \to \mathcal{L}(E)$ be a graded representation on a Hilbert B-module. For a finite subset X of A_1, $\varepsilon > 0$, and $[p] \in KK^\pi_{\mathcal{P}}(A, B)$, define the *$X$-$\varepsilon$ neighborhood* of $[p]$ to be

$$V\big([p]; X, \varepsilon\big) := \big\{[q] \in KK^\pi_{\mathcal{P}}(A, B) \mid q \in U(p; X, \varepsilon)\big\}.$$

(note that $V([p]; X, \varepsilon)$ does not depend on the representative of the class $[p]$ by Lemma 5.3). The *asymptotic topology* on $KK^\pi_{\mathcal{P}}(A, B)$ is the topology generated by the subsets $V([p]; X, \varepsilon)$ of $KK^\pi_{\mathcal{P}}(A, B)$ as X ranges over finite subsets of A_1, ε over $(0, \infty)$, and p over $\mathcal{P}^\pi(A, B)$.

Lemma 5.5. *Let $\pi: A \to \mathcal{L}(E)$ be a graded representation of A on a Hilbert B-module. For any $[p] \in KK^\pi_{\mathcal{P}}(A, B)$, the collection of sets $V([p]; X, \varepsilon)$ as X ranges over finite subsets of A_1 and ε over $(0, \infty)$ form a neighborhood base of $[p]$. Moreover, the asymptotic topology is first countable.*

Proof. Using Lemma 5.3, the asymptotic topology on $KK^\pi_{\mathcal{P}}(A, B)$ is the quotient topology induced by the canonical surjection $\mathcal{P}^\pi(A, B) \to KK^\pi_{\mathcal{P}}(A, B)$, where $\mathcal{P}^\pi(A, B)$ is equipped with the topology generated by the sets $U(p; X, \varepsilon)$; moreover, the quotient map is open. It thus suffices to show that the family

$$\big\{U(p; X, \varepsilon) \mid X \subseteq A_1 \text{ finite}, \varepsilon > 0\big\}$$

form a neighborhood basis of $p \in \mathcal{P}^\pi(A, B)$, and that this topology on $\mathcal{P}^\pi(A, B)$ is first countable.

For the neighborhood base claim, we must show that whenever $q_1, \dots, q_n$, $X_1, \dots, X_n$ and $\varepsilon_1, \dots, \varepsilon_n$ are such that $p \in \bigcap_{i=1}^n U(q_i; X_i, \varepsilon_i)$, then there exist X, ε with

$$U(p; X, \varepsilon) \subseteq \bigcap_{i=1}^n U(q_i; X_i, \varepsilon_i).$$

As whenever $Y \supseteq X$ and $\delta \leq \varepsilon$, we have that $U(p; Y, \delta) \subseteq U(p; X, \varepsilon)$, it suffices to prove this for $n = 1$. Assume then we are given $q \in \mathcal{P}^\pi(A, B)$, a finite subset $X \subseteq A_1$, and $\varepsilon > 0$ such that $p \in U(q; X, \varepsilon)$. We claim that $U(p; X, \varepsilon) \subseteq U(q; X, \varepsilon)$, which will suffice to complete the neighborhood base part of the proof. Indeed, say r is in $U(p; X, \varepsilon)$. Then there exists $t_r \geq \max\{\tau_{X,\varepsilon}(p), \tau_{X,\varepsilon}(r)\}$ and a homotopy $(r^s)_{s\in[0,1]}$ passing through $\mathcal{P}^\pi_\varepsilon(X, B)$

connecting p_{t_r} and r_{t_r}. Similarly, there exists $t_q \geq \max\{\tau_{X,\varepsilon}(p), \tau_{X,\varepsilon}(q)\}$ and a homotopy $(q^s)_{s\in[0,1]}$ passing through $\mathcal{P}^{\pi}_{\varepsilon}(X, B)$ connecting q_{t_q} and p_{t_q}. Let I be the closed interval bounded by t_r and t_q. Then concatenating the three paths $(q^s)_{s\in[0,1]}$, $(p_t)_{t\in I}$, and $(r^s)_{s\in[0,1]}$ shows that r is in $U(q; X, \varepsilon)$, so we are done.

We now show first countability. As A is separable, there exists a nested sequence $X_1 \subseteq X_2 \subseteq \cdots$ of finite subsets of the unit ball A_1 with dense union. Fix a point $p \in \mathcal{P}^{\pi}(A, B)$. We claim that the sets $U(p; X_n, 1/n)$ form a neighborhood basis at p. Indeed, given what we have already proved, it suffices to show that for any finite $X \subseteq A_1$ and any $\varepsilon > 0$ there exists n with $U(p; X_n, 1/n) \subseteq U(p; X, \varepsilon)$. Let n be so large so that for all $a \in X$ there is $a' \in X_n$ with $\|a - a'\| < \varepsilon/2$, and also so that $1/n < \varepsilon/2$. From the choice of n, it follows that $\mathcal{P}^{\pi}_{1/n}(X_n, B) \subseteq \mathcal{P}^{\pi}_{\varepsilon}(X, B)$, from which the inclusion $U(p; X_n, 1/n) \subseteq U(p; X, \varepsilon)$ follows. ■

We now recall the canonical topology on $KK(A, B)$, which has been introduced and studied in different pictures by several authors: see for example the discussion in [6] for background and references. Dadarlat[23] showed in [6, Lemma 3.1] that this topology is characterized by the following property, and used this to show that the various different descriptions that had previously appeared in the literature agree.

Proposition 5.6. *Let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ be the one point compactification of the natural numbers, and for each $n \in \overline{\mathbb{N}}$, let $e^n : C(\overline{\mathbb{N}}, B) \to B$ be the $*$-homomorphism defined by evaluation at n. Then the canonical topology on $KK(A, B)$ is characterized by the following conditions.*

(i) *It is first countable.*

(ii) *A sequence (x_n) in $KK(A, B)$ converges to x_∞ in $KK(A, B)$ if and only if there is an element $x \in KK(A, C(\overline{\mathbb{N}}, B))$ such that $e^n_*(x) = x_n$ for all $n \in \overline{\mathbb{N}}$.* ■

Theorem 5.7. *Let $\pi : A \to \mathcal{L}(E)$ be a graded, balanced, and strongly absorbing representation of A on a Hilbert B-module. Then the isomorphism of Theorem 4.14 is a homeomorphism between the asymptotic topology on $KK^{\pi}_{\mathcal{P}}(A, B)$ of Definition 5.4 and the canonical topology on $KK(A, B)$ of Proposition 5.6.*

We need an ancillary lemma.

Lemma 5.8. *Let $\pi : A \to \mathcal{L}(E)$ be a graded representation of A on a Hilbert B-module. For any $\varepsilon > 0$ and any finite $X \subseteq A_1$, if $p, q \in \mathcal{P}^{\pi}_{\varepsilon/2}(X, B)$ satisfy $\|p - q\| < \varepsilon/6$, then there exists a homotopy $(p^s)_{s\in[0,1]}$ connecting p and q and passing through $\mathcal{P}^{\pi}_{\varepsilon}(X, B)$.*

Proof. A straight line homotopy between p and q works. We leave the direct checks to the reader. ■

Proof of Theorem 5.7. We have already seen that the asymptotic topology is first countable in Lemma 5.5. Hence by Proposition 5.6, it suffices to show that sequential convergence in the asymptotic topology is characterized by condition (ii) from Proposition 5.6.

[23] Dadarlat attributes some of the idea here to unpublished work of Pimsner.

Assume first that $([p^n])_{n\in\overline{\mathbb{N}}}$ is a collection of elements of $KK^{\pi}_{\mathcal{P}}(A,B)$. Let $1\otimes\pi$ be the amplification of π to the Hilbert $C(\overline{\mathbb{N}})\otimes B$-module $C(\overline{\mathbb{N}})\otimes E$, and let $q\in\mathcal{P}^{1\otimes\pi}(A,C(\overline{\mathbb{N}},B))$ be such that for all $n\in\overline{\mathbb{N}}$ we have $e^n_*[q]=[p^n]$. We want to show that the sequence $([p^n])_{n\in\mathbb{N}}$ converges to $[p^\infty]$ in the asymptotic topology. For this, it follows from Lemmas 5.3 and 5.5 that it suffices to fix a finite subset X of A_1 and $\varepsilon>0$, and show that p^n is in $U(p^\infty;X,\varepsilon)$ for all suitably large n.

Recall the function $\tau_{X,\varepsilon}$ of Definition 5.2. As q is an element of $\mathcal{P}^{1\otimes\pi}(A,C(\overline{\mathbb{N}},B))$, the number $\tau:=\sup_{n\in\overline{\mathbb{N}}}\tau_{X,\varepsilon/2}(q^n)$ is finite. As q is in $\mathcal{P}^{1\otimes\pi}(A,C(\overline{\mathbb{N}},B))$ we also see from Lemma 4.4 that there exists N such that $\|q^n_\tau-q^\infty_\tau\|<\varepsilon/6$ for all $n\geq N$. We claim that p^n is in $U(p^\infty;X,\varepsilon)$ for all $n\geq N$, which will complete the first half of the proof.

Using Lemma 4.4, we may identify q with a collection $(q^n)_{n\in\overline{\mathbb{N}}}$ of elements of $\mathcal{P}^\pi(A,B)$ (satisfying certain conditions). Now let $n\geq N$ and consider the following homotopies.

(i) As $e^\infty_*[q]=[p^\infty]$, Theorem 4.14 implies that $q^\infty\sim p^\infty$, and thus there is $t_\infty\geq\max\{\tau,\tau_{X,\varepsilon}(p^\infty)\}$ and a homotopy passing through $\mathcal{P}^\pi_\varepsilon(X,B)$ and connecting $p^\infty_{t_\infty}$ and $q^\infty_{t_\infty}$.

(ii) Similarly to (i), there is $t_n\geq\max\{\tau,\tau_{X,\varepsilon}(p^n)\}$ and a homotopy passing through $\mathcal{P}^\pi_\varepsilon(X,B)$ and connecting $p^n_{t_n}$ and $q^n_{t_n}$.

(iii) As $\|q^n_\tau-q^\infty_\tau\|<\varepsilon/6$ for all $n\geq N$ and as $\tau=\sup_{n\in\overline{\mathbb{N}}}\tau_{X,\varepsilon/2}(q^n)$, Lemma 5.8 gives a homotopy passing through $\mathcal{P}^\pi_\varepsilon(X,B)$ and connecting q^∞_τ and q^n_τ.

(iv) The path $(q^n_t)_{t\in[\tau,t_n]}$ is a homotopy passing through $\mathcal{P}^\pi_\varepsilon(X,B)$ that connects q^n_τ and $q^n_{t_n}$.

(v) The path $(q^\infty_t)_{t\in[\tau,t_\infty]}$ is a homotopy passing through $\mathcal{P}^\pi_\varepsilon(X,B)$ that connects q^∞_τ and $q^\infty_{t_\infty}$.

Now let $t_{\max}=\max\{t_n,t_\infty\}$. Concatenating the five homotopies above with the homotopies $(p^n_t)_{t\in[t_n,t_{\max}]}$ and $(p^\infty_t)_{t\in[t_\infty,t_{\max}]}$ (which pass through $P^\pi_\varepsilon(X,B)$) shows that p^n is in $U(p^\infty;X,\varepsilon)$ for $n\geq N$.

For the converse, fix a sequence $X_1\subseteq X_2\subseteq\cdots$ of nested finite subsets of A_1 with dense union. Let us assume that $([p^n])_{n\in\mathbb{N}}$ is a sequence in $KK^\pi_{\mathcal{P}}(A,B)$ that converges to $[p^\infty]$ in the asymptotic topology. We want to construct an element $q\in\mathcal{P}^{1\otimes\pi}(A,C(\overline{\mathbb{N}},B))$ such that $e^n_*[q]=[p^n]$ for each n in $\overline{\mathbb{N}}$. We will define new representatives of the classes $[p^n]$ as follows. For each m, let $N_m\in\mathbb{N}$ be the smallest natural number such that p^n is in $U(p^\infty;X_m,1/m)$ for all $n\geq N_m$; as $[p^n]$ converges to $[p^\infty]$ in the asymptotic topology, such an N_m exists, and the sequence $N_1,N_2,\ldots$ is non-decreasing.

Choose a sequence $t_1\leq t_2\leq\cdots$ in $[1,\infty)$ that tends to infinity in the following way. For $n<N_1$, let $t_n=1$. Otherwise, let m be as large as possible so that $n\geq N_m$. Let $t_n=\max\{\tau_{X_m,1/m}(p^n),\tau_{X_m,1/m}(p^\infty),t_1,\ldots,t_{n-1}\}+1$, and note the choice of N_m implies that $p^n\in U(p^\infty;X_m,1/m)$, so there exists a homotopy between $p^n_{t_n}$ and $p^\infty_{t_n}$ parametrized as usual by $[0,1]$ that passes through $\mathcal{P}^\pi_{1/m}(X_m,B)$. Approximating this homotopy by a piecewise-linear homotopy, we may assume that it is Lipschitz, and still passing through $\mathcal{P}^\pi_{1/m}(X_m,B)$. Moreover, by lengthening the interval parametrizing the homotopy, we

may assume that it is 1-Lipschitz. In conclusion, for some suitably large $r_n \in [1,\infty)$, we may assume that we have a 1-Lipschitz homotopy $(p^{n,t})_{t\in[t_n,t_n+r_n]}$ between $p^n_{t_n}$ and $p^\infty_{t_n}$. Define for each $n \in \mathbb{N}$

$$q^n := \begin{cases} p^\infty_t & t \in [1,t_n], \\ p^{n,t} & t \in [t_n, t_n + r_n], \\ p^n_t & t \geq t_n + r_n, \end{cases}$$

and note that $[q^n] = [p^n]$ for all $n \in \mathbb{N}$ using Lemma 4.6. Define $q^\infty = p^\infty$. Using the characterization of Lemma 4.4, one checks directly that $q = (q^n)_{n\in\bar{\mathbb{N}}}$ defines an element of $\mathcal{P}^{1\otimes\pi}(A, C(\bar{\mathbb{N}}, B))$. This element satisfies $e^n_*[q] = [p^n]$ by construction, so we are done. ∎

6. Controlled KK-theory and KL-theory

Throughout this section, A and B refer to separable C^*-algebras. All Hilbert modules are countably generated, and all are over B unless explicitly stated otherwise. All representations of A are on Hilbert B-modules unless explicitly stated otherwise.

Recall from the introduction that, following Dadarlat [6, Section 5], we define the *KL-group of* (A,B) by

$$KL(A,B) := KK(A,B)/\overline{\{0\}}, \tag{6.1}$$

i.e., the quotient of $KK(A,B)$ by the closure of zero for the topology from Proposition 5.6. This makes $KL(A,B)$ into a Hausdorff topological group when equipped with the quotient topology. As already mentioned in the introduction, $KL(A,B)$ was originally introduced by Rørdam [20, Section 5] using a purely algebraic definition that agrees with the intrinsic topological definition above under a UCT assumption on A.

Our goal in this section is to define *controlled K-homology groups* $KK^\pi_\varepsilon(X,B)$, arrange these into an inverse system, and show that the inverse limit of these is canonically isomorphic to $KL(A,B)$; the isomorphism moreover holds on the level of topological groups when the inverse limit is taken in the category of topological groups and each $KK^\pi_\varepsilon(X,B)$ is given the discrete topology.

We start with the controlled KK-theory groups. For the next definition, recall the notion of a graded representation (plus other conditions on representations) from Definition 4.1, and the set $\mathcal{P}^\pi_\varepsilon(X,B)$ from Definition 5.1.

Definition 6.1. Let $\pi\colon A \to \mathcal{L}(E)$ be a graded representation of A on a Hilbert B-module. Let $X \subseteq A_1$ be a finite subset of the unit ball A_1 of A and let $\varepsilon > 0$. Equip $\mathcal{P}^\pi_\varepsilon(X,B)$ with the norm topology it inherits from $\mathcal{L}(E)$, and define $KK^\pi_\varepsilon(X,B) := \pi_0(\mathcal{P}^\pi_\varepsilon(X,B))$ to be the associated set of path components.

Our first goal is to define a group structure on $KK^\pi_\varepsilon(X,B)$. For this, let us assume that the representation (π,E) is graded, balanced, and infinite multiplicity (see Definition 4.1 for terminology), and fix two isometries s_1, s_2 in $\mathcal{B}(\ell^2)$ satisfying the Cuntz relation

$s_1s_1^* + s_2s_2^* = 1$. Using the inclusion $\mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ from line (4.2) in Lemma 4.2, we think of s_1 and s_2 as isometries in $\mathcal{L}(E)$ that commute with the subalgebra $\pi(A)$ and the neutral projection e. We define an operation on $KK_\varepsilon^\pi(X, B)$ by

$$[p] + [q] := [s_1 p s_1^* + s_2 q s_2^*]. \tag{6.2}$$

The following lemma can be proved in exactly the same way as Lemma 4.7: we leave the direct checks involved to the reader.

Lemma 6.2. *With notation as above, the set $KK_\varepsilon^\pi(X, B)$ is a commutative semigroup. The group operation does not depend on the choice of s_1 and s_2.* ■

In order to show that $KK_\varepsilon^\pi(X, B)$ is a monoid, we need an analogue of Lemma 4.9.

Lemma 6.3. *Let $\pi\colon A \to \mathcal{L}(E)$ be a graded, balanced, and infinite multiplicity representation of A on a Hilbert B-module. Let X be a finite subset of A_1, let $\varepsilon > 0$, let p be an element of $\mathcal{P}_\varepsilon^\pi(X, B)$, and let v be an isometry in the canonical copy $\mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ from line (4.2) from Lemma 4.2. Then the formula*

$$vpv^* + (1 - vv^*)e$$

defines an element of $\mathcal{P}_\varepsilon^\pi(X, B)$ in the same path component as p.

Proof. The fact that $vpv^* + (1 - vv^*)e$ is an element of $\mathcal{P}_\varepsilon^\pi(X, B)$ follows from the fact that v commutes with A. We fix $\delta \in (0, 1)$, to be determined in the course of the proof by X, p, and ε. As $p - e$ is in $\mathcal{K}(E)$, there exists an infinite rank projection $r \in \mathcal{B}(\ell^2)$ such that $1 - r$ also has infinite rank, and such that

$$\|(1 - r)(p - e)\| < \delta. \tag{6.3}$$

Note that as r commutes with e, line (6.3) implies that

$$\|[r, p]\| < 2\delta. \tag{6.4}$$

As r is a projection and commutes with elements of A, and as p is a contraction, this implies that for any $a \in X$,

$$\|a((rpr)^2 - rpr)\| \leq \|r[p, r]pr\| + \|ra(p^2 - p)r\| < 2\delta + \max_{a \in X} \|a(p^2 - p)\|. \tag{6.5}$$

Define now $q := rpr + (1 - r)e$, which is a self-adjoint contraction. Note that $q - e = rpr - re = r(p - e)r$, so $q - e$ is in $\mathcal{K}(E)$. We have $q^2 - q = (rpr)^2 - rpr$, and so line (6.5) implies that for all $a \in X$,

$$\|a(q^2 - q)\| < 2\delta + \max_{a \in X} \|a(p^2 - p)\|.$$

Moreover,

$$\|q - p\| = \|rpr - rp + (1 - r)e - (1 - r)p\| \leq \|[r, p]\| + \|(1 - r)(p - e)\| < 3\delta$$

by lines (4.3) and (6.4). Hence as long as δ is suitably small, depending on ε and on

$$\varepsilon - \max_{a\in X} \|a(p^2-p)\|,$$

we see that q is in $\mathcal{P}_\varepsilon^\pi(X,B)$. Moreover, for suitably small δ, we have that the path

$$[0,1] \longmapsto \mathcal{L}(E), \quad s \longmapsto sp + (1-s)q$$

is norm continuous and passes through $\mathcal{P}_\varepsilon^\pi(X,B)$, and so shows that $p \sim q$. Hence it suffices to prove the result with p replaced by q.

