Doc. Math. 31 (2026), 119–140
DOI 10.4171/DM/1041

© 2025 Deutsche Mathematiker-Vereinigung
Published by EMS Press
This work is licensed under a CC BY 4.0 license

Twisted tropical Hurwitz numbers for elliptic curves

Marvin Anas Hahn and Hannah Markwig

Abstract. Hurwitz numbers enumerate branched morphisms between Riemann surfaces. For a fixed elliptic target, Hurwitz numbers are intimately related to mirror symmetry following work of Dijkgraaf. In recent work of Chapuy and Dołęga, a new variant of Hurwitz numbers with fixed genus 0 target was introduced that includes maps between non-orientable surfaces. These numbers are called b-Hurwitz numbers and are polynomials in a parameter b which measures the non-orientability of the involved maps. An interpretation in terms of factorisations of b-Hurwitz numbers for $b = 1$, so-called twisted Hurwitz numbers, was found in work of Burman and Fesler. In previous work, the authors derived a tropical geometry interpretation of these numbers. In this paper, we introduce a natural generalisation of twisted Hurwitz numbers with elliptic targets within the framework of symmetric groups. We derive a tropical interpretation of these invariants, relate them to Feynman integrals and derive an expression as a matrix element of an operator in the bosonic Fock space.

1. Introduction

Hurwitz numbers count branched covers of Riemann surfaces with fixed numerical data. They originate from Hurwitz' original work in [18] and have developed to important invariants in enumerative geometry. There are various equivalent definitions of Hurwitz numbers arising from different fields of mathematics. The one most important for this work is its interpretation via monodromy representations as an enumeration of factorisations in the symmetric group. As elliptic curves are the simplest cases of Calabi–Yau varieties, Hurwitz numbers of elliptic curves play a role in mirror symmetry. Dijkgraaf studied the relation between generating functions of Hurwitz numbers of an elliptic curve and Feynman integrals [12].

In recent work of Chapuy and Dołęga [11], a new class of Hurwitz numbers was introduced, called b-Hurwitz numbers depending on a parameter b. For $b = 0$ one obtains classical Hurwitz numbers, while for $b = 1$ these invariants specialise to an enumeration of covers between possibly non-orientable surface. Following [6], this enumeration gives rise to *twisted Hurwitz numbers* which were proved to admit a definition in terms of counting factorisations in the symmetric group in *loc. cit.* that mirrors its classical counterpart. In previous work [16], the authors developed a tropical geometry framework for the study of twisted Hurwitz numbers. So far, twisted Hurwitz numbers (and b-Hurwitz numbers)

Mathematics Subject Classification 2020: 14T15 (primary); 14N10, 57M12, 05C30 (secondary).
Keywords: tropical geometry, Hurwitz numbers.

have only been studied for the enumeration of maps with a genus 0 target. In this work, we introduce twisted Hurwitz numbers of an elliptic curve, also in terms of analogous factorisations in the symmetric group. Motivated by the fact that tropical geometry also provides a natural framework for the study of covers of an elliptic curve [4, 17], we also study the tropical geometry of our twisted Hurwitz numbers of an elliptic curve.

1.1. Elliptic Hurwitz numbers

We first introduce the class of Hurwitz numbers of an elliptic curve showing up in the mirror symmetry relation involving Feynman integrals.

Definition 1 (Hurwitz numbers of an elliptic curve). Let E be an elliptic curve (i.e., a Riemann surface of genus 1), $g \geq 1$ a non-negative integer, and $d > 0$ a positive integer. Moreover, we fix $p_1, \dots, p_{2g-2} \in E$. Then, we consider covers of degree d, $f: S \to E$, such that

- S is a Riemann surface of genus g,
- the ramification profile of $p_1, \dots, p_{2g-2}$ is $(2, 1, \dots, 1)$.

Two covers $f: S \to E$ and $f': S' \to E$ are called equivalent if there exists a homeomorphism $h: S \to S'$, such that $f = f' \circ h$.

Then, we define the *Hurwitz number of the elliptic curve* E as

$$h_{d,g} = \sum_{[f]} \frac{1}{|\mathrm{Aut}(f)|},$$

where the sum runs over all equivalence classes of covers as above.

Such Hurwitz numbers are in fact topological invariants, i.e., they do not depend on the algebraic structure of the Riemann surfaces. Via monodromy representations (see, e.g., [10]), Hurwitz numbers of an elliptic curve can be computed in terms of factorisations in the symmetric group.

Lemma 2. *The Hurwitz number $h_{d,g}$ equals $\frac{1}{d!}$ times the number of tuples*

$$(\sigma, \tau_1, \dots, \tau_{2g-2}, \alpha) \in (S_d)^{2g}$$

that satisfy the following:

(1) *each τ_i is a transposition,*

(2) *the product of these permutations satisfies the following equation:*

$$\tau_{2g-2} \cdots \tau_1 \sigma = \alpha \sigma \alpha^{-1},$$

(3) *the subgroup*

$$\langle \sigma, \tau_1, \dots, \tau_{2g-2}, \alpha \rangle$$

acts transitively on the set $\{1, \dots, d\}$.

The idea for the proof of Lemma 2 is to lift loops in the fundamental group of the elliptic curve to paths in the covering surface, see Figure 1.

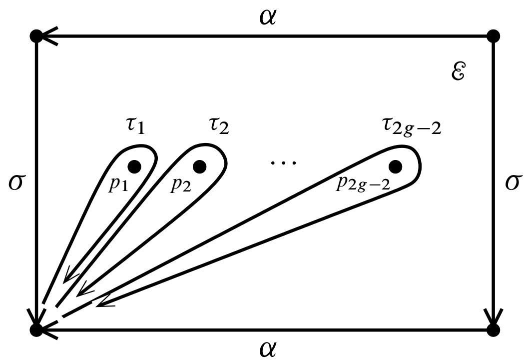

Figure 1. A sketch of a cut open elliptic curve and the paths in its fundamental group which create the tuples to be counted to obtain a Hurwitz number via monodromy representations.

1.2. Twisted elliptic Hurwitz numbers

We fix the involution

$$\tau = (1\,d+1)(2\,d+2)\cdots(d\,2d) \in S_{2d}$$

and use the notation

$$B_d = C(\tau) = \{\sigma \in S_{2d} \mid \sigma\tau\sigma^{-1} = \tau\},$$
$$C^{\sim}(\tau) = \{\sigma \in S_{2d} \mid \tau\sigma\tau^{-1} = \tau\sigma\tau = \sigma^{-1}\}.$$

We further define the subset $B_d^{\sim} \subset C^{\sim}(\tau)$ consisting of those permutations that have no self-symmetric cycles (see [6, Lemma 2.1]). We define the *twisted Hurwitz numbers of an elliptic curve* in terms of the symmetric group.

Definition 3 (Twisted Hurwitz numbers of an elliptic curve). Fix a genus g and a degree d. Then the twisted Hurwitz number $\tilde{h}_{d,g}$ of degree d and genus g of an elliptic curve is defined to be $\frac{1}{(2d)!!}$ times the number of tuples

$$(\sigma, \eta_1, \dots, \eta_{g-1}, \alpha) \in (S_{2d})^{g+1}$$

that satisfy the following conditions:

(1) each η_s is a transposition, $\eta_s = (i_s \;\; j_s)$ such that $j_s \neq \tau(i_s)$,

(2) $\sigma \in B_d^{\sim}$,

(3) $\alpha \in B_d$,

(4) the product of these permutations satisfies the following equation:

$$\eta_1 \cdots \eta_{g-1}\sigma(\tau\eta_{g-1}\tau)\cdots(\tau\eta_1\tau) = \alpha\sigma\alpha^{-1},$$

(5) the subgroup

$$\langle \sigma, \eta_1, \dots, \eta_{g-1}, (\tau\eta_{g-1}\tau), \dots, (\tau\eta_1\tau), \alpha \rangle$$

acts transitively on the set $\{1, \dots, 2d\}$.

The motivation to call these counts of factorisations in the symmetric group twisted Hurwitz numbers of an elliptic curve is coming from Lemma 2. We can also drop the transitivity condition. On the tropical side, this corresponds to allowing disconnected source curves for our covers. We denote these numbers by $\tilde{h}^{\bullet}_{d,g}$.

Remark 4. The following twisted Hurwitz number was computed with GAP: $\tilde{h}_{2,3} = 16$.

Remark 5. It is tempting to expect that $\tilde{h}_{d,g}$ admits a geometric interpretation similar to twisted Hurwitz numbers with genus 0 target (see [11, Section 2.2] and the discussion after [16, Remark 6]). A reasonable expectation could be that elliptic twisted Hurwitz numbers count maps to an elliptic target E with an orientation reversing involution that respects an orientation reversing involution on E. We leave the question of such a geometric interpretation as an open problem.

