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On the finite generation of the cohomology of abelian extensions of Hopf algebras

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Abstract. A finite-dimensional Hopf algebra is called quasi-split if it is Morita equivalent to a split abelian extension of Hopf algebras. Combining results of Schauenburg and Negron, it is shown that every quasi-split finite-dimensional Hopf algebra satisfies the finite generation cohomology conjecture of Etingof and Ostrik. This is applied to a family of pointed Hopf algebras in odd characteristic introduced by Angiono, Heckenberger and the first author, proving that they satisfy the aforementioned conjecture.

1. Introduction

1.1. The problem

Let $\Bbbk$ be an algebraically closed field. We say that an augmented $\Bbbk$-algebra A has finitely generated cohomology (fgc for short) when

(a) the cohomology ring $\mathrm{H}(A, \Bbbk) = \bigoplus_{n \in \mathbb{N}_0} \operatorname{Ext}_A^n(\Bbbk, \Bbbk)$ is finitely generated, and

(b) for any finitely generated A-module M, $\mathrm{H}(A, M) = \bigoplus_{n \in \mathbb{N}_0} \operatorname{Ext}_A^n(\Bbbk, M)$ is a finitely generated $\mathrm{H}(A, \Bbbk)$-module.

This definition was extended in [17] as follows. A finite tensor category $\mathcal{C}$ (with unit object $\mathbb{1}$) has finite generation of cohomology (or fgc) when

(a) the $\Bbbk$-algebra $\mathrm{H}(\mathcal{C}) = \bigoplus_{n \in \mathbb{N}_0} \operatorname{Ext}^n(\mathbb{1}, \mathbb{1})$ is finitely generated, and

(b) for any object V in $\mathcal{C}$ the $\mathrm{H}(\mathcal{C})$-module $\mathrm{H}(V) = \bigoplus_{n \in \mathbb{N}_0} \operatorname{Ext}^n(\mathbb{1}, V)$ is finitely generated.

In [17], P. Etingof and V. Ostrik, drawing on fundamental results of [1, 18, 20–23, 45], conjectured that finite tensor categories have fgc, in particular that finite-dimensional Hopf algebras have fgc; this was verified in many cases [5, 10, 13–15, 19, 24, 31, 34–36, 41]; see [5, Section 1.1] for background.

1.2. Morita equivalence

Let H and U be finite-dimensional Hopf algebras. By [35, Theorem 3.4], see also [5, Theorem 3.2.1], if the Drinfeld double $D(H)$ has fgc, then H has fgc. The following argument was intensively used in [5].

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We say that H and U are *Morita equivalent*, denoted $H \sim_{\mathrm{Mor}} U$, if the finite tensor categories $\operatorname{rep} H$ and $\operatorname{rep} U$ are Morita equivalent as in [16, 33]. Equivalently, $H \sim_{\mathrm{Mor}} U$ if there exists an equivalence of braided tensor categories between the Drinfeld centers $\mathcal{Z}(\operatorname{rep} H)$ and $\mathcal{Z}(\operatorname{rep} U)$ [16]. In other words, $H \sim_{\mathrm{Mor}} U$ iff $D(H)$ and $D(U)$ are twist equivalent quasitriangular Hopf algebras (this does not imply that H and U are Morita equivalent as algebras). Thus to prove that H has fgc it suffices to find U such that

(i) $D(U)$ has fgc, and

(ii) $H \sim_{\mathrm{Mor}} U$.

1.3. Extensions

In this paper, we consider the question of fgc for a Hopf algebra H fitting into an exact sequence of finite-dimensional Hopf algebras

$$\Bbbk \to K \to H \to L \to \Bbbk. \tag{1.1}$$

If H has fgc, then K has fgc by [5, Theorem 3.2.1] while it is unclear whether we could infer that L has fgc. Thus, it is natural to ask the following question.

Question 1.1. Given an extension (1.1) such that K and L have fgc, does H also have fgc?

For instance, if K is semisimple and L has fgc, then H has fgc, see [5, Lemma 3.2.5]. The proof uses a variation of the classical Hochschild–Serre spectral sequence but it is not clear (to us) how to proceed when the semisimplicity assumption on K is dropped.

The extension (1.1) is *abelian* when K is commutative and L is cocommutative. In this case there are suitable actions of L on K and of K^* on L, and a pair (σ, τ) of compatible cocycles that determine the possible extensions H. The suitable actions give rise to a double complex $\mathcal{E}$. The abelian extension (1.1) is *split* if the pair (σ, τ) is trivial in the total complex associated to $\mathcal{E}$. Furthermore, by an argument due to P. Schauenburg, extending previous work of G. I. Kac, there is a long exact sequence (the Kac exact sequence) in which the pair (σ, τ) is sent to a Sweedler 3-cocycle on the Hopf algebra $L \bowtie K^*$ with coefficients in $\Bbbk$ as in Proposition 2.6, see [32, Remark 1.11 (3)] for details.

We shall say that an abelian extension is *quasi-split* if it is Morita equivalent to the split extension (with respect to the same suitable actions); see Definition 3.1. Tautologically, split abelian extensions are quasi-split; also, abelian extensions for which the Kac 3-cocycle is trivial are quasi-split (see Corollary 3.5). The starting point of the paper is the following result.

Theorem 1.2. *If (1.1) is a quasi-split abelian extension, then H and $D(H)$ have fgc.*

See Theorem 3.6 for a precise formulation. As a consequence, the dual, any twist and any cocycle deformation of H have fgc. In characteristic 0, Theorem 1.2 is trivial, as a finite-dimensional abelian extension is semisimple.

Theorem 1.2 follows by combining the next two results due to C. Negron and P. Schauenburg respectively.

Theorem 1.3 ([34]). *If U is a cocommutative Hopf algebra,* $\dim U < \infty$*, then $D(U)$ has fgc.*

Notice that Theorem 1.3 is stated in [34] in the language of finite group schemes.

Theorem 1.4 ([39]). *Let* (1.1) *be a split abelian extension of finite-dimensional Hopf algebras. Then there exists a cocommutative Hopf algebra U such that $H \sim_{\mathrm{Mor}} U$.*

Observe that the Drinfeld double of a finite-dimensional cocommutative Hopf algebra is a split abelian extension.

1.4. Pointed Hopf algebras

In the paper [4] (assuming $\operatorname{char} \Bbbk = 0$) Nichols algebras over abelian groups with finite Gelfand–Kirillov dimension of a certain kind were classified. It was observed later in [3] that many of the Nichols algebras over the analogous braided vector spaces in odd characteristic have finite dimension, hence give rise to finite dimensional pointed Hopf algebras by bosonization with group algebras of finite abelian groups. See e.g. [37, Chapter 11] for the notion of bosonization, named there biproduct. Theorem 6.7, the main result of this paper, shows that these pointed Hopf algebras have fgc when the groups are well chosen.

In this paper we consider Nichols algebras as braided Hopf algebras in the sense of [44], i.e., braided vector spaces with algebra and coalgebra structures suitably compatible with the braiding. Recall that a *realization* of a braided Hopf algebra R in the category ${}^{H}_{H}\mathcal{YD}$ of Yetter–Drinfeld modules over a Hopf algebra H is the data of an action and a coaction of H on R such that R becomes a Hopf algebra in the category ${}^{H}_{H}\mathcal{YD}$, with the initial braiding of R equal to the categorical one. The same braided Hopf algebra might have many realizations, thus giving rise to different Hopf algebras by bosonization.

Concretely, in the notation of Section 6, we consider a family of braided vector spaces $\mathcal{V}(\mathfrak{q}, \mathbf{a})$, we fix a suitable finite group Γ such that the Nichols algebra $\mathcal{B}(\mathcal{V}(\mathbf{1}, \mathbf{a}))$ can be realized as a Hopf algebra in the category ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ of Yetter–Drinfeld modules over Γ and proceed in two stages:

(i) we show that the bosonization $H = \mathcal{B}(\mathcal{V}(\mathbf{1}, \mathbf{a}))\#\Bbbk\Gamma$ fits into a *split* abelian exact sequence, hence $D(H)$ and H have fgc by Theorem 1.2;

(ii) for a general $\mathfrak{q}$, we present $\mathcal{B}(\mathcal{V}(\mathfrak{q}, \mathbf{a}))\#\Bbbk\Gamma$ as a cocycle deformation of H, hence $\mathcal{B}(\mathcal{V}(\mathfrak{q}, \mathbf{a}))\#\Bbbk\Gamma$ has fgc too.

We observe:

- Not all Nichols algebras in [3] belong to the family treated here; most of the remaining ones arise from abelian extensions of Hopf superalgebras and will be dealt with elsewhere.
- Many realizations of $\mathcal{B}(\mathcal{V}(\mathbf{1}, \mathbf{a}))$ fit into abelian exact sequences which are not split.
- Many realizations of $\mathcal{B}(\mathcal{V}(\mathfrak{q}, \mathbf{a}))$ fit into exact sequences which are not abelian.

1.5. Organization

Abelian extensions are discussed in Section 2. Section 3 is devoted to Theorem 1.4. The analysis of the Nichols algebras $\mathcal{B}(\mathcal{V}(\mathfrak{q}, \mathbf{a}))$ is delicate as it involves a number of distinct combinatorial features developped in [4]. For clarity, we first deal with the two simplest examples, namely the restricted Jordan plane and the first Laestrygonian algebra, in Sections 4 and 5, respectively. In Section 6, we work out the strategy outlined above for the general Nichols algebras $\mathcal{B}(\mathcal{V}(\mathfrak{q}, \mathbf{a}))$.

1.6. Conventions

For $\ell < \theta \in \mathbb{N}_0$, we set $\mathbb{I}_{\ell,\theta} = \{\ell, \ell+1, \dots, \theta\}$, $\mathbb{I}_\theta = \mathbb{I}_{1,\theta}$. Let $\mathbb{G}_N$ be the group of roots of unity of order N in $\Bbbk$ and $\mathbb{G}'_N$ the subset of primitive roots of order N; $\mathbb{G}_\infty = \bigcup_{N\in\mathbb{N}} \mathbb{G}_N$ and $\mathbb{G}'_\infty = \mathbb{G}_\infty - \{1\}$. If $L \in \mathbb{N}$ and $q \in \Bbbk^\times$, then $(L)_q := \sum_{j=0}^{L-1} q^j$.

All vector spaces, algebras and tensor products are over $\Bbbk$. We use V^* to denote the linear dual of a vector space V, $V^* = \operatorname{Hom}_k(V, k)$.

By abuse of notation, $\langle a_i : i \in I\rangle$ denotes either the group, the subgroup or the vector subspace generated by all a_i for i in an indexing set I, the meaning being clear from the context. Instead, the subalgebra generated by all a_i for $i \in I$ is denoted by $\Bbbk\langle a_i : i \in I\rangle$.

The notation for Hopf algebras is standard: Δ, ε, $\mathcal{S}$ denote the comultiplication, the counit, the antipode (always assumed bijective), respectively. We use the Sweedler notation for the comultiplication and the coactions. Our reference for the theory of Hopf algebras is [37]. Generalities on Nichols algebras can be found in [2].

2. Extensions of Hopf algebras

This section contains a crash exposition of extensions of Hopf algebras.

2.1. Exact sequences

Recall that a sequence of morphisms of Hopf algebras

$$\Bbbk \to A \xrightarrow{\iota} C \xrightarrow{\pi} B \to \Bbbk \tag{2.1}$$

is exact [7, 25, 40] if the following conditions holds:

(i) ι is injective.

(ii) π is surjective.

(iii) $\ker \pi = C\iota(A)^+$.

(iv) $\iota(A) = C^{\operatorname{co}\pi}$.

Remark 2.1. The definition has a simpler shape if we assume that $A \overset{\iota}{\hookrightarrow} C$ is faithfully flat. In this case, if also $\iota(A)$ is stable by the left adjoint action of C, then (i), (ii) and (iii) imply (iv), see [7, Corollaries 1.2.5 and 1.2.14], [40]. Notice that a finite-dimensional Hopf algebra is always free, hence faithfully flat, over any Hopf subalgebra by the Nichols–Zöller theorem.

The exact sequence (2.1) is *abelian* if A is commutative and B is cocommutative. We shall also refer to C in (2.1) as an (abelian when it corresponds) extension of B by A.

The exact sequence (2.1) is *cleft* (see [7, Definition 3.2.13]) if there exist

(i) a unit preserving right B-colinear section $s\colon B \to C$ of the projection π and

(ii) a counit preserving left A-linear retraction $r\colon C \to A$ of the inclusion ι,

both s and r being invertible with respect to the convolution product, such that the following equivalent conditions hold, for all $c \in C$:

(a) $s^{-1}(\pi(c)) = \mathcal{S}(c_{(1)})r(c_{(2)})$.

(b) $s(\pi(c)) = r^{-1}(c_{(1)})c_{(2)}$.

(c) $r^{-1}(c) = s(\pi(c_{(1)}))\mathcal{S}(c_{(2)})$.

(d) $r(c) = c_{(1)}s^{-1}(\pi(c_{(2)}))$.

(e) $rs = \varepsilon_B 1_A$.

The maps s and r are called *cleaving maps* of (2.1).

Remark 2.2 ([40]). If C is finite dimensional, then the extension (2.1) is cleft.

Definition 2.3 ([39, Definition 6.5.2]). The exact sequence (2.1) is called *split* if there exist cleaving maps s and r as above, called *splittings* of (2.1), such that s is an algebra map and r is a coalgebra map.

Cleft extensions can be described using suitable linear maps. A *bicrossed product datum* of Hopf algebras is a collection $(A, B, \rightharpoonup, \rho, \sigma, \tau)$ where A and B are Hopf algebras,

$$\rightharpoonup\colon B \otimes A \to A, \quad \rho\colon B \to B \otimes A, \quad \sigma\colon B \otimes B \to A \quad \text{and} \quad \tau\colon B \to A \otimes A$$

are maps, called, respectively, a weak action, a weak coaction, a cocycle and a dual cocycle, obeying the conditions in [7, Theorem 2.20].

