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Solitons to mean curvature flow in the hyperbolic 3-space

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Abstract. We consider translators (i.e., initial condition of translating solitons) to mean curvature flow (MCF) in the hyperbolic 3-space $\mathbb{H}^3$, providing existence and classification results. More specifically, we show the existence and uniqueness of two distinct one-parameter families of complete translators in $\mathbb{H}^3$, one containing catenoid-type translators, and the other parabolic cylindrical ones. We establish a tangency principle for translators in $\mathbb{H}^3$ and apply it to prove that properly immersed translators to MCF in $\mathbb{H}^3$ are not cylindrically bounded. As a further application of the tangency principle, we prove that any horoconvex translator which is complete or transversal to the z-axis is necessarily an open set of a horizontal horosphere. In addition, we classify all translators in $\mathbb{H}^3$ which have constant mean curvature. We also consider rotators (i.e., initial condition of rotating solitons) to MCF in $\mathbb{H}^3$ and, after classifying the rotators of constant mean curvature, we show that there exists a one-parameter family of complete rotators which are all helicoidal, bringing to the hyperbolic context a distinguished result by Halldorsson, set in $\mathbb{R}^3$.

1. Introduction

The last decades flourished with great regard to the theory of extrinsic geometric flows in Riemannian manifolds, especially to mean curvature flow in Euclidean spaces, giving rise to a vast literature on the subject (cf. [18] and the references therein). Extrinsic geometric flows constitute evolution equations that describe hypersurfaces of a Riemannian manifold evolving in the normal direction with velocity given by the corresponding extrinsic curvature.

A special class of solutions is that of the *solitons*, also known as the *self-similar* solutions, which are characterized for being generated by a one-parameter subgroup of isometries or dilations of the ambient manifold. When these isometries are translations along a geodesic, we call the corresponding self-similar solutions *translating solitons* to the giving flow, and the initial hypersurfaces are known as *translators*. A main feature of translators in Euclidean spaces is that they naturally appear as type II singularities of certain compact solutions to mean curvature flow (cf. Theorem 4.1 in [11]).

There exist many examples of translators to mean curvature flow (MCF, for short) in Euclidean space $\mathbb{R}^3$. Three of the best known are the cylinder over the graph of the

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function $f(t) = -\log(\cos t)$, $t \in (-\pi/2, \pi/2)$, called the *grim reaper*, the rotational entire graph over $\mathbb{R}^2$ obtained by Altschuler and Wu [1] known as the *translating paraboloid* or *bowl soliton*, and the one-parameter family of rotational annuli obtained by Clutterbuck, Schnürer and Schulze [3], the so-called *translating catenoids*. On the other hand, little is known about translators in hyperbolic spaces.

In this paper, we consider solitons to MCF in hyperbolic space $\mathbb{H}^3$, and first we focus on the case of translators which move by hyperbolic translations along a fixed geodesic. We classify all such surfaces with constant mean curvature (Theorems 3.2 and 3.4) and also obtain new families of examples (see Figure 1), which, by some similarities with the translators in $\mathbb{R}^3$ described above, will be called the *translating catenoid* (Theorem 3.10) and the *grim reaper* (Theorem 3.20). These translators are then proven to be unique with respect to their fundamental properties (Theorems 3.17 and 3.23).

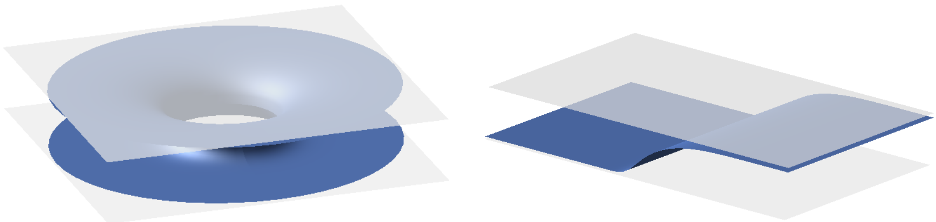

Figure 1. A translating catenoid and a grim reaper in the half-space model of $\mathbb{H}^3$.

A main tool in establishing uniqueness results for translators in Euclidean space is the tangency principle. It asserts that two such translators which are tangent at a point, with one on one side of the other in a neighbourhood of this point, must coincide in this neighbourhood. In fact, this result is a direct consequence of the tangency principle for minimal surfaces, since translators in Euclidean space become minimal surfaces when the ambient space is endowed with a suitable metric.

Apparently, for translators in hyperbolic space, such a metric is not available. Nevertheless, we prove that a tangency principle holds for translators in $\mathbb{H}^3$ as well (cf. Section 3.3). Then, by using the family of translating catenoids as barriers, we apply it to prove that properly immersed translators in $\mathbb{H}^3$ are never cylindrically bounded and, in particular, never closed. As a further application, we show that any horoconvex translator which is complete or transversal to the axis of the translation is necessarily an open set of a horosphere (Theorem 3.26).

We study, as well, *rotators* to MCF, that is, initial data of solitons whose associated isometries are rotations around a geodesic. In [9], Halldorsson considered rotators in $\mathbb{R}^3$, obtaining a one-parameter family of complete helicoidal rotators in $\mathbb{R}^3$ which are also translators. Inspired by Halldorsson's work, we obtain here an analogous result (Theorem 4.4), in which we construct a one-parameter family of helicoidal surfaces in $\mathbb{H}^3$ that, under mean curvature flow, rotates around its axis and translates downwards with velocity that equals its pitch, see Figure 2. As in the case of translators, we also classify all rotators of constant mean curvature in $\mathbb{H}^3$ (Theorem 4.1).

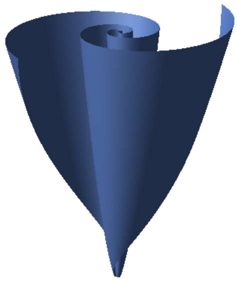

Figure 2. A helicoidal rotational soliton.

The paper is organized as follows. In Section 2, we set some notation and formulae. In Section 3, we introduce translators to MCF in $\mathbb{H}^3$ and establish the aforementioned results related to them. In Section 4, we deal with rotators to MCF in $\mathbb{H}^3$, and in the final Section 5, we provide the classification of minimal translators in $\mathbb{H}^3$.

2. Preliminaries

Throughout the paper, we shall consider the upper half-space model of $\mathbb{H}^3$, that is, $\mathbb{H}^3 := (\mathbb{R}^3_+, ds^2)$, where $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$, $ds^2 := d\bar{s}^2/z^2$ and $d\bar{s}^2$ is the standard Euclidean metric of $\mathbb{R}^3_+$. We will also denote ds^2 by $\langle\, , \rangle$.

Let Σ be an oriented surface in a Riemannian 3-manifold $\overline{M}$. Set $\overline{\nabla}$ for the Levi-Civita connection of $\overline{M}$, η for the unit normal field of Σ, and A for its shape operator with respect to η, so that

$$AX = -\overline{\nabla}_X \eta, \quad X \in T\Sigma,$$

where $T\Sigma$ stands for the tangent bundle of Σ. The principal curvatures of Σ, that is, the eigenvalues of A, will be denoted by k_1 and k_2, and the mean curvature H of Σ is expressed by

$$H = \frac{k_1 + k_2}{2}.$$

The mean curvature vector of Σ is

$$\mathbf{H} = H\eta,$$

which is invariant under the choice of orientation $\eta \to -\eta$ and satisfies $\|\mathbf{H}\| = |H|$.