1.3. Main results

In Section 2, we develop a tropical approach to elliptic twisted Hurwitz numbers. We introduce tropical elliptic twisted Hurwitz numbers as enumerations of tropical coverings of a tropical elliptic curve, i.e., as certain maps between graphs. As the main result of this section, in Theorem 21 we prove a correspondence theorem stating that elliptic twisted Hurwitz numbers and their tropical counterparts coincide. This tropical interpretation then allows us to derive an expression in Theorem 31 of elliptic twisted Hurwitz numbers in terms of Feynman diagrams in Section 3. We note that the computation of elliptic Hurwitz numbers in terms of Feynman diagrams was first derived in [12] and a new proof employing tropical techniques was given in [4]. Finally, we follow the slogan *bosonification is tropicalisation* which has now been established in a plethora of works [3,8,9,14,15] to express elliptic twisted Hurwitz numbers as matrix element on the bosonic Fock space in Section 4.

2. Twisted tropical covers of an elliptic curve

In [16], we have introduced twisted versions of tropical Hurwitz numbers. We now generalize to consider twisted versions of tropical covers of an elliptic curve and their counts, building on [1, 4, 7]. We start by recalling the basic notions of tropical curves and covers. Then we introduce twisted tropical covers of an elliptic curve, which can roughly be viewed as tropical covers with an involution. By fixing branch points, we produce a finite count of twisted tropical covers of an elliptic curve for which we show in the following that it coincides with the corresponding twisted Hurwitz number of an elliptic curve. Readers with a background in the theory of tropical curves are pointed to the fact that we only consider explicit tropical curves in the following, i.e., there is no genus hidden at vertices.

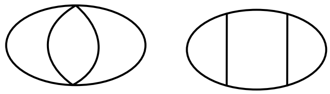

Figure 2. Two abstract tropical curves of genus 3. Edge lengths are not specified in the picture.

Definition 6 (Abstract tropical curves). An abstract tropical curve is a connected graph Γ with the following data:

(1) The vertex set of Γ is denoted by $V(\Gamma)$ and the edge set of Γ is denoted by $E(\Gamma)$.

(2) The 1-valent vertices of Γ are called *leaves* and the edges adjacent to leaves are called *ends*.

(3) The set of edges $E(\Gamma)$ is partitioned into the set of ends $E^{\infty}(\Gamma)$ and the set of *internal edges* $E^0(\Gamma)$.

(4) There is a length function

$$\ell\colon E(\Gamma) \to \mathbb{R} \cup \{\infty\},$$

such that $\ell^{-1}(\infty) = E^{\infty}(\Gamma)$.

The genus g of an abstract tropical curve Γ is defined as the first Betti number of the underlying graph, i.e., $g = 1 - |V(\Gamma)| + |E(\Gamma)|$. An isomorphism of abstract tropical curves is an isomorphism of the underlying graphs that respects the length function. The combinatorial type of an abstract tropical curve is the underlying graph without the length function.

Definition 7. A tropical elliptic curve E is a circle of a given length. It may have several two-valent vertices. In the following we will refer to any tropical elliptic curve as E and do not specify the number of vertices when it is clear from the context.

Notice that a tropical elliptic curve is of genus 1.

Example 8. Figure 2 shows two abstract tropical curves of genus 3. We have not specified edge lengths in the picture.

Next, we define the notion of tropical covers. We restrict to the case where the target is either a tropical elliptic curve E or a subdivided version of $\mathbb{R}$, i.e., a line with some 2-valent vertices.

Definition 9 (Tropical covers). Let the target Γ_2 be either a tropical elliptic curve E or a subdivided version of $\mathbb{R}$. A tropical cover between abstract tropical curves $\pi\colon \Gamma_1 \to \Gamma_2$ is a surjective harmonic map, i.e.:

(1) We have $\pi(V(\Gamma_1)) \subset V(\Gamma_2)$.

(2) Let $e \in E(\Gamma_1)$. Then, we interpret e and $\pi(e)$ as intervals $[0, \ell(e)]$ and $[0, \ell(\pi(e))]$ respectively. We require π restricted to e to be a bijective integer linear function $[0, \ell(e)] \to [0, \ell(\pi(e))]$ given by $t \mapsto \omega(e) \cdot t$, with $\omega(e) \in \mathbb{Z}$. If $\pi(e) \in V(\Gamma_2)$, we define $\omega(e) = 0$. We call $\omega(e)$ the *weight* of e.

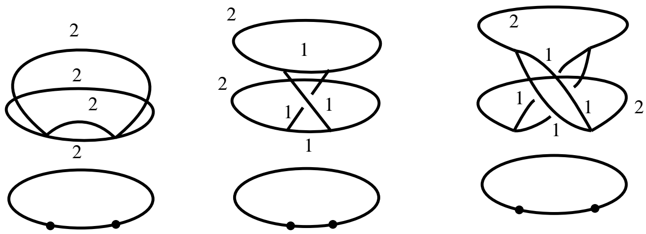

Figure 3. Three twisted tropical covers of E of degree 2 and genus 3. The labels on the edges denote the edge weight. Edges which are not labeled have weight one. The involution ι is supposed to exchange respective edges on top resp. bottom of the picture.

(3) For a vertex $v \in V(\Gamma_1)$, we denote by $\mathrm{Inc}(v)$ the set of incoming edges at v (edges adjacent to v mapping to the left of $\pi(v)$) and by $\mathrm{Out}(v)$ the set of outgoing edges at v (edges adjacent to v mapping to the right of $\pi(v)$). We then require

$$\sum_{e \in \mathrm{Inc}(v)} \omega(e) = \sum_{e \in \mathrm{Out}(v)} \omega(e).$$

This number is called the local degree of π at v. We call this equality the *harmonicity* or *balancing condition*. For a point v in the interior of an edge e of Γ_1, the local degree of π at v is defined to be the weight $\omega(e)$.

Moreover, we define the *degree* of π as the half of the sum of local degrees at all vertices and internal points of Γ_1 in the preimage of a given vertex of Γ_2. By the harmonicity condition, the degree is independent of the choice of vertex of Γ_2.

For any end e of Γ_2, we define a partition μ_e as the partition of weights of ends of Γ_1 mapping to e. We call μ_e the *ramification profile* of e.

We call two tropical covers $\pi_1 \colon \Gamma_1 \to \Gamma_2$ and $\pi_2 \colon \Gamma_1' \to \Gamma_2$ equivalent if there exists an isomorphism $g \colon \Gamma_1 \to \Gamma_1'$ of metric graphs, such that $\pi_2 \circ g = \pi_1$.

We are now ready to give a definition of twisted tropical covers, which may be viewed as tropical covers admitting an involution with specified locus.

Definition 10 (Twisted tropical covers of E). We define a twisted topical cover of E to be a tropical cover $\pi \colon \Gamma_1 \to E$ with an involution $\iota \colon \Gamma_1 \to \Gamma_1$ which respects the cover π, such that:

- we have $g - 1$ branch points $p_1, \dots, p_{g-1} \in E$ which we set as vertices,
- in the preimage of each branch point p_i, there are either two 3-valent vertices or one 4-valent vertex,
- the edges adjacent to a 4-valent vertex all have the same weight,
- the fixed locus of ι is exactly the set of 4-valent vertices.

Example 11. Using the two source curves from Example 8 (see Figure 2), we can build twisted tropical covers of degree 2, see Figure 3.

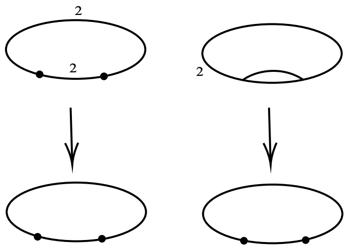

Figure 4. The quotient covers of the three twisted covers from Example 11, see Figure 3.

As usually for Hurwitz numbers, our enumeration of twisted tropical covers will take automorphisms into account. We give the following definition which specifies automorphisms that take the involution into account.

Definition 12 (Automorphisms). Let $\pi : \Gamma \to E$ be a twisted tropical cover with involution $\iota : \Gamma \to \Gamma$. An automorphism of π is a morphism of abstract tropical curves (i.e., a map of metric graphs) $f : \Gamma \to \Gamma$ respecting the cover and the involution, i.e., $\pi \circ f = \pi$ and $f \circ \iota = \iota \circ f$. We denote the group of automorphisms of π by $\mathrm{Aut}(\pi)$.

Example 13. The twisted tropical cover depicted in Figure 3 on the left has an automorphism group of size 4: we can independently exchange the two pairs of weight 2 edges mapping to the same segment of E. The twisted tropical cover in the middle has an automorphism group of size 2 (generated by the involution): we can exchange the two edges of weight 2, with the two pairs of edges of weight 1 following along. The one on the right has an automorphism group of size 4: we can exchange two parallel edges of weight 1 in addition to the involution.