A bicrossed product datum $(A, B, \rightharpoonup, \rho, \sigma, \tau)$ gives rise to a Hopf algebra $A\#_\sigma^\tau B$, called a *bicrossed product*: the underlying vector space is $A \otimes B$, while the multiplication, comultiplication and antipode are determined by the formulas

$$\begin{aligned}
(k\#h)(t\#g) &= k(h_{(1)} \rightharpoonup t)\sigma(h_{(2)}, g_{(1)})\#h_{(3)}g_{(2)},\\
\Delta(k\#h) &= k_{(1)}\tau^{(1)}(h_{(1)})\#\rho(h_{(2)})_i \otimes k_{(2)}\tau^{(2)}(h_{(1)})\rho(h_{(2)})^i\#h_{(3)},\\
\mathcal{S}(a\#b) &= \big(\big(\mathcal{S}\big(\big(\rho(b)_i\big)_{(2)}\big) \rightharpoonup \mathcal{S}\big(\rho(b)^i\big)\big)\#\mathcal{S}\big(\big(\rho(b)_i\big)_{(1)}\big)\big)\mathcal{S}(a)\#1.
\end{aligned}$$

The natural maps $\iota\colon A \to A\#_\sigma^\tau B$ and $\pi\colon A\#_\sigma^\tau B \to B$ fit into a cleft exact sequence

$$\Bbbk \to A \xrightarrow{\iota} A\#_\sigma^\tau B \xrightarrow{\pi} B \to \Bbbk,$$

of Hopf algebras, with cleaving maps

$$s\colon B \to A\#_\sigma^\tau B, \quad s(b) = 1\#b, \quad r\colon A\#_\sigma^\tau B \to A, \quad r(a\#b) = \varepsilon(b)a, \quad a \in A,\ b \in B.$$

Conversely, given a cleft exact sequence (2.1), there exists a bicrossed product datum $(A, B, \rightharpoonup, \rho, \sigma, \tau)$ such that $C \cong A\#_{\sigma}^{\tau} B$; it arises from the cleaving maps s and r as follows:

$$b \rightharpoonup a = s(b_{(1)})as^{-1}(b_{(2)}), \quad \sigma(b\#b') = s(b_{(1)})s(b'_{(1)})s^{-1}(b_{(2)}b'_{(2)}),$$
$$\rho\big(\pi(c)\big) = \pi(c_{(2)}) \otimes r^{-1}(c_{(1)})r(c_{(3)}), \quad \tau\big(\pi(c)\big) = \Delta\big(r^{-1}(c_{(1)})\big)\, r(c_{(2)}) \otimes r(c_{(3)}),$$

for all $a \in A, b \in B, c \in C$. See [7, Theorem 3.2.14].

A special instance of a bicrossed product datum $(A, B, \rightharpoonup, \rho, \sigma, \tau)$ occurs where the cocycle σ and the dual cocycle τ are trivial maps; in this case, we omit the mention of σ and τ and call the bicrossed product datum $(A, B, \rightharpoonup, \rho)$ a *bismash datum*. Notice that in a bismash datum, A is a left B-module algebra and B is a right A-comodule algebra with action $\rightharpoonup$ and coaction ρ.

Given a bismash datum $(A, B, \rightharpoonup, \rho)$, the associated bicrossed product is denoted $A\#B$. The canonical cleaving maps imply that the associated exact sequence

$$\Bbbk \to A \xrightarrow{\iota} A\#B \xrightarrow{\pi} B \to \Bbbk \tag{2.2}$$

is split. Conversely, if (2.1) is a split exact sequence, then the corresponding bicrossed product datum $(A, B, \rightharpoonup, \rho, \sigma, \tau)$ is in fact a bismash datum $(A, B, \rightharpoonup, \rho)$ such that $C \cong A\#B$.

A bismash datum $(A, B, \rightharpoonup, \rho)$ is called a *Singer pair* if A is commutative and B is cocommutative. In this case (2.2) is a split abelian extension of Hopf algebras.

Remark 2.4. Observe that every abelian cleft exact sequence (2.1) gives rise to a Singer pair through the actions $\rightharpoonup$, ρ.

2.2. Matched pairs

We now present a way to produce extensions due to G. I. Kac for abelian extensions and to S. Majid in general. We start by the definition, see [29, Section 7.2]. An *exact factorization* of a Hopf algebra S consists of a pair (G, L) of Hopf subalgebras of S such that the restriction of the multiplication map

$$\text{mult}\colon G \otimes L \to S$$

is a linear isomorphism. Exact factorizations are classified through the following notion.

Definition 2.5 ([29, Definition 7.2.1]). A *matched pair* of Hopf algebras is a collection $(G, L, \triangleright, \triangleleft)$ where L and G Hopf algebras, G is a left L-module coalgebra with action $\triangleright$, L is a right G-module coalgebra with action $\triangleleft$ such that for all $\ell, m \in L$, $x, y \in G$:

$$(\ell m) \triangleleft x = \big(\ell \triangleleft (m_{(1)} \triangleright x_{(1)})\big)(m_{(2)} \triangleleft x_{(2)}), \tag{2.3}$$
$$\ell \triangleright (xy) = \big(\ell_{(1)} \triangleright x_{(1)}\big)\big((\ell_{(2)} \triangleleft x_{(2)}) \triangleright y\big), \tag{2.4}$$
$$\ell_{(1)} \triangleleft x_{(1)} \otimes \ell_{(2)} \triangleright x_{(2)} = \ell_{(2)} \triangleleft x_{(2)} \otimes \ell_{(1)} \triangleright x_{(1)}. \tag{2.5}$$

Proposition 2.6. *Let G and L be Hopf algebras.*

[29, Theorem 7.2.2] *Given a matched pair $(G, L, \triangleright, \triangleleft)$, the coalgebra $G \otimes L$ with the multiplication*

$$(x \otimes \ell)(y \otimes m) = x(\ell_{(1)} \triangleright y_{(1)}) \otimes (\ell_{(2)} \triangleleft y_{(2)})m, \quad \ell, m \in L,\ x, y \in G,$$

is a Hopf algebra denoted $G \bowtie L$. The natural inclusions $G \hookrightarrow G \bowtie L \hookleftarrow L$ form an exact factorization of $G \bowtie L$.

[29, Theorem 7.2.3] *If (G, L) is an exact factorization of a Hopf algebra S, then there are actions $\triangleright$ and $\triangleleft$ such that $(G, L, \triangleright, \triangleleft)$ is a matched pair and*

$$S \simeq G \bowtie L.$$

The Hopf algebra $G \bowtie L$ is called the *double crossproduct* associated to the actions $\triangleleft, \triangleright$. Matched pairs of Hopf algebras give rise to split exact sequences in the following way.

Proposition 2.7 ([29, Proposition 7.2.4]). *If $(G, L, \triangleright, \triangleleft)$ is a matched pair of a Hopf algebras such that* $\dim L < \infty$*, then $(L^*, G, \rightharpoonup, \rho)$ is a bismash datum, where $\rightharpoonup$ and ρ are obtained by dualization; and vice versa.*

In conclusion, finite-dimensional split abelian extensions are determined by exact factorizations of *cocommutative* Hopf algebras (that can be thought of as finite group schemes). We illustrate these notions with some examples, see Examples 3.7 and 3.8 for a discussion of the fgc property.

Group algebras. The exact factorizations of a group algebra $\Bbbk\Sigma$ are in bijective correspondence with the exact factorizations of the group Σ, [27, 28, 43]. Also, the matched pairs of Hopf algebras $(\Bbbk\Gamma, \Bbbk\Lambda, \triangleright, \triangleleft)$ are the linearizations of the matched pairs of groups $(\Gamma, \Lambda, \triangleright, \triangleleft)$.

Lie algebras. The exact factorizations of an enveloping algebra $U(\mathfrak{s})$ are in bijective correspondence with those of the Lie algebra $\mathfrak{s}$; matched pairs of Hopf algebras $(U(\mathfrak{g}), U(\mathfrak{l}), \triangleright, \triangleleft)$ are in bijective correspondence with matched pairs of Lie algebras $(\mathfrak{g}, \mathfrak{l}, \triangleright, \triangleleft)$. See [29, Section 8.3]. More precisely,

- an exact factorization of a Lie algebra $\mathfrak{s}$ consists of a pair $(\mathfrak{g}, \mathfrak{l})$ of Lie subalgebras such that $\mathfrak{s} = \mathfrak{g} \oplus \mathfrak{l}$ (as vector spaces);
- a *matched pair* of Lie algebras is a collection $(\mathfrak{g}, \mathfrak{l}, \triangleright, \triangleleft)$ where $\mathfrak{g}$ and $\mathfrak{l}$ are Lie algebras, $\triangleright$ and $\triangleleft$ are left and right actions $\mathfrak{l} \xleftarrow{\triangleleft} \mathfrak{l} \times \mathfrak{g} \xrightarrow{\triangleright} \mathfrak{g}$ satisfying (2.11) and (2.12) below.

Given such a matched pair, $\mathfrak{g} \bowtie \mathfrak{l} := \mathfrak{g} \oplus \mathfrak{l}$ with the multiplication given by (2.15) is a Lie algebra. Up to identifications, $(\mathfrak{g}, \mathfrak{l})$ is an exact factorization of $\mathfrak{g} \bowtie \mathfrak{l}$. Also, (2.15) is equivalent to $\mathfrak{g}$ and $\mathfrak{l}$ being Lie subalgebras and

$$[\ell, y] = \ell \triangleright y + \ell \triangleleft y, \quad \ell \in \mathfrak{l},\ y \in \mathfrak{g}. \tag{2.6}$$

Conversely if $(\mathfrak{g}, \mathfrak{l})$ is an exact factorization of a Lie algebra $\mathfrak{s}$, then (2.6) defines the actions $\triangleright$ and $\triangleleft$, $(\mathfrak{g}, \mathfrak{l}, \triangleright, \triangleleft)$ is a matched pair and $\mathfrak{s} \simeq \mathfrak{g} \bowtie \mathfrak{l}$.

Restricted Lie algebras. In this example $\operatorname{char} \Bbbk = p > 0$. Let $\mathfrak{s}$ be a restricted Lie algebra with p-operation $s \mapsto s^{[p]}$, see [26, (65), p. 187]. Concretely, given $i \in \mathbb{I}_{p-1}$, recall that $\mathtt{s}_i : \mathfrak{s} \times \mathfrak{s} \to \mathfrak{s}$ is the homogeneous polynomial of degree p defined on a pair (s, t) as the coefficient of X^{i-1} in $\operatorname{ad}(Xs + t)^{p-1}(s)$, where X is a formal variable. Then

$$(s + t)^p = s^p + t^p + \sum_{i=1}^{p-1} \frac{\mathtt{s}_i(s, t)}{i}, \quad s, t \in \mathfrak{s}. \tag{2.7}$$

By definition the p-operation satisfies for all $k \in \Bbbk$, $s, t \in \mathfrak{s}$ the identities

$$(ks)^{[p]} = k^p s^{[p]}, \tag{2.8}$$

$$\operatorname{ad}(s^{[p]}) = \operatorname{ad}(s)^p, \tag{2.9}$$

$$(s + t)^{[p]} = s^{[p]} + t^{[p]} + \sum_{i=1}^{p-1} \frac{\mathtt{s}_i(s, t)}{i}. \tag{2.10}$$

The following definitions are natural.

- An *exact factorization* of $\mathfrak{s}$ is a pair $(\mathfrak{g}, \mathfrak{l})$ of restricted Lie subalgebras such that $\mathfrak{s} = \mathfrak{g} \oplus \mathfrak{l}$ (as vector spaces).
- A *matched pair* of restricted Lie algebras is a collection $(\mathfrak{g}, \mathfrak{l}, \triangleright, \triangleleft)$ where $\mathfrak{g}$ and $\mathfrak{l}$ are restricted Lie algebras, $\triangleright$ and $\triangleleft$ are left and right p-actions $\mathfrak{l} \overset{\triangleleft}{\leftarrow} \mathfrak{l} \times \mathfrak{g} \overset{\triangleright}{\rightarrow} \mathfrak{g}$ satisfying for all $\ell, m \in \mathfrak{l}$, $x, y \in \mathfrak{g}$ the identities

$$[\ell, m] \triangleleft x = [\ell \triangleleft x, m] + [\ell, m \triangleleft x] + \ell \triangleleft (m \triangleright x) - m \triangleleft (\ell \triangleright x), \tag{2.11}$$

$$\ell \triangleright [x, y] = [\ell \triangleright x, y] + [x, \ell \triangleright y] + (\ell \triangleleft x) \triangleright y - (\ell \triangleleft y) \triangleright x, \tag{2.12}$$

$$\ell^{[p]} \triangleleft y = \sum_{1 \le i \le p-1} (\operatorname{ad} \ell)^i \big(\ell \triangleleft (\ell^{p-i} \triangleright y)\big), \tag{2.13}$$

$$\ell \triangleright y^{[p]} = \sum_{1 \le i \le p-1} (-1)^{p-i} (\operatorname{ad} y)^i \big((\ell \triangleleft y^{p-1-i}) \triangleright y\big). \tag{2.14}$$

Lemma 2.8. *Let $\mathfrak{g}$ and $\mathfrak{l}$ be restricted Lie algebras.*

(i) *If $(\mathfrak{g}, \mathfrak{l}, \triangleright, \triangleleft)$ is a matched pair of restricted Lie algebras, then $\mathfrak{g} \bowtie \mathfrak{l} := \mathfrak{g} \oplus \mathfrak{l}$ is a restricted Lie algebra with the multiplication*

$$\big[(x, \ell), (y, m)\big] = \big([x, y] + \ell \triangleright y - m \triangleright x, [\ell, m] + \ell \triangleleft y - m \triangleleft x\big) \tag{2.15}$$

$\ell, m \in \mathfrak{l}$, $x, y \in \mathfrak{g}$; and with p-operation extending those of $\mathfrak{g}$ and $\mathfrak{l}$ and

$$(y + \ell)^{[p]} = y^{[p]} + \ell^{[p]} + \sum_{i=1}^{p-1} \frac{\mathtt{s}_i(y, \ell)}{i}, \quad y \in \mathfrak{g},\ \ell \in \mathfrak{l}, \tag{2.16}$$

where $\mathtt{s}_i : \mathfrak{g} \bowtie \mathfrak{l} \times \mathfrak{g} \bowtie \mathfrak{l} \to \mathfrak{g} \bowtie \mathfrak{l}$ is defined from the Lie bracket (2.15).

(ii) *Let* $(\mathfrak{g}, \mathfrak{l})$ *be an exact factorization of a restricted Lie algebra* $\mathfrak{s}$. *Then* $(\mathfrak{g}, \mathfrak{l}, \triangleright, \triangleleft)$ *with the actions given by* (2.6) *is a matched pair of restricted Lie algebras and* $\mathfrak{s} \simeq \mathfrak{g} \bowtie \mathfrak{l}$ *as restricted Lie algebras.*

Proof. (i): Since $\triangleright, \triangleleft$ are actions that satisfy (2.11) and (2.12), $\mathfrak{g} \bowtie \mathfrak{l}$ with the bracket (2.15) is a Lie algebra [29, Section 8.3]. Thus the maps $\mathtt{s}_i$ are defined and we just have to check that (2.16) gives a p-operation. (2.8) holds because $\mathtt{s}_i$ is homogeneous of degree p. We first verify (2.9) for $s = y \in \mathfrak{g}$. Since both sides are linear operators and the restriction to $\mathfrak{g}$ is a p-operation, it is enough to see that $\operatorname{ad}(y^{[p]})(\ell) \overset{\star}{=} \operatorname{ad}(y)^p(\ell)$ for $\ell \in \mathfrak{l}$. Arguing by induction from (2.6) we prove that for every $N \in \mathbb{N}$ and $\ell \in \mathfrak{l}$

$$(\operatorname{ad} \ell)^N(y) = \ell^N \triangleright y + \sum_{0 \le i \le N-1} (\operatorname{ad} \ell)^i \big(y \triangleleft (\ell^{N-1-i} \triangleright y)\big).$$

Taking $N = p$ and again by (2.6), we conclude that $\star$ holds iff (2.13) is true. Similarly we prove that for every $N \in \mathbb{N}$, $y \in \mathfrak{g}$ and $\ell \in \mathfrak{l}$

$$(\operatorname{ad} y)^N(\ell) = \sum_{0 \le i \le N-1} (-1)^{N+i} (\operatorname{ad} y)^i \big((\ell \triangleleft y^{N-1-i}) \triangleright y\big) + (-1)^N \ell \triangleleft y^N.$$

(Here (2.6) says that $(\operatorname{ad} y)(\ell) = -\ell \triangleright y - \ell \triangleleft y$.) Taking $N = p$, we conclude that $\operatorname{ad}(\ell^{[p]})(y) = \operatorname{ad}(\ell)^p(y)$ holds iff (2.14) is true. Finally

$$\begin{aligned}
\operatorname{ad}(y + \ell)^p &= \operatorname{ad}(y)^p + \operatorname{ad}(\ell)^p + \sum_{i=1}^{p-1} \frac{\operatorname{ad} \mathtt{s}_i(y, \ell)}{i} \\
&= \operatorname{ad}(y^{[p]}) + \operatorname{ad}(\ell^{[p]}) + \sum_{i=1}^{p-1} \frac{\operatorname{ad} \mathtt{s}_i(y, \ell)}{i} \\
&= \operatorname{ad}\big((y + \ell)^{[p]}\big),
\end{aligned}$$

where the first equality is by (2.7), the second because these are p-operations on the subalgebras and the third is by definition. Thus (2.9) holds.