Given an oriented surface $\Sigma \subset \mathbb{R}^3_+$, let $\bar{\eta} = (\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3)$ be the unit normal of Σ with respect to the induced Euclidean metric $d\bar{s}^2$. It is easily checked that

$$\eta(p) = z\bar{\eta}(p), \quad p = (x, y, z) \in \Sigma,$$

defines a unit normal of Σ with respect to the hyperbolic metric ds^2. With these orientations, if we denote by $\overline{H}$ (respectively, H) the mean curvature of Σ with respect to the Euclidean metric (respectively, hyperbolic metric) of $\mathbb{R}^3_+$, we have that $\overline{H}$ and H satisfy the following relation (cf. Lemma 10.1.1 in [14]):

$$H(p) = z\overline{H}(p) + \bar{\eta}_3(p) \quad \text{for all } p = (x, y, z) \in \Sigma. \tag{2.1}$$

2.1. Mean curvature flow

We say that a family of oriented surfaces $\Sigma_t = X_t(M)$ of a Riemannian 3-manifold $\bar{M}$ *evolves under mean curvature flow* if the corresponding one-parameter family of immersions

$$X_t : M \to \bar{M}, \quad t \in [0, \delta), \quad 0 < \delta \leq +\infty,$$

satisfies the following condition:

$$\frac{\partial X_t}{\partial t}^{\perp}(p) = H_t(p)\eta_t(p) \quad \text{for all } p \in M, \tag{2.2}$$

where η_t is the unit normal to X_t, H_t is the mean curvature of X_t with respect to η_t, and $(\partial X_t/\partial t)^{\perp}$ denotes the normal component of $\partial X_t/\partial t$, that is,

$$\frac{\partial X_t}{\partial t}^{\perp} = \left\langle \frac{\partial X_t}{\partial t}, \eta_t \right\rangle \eta_t.$$

In particular, the equality (2.2) is equivalent to

$$\left\langle \frac{\partial X_t}{\partial t}, \eta_t \right\rangle = H_t.$$

We call such a family $X_t : M \to \bar{M}$, $t \in [0, \delta)$, a *mean curvature flow* (MCF, for short) in $\bar{M}$ with initial data X_0. In this setting, we say that $\Sigma_t = X_t(M)$ is a *soliton* or a *self-similar solution* to MCF if there exists a one-parameter subgroup $\mathscr{G} := \{\Gamma_t \mid t \in \mathbb{R}\}$ of the group of isometries or dilations of $\bar{M}$ such that Γ_0 is the identity map of $\bar{M}$ and

$$\Sigma_t = \Gamma_t(\Sigma) \quad \text{for all } t \in \mathbb{R}$$

is a MCF. More specifically, we shall call such a family Σ_t a $\mathscr{G}$*-soliton*.

Let ξ be the Killing field determined by a subgroup $\mathscr{G}$ of isometries of $\bar{M}$, that is, for any $p \in \bar{M}$,

$$\xi(p) := \frac{\partial}{\partial t}\Gamma_t(p) \quad \text{at } t = 0.$$

It can be proved (see, e.g., [12]) that the surface $\Sigma = X_0(M)$ with unit normal η is the initial condition of a $\mathscr{G}$*-soliton* generated by ξ in $\bar{M}$ if and only if the equality

$$H = \langle \xi, \eta \rangle \tag{2.3}$$

holds everywhere on Σ. So, in the class of solitons, equation (2.2) is in fact a prescribed mean curvature problem.

3. Translators to MCF in $\mathbb{H}^3$

Consider in hyperbolic space $\mathbb{H}^3$ the group $\mathscr{G} = \{\Gamma_t \mid t \in \mathbb{R}\} \subset \mathrm{Iso}(\mathbb{H}^3)$ of hyperbolic translations along the z-axis, defined by

$$\Gamma_t(p) = e^t p, \quad p \in \mathbb{H}^3.$$

In this setting, an initial condition of a $\mathscr{G}$-soliton will be called a *translating soliton* or simply a *translator*. Using the abuse of notation

$$p = (x, y, z) \in \mathbb{H}^3 \longleftrightarrow x\partial_x + y\partial_y + z\partial_z \in T_p\mathbb{H}^3,$$

the Killing field associated to $\mathscr{G}$ is $\xi(p) = p$, $p \in \mathbb{H}^3$. Thus, it follows from (2.3) that a surface $\Sigma \subset \mathbb{H}^3$ is a translator to MCF if and only if

$$H(p) = \langle p, \eta(p)\rangle \quad \text{for all } p \in \Sigma. \tag{3.1}$$

Example 3.1. Let Π be a totally geodesic vertical plane of $\mathbb{H}^3$ which contains $(0, 0, 1)$. Since H vanishes on Π, it is clear that (3.1) holds for $\Sigma = \Pi$. Thus, Π is a stationary translator to MCF in $\mathbb{H}^3$.

In fact, equation (2.3) implies that a minimal surface $\Sigma \subset \mathbb{H}^3$ is a (stationary) translator to MCF if and only if it is invariant under the group $\mathscr{G}$ of hyperbolic isometries as above. A complete classification of such surfaces is given by the following description.

Theorem 3.2. *There exists a one-parameter family Σ_θ, $\theta \in (0, \pi]$, of properly embedded minimal surfaces in $\mathbb{H}^3$ with the following properties*:

(i) *Σ_θ is invariant under the one-parameter group $\{\Gamma_t\}_{t\in\mathbb{R}}$ of hyperbolic translations*

$$p \in \mathbb{H}^3 \mapsto \Gamma_t(p) := e^t p \in \mathbb{H}^3,$$

and so it is a stationary translator to MCF in $\mathbb{H}^3$.

(ii) *$\partial_\infty\Sigma_\theta \cap \mathbb{R}^2$ is the union of two half-lines making an angle θ.*

(iii) *Σ_π is a vertical plane.*

Conversely, if Σ is a properly embedded minimal surface of $\mathbb{H}^3$ which is invariant under the group Γ_t, then $\Sigma = \Sigma_\theta$ for some $\theta \in (0, \pi]$.

The proof of Theorem 3.2, for convenience, will be presented separately in Section 5. Concerning the case of translators with nonzero constant mean curvature, we start with the next example.

Example 3.3. Let $\mathcal{H}_h$ be the horosphere of $\mathbb{H}^3$ at height $h > 0$, i.e.,

$$\mathcal{H}_h = \{(x, y, h) \in \mathbb{H}^3 \mid x, y \in \mathbb{R}\}.$$

At any point $p = (x, y, h) \in \mathcal{H}_h$, we have that $H(p) = 1$ and $\eta(p) = he_3$, so that

$$\langle p, \eta(p)\rangle = \frac{1}{h^2}h^2 = 1 = H(p) \quad \text{for all } p \in \mathcal{H}_h.$$

Hence, $\mathcal{H}_h$ is a translator to MCF in $\mathbb{H}^3$.

In our next result we show that horospheres are the only translators to MCF which have nonzero constant mean curvature. In the proof, we shall use the following evolution formula for the mean curvature H_t (notation as in Section 2) of a mean curvature flow $X_t \colon M \to \overline{M}$:

$$\frac{\partial H_t}{\partial t} = \Delta H_t + H_t(\|A_t\|^2 + \overline{\mathrm{Ric}}(\eta_t, \eta_t)), \tag{3.2}$$

where $\overline{\mathrm{Ric}}$ denotes the Ricci tensor of $\overline{M}$ (see Theorem 3.2 (v) in [10]).

Theorem 3.4. *Let Σ be a connected translator to MCF in $\mathbb{H}^3$ which has nonzero constant mean curvature. Then Σ is an open subset of a horosphere.*

Proof. After a change of orientation, we may assume without loss of generality that the mean curvature H of Σ is positive. Let $X_t\colon M \to \mathbb{H}^3$, $t > 0$, be the MCF such that $X_0(M) = \Sigma$ and

$$X_t(p) = e^t X_0(p), \quad p \in M.$$

Since $X_t(M)$ differs from $X_0(M)$ by an ambient isometry, $H_t = H > 0$ is constant in space and time, thus $\partial H_t/\partial t = \Delta H_t = 0$. Also, in $\mathbb{H}^3$, $\overline{\mathrm{Ric}}(\eta_t, \eta_t) = -2$. Then formula (3.2) yields $\|A_t\|^2 = 2$ for all $t \geq 0$. Taking $t = 0$, we conclude that the principal curvatures k_1 and k_2 of Σ satisfy

$$\begin{cases} k_1 + k_2 = 2H, \\ k_1^2 + k_2^2 = 2, \end{cases}$$

from where it follows that $H \in (0, 1]$ and, after possibly reindexing,

$$k_1 = H + \sqrt{1 - H^2} \quad \text{and} \quad k_2 = H - \sqrt{1 - H^2}.$$

Since H is constant, both k_1 and k_2 are constant, so Σ is isoparametric. The isoparametric surfaces of $\mathbb{H}^3$ are classified (see Theorem 3.14 in [2]) and the fact that $H \in (0, 1]$ imply that Σ is either an open subset of a horosphere or of an equidistant surface to a totally geodesic plane. However, $k_1^2 + k_2^2 = 2$ only holds when Σ is contained in a horosphere, which finishes the proof of the theorem. ∎

Remark 3.5. Since (3.2) holds for any MCF, the proof of Theorem 3.4 applies to show that any initial condition of a MCF in $\mathbb{H}^3$ whose mean curvature is constant in space and time is necessarily an open subset of a horosphere.