Definition 14 (Quotient graph Γ/ι, see [16]). Let $\pi\colon \Gamma \to E$ be a twisted tropical cover with involution $\iota\colon \Gamma \to \Gamma$. The involution ι induces a symmetric relation on the vertex and edge sets of Γ: We define for $v, v' \in V(\Gamma)$ (resp. $e, e' \in E(\Gamma)$) that $v \sim v'$ (resp. $e \sim e'$) if and only if $\iota(v) = v'$ (resp. $\iota(e) = e'$). We define Γ/ι as the graph with vertex set $V(\Gamma)/\sim$ and edge set $E(\Gamma)/\sim$ with natural identifications. For $e = [e', e''] \in E(\Gamma/\iota)$ we define the length $\ell(e)$ as $\ell(e') = \ell(e'')$ and its weight $\omega(e)$ with respect to π to be the weight $\omega(e') = \omega(e'')$. In this way, we obtain a tropical cover from the quotient graph Γ/ι to E, which has 2-valent vertices coming from the 4-valent vertices of Γ, and 3-valent vertices else.

Example 15. The quotient covers of the three twisted covers from Figure 3 are depicted in Figure 4. The middle and right tropical cover have the same quotient graph.

Proposition 16. *Let $\overline{\pi} : \overline{\Gamma} \to E$ be a (connected) quotient of a twisted tropical cover (see Definition 14). Assume $\overline{\Gamma}$ has c 2-valent vertices and is of genus g'. Then*

$$\sum_{\pi} \frac{1}{\sharp\,\mathrm{Aut}(\pi)} = \frac{2^{g'} - \delta_{0c}}{2^{c+1} \cdot \sharp\,\mathrm{Aut}(\overline{\pi})},$$

where the sum goes over all (connected) twisted tropical covers $\pi : \Gamma \to E$ *with involution* ι *whose quotient* $\Gamma/\iota \to E$ *equals* $\overline{\pi} : \overline{\Gamma} \to E$ *and* $\delta_{0c} = 1$ *if* $c = 0$ *and* 0 *else.*

Proof. Every summand on the left hand side comes with a factor of $\frac{1}{2}$, which arises due to the involution. This is true since the involution yields a contribution to the automorphism group of each twisted tropical cover whose quotient equals $\overline{\pi}$, but descends to the identity on the quotient. This accounts for one factor of 2 in the denominator of the right hand side. We now discuss the equality up to this factor of $\frac{1}{2}$, which arises on both sides.

Assume first $\overline{\Gamma}$ has no 2-valent vertices. First, we remove g' edges $e_1, \dots, e_{g'}$ of $\overline{\Gamma}$ such that we obtain a tree which we call $\overline{\Gamma'}$.

First, we would like to understand the preimages of $\overline{\Gamma'}$ under taking a quotient with respect to an involution ι. That means, we take two copies of every edge of $\overline{\Gamma'}$ such that the involution exchanges the edges in the pair. When drawing a picture, we like to draw one edge on the top and one on the bottom. Now let us consider an adjacent vertex, and another edge starting from this vertex. Again, we take two copies, but since the involution exchanges the two it does not matter which we draw on top and which we draw on the bottom. Thus, for the whole tree $\overline{\Gamma'}$, we have a unique preimage under taking the quotient which just consists of two disjoint copies of $\overline{\Gamma'}$.

To understand the preimage of taking the quotient for $\overline{\Gamma}$, we can start by taking the preimage of the tree $\overline{\Gamma'}$ and putting in two copies for each of the missing edges $e_1, \dots, e_{g'}$. As discussed above, the preimage of the tree just consists of two copies which we call the top and the bottom part. When we insert a pair of edges for e_1, we now have two options: we can either let one connect top with top, and the other bottom with bottom, or both can connect top with bottom. We have the same choice for all g' edges, yielding $2^{g'}$ choices for preimages under taking the quotient. However, one of these (where we always connect top with top and bottom with bottom) is disconnected and should therefore be discarded. Also, not all the $2^{g'}$ choices have to be distinct, and this happens in the presence of automorphisms of $\overline{\pi}$: the latter arise due to parallel edges which are mapped in the same way (in particular, with the same weight). This is true, since an automorphism of the cover is an automorphism of the source graph which is compatible with the covering map, thus only edges with the same image can be permuted. Since $\overline{\Gamma}$ is 3-valent and because of the balancing condition, locally at every vertex, at most two edges can be permuted, which then must form a pair of parallel edges of the same weight. Every such pair of parallel edges with the same weight produces a factor of 2 for the size of the automorphism group of $\overline{\pi}$ (see also [7, Corollary 4.4]).

We now discuss the effect of the presence of such automorphisms of the quotient cover on the a priori $2^{g'}$ choices for preimages under taking the quotient discussed above. For this purpose, we focus on one set of parallel edges, the argument has to be repeated if there are more. Earlier, we chose g' edges $e_1, \dots, e_{g'}$ to remove, obtaining a tree. Since parallel edges form a cycle themselves, one of two parallel edges must be among these. We assume it is e_1. So, assume e_1 is one of two parallel edges connecting the vertices v_1 and v_2. Consider a path connecting v_2 with v_1 (avoiding e_1 and its parallel edge). Since

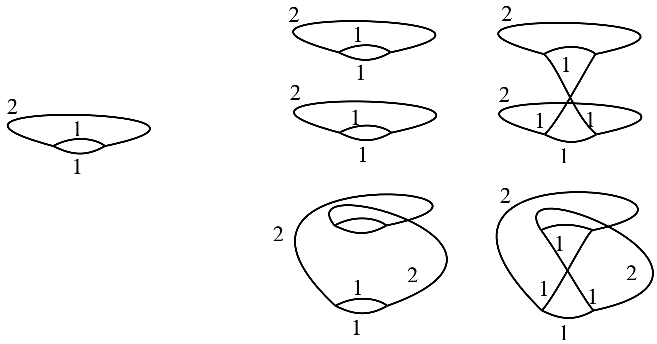

Figure 5. On the left, a quotient cover $\overline{\pi}$ of genus 2. On the right, the 4 choices of preimage under taking the quotient, as in the proof of Proposition 16. Because of the automorphism of the quotient cover, the upper right and the lower right choice get identified. The top middle is disconnected and should be discarded. The lower middle has an automorphism group of size 4 due to the automorphism of $\overline{\pi}$. Altogether, we have $\frac{1}{2} + \frac{1}{4} = \frac{4-1}{2\cdot 2}$ preimages counted with one over the size of their automorphism group, as predicted by Proposition 16.

$\overline{\Gamma}$ is mapped to an elliptic curve E, such a path must exist. We can assume that e_2 is an edge of this path (as this path, together with the parallel edge of e_1 forms another cycle). Then we insert two copies of each edge to the top and bottom part of two copies of the thus resulting tree. If we choose the two copies of e_1 to both connect top and bottom, and the two copies of e_2 to connect top with top and bottom with bottom, we obtain the same twisted cover as when we choose the two copies of e_1 to connect top with top and bottom with bottom, and the two copies of e_2 to both connect top with bottom. This is true since we can switch the role of top and bottom for the two preimages of v_2, and accordingly for the path from v_2 to v_1 until we reach the two copies of e_2, for which we switch again. For an example, see Figure 5.

To sum up, for each such pair of parallel edges, we obtain an action of S_2 on the set of the $2^{g'}$ choices from above. The orbit-stabilizer theorem tells us that we either identify two of the choices (which are in the same orbit), or we have an extra automorphism working on one choice. Thus, dividing by the size of the automorphism group of $\overline{\pi}$, we even out our overcountings, taking into account the automorphisms of the preimages, too.

Finally, let us consider what happens in the presence of 2-valent vertices. First, every choice of preimage will be connected, so we do not have to subtract one when counting possibilities. Second, for every 4-valent vertex in each preimage π, we can exchange the edges of one adjacent twisted pair, leading to an extra automorphism of π which descends to the identity on the quotient $\overline{\pi}$. To be more precise, we can follow the argument from above where we took out g' edges, consider the unique preimage of the tree, and reinsert the g' twisted pairs. When reinserting for an edge between two 3-valent vertices, the argument is the same. For an edge connecting a 2-valent to a 3-valent vertex, we do not have the two choices of connecting top with top and bottom with bottom or top with bottom twice. We have only one choice: to connect the 4-valent vertex with top, and with bottom.