Now, let $v \in \mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ be an isometry as in the original statement. Choose a partial isometry $w \in \mathcal{B}(\ell^2)$ such that $ww^* = 1-r$ and $w^*w = 1 - vv^* + v(1-r)v^*$; such exists as the operators appearing on the right-hand sides of these equations are infinite rank projections. Define

$$u := vr + w^* \in \mathcal{B}(\ell^2) \subseteq \mathcal{L}(E).$$

Then one checks that u is unitary, and moreover that $uqu^* = vqv^* + (1-vv^*)e$. Let $(u_s)_{s\in[0,1]}$ be any norm continuous path of unitaries in $\mathcal{B}(\ell^2)$ connecting u to the identity. Then if we write "$r \sim s$" to mean that $r, s \in \mathcal{P}_\varepsilon^\pi(X,B)$ are in the same path component, the homotopy $(u_squ_s^*)_{s\in[0,1]}$ shows that $q \sim vqv^* + (1-vv^*)e$. In conclusion, we have that

$$p \sim q \sim vqv^* + (1-vv^*)e \sim vpv^* + (1-vv^*)e,$$

where the last "$\sim$" follows from the homotopy $(v(sp+(1-s)q)v^* + (1-vv^*)e)_{s\in[0,1]}$. ∎

Corollary 6.4. *Let $\pi\colon A \to \mathcal{L}(E)$ be a graded, balanced, and infinite multiplicity representation of A on a Hilbert B-module. Let X be a finite subset of A_1, and let $\varepsilon > 0$. Then the commutative semigroup $KK_\varepsilon^\pi(X,B)$ is a commutative monoid with identity element $[e]$.*

Proof. This follows from Lemma 6.2, and Lemma 6.3 with $v = s_1$ (whence $1 - vv^* = s_2s_2^*$). ∎

Proposition 6.5. *Let $\pi\colon A \to \mathcal{L}(E)$ be a graded, balanced, and infinite multiplicity representation. Let X be a finite subset of A_1 and let $\varepsilon > 0$. Then the monoid $KK_\varepsilon^\pi(X,B)$ is a group.*

Proof. For simplicity of notation, if $p, q \in \mathcal{P}_\varepsilon^\pi(X,B)$, write $p \sim q$ if p, q are in the same path component, i.e., if they represent the same element of $KK_\varepsilon^\pi(X,B)$.

Let p be an element of $\mathcal{P}_\varepsilon^\pi(X,B)$. According to Lemma 6.2 and Corollary 6.4 it suffices to show that p has an inverse, i.e., to find an element $q \in \mathcal{P}_\varepsilon(X,B)$ such that

$$s_1qs_1^* + s_2ps_2^* \sim e. \tag{6.6}$$

Let $M_2(\mathbb{C})$ be unitally included in $\mathcal{L}(E)$ as in line (4.2) from Lemma 4.2, and let u be the element $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in M_2(\mathbb{C})$, so $ueu = 1-e$. The self-adjoint contraction

$$q := s_1es_1^* + s_2u(1-p)us_2^*$$

then defines an element of $\mathcal{P}_\varepsilon^\pi(X,B)$, and we claim this satisfies the condition in line (6.6) above.

We first define

$$v := s_2 s_1^* s_1^* + s_1 s_1 s_2^* + s_1 s_2 s_2^* s_1^*,$$

which is unitary in $\mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$. Computing,

$$v(s_1 q s_1^* + s_2 p s_2^*)v^* = s_1\big(s_1 p s_1^* + s_2 u(1-p)u s_2^*\big)s_1^* + s_2 e s_2^*. \tag{6.7}$$

On the other hand, Lemma 6.3 (with $v = s_1$) implies that

$$s_1\big(s_1 p s_1^* + s_2 u(1-p)u s_2^*\big)s_1^* + s_2 e s_2^* \sim s_1 p s_1^* + s_2 u(1-p)u s_2^*. \tag{6.8}$$

Moreover, v is connected to the identity in the unitary group of $\mathcal{B}(\ell^2)$; as $\mathcal{B}(\ell^2)$ commutes with A and e, this implies that

$$vrv^* \sim r \quad \text{for any } r \in \mathcal{P}_\varepsilon^\pi(X, B). \tag{6.9}$$

Combining lines (6.7), (6.8), and (6.9) we get that

$$s_1 q s_1^* + s_2 p s_2^* \sim s_1 p s_1^* + s_2 u(1-p)u s_2^*.$$

Comparing this with line (6.6), it thus suffices to show that

$$s_1 p s_1^* + s_2 u(1-p)u s_2^* \sim e. \tag{6.10}$$

We will show this in two steps:

(i) construct a homotopy $(p_t)_{t\in[0,1]}$ between $s_1 p s_1^* + s_2 u(1-p)u s_2^*$ and $s_1 s_1^*$ such that $\|[p_t, a]\| < \varepsilon$ and $\|a(p_t^2 - p_t)\| < \varepsilon$ for all t and all $a \in X$;

(ii) show how to "fix" this homotopy so that it also satisfies $p_t - e \in \mathcal{K}(E)$ for all $t \in [0,1]$.

Let us start on step (i) above. Connect u to the identity through unitary elements of $M_2(\mathbb{C})$. This gives a path, say $(p_t^{(0)})_{t\in[0,1]}$ connecting $s_1 p s_1^* + s_2 u(1-p)u s_2^*$ to $s_1 p s_1^* + s_2(1-p)s_2^*$ and that satisfies $\|[p_t^{(0)}, a]\| < \varepsilon$ and $\|a((p_t^{(0)})^2 - p_t^{(0)})\| < \varepsilon$ for all t and all $a \in X$.

At this point, to simplify notation, let us write elements of $\mathcal{L}(E)$ as 2×2 matrices with respect to the matrix units $e_{ij} := s_i s_j^*$. With this notation,[24] consider the path $(p_t^{(1)})_{t\in[0,\pi/2]}$ defined by

$$p_t^{(1)} := \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1-p \end{pmatrix}\begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}.$$

[24]In more formal notation,

$$\begin{aligned} p_t^{(1)} &= s_1\big(p + \sin^2(t)(1-p)\big)s_1^* + s_1\cos(t)\sin(t)(1-p)s_2^* + s_2\cos(t)\sin(t)(1-p)s_1^* \\ &\quad + s_2\cos^2(t)(1-p)s_2^*. \end{aligned}$$

One computes that

$$(p_t^{(1)})^2 - p_t^{(1)} = \begin{pmatrix} \cos(t)(p^2-p) & 0 \\ 0 & \cos^2(t)(p^2-p) \end{pmatrix},$$

whence $\|a((p_t^{(1)})^2 - p_t^{(1)})\| < \varepsilon$ for all $t \in [0, \pi/2]$ and all $a \in X$. Another computation gives that for any $a \in A$ and $t \in [0, \pi/2]$,

$$[a, p_t^{(1)}] = [a, p] \begin{pmatrix} \cos^2(t) & -\cos(t)\sin(t) \\ -\cos(t)\sin(t) & -\cos^2(t) \end{pmatrix}.$$

The norm of the matrix appearing on the right-hand side is $|\cos(t)|$, and therefore

$$\big\|[a, p_t^{(1)}]\big\| < \varepsilon$$

for all $a \in X$ and all $t \in [0, \pi/2]$.

Concatenating the paths $(p_t^{(0)})_{t\in[0,1]}$ and $(p_t^{(1)})_{t\in[0,\pi/2]}$, and reparametrizing, we get a new path $(p_t)_{t\in[0,1]}$ connecting $s_1 p s_1^* + s_2 u(1-p)us_2^*$ and $s_1 s_1^*$. This completes step (i) above.

For step (ii), let $\varpi\colon \mathcal{L}(E) \to \mathcal{L}(E)/\mathcal{K}(E)$ be the quotient map. With respect to the decomposition in Lemma 4.2, ϖ is injective on the canonical copy of $\mathcal{B}(\mathbb{C}^2 \otimes \ell^2) \subseteq \mathcal{L}(E)$. As $p - e \in \mathcal{K}(E)$ and e is a projection, we see that the path $(\varpi(p_t))_{t\in[0,1]}$ passes through projections in $\mathcal{B}(\mathbb{C}^2 \otimes \ell^2)$, and it connects e and $s_1 s_1^*$. Hence using Lemma 4.8, there exists a continuous path of unitaries $(w_t)_{t\in[0,1]}$ in $\mathcal{B}(\mathbb{C}^2 \otimes \ell^2)$ with $w_0 = 1$ and such that $\varpi(p_t) = w_t \varpi(p_0) w_t^*$ for all t. The path $(w_t^* p_t w_t)_{t\in[0,1]}$ then lies in $\mathcal{P}_\varepsilon^\pi(X, B)$, and connects $s_1 p s_1^* + s_2 u(1-p)us_2^*$ and e. This completes step (ii), and the proof. ∎

We now have that $KK_\varepsilon^\pi(X, B)$ is a group whenever π is graded, balanced, and infinite multiplicity. Under a slightly stronger assumption on the representations involved, the groups $KK_\varepsilon^\pi(X, B)$ are independent of the representation π.

Lemma 6.6. *Let $\pi\colon A \to \mathcal{L}(E)$ and $\sigma\colon A \to \mathcal{L}(F)$ be graded, balanced, and strongly absorbing representations of A on Hilbert modules in the sense of Definition 4.1. Then the groups $KK_\varepsilon^\pi(X, B)$ and $KK_\varepsilon^\sigma(X, B)$ are isomorphic.*

Proof. Analogously to line (4.1) above, we may write

$$(\pi, E) = (1_{\mathbb{C}^2 \otimes \ell^2} \otimes \pi_0, \mathbb{C}^2 \otimes \ell^2 \otimes E_0)$$

where $\pi_0\colon A \to \mathcal{L}(E_0)$ is a strongly absorbing representation of A, and analogously for (σ, F). Let $e^E = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \otimes 1_{\ell^2 \otimes E_0}$ denote the neutral projection for (π, E) as in Definition 4.1, and choose a pair of isometries in $s_1^E, s_2^E \in 1_{\mathbb{C}^2} \otimes \mathcal{B}(\ell^2) \otimes 1_{F_0}$ satisfying the Cuntz relation and implementing the group operation on $KK_\varepsilon^\pi(X, B)$ as in Lemma 6.2. Let e^F be defined similarly, and let s_1^F and s_2^F be chosen compatibly with s_1^E and s_2^E.

Proposition 2.6 gives us a path of unitaries

$$u^0 = (u_t^0)_{t\in[1,\infty)} \in C_b\big([1,\infty), \mathcal{L}(F_0, E_0)\big)$$

such that $(u_t^0)^*\pi_0(a)u_t^0 - \sigma_0(a) \to 0$ as $t \to \infty$, and such that $(u_t^0)^*\pi(a)u_t^0 - \sigma(a) \in \mathcal{K}(F_0)$ for all t. Let $u_t = 1_{\mathbb{C}^2\otimes\ell^2} \otimes u_t^0 \in \mathcal{L}(F, E)$.

Now, let $p \in \mathcal{P}_\varepsilon^\pi(X, B)$. One checks that there exists $t_p \geq 1$ such that for all $t \geq t_p$ suitably large, $u_t^* p u_t \in \mathcal{P}_\varepsilon^\sigma(X, B)$. Provisionally define a group homomorphism by

$$KK_\varepsilon^\pi(X, B) \longrightarrow KK_\varepsilon^\sigma(X, B), \quad [p] \longmapsto [u_t^* p u_t] \text{ for any } t \geq t_p. \tag{6.11}$$

Note that the class $[u_t^* p u_t]$ does not depend on the choice of t: indeed, if $t, s \geq t_p$ and I is the interval between them, then $(u_t^* p u_t)_{t\in I}$ defines a homotopy between $u_t^* p u_t$ and $u_s^* p u_s$ in $\mathcal{P}_\varepsilon^\sigma(X, B)$. Note also that this map does not depend on the choice of representative p of the class $[p]$. Indeed, if p^0 and p^1 are two such representatives, then there is a homotopy $\mathbf{p} = (p^s)_{s\in[0,1]}$ between them in $\mathcal{P}_\varepsilon^\pi(X, B)$. A compactness argument gives $t_{\mathbf{p}} \geq 1$ such that for all $t \geq t_{\mathbf{p}}$ and all $s \in [0, 1]$, $u_t^* p^s u_t \in \mathcal{P}_\varepsilon^\sigma(X, B)$. Hence for any $t \geq t_{\mathbf{p}}$, $(u_t^* p^s u_t)_{s\in[0,1]}$ defines a homotopy between $u_t^* p^0 u_t$ and $u_t^* p^1 u_t$.

From the discussion above, the map in line (6.11) is well defined. The choice of u_t guarantees that $u_t^* s_i^E u_t = s_i^F$ for $i \in \{0, 1\}$ and all t, and also that $u_t e^E u_t^* = e^F$ for all t. Hence the map in line (6.11) is a group homomorphism. Switching the roles of u_t and u_t^* gives a group homomorphism in the other direction; direct checks show that these homomorphisms are mutually inverse, completing the proof that $KK_\varepsilon^\pi(X, B)$ is isomorphic to $KK_\varepsilon^\sigma(X, B)$. ∎

Given Lemma 6.6, the following definition is reasonable.

Definition 6.7. Let $\pi\colon A \to \mathcal{L}(E)$ be a graded, balanced, and strongly absorbing representation of A on a Hilbert B-module. We call $KK_\varepsilon^\pi(X, B)$ the *controlled KK-theory group* of A associated to X and ε.

Having established that each $KK_\varepsilon^\pi(X, B)$ is a group, we now arrange these groups into an inverse system, and show that the resulting inverse limit agrees with $KL(A, B)$.

Definition 6.8. Let $\mathcal{X}$ be the set of all pairs (X, ε) where X is a finite subset of A_1, and $\varepsilon > 0$. We equip $\mathcal{X}$ with the partial order defined by $(X, \varepsilon) \leq (Y, \delta)$ if for any graded representation $\pi\colon A \to \mathcal{L}(E)$ of A on a Hilbert B-module we have that

$$\mathcal{P}_\delta^\pi(Y, B) \subseteq \mathcal{P}_\varepsilon^\pi(X, B).$$

Remark 6.9. We record some basic properties of the partially ordered set $\mathcal{X}$.

(i) Note that $(X, \varepsilon) \leq (Y, \delta)$ if $X \subseteq Y$ and $\delta \leq \varepsilon$.

(ii) It follows from point (i) that $\mathcal{X}$ is directed: an upper bound for (X_1, ε_1) and (X_2, ε_2) is given by $(X_1 \cup X_2, \min\{\varepsilon_1, \varepsilon_2\})$.

(iii) Recall that a subset S of a directed set I is *cofinal* if for all $i \in I$ there is $s \in S$ with $s \geq i$. The partial order in Definition 6.8 contains a lot more comparable pairs than those arising from the "naive ordering" on the set $\mathcal{X}$ defined by "$(X, \varepsilon) \leq (Y, \delta)$ if $X \subseteq Y$ and $\delta \leq \varepsilon$" as in (i) above. For example, the naive ordering never contains cofinal sequences (even for $A = \mathbb{C}$), while the ordering from Definition 6.8 always does. To see this, let $(a_n)_{n=1}^{\infty}$ be a dense sequence in A_1, and define $X_n := \{a_1, \ldots, a_n\}$. Then the sequence $(X_n, 1/n)_{n=1}^{\infty}$ is cofinal in $\mathcal{X}$ for the ordering from Definition 6.8.

(iv) If A is generated as a C^*-algebra by a finite set $X \subseteq A_1$, then the sequence $(X, 1/n)_{n=1}^{\infty}$ is cofinal in $\mathcal{X}$.

As it is not so widely used in C^*-algebra theory, and to establish conventions, we recall the definition of an inverse limit in the category of topological abelian groups. We will need both a purely algebraic notion and a notion that incorporates a topology: for the purely algebraic notion, just omit all the words in parentheses, and include them all for the version incorporating a topology.

Definition 6.10. Let I be a directed set. An *inverse system* of (topological) abelian groups indexed by I is a collection $(G_i)_{i \in I}$ of (topological) abelian groups, and (continuous) group homomorphisms $\phi_{ij} \colon G_j \to G_i$ for each $j \geq i$ such that $\phi_{ii} = \mathrm{id}_{G_i}$ for each i, and such that $\phi_{ij} \circ \phi_{jk}$ whenever $k \geq j \geq i$.

Given such an inverse system, its *inverse limit* is defined to be

$$\varprojlim G_i := \Big\{ (g_i) \in \prod_{i \in I} G_i \mid \phi_{ij}(g_j) = g_i \text{ for all } j \geq i \Big\},$$

equipped with the group operations (and the topology that it inherits from the product topology on $\prod_{i \in I} G_i$).

We recall some basic facts about inverse limits; we leave the direct checks involved to the reader.

Remark 6.11. The following hold for inverse limits.

(i) The inverse limit $\varprojlim G_i$ of an inverse system (G_i) of (topological) abelian groups has the following universal property. First, note that restricting to the ith coordinate gives a canonical (continuous) homomorphism $\phi_i \colon \varprojlim G_j \to G_i$ for each i. Then the universal property is: for any (topological) abelian group H equipped with a family of (continuous) homomorphisms $\psi_i \colon H \to G_i$ such that the diagrams

$$\begin{tikzcd} & H \arrow[dl, "\psi_j"'] \arrow[dr, "\psi_i"] & \\ G_j \arrow[rr, "\phi_{ij}"] & & G_i \end{tikzcd}$$

commute for each $j \geq i$, there is a unique (continuous) homomorphism $H \to$

$\varprojlim G_i$ making the following diagrams

$$\begin{tikzcd}
& H \arrow[d] \arrow[ddl, bend right, "\psi_j"'] \arrow[ddr, bend left, "\psi_i"] & \\
& \varprojlim G_i \arrow[dl, "\phi_j"'] \arrow[dr, "\phi_i"] & \\
G_j \arrow[rr, "\phi_{ij}"] & & G_i
\end{tikzcd}$$

commute for all $j \geq i$.

(ii) Any cofinal subset J of a directed set I defines the same inverse limit. Precisely, the canonical “restriction” map

$$\prod_{i \in I} G_i \longrightarrow \prod_{i \in J} G_i$$

defined by forgetting all coordinates outside of J restricts to an isomorphism

$$\varprojlim_{i \in I} G_i \longrightarrow \varprojlim_{i \in J} G_i\,,$$

which is also a homeomorphism in the topological setting.

We now come to the inverse limit we are interested in.