3.1. Rotational translators

In this section, we focus on translators to MCF in $\mathbb{H}^3$ which are invariant under rotations about the z-axis. With this purpose, we first consider vertical rotational graphs. More precisely, let ϕ be a positive smooth function on an open interval $I \subset (0, +\infty)$ and

$$X(\theta, s) = (s\cos\theta, s\sin\theta, \phi(s)), \quad (\theta, s) \in U := \mathbb{R} \times I \subset \mathbb{R}^2.$$

We shall call $\Sigma = X(U)$ the *rotational vertical graph determined by* ϕ, and Lemma 3.6 below provides the equation that ϕ satisfies in order for Σ to be a translator to MCF.

Lemma 3.6. *A vertical rotational graph determined by a smooth function ϕ is a translator to MCF in $\mathbb{H}^3$ if and only if ϕ satisfies the second order ODE*

$$\phi'' = -\phi'(1 + (\phi')^2)\left(\frac{2s}{\phi^2} + \frac{1}{s}\right). \tag{3.3}$$

Proof. For a rotational graph Σ as above, a direct computation gives that

$$\bar\eta := (\bar\eta_1, \bar\eta_2, \bar\eta_3) = \varrho(-\phi'\cos\theta, -\phi'\sin\theta, 1), \quad \varrho := \frac{1}{\sqrt{1 + (\phi')^2}},$$

is a unit normal with respect to the induced Euclidean metric, and that the corresponding Euclidean mean curvature is

$$\bar{H} = \frac{\varrho}{2}\Big(\frac{\phi''}{1+(\phi')^2} + \frac{\phi'}{s}\Big).$$

Thus, from (2.1), the mean curvature H of Σ in $\mathbb{H}^3$ with respect to $\eta := \phi\bar{\eta}$ is

$$H = \phi\bar{H} + \bar{\eta}_3 = \varrho\Big(\frac{\phi}{2}\Big(\frac{\phi''}{1+(\phi')^2} + \frac{\phi'}{s}\Big) + 1\Big). \tag{3.4}$$

It is also straightforward to see that the equality

$$\langle X, \eta\rangle = \frac{\varrho}{\phi}\,(\phi - s\phi') \tag{3.5}$$

holds everywhere on Σ.

From (3.4) and (3.5), we conclude that equation (3.1) for the vertical graph Σ is equivalent to the second order ODE:

$$\phi'' = -\phi'(1+(\phi')^2)\Big(\frac{2s}{\phi^2} + \frac{1}{s}\Big),$$

which proves the lemma. ■

Our next arguments will rely on qualitative analysis of ordinary differential equations for establishing some properties of the solutions to (3.3).

Lemma 3.7. *For any $s_0, z_0 > 0$ and any $\lambda \in \mathbb{R}$, the initial value problem*

$$\begin{cases} f'' = -f'(1+(f')^2)\Big(\dfrac{2s}{f^2} + \dfrac{1}{s}\Big), \\ f(s_0) = z_0, \\ f'(s_0) = \lambda \end{cases} \tag{3.6}$$

has a unique smooth solution ϕ on $[s_0, +\infty)$ which has the following properties:

(i) *ϕ is constant if $\lambda = 0$.*

(ii) *ϕ is increasing, concave and bounded above by a positive constant if $\lambda > 0$.*

(iii) *ϕ is decreasing, convex and bounded below by a positive constant if $\lambda < 0$.*

Proof. For $\Omega := (0, +\infty) \times (0, +\infty) \times \mathbb{R}$, since the function

$$(u, v, w) \in \Omega \mapsto -w(1+w^2)\Big(\frac{2u}{v^2} + \frac{1}{u}\Big)$$

is C^∞ in Ω, the standard results on solutions for ODEs ensure the existence and uniqueness of a C^∞ solution ϕ defined in a maximal interval $I_{\max} := [s_0, x_{\max})$, $x_{\max} \le +\infty$, in the sense that the equality

$$\phi'' = -\phi'(1+(\phi')^2)\Big(\frac{2x}{\phi^2} + \frac{1}{x}\Big) \tag{3.7}$$

holds in $I_{\max}$.

If $\lambda = 0$, it is clear from (3.7) that the solution ϕ is constant, in which case $x_{\max} = +\infty$. This proves (i).

Assume now that $\lambda > 0$. Then ϕ is increasing near s_0. Also, from property (i) and the uniqueness of solutions, ϕ has no critical points. Hence, ϕ is increasing in $I_{\max}$. In addition, equality (3.7) gives that ϕ is concave in $I_{\max}$, which yields $x_{\max} = +\infty$.

To prove that ϕ is bounded above, we will make use of an estimate that will be recurrent in the arguments of this section, for which reason it is presented separately (without proof due to its simplicity).

Claim 3.8. For a given $v > 0$, let $f_v\colon (0,\infty) \to \mathbb{R}$ be defined as $f_v(u) = 2u/v^2 + 1/u$. Then f_v has a unique critical point at $u = v/\sqrt{2}$, being decreasing in $(0, v/\sqrt{2}]$ and increasing in $[v/\sqrt{2}, +\infty)$. In particular, for any $u, v > 0$,

$$\frac{2u}{v^2} + \frac{1}{u} \geq \frac{2\sqrt{2}}{v}. \tag{3.8}$$

Using estimate (3.8) in (3.7), we obtain that

$$\frac{\phi''}{1+(\phi')^2} \leq -\phi' \, \frac{2\sqrt{2}}{\phi}.$$

Thus, for $s > s_0$, we may integrate over $[s_0, s]$ to obtain

$$\arctan(\phi') - \arctan(\lambda) \leq -2\sqrt{2} \log\left(\frac{\phi}{z_0}\right),$$

so

$$2\sqrt{2} \log\left(\frac{\phi}{z_0}\right) \leq \arctan(\lambda) - \arctan(\phi') < \frac{\pi}{2},$$

which implies that ϕ is bounded above, therefore proving (ii).

To prove (iii), we can argue as in the proof of (ii) to conclude that ϕ is decreasing and convex in $I_{\max}$ if $\lambda < 0$. Once again we may use (3.8) in (3.7), observing that in this situation $\phi' < 0$, to obtain

$$\frac{\phi''}{1+(\phi')^2} \geq -\phi' \, \frac{2\sqrt{2}}{\phi},$$

and proceed as in the previous case to arrive at

$$2\sqrt{2} \log\left(\frac{\phi}{z_0}\right) \geq \arctan(\lambda) - \arctan(\phi') > \arctan(\lambda),$$

so

$$\phi > z_0 \, e^{\arctan(\lambda)/(2\sqrt{2})},$$

proving (iii) and finishing the proof of the lemma. ∎

Lemmas 3.6 and 3.7 already imply the existence of rotational translators. However, to improve the description of these examples, we next consider rotational surfaces which are also horizontal graphs. More precisely, given a rotational surface $\Sigma \subset \mathbb{H}^3$ with axis $\ell := \{(0,0)\} \times (0, +\infty)$, let us consider $\gamma = \Sigma \cap \{x = 0\}$ as the profile curve of Σ and assume that the tangent plane of Σ at a given point $p \in \gamma$ is not orthogonal to ℓ. If we let d denote the Euclidean distance function from γ to ℓ on $\mathbb{R}^3_+$ and let the z coordinate parametrize γ, then, in a neighbourhood of p, Σ can be parametrized as

$$X(x,z) := (x, \sqrt{d^2(z) - x^2}, z), \quad (x,z) \in U \subset \mathbb{R} \times (0, +\infty).$$

We shall call $X(U)$ the *horizontal rotational graph determined by d*.