However, just as before, by the orbit-stabilizer theorem there is an additional automorphism exchanging this pair of twisted edges. For an edge connecting two 2-valent vertices, we have to choose two parallel edges which again yields an automorphism. In the end, we produce a factor of 2 for the size of the automorphism group for every 4-valent vertex. ∎

Remark 17. We note that Proposition 16 generalises to covers of $\mathbb{R}$, and disconnected twisted tropical covers as well. Indeed, let $\overline{\pi}\colon \overline{\Gamma} \to \mathbb{R}$ be a quotient of a twisted tropical cover with c 2-valent vertices and r connected components, each of genus g_i. Then, the genus of $\overline{\Gamma}$ is given by $g' = \sum g_i - r + 1$. As in the proof of Proposition 16, we obtain $2^{\sum g_i}$ many preimages under taking the quotient. Now, we aim to count automorphisms. The argument is the same as in the proof above with the difference, that the involutions acts independently on each component and thus, we obtain a factor in the denominator of 2^r. Thus, in total, we obtain

$$\sum_{\pi} \frac{1}{\sharp \mathrm{Aut}(\pi)} = \frac{2^{\sum g_i}}{2^r 2^c \sharp \mathrm{Aut}(\overline{\pi})},$$

where the sum now runs over possibly disconnected twisted covers π with quotient $\overline{\pi}$. To conclude, we observe however that $\sum g_i = g' + r + 1$ and thus, we obtain

$$\sum_{\pi} \frac{1}{\sharp \mathrm{Aut}(\pi)} = \frac{2^{g'}}{2^{c+1} \sharp \mathrm{Aut}(\overline{\pi})}.$$

The only difference to the connected case in Proposition 16 is the factor $\delta_{0,c}$ that ensured connectedness which obviously does not play a role here.

We are now ready to define twisted tropical Hurwitz numbers of an elliptic curve.

Definition 18 (Twisted tropical Hurwitz number of an elliptic curve). We define the twisted tropical Hurwitz number of an elliptic curve $\tilde{h}^{\mathrm{trop}}_{d,g}$ to be the weighted enumeration of equivalence classes of twisted tropical covers of degree d and genus g of a tropical elliptic curve E, such that each equivalence class $[\pi\colon \Gamma \to E]$ is counted with multiplicity

$$2^{g-1} \cdot \frac{1}{|\mathrm{Aut}(\pi)|} \cdot \prod_V (\omega_V - 1) \prod_e \omega(e),$$

where the first product goes over all 4-valent vertices and ω_V denotes the weight of the adjacent edges, while the second product is taken over all edges of the quotient graph Γ/ι and $\omega(e)$ denotes their weights.

Example 19. The twisted tropical Hurwitz number $\tilde{h}^{\mathrm{trop}}_{2,3}$ equals 16. There are five twisted tropical covers for $d = 2$ and $g = 3$. We may obtain all from the sketches in Figure 3. The middle and right depicted maps each give rise to two tropical covers depending on the labelling of the branch points, i.e., for the tropical elliptic curve E on the bottom, either the left vertex is labelled p_1 and the right vertex is labelled p_2 or the left vertex is labelled

p_2 and the right vertex is labelled p_1. The left cover in Figure 3 has multiplicity

$$2^2 \cdot \frac{1}{4} \cdot (2-1) \cdot (2-1) \cdot 2 \cdot 2 = 4.$$

Each cover coming from the middle picture has multiplicity

$$2^2 \cdot \frac{1}{2} \cdot 2 = 4.$$

Each cover coming from the right picture has multiplicity

$$2^2 \cdot \frac{1}{4} \cdot 2 = 2.$$

For the sizes of the automorphism groups, see Example 13. Thus, we obtain $\tilde{h}^{\text{trop}}_{2,3} = 16$ in total. Note that this number coincides with the twisted Hurwitz number we computed in Remark 4.

Remark 20. Notice that, by Proposition 16, twisted tropical Hurwitz numbers of an elliptic curve can also be determined by counting quotient covers directly. If Γ has c 4-valent vertices and is of genus g, then by an Euler characteristics computation the quotient $\overline{\Gamma}$ has genus $g' = \frac{1}{2} \cdot (g - c + 1)$. For the multiplicities of the preimages of a quotient cover, only the factor $\frac{1}{|\operatorname{Aut}(\pi)|}$ differ, all others remain. But the sum of the $\frac{1}{|\operatorname{Aut}(\pi)|}$ is obtained via Proposition 16, and so a quotient cover $\overline{\pi}$ has to be counted with multiplicity

$$\begin{aligned}
&\frac{2^{g'} - \delta_{0c}}{2^{c+1}} \cdot 2^{g-1} \cdot \frac{1}{\left|\operatorname{Aut}(\overline{\pi})\right|} \cdot \prod_V (\omega_V - 1) \prod_e \omega(e) \\
&\quad = (2^{g'} - \delta_{0c}) \cdot 2^{2g'-3} \cdot \frac{1}{\left|\operatorname{Aut}(\overline{\pi})\right|} \cdot \prod_V (\omega_V - 1) \prod_e \omega(e),
\end{aligned}$$

where c denotes the number of 2-valent vertices of the source graph Γ, the first product goes over all 2-valent vertices and ω_V denotes the weight of the adjacent edges, while the second product is taken over all edges and $\omega(e)$ denotes their weights.

The equality observed in Example 19 is no coincidence, as shown in the following theorem.

Theorem 21 (Correspondence theorem). *The twisted Hurwitz number of E equals its tropical counterpart, i.e.,*

$$\tilde{h}^{\text{trop}}_{d,g} = \tilde{h}_{d,g}.$$

Proof. Let $\pi : \Gamma \to E$ be a twisted tropical cover of degree d. We pick a base point p_0 between p_{g-1} and p_1 and cut the elliptic curve open at p_0. We also cut the preimages of p_0 under π, thus obtaining a twisted tropical cover $\tilde{\pi} : \tilde{\Gamma} \to \mathbb{R}$. In the untwisted case, this is explained in detail in [4, Construction 4.4].

The twisted Hurwitz number counts tuples

$$\tilde{h}_{d,g} = \frac{1}{(2d)!!} \sharp\{(\sigma, \eta_1, \ldots, \eta_{g-1}, \alpha)\}$$

as in Definition 1. We can split these tuples and first list tuples of the form $\{(\sigma, \eta_1, \ldots, \eta_{g-1})\}$, combining each such tuple with a list of possible α. Each tuple $\{(\sigma, \eta_1, \ldots, \eta_{g-1})\}$ yields a twisted tropical cover $\tilde{\pi}$ of $\mathbb{R}$ as in [16, Construction/Theorem 14]. This cover has left and right ends of weights given by the cycle lengths of σ. For each twisted tropical cover of $\mathbb{R}$ having ends of the same weights in both directions, the number of tuples of the form above leading to this cover equals its tropical multiplicity by the correspondence theorem (see [16, Proposition 18 and Remark 6]). The tropical multiplicity equals

$$2^{g-1} \prod_V (\omega_V - 1) \prod_e \omega(e) \prod_K \frac{1}{\omega_K} \cdot \frac{1}{|\operatorname{Aut}(\tilde{\pi})|}$$

where the first product goes over all 4-valent vertices V and ω_V denotes the weight of its adjacent edges, the second product goes over all pairs of twisted internal edges of the source $\tilde{\Gamma}$ of the twisted tropical cover $\tilde{\pi}$, and the third over all twisted pairs of components K which consist of a single edge of weight ω_K. Combining tuples of the form $\{(\sigma, \eta_1, \ldots, \eta_{g-1})\}$ with possible α amounts to gluing a twisted tropical cover of $\mathbb{R}$ to obtain a twisted tropical cover of E. To determine the number of such gluings, we pass to the quotient covers on each side. Given a twisted tropical cover π of E and its cut cover $\tilde{\pi}$, we consider the quotient cover $\overline{\pi}$ of E and the quotient cut cover $\overline{\tilde{\pi}}$. Note that taking the quotient and cutting the cover commutes.

When gluing a cut quotient cover of $\mathbb{R}$ to a quotient cover of E, we want to pair up left and right ends that should be glued. Each left end of the quotient cover corresponds to a pair of ends of the twisted cover. Assume the pair of cycles $c_1, \tau \circ c_1 \circ \tau$ corresponds to these two ends, and assume that our gluing merges the left end corresponding to c_1 with the right end corresponding to a cycle c_2 of the same length. We want to count the number of α that satisfy $c_2 = \alpha \circ c_1 \circ \alpha^{-1}$. Let $c_1 = (c_{11}, \ldots, c_{1\ell(c_1)})$, and $c_2 = (c_{21}, \ldots, c_{2\ell(c_1)})$. A choice of α is fixed by setting $\alpha(c_{11}) = c_{2i}$ for any $i = 1, \ldots, \ell(c_1)$. As we require that $\alpha\tau = \tau\alpha$ any element in the twisted cycle $\tau \circ c_1 \circ \tau$ of the form $\tau(c_{1j})$ must be mapped to $\tau(\alpha(c_{1j}))$ via α. Thus, choosing a gluing on one of a pair of twisted ends of the cut cover fixes the gluing on the other.