Definition 6.12. Assume that $(X, \varepsilon) \leq (Y, \delta)$ as elements of the partially ordered set $\mathcal{X}$ from Definition 6.8. Then for any graded representation $\pi\colon A \to \mathcal{L}(E)$ of A on a Hilbert B-module there is a canonical *forget control* map

$$\varphi^{Y,\delta}_{X,\varepsilon} : KK^{\pi}_{\delta}(Y, B) \longrightarrow KK^{\pi}_{\varepsilon}(X, B) \tag{6.12}$$

induced by the inclusion $\mathcal{P}^{\pi}_{\delta}(Y, B) \to \mathcal{P}^{\pi}_{\varepsilon}(X, B)$.

Assume moreover that π is balanced and infinite multiplicity so that each $KK^{\pi}_{\varepsilon}(X, B)$ is an abelian group, which we equip with the discrete topology. Then the collection

$$\big(KK^{\pi}_{\varepsilon}(X, B)\big)_{(X,\varepsilon)\in\mathcal{X}}$$

becomes an inverse system of topological abelian groups and we define $\varprojlim KK^{\pi}_{\varepsilon}(X, B)$ to be its inverse limit as in Definition 6.10.

Our goal in the remainder of this section is to show that with notation as in line (6.1) and Definition 6.12

$$\varprojlim KK^{\pi}_{\varepsilon}(X, B) \cong KL(A, B)$$

whenever π is graded, balanced, and strongly absorbing as in Definition 4.1.

For the next lemma, recall the notation $\tau_{X,\varepsilon}(p)$ from Definition 5.2 above and the notation $KK^{\pi}_{\mathcal{P}}(A, B)$ from Definition 4.5. It is an abelian group under the assumptions in the Lemma by Theorem 4.14.

Lemma 6.13. *Let $\pi \colon A \to \mathcal{L}(E)$ be a graded, balanced, and strongly absorbing representation of A on a Hilbert B-module. For each (X, ε) in the set $\mathcal{X}$ of Definition 6.8 there is a group homomorphism*

$$\psi_{X,\varepsilon} \colon KK^{\pi}_{\mathcal{P}}(A, B) \longrightarrow KK^{\pi}_{\varepsilon}(X, B)$$

defined by sending $[p]$ to the class of $[p_t]$, where $t = \tau_{X,\varepsilon}(p) + 1$. Moreover, the family of homomorphisms $(\psi_{X,\varepsilon})_{(X,\varepsilon)\in\mathcal{X}}$ are compatible with the forget control maps in line (6.12) above in the sense that the diagrams

$$\begin{array}{ccc} KK^{\pi}_{\mathcal{P}}(A, B) & = & KK^{\pi}_{\mathcal{P}}(A, B) \\ \Big\downarrow{\scriptstyle \psi_{Y,\delta}} & & \Big\downarrow{\scriptstyle \psi_{X,\varepsilon}} \\ KK^{\pi}_{\delta}(Y, B) & \xrightarrow{\varphi^{Y,\delta}_{X,\varepsilon}} & KK^{\pi}_{\varepsilon}(X, B) \end{array}$$

commute. In particular, the maps $\psi_{X,\varepsilon}$ determine a group homomorphism

$$\psi : KK^{\pi}_{\mathcal{P}}(A, B) \longrightarrow \varprojlim KK^{\pi}_{\varepsilon}(X, B).$$

Proof. To see that the map $\psi_{X,\varepsilon}$ is well defined, let $(p^s)_{s\in[0,1]}$ be a homotopy between p^0, p^1 in the set $\mathcal{P}^{\pi}(A, B)$ of paths of projections as in Definition 4.5. Let $t_0 = \tau_{X,\varepsilon}(p^0) + 1$, $t_1 = \tau_{X,\varepsilon}(p^1) + 1$, and choose t_2 such that $t_2 \geq \max\{t_0, t_1\}$, and such that $p^s_{t_2}$ is in $\mathcal{P}^{\pi}_{\varepsilon}(X, B)$ for all s (such a number t_2 exists by compactness of $[0, 1]$). Then concatenating the homotopies $(p^0_t)_{t\in[t_0,t_2]}$, $(p^s_{t_2})_{s\in[0,1]}$, and $(p^1_t)_{t\in[t_1,t_2]}$ connects $p^0_{t_0}$ and $p^1_{t_1}$ in $\mathcal{P}^{\pi}_{\varepsilon}(X, B)$, and we get well-definedness.

To see that $\psi_{X,\varepsilon}$ is a group homomorphism, let $s_1, s_2 \in \mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ be a pair of isometries satisfying the Cuntz relation, and used to define the group operations on both $KK^{\pi}_{\mathcal{P}}(A, B)$ and $KK^{\pi}_{\varepsilon}(X, B)$, and let $[p], [q] \in \mathcal{P}^{\pi}(A, B)$. Then $[p] + [q]$ is the class of $[s_1 p s_1^* + s_2 q s_2^*]$, and we have that

$$\psi_{X,\varepsilon} \colon [s_1 p s_1^* + s_2 q s_2^*] \longmapsto [s_1 p_{t_{p+q}} s_1^* + s_1 q_{t_{p+q}} s_1^*]$$

where $t_{p+q} := \tau_{X,\varepsilon}(s_1 p s_1^* + s_2 q s_2^*) + 1$. On the other hand, if we define $t_p := \tau_{X,\varepsilon}(p) + 1$ and $t_q := \tau_{X,\varepsilon}(q)$, then

$$\psi_{X,\varepsilon}[p] + \psi_{X,\varepsilon}[q] = [s_1 p_{t_p} s_1^* + s_2 q_{t_q} s_2^*].$$

Define $t_{p+q} := \max\{t_p, t_q\}$, and say without loss of generality that $t_p \geq t_q$. Then the path $(s_1 p_{t_p} s_1^* + s_2 q_t s_2^*)_{t\in[t_q,t_p]}$ shows that $\psi_{X,\varepsilon}([p] + [q]) = \psi_{X,\varepsilon}[p] + \psi_{X,\varepsilon}[q]$ as required.

Compatibility of the maps $\psi_{X,\varepsilon}$ with the forget control maps in line (6.12) can be shown via similar arguments; we leave the details to the reader. The existence of ψ follows from this and the universal property of the inverse limit as in Remark 6.11, part (i). ∎

Using the ideas in the previous section, we now get the promised relationship to KL.

Theorem 6.14. *Let $\pi\colon A \to \mathcal{L}(E)$ be a graded, balanced, and strongly absorbing representation on a Hilbert B-module. The homomorphism ψ in Lemma 6.13 is surjective and descends to a homeomorphic isomorphism*

$$\psi\colon KL(A,B) \longrightarrow \varprojlim KK_{\varepsilon}^{\pi}(X,B).$$

Proof. Inspecting Definition 6.10, a neighborhood basis of 0 in $\varprojlim KK_{\varepsilon}^{\pi}(X,B)$ consists of the sets

$$W(X,\varepsilon) := \{([p_{Y,\delta}])_{(Y,\delta)\in\mathcal{X}} \mid [p_{X,\varepsilon}] = 0 \text{ in } KK_{\varepsilon}^{\pi}(X,B)\}$$

as (X,ε) varies over the set $\mathcal{X}$ of Definition 6.8. On the other hand, inspecting Definition 5.4, a neighborhood basis of 0 in $KK_{\mathcal{P}}^{\pi}(A,B)$ consists precisely of the sets $V(0;X,\varepsilon)$ as (X,ε) ranges over $\mathcal{X}$. Direct checks show that these sets are in bijective correspondence via the map ψ. It follows from this that the map

$$\psi\colon KK_{\mathcal{P}}^{\pi}(A,B) \longrightarrow \varprojlim KK_{\varepsilon}^{\pi}(X,B)$$

of Lemma 6.13 descends to an open and continuous injection

$$\psi\colon \frac{KK_{\mathcal{P}}^{\pi}(A,B)}{\overline{\{0\}}} \longrightarrow \varprojlim KK_{\varepsilon}^{\pi}(X,B).$$

The left-hand side identifies with $KL(A,B)$ by Theorem 4.14, Theorem 5.7 and the definition of $KL(A,B)$ as in line (6.1) above, so we have an open continuous injection

$$\psi\colon KL(A,B) \longrightarrow \varprojlim KK_{\varepsilon}^{\pi}(X,B)$$

To see that ψ is a homeomorphic isomorphism, it remains to show that it is surjective.

For this, let us choose a cofinal sequence $(X_n, 1/n)_{n=1}^{\infty}$ of $\mathcal{X}$ as in Remark 6.9 part (iii), whence there is a canonical isomorphism

$$\varprojlim KK_{\varepsilon}^{\pi}(X,B) = \varprojlim KK_{1/n}^{\pi}(X_n,B)$$

as in Remark 6.11 part (ii). Hence it suffices to prove surjectivity of the induced map

$$KK_{\mathcal{P}}^{\pi}(A,B) \longrightarrow \varprojlim KK_{1/n}^{\pi}(X_n,B).$$

As in Definition 6.10 above, let $([p^n])_{n=1}^{\infty}$ define an element of $\varprojlim KK_{1/n}^{\pi}(X_n,B)$ with $p^n \in \mathcal{P}_{1/n}^{\pi}(X_n,B)$ for each n. As this sequence defines an element of the inverse limit we must have that for each n, the forget control map

$$KK_{1/(n+1)}^{\pi}(X_{n+1},B) \longrightarrow KK_{1/n}^{\pi}(X_n,B)$$

sends $[p^{n+1}]$ to $[p^n]$. This implies that there is a homotopy $(p_s^n)_{s\in[0,1]}$ of elements in $\mathcal{P}_{1/n}^{\pi}(X_n,B)$ with $p_0^n = p^n$ and $p_1^n = p^{n+1}$. Define $p\colon [1,\infty) \to \mathcal{L}(E)$ by setting $p_t := p_{t-n}^n$ whenever t is in $[n,n+1]$, and note that p is then an element of $\mathcal{P}^{\pi}(A,B)$.

We claim that $\pi[p] = ([p^n])_{n=1}^{\infty}$, which will complete the proof. Indeed, it suffices to fix n and show that $\psi_{X_n,1/n}[p] = [p^n]$. We have $\psi_{X_n,1/n}[p] = [p_{t_p}]$, where $t_p := \tau_{X_n,1/n}(p)$. By definition of p and of $\tau_{X_n,1/n}$, the interval I with endpoints n and t_p is such that the path $(p_t)_{t\in I}$ lies entirely in $\mathcal{P}^{\pi}_{1/n}(X_n, B)$. Hence

$$\psi_{X_n,1/n}[p] = [p_{t_p}] = [p_n] = [p^n]$$

and we are done. ∎

7. The closure of zero and derived inverse limit groups

Throughout this section, A and B refer to separable C^*-algebras. All Hilbert modules are countably generated, and all are over B unless explicitly stated otherwise. All representations of A are on Hilbert B-modules unless explicitly stated otherwise.

Our goal in this section is to concretely identify the closure of zero in the asymptotic topology on $KK^{\pi}_{\mathcal{P}}(A, B)$ (see Definitions 4.5 and 5.4) in terms of the controlled KK-theory groups $KK^{\pi}_{\varepsilon}(X, B)$ (see Definition 6.7). The precise statement we are aiming for is that the *derived inverse limit group*, or $\varprojlim^1$ *group*,

$$\varprojlim{}^1 KK^{1\otimes\pi}_{\varepsilon}(X, SB) \tag{7.1}$$

of the inverse system from Definition 6.12 is isomorphic to the closure of zero inside $KK^{\pi}_{\mathcal{P}}(A, B)$. This will complete the proof of Theorem 1.1 from the introduction.

The general definition of $\varprojlim^1$ is as follows: see [13, pp. 1–3] for details, and see for example [30, Section 2.5] for general background on the theory of right derived functors being used here. Let I be a directed set, and let $(G_i)_{i\in I}$ be an inverse system of abelian groups indexed by I in the sense of Definition 6.10 (we will not assume the groups are equipped with a topology). The collection Ab^I of all such inverse systems can be arranged into an abelian category, and taking inverse limits defines a left exact functor

$$\varprojlim\colon \mathrm{Ab}^I \longrightarrow \mathrm{Ab}$$

from Ab^I to the category Ab of abelian groups. One can show that Ab^I has enough injective objects so that the right derived functors of $\varprojlim$ make sense. Then by definition $\varprojlim^1$ is the first right derived functor of $\varprojlim$; in particular $\varprojlim^1$ is a functor taking inverse systems of abelian groups indexed by I to abelian groups.

We gave the general definition of $\varprojlim^1$ above for completeness, but will not need to use it in the proofs below. There is a more concrete picture that is available when the index set is $\mathbb{N}$ and that is more useful for computations: see for example [30, Section 3.5] for a detailed exposition of the concrete definition below.

Definition 7.1. Let $(G_n)_{n=1}^{\infty}$ be an inverse system of abelian groups indexed by $\mathbb{N}\setminus\{0\}$ as in Definition 6.10 above, and write $\phi : G_{n+1} \to G_n$ for the connecting maps. Define

$$\delta\colon \prod_{n=1}^{\infty} G_n \longrightarrow \prod_{n=1}^{\infty} G_n, \quad (g_n)_{n=1}^{\infty} \longmapsto \big(g_n - \phi(g_{n+1})\big)_{n=1}^{\infty}. \tag{7.2}$$

The *derived inverse limit group*, or $\varprojlim^1$ *group*, of (G_n), denoted $\varprojlim^1 G_n$ or $\varprojlim^1_{\mathbb{N}} G_n$, is defined to be the cokernel of δ.

We record some basic facts as a lemma. For the statement, recall that a subset J of a directed set I is *cofinal* if for all $i \in I$ there is $j \in J$ with $j \geq i$.

Lemma 7.2. *Let $(G_i)_{i\in I}$ be an inverse system of abelian groups as in Definition 6.10, and let $J \subseteq I$ be a cofinal subset. Then there is a canonical isomorphism*

$$\varprojlim{}^1_I G_i \cong \varprojlim{}^1_J G_j .$$

In particular, if $((X_n, \varepsilon_n))_{n=1}^{\infty}$ is a cofinal subsequence[25] *of the directed set $\mathcal{X}$ of Definition 6.8, and $\pi\colon A \to \mathcal{L}(E)$ is a balanced, graded, infinite multiplicity representation of A on a Hilbert B-module, then there is a canonical isomorphism*

$$\varprojlim{}^1_{\mathcal{X}} KK^{\pi}_{\varepsilon}(X, B) \cong \varprojlim{}^1_{\mathbb{N}} KK^{\pi}_{\varepsilon_n}(X_n, B)$$

where the left-hand side is the $\varprojlim^1$ group over the directed set $\mathcal{X}$ of Definition 6.8, and the right-hand side is the $\varprojlim^1$ group over the sequence $((X_n, \varepsilon_n))_{n=1}^{\infty}$, computed using Definition 7.1 above.

Proof. As pointed out on [13, p. 11], the general statement about cofinal subsets is a consequence of [13, Théorème 1.9 and Lemme 1.5]. The second part is a special case of the first part, combined with the discussion on [13, pp. 13–14] which shows that the general definition of the $\varprojlim^1$ group agrees with the concrete version from Definition 7.1 in situations where both make sense. ∎

We will need an analogue of (a special case of) Lemma 4.4 in the controlled setting; the proof is very similar, and left to the reader. For the statement, recall the notation $\mathcal{P}^{\pi}_{\varepsilon}(X, B)$ from Definition 5.1, and recall that $SB := C_0(0, 1) \otimes B$ denotes the suspension of B.

Lemma 7.3. *Let $\pi\colon A \to \mathcal{L}(E)$ be a graded representation of A on a Hilbert B-module, and let $1 \otimes \pi$ be the amplified representation of A on the SB-module $C_0(0, 1) \otimes E$. For any finite subset X of A_1 and $\varepsilon > 0$, elements of $\mathcal{P}^{1\otimes\pi}_{\varepsilon}(X, SB)$ can be identified with norm continuous functions*

$$p\colon [0, 1] \longrightarrow \mathcal{L}(E), \quad t \longmapsto p_t$$

such that:

(i) $p_0 = p_1 = e$;

(ii) $p_t - e \in \mathcal{K}(E)$ *for all* $t \in [0, 1]$;

(iii) $\|a(p_t^2 - p_t)\| < \varepsilon$ *and* $\|[p_t, a]\| < \varepsilon$ *for all* $a \in X$ *and all* $t \in [0, 1]$. ∎

[25] Cofinal subsequences of $\mathcal{X}$ always exist by Remark 6.9, part (iii).

Proposition 7.4. *Let $\pi: A \to \mathcal{L}(E)$ be a graded, balanced, and infinite multiplicity representation of A on a Hilbert B-module, and let $((X_n, \varepsilon_n))_{n=1}^{\infty}$ be a cofinal subsequence of the directed set $\mathcal{X}$ of Definition 6.8.*

For each n, let p^n be an element of the space $\mathcal{P}_{\varepsilon_n}^{1\otimes\pi}(X_n, SB)$ of Definition 5.1. Use the identification in Lemma 7.3 to consider each p^n as a function $p^n: [0,1] \to \mathcal{L}(E)$, and define $p: [1,\infty) \to \mathcal{L}(E)$ by setting $p_t := p^n_{t-n}$ whenever $t \in [n, n+1]$. Then p is in the space $\mathcal{P}^{\pi}(A, B)$ of Definition 4.3, and the formula

$$\psi : \prod_n KK_{\varepsilon_n}^{1\otimes\pi}(X_n, SB) \longrightarrow KK_{\mathcal{P}}^{\pi}(A, B), \quad \big([p^n]\big)_{n=1}^{\infty} \longmapsto [p]$$

gives a well-defined homomorphism.

Moreover, the homomorphism ψ takes image in the closure $\overline{\{0\}}$ of the zero element for the asymptotic topology (see Definition 5.4), and descends to a well-defined homomorphism

$$\psi: \varprojlim{}^1 KK_{\varepsilon_n}^{1\otimes\pi}(X_n, SB) \longrightarrow KK_{\mathcal{P}}^{\pi}(A, B)$$

on the $\lim^1$*-group.*

Proof. We leave the direct check that p is an element of $\mathcal{P}^{\pi}(A, B)$ to the reader: for this purpose note that $\varepsilon_n \to 0$ as $n \to \infty$, and also that for each n, $p^n(0) = p^n(1) = e$.

To see that ψ is well defined on the product $\prod_n KK_{\varepsilon_n}^{1\otimes\pi}(X_n, SB)$, let $([p^{n,(0)}])_{n=1}^{\infty}$ and $([p^{n,(1)}])_{n=1}^{\infty}$ be sequences in $KK_{\varepsilon_n}^{1\otimes\pi}(X_n, SB)$ representing the same class in the product $\prod_n KK_{\varepsilon_n}^{1\otimes\pi}(X_n, SB)$, and with images $[p^{(0)}]$ and $[p^{(1)}]$ in $KK_{\mathcal{P}}^{\pi}(A, B)$. Then for each n there is a homotopy $(p^{n,(s)})_{s\in[0,1]}$ between $p^{n,(0)}$ and $p^{n,(1)}$. Using the identification in Lemma 4.4, define a new function

$$p: [1,\infty) \longrightarrow \mathcal{L}\big(C[0,1] \otimes E\big)$$

by $p_t^{(s)} := p_{t-n}^{n,s}$ for $t \in [n, n+1]$. Then direct checks using the conditions in Lemma 4.4 show that $(p^{(s)})_{s\in[0,1]}$ is a homotopy between $p^{(0)}$ and $p^{(1)}$, whence $[p^{(0)}] = [p^{(1)}]$ and we have well-definedness.