Lemma 3.9. *A horizontal rotational graph determined by a smooth function d is a translator to MCF in $\mathbb{H}^3$ if and only if the function d satisfies the ODE*

$$d'' = (1 + (d')^2)\Big(\frac{2d}{z^2} + \frac{1}{d}\Big).$$

In particular, such a solution d is strictly convex.

Proof. Writing $\varphi(x,z) := \sqrt{d^2(z) - x^2}$, we have that a Euclidean unit normal to Σ is

$$\bar\eta := (\bar\eta_1, \bar\eta_2, \bar\eta_3) = \varrho(-\varphi_x, 1, -\varphi_z), \quad \varrho := \frac{1}{\sqrt{1 + \varphi_x^2 + \varphi_z^2}},$$

and the corresponding Euclidean mean curvature is

$$\bar H(X(x,z)) = \frac{\varrho^3(x,z)}{2} \Lambda(x,z),$$

where Λ is the function

$$\Lambda := \varphi_{xx}(1 + \varphi_z^2) - 2\varphi_{xz}\varphi_x\varphi_z + \varphi_{zz}(1 + \varphi_x^2).$$

Hence, the hyperbolic mean curvature H of Σ is

$$H = z\bar H + \bar\eta_3 = \varrho\Big(\frac{z\varrho^2}{2}\Lambda - \varphi_z\Big), \tag{3.9}$$

and its hyperbolic unit normal is $\eta := z\bar\eta$, so that

$$\langle X, \eta\rangle = \frac{\varrho}{z}(\varphi - x\varphi_x - z\varphi_z). \tag{3.10}$$

From (3.9) and (3.10), after noticing that $\varphi_x = -x/\varphi$, we have that the translating soliton equation $\langle X, \eta\rangle = H$ for Σ is equivalent to

$$\Lambda = \frac{2d^2}{z^2\varphi\varrho^2}. \tag{3.11}$$

After taking all first and second order partial derivatives of φ, we get from a direct and long calculation that

$$\Lambda = \frac{d^2}{\varphi^3}(dd'' - (d')^2 - 1). \tag{3.12}$$

Finally, observing that

$$\frac{\varphi^2}{\varrho^2} = \varphi^2(1+\varphi_x^2+\varphi_z^2) = \varphi^2\,\frac{x^2+(dd')^2+\varphi^2}{\varphi^2} = d^2(1+(d')^2),$$

it follows from (3.11) and (3.12) that

$$d'' = \left(\frac{2d^2}{z^2}+1\right)\frac{1+(d')^2}{d},$$

as we wished to prove. ■

Now, we are in position to prove the existence of properly embedded annular translators to MCF in $\mathbb{H}^3$, which we shall call *translating catenoids*, see Figure 3.

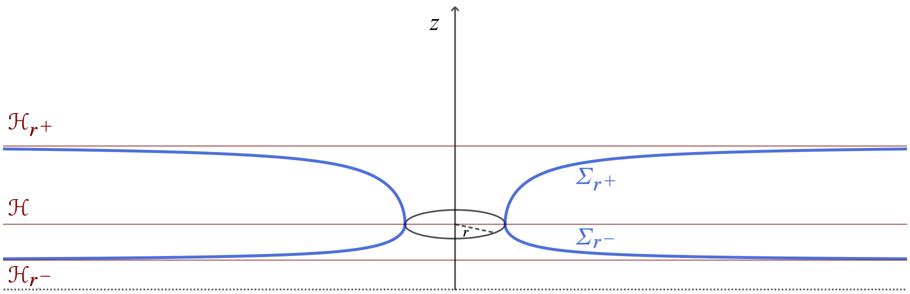

Figure 3. The profile curve of a translating catenoid Σ_r in $\mathbb{H}^3$ bounded by (and asymptotic to) two horospheres $\mathcal{H}_{r-}$ and $\mathcal{H}_{r+}$. $\Sigma_r \setminus \mathcal{H}$ decomposes as two vertical graphs Σ_r^- and Σ_r^+ over the complement of the Euclidean disk of radius r centred at the rotation axis z in the horosphere $\mathcal{H}$.

Theorem 3.10. *There exists a one-parameter family $\mathcal{C} := \{\Sigma_r \mid r > 0\}$ of noncongruent, properly embedded rotational annular translators in $\mathbb{H}^3$ (to be called* translating catenoids*). For each $r > 0$, the surface $\Sigma_r \in \mathcal{C}$ satisfies the following:*

(i) *Σ_r is contained in a slab determined by two horospheres $\mathcal{H}_{r-}$ and $\mathcal{H}_{r+}$. In particular, the asymptotic boundary of Σ_r is the point at infinity of the horosphere $\mathcal{H}$ at height* 1.

(ii) *Σ_r is the union of two vertical graphs Σ_r^- and Σ_r^+ over the complement of the Euclidean r-disk $\mathcal{D}_r$ centred at the rotation axis in the horosphere $\mathcal{H}$.*

(iii) *The graphs Σ_r^- and Σ_r^+ lie in distinct connected components of $\mathbb{H}^3 - \mathcal{H}$ with common boundary the r-circle that bounds $\mathcal{D}_r$ in $\mathcal{H}$, being Σ_r^- asymptotic to $\mathcal{H}_{r-}$ and Σ_r^+ asymptotic to $\mathcal{H}_{r+}$.*

In addition, when $r \to 0$ or when $r \to \infty$, both r^+ and r^- converge to 1 *and the limiting behaviour of Σ_r is as follows:*

(iv) *As $r \to 0$, Σ_r converges (on the $C^{2,\alpha}$-norm, on compact sets outside* $(0,0,1)$*) to a double copy of $\mathcal{H}$.*

(v) *As $r \to +\infty$, Σ_r escapes to infinity.*

Proof. Given $r > 0$, let $d_r : (1-\delta, 1+\delta) \to (0, +\infty)$ be the local solution to the following initial value problem:

$$\begin{cases} f'' = (1+(f')^2)\Big(\dfrac{2f}{z^2} + \dfrac{1}{f}\Big), \\ f(1) = r, \\ f'(1) = 0. \end{cases} \tag{3.13}$$

By Lemma 3.9, the rotational horizontal graph Σ_r determined by d_r is a translator to MCF in $\mathbb{H}^3$. Since d_r is strictly convex, $z = 1$ is a strict local minimum of d_r and $\Sigma_r - \mathcal{H}$ is the union of two disjoint rotational vertical graphs Σ_r^- and Σ_r^+ over an open set contained in $\mathcal{H} - \mathcal{D}_r$. Let us index Σ_r^+ as being the component contained in the horoball $\{z > 1\}$, and let G_r^+ and G_r^- denote the closure of the respective generating curves to Σ_r^+ and Σ_r^- in the yz plane.

Lemmas 3.6 and 3.7 apply to G_r^+ to show that there exists an increasing, concave function $\phi_r\colon [r, +\infty) \to \mathbb{R}$ so that we may extend G_r^+ to assume it is a complete curve

$$G_r^+ = \{(s, \phi_r(s)) \mid s \geq r\}.$$

On $(r, +\infty)$, ϕ_r is smooth and satisfies (3.7), and it holds that $\lim_{s\to r} \phi_r'(s) = +\infty$, see Figure 4. In particular, Σ_r^+ is a vertical graph over $\mathcal{H} - \mathcal{D}_r$. Also, item (ii) of Lemma 3.7 implies that ϕ_r is bounded above by a positive constant $r^+ = \lim_{s\to\infty} \phi_r(s)$, so Σ_r^+ is asymptotic to the horosphere $\mathcal{H}_{r^+}$.