By the same argument as in [4, Proposition 4.9] the number of such α is given by

$$\prod_{e'} \omega(e')^{c_{e'}} \cdot \frac{|\operatorname{Aut}(\tilde{\pi})|}{|\operatorname{Aut}(\pi)|}, \tag{2.1}$$

where the product goes over all pairs of twisted edges e' of Γ that contain a preimage of the base point p_0 of E and $c_{e'}$ denotes the number of preimages in e', $c_{e'} = \#(\pi^{-1}(p_0) \cap e')$.

We can group the tuples in the set according to the twisted tropical cover $\pi : C \to E$ they provide under the cut-and-join construction, see [16, Construction/Theorem 14]. Thus we can write $\tilde{h}_{d,g}$ as

$$\tilde{h}_{d,g} = \frac{1}{(2d)!!} \cdot \sum_{\pi} \#\big\{(\sigma, \eta_1, \ldots, \eta_{g-1}, \alpha) \text{ yielding the cover } \pi\big\}.$$

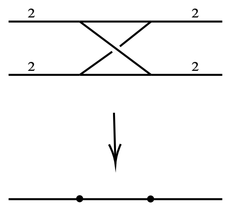

Figure 6. The twisted tropical cover on the right in Figure 3 cut open at the back.

For a fixed cover π, instead of counting tuples yielding π, we can count tuples $(\sigma, \eta_1, \dots, \eta_{g-1})$ yielding the cut twisted tropical cover $\tilde{\pi}$ and then multiply with the number of appropriate α, which we denote by $n_{\tilde{\pi},\pi}$:

$$\tilde{h}_{d,g} = \frac{1}{(2d)!!} \cdot \sum_{\pi} \#\{(\sigma, \eta_1, \dots, \eta_{g-1}) \text{ that provide the cover } \tilde{\pi}\} \cdot n_{\tilde{\pi},\pi}.$$

By the above, the count of the tuples yielding a cover $\tilde{\pi}$ divided by $(2d)!!$ equals

$$2^{g-1} \frac{1}{|\operatorname{Aut}(\tilde{\pi})|} \cdot \prod_V (\omega_V - 1) \cdot \prod_{\tilde{e}} \omega(\tilde{e}) \cdot \prod_K \frac{1}{\omega_K}$$

where the first product goes over the 4-valent vertices, the second over all pairs of twisted internal edges $\tilde{e}$ of $\tilde{\Gamma}$ of weight $\omega(\tilde{e})$ and the third over all twisted pairs of components K consisting of a single edge of weight ω_K. From the above, the number $n_{\tilde{\pi},\pi}$ can be substituted by the expression in eq. (2.1).

We obtain

$$\tilde{h}_{d,g} = \sum_{\pi} \frac{1}{|\operatorname{Aut}(\tilde{\pi})|} 2^{g-1} \cdot \prod_V (\omega_V - 1) \cdot \prod_{\tilde{e}} \omega(\tilde{e}) \cdot \prod_K \frac{1}{\omega_K} \cdot \prod_{e'} \omega(e')^{c_{e'}} \cdot \frac{|\operatorname{Aut}(\tilde{\pi})|}{|\operatorname{Aut}(\pi)|}.$$

A pair of twisted edges e' of Γ of weight $\omega(e')$ having $c_{e'}$ preimages over the base point provides exactly $c_{e'} - 1$ pairs of single-edge-components of weight $\omega(e')$ in the cut cover $\tilde{\pi}$. Vice versa, each such pair of components comes from a pair of edges with multiple preimages over the base point. Therefore the expression $\prod_K \frac{1}{\omega_K} \cdot \prod_{e'} \omega(e')^{c_{e'}}$ simplifies to $\prod_{e'} \omega(e')$. We obtain

$$\tilde{h}_{d,g} = \sum_{\pi} 2^{g-1} \frac{1}{|\operatorname{Aut}(\pi)|} \cdot \prod_V (\omega_V - 1) \cdot \prod_e \omega(e) = \tilde{h}^{\text{trop}}_{d,g}$$

and the theorem is proved. ■

Example 22. Consider the twisted tropical cover π depicted in Figure 3 in the middle with the vertex on the left labelled p_1 and the vertex on the right labelled p_2. Cutting it open, we obtain the twisted tropical cover $\tilde{\pi}$ of $\mathbb{R}$ depicted in Figure 6.

By the correspondence theorem for twisted tropical covers of $\mathbb{R}$ in [16, Theorem 22], it accounts for $(2d)!! = 8$ times its tropical multiplicity many tuples. The cover has an automorphism group of size $2^2 = 4$, as we can independently exchange both pairs of twisted

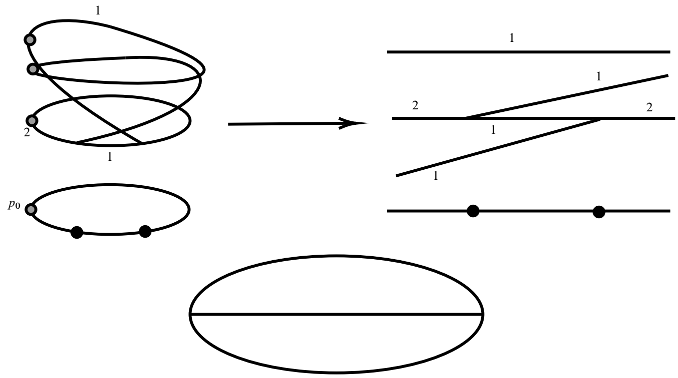

Figure 7. On the top left, there is the quotient cover $\pi\colon \overline{\Gamma} \to E$ of an elliptic twisted tropical cover. Cutting the cover at p_0 and the preimages, we obtain the twisted tropical cover on the top right. At the bottom is the graph $\overline{\Gamma}$.

edges of weight 2, with the edges of weight 1 following along. Its tropical multiplicity thus equals

$$2^2 \cdot \frac{1}{4} = 1.$$

This twisted tropical cover of $\mathbb{R}$ thus accounts for 8 tuples of the form $(\sigma_1, \eta_1, \eta_2)$. By [16, Lemma 16], there are 2 permutations suitable for σ_1, (14)(23) and (12)(34). Fix $\sigma_1 =$ (14)(23) momentarily, the other choice is analogous. Then there are two choices for η_1, (14) of (23). For the next branch point, there are 2 more choices for η_2, (12) or (34). Altogether, we obtain the 8 tuples as expected.

The extra automorphism that the cut cover obtains (we have $\frac{|\operatorname{Aut}(\tilde{\pi})|}{|\operatorname{Aut}(\pi)|} = \frac{4}{2} = 2$) allows to make an additional choice which left end should be glued to which right end. Let us momentarily fix one of our 8 tuples, ((14)(23), (14), (12)). If we label all edges with the corresponding permutations, the two right ends are labelled with (12) and (34). Because of the extra automorphism, we can glue the left end labelled (14) either to (12) or to (34). For each choice, we obtain as many α satisfying $\alpha\tau = \tau\alpha$ as the weight of one end, i.e., 2. We thus obtain 4 possible α to add to each of the 8 tuples, yielding 32 tuples of the form $(\sigma_1, \eta_1, \eta_2, \alpha)$. For the tuple fixed above, the 4 possible α we can add are

$$\big\{(24), (1234), (13), (1432)\big\}.$$

Dividing the 32 tuples by $(2d)!! = 8$, we expect the tropical multiplicity of the right cover of E in Figure 3 to be 4. Indeed, in Example 19 we already computed its tropical multiplicity to be

$$2^2 \cdot \frac{1}{2} \cdot 2 = 4.$$

Example 23. We illustrate another example in Figure 7. On the top left, we have the quotient cover $\overline{\pi}\colon \overline{\Gamma} \to E$ of an elliptic twisted tropical cover of degree 4. Note that $\overline{\Gamma}$, as illustrated at the bottom of Figure 7 has three edges, two of weight 1 and one of weight 2.

On the top right in this figure, we have the twisted tropical cover obtained by cutting $\overline{\pi}$ at p_0 and its preimages. In particular, we obtain two edges of weight 1 arising from the same edge of $\overline{\Gamma}$. This is because this edge curls twice before reattaching again at the bottom to join to an edge of weight 2.

3. Generating series in terms of Feynman integrals

In this section, we express elliptic twisted Hurwitz numbers as Feynman integrals. We assume that $g > 2$ in the following. Consequently, in the quotient covers there cannot be loop edges.

In our context, the following definition of Feynman graph will be needed. These are exactly the graphs that appear as sources for quotients of twisted covers, up to labelling.

Definition 24 (Feynman graph). A Feynman graph is a graph with 2- and 3-valent vertices whose edges are labelled with $q_1, \dots, q_r$ and whose vertices are labelled with $x_1, \dots, x_s$.