We proceed with (2.10). Let $s = x + \ell$ and $t = y + m$, where $x, y \in \mathfrak{g}$, $\ell, m \in \mathfrak{l}$. Then

$$\begin{aligned}
(s + t)^{[p]} &= \big((x + y) + (\ell + m)\big)^{[p]} \\
&= (x + y)^{[p]} + (\ell + m)^{[p]} + \sum_{i=1}^{p-1} \frac{\mathtt{s}_i\big((x + y), (\ell + m)\big)}{i} \\
&= x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} \frac{\mathtt{s}_i(x, y)}{i} + \ell^{[p]} + m^{[p]} \\
&\quad + \sum_{i=1}^{p-1} \frac{\mathtt{s}_i(\ell, m)}{i} + \sum_{i=1}^{p-1} \frac{\mathtt{s}_i\big((x + y), (\ell + m)\big)}{i}.
\end{aligned}$$

On the other hand,

$$s^{[p]} + t^{[p]} + \sum_{i=1}^{p-1} \frac{\mathrm{s}_i(s,t)}{i} = x^{[p]} + \ell^{[p]} + \sum_{i=1}^{p-1} \frac{\mathrm{s}_i(x,\ell)}{i} + y^{[p]} + m^{[p]} + \sum_{i=1}^{p-1} \frac{\mathrm{s}_i(y,m)}{i} + \sum_{i=1}^{p-1} \frac{\mathrm{s}_i(s,t)}{i}.$$

To show that these two expressions are equal just apply (2.7) to both sides of the equality $((x+y)+(\ell+m))^p = ((x+\ell)+(y+m))^p$. ■

Let $\mathfrak{u}(\mathfrak{s})$ be the restricted enveloping algebra of the restricted Lie algebra $\mathfrak{s}$; it has a PBW-basis with the powers of the generators truncated at p.

Lemma 2.9. *Let $\mathfrak{s}$ be a restricted Lie algebra. The following are equivalent:*

(i) *Exact factorizations of the Hopf algebra $\mathfrak{u}(\mathfrak{s})$.*

(ii) *Exact factorizations of the restricted Lie algebra $\mathfrak{s}$.*

Proof. (i)⇒(ii). If $(\mathfrak{G}, \mathfrak{L})$ is an exact factorization of $\mathfrak{u}(\mathfrak{s})$, then take $\mathfrak{g} = \mathcal{P}(\mathfrak{G})$, $\mathfrak{l} = \mathcal{P}(\mathfrak{L})$. As a consequence of [42, Proposition 13.2.3], $\mathfrak{G} \simeq \mathfrak{u}(\mathfrak{g})$, $\mathfrak{L} \simeq \mathfrak{u}(\mathfrak{l})$, and by the PBW theorem, $(\mathfrak{g}, \mathfrak{l})$ is an exact factorization of $\mathfrak{s}$.

(i)⇐(ii). If $(\mathfrak{g}, \mathfrak{l})$ is an exact factorization of $\mathfrak{s}$, then the multiplication $\mathfrak{u}(\mathfrak{g}) \otimes \mathfrak{u}(\mathfrak{l}) \to \mathfrak{u}(\mathfrak{s})$ is a linear isomorphism–apply the PBW-theorem to the union of bases of $\mathfrak{g}$ and $\mathfrak{l}$. ■

Here are some concrete examples of factorizations of restricted Lie algebras.

Restricted Lie bialgebras. Let us say that a finite-dimensional Lie bialgebra $\mathfrak{b}$ is restricted if and only if its Manin triple $(\mathfrak{p}, \mathfrak{b}, \mathfrak{b}^*)$ is restricted, meaning that $\mathfrak{p}$ is restricted and $\mathfrak{b}$, $\mathfrak{b}^*$ are restricted subalgebras. In particular $(\mathfrak{b}, \mathfrak{b}^*)$ is an exact factorization of $\mathfrak{p}$.

Restricted Lie algebras with triangular decompositions. A triangular decomposition of a Lie algebra $\mathfrak{a}$ is a collection $(\mathfrak{a}_0, \mathfrak{a}_+, \mathfrak{a}_-, (\,|\,))$ where $\mathfrak{a}_0, \mathfrak{a}_-, \mathfrak{a}_+$ are subalgebras of $\mathfrak{a}$ and $(\,|\,) : \mathfrak{a} \times \mathfrak{a} \to \Bbbk$ is a non-degenerate symmetric $\mathfrak{a}$-invariant bilinear form such that $\mathfrak{a}_0$ is abelian,

$$\mathfrak{a} = \mathfrak{a}_- \oplus \mathfrak{a}_0 \oplus \mathfrak{a}_+, \ [\mathfrak{a}_\pm, \mathfrak{a}_0] \subset \mathfrak{a}_\pm, \text{ and } (\mathfrak{a}_+|\mathfrak{a}_+) = (\mathfrak{a}_-|\mathfrak{a}_-) = (\mathfrak{a}_+|\mathfrak{a}_0) = (\mathfrak{a}_0|\mathfrak{a}_-) = 0.$$

A triangular decomposition gives rise to a Manin triple $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ defined by

$$\mathfrak{p} = \mathfrak{a} \times \mathfrak{a}, \ \mathfrak{p}_1 = \operatorname{diag} \mathfrak{a}, \text{ and } \mathfrak{p}_2 = \big\{(a_- + a_0, a_+ - a_0) \colon a_\star \in \mathfrak{a}_\star, \ \star \in \{+, 0, -\}\big\}.$$

If $\mathfrak{a}$ is restricted and $\mathfrak{a}_0, \mathfrak{a}_-, \mathfrak{a}_+$ are restricted subalgebras, then $\mathfrak{p}$ is restricted and $(\mathfrak{p}_1, \mathfrak{p}_2)$ is an exact factorization. There are other factorizations:

- $(\mathfrak{a}_\pm, \mathfrak{a}_0 \oplus \mathfrak{a}_\mp)$ are exact factorizations of $\mathfrak{a}$;
- $(\mathfrak{a}_- \oplus \mathfrak{a}_0, \mathfrak{a}_0 \oplus \mathfrak{a}_+)$ is an exact factorization of $\mathfrak{a} \times \mathfrak{a}_0$.

Restricted $\mathbb{Z}$-graded Lie algebras. A finite-dimensional $\mathbb{Z}$-graded Lie algebra $\mathfrak{s} = \bigoplus_{i=-r}^{t} \mathfrak{s}_i$ (where $r, t \in \mathbb{N}_0$) is *restricted* if the underlying Lie algebra $\mathfrak{s}$ is restricted and $\mathfrak{s}_i^{[p]} \subseteq \mathfrak{s}_{pi}$ for all i. Then $\mathfrak{s}_+ := \bigoplus_{i=1}^{t} \mathfrak{s}_i$ and $\mathfrak{s}_{\leq 0} := \bigoplus_{i=-r}^{0} \mathfrak{s}_i$ form an exact factorization of $\mathfrak{s}$.

3. Quasi-split extensions

3.1. Morita equivalence

Two finite-dimensional Hopf algebras H and U are *Morita equivalent* ($H \sim_{\mathrm{Mor}} U$) iff there exists an equivalence of braided tensor categories between the Drinfeld centers $\mathcal{Z}(\operatorname{rep} H)$ and $\mathcal{Z}(\operatorname{rep} U)$, iff $D(H)$ and $D(U)$ are twist equivalent quasitriangular Hopf algebras.[1]

In other words, $H \sim_{\mathrm{Mor}} U$ if the tensor categories $\operatorname{rep} H$ and $\operatorname{rep} U$ are Morita equivalent. Notice that our defining condition is in fact a characterization of the original definition of Morita equivalence of tensor categories in [16, 33].

Instances of situations when two Hopf algebras H and U are Morita equivalent occur when $U \simeq H^*$, or when $U \simeq H^J$ is obtained from H by twisting the comultiplication by $J \in H \otimes H$, or when $U \simeq H_\sigma$ is obtained from H by twisting the multiplication by $\sigma: H \otimes H \to \Bbbk$.

From now on, we assume that all Hopf algebras in (2.1) are finite-dimensional. In this section, we study the following notion.

Definition 3.1. We shall say that a cleft abelian exact sequence (2.1) is *quasi-split* if, for any choice of cleaving maps, the Hopf algebra C is Morita equivalent to the bismash product $A\#B$ associated to the induced Singer pair (see Remark 2.4).

3.2. Coquasi-Hopf algebras

Recall that quasi-Hopf algebras were introduced by Drinfeld as generalizations of Hopf algebras, where the main difference is that the coassociativity of the comultiplication holds up to a 3-tensor called the associator [12]. Dually, a coquasi-bialgebra or coquasi-Hopf algebra H is a generalization of a bialgebra or a Hopf algebra where the main difference is that the associativity of the multiplication holds up to a dual 3-tensor $\varphi: H \otimes H \otimes H \to \Bbbk$, called the coassociator. See e.g. [39] for details.

Here is the starting point of our analysis.

Definition 3.2 ([39]). Let K and Q be Hopf algebras. A *generalized product coquasi-Hopf algebra* of K and Q is a co-quasi Hopf algebra H together with coquasi-Hopf algebra maps

$$i: K \to H \quad \text{and} \quad j: Q \to H \quad \text{such that } \operatorname{mult}(i \otimes j): K \otimes Q \to H$$

is a linear isomorphism.

[1]Observe that this differs from [5], where it was claimed that $H \sim_{\mathrm{Mor}} U \Leftrightarrow D(H) \simeq D(U)$ as quasi-triangular Hopf algebras; we point out that this discrepancy does not affect the results of *loc. cit.*

Let us say that a finite-dimensional Hopf algebra L and a coquasi-Hopf algebra U are *Morita equivalent* iff there exists an equivalence of braided tensor categories between $\mathcal{Z}(\operatorname{corep} L)$ and $\mathcal{Z}(\operatorname{corep} U)$ or equivalently that the quantum doubles $D(L)$ and $D(U)$ are twist-equivalent. This extends the notion introduced at the beginning of Section 3.1; also in this case we have that L and U are Morita equivalent if and only if their tensor categories of finite dimensional corepresentations are Morita equivalent.

By the results of [39, Section 6], a cleft exact sequence of Hopf algebras (2.1) gives rise to a generalized product coquasi-Hopf algebra H of A^* and B, where

- $H = B \otimes A^*$ as a coalgebra;
- the multiplication is given by
$$(x \otimes \ell)(y \otimes m) = x(\ell_{(1)} \triangleleft y_{(1)}) \otimes (\ell_{(2)} \triangleright y_{(2)})m, \quad \ell, m \in A^*,\ x, y \in B,$$
where the maps $\triangleleft\colon A^* \otimes B \to A^*$ and $\triangleright\colon A^* \otimes B \to B$ are determined by the associated weak action $\rightharpoonup$ and the weak coaction ρ by
$$(\ell \triangleleft x)(a) = \ell(x \rightharpoonup a), \quad \ell \triangleright x = \rho(x)_i \ell\big(\rho(x)^i\big), \quad \ell \in A^*,\ x \in B;$$
- the coassociator $\varphi\colon H \otimes H \otimes H \to \Bbbk$ is determined by the cocycle σ and the dual cocycle τ in the form
$$\begin{aligned}&\varphi(x \otimes \ell \otimes y \otimes m \otimes z \otimes r)\\ &\quad = \varepsilon(x)\, \ell\big(y \rightharpoonup \tau^{(1)}(z_{(1)})\sigma\big(y_{(2)}, \rho(z_{(2)})_i\big)\, m\big(\tau^{(1)}(z_{(1)})\rho(z_{(2)})^i\big)\big)\, \varepsilon(r).\end{aligned}$$

Proposition 3.3. *Given a cleft exact sequence of Hopf algebras* (2.1)*, the Hopf algebra C is Morita equivalent to the coquasi-Hopf algebra H.*

Proof. The main result of [39] implies the existence of an equivalence of monoidal categories
$${}_A(\operatorname{corep} C)_A \simeq \operatorname{corep} H,$$
which amounts to the Morita equivalence of the categories $\operatorname{corep} C$, $\operatorname{corep} H$, and *a fortiori* of C and H. Indeed, an equivalence of braided tensor categories between the Drinfeld centers of $\operatorname{corep} C$ and $\operatorname{corep} H$ was established in [38]. ∎

Combining Propositions 2.7 and 3.3, we obtain the following result.

Corollary 3.4. *Let S be a finite-dimensional Hopf algebra and suppose that S is a double crossproduct of its Hopf subalgebras G and L. Then S is Morita equivalent to a bismash product $L^*\#G$. In particular, if $D(S)$ has fgc so does $L^*\#G$.*

Proof. By Proposition 2.7, the matched pair defining S gives rise to a bismash datum $(L^*, G, \rightharpoonup, \rho)$ hence to a cleft exact sequence of Hopf algebras $\Bbbk \to L^* \xrightarrow{\iota} L^*\#G \xrightarrow{\pi} G \to \Bbbk$, which is split. Let H be the coquasi-Hopf algebra described just before Proposition 3.3; then by this Proposition, $L^*\#G$ is Morita equivalent to H. But by definition H is isomorphic in this case to the double crossproduct S. ∎

3.3. Abelian extensions

Suppose now that the exact sequence (2.1) is abelian. Then the coquasi-Hopf algebra H described above turns out to be the double crossproduct associated to the Singer pair (A, B) with a possibly nontrivial coassociator determined by the cocycles σ and τ. However, the coalgebra H is cocommutative in this case.

Corollary 3.5. *An abelian extension of B by A is quasi-split provided that the coassociator φ is trivial.*

Proof. It follows from the fact that the split extension $A\#B$ is Morita equivalent to $A^* \bowtie B$ (with trivial associator) by Proposition 3.3. ∎

Exactness of the Kac sequence of [39] implies that, since A is finite-dimensional, H is isomorphic as a coquasi-Hopf algebra to the double crossproduct $A^* \bowtie B$ (as generalized products of A^* and B) if and only if (2.1) is isomorphic (as a B-extension of A) to a twisting $(A\#B)_\chi^J$ of the bismash product, where

- $J \in A \otimes A$ is a twist in A, and
- $\chi\colon B \otimes B$ is a 2-cocycle on B,

regarded respectively as a twist in C and a 2-cocycle on C, see [39, Theorem 6.3.6].