The fact that ψ is a homomorphism follows as we may assume the group operations are all defined using the same pair of isometries (s_1, s_2) satisfying the Cuntz relation.

We now show that the image of ψ is contained in $\overline{\{0\}}$. Using the definition of the asymptotic topology (see Definition 5.4 above) we must show that if $[p]$ is in the image, then for every finite subset $X \subseteq A_1$ and $\varepsilon > 0$ there is $t \geq \tau_{X,\varepsilon}(p)$ (see Definition 5.2) and a homotopy passing through $\mathcal{P}_{\varepsilon}^{\pi}(X, B)$ connecting p_t to e. Indeed, by construction of p, there is a sequence (t_n) tending to infinity such that $p_{t_n} = e$ for all n. Choose then n such that $t_n \geq \tau_{X,\varepsilon}(p)$, and set $t = t_n$; the constant homotopy connecting p_t to e works.

For the statement that ψ descends to the $\lim^1$ group, we must show that if $([p^n])_{n=1}^{\infty}$ is a sequence in $\prod_n KK_{\varepsilon_n}^{1\otimes\pi}(X_n, SB)$, then the image of $([p^n])$ is the same as that of the sequence $([p^{n+1}])_{n=1}^{\infty}$. Indeed, say the image of the former is p and the image of the latter is q. Then by construction we have that $q_t = p_{t-1}$ for all $t \geq 2$. The path $(p^s)_{s\in[0,1]}$ defined by $p_t^s := p_{t-s}$ is a homotopy between p and q, so we are done. ∎

Our goal in the remainder of this section is to show that ψ as in Proposition 7.4 actually defines an isomorphism between $\varprojlim^1 KK^{1\otimes\pi}_{\varepsilon_n}(X_n, SB)$ and the closure of 0 in $KK^{\pi}_{\mathcal{P}}(A, B)$.

The next definition and lemma give an alternative description of the group operation on $KK^{1\otimes\pi}_{\varepsilon}(X, SB)$.

Definition 7.5. Let $\pi\colon A \to \mathcal{L}(E)$ be a graded representation of A on a Hilbert B-module, let $X \subseteq A_1$ be finite, and let $\varepsilon > 0$. Let $p, q \in \mathcal{P}^{1\otimes\pi}_{\varepsilon}(X, SB)$ be represented by paths as in Corollary 7.3. Define their *concatenation*, denoted $p \cdot q$, to be the path that follows p then q: precisely

$$(p \cdot q)_t := \begin{cases} p_{2t} & 0 \leq t \leq 1/2, \\ q_{2t-1} & 1/2 < t \leq 1. \end{cases}$$

Lemma 7.6. *Let $\pi\colon A \to \mathcal{L}(E)$ be a graded, balanced, and infinite multiplicity representation of A on a Hilbert B-module, let X be a finite subset of A_1, let $\varepsilon > 0$, and let SB be the suspension of B. Then for any $[p], [q] \in KK^{1\otimes\pi}_{\varepsilon}(X, SB)$ we have that $[p] + [q] = [p \cdot q]$. Moreover, $-[p]$ is represented by the path $\bar{p}$ that traverses p in the opposite direction.*

Proof. Up to homotopy, we may assume that $p_t = e$ for all $t \in [1/3, 1]$, and that $q_t = e$ for all $t \in [0, 2/3]$. The sum $[p] + [q]$ is then represented by a function of the form

$$(s_1 p s_1^* + s_2 q s_2^*)_t = \begin{cases} s_1 p_t s_1^* + s_2 e s_2^* & t \in [0, 1/3], \\ e & t \in [1/3, 2/3], \\ s_1 e s_1^* + s_2 q_t s_2^* & t \in [2/3, 1]. \end{cases}$$

Let $v = s_1 s_2^* + s_2 s_1^*$, which is a unitary in $\mathcal{B}(\ell^2)$. We identity $\mathcal{B}(\ell^2)$ with a unital C^*-subalgebra of $\mathcal{L}(E)$ as in line (4.2) from Lemma 4.2. As the unitary group of $\mathcal{B}(\ell^2)$ is connected, there is a path $u = (u_t)_{t\in[0,1]}$ of unitaries in $\mathcal{B}(\ell^2)$ such that $u_t = 1$ for $t \leq 1/3$ and $u_t = v$ for $t \geq 2/3$. We may consider u as an element of the unitary group of $\mathcal{L}(C_0(0,1) \otimes \ell^2)$, which we identify with a unital C^*-subalgebra of $\mathcal{L}(C_0(0,1) \otimes E)$ compatibly with the inclusion $\mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ from line (4.2) again. Using that u commutes with e, we have then that

$$(u(s_1 p s_1^* + s_2 q s_2^*)u^*)_t = \begin{cases} s_1 p_t s_1^* + s_2 e s_2^* & t \in [0, 1/3], \\ e & t \in [1/3, 2/3], \\ s_1 q_t s_1^* + s_2 e s_2^* & t \in [2/3, 1]. \end{cases} \tag{7.3}$$

On the other hand, the unitary group of $\mathcal{L}(C_0(0,1) \otimes \ell^2)$ is connected (even contractible) by [5, Theorem on p. 433], so we may connect u to the identity via some norm continuous path in this unitary group. As the unitary group of $\mathcal{L}(C_0(0,1) \otimes \ell^2)$ commutes with both e and A, this gives a homotopy showing that $u(s_1 p s_1^* + s_2 q s_2^*)u^*$ defines the same element of $KK^{1\otimes\pi}_{\varepsilon}(X, SB)$ as $s_1 p s_1^* + s_2 q s_2^*$. From the description in line (7.3), we have also that $u(s_1 p s_1^* + s_2 q s_2^*)u^*$ and $s_1(p \cdot q)s_1^* + s_2 e s_2^*$ define the same element of $KK_\varepsilon^{1\otimes\pi}(X, SB)$. The latter element is homotopic to $p \cdot q$ by Lemma 6.3, so we are done with the proof that $[p] + [q] = [p \cdot q]$.

The fact that $-[p] = [\bar{p}]$ is a consequence of the first part: indeed, $[p] + [\bar{p}] = [p \cdot \bar{p}]$, and $p \cdot \bar{p}$ is easily seen to be homotopic to the constant path e, which represents the identity in $KK_\varepsilon^{1\otimes\pi}(X, SB)$ by Corollary 6.4. ∎

We need one more technical lemma before we can establish the main result. This is elementary, but a little fiddly: unfortunately, we could not find a more conceptual proof of what we need. For the proof of the lemma, let us explicitly adopt the convention that $\mathbb{N}$ does not contain zero.

Lemma 7.7. *Let $(G_n)_{n=1}^\infty$ be an inverse system of abelian groups indexed by $\mathbb{N}$. For notational simplicity, let us write $\phi\colon G_{n+1} \to G_n$ for the connecting map, and for $m \geq n$ write $\phi^{m-n}\colon G_m \to G_n$ for the connecting map.*[26] *Let $a = (a_n) \in \prod_{n=1}^\infty G_n$, and assume that there exists a sequence $(m_n)_{n=1}^\infty$ in $\mathbb{N}$ and elements $b_n \in G_{m_n}$ with the following properties:*

(i) *for each n, $m_n \in \{1, \dots, n\}$;*

(ii) *$m_1 \leq m_2 \leq \cdots$;*

(iii) *$m_n \to \infty$ as $n \to \infty$;*

(iv) *for each n, there exists $b_n \in G_{m_n}$ such that*

$$\phi^{n-m_n}(a_n) = b_n - \phi^{m_{n+1}-m_n}(b_{n+1}). \tag{7.4}$$

Then (a_n) is in the image of the map δ from line (7.2).

Proof. Given $g \in G_n$, we abuse notation slightly by also writing g for the element of $\prod G_m$ with g in the nth slot, and zero elsewhere. For an element $a = (a_n) \in \prod G_n$, the *support* of a is the set $\{n \in \mathbb{N} \mid a_n \neq 0\}$. For integers $k, l \in \mathbb{N}$, we write $[k, l]$ for the set $\{k, k+1, \dots, l\}$ if $k \leq l$, with the convention that $[k, l] = \varnothing$ if $k > l$; we call such a set $[k, l]$ an *interval*. We also adopt the convention that a sum indexed by the empty set is zero. We say that a possibly infinite formal sum

$$\sum_{i \in I} a^{(i)}$$

of elements of $\prod G_n$ is *locally finite* if each $n \in \mathbb{N}$ appears in the support of only finitely many $a^{(i)}$s, and we note that locally finite sums give well-defined elements of $\prod G_n$.

Write $\mu\colon \mathbb{N} \to \mathbb{N}$ for the map $n \mapsto m_n$. The properties of the sequence (m_n) guarantee that for each $m \in \mathbb{N}$, $\mu^{-1}(m)$ is a (possibly empty) finite interval; moreover, if it is non-empty, then it is of the form $[\alpha(m), \beta(m)]$ with $\alpha(m) \geq m$ as in Figure 1.

[26] This is consistent with standard notation for composition of function: in particular, $\phi^1 = \phi$, and ϕ^0 is the identity map.

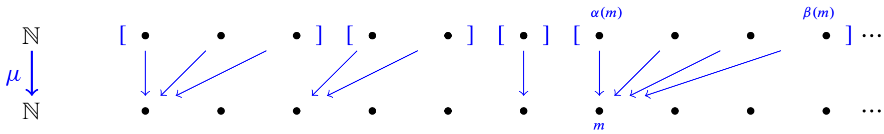

Figure 1. Schematic of the map μ.

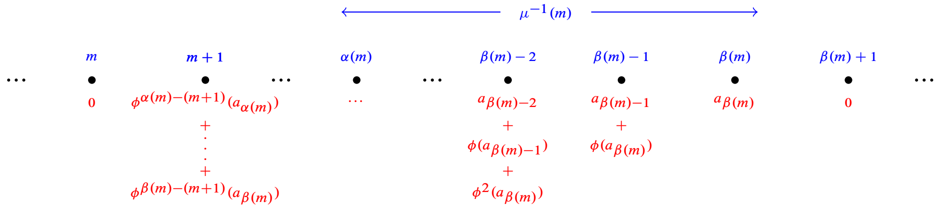

Figure 2. Schematic of the element $d^{(m)}$.

For each $m \in \mathbb{N}$, define

$$c^{(m)} = \sum_{n \in \mu^{-1}(m)} \phi^{n-m}(a_n).$$

In words, $c^{(m)}$ takes all the elements a_n with $m_n = m$, moves them all down to be supported at m, and sums them. Note that each $c^{(m)}$ is supported on the singleton $\{m\}$, or has empty support if $\mu^{-1}(m)$ is empty. Hence the sum $c := \sum_{m \in \mathbb{N}} c^{(m)}$ is locally finite, so c is a well-defined element of $\prod G_n$.

We first claim that the element $a - c$ is contained in the image of the map δ from line (7.2). To see this, for each $m \in \mathbb{N}$, define $d^{(m)} \in \prod G_n$ by the conditions below. First, if $\mu^{-1}(m) = \varnothing$, define $d^{(m)} = 0$. Second, if $m = \alpha(m) = \beta(m)$, define $d^{(m)} = 0$. Third, in all other cases define

$$d_n^{(m)} = \begin{cases} \sum_{i=n}^{\beta(m)} \phi^{i-n}(a_i) & n \in [\alpha(m)+1, \beta(m)], \\ \sum_{i=\alpha(m)}^{\beta(m)} \phi^{i-n}(a_i) & n \in [m+1, \alpha(m)], \\ 0 & \text{otherwise.} \end{cases}$$

The schematic in Figure 2 pictures $d^{(m)}$ in the case that $\mu^{-1}(m) \neq \varnothing$ and $\beta(m) > \alpha(m) > m$; the blue text labels each integer, and the red text gives the value of $d^{(m)}$ at that point.

One checks that for each m,

$$\delta(d^{(m)}) = \left(\sum_{k=\alpha(m)}^{\beta(m)} a_k \right) - c^{(m)}. \tag{7.5}$$

As $m_n \to \infty$, the sum $d := \sum_{m \in \mathbb{N}} d^{(m)}$ is locally finite and makes sense as an element of $\prod G_n$. The computation in line (7.5) above shows that one has $\delta(d) = a - c$, so $a - c$ is in the image of δ as claimed.

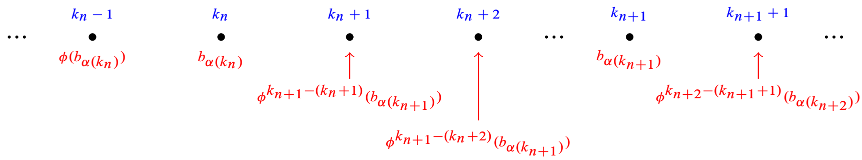

Figure 3. Schematic of the element e.

To complete the proof, it therefore suffices to show that c is in the image of δ. Now, for each m such that $\mu^{-1}(m) \neq \varnothing$,

$$c^{(m)} = \sum_{n \in \mu^{-1}(m)} \phi^{n-m}(a_n) = \sum_{n=\alpha(m)}^{\beta(m)} \phi^{n-m_n}(a_n).$$

Using the assumption in line (7.4), we thus see that

$$c^{(m)} = \sum_{n=\alpha(m)}^{\beta(m)} \big(b_n - \phi^{m_{n+1}-m_n}(b_{n+1})\big). \tag{7.6}$$

For $n \in [\alpha(m), \beta(m) - 1]$, $m_{n+1} = m_n$, and so the sum in line (7.6) above telescopes, leaving just

$$c^{(m)} = b_{\alpha(m)} - \phi^{m_{\beta(m)+1}-m_{\beta(m)}}(b_{\beta(m)+1}). \tag{7.7}$$

Let us write the elements of $\mu(\mathbb{N})$ as $k_1 < k_2 < k_3 < \cdots$. For $m \in \mathbb{N}$, let $k(m)$ be the smallest k_n such that $m \leq k_n$, and define $e \in \prod G_m$ by

$$e_m := \phi^{k(m)-m}(b_{\alpha(k(m))})$$

as in Figure 3; the blue text labels each integer, and the red text gives the value of e_m at that point. Using the formula in line (7.7) and the fact that $\beta(k_n) + 1 = \alpha(k_{n+1})$ one checks that $\delta(e) = c$, so we are done. ∎

We are now ready for the main result of this section. As already commented above, it completes the proof of Theorem 1.1.

Theorem 7.8. *Let $\pi\colon A \to \mathcal{L}(E)$ be a graded, balanced, and strongly absorbing representation of A on a Hilbert B-module, and let $((X_n, \varepsilon_n))_{n=1}^{\infty}$ be a cofinal subsequence of the directed set $\mathcal{X}$ of Definition 6.8. Then the map*

$$\psi\colon \varprojlim{}^1 KK^{1\otimes\pi}_{\varepsilon_n}(X_n, SB) \longrightarrow KK^{\pi}_{\mathcal{P}}(A, B)$$

from Proposition 7.4 is an isomorphism onto the closure of zero in $KK^{\pi}_{\mathcal{P}}(A, B)$. In particular, we have an isomorphism

$$\varprojlim{}^1 KK^{\pi}_{\varepsilon}(X, SB) \cong \overline{\{0\}}$$

where the $\varprojlim{}^1$ group on the left is taken over the directed set $\mathcal{X}$ of Definition 6.8.

Proof. The last statement follows from Lemma 7.2 and the statement that ψ is an isomorphism. We focus then on the statement that ψ is an isomorphism.

To see that the map ψ is surjective, let $p \in \mathcal{P}^{\pi}(A, B)$ (see Definition 4.3) be an element so that $[p]$ is in the closure of zero. Using the description of neighborhood bases from Lemma 5.5, we may find an increasing sequence (t_n) in $[1, \infty)$ such that $t_n \to \infty$, such that: $t_n \geq \tau_{X_n,\varepsilon_n}(p)$ for all n; such that for each n there is a homotopy $(q^n_s)_{s\in[0,1]}$ such that $q^n_0 = p_{t_n}$ and $q^n_1 = e$; and such that q^n_s is in $\mathcal{P}^{\pi}_{\varepsilon_n}(X_n, B)$ for all s. For each n, build a path $r^n\colon [0, 1] \to \mathcal{L}(E)$ by concatenating the paths $(q^n_{1-s})_{s\in[0,1]}$, $(p_t)_{t\in[t_n,t_{n+1}]}$, and $(q^{n+1}_s)_{s\in[0,1]}$, and reparametrizing to get the domain equal to $[0, 1]$. Note that the path $(r^n_s)_{s\in[0,1]}$ starts and ends at e, and has image contained in $\mathcal{P}^{\pi}_{\varepsilon_n}(X_n, B)$. One checks directly that r^n lies in $\mathcal{P}^{1\otimes\pi}_{\varepsilon_n}(X_n, SB)$ using the conditions in Corollary 7.3, and thus we get a class $([r^n]) \in \varprojlim^1 KK^{\pi}_{\varepsilon_n}(X_n, SB)$. We claim the image of $([r^n])$ in $KK^{\pi}_{\mathcal{P}}(A, B)$ is $[p]$.

Indeed, up to reparametrizations (which do not affect the resulting class in $KK^{\pi}_{\mathcal{P}}(A,B)$), the image of $([r^n])$ is represented by concatenating the paths

$$(q^1_{1-s})_{s\in[0,1]},\ (p_t)_{t\in[t_1,t_2]},\ (q^2_s)_{s\in[0,1]},\ (q^2_{1-s})_{s\in[0,1]},$$
$$(p_t)_{t\in[t_2,t_3]},\ (q^3_s)_{s\in[0,1]},\ (q^3_{1-s})_{s\in[0,1]},\ (p_t)_{t\in[t_3,t_4]},\ \dots$$

As each pair $(q^n_s)_{s\in[0,1]}$, $(q^n_{1-s})_{s\in[0,1]}$ consists of the same path traversed in opposite directions, a homotopy removes all these pairs, so we are left with the concatenation of the paths

$$(q^1_{1-s})_{s\in[0,1]},\ (p_t)_{t\in[t_1,t_2]},\ (p_t)_{t\in[t_2,t_3]},\ (p_t)_{t\in[t_3,t_4]},\ \dots$$

or in other words of $(q^1_{1-s})_{s\in[0,1]}$ and $(p_t)_{t\geq t_1}$. As any element $q \in \mathcal{P}^{\pi}(A, B)$ is homotopic to the path defined by $t \mapsto q_{t+L}$ for any fixed $L > 0$, this path is homotopic to the original p and we are done with surjectivity.