A Feynman integral depends on a Feynman graph and the choice of an order Ω of the vertices.

Definition 25 (Edge propagator). Let q_k be an edge of a Feynman graph, adjacent to two vertices x_{k_1} and x_{k_2}, where we assume that $x_{k_1} < x_{k_2}$ in the order Ω.

Given $w \in \mathbb{N}$, we define the coefficient c_w of the following propagator function to be

$$c_w := \begin{cases} (w-1)\cdot w & \text{if } x_{k_1} \text{ and } x_{k_2} \text{ are 2-valent} \\ \sqrt{w-1}\cdot w & \text{if } x_{k_1} \text{ or } x_{k_2} \text{ is 2-valent} \\ w & \text{if neither } x_{k_1} \text{ nor } x_{k_2} \text{ are 2-valent.} \end{cases}$$

We then define the propagator function of the edge q_k to be

$$P(q_k) = \sum_{w=1}^{\infty} c_w \left(\frac{x_{k_1}}{x_{k_2}}\right)^w + \sum_{a_k=1}^{\infty} \Big(\sum_{w|a_k} c_w \Big(\Big(\frac{x_{k_1}}{x_{k_2}}\Big)^w + \Big(\frac{x_{k_2}}{x_{k_1}}\Big)^w\Big)\Big) q_k^{a_k}.$$

Definition 26 (Feynman integral). Let Γ be a Feynman graph and Ω be an order of its vertices. For each edge q_k, we denote its adjacent vertices by x_{k_1} and x_{k_2}, where we assume that $x_{k_1} < x_{k_2}$ in the order Ω. We define the Feynman integral $I_{\Gamma,\Omega}(q_1, \dots, q_r)$ to be

$$I_{\Gamma,\Omega}(q_1, \dots, q_r) = \operatorname{coef}_{[x_1^0 \cdots x_s^0]} \prod_{k=1}^{r} P(q_k).$$

Setting all q_k equal to one variable q, we obtain the Feynman integral $I_{\Gamma,\Omega}(q)$.

Remark 27. Here, we consider Feynman integrals merely as formal power series. In the relation involving (usual) Hurwitz numbers of an elliptic curve, the propagator series can, using a coordinate change, be transformed into a linear combination of the Weierstraß-$\wp$-function and an Eisenstein series. After this coordinate change, the Feynman integral can be viewed as a complex analytic path integral.

Remark 28. Fix a genus $g > 2$. A 3-valent graph of genus 2 has 2 vertices, increasing the genus by one yields 2 more vertices. It follows that a graph of genus g has $2g - 2$ vertices if it is 3-valent. Every 4-valent vertex can be viewed as a merging of 2 3-valent vertices, thus a graph of genus g with c 4-valent vertices and only 3-valent vertices else has $2g - 2 - c$ vertices. It follows that the source of a twisted tropical cover of E has $2g - 2 - c$ vertices, where c denotes the number of 4-valent vertices. That is, $2g - 2 - 2c$ vertices are 3-valent and c are 4-valent. When passing to the quotient cover, its source graph has $g - 1 - c$ many 3-valent vertices and c many 2-valent vertices. Its total number of vertices is thus $g - 1$, independent of the number of 4-valent vertices in the twisted cover.

Definition 29 (Labelled quotient covers). A *labelled quotient cover* is a quotient of a twisted tropical cover for which the vertices and edges of its source are labelled like a Feynman graph.

Fix a base point p_0 of a tropical elliptic curve E. Fix a genus $g > 2$, and $g - 1$ branch points $p_1, \ldots, p_{g-1}$ in E.

Given a labeled quotient cover $\overline{\pi}$, we can define its *multidegree* $a \in \mathbb{N}^r$ to be the tuple whose k-th entry equals the sum of the weights of the preimages of the base point p_0 in the edge q_k. Fix an order Ω on $g - 1$ elements $x_1, \ldots, x_{g-1}$. Let Γ be a Feynman graph.

We define $\tilde{h}_{\Gamma,\Omega,a}$ to be the weighted number of labelled quotient covers whose source is of combinatorial type Γ, whose multidegree equals a and such that the order given by the preimages of the branch points $\overline{\pi}^{-1}(p_1) < \cdots < \overline{\pi}^{-1}(p_{g-1})$ equals Ω. Each such cover is weighted by $\frac{2^{g'} - \delta_{0c}}{2^{c+1}} \cdot 2^{g-1} \cdot \prod_V (\omega_V - 1) \cdot \omega(e)$. Since there are no non-trivial automorphisms in the presence of labels, this equals the multiplicity given in Remark 20.

Proposition 30. *Let Γ be a Feynman graph of genus $g > 2$ and c be its number of 2-valent vertices. Let a be a multidegree and Ω an order. The count of labelled quotient covers equals a coefficient of a Feynman integral:*

$$\tilde{h}_{\Gamma,\Omega,a} = \frac{2^{g'} - \delta_{0c}}{2^{c+1} \cdot 2^{g-1}} \cdot \operatorname{coef}_{[q_1^{a_1} \cdots q_r^{a_r}]} I_{\Gamma,\Omega}(q_1, \ldots, q_r).$$

The proof follows ideas of [4, 5].

Proof. Expanding the product $\prod_{k=1}^r P(q_k)$, the summands are equal to products of the form

$$\prod_{k=1}^r c_{w_k} \left(\frac{x_i}{x_j}\right)^{w_k} \cdot q_k^{a_k}.$$

If a_k is zero, w_k can be any element in $\mathbb{N}$, and $i = k_1$, $j = k_2$. If $a_k > 0$, $w_k | a_k$ and i can be either k_1 or k_2, and j the remaining. To each such summand, we associate a labelled quotient cover in the following way: We start by fixing as preimages of the branch points the vertices x_i as imposed by the order Ω.

For a factor $c_{w_k} (\frac{x_i}{x_j})^{w_k} \cdot q_k^{a_k}$ with $a_k = 0$, we draw an edge labelled q_k which goes from the vertex x_{k_1} to x_{k_2} without crossing over the base point. This is possible since the x_i respect the order Ω. We fix the weight of our edge to be w_k.

For a factor $c_{w_k}(\frac{x_i}{x_j})^{w_k} \cdot q_k^{a_k}$ with $a_k > 0$, we draw an edge labelled q_k: If $i = k_1$ we let it start at x_{k_1} and connect with x_{k_2} (where we think of the edges of our cover as oriented in the way imposed by the order Ω). If $i = k_2$, we let it start at x_{k_2} at connect it with x_{k_1}. We "curl" this edge in such a way that is passes $\frac{a_k}{w_k}$ times over the base point p_0. The weight of the edge in each case is defined to be w_k.

We claim that in this way, we produce a labelled quotient cover contributing to $\tilde{h}_{\Gamma,\Omega,a}$ with $a = (a_1, \ldots, a_r)$. Since we used the edge q_k to connect its neighbouring vertices in Γ, the source of the covers is of combinatorial type Γ by construction. The multidegree is a, since for each k with $a_k = 0$, we let our edge not pass over the base point, whereas for each k with $a_k > 0$ the edge of weight w_k passes $\frac{a_k}{w_k}$ times over the base point, leading to the entry a_k in the multidegree. The order Ω is also respected by construction.

What remains to be seen is that we obtained indeed a cover, i.e., the balancing condition has to be satisfied. This holds true since a product as above only contributes to the Feynman integral $I_{\Gamma,\Omega}(q_1, \ldots, q_r)$ if its total degree in the x_i vanishes. The total power of x_i equals, by construction, the signed sum of the weights of its adjacent edges. The fact that the degree in x_i is zero is thus equivalent to the balancing condition at vertex x_i.

In this way, we obtain a bijection between summands contributing to the Feynman integral and labelled quotient covers. What about multiplicities? In the Feynman integral, a summand contributes $\prod_k c_{w_k}$. Thus, the summand contributes $\frac{2^{g'}-\delta_{0c}}{2^{c+1}} \cdot 2^{g-1} \cdot \prod_k c_{w_k}$ to the right hand side. We have to show that this equals the multiplicity with which the labelled quotient cover is counted in Remark 20, i.e., that $\prod_k c_{w_k} = \prod_V (\omega_V - 1) \cdot \prod_e \omega_e$. Recall that the weight of the edge q_k equals w_k. For a 2-valent vertex V, we have a factor of $\omega_V - 1$, where ω_V denotes the weight of the adjacent edges. We can thus part this contribution into two factors of $\sqrt{\omega_V - 1}$ and shift those towards the adjacent edges.