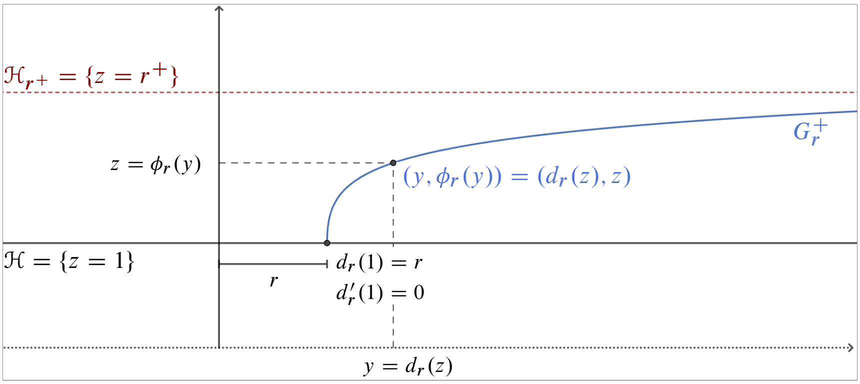

Figure 4. In the yz plane, the curve G_r^+ can be seen both as a horizontal graph $z = d_r(x)$ or as a vertical graph $y = \phi_r(z)$.

Analogously, Lemmas 3.6 and 3.7 give that Σ_r^- can be extended and is asymptotic to the horosphere $\mathcal{H}_{r^-}$ of $\mathbb{H}^3$, where $r^- = \lim_{s\to\infty} \varphi_r(s)$ and $\varphi_r\colon [r, +\infty) \to \mathbb{R}$ is the graphing function of the profile curve of Σ_r^- when it is considered as a rotational vertical graph. Since $\Sigma_r = \overline{\Sigma}_r^- \cup \overline{\Sigma}_r^+$, we have that Σ_r is an annular properly embedded translator to MCF in $\mathbb{H}^3$. This proves assertions (i)–(iii).

In the remainder of the proof, we will use the functions ϕ_r, φ_r and d_r as defined above, and we notice that the previous arguments imply that the maximal interval of definition of d_r is (r^-, r^+). For $r^* = 0$ or ∞, we will prove that

$$\lim_{r \to r^*} r^+ = 1 = \lim_{r \to r^*} r^-. \tag{3.14}$$

From there, item (iv) follows directly, since Σ_r is a bygraph of functions that will both converge to 1 uniformly in the C^2 norm on compact sets that do not contain $(0, 0, 1)$ (the $C^{2,\alpha}$ convergence can be obtained by the standard theory of elliptic partial differential equations, after observing that the graphing functions of Σ_r^+ and Σ_r^- satisfy an equation as presented in (3.31)). We notice that our proof of (3.14) follows a series of steps, each of which will be presented separately. Our first claim, stated below, shows that r^+ is uniformly bounded.

Claim 3.11. For any $r > 0$, $r^+ \le e^{\pi/(4\sqrt{2})}$.

Proof of Claim 3.11. Let $r > 0$ be given. For any $z \in [1, r^+)$, Claim 3.8 implies that $2d_r/z^2 + 1/d_r \ge 2\sqrt{2}/z$. In particular, since d_r is a solution to (3.13), one has

$$\frac{d_r''}{1 + (d_r')^2} \ge \frac{2\sqrt{2}}{z}.$$

Integrating over $[1, z]$, we obtain

$$\int_1^z \frac{d_r''(t)}{1 + (d_r'(t))^2}\,dt \ge \int_1^z \frac{2\sqrt{2}}{t}\,dt.$$

Since $d_r'(1) = 0$, it follows that

$$\arctan(d_r'(z)) \ge 2\sqrt{2}\log(z) \implies 2\sqrt{2}\log(z) \le \frac{\pi}{2}.$$

Since z can be chosen arbitrarily close to r^+, this proves the claim. ■

Claim 3.12. With the above notation, it holds that

$$\lim_{r \to +\infty} r^+ = 1 = \lim_{r \to +\infty} r^-.$$

Proof of Claim 3.12. For a given $r > 0$, we have that $d_r(z) \ge r$ for any $z \in (r^-, r^+)$. This, together with the fact that d_r is a solution to (3.13), implies that

$$\frac{d_r''}{1 + (d_r')^2} = \frac{2d_r}{z^2} + \frac{1}{d_r} > \frac{2r}{z^2}. \tag{3.15}$$

As in the proof of Claim 3.11, given $z \in (1, r^+)$, we may integrate both sides of (3.15) over $[1, z]$ to obtain

$$\arctan(d_r'(z)) > -2r\Big(\frac{1}{z} - 1\Big).$$

Passing to the limit when $z \to r^+$, we find the inequality

$$\frac{\pi}{2} \ge 2r\,\frac{r^+ - 1}{r^+}. \tag{3.16}$$

By (3.16) and Claim 3.11, $r(r^+ - 1)$ is uniformly bounded by a constant, from where it follows that $\lim_{r \to +\infty} r^+ = 1$.

An analogous argument can be used to show that $\lim_{r\to+\infty} r^- = 1$. Indeed, for $z \in (r^-, 1)$, we may integrate (3.15) over $[z, 1]$ and take the limit when $z \to r^-$ to obtain

$$-\arctan(d_r'(z)) > -2r\Big(1 - \frac{1}{z}\Big) \implies 2r\Big(\frac{1-r^-}{r^-}\Big) \le \frac{\pi}{2}.$$

When $r \to +\infty$, since $r^- \in (0, 1)$, we have that the left-hand side becomes arbitrarily large, unless $r^- \to 1$. ■

Next, we analyse the limits of r^+ and r^- when $r \to 0$.

Claim 3.13. Let $(r_n)_{n\in\mathbb{N}}$ be a sequence so that $\lim r_n = 0$ and for which $\lim r_n^+ = h > 1$. Then, for any $h_0 \in (1, h)$, there exists $\delta > 0$, depending on h_0, such that

$$\liminf d_{r_n}(h_0) > \delta.$$

Proof of Claim 3.13. Take $h_0 \in (1, h)$. For any sufficiently large n, the domain of d_{r_n} contains $[1, h_0]$. Then, as in the proof of Claim 3.11, we may use the inequality

$$\frac{2d_{r_n}}{z^2} + \frac{1}{d_{r_n}} \ge \frac{2\sqrt{2}}{z}$$

to conclude that, for any $z \in [1, h_0]$,

$$\arctan(d_{r_n}'(z)) \ge 2\sqrt{2}\log(z) \implies d_{r_n}'(z) \ge \tan(2\sqrt{2}\log(z)).$$

Integrating over $[1, h_0]$, we arrive at

$$d_{r_n}(h_0) - r_n > \int_1^{h_0} \tan(2\sqrt{2}\log(z))\,dz.$$

Setting $2\delta = \int_1^{h_0} \tan(2\sqrt{2}\log(z))\,dz > 0$ proves the claim. ■

As the proof continues, for $r > 0$, we will make use of the functions $\phi_r\colon [r, +\infty) \to [1, r^+)$, and recall that, by Lemma 3.6,

$$\phi_r'' = -\phi_r'(1 + (\phi_r')^2)\Big(\frac{2s}{\phi_r^2} + \frac{1}{s}\Big). \tag{3.17}$$

Claim 3.14. Let $\delta > 0$ be given. Then the following inequality holds for any $r \in (0, \delta)$:

$$\arctan(\phi_r'(\delta)) \ge 2\delta\,\frac{r^+ - \phi_r(\delta)}{r^+\phi_r(\delta)}. \tag{3.18}$$

Proof of Claim 3.14. For $\delta > r > 0$, as stated, since ϕ_r' is nonnegative, (3.17) gives the inequality

$$\frac{-\phi_r''(s)}{1 + (\phi_r'(s))^2} > 2\delta\,\frac{\phi_r'(s)}{\phi_r^2(s)} \quad \text{for all } s > \delta. \tag{3.19}$$