The next theorem is the main result of this section.

Theorem 3.6. *Let (A, B) be a Singer pair of finite-dimensional Hopf algebras. Given a quasi-split abelian extension C of B by A, the double $D(C)$ and a fortiori C have fgc. In particular, $A\#B$ has fgc.*

Proof. We have that C is Morita equivalent to the cocommutative Hopf algebra $A^* \bowtie B$. Whence $D(C)$ is twist equivalent to $D(A^* \bowtie B)$, which has fgc by the main result of [34]. Hence $D(C)$ and therefore also C have fgc. ∎

Example 3.7. Let Λ and Γ be finite groups. Consider an exact sequence of Hopf algebras

$$\Bbbk \to \Bbbk^{\Lambda} \to C \to \Bbbk\Gamma \to \Bbbk$$

where $\Bbbk^\Lambda$ is the algebra of functions on Λ. Then $C \simeq \Bbbk^\Lambda \#_\sigma^\tau \Bbbk\Gamma$ is a bicrossed product. The relevant (weak) actions in this case are determined by actions by permutations $\triangleright : \Gamma \times \Lambda \to \Lambda$ and $\triangleleft\colon \Gamma \times \Lambda \to \Gamma$ that make $(\Gamma, \Lambda, \triangleright, \triangleleft)$ into a matched pair of finite groups. Let $\Sigma = \Gamma \bowtie \Lambda$ be the associated double crossproduct group.

The Hopf algebra C is Morita equivalent to a quasi-Hopf algebra $(\Bbbk\Sigma, \omega)$, where $\omega \in H^3(\Sigma, \Bbbk^\times)$ is the 3-cocycle attached to the class of C under the Kac exact sequence (hence in particular, the restriction of ω to Γ and Λ is trivial). It follows from [34] that C has fgc whenever ω is trivial.

For instance, we have $\Bbbk\Sigma \sim_{\mathrm{Mor}} \Bbbk^\Lambda \# \Bbbk\Gamma$, hence $\Bbbk^\Lambda \# \Bbbk\Gamma$ has fgc. This is evident if $\operatorname{char} \Bbbk$ is 0 or coprime to $|\Sigma|$; otherwise it follows alternatively from [5, Lemma 3.2.5] since $\Bbbk^\Lambda$ is semisimple.

Example 3.8. Let $(\mathfrak{g}, \mathfrak{l}, \triangleright, \triangleleft)$ be a matched pair of restricted Lie algebras and $\mathfrak{s} = \mathfrak{g} \bowtie \mathfrak{l}$. The corresponding matched pair of Hopf algebras $(\mathfrak{u}(\mathfrak{g}), \mathfrak{u}(\mathfrak{l}), \triangleright, \triangleleft)$ gives rise to an exact sequence

$$\Bbbk \to \mathfrak{u}(\mathfrak{l})^* \to \mathfrak{u}(\mathfrak{l})^* \# \mathfrak{u}(\mathfrak{g}) \to \mathfrak{u}(\mathfrak{g}) \to \Bbbk.$$

We have $\mathfrak{u}(\mathfrak{s}) \sim_{\mathrm{Mor}} \mathfrak{u}(\mathfrak{l})^* \# \mathfrak{u}(\mathfrak{g})$, hence $\mathfrak{u}(\mathfrak{l})^* \# \mathfrak{u}(\mathfrak{g})$ has fgc by [34].

4. The restricted Jordan plane

In this section and the next two, char $\Bbbk = p$ *is an odd prime* (except when explicitly stated otherwise). In Section 6, we consider a subclass of the finite-dimensional Nichols algebras introduced in [3] and show that their bosonizations with suitable group algebras are split abelian extensions, therefore they have fgc. In this section and in the next, we work out the two simplest examples for illustration.

4.1. The property fgc for bosonizations

In this subsection, char $\Bbbk$ is arbitrary. We record a useful result, a variation of [5, Theorem 3.1.6].

Theorem 4.1. *Let F be a finite group.*

(i) *Assume that $\Bbbk F$ is semisimple. If R is a finite-dimensional Hopf algebra in ${}^{\Bbbk F}_{\Bbbk F}\mathcal{YD}$ that has fgc, then $R \# \Bbbk F$ has fgc.*

(ii) *If R is a finite-dimensional Hopf algebra in ${}^{\Bbbk^F}_{\Bbbk^F}\mathcal{YD}$ that has fgc, then $R \# \Bbbk^F$ has fgc.*

Proof. We sketch the proof for the reader's convenience. Let K be either $\Bbbk F$ as in (i) or $\Bbbk^F$ as in (ii), so clearly K is semisimple. Let R be as in (i) or (ii) accordingly.

Since the proofs of [31, Corollary 3.13] and [5, Lemma 3.1.4] just require that $\Bbbk$ is a field, we conclude that the algebra $\mathrm{H}(R, \Bbbk)$ is Noetherian. Now [5, Lemma 3.1.1] also holds for any field, hence $\mathrm{H}(R, \Bbbk)^K$ is finitely generated.

On the other hand, there is an isomorphism $\mathrm{H}(R\#K, \Bbbk) \simeq \mathrm{H}(R, \Bbbk)^K$, see [41, Theorem 2.17]. Hence $\mathrm{H}(R\#K, \Bbbk)$ is finitely generated.

Next, given a finitely generated $R\#K$-module M, one can prove that $\mathrm{H}(R\#K, M)$ is finitely generated as an $\mathrm{H}(R\#K, \Bbbk)$-module repeating word-by-word the proof of the analogous fact in [5, Theorem 3.1.6]. ∎

For further developments, it would be useful to extend Theorem 4.1 to an arbitrary semisimple Hopf algebra K.

Lemma 4.2. *Let A be a finite-dimensional Hopf algebra and $U \in {}^A_A\mathcal{YD}$ such that $\mathcal{B}(U)$ is finite-dimensional. If $\mathcal{B}(U)\#A$ has fgc, then so does the Nichols algebra $\mathcal{B}(U)$.*

Proof. Since $\mathcal{B}(U)\#A$ is free over $\mathcal{B}(U)$, [5, Theorem 3.2.1] implies the claim. ∎

4.2. The minimal bosonization

We begin with the following basic example. The *block* $\mathcal{V}(1,2)$ is the braided vector space with a basis $\{x, y\}$ such that

$$\begin{aligned} c(x \otimes x) &= x \otimes x, & c(y \otimes x) &= x \otimes y, \\ c(x \otimes y) &= (y + x) \otimes x, & c(y \otimes y) &= (y + x) \otimes y. \end{aligned} \tag{4.1}$$

The Nichols algebra $\mathcal{B}(\mathcal{V}(1,2))$ is called the *restricted Jordan plane*.

Lemma 4.3 ([11]). *The restricted Jordan plane is generated by x, y with relations*

$$yx - xy + \tfrac{1}{2}x^2, \quad x^p, \quad y^p. \tag{4.2}$$

The set $\{x^a y^b : 0 \le a, b < p\}$ is a basis of $\mathcal{B}(\mathcal{V}(1,2))$, so $\dim \mathcal{B}(\mathcal{V}(1,2)) = p^2$.

The minimal bosonization of $\mathcal{V}(1,2)$ arises as follows. Let $\Gamma = \mathbb{Z}/p = \langle g \rangle$. We realize $\mathcal{V}(1,2)$ in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ by

$$g \cdot x = x, \quad g \cdot y = y + x, \quad \deg x = \deg y = g.$$

Thus the Hopf algebra $H = \mathcal{B}(\mathcal{V}(1,2))\#\Bbbk\Gamma$ has dimension p^3. We get a presentation of H by generators x, y, g, where we identify $x = x \otimes 1$, $y = y \otimes 1$, $g = 1 \otimes g \in H$, with defining relations

$$x^p = y^p = 0, \quad g^p = 1, \quad gx = xg, \quad gy = yg + xg, \quad yx = xy - \tfrac{1}{2}x^2.$$

The comultiplication of H is determined by

$$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x, \quad \Delta(y) = y \otimes 1 + g \otimes y.$$

In addition, the monomials $g^i x^j y^\ell$, $0 \le i, j, \ell \le p-1$, form a basis of H.

Let $K = \Bbbk\langle x, g\rangle \subset H$ and $L = \Bbbk[\zeta]/(\zeta^p)$ with ζ primitive.

Lemma 4.4. *The Hopf algebra H fits into a split abelian extension $\Bbbk \to K \xrightarrow{\iota} H \xrightarrow{\pi} L \to \Bbbk$, where ι is the inclusion and π is defined by $\pi(x) = 0$, $\pi(g) = 1$ and $\pi(y) = \zeta$.*

Proof. The defining relations of H imply that K is commutative. Furthermore L is generated by the primitive element ζ, whence cocommutative. Clearly π is well-defined and $\ker \pi = HK^+$. We thus obtain an abelian exact sequence $\Bbbk \to K \xrightarrow{\iota} H \xrightarrow{\pi} L \to \Bbbk$.

Since $y^p = 0 = -\mathcal{S}(y)^p$, there exists a unique algebra map $s\colon L \to H$ such that $s(\zeta) = -\mathcal{S}(y) = g^{-1}y$. Clearly, $\pi s = \mathrm{id}_L$. Being an algebra map, s is automatically invertible for the convolution product. The L-colinearity of s follows from the relation

$$\begin{aligned} (s \otimes \mathrm{id})\Delta(\zeta) &= g^{-1}y \otimes 1 + 1 \otimes \zeta = (\mathrm{id} \otimes \pi)\Delta s(\zeta) \\ &= (\mathrm{id} \otimes \pi)(g^{-1}y \otimes g^{-1} + 1 \otimes g^{-1}y). \end{aligned}$$

Dually, let $r\colon H \to K$ be the linear map defined by

$$r(g^i x^j y^\ell) = g^i x^j \text{ if } \ell = 0, \quad \text{and} \quad r(g^i x^j y^\ell) = 0 \text{ if } \ell > 0.$$

It is clear that r is K-linear, $r\iota = \mathrm{id}_K$ and $rs = \varepsilon_L 1_K$. We next show that r is a coalgebra map. Since $r|_K = \mathrm{id}_K$, we have $\Delta r(x) = (r \otimes r)\Delta(x)$, for all $x \in K$. Let $I = \ker r \subseteq H$, in other words, I is the linear span of all monomials $g^i x^j y^\ell$ with $\ell > 0$. The relation $\Delta(y) = y \otimes 1 + g \otimes y$ implies that $\Delta(I) \subseteq I \otimes H + H \otimes I$. Therefore $(r \otimes r)\Delta(I) = 0$, implying that r is a coalgebra map. This shows that (s, r) is a splitting and finishes the proof of the lemma. ∎

Remark 4.5 ([8]). The Drinfeld double of H fits into an abelian exact sequence

$$\Bbbk \to \mathbf{R} \to D(H) \to \mathfrak{u}\big(\mathfrak{sl}_2(\Bbbk)\big) \to \Bbbk,$$

where $\mathbf{R}$ is a local commutative Hopf algebra.

The following result appeared already in [36] with a different proof.

Proposition 4.6. *The Hopf algebra $\mathcal{B}(\mathcal{V}(1,2))\#\Bbbk\Gamma$ and the Nichols algebra $\mathcal{B}(\mathcal{V}(1,2))$ have fgc.*

Proof. By Lemma 4.4, we may apply Theorem 1.2 and Lemma 4.2. ∎

4.3. More bosonizations

To deal with different realizations of the Jordan plane, we recall the notions of YD-pairs and YD-triples that are available in any characteristic. Let A be a Hopf algebra; as usual $G(A)$ denotes the group of group-like elements of A.

- A pair $(g, \chi) \in G(A) \times \mathrm{Alg}(A, \Bbbk)$ is called a *YD-pair* for A if

$$\chi(h)\, g = \chi(h_{(2)})h_{(1)}\, g\, \mathcal{S}(h_{(3)}), \quad h \in A. \tag{4.3}$$

A YD-pair (g, χ) gives rise to $\Bbbk_g^\chi \in {}^A_A\mathcal{YD}$ of dimension 1, with action and coaction given by χ and g, respectively. Any one-dimensional object in ${}^A_A\mathcal{YD}$ is like this.

- A collection (g, χ, η) where (g, χ) is a YD-pair for A and $\eta \in \mathrm{Der}_{\chi,\chi}(A, \Bbbk)$ is called a *YD-triple* for A if

$$\eta(h)g = \eta(h_{(2)})h_{(1)}g\mathcal{S}(h_{(3)}), \quad h \in A, \tag{4.4}$$

$$\text{and } \chi(g) = \eta(g) = 1. \tag{4.5}$$

Notice that the existence of a YD-triple for A when $\dim A < \infty$ forces that $\operatorname{char}\Bbbk > 0$, since $\eta \in A^*$ is a non-zero (χ, χ)- primitive.

A YD-triple (g, χ, η) gives rise to $\mathcal{V}_g(\chi, \eta) \in {}^A_A\mathcal{YD}$, defined as the vector space with a basis $\{x, y\}$, whose A-action and A-coaction are given by

$$h \cdot x = \chi(h)x, \quad h \cdot y = \chi(h)y + \eta(h)x, \quad h \in A; \quad \delta(x) = g \otimes x, \quad \delta(y) = g \otimes y.$$

By assumption (4.5), $\mathcal{V}_g(\chi, \eta) \simeq \mathcal{V}(1, 2)$ as a braided vector space.

We now turn back to the assumption that $\operatorname{char}\Bbbk = p > 2$.

Question 4.7. Let (g, χ, η) be a YD-triple for A where $\dim A < \infty$. Does $\mathcal{B}(\mathcal{V}_g(\chi, \eta))\#A$ have fgc?

We point out that Theorem 4.1 does not apply in the present situation, by the following observation.

Remark 4.8. Let A be a finite-dimensional Hopf algebra that admits a YD-triple (g, χ, η). Then A is not semisimple.

Proof. Observe that the restriction $\eta\colon \langle g\rangle \to \Bbbk$ is a morphism of abelian groups, hence $\eta(g^p) = 0$, which implies that p divides the order of g. Thus $\Bbbk\langle g\rangle$ is not semisimple, and a fortiori A is not semisimple by [37, Proposition 10.3.4]. ■

There are examples fitting into abelian exact sequences that are not necessarily split.

Remark 4.9. Let F be a finite group and let (g, χ, η) be a YD-triple for $\Bbbk F$. It consists of $g \in Z(F)$, $\chi \in \widehat{F} := \operatorname{Hom}_{\text{gps}}(F, \Bbbk^\times)$ and $\eta \in \operatorname{Der}_{\chi,\chi}(\Bbbk F, \Bbbk)$ such that $\chi(g) = \eta(g) = 1$. Let $N := \ker \chi \cap Z(F) \lhd F$. On one hand we consider the subalgebra of $H = \mathcal{B}(\mathcal{V}_g(\chi, \eta))\#\Bbbk F$:

$$K := \Bbbk\langle x, y\colon y \in N\rangle \simeq \Bbbk\langle x\rangle\#\Bbbk N.$$

Here K is commutative but not necessarily cocommutative. On the other hand, χ induces a character $\overline{\chi}$ of F/N. Let $\Bbbk\zeta \in {}^{\Bbbk(F/N)}_{\Bbbk(F/N)}\mathcal{YD}$ corresponding to the YD-pair $(e, \overline{\chi})$. Then

$$L := \mathcal{B}(\zeta)\#\Bbbk(F/N) \simeq \Bbbk[\zeta]/(\zeta^p)\#\Bbbk(F/N);$$

here ζ is primitive. Clearly L is cocommutative but not necessarily commutative.