By definition, if q_k connects two 2-valent vertices, c_{w_k} equals $(w_k - 1) \cdot w_k$—it obtains two factors of $\sqrt{w_k - 1}$ from both its adjacent vertices, and it also contributes its own weight, as every edge does. If q_k connects a 2-valent with a 3-valent vertex, it obtains only one factor of $\sqrt{w_k - 1}$. If both vertices of q_k are 3-valent, it obtains no such factor. It follows that if we reinterpret the product $\prod_V (\omega_V - 1) \cdot \prod_e \omega_e$ as a product over edges by shifting the vertex contributions as square roots into both adjacent edges, we get exactly the contribution c_{w_k} which is used to define the propagator function for the Feynman integral.

Thus the multiplicity with which a labelled quotient cover contributes to $\tilde{h}_{\Gamma,\Omega,a}$ exactly equals the contribution of its corresponding summand in the Feynman integral (up to the factor of $\frac{2^{g'}-\delta_{0c}}{2^{c+1}} \cdot 2^{g-1}$), and the equality holds. ■

The following is the main theorem of this section and expresses the generation function of elliptic twisted Hurwitz numbers as a finite sum over Feynman integrals.

Theorem 31. *Fix a genus $g > 2$. The generating series of twisted Hurwitz numbers can be expressed in terms of Feynman integrals:*

$$\sum_d \tilde{h}_{d,g} q^d = 2^{g-1} \cdot \sum_\Gamma \frac{2^{\frac{1}{2}\cdot(g-c_\Gamma+1)} - \delta_{0c_\Gamma}}{2^{c_\Gamma+1}} \cdot \sharp\operatorname{Aut}(\Gamma) \sum_\Omega I_{\Gamma,\Omega}(q).$$

Here, the first sum on the right hand side goes over all Feynman graphs of genus g and c_Γ denotes their number of 2-valent vertices, while the second sum goes over all orders Ω.

Proof. For a fixed graph Γ, let $\tilde{h}_{d,\Gamma}$ be the number of (unlabelled) quotient covers of degree d for which the combinatorial type of the source curve is Γ. As in Remark 20, each cover $\overline{\pi}$ is counted with multiplicity

$$\frac{2^{g'}-\delta_{0c}}{2^{c+1}}\cdot 2^{g-1}\cdot\frac{1}{\left|\operatorname{Aut}(\overline{\pi})\right|}\prod_V(\omega_V-1)\prod_e\omega(e).$$

There exists a forgetful map ft from the set of labelled quotient covers to the set of unlabelled covers by just forgetting the labels. For an (unlabelled) quotient cover $\overline{\pi}$ whose source is of combinatorial type Γ, the automorphism group of Γ, $\operatorname{Aut}(\Gamma)$, acts transitively on the fibre $\operatorname{ft}^{-1}(\overline{\pi})$ by relabelling vertices and edges. So, to determine the cardinality of the set $\operatorname{ft}^{-1}(\overline{\pi})$, we think of it as the orbit under this action and obtain $\sharp\operatorname{ft}^{-1}(\overline{\pi})=\frac{\sharp\operatorname{Aut}(\Gamma)}{\sharp\operatorname{Aut}(\overline{\pi})}$, since the stabilizer of the action equals the set of automorphisms of $\overline{\pi}$. Each labelled quotient cover in the set $\operatorname{ft}^{-1}(\pi)$ is counted with the same multiplicity

$$\frac{2^{g'}-\delta_{0c}}{2^{c+1}}\cdot 2^{g-1}\cdot\prod_V(\omega_V-1)\prod_e\omega(e).$$

The sum $\sum_{a\mid\sum a_i=d}\sum_\Omega\tilde{h}_{\Gamma,\Omega,a}$ can be reorganized as a sum over unlabelled quotient covers, where for each unlabelled cover, we have to sum the multiplicities for each labelled quotient cover in the fibre under ft. As the multiplicity is the same for each element in the fibre, and there are $\frac{\sharp\operatorname{Aut}(\Gamma)}{\sharp\operatorname{Aut}(\overline{\pi})}$ elements in the fibre, we can see that this sum equals $\sharp\operatorname{Aut}(\Gamma)\cdot\tilde{h}_{d,\Gamma}$.

We conclude

$$\sum_d\tilde{h}_{d,g}q^d=\sum_d\sum_\Gamma\tilde{h}_{d,\Gamma}q^d=\sum_d\sum_\Gamma\frac{1}{\sharp\operatorname{Aut}(\Gamma)}\sum_{a\mid\sum a_i=d}\sum_\Omega\tilde{h}_{\Gamma,\Omega,a}q^d.$$

Now we can replace $\tilde{h}_{\Gamma,\Omega,a}$ by $\frac{2^{g'}-\delta_{0c}}{2^{c+1}}\cdot 2^{g-1}$ times the coefficient of q^a in $I_{\Gamma,\Omega}(q_1,\ldots,q_r)$ by Proposition 30. If we insert $q_k=q$ for all k we can conclude that the coefficient of q^d in $I_{\Gamma,\Omega}(q)$ equals $\frac{2^{g'}-\delta_{0c}}{2^{c+1}}\cdot 2^{g-1}$ times $\sum_{a\mid\sum a_i=d}\tilde{h}_{\Gamma,\Omega,a}$. Thus the generating series above equals

$$\sum_d\tilde{h}_{d,g}q^d=2^{g-1}\cdot\sum_\Gamma\frac{2^{\frac{1}{2}\cdot(g-c_\Gamma+1)}-\delta_{0c_\Gamma}}{2^{c_\Gamma+1}}\cdot\sharp\operatorname{Aut}(\Gamma)\sum_\Omega I_{\Gamma,\Omega}(q).$$

4. The Fock space approach

We shortly review the bosonic Fock space approach for generating series of Hurwitz numbers.

The bosonic Heisenberg algebra $\mathcal{H}$ is the Lie algebra with basis α_n for $n \in \mathbb{Z}$ such that for $n \neq 0$ the following commutator relations are satisfied:

$$[\alpha_n, \alpha_m] = (n \cdot \delta_{n,-m})\alpha_0,$$

where $\delta_{n,-m}$ is the Kronecker symbol and $[\alpha_n, \alpha_m] := \alpha_n\alpha_m - \alpha_m\alpha_n$. The bosonic Fock space F is a representation of $\mathcal{H}$. It is generated by a single "vacuum vector" $v_\emptyset$. The positive generators annihilate $v_\emptyset$: $\alpha_n \cdot v_\emptyset = 0$ for $n > 0$, α_0 acts as the identity and the negative operators act freely. That is, F has a basis b_μ indexed by partitions, where

$$b_\mu = \alpha_{-\mu_1} \cdots \alpha_{-\mu_m} \cdot v_\emptyset.$$

We define an inner product on F by declaring $\langle v_\emptyset | v_\emptyset \rangle = 1$ and α_n to be the adjoint of α_{-n}.

We write $\langle v|A|w\rangle$ for $\langle v|Aw\rangle$, where $v, w \in F$ and the operator A is a product of elements in $\mathcal{H}$, and $\langle A\rangle$ for $\langle v_\emptyset|A|v_\emptyset\rangle$. The first is called a *matrix element*, the second a *vacuum expectation*. We introduce a new formal variable z to keep track of 4-valent vertices, as their number influences the prefactor with which we have to count quotient covers by Remark 20, i.e., $\frac{2^{g'-\delta_{0c}}}{2^{c+1}} \cdot 2^{g-1}$.

Definition 32. The vertex operator is defined by:

$$M = 2 \cdot \left(\sum_{k>0} (k-1) \cdot \alpha_{-k}\alpha_k \cdot z + \frac{1}{2} \sum_{k>0} \sum_{\substack{0<i,j \\ i+j=k}} \alpha_{-j}\alpha_{-i}\alpha_k + \alpha_{-k}\alpha_i\alpha_j \right) \tag{4.1}$$

We note that unlike in the Feynman diagram approach here we do not have to shift vertex contributions into neighbouring edges. Moreover, the global factor of 2 is to take the number of branch points into account. We can also view it as vertex contribution.

We obtain the following result.

Proposition 33. *The twisted double Hurwitz number* $\tilde{h}_g^{\bullet}(\mu, \nu)$ *(see* [16, Definition 4]*) equals a matrix element on the bosonic Fock space:*

$$\tilde{h}_g^{\bullet}(\mu, \nu) = \frac{1}{\prod_i \mu_i \cdot \prod_j \nu_j} \sum_{c=0}^{g-1} \operatorname{coef}_{[z^c]} \left(\langle b_\mu | M^{g-1} | b_\nu \rangle \right) \cdot \frac{2^{\frac{1}{2}(g-c+1)}}{2^{c+1}} \cdot 2^{g-1}.$$

Proof. This statement follows from the cut-and-join equation for twisted double Hurwitz numbers [11, Theorem 6.5] or by combining Wick's theorem with the Correspondence theorem for twisted double Hurwitz numbers in [16]: Wick's theorem ([8, Theorem 5.4.3], [2, Proposition 5.2], [20]) expresses a matrix element as a weighted count of graphs that are obtained by completing local pictures. It turns out that the graphs in question are exactly the quotient covers we enumerate to obtain $\tilde{h}_g^{\bullet}(\mu, \nu)$, multiplied with the factor depending on the number c of 4-valent vertices, as described in Remark 20.