Then integrating (3.19) over $[\delta, s]$ yields

$$\arctan(\phi_r'(\delta)) - \arctan(\phi_r'(s)) > 2\delta\,\frac{\phi_r(s) - \phi_r(\delta)}{\phi_r(s)\phi_r(\delta)} \quad \text{for all } s > \delta.$$

When $s \to +\infty$, $\phi_r'(s) \to 0$ and $\phi_r(s) \to r^+$, proving (3.18). ■

Claim 3.15. We have

$$\lim_{r\to 0} r^+ = 1.$$

Proof of Claim 3.15. We argue by contradiction and assume that there exists a sequence $(r_n)_{n\in\mathbb{N}}$, $\lim_{n\to\infty} r_n = 0$, such that $\lim_{n\to\infty} r_n^+ = h > 1$. Choose any $h_0 \in (1, h)$ and let δ be given by Claim 3.13. It then follows from Claim 3.14 that, for any sufficiently large $n \in \mathbb{N}$ such that $r_n < \delta$, the following inequality holds:

$$\arctan(\phi'_{r_n}(\delta)) \geq 2\delta\,\frac{r_n^+ - \phi_{r_n}(\delta)}{r_n^+ \phi_{r_n}(\delta)}. \tag{3.20}$$

Next, we prove that $\lim_{n\to\infty} \phi'_{r_n}(\delta) = 0$. To that end, first notice that $\phi'_{r_n}(\delta) < \phi'_{r_n}(s)$ for all $s \in (r_n, \delta)$, since ϕ_{r_n} is concave. Therefore,

$$\phi'_{r_n}(s)\Big(\frac{2s}{\phi^2_{r_n}(s)} + \frac{1}{s}\Big) > \phi'_{r_n}(\delta)\,\frac{1}{s} \quad \text{for all } s \in (r_n, \delta). \tag{3.21}$$

Together with (3.17), (3.21) implies

$$\arctan(\phi'_{r_n}(s)) - \arctan(\phi'_{r_n}(\delta)) > \phi'_{r_n}(\delta)\log\Big(\frac{\delta}{s}\Big) \quad \text{for all } s \in (r_n, \delta).$$

By letting $s \to r_n$, we finally obtain

$$\frac{\pi}{2} > \frac{\pi}{2} - \arctan(\phi'_{r_n}(\delta)) > \phi'_{r_n}(\delta)\log\Big(\frac{\delta}{r_n}\Big). \tag{3.22}$$

Since δ is fixed and $\lim_{n\to\infty} r_n = 0$, we have from (3.22) that $\lim_{n\to\infty} \phi'_{r_n}(\delta) = 0$, as we wished to prove. Therefore, by taking the limit as $n \to \infty$ in (3.20), it follows that $\lim_{n\to\infty} \phi_{r_n}(\delta) = h > h_0$.

To finish the proof of the claim, we notice that there exists $n_0 > 0$ such that

$$\phi_{r_n}(\delta) = h_n > h_0 \quad \text{for all } n \geq n_0.$$

Since $d_{r_n}(h_n) = \delta$ and $h_n > h_0$, the fact that d_{r_n} is increasing in $[1, r_n^+)$ implies that, for all $n \geq n_0$, $d_{r_n}(h_0) < \delta$, contradicting the defining property of δ. ■

The next claim will finish the proof of (3.14). We will follow the same arguments presented in Claims 3.13, 3.14 and 3.15, and due to the similarity on the reasoning, only the main differences will be discussed.

Claim 3.16. We have

$$\lim_{r\to 0} r^- = 1.$$

Proof of Claim 3.16. By contradiction, assume that there exists a sequence $(r_n)_{n\in\mathbb{N}}$ converging to zero for which $\lim r_n^- = h < 1$. As in Claim 3.13, fix $h_0 \in (h, 1)$ and consider the functions $d_{r_n}\colon (r_n^-, 1] \to [r_n, +\infty)$. Then we may use the inequality

$$\frac{2d_{r_n}}{z^2} + \frac{1}{d_{r_n}} \geq \frac{2\sqrt{2}}{z}$$

to obtain, for any $z \in [h_0, 1]$,

$$d'_{r_n}(z) \le \tan(2\sqrt{2}\log(z)).$$

Integrating over $[h_0, 1]$ and setting $2\delta = -\int_{h_0}^1 \tan(2\sqrt{2}\log(z))\,dz > 0$, we arrive at

$$\liminf d_{r_n}(h_0) > \delta. \tag{3.23}$$

Next, for given $r > 0$, we will make use of the graphing function $\varphi_r : [r, +\infty) \to (r^-, 1]$ to the profile curve of Σ_r^-, which by Lemma 3.6 satisfies

$$\varphi_r'' = -\varphi_r'(1 + (\varphi_r')^2)\Big(\frac{2s}{\varphi_r^2} + \frac{1}{s}\Big). \tag{3.24}$$

Following the arguments in Claim 3.14, for any $s \ge \delta$, we have $2s/\varphi_{r_n}^2 + 1/s > 2\delta/\varphi_{r_n}^2$. Since $\varphi'_{r_n}(s) < 0$ for all $s \ge \delta$, this allows us to obtain

$$\frac{\varphi''_{r_n}}{1 + (\varphi'_{r_n})^2} > -\varphi'_{r_n}\frac{2\delta}{\varphi_{r_n}^2}.$$

Integrating over $[\delta, s]$ and letting $s \to \infty$ gives

$$-\arctan(\varphi'_{r_n}(\delta)) \ge 2\delta\Big(\frac{1}{r_n^-} - \frac{1}{\varphi_{r_n}(\delta)}\Big). \tag{3.25}$$

Next, we will prove that $\lim \varphi'_{r_n}(\delta) = 0$. To do so, consider $s \in (r_n, \delta)$ and observe that

$$\frac{\varphi''_{r_n}}{1 + (\varphi'_{r_n})^2} = -\varphi'_{r_n}\Big(\frac{2s}{\varphi_{r_n}^2} + \frac{1}{s}\Big) > -\varphi'_{r_n}(\delta)\,\frac{1}{s},$$

so integrating over $[s, \delta]$ and letting $s \to r_n^-$ yields

$$\frac{\pi}{2} > \arctan(\varphi'_{r_n}(\delta)) + \frac{\pi}{2} > -\varphi'_{r_n}(\delta)\log\Big(\frac{\delta}{r_n^-}\Big),$$

which proves that $\lim \varphi'_{r_n}(\delta) = 0$. Using this limit in (3.25) implies that $\lim \varphi_{r_n}(\delta) = h$, which, analogously to the final step in Claim 3.15, contradicts (3.23). ■

Claims 3.12, 3.15 and 3.16 prove (3.14), finishing the proof of Theorem 3.10. ■

Let Σ be a connected rotational translator in $\mathbb{H}^3$ with (possibly empty) boundary. If Σ is not a horosphere, Lemmas 3.6 and 3.9, together with the uniqueness of solutions of ODEs with given initial conditions, imply that the profile curve of Σ coincides, up to its boundary, with the profile curve of some translating catenoid Σ_r obtained in Theorem 3.10. Therefore, we have the following uniqueness result.

Theorem 3.17. *Any connected rotational translator of $\mathbb{H}^3$ is either an open subset of a horosphere or of some translating catenoid.*

3.2. Parabolic translators

Having considered rotational translators in the previous section, we now look at translators which are invariant by a 1-parameter group of *parabolic* isometries of $\mathbb{H}^3$, i.e., isometries of $\mathbb{H}^3$ that fix parallel families of horospheres. Horizontal cylinders over curves on vertical totally geodesic planes of $\mathbb{H}^3$ (to be called *parabolic cylinders*) are the simplest examples of surfaces which are invariant by parabolic translations. When these generating curves are graphs on the whole of $\mathbb{R}$, such a surface can be parametrized by a map $X\colon\mathbb{R}^2\to\mathbb{R}^3_+$ defined by

$$X(x,y)=(x,y,\phi(y)),\quad (x,y)\in\mathbb{R}^2,$$

where ϕ is a smooth positive function on $\mathbb{R}$. We shall call $\Sigma := X(\mathbb{R}^2)$ the *parabolic cylinder determined by* ϕ.