Let $\iota\colon K \to H$ be the inclusion and let $\pi\colon H \to L$ be the map defined by $\pi(x) = 0$, $\pi(y) = \zeta$ and $\pi(\gamma) =$ the class of γ in F/N. Then H fits into the abelian exact sequence $\Bbbk \to K \xrightarrow{\iota} H \xrightarrow{\pi} L \to \Bbbk$, which is not split, for instance, when the exact sequence of groups $1 \to N \to F \to F/N \to 1$ is not split.

In the setting of the previous remark, the Hopf algebra H^* has fgc.

Proposition 4.10. *Let $H = \mathcal{B}(\mathcal{V}_g(\chi, \eta))\#\Bbbk F$, where F is a finite group and (g, χ, η) is a YD-triple for $\Bbbk F$. Then H^* has fgc.*

Proof. Arguing as in [8, Lemma 1.5], we see that $H^* \simeq \mathcal{R}\#\Bbbk^F$, where $\mathcal{R} \simeq \mathcal{B}(W)$ is isomorphic to $\mathcal{B}(\mathcal{V}(1, 2))$ as algebras (although not as braided Hopf algebras). Anyway, $\mathcal{R}$ has fgc by Proposition 4.6, hence H^* has fgc by Theorem 4.1. ■

5. The first Laestrygonian algebra $\mathcal{B}(\mathfrak{L}_q(1, \mathscr{G}))$

5.1. The Nichols algebra $\mathcal{B}(\mathfrak{L}_q(1, \mathscr{G}))$

The next example of interest to us depends on the data: $q \in \Bbbk^\times$ and $a \in \mathbb{F}_p{}^\times$. Let

$$\mathtt{r} \in \{1 - p, 2 - p, \dots, -2, -1\} \text{ such that } \mathtt{r} \equiv 2a \mod p.$$

The *ghost* is the integer $\mathscr{G} := -\mathtt{r} \in \mathbb{I}_{p-1}$; since p is odd, $\mathscr{G}$ determines a. To this data, we attach the braided vector space $\mathfrak{L}_q(1,\mathscr{G})$ with basis $\mathfrak{b} = \{x_1, y_1, x_2\}$ and braiding given by

$$\big(c(b \otimes b')\big)_{b,b' \in \mathfrak{b}} = \begin{pmatrix} x_1 \otimes x_1 & (y_1 + x_1) \otimes x_1 & q\, x_2 \otimes x_1 \\ x_1 \otimes y_1 & (y_1 + x_1) \otimes y_1 & q\, x_2 \otimes y_1 \\ q^{-1} x_1 \otimes x_2 & q^{-1}(y_1 + a x_1) \otimes x_2 & x_2 \otimes x_2 \end{pmatrix}. \tag{5.1}$$

Thus $V_1 := \Bbbk x_1 + \Bbbk y_1 \simeq \mathcal{V}(1,2)$ and $V_2 := \Bbbk x_2$ satisfy $c\colon V_i \otimes V_j = V_j \otimes V_i$, $i, j \in \{1,2\}$; in particular V_1 and V_2 are braided subspaces of V. We introduce

$$z_0 := x_2, \quad z_{n+1} := y_1 z_n - q z_n y_1, \quad n > 0. \tag{5.2}$$

Lemma 5.1 ([3, Section 4.3.1]). *The algebra* $\mathcal{B}(\mathfrak{L}_q(1,\mathscr{G}))$ *is presented by generators* x_1, y_1, x_2 *and relations* (4.2)*, and, in the notation* (5.2)*,*

$$x_1 x_2 = q\, x_2 x_1, \tag{5.3}$$
$$z_{1+\mathscr{G}} = 0, \tag{5.4}$$
$$z_t z_{t+1} = q^{-1} z_{t+1} z_t, \quad 0 \le t < \mathscr{G}, \tag{5.5}$$
$$z_t^p = 0, \quad 0 \le t \le \mathscr{G}. \tag{5.6}$$

The algebra $\mathcal{B}(\mathfrak{L}_q(1,\mathscr{G}))$ *has a PBW-basis*

$$B = \big\{x_1^{m_1} y_1^{m_2} z_{\mathscr{G}}^{n_{\mathscr{G}}} \cdots z_1^{n_1} z_0^{n_0} \colon m_i, n_j \in \mathbb{I}_{0,p}\big\}, \tag{5.7}$$

hence $\dim \mathcal{B}(\mathfrak{L}_q(1,\mathscr{G})) = p^{\mathscr{G}+3}$.

5.2. A suitable realization

In order to realize $\mathfrak{L}_q(1,\mathscr{G})$ in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ for some finite group Γ, we need to assume that q is a root of 1. Set $d := \operatorname{ord} q$; then d is coprime to $p = \operatorname{char} \Bbbk$. Fix a positive integer f which is a multiple of pd. A suitable choice of group is

$$\Gamma = \mathbb{Z}/f \times \mathbb{Z}/f = \langle g_1 \rangle \oplus \langle g_2 \rangle, \quad \text{where } \operatorname{ord} g_1 = \operatorname{ord} g_2 = f.$$

It is not difficult to see that $\mathfrak{L}_q(1,\mathscr{G})$ can be realized in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ by

$$\begin{aligned} &g_1 \cdot x_1 = x_1, && g_1 \cdot y_1 = y_1 + x_1, && g_1 \cdot x_2 = q x_2, \\ &g_2 \cdot x_1 = q^{-1} x_1, && g_2 \cdot y_1 = q^{-1}(y_1 + a x_1), && g_2 \cdot x_2 = x_2, \\ &\deg x_1 = g_1, && \deg y_1 = g_1, && \deg x_2 = g_2. \end{aligned} \tag{5.8}$$

Therefore the Hopf algebra $\widetilde{H} := \mathcal{B}(\mathfrak{L}_q(1,\mathscr{G}))\#\Bbbk\Gamma$ is presented by generators x_1, y_1, x_2, g_1, g_2, with relations (4.2), (5.3), (5.4), (5.5), (5.6), and

$$g_1^f = 1, \qquad g_2^f = 1, \qquad g_1 g_2 = g_2 g_1, \tag{5.9}$$
$$g_1 x_1 = x_1 g_1, \qquad g_1 y_1 = y_1 g_1 + g_1 x_1, \qquad g_1 x_2 = q x_2 g_1, \tag{5.10}$$
$$g_2 x_1 = q^{-1} x_1 g_2, \qquad g_2 y_1 = q^{-1}(y_1 g_2 + a x_1 g_2), \qquad g_2 x_2 = x_2 g_2. \tag{5.11}$$

The comultiplication of $\widetilde{H}$ is determined by $\Delta(g_i) = g_i \otimes g_i$, $i = 1, 2$, and

$$\Delta(x_1) = x_1 \otimes 1 + g_1 \otimes x_1, \ \Delta(y_1) = y_1 \otimes 1 + g_1 \otimes y_1, \ \Delta(x_2) = x_2 \otimes 1 + g_2 \otimes x_2. \quad (5.12)$$

Clearly $\dim \widetilde{H} = p^{\mathcal{G}+3} f^2$.

5.3. The split case

In this subsection, we deal with $\mathcal{B}(\mathfrak{L}_1(1, \mathcal{G}))$; Let Γ be as above with f divisible by pd with $d = \operatorname{ord} q$. Let $K \subseteq H := \mathcal{B}(\mathfrak{L}_1(1, \mathcal{G}))\#\Bbbk\Gamma$ be the Hopf subalgebra

$$K = \Bbbk\langle x_1, g_1, g_2 \rangle.$$

We shall consider the restricted enveloping algebra

$$L = \mathfrak{u}(\mathfrak{l}),$$

where $\mathfrak{l}$ is the restricted Lie algebra defined as follows. Let $V(\mathcal{G})$ be the simple $\mathfrak{sl}(2)$-module of highest weight $\mathcal{G}$. Let $E = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Pick a basis $(v_n)_{n \in \mathbb{I}_{0,\mathcal{G}}}$ of $V(\mathcal{G})$ such that

$$E \cdot v_n = v_{n+1}, \quad n \in \mathbb{I}_{0,\mathcal{G}-1}, \quad E \cdot v_{\mathcal{G}} = 0.$$

Let $\mathfrak{l} = V(\mathcal{G}) \rtimes \Bbbk E$, a Lie subalgebra of the motion Lie algebra $V(\mathcal{G}) \rtimes \mathfrak{sl}(2)$; it follows from Lemma 2.8 that $\mathfrak{l}$ is restricted with p-operation equal to 0. That is, L is presented by generators v_0 and E with defining relations, in terms of $v_{n+1} = Ev_n - v_n E$, $n \in \mathbb{I}_{0,\mathcal{G}-1}$,

$$v_m v_n = v_n v_m, \quad v_n^p = 0, \quad m, n \in \mathbb{I}_{0,\mathcal{G}}, \quad E v_{\mathcal{G}} - v_{\mathcal{G}} E = 0, \quad E^p = 0.$$

Proposition 5.2. *The Hopf algebra $H = \mathcal{B}(\mathfrak{L}_1(1, \mathcal{G}))\#\Bbbk\Gamma$ fits into a split abelian exact sequence $\Bbbk \to K \xrightarrow{\iota} H \xrightarrow{\pi} L \to \Bbbk$, where ι is the inclusion and π is defined by*

$$\pi(g_1) = 1, \quad \pi(g_2) = 1, \quad \pi(x_1) = 0, \quad \pi(y_1) = E, \quad \pi(x_2) = v_0. \quad (5.13)$$

Proof. By assumption, the Hopf subalgebra K is commutative. In addition L is cocommutative. Notice that $\dim K = pf^2$. We see by inspection that K is normal, i.e., $HK^+ = K^+H$. Thus we have an abelian exact sequence

$$\Bbbk \to K \xrightarrow{\iota} H \to H/HK^+ \to \Bbbk.$$

Since $\dim H = p^{\mathcal{G}+3} f^2$, $\dim H/HK^+ = p^{\mathcal{G}+2} = \dim L$. Now (5.13) determines an algebra map $\pi \colon H \to L$ by (5.2) and the defining relations of $\mathcal{B}(\mathfrak{L}_1(1, \mathcal{G}))$. The map π is surjective, has $HK^+ \subseteq \ker \pi$ and preserves the comultiplication since the classes of y_1 and x_2 are primitive in H/HK^+. By dimension counting, π induces an isomorphism of Hopf algebras $H/HK^+ \to L$.

Observe that $\pi(z_n) = v_n$, for all $n \geq 0$. To define a splitting, we argue as in the proof of Lemma 4.4. The universal property of $\mathfrak{u}(\mathfrak{l})$ implies the existence of a unique algebra map $s \colon L = \mathfrak{u}(\mathfrak{l}) \to H$ such that

$$s(E) = -\mathcal{S}(y_1), \quad s(v_n) = -\mathcal{S}(z_n), \quad \forall n = 0, 1, \dots.$$

Clearly $\pi s = \mathrm{id}_L$. To see that s is L-colinear, it will be enough to verify that the maps $(s \otimes \mathrm{id}_L)\Delta$ and $(\mathrm{id} \otimes \pi)\Delta s$ agree on E and v_0, because both $(s \otimes \mathrm{id}_L)\Delta$ and $(\mathrm{id}_H \otimes \pi)\Delta s$ are algebra maps and E and v_0 generate L as an algebra. This follows at once from (5.12).

Let now $r: H \to K$ be the linear map defined on the basis of H arising from (5.7) by

$$r\big(g_1^c g_2^b x_1^{m_1} y_1^{m_2} z_{\mathfrak{G}}^{n_{\mathfrak{G}}} \cdots z_1^{n_1} z_0^{n_0}\big) = \begin{cases} g_1^c g_2^b x_1^{m_1}, & \text{if } m_2 + n_{\mathfrak{G}} + \cdots + n_0 = 0, \\ 0, & \text{otherwise.} \end{cases}$$

As in the proof of Lemma 4.4, we see that r is a K-linear retraction of ι and that $rs = \varepsilon_L 1_K$. Moreover, r is a coalgebra map. To see this, we consider the subspace $I = \ker r$ which coincides with the linear span of all monomials $g_1^c g_2^b x_1^{m_1} y_1^{m_2} z_{\mathfrak{G}}^{n_{\mathfrak{G}}} \ldots z_1^{n_1} z_0^{n_0}$ such that $m_2 + n_{\mathfrak{G}} + \cdots + n_0 > 0$. The defining relations of H imply that I is a left ideal. We claim that $\Delta(I) \subseteq I \otimes H + H \otimes I$. Indeed, (5.12) implies that $\Delta(y_1), \Delta(x_2) \in I \otimes H + H \otimes I$. By [4, Lemma 4.2.5] and the comultiplication formula of the bosonization, there exist $\nu_{j,n} \in \Bbbk$ such that

$$\Delta(z_n) = z_n \otimes 1 + \sum_{j=0}^{n} \nu_{j,n}\, x_1^{n-j} g_1^j g_2 \otimes z_j, \quad \forall n \in \mathbb{I}_{0,\mathfrak{G}}.$$

Thus $\Delta(z_n) \in I \otimes H + H \otimes I$, for all $n \geq 0$. Then the claim follows. Since $r|_K = \mathrm{id}_K$, we conclude that r is a coalgebra map. Thus (s, r) is a splitting and the proof of the Proposition is complete. ■

Proposition 5.3. *The Hopf algebra $H = \mathcal{B}(\mathfrak{L}_1(1,\mathfrak{G}))\#\Bbbk\Gamma$ and its double $D(H)$ have fgc.*

Proof. This follows from Proposition 5.2 and Theorem 3.6. ■

5.4. The general case

Recall that $q \in \Bbbk^\times$ has order d, that f is a multiple of pd and that $\Gamma = \langle g_1, g_2 \colon g_1^f = g_2^f = 1,\ g_1 g_2 = g_2 g_1 \rangle \simeq \mathbb{Z}/f \times \mathbb{Z}/f$.