Notice that we have to use the disconnected theory here (•), since the matrix element encodes *all* graphs completing the local pictures and cannot distinguish connected and disconnected graphs.

The local pictures are built as follows: we draw one vertex for each vertex operator. For an α_n with $n > 0$, we draw an edge germ of weight n pointing to the right. If $n < 0$, we draw an edge germ of weight n pointing to the left. For the two Fock space elements b_μ and b_ν, we draw germs of ends: of weights μ_i on the left pointing to the right, of weights ν_i on the right pointing to the left. Wick's theorem states that the matrix element $\langle b_\mu | M^n | b_\nu \rangle$ equals a sum of graphs completing all possible local pictures, where each graph contributes the product of the weights of all its edges (including the ends) and the vertex contributions arising from the vertex operator. A completion of the local pictures can be interpreted as a quotient cover of $\mathbb{R}$ (with suitable metrization).

The vertex operator sums over all the possibilities of the local pictures for the graphs, i.e., it sums over all possibilities how a vertex of a quotient cover can look like. The variable z takes care of how many 4-valent vertices there are. ∎

Combining Proposition 33 with the relation we obtain via cutting in the proof of Theorem 21, we can express twisted Hurwitz numbers of the elliptic curve in terms of matrix elements:

Proposition 34. *A twisted Hurwitz number of the elliptic curve equals a weighted sum of twisted double Hurwitz numbers:*

$$\tilde{h}^{\bullet}_{d,g} = \sum_{\mu \vdash d} \frac{\prod_i \mu_i}{|\operatorname{Aut}(\mu)|} \tilde{h}^{\bullet}_{g}(\mu, \mu).$$

Here, the sum goes over all partitions μ of d.

Proposition 34 is a corollary of the two Correspondence Theorems: given a tropical cover of E, let μ be the partition encoding the weights of the edges mapping to the base point p_0. We mark the preimages of p_0, for which we have $|\operatorname{Aut}(\mu)|$ choices. For each choice, we cut off E at p_0 and the covering curve at the preimages of p_0, obtaining a cover of $\mathbb{R}$ with ramification profiles μ and μ above $\pm\infty$. The cut off tropical cover contributes to $\tilde{h}^{\bullet}_{g}(\mu, \nu)$, but its multiplicity differs from the multiplicity of the cover of E by a factor of $\prod \mu_i$, since the edges we cut off are no longer bounded.

Finally, we obtain an expression of elliptic twisted Hurwitz numbers as a matrix element on the bosonic Fock space as a corollary of Propositions 33 and 34.

Corollary 35. *A twisted Hurwitz number of the elliptic curve E equals a sum of matrix elements on the bosonic Fock space:*

$$\tilde{h}^{\bullet}_{g,d} = \sum_{\mu \vdash d} \frac{1}{|\operatorname{Aut}(\mu)| \prod_i \mu_i} \sum_{c=0}^{g-1} \operatorname{coef}_{[z^c]} \left(\langle b_\mu | M^{g-1} | b_\mu \rangle \right) \cdot \frac{2^{\frac{1}{2}(g-c+1)}}{2^{c+1}} \cdot 2^{g-1}. \quad ∎$$

Acknowledgements. The authors thanks Raphaël Fesler for many useful discussions. Computations have been made using the Computer Algebra System GAP and OSCAR [13, 19]. We thank an anonymous referee for useful comments.

Funding. H. Markwig acknowledges support by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation), Project-ID 286237555, TRR 195.

References

[1] B. Bertrand, E. Brugallé, and G. Mikhalkin, Tropical open Hurwitz numbers. *Rend. Semin. Mat. Univ. Padova* **125** (2011), 157–171 Zbl 1226.14066 MR 2866125

[2] F. Block and L. Göttsche, Fock spaces and refined Severi degrees. *Int. Math. Res. Not. IMRN* **2016** (2016), no. 21, 6553–6580 Zbl 1404.14011 MR 3579972

[3] F. Block and L. Göttsche, Refined curve counting with tropical geometry. *Compos. Math.* **152** (2016), no. 1, 115–151 Zbl 1348.14125 MR 3453390

[4] J. Böhm, K. Bringmann, A. Buchholz, and H. Markwig, Tropical mirror symmetry for elliptic curves. *J. Reine Angew. Math.* **732** (2017), 211–246 Zbl 1390.14191 MR 3717092

[5] J. Böhm, C. Goldner, and H. Markwig, Tropical mirror symmetry in dimension one. *SIGMA Symmetry Integrability Geom. Methods Appl.* **18** (2022), article no. 046 Zbl 1495.14056 MR 4444300

[6] Y. Burman and R. Fesler, Ribbon decomposition and twisted Hurwitz numbers. *Math. Res. Rep.* **5** (2024), 1–19 Zbl 1541.57016 MR 4705806

[7] R. Cavalieri, P. Johnson, and H. Markwig, Tropical Hurwitz numbers. *J. Algebraic Combin.* **32** (2010), no. 2, 241–265 Zbl 1218.14058 MR 2661417

[8] R. Cavalieri, P. Johnson, H. Markwig, and D. Ranganathan, A graphical interface for the Gromov–Witten theory of curves. In *Algebraic geometry: Salt Lake City 2015*, pp. 139–167, Proc. Sympos. Pure Math. 97, American Mathematical Society, Providence, RI, 2018 Zbl 1451.14177 MR 3821170

[9] R. Cavalieri, P. Johnson, H. Markwig, and D. Ranganathan, Counting curves on Hirzebruch surfaces: Tropical geometry and the Fock space. *Math. Proc. Cambridge Philos. Soc.* **171** (2021), no. 1, 165–205 Zbl 1477.14088 MR 4268808

[10] R. Cavalieri and E. Miles, *Riemann surfaces and algebraic curves. A first course in hurwitz theory*. London Math. Soc. Stud. Texts 87, Cambridge University Press, Cambridge, 2016 Zbl 1354.14001 MR 3585685

[11] G. Chapuy and M. Dołęga, Non-orientable branched coverings, b-Hurwitz numbers, and positivity for multiparametric Jack expansions. *Adv. Math.* **409** (2022), article no. 108645 Zbl 1498.05278 MR 4477016

[12] R. Dijkgraaf, Mirror symmetry and elliptic curves. In *The moduli space of curves (Texel Island, 1994)*, pp. 149–163, Progr. Math. 129, Birkhäuser, Boston, MA, 1995 Zbl 0913.14007 MR 1363055

[13] GAP – Groups, Algorithms, and Programming, Version 4.7.8, https://www.gap-system.org/ visited on 28 july 2025

[14] M. A. Hahn and D. Lewański, Wall-crossing and recursion formulae for tropical Jucys covers. *Trans. Amer. Math. Soc.* **373** (2020), no. 7, 4685–4711 Zbl 1451.14155 MR 4127859

[15] M. A. Hahn and D. Lewański, Tropical Jucys covers. *Math. Z.* **301** (2022), no. 2, 1719–1738 Zbl 1487.14114 MR 4418336

[16] M. A. Hahn and H. Markwig, Twisted Hurwitz numbers: Tropical and polynomial structures. *Algebr. Comb.* **7** (2024), no. 4, 1075–1101 Zbl 7921886 MR 4804584

[17] M. A. Hahn, J.-W. M. van Ittersum, and F. Leid, Triply mixed coverings of arbitrary base curves: Quasimodularity, quantum curves and a mysterious topological recursion. *Ann. Inst. Henri Poincaré D* **9** (2022), no. 2, 239–296 Zbl 1499.14090 MR 4450015

[18] A. Hurwitz, Ueber algebraische Gebilde mit eindeutigen Transformationen in sich. *Math. Ann.* **41** (1892), no. 3, 403–442 Zbl 24.0380.02 MR 1510753

[19] OSCAR – Open Source Computer Algebra Research system, Version 0.8.3-DEV, https://www.oscar-system.org/ visited on 28 july 2025

[20] G. C. Wick, The evaluation of the collision matrix. *Phys. Rev. (2)* **80** (1950), 268–272 Zbl 0040.13006 MR 0038281

Communicated by Roland Speicher

Received 3 March 2024; revised 29 March 2025.

Marvin Anas Hahn
School of Mathematics, Trinity College Dublin, 17 Westland Row, D02 PN40 Dublin, Ireland; hahnma@tcd.ie

Hannah Markwig
Fachbereich Mathematik, Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany; hannah@math.uni-tuebingen.de