Defining

$$\varrho(y):=(1+(\phi'(y))^2)^{-1/2},$$

we have that

$$\bar\eta:=\varrho(0,-\phi',1)$$

is a unit normal to Σ with respect to the induced Euclidean metric of $\mathbb{R}^3_+$. With this orientation, the Euclidean mean curvature $\bar H$ of Σ is

$$\bar H=\frac{\varrho^3\phi''}{2}\cdot$$

From this last equality and (2.1), we have that the hyperbolic mean curvature H of Σ with respect to the orientation $\eta:=\phi\bar\eta$ is

$$H=\varrho\Big(\frac{\varrho^2\phi\phi''}{2}+1\Big).$$

Since $\langle\eta,X\rangle=\varrho(\phi-y\phi')/\phi$, we also have that the identity (3.1) for the parabolic cylinder $\Sigma=X(\mathbb{R}^2)$ is equivalent to the following second order ODE:

$$\phi''=-\phi'(1+(\phi')^2)\,\frac{2y}{\phi^2}\cdot$$

The above considerations yield the following lemma.

Lemma 3.18. *A parabolic cylinder determined by a smooth function ϕ is a translator to MCF in $\mathbb{H}^3$ if and only if $\phi=\phi(y)$ is a solution to the second order ODE*

$$f''=-f'(1+(f')^2)\,\frac{2y}{f^2}\cdot \tag{3.26}$$

The solutions of (3.26) are all increasing on $\mathbb{R}$ and their graphs are "S-shaped", as attested by the following result, see Figure 5.

Lemma 3.19. *Given $\lambda\geq 0$, the initial value problem*

$$\begin{cases} f''=-f'(1+(f')^2)\,\dfrac{2y}{f^2},\\ f(0)=1,\\ f'(0)=\lambda. \end{cases} \tag{3.27}$$

has a unique smooth solution $\phi\colon \mathbb{R} \to (0, +\infty)$ *which has the following properties*:

(i) ϕ *is constant if* $\lambda = 0$.

(ii) ϕ *is increasing, convex in* $(-\infty, 0)$*, and concave in* $(0, +\infty)$ *if* $\lambda > 0$.

(iii) ϕ *is bounded from above by a positive constant.*

(iv) ϕ *is bounded from below by a positive constant.*

Proof. Once again, the existence and uniqueness of local solutions follow directly from the fact that

$$(u, v, w) \in \mathbb{R} \times (0, \infty) \times \mathbb{R} \mapsto -w(1 + (w)^2)\,\frac{2u}{v^2}$$

is smooth. In particular, assertion (i) is immediate.

Let ϕ denote a local solution to (3.27) for some $\lambda > 0$, defined in its maximal domain $I_{\max} := (y_{\min}, y_{\max})$, $-\infty \leq y_{\min} < 0 < y_{\max} \leq +\infty$. From uniqueness of solutions, ϕ cannot admit any critical point. Hence, we may argue as in the proof of Lemma 3.7 to observe that ϕ is strictly increasing. In particular, the equality

$$\phi'' = -\phi'(1 + (\phi')^2)\,\frac{2y}{\phi^2} \tag{3.28}$$

implies that ϕ is convex in $(y_{\min}, 0)$ and concave in $(0, y_{\max})$, which yields $y_{\max} = +\infty$.

Next, we prove that the solution ϕ is bounded above. We will argue by contradiction and assume that $\lim_{y\to\infty} \phi(y) = +\infty$.

The fact that $\phi'(y) > 0$ and $\phi''(y) < 0$, for all $y > 0$, implies the existence of some $c \geq 0$ such that $\lim_{y\to+\infty} \phi'(y) = c$. We will derive a contradiction by ruling out the two possible cases: $c = 0$ and $c > 0$. First, fix any $y_0 > 0$. It follows from (3.28) that

$$\frac{-\phi''(y)}{1 + (\phi')^2(y)} = 2y\,\frac{\phi'(y)}{\phi^2(y)} \geq 2y_0\,\frac{\phi'(y)}{\phi^2(y)} \quad \text{for all } y \geq y_0.$$

Integrating over $[y_0, y]$ gives that

$$\arctan(\phi'(y_0)) - \arctan(\phi'(y)) \geq 2y_0\left(\frac{1}{\phi(y_0)} - \frac{1}{\phi(y)}\right).$$

Then, taking the limit when $y \to +\infty$, the hypothesis that $c = 0$ implies that

$$\frac{2y_0}{\phi(y_0)} \leq \arctan(\phi'(y_0)) < \frac{\pi}{2}.$$

However, it follows from L'Hôpital's rule that

$$\frac{\pi}{2} \geq \lim_{y_0\to+\infty} \frac{2y_0}{\phi(y_0)} = \lim_{y_0\to+\infty} \frac{2}{\phi'(y_0)} = +\infty,$$

a contradiction.

To rule out the case when $c > 0$, once again we let $y_0 > 0$ be given and, for $y \geq y_0$, we use the inequality $1 + (\phi'(y))^2 \geq 2\phi'(y)$ in (3.28) to obtain

$$\frac{-\phi''(y)}{\phi'(y)} = (1 + (\phi'(y))^2)\,\frac{2y}{\phi^2(y)} \geq 4y_0\,\frac{\phi'(y)}{\phi^2(y)} \quad \text{for all } y \geq y_0.$$

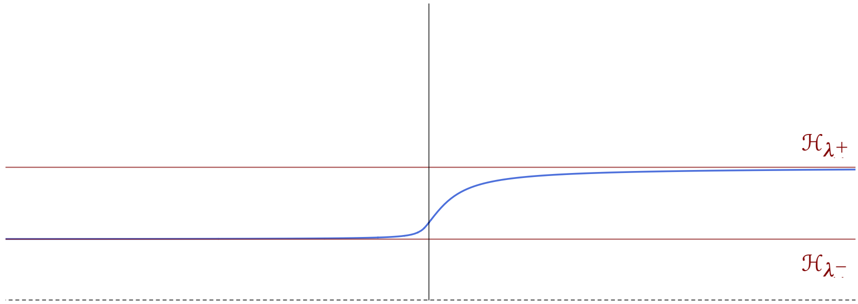

Figure 5. The profile curve of a hyperbolic grim reaper in $\mathbb{H}^3$ lying between two horospheres $\mathcal{H}_{\lambda^-}$ and $\mathcal{H}_{\lambda^+}$.

Integrating over $[y_0, y]$ and letting $y \to +\infty$ gives

$$\log\left(\frac{\phi'(y_0)}{c}\right) \geq \frac{4y_0}{\phi(y_0)} > 0,$$

which immediately implies that $\lim_{y_0\to+\infty} \frac{y_0}{\phi(y_0)} = 0$. This is a contradiction, since

$$\lim_{y_0\to+\infty} \frac{y_0}{\phi(y_0)} = \lim_{y_0\to+\infty} \frac{1}{\phi'(y_0)} = \frac{1}{c} > 0.$$

Hence, this proves item (iii) of the lemma.

To prove (iv), fix any $y_0 \in (y_{\min}, 0)$ and, for $y_{\min} < y \leq y_0$, apply (3.28) to obtain

$$\frac{-\phi''(y)}{1 + (\phi'(y))^2} \leq 2y_0 \frac{\phi'(y)}{\phi^2(y)} \quad \text{for all } y \in (y_{\min}, y_0),$$

which implies that

$$\arctan(\phi'(y)) - \arctan(\phi'(y_0)) \leq 2y_0\left(\frac{1}{\phi(y)} - \frac{1}{\phi(y_0)}\right) \quad \text{for all } y \in (y_{\min}, y_0). \tag{3.29}$$

Since the left-hand side of (3.29) is bounded from below by $-\pi$ and $y_0 < 0$, a simple algebraic manipulation gives

$$\frac{1}{\phi(y_0)} - \frac{\pi}{2y_0} \geq \frac{1}{\phi(y)} \quad \text{for all } y \in (y_{\min}, y_0),$$

from which we conclude that $\lim_{y\to y_{\min}} \phi(y) > 0$. This shows that $y_{\min} = -\infty$ and proves item (iv). ■

Lemmas 3.18 and 3.19 immediately give the following result (see Figure 5).