For injectivity, let $([p^n])_{n=1}^{\infty}$ be a sequence in $\prod_n KK^{1\otimes\pi}_{\varepsilon_n}(X_n, SB)$ that maps to zero in $KK^{\pi}_{\mathcal{P}}(A, B)$. The image of $([p^n])_{n=1}^{\infty}$ is represented by the path p, which is given by concatenating the functions $p^n\colon [0,1] \to \mathcal{L}(E)$; in symbols, for $t \in [n, n+1]$, $p_t := p^n_{t-n}$. As $([p^n])$ maps to zero, there is a homotopy $(\mathbf{p}^s)_{s\in[0,1]}$ connecting p to e. For each n, let $(q^n_s)_{s\in[0,1]}$ be the path in $\mathcal{L}(E)$ defined by $q^n_s := p^s_n$. Schematically, we have the rectangle in Figure 4: the red arrow at the top is p, built from stringing together the p^n (each of which starts and ends at e); at the bottom of the rectangle is the constant path with value e; the vertical blue arrows are the paths q^n; the whole rectangle represents the homotopy $(\mathbf{p}^s_t)_{s\in[0,1],t\in[1,\infty)}$; the green square will be used later. For each n, let $m_n \in \{1, \dots, n\}$ be as large as possible subject to the condition that the elements $(\mathbf{p}^s_t)_{s\in[0,1],t\in[n,\infty)}$ are all in $\mathcal{P}^{\pi}_{\varepsilon_{m(n)}}(X_{m_n}, B)$. Note that $m_1 \leq m_2 \leq \cdots$, and that $m_n \to \infty$ as $n \to \infty$ by definition of a homotopy. Note moreover that q^n_s is in $\mathcal{P}^{\pi}_{\varepsilon_{m_n}}(X_{m_n}, B)$ for all n and s.

Now, for each n, consider $-[q^n]+[p^n]+[q^{n+1}]$; this is in $KK^{\pi}_{\varepsilon_{m_n}}(X_{m_n}, SB)$ by choice of m_n. This element is represented by the concatenation $\overline{q^n} \cdot p^n \cdot q^{n+1}$ by Lemma 7.6, so it forms three sides of the green square $(\mathbf{p}^s_t)_{s\in[0,1],\,t\in[n,n+1]}$ from Figure 4. The fourth side is the constant function with value e, so $-[q^n] + [p^n] + [q^{n+1}] = [e]$ in $KK^{1\otimes\pi}_{\varepsilon_{m_n}}(X_{m_n}, SB)$.

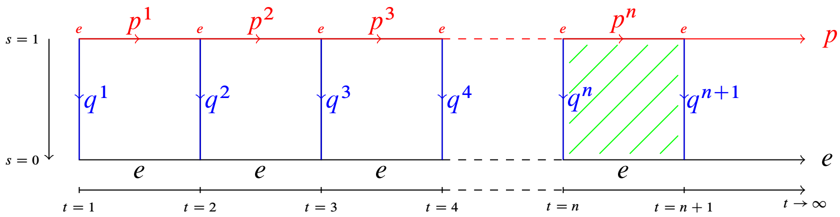

Figure 4. Schematic of ps and qs.

Moreover, $[e]=0$ by Corollary 6.4, so

$$[p^n]=[q^n]-[q^{n+1}] \quad \text{in } KK^{1\otimes\pi}_{\varepsilon_{m_n}}(X_{m_n}, SB) \tag{7.8}$$

for all n; here we abuse notation slightly, and use the symbol $[p^n]$ for both the original element of $KK^{1\otimes\pi}_{\varepsilon_n}(X_n, SB)$ and its image under the forgetful map

$$KK^{1\otimes\pi}_{\varepsilon_n}(X_n, SB) \longrightarrow KK^{1\otimes\pi}_{\varepsilon_{m_n}}(X_{m_n}, SB),$$

and similarly for the $[q^n]$.

We are now in the situation of the purely algebraic Lemma 7.7, with $([p^n])$ playing the role of a, and the $[q^n]$s playing the role of the b_ns. Hence $([p^{(n)}])$ is in the image of the map δ from line (7.2), which is exactly what we wanted to show. ∎

We conclude this section with a corollary on the general structure of the KK groups; this is not connected to the rest of the paper, but seems of interest in itself. We thank Claude Schochet for pointing it out to us.

Corollary 7.9. *For any separable C^*-algebras A and B, the closure of $\{0\}$ in $KK(A,B)$ is either $\{0\}$ or uncountable.*

Proof. Let $\pi\colon A\to\mathcal{L}(E)$ be a unitally strongly absorbing representation of A on a Hilbert B-module. Standard separability arguments show that for each $\varepsilon>0$ and finite $X\subseteq A_1$, the group $KK^{1\otimes\pi}_{\varepsilon}(X, SB)$ is countable. It follows from an argument of Gray [11, p. 242] that a $\lim^1$ group associated to a sequence of countable groups is either zero or uncountable. Hence $\varprojlim^1 KK^{\pi}_{\varepsilon}(X, SB)$ is either zero or uncountable (here we also use Lemma 7.2 to compute this group using a cofinal sequence in $\mathcal{X}$). The corollary now follows from Theorem 7.8. ∎

A. Alternative cycles for controlled KK-theory

Throughout this appendix, A and B refer to separable C^*-algebras. All Hilbert modules are countably generated, and all are over B unless explicitly stated otherwise. All representations of A are on Hilbert B-modules unless explicitly stated otherwise.

In this appendix, we discuss some technical variants of the groups $KK^\pi_\varepsilon(X,B)$ that will be useful in sequels to this work. In each case, we will define new controlled KK-theory groups, and show that these do not affect the resulting inverse limit or $\lim^1$ groups. The following purely algebraic lemma will be a useful technical tool.

Lemma A.1. *Let (G_n) and (H_n) be inverse sequences of abelian groups indexed by $\mathbb{N}$. Assume moreover that for each n there are $\phi(n)\in\mathbb{N}$ and a homomorphism $\Phi\colon G_n\to H_{\phi(n)}$, and also $\psi(n)\in\mathbb{N}$ and a homomorphism $\Psi\colon H_n\to G_{\psi(n)}$, with the following properties:*

(i) *for each n, there exists $m\geq\psi(n)$ such that $\phi(m)\geq n$, and such that the diagram*

$$\begin{tikzcd} G_m \arrow[rr] \arrow[dr, "\Phi"'] & & G_{\psi(n)} \\ & H_{\phi(m)} \arrow[r] & H_n \arrow[u, "\Psi"'] \end{tikzcd}$$

commutes (the unlabeled arrows are those from the inverse system);

(ii) *for each m, there exists $k\geq\phi(m)$ such that $\psi(k)\geq m$, and such that the diagram*

$$\begin{tikzcd} & G_{\psi(k)} \arrow[r] & G_m \arrow[dr, "\Phi"] & \\ H_k \arrow[ur, "\Psi"] \arrow[rrr] & & & H_{\phi(m)} \end{tikzcd}$$

commutes (the unlabeled arrows are those from the inverse system).

Then the maps Φ and Ψ induce mutually inverse isomorphisms

$$\varprojlim G_n\cong\varprojlim H_n \quad \textit{and} \quad \varprojlim{}^1 G_n\cong\varprojlim{}^1 H_n.$$

Moreover, if the inverse systems are in the category of topological abelian groups, then the isomorphism $\varprojlim G_n\cong\varprojlim H_n$ of inverse limits is a homeomorphism.

Proof. We inductively construct sequences (n_k) and (m_k) tending to infinity in $\mathbb{N}$, and a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & G_{n_3} \arrow[rr] \arrow[dr, "\Phi'"'] & & G_{n_2} \arrow[rr] \arrow[dr, "\Phi'"'] & & G_{n_1} \arrow[dr, "\Phi'"'] & \\ & \cdots \arrow[r] & H_{m_3} \arrow[ur, "\Psi'"] \arrow[rr] & & H_{m_2} \arrow[ur, "\Psi'"] \arrow[rr] & & H_{m_1} \end{tikzcd} \tag{A.1}$$

where: the horizontal arrows are the maps from the original inverse systems; the down-right diagonal arrows Φ' are compositions of the maps Φ and the maps from the original inverse system; and the up-right diagonal arrows Ψ' are compositions of the maps Ψ and the maps from the original inverse system. Let δ be as in Definition 7.1 for either of the inverse systems $(G_{n_k})_{k=1}^{\infty}$ and $(H_{m_k})_{k=1}^{\infty}$. We then get a commutative diagram of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \varprojlim G_{n_k} \arrow[r] & \prod_k G_{n_k} \arrow[r, "\delta"] \arrow[d, shift left] & \prod_k G_{n_k} \arrow[r] \arrow[d, shift left] & \varprojlim^1 G_{n_k} \arrow[r] & 0 \\ 0 \arrow[r] & \varprojlim H_{m_k} \arrow[r] & \prod_k H_{m_k} \arrow[r, "\delta"] \arrow[u, shift left] & \prod_k H_{m_k} \arrow[r] \arrow[u, shift left] & \varprojlim^1 H_{m_k} \arrow[r] & 0 \end{tikzcd} \tag{A.2}$$

where the vertical arrows are the homomorphisms induced on products by the diagonal arrows in line (A.1): precisely, the downwards arrows are both

$$(\cdots, g_{n_3}, g_{n_2}, g_{n_1}) \longmapsto \big(\cdots, \Phi'(g_{n_3}), \Phi'(g_{n_2}), \Phi'(g_{n_1})\big)$$

and the upwards arrows are both

$$(\cdots, h_{n_3}, h_{n_2}, h_{n_1}) \longmapsto \big(\cdots, \Psi'(h_{n_3}), \Psi'(h_{n_2})\big).$$

It follows from commutativity of the diagram in line (A.1) that the vertical maps in line (A.2) induce mutually inverse isomorphisms $\varprojlim G_{n_k} \cong \varprojlim H_{m_k}$ and $\varprojlim^1 G_{n_k} \cong \varprojlim^1 H_{m_k}$; moreover, the first of these is a homeomorphism if we are in the topological category. However both the inverse limit and $\lim^1$ functors are insensitive to passing to (cofinal) subsequences (see Remark 6.11, part (ii) and Lemma 7.2 above), so this completes the proof. ∎

A.1. Controlled KK-groups in the unital case

In this subsection, we specialize to the case that A is unital. The main goal is to show that one can use honest projections to define the controlled KK-theory groups rather than just elements satisfying "$\|a(p^2 - p)\| < \varepsilon$" (for suitable $a \in A$ and $\varepsilon > 0$) as in Definition 5.1.

Definition A.2. Let $\pi: A \to \mathcal{L}(E)$ be a graded representation of A on a Hilbert B-module. Let $e \in \mathcal{L}(E)$ be the associated neutral projection as in Definition 4.1. Let X be a finite subset of the unit ball A_1 of A, and let $\varepsilon > 0$. Define $\mathcal{P}_{\varepsilon}^{\pi,u}(X, B)$[27] to be the set of projections in $\mathcal{L}(E)$ satisfying the following conditions:

(i) $p - e$ is in $\mathcal{K}(E)$;

(ii) $\|[p, a]\| < \varepsilon$ for all $a \in X$.

Equip $\mathcal{P}_{\varepsilon}^{\pi,u}(X, B)$ with the norm topology it inherits from $\mathcal{L}(E)$, and define

$$KK_{\varepsilon}^{\pi,u}(X, B) := \pi_0\big(\mathcal{P}_{\varepsilon}^{\pi,u}(X, B)\big),$$

i.e., $KK_{\varepsilon}^{\pi,u}(X, B)$ is the set of path components of $\mathcal{P}_{\varepsilon}^{\pi,u}(X, B)$.

[27] The "u" is for "unital".

Definition A.3. Let $\pi \colon A \to \mathcal{L}(E)$ be a graded, balanced, and infinite multiplicity representation of A on a Hilbert B-module, let X be a finite subset of A_1, let $\varepsilon > 0$, and let $KK_\varepsilon^{\pi,u}(X, B)$ be as in Definition A.2. Let $s_1, s_2 \in \mathcal{B}(\ell^2)$ be a pair of isometries satisfying the Cuntz relation $s_1 s_1^* + s_2 s_2^* = 1$, considered as elements of $\mathcal{L}(E)$ via the inclusion $\mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ from line (4.2) of Lemma 4.2.

Define a binary operation on $KK_\varepsilon^{\pi,u}(X, B)$ by

$$[p] + [q] := [s_1 p s_1^* + s_2 q s_2^*] \tag{A.3}$$

(it is clear that this definition respects path components, so really does define an operation on $KK_\varepsilon^{\pi,u}(X, B)$).

To show that $KK_\varepsilon^{\pi,u}(X, B)$ is a group, we will need an analogue of Lemma 6.3.

Lemma A.4. *Fix notation as in Definition A.3. Let $e \in \mathcal{L}(E)$ be the neutral projection, let p be an element of the set $\mathcal{P}_\varepsilon^{\pi,u}(X, B)$ from Definition A.2, and let v be an isometry in the canonical copy of $\mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ from line (4.2) from Lemma 4.2. Then the formula*

$$vpv^* + (1 - vv^*)e$$

defines an element of $\mathcal{P}_\varepsilon^{\pi,u}(X, B)$ that is in the same path component as p.

Proof. The proof is essentially the same as that of Lemma 6.3, so we just give a brief sketch, pointing out differences where necessary. As in the proof of Lemma 6.3, we fix $\delta > 0$, and choose an infinite rank projection $r \in \mathcal{B}(\ell^2)$ such that $\|(1 - r)(p - e)\| < \delta$. Let $\chi \colon \mathbb{R} \to \{0, 1\}$ be the characteristic function of $(1/2, \infty)$, and define $q := \chi(rpr + (1 - r)e)$, which is an element of $\mathcal{P}_\varepsilon^{\pi,u}(X, B)$ for suitably small δ by the computations in the proof of Lemma 6.3 . Moreover, for δ suitably small, the homotopy

$$[0, 1] \longrightarrow \mathcal{L}(E), \quad s \longmapsto \chi\big(sp + (1 - s)q\big) \tag{A.4}$$

shows that p and q are in the same path component of $\mathcal{P}_\varepsilon^{\pi,u}(X, B)$ (here we use that there is some $\gamma = \gamma(\delta)$ such that $\gamma \to 0$ as $\delta \to 0$, and such that $\|\chi(sp + (1 - s)q) - p\| < \gamma$ for all s). The proof is now finished analogously to that of Lemma 6.3 by considering the element $u := vr + w^* \in \mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ defined just as in that proof, using the element q above in place of the element q from the proof of Lemma 6.3, and using the homotopy in line (A.4) in place of the homotopy $s \mapsto sp + (1 - s)q$ from the proof of Lemma 6.3. ∎

Lemma A.5. *Fix notation as in Definition A.3. Then $KK_\varepsilon^{\pi,u}(X, B)$ is an abelian group, and does not depend on the choice of Cuntz isometries s_1 and s_2.*

Proof. The fact that $KK_\varepsilon^{\pi,u}(X, B)$ is an abelian semigroup with operation not depending on the choice of s_1, s_2 proceeds in exactly the same way as Lemma 4.7. The fact that it is a monoid with identity element $[e]$ follows directly from Lemma A.4 similarly to Corollary 6.4. The proof that inverses exist carries over essentially verbatim from the proof of Proposition 6.5 (with slight simplifications, as estimates of the form "$\|a(p^2 - p)\| < \varepsilon$" no longer need to be checked). ∎

Let now A be a unital C^*-algebra, and let $\pi: A \to \mathcal{L}(E)$ be a representation of A on a Hilbert B-module. We write π_1 for the corestriction of the representation to a representation $\pi_1: A \to \mathcal{L}(1_A \cdot E)$. Note that if π is graded, balanced, and infinite multiplicity (see Definition 4.1), then π_1 is too.

Definition A.6. Let A be unital. Let $\pi: A \to \mathcal{L}(E)$ be a graded, balanced, and infinite multiplicity representation of A on a Hilbert B-module and let π_1 be the corestriction of π to $1_A \cdot E$. Let X be a finite subset of A_1, let $\varepsilon > 0$, and let $\mathcal{P}_\varepsilon^{\pi_1,u}(X, B)$ be as in Definition A.2. Provisionally define

$$\phi: \mathcal{P}_\varepsilon^{\pi_1,u}(X, B) \longrightarrow \mathcal{P}_\varepsilon^{\pi}(X, B), \quad p \longmapsto p + (1 - 1_A)e.$$

Lemma A.7. *The map ϕ from Definition A.6 is well defined, and descends to a homomorphism*

$$\phi_*: KK_\varepsilon^{\pi_1,u}(X, B) \longrightarrow KK_\varepsilon^{\pi}(X, B).$$

Proof. It is straightforward to see that ϕ is a well-defined map that takes homotopies to homotopies and so descends to a well-defined set map

$$\phi_*: KK_\varepsilon^{\pi_1,u}(X, B) \longrightarrow KK_\varepsilon^{\pi}(X, B).$$

Let $s_1, s_2 \in \mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ be Cuntz isometries inducing the group operation on $KK_\varepsilon^{\pi}(X, B)$ as in line (6.2). Define $t_1 := 1_A s_1$ and $t_2 := 1_A s_2$, which are a pair of Cuntz isometries in $\mathcal{L}(1_A E)$ that we may use to define the group operation on $KK_\varepsilon^{\pi_1,u}(X, B)$ as in line (A.3). We compute that for $p, q \in \mathcal{P}_\varepsilon^{\pi_1,u}(X, B)$

$$t_1 p t_1^* + t_2 q t_2^* + (1 - 1_A)e = s_1\big(p + (1 - 1_A)e\big)s_1^* + s_2\big(q + (1 - 1_A)e\big)s_2^*$$

which implies that $\phi_*([p] + [q]) = \phi_*[p] + \phi_*[q]$ as claimed. ∎

Definition A.8. Let A be unital. Let $\pi: A \to \mathcal{L}(E)$ be a graded, balanced, and infinite multiplicity representation of A on a Hilbert B-module and let π_1 be the corestriction of π to $1_A \cdot E$. Let X be a finite subset of A_1, let $\varepsilon > 0$, and let $\mathcal{P}_\varepsilon^{\pi_1,u}(X, B)$ be as in Definition A.2. Assume moreover that $\varepsilon < 1/8$ and that X contains the unit of A. Let χ be the characteristic function of $(1/2, \infty)$. Provisionally define

$$\psi: \mathcal{P}_\varepsilon^{\pi}(X, B) \longrightarrow \mathcal{P}_{4\sqrt{\varepsilon}}^{\pi_1,u}(X, B), \quad p \longmapsto \chi(1_A p 1_A).$$

Lemma A.9. *The map ψ from Definition A.8 is well defined and descends to a well-defined homomorphism*

$$\psi_*: KK_\varepsilon^{\pi}(X, B) \longrightarrow KK_{4\sqrt{\varepsilon}}^{\pi_1,u}(X, B).$$

Proof. First, we check that ψ is well defined, and takes image where we claim. Let p be an element of $\mathcal{P}_\varepsilon^{\pi}(X, B)$. As we are assuming that 1_A is in X, we have that

$$\big\|[p, 1_A]\big\| < \varepsilon. \tag{A.5}$$

Hence

$$\begin{aligned}\left\|(1_A p 1_A)^2-(1_A p 1_A)\right\| &\leq \|1_A p 1_A p-1_A p\|\|1_A\| \\ &\leq \|1_A\|\left\|[1_A, p]\right\|\|p\|+\left\|1_A(p^2-p)\right\| \\ &< 2\varepsilon .\end{aligned}$$

The polynomial spectral mapping theorem thus implies that the spectrum of $1_A p 1_A$ is contained in the $\sqrt{2\varepsilon}$-neighborhood of $\{0,1\}$ in $\mathbb{R}$. As $\varepsilon<1/8$, we have that $\sqrt{2\varepsilon}<1/2$ and so the characteristic function χ of $(1/2, \infty)$ is continuous on the spectrum of $1_A p 1_A$. Hence $\chi(1_A p 1_A)$ makes sense by the continuous functional calculus and moreover

$$\left\|1_A p 1_A-\chi(1_A p 1_A)\right\|<\sqrt{2\varepsilon}. \tag{A.6}$$

Hence we see that for any $a \in X$,

$$\left\|\left[\chi(1_A p 1_A), a\right]\right\| \leq \left\|[\chi(1_A p 1_A)-1_A p 1_A, a]\right\|+\left\|[1_A p 1_A, a]\right\|<2 \sqrt{2\varepsilon}+\varepsilon . \tag{A.7}$$

Putting the discussion so far together, $\chi(1_A p 1_A)$ is a projection in $\mathcal{L}(1_A E)$ such that

$$\left\|\left[\chi(1_A p 1_A), a\right]\right\|<4 \sqrt{\varepsilon} \quad \text{for all } a \in X .$$

We have moreover that $1_A p 1_A-1_A e=1_A(p-e) 1_A \in \mathcal{K}(1_A E)$, whence also $\chi(1_A p 1_A)-1_A e \in \mathcal{K}(1_A E)$. In conclusion, we see that $\chi(1_A p 1_A)$ defines an element of $\mathcal{P}_{4 \sqrt{\varepsilon}}^{\pi_1, u}(X, B)$. We have thus shown that ψ is well defined.