Theorem 5.4. *The Hopf algebra $\widetilde{H} = \mathcal{B}(\mathfrak{L}_q(1,\mathfrak{G}))\#\Bbbk\Gamma$ has fgc.*

Proof. Consider the bilinear form $\sigma\colon \Gamma \times \Gamma \to \Bbbk^\times$ determined by

$$\sigma(g_i, g_j) = \begin{cases} 1, & i = j, \\ q, & i = 1,\ j = 2, \\ 1, & i = 2,\ j = 1. \end{cases}$$

Then σ is a 2-cocycle in Γ. Let $\vartheta\colon \Gamma \times \Gamma \to \Bbbk^\times$, $\vartheta(g,h) = \sigma(g,h)\sigma(h,g)^{-1}$ be the associated antisymmetric bilinear form. Thus,

$$\vartheta(g_i, g_j) = \begin{cases} 1, & i = j, \\ q, & i = 1,\ j = 2, \\ q^{-1}, & i = 2,\ j = 1. \end{cases}$$

As in Section A.3 of the Appendix, let $\mathcal{F}_\sigma : {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD} \to {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ be the monoidal functor associated to σ. The image under $\mathcal{F}_\sigma$ of the braided vector space $V = \mathfrak{L}_1(1, \mathcal{G})$ in Section 5.2 is $\mathcal{F}_\sigma(V) = V$ with the same grading as V and Γ-action (A.4); that is,

$$\begin{aligned} &g_1 \cdot x_1 = x_1, && g_1 \cdot y_1 = y_1 + x_1, && g_1 \cdot x_2 = q x_2, \\ &g_2 \cdot x_1 = q^{-1} x_1, && g_2 \cdot y_1 = q^{-1}(y_1 + a x_1), && g_2 \cdot x_2 = x_2, \\ &\deg x_1 = g_1, && \deg y_1 = g_1, && \deg x_2 = g_2. \end{aligned} \tag{5.14}$$

Hence $\mathcal{F}_\sigma(V) = \mathfrak{L}_q(1, \mathcal{G})$. By Lemma A.4. the bosonization $\widetilde{H} = \mathcal{B}(\mathfrak{L}_q(1, \mathcal{G}))\#\Bbbk\Gamma$ is a cocycle deformation of the bosonization $H = \mathcal{B}(\mathfrak{L}_1(1, \mathcal{G}))\#\Bbbk\Gamma$. By Proposition 5.3, the double $D(H)$ has fgc. Hence $\widetilde{H}$ also has fgc, as claimed. ∎

Corollary 5.5. *Let $q \in \mathbb{G}_\infty$. The Nichols algebra $\mathcal{B}(\mathfrak{L}_q(1, \mathcal{G}))$ has fgc.*

6. Pointed Hopf algebras over abelian groups

6.1. A class of braided vector spaces

We now proceed with a family of Hopf algebras which are bosonizations of some Nichols algebras introduced in [3], analogues in odd characteristic of Nichols algebras appearing in [4]. We show that, up to a cocycle deformation, the Hopf algebra $H = \mathcal{B}(\mathcal{V}(\mathfrak{q}, \mathbf{a}))\#\Bbbk\Gamma$ fits into a split abelian exact sequence $\Bbbk \to K \to H \to \mathfrak{u}(\mathfrak{l}) \to \Bbbk$, where the restricted Lie algebra $\mathfrak{l}$ is determined explicitly.

Data. We shall consider braided vector spaces depending on:

- ∘ two positive integers $t < \theta$;
- ∘ a matrix $\mathfrak{q} = (q_{ij})_{i,j \in \mathbb{I}_\theta}$ such that

$$q_{ij} q_{ji} = 1, \quad q_{ii} = 1, \quad i, j \in \mathbb{I}_\theta,\ i \neq j. \tag{6.1}$$

- ∘ a family $\mathbf{a} = (a_{ij})_{\substack{i \in \mathbb{I}_{t+1,\theta},\\ j \in \mathbb{I}_t}}$ with entries in $\mathbb{F}_p$. We lift this family to $\mathbb{Z}$ as follows:

- ◇ when $a_{ij} \neq 0$, we take

$$\mathtt{r}_{ij} \in \{1 - p, 2 - p, \dots, -2, -1\} \text{ such that } \mathtt{r}_{ij} \equiv 2a_{ij} \mod p, \tag{6.2}$$

  and then we set $\mathcal{G}_{i,j} := -\mathtt{r}_{ij} \in \mathbb{I}_{p-1}$.
- ◇ when $a_{ij} = 0$, we lift it to $\mathcal{G}_{i,j} = 0$.

The family $\mathcal{G} := (\mathcal{G}_{i,j})_{\substack{i \in \mathbb{I}_{t+1,\theta},\\ j \in \mathbb{I}_t}}$, equivalent to $\mathbf{a}$, is the *ghost*; both $\mathbf{a}$ and $\mathcal{G}$ are needed.

Definition. We define a braided vector space $(\mathcal{V}(\mathfrak{q}, \mathbf{a}), c)$ from the data above. First, it has a decomposition $\mathcal{V}(\mathfrak{q}, \mathbf{a}) = V_1 \oplus \cdots \oplus V_t \oplus \cdots \oplus V_\theta$ such that

$$c(V_i \otimes V_j) = V_j \otimes V_i, \quad i, j \in \mathbb{I}_\theta.$$

Then we assume:

- If $j \in \mathbb{I}_t$, then $V_j \simeq \mathcal{V}(1, 2)$ (the blocks). Let $\{x_j, y_j\}$ be a basis of V_j realizing (4.1):

$$\begin{aligned} c(x_j \otimes x_j) &= x_j \otimes x_j, & c(y_j \otimes x_j) &= x_j \otimes y_j, \\ c(x_j \otimes y_j) &= (y_j + x_j) \otimes x_j, & c(y_j \otimes y_j) &= (y_j + x_j) \otimes y_j. \end{aligned} \tag{6.3}$$

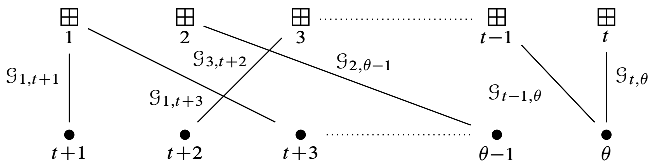
- If $i \in \mathbb{I}_{t+1,\theta}$, then $\dim V_i = 1$; these are the points. We fix a basis $\{x_i\}$ of V_i. Thus, $\{x_i : i \in \mathbb{I}_\theta\} \amalg \{y_j : j \in \mathbb{I}_t\}$ is a basis of $\mathcal{V}(\mathfrak{q}, \mathbf{a})$.

  The braidings $c_{ij} = c_{|V_i \otimes V_j}$, $i, j \in \mathbb{I}_\theta$ are given by the data $\mathfrak{q}$ and $\mathbf{a}$ as follows:

  - If $i, j \in \mathbb{I}_\theta$, then
  $$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i. \tag{6.4}$$
  - If $i, j \in \mathbb{I}_t$ are blocks, then $c_{ij} = q_{ij}\tau$, τ being the flip, for $i \neq j$ while c_{jj} is given by (6.3).
  - If $j \in \mathbb{I}_t$ a block and $i \in \mathbb{I}_{t+1,\theta}$ is a point, then $c_{|(V_i \oplus V_j) \otimes (V_i \oplus V_j)}$ is given by (6.4) and

$$c(y_j \otimes x_i) = q_{ji} x_i \otimes y_j, \quad c(x_i \otimes y_j) = q_{ij}(y_j + a_{ij} x_j) \otimes x_i. \tag{6.5}$$

  As in [4], $\mathcal{V}(\mathfrak{q}, \mathbf{a})$ is described by a diagram of the following shape:



That is, there are t blocks, $\theta - t$ points and a line decorated by $\mathcal{G}_{k,\ell}$ when $\mathcal{G}_{k,\ell} \neq 0$, joining the block k with the point ℓ; this graph is admissible in the sense of [4, Definition 1.3.7].

Remark 6.1. The data above is an $\mathfrak{ab}$-triple $\mathcal{T} = (\mathbf{n}, \mathfrak{q}, \mathbf{a})$ of rank θ, cf. Section A.1, as follows:

- $\mathbf{n} = (n_j)_{j \in \mathbb{I}_\theta}$ is given by $n_j = 2$, if $1 \leq j \leq t$; and $n_j = 1$, if $t < j \leq \theta$.
- the matrix $\mathfrak{q} = (q_{ij})_{i,j \in \mathbb{I}_\theta}$ is as given above with the constraint (6.1);
- the family $\mathbf{t} = (\mathbf{t}_{ij})_{i,j \in \mathbb{I}_\theta}$, where $\mathbf{t}_{ij} \in \operatorname{End} \Bbbk^{n_j}$ is given by

$$\begin{aligned} \mathbf{t}_{ij} &= 0, && \text{when } t < j \leq \theta \text{ or } 1 \leq i, j \leq t; \\ \mathbf{t}_{ij}(x_j) = 0, \quad \mathbf{t}_{ij}(y_j) &= a_{ij} x_j, && \text{when } 1 \leq j \leq t \text{ and } t < i \leq \theta. \end{aligned}$$

If $t = \theta = 1$, then we have the restricted Jordan plane as in Section 4. If $t = 1$ and $\theta = 2$, then we recover $\mathfrak{L}_q(1, \mathcal{G})$ as in Section 5.

Grading. The subspace $\mathcal{U}$ of $\mathcal{V} = \mathcal{V}(\mathfrak{q}, \mathbf{a})$ spanned by $(x_i)_{i\in\mathbb{I}_\theta}$ is a braided subspace and $\mathcal{U} \subseteq \mathcal{V}$ is a filtration of braided vector spaces. The associated graded vector space $\operatorname{gr}\mathcal{V} = \mathcal{U} \oplus \mathcal{V}/\mathcal{U}$ is of diagonal type; indeed, if $\bar{y}_j$ is the class of y_j in $\mathcal{V}/\mathcal{U}$, $j \in \mathbb{I}_t$, then $(x_i)_{i\in\mathbb{I}_\theta} \amalg (\bar{y}_j)_{j\in\mathbb{I}_t}$ is a basis of $\operatorname{gr}\mathcal{V}$ and the braiding of $\operatorname{gr}\mathcal{V}$ is given by

$$\begin{aligned} c(x_i \otimes x_k) &= q_{ik}\, x_k \otimes x_j, & c(\bar{y}_j \otimes x_i) &= q_{ji}\, x_i \otimes \bar{y}_j,\\ c(x_i \otimes \bar{y}_j) &= q_{ij}\, \bar{y}_j \otimes x_i, & c(\bar{y}_j \otimes \bar{y}_\ell) &= q_{j\ell}\, \bar{y}_\ell \otimes \bar{y}_j, \end{aligned} \qquad \text{for all } i,k \in \mathbb{I}_\theta,\ j,\ell \in \mathbb{I}_t.$$

Precisely, let $Q \simeq \mathbb{Z}^{\theta+t}$ be the free abelian group with basis $(\alpha_i)_{i\in\mathbb{I}_\theta} \amalg (\beta_j)_{j\in\mathbb{I}_t}$; then the braiding is given by the bilinear form $\mathbf{p}\colon Q \times Q \to \Bbbk^\times$ defined by

$$\mathbf{p}(\alpha_i \otimes \alpha_j) = \mathbf{p}(\alpha_i \otimes \beta_j) = \mathbf{p}(\beta_i \otimes \alpha_j) = \mathbf{p}(\beta_i \otimes \beta_j) = q_{ij}$$

whenever i and j make sense. By (6.1), $\mathcal{B}(\operatorname{gr}\mathcal{V})$ is a quantum linear space and we have

$$\mathbf{p}(\gamma \otimes \delta) = \mathbf{p}(\delta \otimes \gamma) \quad \forall \gamma, \delta \in Q.$$

Now consider the grading of $\mathcal{V}$ and its extension to $T(\mathcal{V})$ given by

$$\deg x_i = \alpha_i, \quad \deg y_j = \beta_j.$$

Given points $h, \ell \in \mathbb{I}_{t+1,\theta}$ we set

$$ш_{h,\mathbf{n}} := (\operatorname{ad}_c y_1)^{n_1} \cdots (\operatorname{ad}_c y_t)^{n_t} x_h, \quad \mathbf{n} = (n_1, \dots, n_t) \in \mathbb{N}_0^t; \tag{6.6}$$

$$\mathcal{A}_h := \big\{\mathbf{n} \in \mathbb{N}_0^t : 0 \le \mathbf{n} \le \mathcal{G}_h = (\mathcal{G}_{h,1}, \dots, \mathcal{G}_{h,t})\big\}, \text{ ordered lexicographically.} \tag{6.7}$$

In terms of the bilinear form $\mathbf{p}$, we also set

$$\mathbf{p}_{h,\ell;\mathbf{m},\mathbf{n}} := \mathbf{p}_{\deg ш_{h,\mathbf{m}}, \deg ш_{\ell,\mathbf{n}}} = \Big(\prod_{k,j\in\mathbb{I}_t} q_{kj}^{m_k n_j}\Big)\Big(\prod_{k\in\mathbb{I}_t} q_{k\ell}^{m_k}\Big)\Big(\prod_{j\in\mathbb{I}_t} q_{hj}^{n_j}\Big) q_{h\ell},$$

for $\mathbf{m} \in \mathcal{A}_h$, $\mathbf{n} \in \mathcal{A}_\ell$.

Remark 6.2. In the definition (6.6), ad_c means the braided adjoint. However, it could be replaced by a sequence of q-commutators (with various q). More precisely, given $h \in \mathbb{I}_{t+1,\theta}$ and $\mathbf{n} = (n_1, \dots, n_t) \in \mathbb{N}_0^t$, set $ш = ш_{h,\mathbf{n}}$. Then

$$ш = y_j\, \widetilde{ш} - q\, \widetilde{ш}\, y_j, \tag{6.8}$$

where $j = \min\{i \in \mathbb{I}_t : n_i \neq 0\}$, $\tilde{\mathbf{n}} = (0, \dots, 0, n_j - 1, n_{j+1}, \dots, n_t)$, $\widetilde{ш} = ш_{h,\tilde{\mathbf{n}}}$ and

$$q = q_{jh} \prod_{j \le i \le t} q_{ji}^{n_i}.$$

See the proof of [4, Lemma 7.2.3]. In particular, if $\mathfrak{q} = \mathbf{1}$ is the matrix with all entries equal to 1, then $q = 1$ and we can replace ad_c by the usual adjoint in (6.6).

The Nichols algebra $\mathcal{B}(\mathcal{V}(\mathfrak{q}, \mathbf{a}))$. The following result is not included in [3] but its proof is similar to that of [3, Sections 4.3.1 and 6.1].