It is straightforward to check that homotopies pass through the above construction, so that ψ induces a well-defined map of sets

$$\psi_*: K K_{\varepsilon}^{\pi}(X, B) \longrightarrow K K_{4 \sqrt{\varepsilon}}^{\pi_1, u}(X, B) .$$

Finally, to see that ψ_* is a homomorphism, we fix Cuntz isometries s_1, s_2 inducing the group operation in $K K_{\varepsilon}^{\pi}(X, B)$. As in the proof of Lemma A.7, we may use the Cuntz isometries $t_1:=1_A s_1$ and $t_2:=1_A s_2$ to define the group operation on $K K_{4 \sqrt{\varepsilon}}^{\pi_1, u}(X, B)$. Using naturality of the functional calculus and the fact that s_1 and s_2 commute with 1_A, we see that for $p, q \in K K_{\varepsilon}^{\pi}(X, B)$ we have that

$$\chi\big(1_A(s_1 p s_1^*+s_2 q s_2^*) 1_A\big)=t_1 \chi(1_A p 1_A) t_1^*+t_2 \chi(1_A q 1_A) t_2^*,$$

and thus that $\psi_*([p]+[q])=\psi_*[p]+\psi_*[q]$, completing the proof. ∎

Lemma A.10. *Fix notation as in Definition A.3. Assume moreover that $\varepsilon<1/8$ and that X contains the unit of A. Consider the diagrams*

$$\begin{tikzcd} K K_{\varepsilon}^{\pi}(X, B) \arrow[dr, "\psi_*"] & \\ K K_{\varepsilon}^{\pi_1, u}(X, B) \arrow[u, "\phi_*"] \arrow[r] & K K_{4 \sqrt{\varepsilon}}^{\pi_1, u}(X, B) \end{tikzcd} \tag{A.8}$$

and

$$\begin{tikzcd}
KK^{\pi}_{\varepsilon}(X,B) \arrow[rr] \arrow[d, "\psi_*"] & & KK^{\pi}_{8\sqrt{\varepsilon}}(X,B) \\
& KK^{\pi}_{4\sqrt{\varepsilon}}(X,B) \arrow[ur] & \\
KK^{\pi_1,u}_{4\sqrt{\varepsilon}}(X,B) \arrow[ur, "\phi_*"] & &
\end{tikzcd} \tag{A.9}$$

where the unlabeled arrows are the canonical forget control maps (compare Definition 6.12*). These both commute.*

Proof. For any $p \in \mathcal{P}^{\pi_1,u}_{\varepsilon}(X,B)$ we have that $\psi(\phi(p)) = p$, and so the diagram in line (A.8) clearly commutes. For the diagram in line (A.9), we need to show that if $p \in \mathcal{P}^{\pi}_{\varepsilon}(X,B)$, then the classes of p and of $\chi(1_A p 1_A) + (1-1_A)e$ in $KK^{\pi}_{8\sqrt{\varepsilon}}(X,B)$ are the same. For this, we concatenate two homotopies. First, consider the homotopy

$$t \longmapsto p_t := \chi(1_A p 1_A) + (1-1_A)\big(te + (1-t)p\big), \quad t \in [0,1].$$

As $ap_t = a\chi(1_A p 1_A)$ for all $a \in A$ and all $t \in [0,1]$, we see that $a(p_t^2 - p_t) = 0$. Moreover, as A commutes with e, as $\|[p,a]\| < \varepsilon$ for all $a \in X$, and as $\|[\chi(1_A p 1_A), a]\| < 4\sqrt{\varepsilon}$ for all $a \in X$, we see that $\|[p_t,a]\| < 4\sqrt{\varepsilon} + \varepsilon < 5\sqrt{\varepsilon}$ for all $a \in X$. Hence this homotopy passes through $\mathcal{P}^{\pi}_{5\sqrt{\varepsilon}}(X,B)$, and connects $\chi(1_A p 1_A) + (1-1_A)e$ and $\chi(1_A p 1_A) + (1-1_A)p$.

For the second homotopy, note first that lines (A.6) and (A.5) imply that

$$\big\|\chi(1_A p 1_A) - 1_A p\big\| \le \big\|\chi(1_A p 1_A) - 1_A p 1_A\big\| + \big\|1_A[1_A, p]\big\| < \sqrt{2\varepsilon} + \varepsilon. \tag{A.10}$$

Consider now the homotopy

$$t \longmapsto q_t := (1-t)\chi(1_A p 1_A) + t1_A p + (1-1_A)p, \quad t \in [0,1]. \tag{A.11}$$

Write $r_t := (1-t)\chi(1_A p 1_A) + t1_A p$, so we have $\|r_t - \chi(1_A p 1_A)\| < \sqrt{2\varepsilon} + \varepsilon$ for all t by line (A.10). Hence for any $a \in A$,

$$\begin{aligned}
\big\|a(q_t^2 - q_t)\big\| &= \big\|a(r_t^2 - r_t)\big\| \\
&\le \big\|r_t\big(r_t - \chi(1_A p 1_A)\big)\big\| + \big\|\big(\chi(1_A p 1_A) - r_t\big)\chi(1_A p 1_A)\big\| + \big\|r_t - \chi(1_A p 1_A)\big\| \\
&< 3(\sqrt{2\varepsilon} + \varepsilon).
\end{aligned}$$

Moreover, for any $a \in X$, lines (A.7) and (A.5) give that

$$\big\|[q_t, a]\big\| \le \big\|\big[\chi(1_A p 1_A), a\big]\big\| + \big\|[1_A p, a]\big\| + \big\|\big[(1-1_A)p, a\big]\big\| \le 4\sqrt{\varepsilon} + 2\varepsilon.$$

Putting all this together, the homotopy $t \mapsto q_t$ from line (A.11) passes through $KK^{\pi}_{8\sqrt{\varepsilon}}(X,B)$. As this homotopy connects $\chi(1_A p 1_A) + (1-1_A)p$ and p, this completes the proof. ∎

We are now in a position to establish the following, which is the main goal of this subsection.

Proposition A.11. *Let A and B be separable C^*-algebras with A unital. Let $\pi\colon A \to \mathcal{L}(E)$ be a graded, balanced, and strongly absorbing representation of A on a Hilbert B-module, and let $\pi_1\colon A \to \mathcal{L}(1_A E)$ denote the associated corestriction. Then with notation as in Definition A.2 above there are isomorphisms*

$$KL(A,B) \longrightarrow \varprojlim KK_{\varepsilon}^{\pi_1,u}(X,B). \tag{A.12}$$

and

$$\varprojlim{}^1 KK_{\varepsilon}^{1\otimes\pi_1,u}(X,SB) \longrightarrow \overline{\{0\}},$$

where the limits are taken over the directed set $\mathcal{X}$ of Definition 6.8 and $\overline{\{0\}}$ is the closure of 0 in $KK(A,B)$. Moreover, the isomorphism in line (A.12) is a homeomorphism when the right-hand side is equipped with the inverse limit topology.

Finally there is a short exact sequence

$$0 \longrightarrow \varprojlim{}^1 KK_{\varepsilon}^{1\otimes\pi_1,u}(X,SB) \longrightarrow KK(A,B) \longrightarrow \varprojlim KK_{\varepsilon}^{\pi_1,u}(X,B) \longrightarrow 0.$$

Proof. Thanks to Theorems 6.14 and 7.8 respectively, it will suffice to show that

$$\varprojlim KK_{\varepsilon}^{\pi_1,u}(X,B) \cong \varprojlim KK_{\varepsilon}^{\pi}(X,B) \tag{A.13}$$

and

$$\varprojlim{}^1 KK_{\varepsilon}^{1\otimes\pi_1,u}(X,SB) \cong \varprojlim{}^1 KK_{\varepsilon}^{\pi}(X,SB). \tag{A.14}$$

We may compute both sides by passing to a cofinal subsequence of pairs (X_n,ε_n) such that each X_n contains 1_A. The result now follows directly from Lemmas A.1 and A.10. ∎

A.2. Unitally absorbing representations

In Proposition A.11 above, we established isomorphisms

$$KL(A,B) \longrightarrow \varprojlim KK_{\varepsilon}^{\pi_1,u}(X,B).$$

and

$$\varprojlim{}^1 KK_{\varepsilon}^{1\otimes\pi_1,u}(X,SB) \longrightarrow \overline{\{0\}},$$

where π is a graded, balanced, and strongly absorbing representation, and π_1 is the associated corestriction to a unital representation. This is a little unnatural, however: it would be better to establish these isomorphisms with π_1 replaced by a unital representation satisfying appropriate assumptions without assuming in advance that it is any sort of corestriction. Our goal in this section is to make this precise.

First, we recall a definition, which is essentially [29, Definition 2.2] (compare also condition (2) from [29, Theorem 2.1]). It should be compared to Definition 2.1 above.

Definition A.12. A unital representation $\pi\colon A \to \mathcal{L}(F)$ of a unital C^*-algebra A on a Hilbert B-module is *unitally absorbing* (for the pair (A, B)) if for any Hilbert B-module E and ucp map $\sigma\colon A \to \mathcal{L}(E)$, there is a sequence (v_n) of isometries in $\mathcal{L}(E, F)$ such that:

(i) $\sigma(a) - v_n^*\pi(a)v_n \in \mathcal{K}(E)$ for all $a \in A$ and $n \in \mathbb{N}$;

(ii) $\|\sigma(a) - v_n^*\pi(a)v_n\| \to 0$ as $n \to \infty$ for all $a \in A$.

The representation π is *strongly unitally absorbing* if it is an infinite amplification of a unitally absorbing representation.

The next lemma, which follows ideas of Kasparov [14] (compare also [29, Theorem 2.1]), says that unitally absorbing representations are essentially unique. It is well known; we give a proof as we could not find the precise statement in the literature.

Lemma A.13. *Let $\pi\colon A \to \mathcal{L}(F)$ and $\sigma\colon A \to \mathcal{L}(E)$ be unitally absorbing representations of a unital C^*-algebra A on Hilbert B-modules. Then there is a sequence (u_n) of unitaries in $\mathcal{L}(E, F)$ such that*

(i) $\sigma(a) - u_n^*\pi(a)u_n \in \mathcal{K}(E)$ *for all $a \in A$ and $n \in \mathbb{N}$;*

(ii) $\|\sigma(a) - u_n^*\pi(a)u_n\| \to 0$ *as $n \to \infty$ for all $a \in A$.*

Proof. Let $(\sigma^\infty, E^\infty)$ be the infinite amplification of (σ, E). As π is unitally absorbing there is a sequence (v_n) of isometries in $\mathcal{L}(E^\infty, F)$ such that $v_n^*\pi(a)v_n - \sigma^\infty(a) \to 0$ for all $a \in A$, and such that $v_n^*\pi(a)v_n - \sigma^\infty(a) \in \mathcal{K}(E^\infty)$ for all $a \in A$ and all n. Analogously to Lemma 2.7, we also have $\pi(a)v_n - v_n\sigma^\infty(a) \in \mathcal{K}(E^\infty, F)$ for all n and all $a \in A$, and that $\|\pi(a)v_n - v_n\sigma^\infty(a)\| \to 0$ as $n \to \infty$ for all $a \in A$.

Now, for representations $\phi\colon A \to \mathcal{L}(G)$ and $\psi\colon A \to \mathcal{L}(H)$ on Hilbert B-modules, let us write $\phi \sim \psi$ if there is a sequence of unitaries (u_n) in $\mathcal{L}(G, H)$ such that $\phi(a) - u_n^*\psi(a)u_n \in \mathcal{K}(G)$ for all $a \in A$ and $n \in \mathbb{N}$ and $\|\phi(a) - u_n^*\psi(a)u_n\| \to 0$ as $n \to \infty$ for all $a \in A$. Let $u_n^F \in \mathcal{L}(F, E \oplus F)$ be the unitary built from v_n as in Lemma 2.8. Then the sequence (u_n^F) in $\mathcal{L}(F, E \oplus F)$ shows that $\pi \sim \sigma \oplus \pi$. As the situation is symmetric in σ and π, we also see that $\sigma \sim \sigma \oplus \pi$. As $\sim$ is transitive, we see that $\pi \sim \sigma$ and are done. ∎

Corollary A.14. *Let A and B be separable C^*-algebras with A unital, and let $\pi\colon A \to \mathcal{L}(E)$ be a unitally absorbing representation. Then $E \cong \ell^2 \otimes B$.*

Proof. Using [29, Theorem 2.4], if A and B are separable with A unital, there always exists a unitally absorbing representation $\pi\colon A \to \mathcal{L}(\ell^2 \otimes B)$. Hence if $\sigma\colon A \to \mathcal{L}(E)$ is any unitally absorbing representation, Lemma A.13 implies that there exists a unitary isomorphism $u\colon E \to \ell^2 \otimes B$. ∎

Remark A.15. The same conclusion as in Corollary A.14 holds if A is not necessarily unital, and $\pi\colon A \to \mathcal{L}(E)$ is absorbing; essentially the same argument works.

We will need a lemma relating unitally absorbing representations to absorbing representations; again this is well known, but we give a proof as we could not find one in the literature.

Lemma A.16. *Let A be unital.*

Let $\pi\colon A \to \mathcal{L}(F)$ be an absorbing representation of A on a Hilbert B-module. Then the corestriction of π_1 of π to a unital representation $\pi_1\colon A \to \mathcal{L}(1_A F)$ is a unitally absorbing representation.

Conversely, if $\pi\colon A \to \mathcal{L}(F)$ is a unitally absorbing representation of A on a Hilbert B-module, then the representation $\pi \oplus 0\colon A \to \mathcal{L}(F \oplus F)$ is absorbing.

Proof. Let $\pi\colon A \to \mathcal{L}(F)$ be absorbing, and let $\sigma\colon A \to \mathcal{L}(E)$ be a ucp map with E a Hilbert B-module. As π is absorbing, there is a sequence (v_n) of isometries in $\mathcal{L}(E, F)$ such that

$$\sigma(a) - v_n^*\pi(a)v_n \in \mathcal{K}(E)$$

for all $a \in A$ and $n \in \mathbb{N}$, and such that

$$\|\sigma(a) - v_n^*\pi(a)v_n\| \longrightarrow 0$$

as $n \to \infty$ for all $a \in A$. As σ is unital we in particular have that $\|1_E - v_n^*\pi(1_A)v_n\| \to 0$ as $n \to \infty$. Set $w_n := \pi(1_A)v_n \in \mathcal{L}(E, 1_A F) \subseteq \mathcal{L}(E, F)$. We compute that

$$w_n^* w_n - 1_E = v_n^*\pi(1_A)v_n - 1_E$$

so $w_n^* w_n$ is a compact perturbation of 1_E, and $\|w_n^* w_n - 1_E\| \to 0$ as $n \to \infty$. Passing to a subsequence, we may assume in particular that $w_n^* w_n$ is invertible for all n. Note then that for all n

$$(w_n^* w_n)^{-1/2} - 1_E \in \mathcal{K}(E), \quad \text{and} \quad (w_n^* w_n)^{-1/2} - 1_E \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{A.15}$$

Define $t_n := w_n(w_n^* w_n)^{-1/2}$. Then (t_n) is a sequence of isometries in $\mathcal{L}(E, 1_A F)$ such that

$$\sigma(a) - t_n^*\pi(a)t_n = \sigma(a) - (w_n^* w_n)^{-1/2} v_n \pi(a) v_n (w_n^* w_n)^{-1/2}$$

for all $a \in A$. This computation combined with line (A.15) shows that (t_n) has the properties needed to show that π_1 is unitally absorbing.

Conversely, say $\pi\colon A \to \mathcal{L}(F)$ is unitally absorbing, and let $\sigma\colon A \to \mathcal{L}(E)$ be a ccp map. As in [2, Proposition 2.2.1], σ extends uniquely to a ucp map $\sigma^+\colon A^+ \to \mathcal{L}(E)$ with domain the unitization A^+ of A. Let $(\pi \oplus 0)^+\colon A^+ \to \mathcal{L}(F \oplus F)$ be the usual unitization of $\pi \oplus 0$, so $(\pi \oplus 0)^+(1_A)$ is the unit of the first copy of F, and $(\pi \oplus 0)^+(1_{A^+} - 1_A)$ is the unit of the second copy.

We claim that $(\pi \oplus 0)^+$ is unitally absorbing as a representation of A^+. Indeed, with respect to the usual isomorphism $A^+ \cong A \oplus \mathbb{C}$ for a unital C^*-algebra A (the copy of $\mathbb{C}$ is generated by $1_{A^+} - 1_A$), $(\pi \oplus 0)^+$ splits as a direct sum of representations $\pi \oplus \tau\colon A \oplus \mathbb{C} \to \mathcal{L}(F) \oplus \mathcal{L}(F)$, where τ is the unital representation of $\mathbb{C}$ on $\mathcal{L}(F)$. The representation τ is unitally absorbing by Kasparov's stabilization theorem [14, Theorem 2] and Corollary A.14 applied to F. On the other hand, a direct sum of unitally absorbing representations is unitally absorbing by the equivalence of (2) and (3) from [29, Theorem 2.1], completing the proof of the claim.

It follows that there is a sequence of isometries $v_n\colon E \to F \oplus F$ such that $\|\sigma^+(a) - v_n^*(\pi \oplus 0)^+(a)v_n\| \to 0$ and $\sigma^+(a) - v_n^*(\pi \oplus 0)^+(a)v_n \in \mathcal{K}(E)$ for all $a \in A^+$. The same holds if we remove the superscripts "+" and quantify over $a \in A$, so we are done. ■

We will need a unital variant of Definition 4.1.