Lemma 6.3. *The algebra $\mathcal{B}(\mathcal{V}(\mathfrak{q}, \mathbf{a}))$ is presented by generators x_i, $i \in \mathbb{I}_\theta$, y_j, $j \in \mathbb{I}_t$, and relations*

$$x_j^p = 0,\ y_j^p = 0,\ y_j x_j - x_j y_j + \tfrac{1}{2} x_j^2 = 0,\ j \in \mathbb{I}_t; \tag{6.9}$$

$$x_k x_j = q_{kj}\, x_j x_k,\ x_k y_j = q_{kj}\, y_j x_k,\ y_k y_j = q_{kj}\, y_j y_k\ k \neq j \in \mathbb{I}_t; \tag{6.10}$$

$$x_j x_h = q_{jh}\, x_h x_j,\ j \in \mathbb{I}_t,\ h \in \mathbb{I}_{t+1,\theta}; \tag{6.11}$$

$$(\operatorname{ad}_c y_j)^{1+\mathcal{G}_{jh}}(x_h) = 0,\ j \in \mathbb{I}_t,\ h \in \mathbb{I}_{t+1,\theta}; \tag{6.12}$$

$$Ш_{h,\mathbf{n}}^p = 0,\ \mathbf{n} \in \mathcal{A}_h,\ h \in \mathbb{I}_{t+1,\theta}; \tag{6.13}$$

$$Ш_{h,\mathbf{m}} Ш_{\ell,\mathbf{n}} = \mathbf{p}_{h,\ell;\mathbf{m},\mathbf{n}}\, Ш_{\ell,\mathbf{n}} Ш_{h,\mathbf{m}},\ h, \ell \in \mathbb{I}_{t+1,\theta},\ \mathbf{m} \in \mathcal{A}_h,\ \mathbf{n} \in \mathcal{A}_\ell. \tag{6.14}$$

A basis of $\mathcal{B}(\mathcal{V}(\mathfrak{q}, \mathbf{a}))$ is given by $B =$

$$\Big\{ x_1^{m_1} y_1^{m_2} \cdots x_t^{m_{2t-1}} y_t^{m_{2t}} \prod_{\substack{h \in \mathbb{I}_{t+1,\theta} \\ \mathbf{n} \in \mathcal{A}_h}} Ш_{h,\mathbf{n}}^{n_{h,\mathbf{n}}} : 0 \le n_{h,\mathbf{n}},\ m_j < p \text{ if } j \in \mathbb{I}_{2t},\ h \in \mathbb{I}_{t+1,\theta},\ \mathbf{n} \in \mathcal{A}_h \Big\}. \tag{6.15}$$

Hence $\dim \mathcal{B}(\mathcal{V}(\mathfrak{q}, \mathbf{a})) = p^{2t + \sum_{h \in \mathbb{I}_{t+1,\theta}} |\mathcal{A}_h|}$.

Proof. Argue as in the proofs of [3, Sections 4.3.1 and 6.1]; relation (6.14) follows as in the discussion in [4, p. 107]. ∎

6.2. A suitable realization

In order to realize $\mathcal{V}(\mathfrak{q}, \mathbf{a})$ in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ for some finite group Γ, we need to assume that the entries of the matrix $\mathfrak{q}$ are roots of 1. Set

$$d := \operatorname{lcm}\{\operatorname{ord} q_{ij} : i, j \in \mathbb{I}_\theta\};$$

then d is coprime to $p = \operatorname{char} \Bbbk$. Fix a positive integer f which is a multiple of pd. A suitable choice of group is

$$\Gamma = (\mathbb{Z}/f)^\theta = \langle g_1 \rangle \oplus \cdots \oplus \langle g_\theta \rangle, \quad \text{where } \operatorname{ord} g_1 = \operatorname{ord} g_2 = \cdots \operatorname{ord} g_\theta = f.$$

In other words, Γ is generated by $g_1, \dots, g_\theta$ with relations

$$g_i^f = 1, \quad g_i g_j = g_j g_i, \quad i, j \in \mathbb{I}_\theta. \tag{6.16}$$

It is not difficult to see that $\mathcal{V}(\mathfrak{q}, \mathbf{a})$ can be realized in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ by

$$\begin{aligned}
&g_k \cdot V_\ell = q_{k\ell} \operatorname{id}_{V_\ell}, && k \in \mathbb{I}_t,\ \ell \in \mathbb{I}_\theta,\ k \neq \ell; \text{ or } k, \ell \in \mathbb{I}_{t+1,\theta}; \\
&g_i \cdot x_j = q_{ij} x_j, \quad g_i \cdot y_j = q_{ij}(y_j + a_{ij} x_j), && i \in \mathbb{I}_{t+1,\theta},\ j \in \mathbb{I}_t, \\
&g_j \cdot x_j = x_j, \quad g_j \cdot y_j = y_j + x_j, && j \in \mathbb{I}_t, \\
&\deg x_\ell = g_\ell, \quad \deg y_j = g_j, && \ell \in \mathbb{I}_\theta,\ j \in \mathbb{I}_t.
\end{aligned} \tag{6.17}$$

Let $\widetilde{H}$ denote the Hopf algebra $\mathcal{B}(\mathcal{V}(\mathfrak{q}, \mathbf{a}))\#\Bbbk\Gamma$; it has a presentation by generators x_i, y_j, g_k, $i, k \in \mathbb{I}_\theta$, $j \in \mathbb{I}_t$ with relations (6.9), (6.10), (6.11), (6.12), (6.13), (6.14), (6.16) and those induced by (6.17). The comultiplication is given by

$$\Delta(x_i)=x_i\otimes 1+g_i\otimes x_i,\ \Delta(y_j)=y_j\otimes 1+g_j\otimes y_j,\ \Delta(g_i)=g_i\otimes g_i,\ i\in\mathbb{I}_\theta,\ j\in\mathbb{I}_t. \quad (6.18)$$

One has

$$\dim \widetilde{H} = p^{2t+\sum_{h\in\mathbb{I}_{t+1,\theta}}|\mathcal{A}_h|} f^\theta.$$

6.3. The split case

In this subsection we deal with $\mathcal{B}(\mathcal{V}(\mathbf{1}, \mathbf{a}))$, where $\mathbf{1} \in \Bbbk^\theta$ has all entries equal to 1. Let Γ be as above with f divisible by pd with $d = \operatorname{ord} q$. Consider the Hopf subalgebra $K \subseteq H = \mathcal{B}(\mathcal{V}(\mathbf{1}, \mathbf{a}))\#\Bbbk\Gamma$

$$K = \Bbbk\langle x_1, \dots, x_t, g_1, \dots, g_\theta\rangle.$$

We shall consider the restricted enveloping algebra

$$L = \mathfrak{u}(\mathfrak{l}),$$

where $\mathfrak{l}$ is the restricted Lie algebra defined by the following steps.

- $\mathfrak{g} := \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_t$ is the direct sum of r copies $\mathfrak{g}_j \simeq \mathfrak{sl}(2)$, $j \in \mathbb{I}_t$.
- $E_j \in \mathfrak{g}_j$ is the element corresponding to $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$; $\mathfrak{n} := \Bbbk E_1 \oplus \cdots \oplus \Bbbk E_t \hookrightarrow \mathfrak{g}$.
- For any point $h \in \mathbb{I}_{t+1,\theta}$, let $V(\mathcal{G}_{h,j})$ be the simple $\mathfrak{g}_j$-module of highest weight $\mathcal{G}_{h,j}$. Pick a basis $(v_{h,j;n})_{n\in\mathbb{I}_{0,\mathcal{G}_{h,j}}}$ of $V(\mathcal{G}_{h,j})$ such that

$$E_j \cdot v_{h,j;m} = v_{h,j;m+1}, \quad 0 \le m < \mathcal{G}_{h,j}, \quad E_j \cdot v_{h,\mathcal{G}_{hij}} = 0.$$

- For any point $h \in \mathbb{I}_{t+1,\theta}$, let $V(\mathcal{G}_h)$ be the simple $\mathfrak{g}$-module

$$V(\mathcal{G}_h) := V(\mathcal{G}_{h,1}) \otimes \cdots \otimes V(\mathcal{G}_{h,t}).$$

  Recalling the notation (6.7), a basis of $V(\mathcal{G}_h)$ is formed by the elements

$$v_{h,\mathbf{m}} := v_{h,1,[m_1]} \otimes \cdots \otimes v_{h,t,[m_t]}, \quad \mathbf{m} = (m_1, \dots, m_t) \in \mathcal{A}_h.$$

  For $h \in \mathbb{I}_{t+1,\theta}$, set $v_h := v_{h,1,[0]} \otimes \cdots \otimes v_{h,t,[0]}$. Then for any $\mathbf{m} \in \mathcal{A}_h$, we have

$$v_{h,\mathbf{m}} = E_1^{m_1} \cdots E_t^{m_t} \cdot v_h.$$

  Indeed, $V(\mathcal{G}_h)$ is the simple $\mathfrak{g}$-module of highest weight $\mathcal{G}_h = (\mathcal{G}_{h,1}, \dots, \mathcal{G}_{h,t})$.
- Finally, let $\mathfrak{l} = V(\mathcal{G}) \rtimes \mathfrak{n}$, where

$$V(\mathcal{G}) := V(\mathcal{G}_{t+1}) \oplus \cdots \oplus V(\mathcal{G}_\theta).$$

By Lemma 2.8, $\mathfrak{l}$ is a restricted Lie algebra with p-operation equal to 0. The restricted enveloping algebra $L = \mathfrak{u}(\mathfrak{l})$ is presented by generators E_j, $j \in \mathbb{I}_t$, and v_h, $h \in \mathbb{I}_{t+1,\theta}$; set

$$v_{h,\mathbf{m}} = (\operatorname{ad} E_1)^{m_1} \cdots (\operatorname{ad} E_t)^{m_t}(v_h).$$

Then the defining relations are

$$E_j E_k = E_k E_j, \qquad j,k \in \mathbb{I}_t; \tag{6.19}$$

$$(\operatorname{ad} E_j)^{1+\mathcal{G}_{jh}}(v_h) = 0, \qquad j \in \mathbb{I}_t,\ h \in \mathbb{I}_{t+1,\theta}; \tag{6.20}$$

$$v_{h,\mathbf{m}} v_{i,\mathbf{n}} = v_{i,\mathbf{n}} v_{h,\mathbf{m}}, \qquad i,h \in \mathbb{I}_{t+1,\theta},\ \mathbf{m} \in \mathcal{A}_h,\ \mathbf{n} \in \mathcal{A}_i; \tag{6.21}$$

$$E_j^p = 0, \qquad j \in \mathbb{I}_t; \tag{6.22}$$

$$v_{h,\mathbf{m}}^p = 0, \qquad h \in \mathbb{I}_{t+1,\theta},\ \mathbf{m} \in \mathcal{A}_h. \tag{6.23}$$

One has

$$\dim L = p^{t+\sum_{h\in\mathbb{I}_{t+1,\theta}} |\mathcal{A}_h|}.$$

Remark 6.4. Clearly $\mathfrak{l}$ is a Lie subalgebra of $V(\mathcal{G}) \rtimes \mathfrak{g}$; we do not need the reference to $\mathfrak{g}$ here but it will be necessary in further developments.

Proposition 6.5. *The Hopf algebra $H = \mathcal{B}(\mathcal{V}(\mathbf{1},\mathbf{a}))\#\Bbbk\Gamma$ fits into a split abelian exact sequence $\Bbbk \to K \xrightarrow{\iota} H \xrightarrow{\pi} L \to \Bbbk$, where ι is the inclusion and $\pi\colon H \to L$ is determined by*

$$\pi(g_i) = 1,\ i \in \mathbb{I}_\theta,\ \pi(x_j) = 0,\ \pi(y_j) = E_j,\ j \in \mathbb{I}_t,\ \pi(x_\ell) = v_\ell,\ \ell \in \mathbb{I}_{t+1,\theta}. \tag{6.24}$$

The proof has the same pattern as the proof of Proposition 5.2.

Proof. Evidently K is commutative, L is cocommutative and $\dim K = p^t f^\theta$. By inspection, K is normal, i.e., $HK^+ = K^+H$. Thus we have an abelian exact sequence

$$\Bbbk \to K \xrightarrow{\iota} H \to H/HK^+ \to \Bbbk.$$

We have $\dim H/HK^+ = \dim L$. By the defining relations of $\mathcal{B}(\mathcal{V}(\mathbf{1},\mathbf{a}))$, see Remark 6.2, the assignment (6.24) determines a Hopf algebra map $\pi\colon H \to L$, which is surjective and has $HK^+ \subseteq \ker \pi$. Thus π induces an isomorphism of Hopf algebras $H/HK^+ \to L$.

Observe that $\pi(\text{ш}_{h,\mathbf{n}}) = v_{h,\mathbf{n}}$, for all $h, \mathbf{n}$. The section $s\colon L = \mathfrak{u}(\mathfrak{l}) \to H$ is the unique algebra map such that

$$s(E_j) = -\mathcal{S}(y_j) = -g_j^{-1} y_j, \quad s(v_h) = -\mathcal{S}(x_h) = -g_h^{-1} x_h, \quad \forall j \in \mathbb{I}_t,\ h \in \mathbb{I}_{t+1,\theta}.$$

We appeal again to Remark 6.2 to see that s is indeed well defined. As in the proof of Proposition 5.2, we get that s is an L-colinear section of π.

Observe that the generic element of the basis of H arising from (6.15) can be expressed as

$$u = g_1^{b_1} \cdots g_\theta^{b_\theta} x_1^{m_1} \cdots x_t^{m_t} y_1^{n_1} \cdots y_t^{n_t} \prod_{\substack{h\in\mathbb{I}_{t+1,\theta}\\ \mathbf{n}\in\mathcal{A}_h}} \text{ш}_{h,\mathbf{n}}^{n_{h,\mathbf{n}}}.$$

Let $r\colon H \to K$ be the linear map defined on u by the formula

$$r(u) = \begin{cases} g_1^{b_1} \cdots g_\theta^{b_\theta} x_1^{m_1} \cdots x_t^{m_t}, & \text{if } n_k = n_{h,\mathbf{n}} = 0, \ \forall k, h, \mathbf{n}; \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, r is a K-linear retraction of ι and $rs = \varepsilon_L 1_K$. We claim that r is a coalgebra map. To see this, we consider the subspace $I = \ker r$ which is the linear span of the monomials

$$g_1^{b_1} \cdots g_\theta^{b_\theta} x_1^{m_1} \cdots x_t^{m_t} y_1^{n_1} \cdots y_t^{n_t} \prod_{\substack{h \in \mathbb{I}_{t+1,\theta} \\ \mathbf{n} \in \mathcal{A}_h}} ш_{h,\mathbf{n}}^{n_{h,\mathbf{n}}}\colon \quad n_1 + \cdots + n_h + \sum_{\substack{h \in \mathbb{I}_{t+1,\theta} \\ \mathbf{n} \in \mathcal{A}_h}} n_{h,\mathbf{n}} > 0.$$

By the defining relations of H, I is a left ideal.

We claim that $\Delta(I) \subseteq I \otimes H + H \otimes I$. First, $\Delta(y_j), \Delta(x_h) \in I \otimes H + H \otimes I$ for all $j \in \mathbb{I}_t$, $h \in \mathbb{I}_{t+1,\theta}$ by (6.18). Next, by [4, Lemmas 7.2.3 (b) and 7.2.4] and the comultiplication formula of the bosonization, we have

$$\Delta(ш_{h,\mathbf{n}}) \in ш_{h,\mathbf{n}} \otimes 1 + \sum_{0 \le \mathbf{k} \le \mathbf{n}} H \otimes ш_{h,\mathbf{k}} \subset I \otimes H + H \otimes I, \quad \forall h, \mathbf{n}.$$

Let u, v be monomials in I such that $\Delta(u), \Delta(v) \in I \otimes H + H \otimes I$. Then $\Delta(uv) \in I \otimes H + H \otimes I$. By a recursive argument, the second claim follows. Since $r|_K = \mathrm{id}_K$, we get the first claim, i.e., that r is a coalgebra map. Thus (s, r) is a splitting and the proof of the Proposition is complete. ∎

Proposition 6.6. *The Hopf algebra* $H = \mathcal{B}(\mathcal{V}(\mathbf{1}, \mathbf{a}))\#\Bbbk\Gamma$ *and its double* $D(H)$ *have fgc.*

Proof. This follows from Proposition 6.5 and Theorem 3.6. ∎

6.4. The general case

Theorem 6.7. *The Hopf algebra* $\widetilde{H} = \mathcal{B}(\mathcal{V}(\mathfrak{q}, \mathbf{a}))\#\Bbbk\Gamma$ *has fgc.*

Proof. By Corollary A.8, the bosonization $\widetilde{H} = \mathcal{B}(\mathcal{V}(\mathfrak{q}, \mathbf{a}))\#\Bbbk\Gamma$ is a cocycle deformation of the bosonization $H = \mathcal{B}(\mathcal{V}(\mathbf{1}, \mathbf{a}))\#\Bbbk\Gamma$. By Proposition 6.6, the double $D(H)$ has fgc. Hence $\widetilde{H}$ also has fgc, as claimed. ∎

Since H is free over $\mathcal{B}(\mathcal{V}(\mathfrak{q}, \mathbf{a}))$, Theorem 6.7 together with [5, Theorem 3.2.1] implies the following result.

Corollary 6.8. *The Nichols algebra* $\mathcal{B}(\mathcal{V}(\mathfrak{q}, \mathbf{a}))$ *has fgc.*

A. Cocycle-equivalence of Nichols algebras

In this appendix, we describe braided vector spaces over abelian groups and spell out conditions for their Nichols algebras being twist-equivalent. Except in Corollary A.8, char $\Bbbk$ is arbitrary.

A.1. Braided vector spaces over abelian groups

Definition A.1. Let $\theta \in \mathbb{N}$. An $\mathfrak{ab}$-triple (of rank θ) is a collection $\mathcal{T} = (\mathbf{n}, \mathfrak{q}, \mathbf{t})$ where

- $\mathbf{n} = (n_j)_{j\in\mathbb{I}_\theta}$ is a family of positive integers, normalized by $n_1 \geq n_2 \geq \cdots \geq n_\theta$;
- $\mathfrak{q} = (q_{ij})_{i,j\in\mathbb{I}_\theta}$ is a matrix with invertible entries and
- $\mathbf{t} = (\mathbf{t}_{ij})_{i,j\in\mathbb{I}_\theta}$ is a family where $\mathbf{t}_{ij} \in \operatorname{End}\Bbbk^{n_j}$ satisfies $\mathbf{t}_{ij} = 0$ when $\dim V_j = 1$ and $\mathbf{t}_{ik}\mathbf{t}_{jk} = \mathbf{t}_{jk}\mathbf{t}_{ik}$ for all i, j, k.

An $\mathfrak{ab}$-triple is *nilpotent* if every $\mathbf{t}_{ij}$ is nilpotent.

We attach a braided vector space to an $\mathfrak{ab}$-triple $(\mathbf{n}, \mathfrak{q}, \mathbf{t})$ by the following recipe. Let $V = \bigoplus_{j\in\mathbb{I}_\theta} V_j$ be a vector space with a decomposition such that $\dim V_j = n_j$ for all $j \in \mathbb{I}_\theta$; pick a basis of V_j, pull back $\mathbf{t}_{ij}$ to $\mathbf{t}_{ij} \in \operatorname{End} V_j$ and define $c \in GL(V \otimes V)$ by

$$c(x \otimes y) = q_{ij}\big(y + \mathbf{t}_{ij}(y)\big) \otimes x, \quad x \in V_i,\ y \in V_j. \tag{A.1}$$

The proof of the following result is left to the reader.

Lemma A.2. *The pair (V, c) is a braided vector space that can be realized over ${}^{\Bbbk\Lambda}_{\Bbbk\Lambda}\mathcal{YD}$, where $\Lambda \simeq \mathbb{Z}^\theta$ with canonical basis $\alpha_1, \ldots, \alpha_\theta$, by*

$$V_{\alpha_i} = V_i, \quad \alpha_i \rightharpoonup x = q_{ij}\big(x + \mathbf{t}_{ij}(x)\big), \quad x \in V_j,\ i, j \in \mathbb{I}_\theta.$$

A.2. Cocycle equivalence

We start by discussing the relationship between cocycle deformation and bosonization, as it appears in [30]. Let H be a Hopf algebra; let $\sigma\colon H \otimes H \to \Bbbk$ be an invertible 2-cocycle; let H_σ be the Hopf algebra which is H as coalgebra and has multiplication

$$x \cdot_\sigma y = \sigma(x_{(1)}, y_{(1)})x_{(2)}y_{(2)}\sigma^{-1}(x_{(3)}, y_{(3)}), \quad x, y \in H. \tag{A.2}$$

Let now R be a Hopf algebra in ${}^H_H\mathcal{YD}$ and $A := R\#H$ the bosonization with canonical projection and injection $\pi\colon A \to H$ and $\iota\colon H \to A$. Let $\sigma^\pi\colon A \otimes A \to \Bbbk$ be given by $\sigma^\pi := \sigma(\pi \otimes \pi)$; this is an invertible 2-cocycle on A. Then $\pi\colon A_{\sigma^\pi} \to H_\sigma$ and $\iota\colon H_\sigma \to A_{\sigma^\pi}$ are still Hopf algebra maps. Hence $A_{\sigma^\pi} \simeq R_\sigma\#H_\sigma$ where R_σ is a Hopf algebra in ${}^{H_\sigma}_{H_\sigma}\mathcal{YD}$ that coincides with R as vector subspace of A, with multiplication

$$x \cdot_\sigma y = \sigma(x_{(0)}, y_{(0)})x_{(1)}y_{(1)}, \quad x, y \in R_\sigma. \tag{A.3}$$

Lemma A.3 ([9, Lemma 2.13]). *If $R = \bigoplus_{n\in\mathbb{N}_0} R(n)$ is a graded Hopf algebra in ${}^H_H\mathcal{YD}$, then R_σ is a graded Hopf algebra in ${}^{H_\sigma}_{H_\sigma}\mathcal{YD}$ with $R(n) = R_\sigma(n)$ as vector spaces for all $n \geq 0$. Also, R is a Nichols algebra if and only if R_σ is.*

A.3. Cocycles over an abelian group

We fix an abelian group Γ. Let $\sigma\colon \Gamma \times \Gamma \to \Bbbk^\times$ be a group 2-cocycle: $\sigma(gh, k)\sigma(g, h) = \sigma(g, hk)\sigma(h, k)$, for all $g, h, k \in \Gamma$. Then the map $\vartheta\colon \Gamma \times \Gamma \to \Bbbk^\times$ given by

$$\vartheta(g, h) = \sigma(g, h)\sigma^{-1}(h, g), \quad g, h \in \Gamma,$$

is bilinear and antisymmetric. Let $\mathcal{F}_\sigma\colon {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD} \to {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ be the monoidal functor that assigns $V \mapsto \mathcal{F}_\sigma(V) = V$ with the same grading and action

$$g \rightharpoonup_\sigma v = \vartheta(g,h) g \rightharpoonup v, \quad g,h \in \Gamma,\ v \in V_h, \tag{A.4}$$

where $\rightharpoonup$ is the action on V. The braiding c_σ in $\mathcal{F}_\sigma(V) \otimes \mathcal{F}_\sigma(W)$ is given by

$$c_\sigma(v \otimes w) = \vartheta(g,h) c(v \otimes w), \quad v \in V_g,\ w \in W_h,\ g,h \in \Gamma. \tag{A.5}$$

Fix $V \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ and let $A = \mathcal{B}(V)\#\Bbbk\Gamma$; as before $\pi\colon A \to \Bbbk\Gamma$ and $\iota\colon \Bbbk\Gamma \to A$ are the canonical projection and the inclusion. Clearly, the linear extension $\sigma\colon \Bbbk\Gamma \otimes \Bbbk\Gamma \to \Bbbk$ of σ is a 2-cocycle as in the previous subsection. Let $\sigma^\pi = \sigma(\pi \otimes \pi)$ be as above.

Lemma A.4. *The Hopf algebra A_{σ^π} is isomorphic to $\mathcal{B}(V)_\sigma\#\Bbbk\Gamma$ with the same comultiplication of $\mathcal{B}(V)$ and the multiplication given by*

$$x \cdot_{\sigma^\pi} y = \sigma(g,h) xy, \quad x \in \mathcal{B}(V)_g,\ y \in \mathcal{B}(V)_h,\ g,h \in \Gamma. \tag{A.6}$$

Furthermore $\mathcal{B}(V)_\sigma \simeq \mathcal{B}(\mathcal{F}_\sigma(V))$.

Proof. The first part follows from the discussion above; (A.6) is particular instance of (A.3). For the second part we show that the degree one homogenous component of $\mathcal{B}(V)_\sigma$ is $\mathcal{F}_\sigma(V)$ and apply Lemma A.3. If $y \in V_h$, then

$$\begin{aligned} g \cdot_{\sigma^\pi} y &= \sigma\big(g, \pi(y)\big) g \sigma^{-1}(g,1) + \sigma(g,h)\, g y \sigma^{-1}(g,1) + \sigma(g,h)\, g h \sigma^{-1}\big(g, \pi(y)\big) \\ &= \sigma(g,h) g y. \end{aligned}$$

Similarly, $(gy) \cdot_{\sigma^\pi} g^{-1} = \sigma(gh, g^{-1}) g y g^{-1} \sigma^{-1}(g, g^{-1})$ so the action is given by

$$\begin{aligned} (g \cdot_{\sigma^\pi} y) \cdot_{\sigma^\pi} g^{-1} &= \sigma(g,h)(gy \cdot_{\sigma^\pi} g^{-1}) = \sigma(g,h)\sigma(gh, g^{-1}) g y g^{-1} \sigma^{-1}(g, g^{-1}) \\ &= \sigma(g,h)\sigma(h,g)^{-1} g \rightharpoonup y = g \rightharpoonup_{\sigma^\pi} y. \end{aligned}$$

∎

Definition A.5 ([6], [9, Section 2.4]). Two braided Hopf algebras R and S are *cocycle-equivalent* if there exist a Hopf algebra H and an invertible 2-cocycle $\sigma\colon H \otimes H \to \Bbbk$ such that

- R is realizable in ${}^{H}_{H}\mathcal{YD}$;
- S is isomorphic to R_σ as a braided Hopf algebra.

The following definition extends [6, Lemma 4.3] and [9, Section 2.4].

Definition A.6. Two braided vector spaces (V, c) and (V', c') arising from $\mathfrak{ab}$-triples $(\mathbf{n}, \mathfrak{q}, \mathbf{t})$ and $(\mathbf{n}', \mathfrak{q}', \mathbf{t}')$ are *twist-equivalent* if

$$\mathbf{n} = \mathbf{n}', \quad \mathbf{t} = \mathbf{t}', \quad q_{ii} = q'_{ii}, \quad q_{ij} q_{ji} = q'_{ij} q'_{ji}, \quad i,j \in \mathbb{I}_\theta. \tag{A.7}$$

Lemma A.7. *If the braided vector spaces (V, c) and (V', c') arising from the $\mathfrak{ab}$-triples $(\mathbf{n}, \mathfrak{q}, \mathbf{t})$ and $(\mathbf{n}', \mathfrak{q}', \mathbf{t}')$ are twist-equivalent, then the Nichols algebras $\mathcal{B}(V)$ and $\mathcal{B}(V')$ are cocycle-equivalent.*

Proof. By (A.7) there exists a linear isomorphism $\psi\colon V \to V'$ preserving the decompositions $V = \bigoplus_{i\in\mathbb{I}_\theta} V_i$ and $V' = \bigoplus_{i\in\mathbb{I}_\theta} V_i'$ and intertwining the endomorphisms $\mathbf{t}_{ij}$ and $\mathbf{t}'_{ij}$. We realize (V, c) as in Lemma A.2. We consider the unique bilinear form $\sigma\colon \Lambda\times\Lambda \to \Bbbk^\times$, hence a group 2-cocycle, given by

$$\sigma(\alpha_i,\alpha_j) = \begin{cases} q'_{ij}q_{ij}^{-1}, & i \le j,\\ 1, & i > j.\end{cases} \tag{A.8}$$

We claim that $\psi\colon V_\sigma \to V'$ is an isomorphism in ${}^{\Bbbk\Lambda}_{\Bbbk\Lambda}\mathcal{YD}$. Clearly ψ preserves the grading. Assume $i \le j$ and let $x \in V_i$, $y \in V_j$. Then

$$\begin{aligned} \alpha_j \rightharpoonup_\sigma x &\overset{\text{(A.4)}}{=} \sigma(\alpha_j,\alpha_i)\sigma^{-1}(\alpha_i,\alpha_j)q_{ji}\big(x + \mathbf{t}_{ji}(x)\big)\\ &= (q'_{ij})^{-1}q_{ij}q_{ji}\big(x + \mathbf{t}_{ji}(x)\big) \overset{\text{(A.7)}}{=} q'_{ji}\big(x + \mathbf{t}_{ji}(x)\big);\\ \alpha_i \rightharpoonup_\sigma y &\overset{\text{(A.4)}}{=} \sigma(\alpha_i,\alpha_j)\sigma^{-1}(\alpha_j,\alpha_i)q_{ij}\big(y + \mathbf{t}_{ij}(i)\big) = q'_{ij}\big(y + \mathbf{t}_{ij}(i)\big). \end{aligned}$$

Thus ψ preserves the action of Λ, hence it extends to an isomorphism $\Psi\colon \mathcal{B}(V_\sigma) \to \mathcal{B}(V')$ of Hopf algebras in ${}^{\Bbbk\Lambda}_{\Bbbk\Lambda}\mathcal{YD}$. Now $\mathcal{B}(\mathcal{F}_\sigma(V)) \simeq \mathcal{B}(V)_\sigma$ by Lemma A.4. ∎

The following statement is needed in the paper. Assume that $\operatorname{char}\Bbbk = p$ is odd. Let $\mathcal{V}(\mathfrak{q}, \mathbf{a})$ and $\mathcal{V}(\mathfrak{q}', \mathbf{a})$ be two braided vector spaces as in Section 6 with the same θ. By hypothesis, $q_{ii} = q'_{ii} = 1$, and $q_{ij}q_{ji} = q'_{ij}q'_{ji} = 1$, for all $i, j \in \mathbb{I}_\theta$.

Corollary A.8. *Assume that there exists a positive integer f such that*

$$p \text{ divides } f, \tag{A.9}$$

$$\operatorname{ord} q_{ij} \text{ divides } f, \quad \operatorname{ord} q'_{ij} \text{ divides } f, \quad \forall i, j \in \mathbb{I}_\theta. \tag{A.10}$$

Let $\Gamma = (\mathbb{Z}/f)^\theta$. Then $\mathcal{V}(\mathfrak{q}, \mathbf{a})$ and $\mathcal{V}(\mathfrak{q}', \mathbf{a})$ are realizable in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ and there exists an invertible 2-cocycle $\sigma\colon \Bbbk\Gamma\otimes\Bbbk\Gamma \to \Bbbk$ such that $\mathcal{B}(\mathcal{V}(\mathfrak{q}', \mathbf{a}))$ is isomorphic to $\mathcal{B}(\mathcal{V}(\mathfrak{q}, \mathbf{a}))_\sigma$ as Hopf algebras in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$.

Proof. The proof of the first claim on the realizations is straightforward using the hypotheses (A.9) and (A.10). By (A.10), there is a unique bilinear form $\sigma\colon \Gamma\times\Gamma \to \Bbbk^\times$ given by (A.8). Then the second claim follows as in the proof of Lemma A.7. ∎

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