Definition A.17. A representation $\pi\colon A \to \mathcal{L}(E)$ is *graded, balanced, and strongly unitally absorbing* if it comes with a fixed grading $(\pi, E) = (\pi_0 \oplus \pi_0, E_0 \oplus E_0)$ such that (π_0, E_0) is strongly unitally absorbing.

Given this, the following corollary of Lemma A.16 is immediate.

Corollary A.18. *Let A be unital, and let $\pi\colon A \to \mathcal{L}(F)$ be a strongly absorbing representation of A on a Hilbert B-module. Then the corestriction π_1 of π to a unital representation $\pi_1\colon A \to \mathcal{L}(1_A F)$ is a strongly unitally absorbing representation.*

Conversely, if $\pi\colon A \to \mathcal{L}(F)$ is strongly unitally absorbing, then the representation $\pi \oplus 0\colon A \to \mathcal{L}(F \oplus F)$ is strongly absorbing. ■

Our main goal in this section is the following result, which says essentially that any strongly unitally absorbing representation can be used in our Milnor sequence.

Proposition A.19. *Let A and B be separable C^*-algebras with A unital. Then for any graded, balanced, and strongly unitally absorbing representation π of A there are isomorphisms*

$$KL(A, B) \longrightarrow \varprojlim KK_\varepsilon^{\pi,u}(X, B). \tag{A.16}$$

and

$$\varprojlim{}^1 KK_\varepsilon^{1\otimes\pi,u}(X, SB) \longrightarrow \overline{\{0\}},$$

where the limits are taken over the directed set $\mathcal{X}$ of Definition 6.8 and $\overline{\{0\}}$ is the closure of 0 in $KK(A, B)$. Moreover, the isomorphism in line (A.16) is a homeomorphism when the right-hand side is equipped with the inverse limit topology.

Finally there is a short exact sequence

$$0 \longrightarrow \varprojlim{}^1 KK_\varepsilon^{1\otimes\pi,u}(X, SB) \longrightarrow KK(A, B) \longrightarrow \varprojlim KK_\varepsilon^{\pi,u}(X, B) \longrightarrow 0.$$

Proof. Proposition A.11 establishes this in the special case that π_1 is the unital corestriction of a graded, balanced, strongly absorbing representation (this is genuinely a special case by Corollary A.18). Corollary A.18 implies that any graded, balanced, strongly unitally absorbing representation is the unital corestriction of a graded, balanced, strongly absorbing representation, however, so we are done. ■

Remark A.20. Let A be unital, and assume also that at least one of A or B is nuclear. It follows from [14, Theorem 5] that if $\pi\colon A \to \mathcal{B}(\ell^2)$ is a faithful unital representation such that $\pi^{-1}(\mathcal{K}(\ell^2)) = \{0\}$, then the amplification $\pi \otimes 1\colon A \to \mathcal{L}(\ell^2 \otimes B)$ is unitally absorbing. We do not use this remark directly in the body of the paper, but it is needed to justify the simplified picture of controlled KK-theory for nuclear and unital C^*-algebras that we gave in the introduction.

A.3. Matricial representations of controlled KK-groups

In this subsection, we give a formulation of controlled KK-theory in terms of matrices, which is perhaps closer to standard formulations of elementary C^*-algebra K-theory. Although the definitions in the main body of the paper are more convenient for establishing the theory (particularly with regard to the topology on KK), this definition will make computations easier in some subsequent applications [34]. For applications, we are mainly interested in the case where A is unital, so focus on that.

For a representation $\pi\colon A \to \mathcal{L}(E)$ of A on a Hilbert B-module we use the amplifications $1_{M_n} \otimes \pi\colon A \to M_n(\mathcal{L}(E))$ to identify A with a (diagonal) C^*-subalgebra of $M_n(\mathcal{L}(E))$ for all n.

Definition A.21. Let A be unital, and let $\pi\colon A \to \mathcal{L}(E)$ be a unital representation of A on a Hilbert B-module. Let $\mathcal{K}(E)^+$ be the unitization of $\mathcal{K}(E)$.

Let X be a finite subset of A_1, let $\varepsilon > 0$, and let $n \in \mathbb{N}$. Define $\mathcal{P}^{\pi,\mathrm{mx}}_{n,\varepsilon}(X, B)$[28] to be the collection of pairs (p, q) of projections in $M_n(\mathcal{K}(E)^+)$ satisfying the following conditions:

(i) $\|[p, a]\| < \varepsilon$ and $\|[q, a]\| < \varepsilon$ for all $a \in X$;

(ii) the classes $[p], [q] \in K_0(\mathbb{C})$ formed by taking the images of p and q under the canonical quotient map $M_n(\mathcal{K}(E)^+) \to M_n(\mathbb{C})$ are the same.

If (p_1, q_1) is an element of $\mathcal{P}^{\pi,\mathrm{mx}}_{n_1,\varepsilon}(X, B)$ and (p_2, q_2) is an element of $\mathcal{P}^{\pi,\mathrm{mx}}_{n_2,\varepsilon}(X, B)$, define

$$(p_1 \oplus p_2, q_1 \oplus q_2) := \left(\begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}, \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \right) \in \mathcal{P}^{\pi,\mathrm{mx}}_{n_1+n_2,\varepsilon}(X, B).$$

Define

$$\mathcal{P}^{\pi,\mathrm{mx}}_{\infty,\varepsilon}(X, B) := \bigsqcup_{n=1}^{\infty} \mathcal{P}^{\pi,\mathrm{mx}}_{n,\varepsilon}(X, B),$$

i.e., $\mathcal{P}^{\pi,\mathrm{mx}}_{\infty,\varepsilon}(X, B)$ is the disjoint union of all the sets $\mathcal{P}^{\pi,\mathrm{mx}}_{n,\varepsilon}(X, B)$.

Equip each $\mathcal{P}^{\pi,\mathrm{mx}}_{n,\varepsilon}(X, B)$ with the norm topology it inherits from $M_n(\mathcal{L}(E)) \oplus M_n(\mathcal{L}(E))$, and equip $\mathcal{P}^{\pi,\mathrm{mx}}_{\infty,\varepsilon}(X, B)$ with the disjoint union topology. Let $\sim$ be the equivalence relation on $\mathcal{P}^{\pi,\mathrm{mx}}_{\infty,\varepsilon}(X, B)$ generated by the following relations:

(i) $(p, q) \sim (p \oplus r, q \oplus r)$ for any element $(r, r) \in \mathcal{P}^{\pi,\mathrm{mx}}_{\infty,\varepsilon}(X, B)$ with both components the same;

(ii) $(p_1, q_1) \sim (p_2, q_2)$ whenever these elements are in the same path component of $\mathcal{P}^{\pi,\mathrm{mx}}_{\infty,\varepsilon}(X, B)$.[29]

Finally, define $KK^{\pi,\mathrm{mx}}_{\varepsilon}(X, B)$ to be $\mathcal{P}^{\pi,\mathrm{mx}}_{\infty,\varepsilon}(X, B)/\sim$.

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[28] The "mx" is for "matrix".

[29] Equivalently, both are in the same $\mathcal{P}^{\pi,\mathrm{mx}}_{n,\varepsilon}(X, B)$, and are in the same path component of this space.

Lemma A.22. *Let A be unital, let $X \subseteq A_1$ be a finite set, and let $\varepsilon > 0$. If $\pi\colon A \to \mathcal{L}(E)$ is a unital representation of A on a Hilbert B-module, then $KK_\varepsilon^{\pi,\mathrm{mx}}(X, B)$ is an abelian group.*

Proof. It follows directly from the definition that $KK_\varepsilon^{\pi,\mathrm{mx}}(X, B)$ is a monoid with identity element the class $[0, 0]$. A standard rotation homotopy shows that $KK_\varepsilon^{\pi,\mathrm{mx}}(X, B)$ is commutative. To complete the proof, we claim that $[q, p]$ is the inverse of $[p, q]$. Indeed, applying a rotation homotopy to the second variable shows that $(p \oplus q, q \oplus p) \sim (p \oplus q, p \oplus q)$, and the element $(p \oplus q, p \oplus q)$ is trivial by definition of the equivalence relation. ∎

Let $\pi\colon A \to \mathcal{L}(E)$ be a strongly unitally absorbing representation of A on a Hilbert B-module as in Definition A.17. Our goal in this section is to establish isomorphisms

$$KL(A, B) \longrightarrow \varprojlim KK_\varepsilon^{\pi,\mathrm{mx}}(X, B)$$

and

$$\varprojlim{}^1 KK_\varepsilon^{1\otimes\pi,\mathrm{mx}}(X, SB) \longrightarrow \overline{\{0\}}$$

analogously to Propositions A.11 and A.19 above; here the limits are (as usual) taken over the directed set $\mathcal{X}$ of Definition 6.8.

Let then $(\pi, E) = (\pi_0 \oplus \pi_0, E_0 \oplus E_0)$ be a graded, balanced, strongly unitally absorbing representation of A on a Hilbert B-module, so π_0 is a strongly unitally absorbing representation. First, we provisionally define

$$\phi\colon \mathcal{P}_\varepsilon^{\pi,u}(X, B) \longrightarrow \mathcal{P}_{2,\varepsilon}^{\pi_0,\mathrm{mx}}(X), \quad p \longmapsto \left(p, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right), \tag{A.17}$$

where we have used the identification $\mathcal{L}(E) = M_2(\mathcal{L}(E_0))$ to make sense of the right-hand side.

Lemma A.23. *Let $(\pi, E) = (\pi_0 \oplus \pi_0, E_0 \oplus E_0)$ be a graded, balanced, strongly unitally absorbing representation of A on a Hilbert B-module. The map ϕ in line (A.17) above is well defined, and descends to a group homomorphism*

$$\phi_*\colon KK_\varepsilon^{\pi,u}(X, B) \longrightarrow KK_\varepsilon^{\pi_0,\mathrm{mx}}(X, B).$$

Proof. As the neutral projection $e \in \mathcal{L}(E)$ corresponds to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M_2(\mathcal{L}(E_0))$, the image of ϕ is contained in $\mathcal{P}_{2,\varepsilon}^{\pi_0,\mathrm{mx}}(X, B)$. Moreover, ϕ takes homotopies to homotopies, so descends to a well-defined map of sets $\phi_*\colon KK_\varepsilon^{\pi,u}(X, B) \to KK_\varepsilon^{\pi_0,\mathrm{mx}}(X, B)$. It remains to show that this set map is a homomorphism.

For this, let $s_1, s_2 \in \mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ be a pair of Cuntz isometries inducing the operation on $KK_\varepsilon^{\pi,u}(X, B)$ as in Definition A.3. For simplicity of notation, let us write $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M_2(\mathcal{K}(E_0)^+)$. Then for $[p], [q] \in KK_\varepsilon^{\pi,u}(X, B)$, we see that

$$\phi_*\big([p] + [q]\big) = [s_1 p s_1^* + s_2 q s_2^*, e]$$

(the entries on the right should be considered as matrices in $M_2(\mathcal{K}(E_0)^+)$). According to the definition of the equivalence relation defining $KK_\varepsilon^{\pi_0,\mathrm{mx}}(X,B)$, this is the same element as

$$[s_1ps_1^* + s_2qs_2^* \oplus e, e \oplus e].$$

For $t \in [0,\pi/2]$, write

$$u_t := s_1s_1^* \otimes 1_2 + (s_2s_2^* \otimes 1_2)\left(1_{M_2(\mathcal{L}(E_0))} \otimes \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}\right)$$

(here $1_2 \in M_2(\mathbb{C})$, so we are considering each u_t as an element of $\mathcal{L}(E) \otimes M_2(\mathbb{C}) = M_2(\mathcal{L}(E_0)) \otimes M_2(\mathbb{C})$). Consider now the path

$$\big(u_t(s_1ps_1^* + s_2qs_2^* \oplus e)u_t^*, u_t(e \oplus e)u_t^*\big), \quad t \in [0,\pi/2]. \tag{A.18}$$

We have that $u_t(e \oplus e)u_t^* = e \oplus e$ for all t. As

$$\big((s_1ps_1^* + s_2qs_2^*) \oplus e\big) - (e \oplus e)$$

is in $M_4(\mathcal{K}(E_0))$, we thus see that

$$u_t\big((s_1ps_1^* + s_2qs_2^*) \oplus e\big)u_t^* - u_t(e \oplus e)u_t^* = u_t(s_1ps_1^* + s_2qs_2^* \oplus e)u_t^* - e \oplus e.$$

is also in $M_4(\mathcal{K}(E_0))$. It follows from this that $u_t(s_1ps_1^* + s_2qs_2^* \oplus e)u_t^*$ is in $M_4(\mathcal{K}(E_0)^+)$ for all $t \in [0,\pi/2]$, and therefore the path in line (A.18) passes through $\mathcal{P}_{4,\varepsilon}^{\pi_0,\mathrm{mx}}(X,B)$. Therefore we have the identity

$$[s_1ps_1^* + s_2qs_2^* \oplus e, e \oplus e] = [s_1ps_1^* + s_2es_2^* \oplus s_2qs_2^* + s_1es_1^*, e \oplus e]$$

in $KK_\varepsilon^{\pi_0,\mathrm{mx}}(X,B)$. As the left-hand side above is $\phi_*([p]+[q])$ we thus get

$$\phi_*\big([p]+[q]\big) = [s_1ps_1^* + s_2es_2^*, e] + [s_2qs_2^* + s_1es_1^*, e].$$

To complete the proof, it thus suffices to show that $[s_1ps_1^* + s_2es_2^*, e] = \phi_*[p]$ and $[s_2qs_2^* + s_1es_1^*, e] = \phi_*[q]$, i.e., that $[s_1ps_1^* + s_2es_2^*, e] = [p,e]$ and $[s_2qs_2^* + s_1es_1^*, e] = [q,e]$. These identities follow from Lemma A.4 (the first with $v = s_1$ on using the identity $(1 - s_1s_1^*)e = s_2s_2^*e = s_2es_2^*$, and the second similarly with $v = s_2$), which completes the proof. ∎

We now define a map going in the other direction to ϕ; unfortunately, this is more complicated. Recall first that (as throughout the paper) "ℓ^2" is shorthand for $\ell^2(\mathbb{N})$. We write $(\ell^2)^{\oplus n}$ for the direct sum of ℓ^2 with itself n times; of course, this is isomorphic to ℓ^2, but the distinction will help keep track of notation.

Let $(\pi, E) = (\pi_0 \oplus \pi_0, E_0 \oplus E_0)$ be a graded, balanced, strongly unitally absorbing representation of A on a Hilbert B-module E, with π_0 is a strongly unitally absorbing representation on E_0. For each n, fix a unitary isomorphism $u_n : \ell^2 \to (\ell^2)^{\oplus n}$, and let

$v_n := 1_{\mathbb{C}^2} \otimes u_n : \mathbb{C}^2 \otimes \ell^2 \to \mathbb{C}^2 \otimes (\ell^2)^{\oplus n}$. We identity $\mathbb{C}^2 \otimes (\ell^2)^{\oplus n}$ with $(\ell^2)^{\oplus 2n}$ by identifying $\{(1,0)\} \otimes (\ell^2)^{\oplus n}$ (respectively, $\{(0,1)\} \otimes (\ell^2)^{\oplus n}$) with the first (respectively, last) n summands. As in the discussion around Lemma 4.2, we fix an identification of E_0 with $\ell^2 \otimes F$ for some Hilbert module F, and use this to fix an identification of E with $\mathbb{C}^2 \otimes \ell^2 \otimes F$. Using this, identify $\mathcal{B}(\mathbb{C}^2 \otimes \ell^2, (\ell^2)^{\oplus 2n})$ with a subspace of $\mathcal{L}(E, E_0^{\oplus 2n})$, and use this to consider v_n as an element of $\mathcal{L}(E, E_0^{\oplus 2n})$. Up to the canonical identification $\mathcal{L}(E_0^{\oplus 2n}) = M_{2n}(\mathcal{L}(E_0))$, we thus see that $v_n^* M_{2n}(\mathcal{L}(E_0)) v_n = \mathcal{L}(E)$, and that $v_n^* \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) v_n = e$, where the entries of the matrix on the left are understood as $n \times n$ blocks.

Now, let (p, q) be an element of $\mathcal{P}_{n,\varepsilon}^{\pi_0,\mathrm{mx}}(X, B)$ for some n. As the images of p and q under the canonical quotient map $\sigma : M_n(\mathcal{K}(E_0)^+) \to M_n(\mathbb{C})$ are the same in $K_0(M_n(\mathbb{C}))$, there is a unitary $u \in M_n(\mathbb{C})$ such that $\sigma(p) = u\sigma(q)u^*$. Define

$$v = v(q, u, n) := \begin{pmatrix} uqu^* & 1 - uqu^* \\ 1 - uqu^* & uqu^* \end{pmatrix} v_n \in \mathcal{L}(E, E_0^{\oplus 2n}).$$

Provisionally define a map

$$\psi\colon \mathcal{P}_{\infty,\varepsilon}^{\pi_0,\mathrm{mx}}(X, B) \longrightarrow \mathcal{P}_{5\varepsilon}^{\pi,u}(X, B), \quad (p, q) \longmapsto v^* \begin{pmatrix} p & 0 \\ 0 & 1 - uqu^* \end{pmatrix} v. \tag{A.19}$$

Lemma A.24. *The map ψ above is well defined, and descends to a group homomorphism*

$$\psi_*\colon KK_\varepsilon^{\pi_0,\mathrm{mx}}(X, B) \longrightarrow KK_{5\varepsilon}^{\pi,u}(X, B)$$

that does not depend on the choice of u or v_n.

Proof. We first show that the element on the right-hand side of line (A.19) is in $\mathcal{P}_{5\varepsilon}^{\pi,u}(X, B)$. For simplicity of notation, define $q' := uqu^*$, so we have that $p - q'$ is in $M_n(\mathcal{K}(E_0))$. Hence

$$\begin{pmatrix} q' & 1-q' \\ 1-q' & q' \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1-q' \end{pmatrix} \begin{pmatrix} q' & 1-q' \\ 1-q' & q' \end{pmatrix} \\ - \begin{pmatrix} q' & 1-q' \\ 1-q' & q' \end{pmatrix} \begin{pmatrix} q' & 0 \\ 0 & 1-q' \end{pmatrix} \begin{pmatrix} q' & 1-q' \\ 1-q' & q' \end{pmatrix}$$

is in $M_{2n}(\mathcal{K}(E_0))$. Simplifying the second term,

$$\begin{pmatrix} q' & 1-q' \\ 1-q' & q' \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1-q' \end{pmatrix} \begin{pmatrix} q' & 1-q' \\ 1-q' & q' \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is in $M_{2n}(\mathcal{K}(E_0))$. Conjugating by v_n and also recalling the identity $v_n^* \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) v_n = e$, we see that

$$v_n^* \begin{pmatrix} q' & 1-q' \\ 1-q' & q' \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1-q' \end{pmatrix} \begin{pmatrix} q' & 1-q' \\ 1-q' & q' \end{pmatrix} v_n - e$$

is in $M_2(\mathcal{K}(E_0))$. Direct estimates show that the projection

$$\begin{pmatrix} q' & 1-q' \\ 1-q' & q' \end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1-q' \end{pmatrix}\begin{pmatrix} q' & 1-q' \\ 1-q' & q' \end{pmatrix}$$

commutes with elements of X up to error 5ε. At this point, we have that the element on the right-hand side of line (A.19) is indeed in $\mathcal{P}^{\pi,u}_{5\varepsilon}(X,B)$.

We now pass to the set of path components on the right-hand side, so getting a map $\psi_\flat\colon \mathcal{P}^{\pi_0,\mathrm{mx}}_{\infty,\varepsilon}(X,B) \to KK^{\pi,u}_{5\varepsilon}(X,B)$. We will show that this map does not depend on the choices of u or v_n, which will certainly imply the same thing for ψ_* once we show the latter exists. To see that $\psi_\flat$ does not depend on the choice of u such that $\sigma(p) = u\sigma(q)u^*$, note that if $U_n(\mathbb{C})$ is the unitary group of $M_n(\mathbb{C})$, then the collection of all such unitaries is homeomorphic to $\sigma(p)U_n(\mathbb{C})\sigma(q) \times (1-\sigma(p))U_n(\mathbb{C})(1-\sigma(q))$, so path connected. Hence any two such choices give rise to homotopic elements of $\mathcal{P}^{\pi,u}_{5\varepsilon}(X,B)$. One can argue that $\psi_\flat$ does not depend on the choice of v_n similarly: the collection of all possible v_n is path-connected.

We now show that ψ descends to a well-defined map

$$\psi_*\colon KK^{\pi_0,\mathrm{mx}}_{\varepsilon}(X,B) \longrightarrow KK^{\pi,u}_{5\varepsilon}(X,B).$$

First we look at part (ii) of the definition of the equivalence relation defining $KK^{\pi_0,\mathrm{mx}}_{\varepsilon}(X,B)$ from Definition A.21. Let $(p_t,q_t)_{t\in[0,1]}$ be a homotopy in some $\mathcal{P}^{\pi_0,\mathrm{mx}}_{n,\varepsilon}(X,B)$. Using Lemma 4.8 (twice) we may choose a continuous path of unitaries $(u_t)_{t\in[0,1]}$ in $M_n(\mathbb{C})$ such that $\sigma(p_t) = u_t\sigma(q_t)u_t^*$ for all $t\in[0,1]$, and use these to define $\phi_*[p_t,q_t]$ for each t. Having made this choice, ψ takes homotopies to homotopies, so we are done with this part of the equivalence relation.

We now look at part (i) of the equivalence relation from Definition A.21. Let $p,q\in M_n(\mathcal{K}(E_0)^+)$ and $r\in M_k(\mathcal{K}(E_0)^+)$ for some $n,k\in\mathbb{N}$, and let $u\in M_n(\mathbb{C})$ be a unitary such that $\sigma(p) = u\sigma(q)u^*$ in $M_n(\mathbb{C})$. As $(p\oplus r, q\oplus r)$ and $(p\oplus r, uqu^*\oplus r)$ are homotopic, the previous paragraph lets us replace q by uqu^*, and thus assume that $p-q\in M_n(\mathcal{K}(E_0))$. Then ψ sends $(p\oplus r, q\oplus r)$ to

$$v_{n+k}^*\begin{pmatrix} qpq+1-q & 0 & qp(1-q) & 0 \\ 0 & 1 & 0 & 0 \\ (1-q)pq & 0 & (1-q)p(1-q) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} v_{n+k} \tag{A.20}$$

(the odd rows (respectively, columns) have height (resp. width) n, and the even rows (resp. columns) have height (resp. width) k). On the other hand, ψ sends (p,q) to

$$v_n^*\begin{pmatrix} qpq+1-q & qp(1-q) \\ (1-q)pq & (1-q)p(1-q) \end{pmatrix} v_n, \tag{A.21}$$

so we must show that the elements in lines (A.20) and (A.21) define the same class in $KK^{\pi,u}_{5\varepsilon}(X,B)$. Let now $i\colon E_0^{\oplus 2n} \to E_0^{\oplus 2(n+k)}$ be the canonical inclusion given by

$(\xi_1, \xi_2) \mapsto (\xi_1, 0, \xi_2, 0)$, and let $w_n := i v_n \in \mathcal{B}(\mathbb{C}^2 \otimes \ell^2, (\ell^2)^{\oplus 2(n+k)}) \subseteq \mathcal{L}(E, E_0^{2(n+k)})$. Set $t := v_{n+k}^* w_n$, which is an isometry in $1_{\mathbb{C}^2} \otimes \mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$. Looking back at line (A.20), we have that

$$
\begin{aligned}
&v_{n+k}^* \begin{pmatrix} qpq + 1 - q & 0 & qp(1-q) & 0 \\ 0 & 1 & 0 & 0 \\ (1-q)pq & 0 & (1-q)p(1-q) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} v_{n+k} \\
&= v_{n+k}^* \begin{pmatrix} qpq + 1 - q & 0 & qp(1-q) & 0 \\ 0 & 0 & 0 & 0 \\ (1-q)pq & 0 & (1-q)p(1-q) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} v_{n+k} + v_{n+k}^* \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} v_{n+k}.
\end{aligned}
$$

The terms on the left and right above are equal to

$$
t v_n^* \begin{pmatrix} qpq + 1 - q & qp(1-q) \\ (1-q)pq & (1-q)p(1-q) \end{pmatrix} v_n t^* \quad \text{and} \quad (1 - tt^*)e
$$

respectively. Putting all this together, we see that

$$
\psi_*[p \oplus r, q \oplus r] = \left[t v_n^* \begin{pmatrix} qpq + 1 - q & qp(1-q) \\ (1-q)pq & (1-q)p(1-q) \end{pmatrix} v_n t^* + (1 - tt^*)e \right]. \tag{A.22}
$$

Lemma A.4 implies that the class on the right-hand side of line (A.22) equals the class of the element in line (A.21). Hence we are done with this case of the equivalence relation too.

At this point, we know that $\psi_*\colon KK_\varepsilon^{\pi_0,\mathrm{mx}}(X, B) \to KK_{5\varepsilon}^{\pi,u}(X, B)$ is a well-defined set map. It remains to show that ψ_* is a group homomorphism. Let then (p_1, q_1) and (p_2, q_2) be elements of $\mathcal{P}_{n_1,\varepsilon}^{\pi_0,\mathrm{mx}}(X, B)$ and $\mathcal{P}_{n_2,\varepsilon}^{\pi_0,\mathrm{mx}}(X, B)$ respectively. As ψ is insensitive to homotopies, we may assume that $p_i - q_i \in M_{n_i}(\mathcal{K}(E_0))$. The sum $[p_1, q_1] + [p_2, q_2]$ is represented by $[p_1 \oplus p_2, q_1 \oplus q_2]$, and this is mapped by ψ_* to the class of the product

$$
\begin{aligned}
&v_{n_1+n_2}^* \begin{pmatrix} q_1 & 0 & 1-q_1 & 0 \\ 0 & q_2 & 0 & 1-q_2 \\ 1-q_1 & 0 & q_1 & 0 \\ 0 & 1-q_2 & 0 & q_2 \end{pmatrix} \begin{pmatrix} p_1 & 0 & 0 & 0 \\ 0 & p_2 & 0 & 0 \\ 0 & 0 & 1-q_1 & 0 \\ 0 & 0 & 0 & 1-q_2 \end{pmatrix} \\
&\qquad \cdot \begin{pmatrix} q_1 & 0 & 1-q_1 & 0 \\ 0 & q_2 & 0 & 1-q_2 \\ 1-q_1 & 0 & q_1 & 0 \\ 0 & 1-q_2 & 0 & q_2 \end{pmatrix} v_{n_1+n_2}.
\end{aligned} \tag{A.23}
$$

Let now s be the permutation unitary in $\mathcal{B}((\ell^2)^{\oplus 2(n_1+n_2)}) \subseteq \mathcal{L}(E_0^{\oplus 2(n_1+n_2)})$ such that conjugation by s exchanges the second and third rows and columns in the matrices above. Let $w_1 := i_{n_1} v_{n_1}$, where $i_{n_1}\colon (\ell^2)^{\oplus 2n_1} \to (\ell^2)^{\oplus 2(n_1+n_2)}$ is the inclusion

$$
(\xi_1, \xi_2) \longmapsto (\xi_1, 0, \xi_2, 0),
$$

and similarly for w_2. Set $s_1 := v_{n_1+n_2}^* s w_1$ and $s_2 := v_{n_1+n_2}^* s w_2$, so $s_1, s_2 \in 1_{\mathbb{C}^2} \otimes \mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$; as $w_1 w_1^* + w_2 w_2^* = 1$, these elements satisfy the Cuntz relation $s_1 s_1^* + s_2 s_2^*$. According to Lemma A.5, we may use s_1 and s_2 to define the group operation on $KK_{5\varepsilon}^{\pi,u}(X, B)$, and so

$$\begin{aligned}
&\psi_*[p_1, q_1] + \psi_*[p_2, q_2] \\
&= \Bigg[s_1 v_{n_1}^* \begin{pmatrix} q_1 & 1-q_1 \\ 1-q_1 & q_1 \end{pmatrix} \begin{pmatrix} p_1 & 0 \\ 0 & 1-q_1 \end{pmatrix} \begin{pmatrix} q_1 & 1-q_1 \\ 1-q_1 & q_1 \end{pmatrix} v_{n_1} s_1^* \\
&\quad + s_1 v_{n_2}^* \begin{pmatrix} q_2 & 1-q_2 \\ 1-q_2 & q_2 \end{pmatrix} \begin{pmatrix} p_2 & 0 \\ 0 & 1-q_2 \end{pmatrix} \begin{pmatrix} q_2 & 1-q_2 \\ 1-q_2 & q_2 \end{pmatrix} v_{n_2} s_2^* \Bigg].
\end{aligned}$$

A direct computation shows that this equals the element in line (A.23) above, however, so we are done. ■

We need one more technical lemma before we get to our main goal.

Lemma A.25. *Let A be unital. Let $(\pi, E) = (\pi_0 \oplus \pi_0, E_0 \oplus E_0)$ be a graded, balanced, strongly unitally absorbing representation of A on a Hilbert B-module. Let $X \subseteq A_1$ be finite, and let $\varepsilon > 0$. Consider the diagrams*

$$\begin{tikzcd}
KK_{\varepsilon}^{\pi,u}(X, B) \arrow[r] \arrow[d, "\phi_*"] & KK_{5\varepsilon}^{\pi,u}(X, B) \\
KK_{\varepsilon}^{\pi_0,\mathrm{mx}}(X, B) \arrow[ur, "\psi_*"] &
\end{tikzcd} \tag{A.24}$$

and

$$\begin{tikzcd}
KK_{5\varepsilon}^{\pi,u}(X, B) \arrow[dr, "\phi_*"] & \\
KK_{\varepsilon}^{\pi_0,\mathrm{mx}}(X, B) \arrow[r] \arrow[u, "\psi_*"] & KK_{5\varepsilon}^{\pi_0,\mathrm{mx}}(X, B)
\end{tikzcd} \tag{A.25}$$

where the horizontal arrows are the canonical forget control maps. These commute.

Proof. We first look at diagram (A.24). We compute that for $p \in \mathcal{P}_{\varepsilon}^{\pi,u}(X, B)$,

$$\psi\phi(p) = v_2^* \begin{pmatrix} epe + (1-e) & ep(1-e) \\ (1-e)pe & (1-e)p(1-e) \end{pmatrix} v_2.$$

Define

$$i \colon E_0^{\oplus 2} \longrightarrow E_0^{\oplus 4}, \quad (\xi_1, \xi_2) \longmapsto (\xi_1, 0, 0, \xi_2),$$

and define $v := v_2^* i$, which is an isometry inside the copy $1_{\mathbb{C}^2} \otimes \mathcal{B}(\ell^2) \subseteq \mathcal{L}(E)$ from Lemma 4.2. One computes that

$$\psi\phi(p) = vpv^* + (1 - vv^*)e,$$

whence $[\psi\phi(p)] = [p]$ by Lemma A.4, as required.

Now let us look at diagram (A.25). Let (p,q) be an element of $\mathcal{P}_{n,\varepsilon}^{\pi_0,\mathrm{mx}}(X,B)$ for some n. Conjugating q by a unitary in $M_n(\mathbb{C})$, we may assume that $p-q \in M_n(\mathcal{K}(E_0))$. We compute that

$$\phi\psi(p,q) = \left(v_n^*\begin{pmatrix} q & 1-q \\ 1-q & q\end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1-q\end{pmatrix}\begin{pmatrix} q & 1-q \\ 1-q & q\end{pmatrix} v_n, \begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix}\right).$$

Let 0_{2n-2} be the zero element in $M_{2n-2}(\mathcal{K}(E_0)^+)$. Then the element above has the same class in $KK_\varepsilon^{\pi_0,\mathrm{mx}}(X,B)$ as

$$\left(v_n^*\begin{pmatrix} q & 1-q \\ 1-q & q\end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1-q\end{pmatrix}\begin{pmatrix} q & 1-q \\ 1-q & q\end{pmatrix} v_n \oplus 0_{2n-2}, \begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix}\oplus 0_{2n-2}\right). \quad \text{(A.26)}$$

Let $q_n : (\ell^2)^{\oplus 2n} \to (\ell^2)^{\oplus 2} = 1_{\mathbb{C}^2}\otimes \ell^2$ be defined by projecting onto the first two coordinates, and define $w_n := v_n q_n$, which is a co-isometry in $\mathcal{B}((\ell^2)^{\oplus 2n}) \subseteq \mathcal{L}(E_0^{\oplus n})$ with source projection dominating the projection onto the first two coordinates in $E_0^{\oplus 2n}$. Let C_2 be the space of all co-isometries in $\mathcal{B}((\ell^2)^{\oplus 2n})$ whose source projection dominates projection onto the first two coordinates. Then w_n is path-connected within C_2 to an element that acts as the identity on the first two coordinates, whence the element in line (A.26) represents the same class in $KK_\varepsilon^{\pi_0,\mathrm{mx}}(X,B)$ as

$$\Bigg(w_n\left(v_n^*\begin{pmatrix} q & 1-q \\ 1-q & q\end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1-q\end{pmatrix}\begin{pmatrix} q & 1-q \\ 1-q & q\end{pmatrix} v_n \oplus 0_{2n-2}\right)w_n^*,$$
$$w_n\left(\begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix}\oplus 0_{2n-2}\right)w_n^*\Bigg).$$

Computing, this equals

$$\left(\begin{pmatrix} q & 1-q \\ 1-q & q\end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1-q\end{pmatrix}\begin{pmatrix} q & 1-q \\ 1-q & q\end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix}\right), \quad \text{(A.27)}$$

where all blocks in the matrices appearing above are $n \times n$.

Write now

$$r := \frac{1}{2}\left(\begin{pmatrix} q & 1-q \\ 1-q & q\end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}\right),$$

then r is a projection such that $\|[r,a]\| < \varepsilon$ for all $a \in X$. For $t \in [0,\pi]$ define $u_t := r + \exp(it)(1-r)$, so (u_t) is a path of unitaries connecting $\left(\begin{smallmatrix} q & 1-q \\ 1-q & q\end{smallmatrix}\right)$ to the identity, and all (u_t) satisfy $\|[u_t,a]\| < \varepsilon$ for all $a \in X$. Hence the path

$$\left(u_t\begin{pmatrix} q & 1-q \\ 1-q & q\end{pmatrix}\begin{pmatrix} p & 0 \\ 0 & 1-q\end{pmatrix}\begin{pmatrix} q & 1-q \\ 1-q & q\end{pmatrix}u_t^*, u_t\begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix}u_t^*\right)$$

shows that the element in line (A.27) defines the same class in $KK_\varepsilon^{\pi_0,\mathrm{mx}}(X,B)$ as

$$\left(\begin{pmatrix} p & 0 \\ 0 & 1-q\end{pmatrix}, \begin{pmatrix} q & 1-q \\ 1-q & q\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix} q & 1-q \\ 1-q & q\end{pmatrix}\right),$$

which equals $(p \oplus 1-q, q \oplus 1-q)$. This last element defines the same class in $KK_\varepsilon^{\pi_0,\mathrm{mx}}(X,B)$ as (p,q) by definition (see ((i)) from Definition A.21 above), however, so we are done. ∎

We are finally ready for the main result of this subsection; it follows directly from Lemmas A.1 and A.25, quite analogously to Proposition A.11.

Proposition A.26. *Let A and B be separable C^*-algebras with A unital. Let $\pi\colon A \to \mathcal{L}(E)$ be a graded, balanced, and strongly unitally absorbing representation of A on a Hilbert B-module. Write $(\pi, E) = (\pi_0 \oplus \pi_0, E_0 \oplus E_0)$ with (π_0, E_0) strongly unitally absorbing.*

Then there are isomorphisms

$$KL(A,B) \longrightarrow \varprojlim KK_\varepsilon^{\pi_0,\mathrm{mx}}(X,B). \tag{A.28}$$

and

$$\varprojlim{}^1 KK_\varepsilon^{1\otimes\pi_0,\mathrm{mx}}(X,SB) \longrightarrow \overline{\{0\}},$$

where the limits are taken over the directed set $\mathcal{X}$ of Definition 6.8 and $\overline{\{0\}}$ is the closure of 0 in $KK(A,B)$. Moreover, the isomorphism in line (A.28) is a homeomorphism when the right-hand side is equipped with the inverse limit topology.

Finally, there is a short exact sequence

$$0 \longrightarrow \varprojlim{}^1 KK_\varepsilon^{1\otimes\pi_0,\mathrm{mx}}(X,SB) \longrightarrow KK(A,B) \longrightarrow \varprojlim KK_\varepsilon^{\pi_0,\mathrm{mx}}(X,B) \longrightarrow 0. \quad ∎$$

Let us conclude with a final corollary on representation-independence, which is a (simpler) analogue of Proposition A.19 above. It is immediate from Proposition A.26 and the fact that if $\pi\colon A \to \mathcal{L}(E)$ is strongly unitally absorbing, then

$$(\sigma, F) = (\pi \oplus \pi, E \oplus E)$$

is graded, balanced and strongly unitally absorbing, and also satisfies $\sigma_0 = \pi$.

Corollary A.27. *Let A and B be separable C^*-algebras with A unital. Then for any strongly unitally absorbing representation $\pi\colon A \to \mathcal{L}(E)$ we have that*

$$KL(A,B) \longrightarrow \varprojlim KK_\varepsilon^{\pi,\mathrm{mx}}(X,B). \tag{A.29}$$

and

$$\varprojlim{}^1 KK_\varepsilon^{1\otimes\pi,\mathrm{mx}}(X,SB) \longrightarrow \overline{\{0\}},$$

where the limits are taken over the directed set $\mathcal{X}$ of Definition 6.8, and $\overline{\{0\}}$ is the closure of 0 in $KK(A,B)$. Moreover, the isomorphism in line (A.29) is a homeomorphism when the right-hand side is equipped with the inverse limit topology.

Finally, there is a short exact sequence

$$0 \longrightarrow \varprojlim{}^1 KK_\varepsilon^{\pi,\mathrm{mx}}(X,SB) \longrightarrow KK(A,B) \longrightarrow \varprojlim KK_\varepsilon^{\pi,\mathrm{mx}}(X,B) \longrightarrow 0. \quad ∎$$

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