Theorem 3.20. *In hyperbolic space $\mathbb{H}^3$, there exists a one-parameter family*

$$\mathcal{G} := \{\Sigma_\lambda \mid \lambda \in [0, +\infty)\}$$

of noncongruent, complete translators (to be called grim reapers) *which are horizontal parabolic cylinders generated by the solutions of* (3.27). Σ_0 *is the horosphere* $\mathcal{H} \subset \mathbb{H}^3$ *at height one, and for* $\lambda > 0$, *each* $\Sigma_\lambda \in \mathcal{G}$ *is an entire graph over* $\mathbb{R}^2$ *which is contained in a slab determined by two horospheres* $\mathcal{H}_{\lambda-}$ *and* $\mathcal{H}_{\lambda+}$. *Furthermore, there exist open sets* Σ_λ^- *and* Σ_λ^+ *of* Σ_λ *such that* Σ_λ^- *is asymptotic to* $\mathcal{H}_{\lambda-}$, Σ_λ^+ *is asymptotic to* $\mathcal{H}_{\lambda+}$, *and* $\Sigma_\lambda = \text{closure}(\Sigma_\lambda^-) \cup \text{closure}(\Sigma_\lambda^+)$.

If $\Gamma \subset \mathbb{R}^2$ is the graph of the function $t \in (-\pi/2, \pi/2) \mapsto -\log(\cos t)$, then the cylinder $\Sigma = \Gamma \times \mathbb{R} \subset \mathbb{R}^3$ is a translator to MCF contained in a slab $\mathcal{S}$ of $\mathbb{R}^3$, known as the *grim reaper cylinder*. This nomenclature is due to the fact that the curve Γ provides a solution to the curve shortening flow, called *the grim reaper*, which is given by the translation of Γ in $\mathbb{R}^2$ in the $\vec{e}_2$-direction. By the avoidance principle, such a solution "kills" any other solution in the region $(-\pi/2, \pi/2) \times \mathbb{R}$ (see Chapter 2 of [18]). Similarly, two surfaces (one of them compact) in $\mathbb{R}^3$ moving under MCF which are initially disjoint remain so until one of them collapses. Hence, as Σ translates under MCF, it "kills" all solutions to (2.2) in $\mathcal{S}$ with compact initial condition. An analogous process occurs in our case: any surface of the family $\mathcal{G}$ in Theorem 3.20 has this "killing" property. Indeed, by Theorem 4 in [5], the avoidance principle applies to surfaces moving under MCF in $\mathbb{H}^3$. For this reason, we named the elements of $\mathcal{G}$ grim reapers.

Remark 3.21. The symmetry in (3.26) allows us to extend the family $\mathcal{G}$ in Theorem 3.20 for values $\lambda < 0$ by simply defining $\psi(y) = \phi(-y)$ for a given solution ϕ to (3.27) with positive initial data for ϕ'. However, the translator generated by ψ corresponds to a rotation of π around the z-axis of a grim reaper, being therefore congruent to it.

Remark 3.22. At the completion of this manuscript, we became acquainted with the preprint [16], in which the authors consider solitons to MCF generated by conformal fields in $\mathbb{H}^n$, called *conformal solitons*. There, they obtained rotational and cylindrical conformal solitons whose initial conditions are named winglike catenoids and grim reaper cylinder, respectively. However, such solitons are not related to the ones considered here, since their generating fields are not Killing.

Analogously to the rotational case, the uniqueness of solutions of ODEs with given initial conditions yields the following result.

Theorem 3.23. *Any connected translator in* $\mathbb{H}^3$ *which is a parabolic cylinder is, up to an ambient isometry (see Remark* 3.21*), an open subset of a grim reaper or of a totally geodesic plane containing the* z*-axis.*

It would be interesting to determine whether, apart from parabolic cylinders, there exist translators in $\mathbb{H}^3$ that are invariant by parabolic translations. In the negative case, Theorem 3.23 would establish the classification of such invariant translators.

3.3. The tangency principle and applications

A distinguished property of translators to MCF in $\mathbb{R}^3$ is that they are critical points of a weighted area functional and, therefore, they become minimal surfaces when changing the ambient metric in a suitable manner [13]. In particular, the tangency principle applies to them, which allows one to use translators as barriers (cf. [15]). On the other hand, it is unknown to us if translators to MCF in $\mathbb{H}^3$ can be made minimal in a similar fashion. Nevertheless, a tangency principle for translators in $\mathbb{H}^3$ can be established as a direct application of the maximum principle for quasilinear elliptic operators. More precisely, we have the following result.

Tangency principle for translators. *Let Σ_1 and Σ_2 be two translators to MCF in $\mathbb{H}^3$ which are tangent at a point $p \in \operatorname{int}\Sigma_1 \cap \operatorname{int}\Sigma_2$. If Σ_1 lies on one side of Σ_2 in a neighbourhood of p in $\mathbb{H}^3$, then Σ_1 and Σ_2 coincide in a neighbourhood of p in $\Sigma_1 \cap \Sigma_2$. Moreover, if Σ_1 and Σ_2 are both complete and connected, then $\Sigma_1 = \Sigma_2$.*

Proof. Let Σ_1 and Σ_2 be two translators to MCF in $\mathbb{H}^3$, tangent at a point $p \in \Sigma_1 \cap \Sigma_2$, and such that Σ_1 stays locally on one side of Σ_2. If $T_p\Sigma_1$ is not vertical, there exist a domain $\Omega \subset \mathbb{R}^2$ and positive functions $u_1, u_2 \colon \Omega \to \mathbb{R}$ such that neighbourhoods $U_1 \subset \Sigma_1$ and $U_2 \subset \Sigma_2$ containing p are respectively parametrized by

$$U_1 = \{(x, y, u_1(x, y)) \mid (x, y) \in \Omega\} \quad \text{and} \quad U_2 = \{(x, y, u_2(x, y)) \mid (x, y) \in \Omega\}.$$

Furthermore, after reindexing we may assume that $u_1 \geq u_2$ in Ω.

Let Σ_1 and Σ_2 be oriented with respect to vector fields η_1 and η_2 so that $\eta_1(p) = \eta_2(p)$ points upwards. Thus, if Q is the quasilinear elliptic operator

$$Q(u) = u_{xx}(1 + u_y^2) + u_{yy}(1 + u_x^2) - 2u_{xy}u_x u_y, \tag{3.30}$$

it follows from (2.1) that the mean curvature functions H_1 and H_2 of U_1 and U_2 satisfy

$$H_i = u_i \frac{Q(u_i)}{2(1 + (u_i)_x^2 + (u_i)_y^2)^{3/2}} + \frac{1}{(1 + (u_i)_x^2 + (u_i)_y^2)^{1/2}}, \quad i \in \{1, 2\}.$$

Then, after setting $B(x, y, u, Du) = 2(1 + u_x^2 + u_y^2)(xu_x + yu_y)$, where Du denotes the (Euclidean) gradient of u, it follows from (3.1) that

$$(u_i)^2 Q(u_i) + B(x, y, u_i, Du_i) = 0, \quad i \in \{1, 2\}. \tag{3.31}$$

But the operator $u^2 Q(u) + B(x, y, u, Du)$ in (3.31) satisfies the hypothesis of the maximum principle for quasilinear operators, see Theorem 2.2.2 in [19], thus $u_1 = u_2$, which implies $U_1 = U_2$.

The case where $T_p\Sigma_1$ is vertical can be treated analogously. After a rotation about the z-axis (which preserves the property of being a translator to MCF), locally, both Σ_1 and Σ_2 can be parametrized as horizontal graphs

$$\{(x, u_1(x, z), z) \mid (x, z) \in \widehat{\Omega}\} \quad \text{and} \quad \{(x, u_2(x, z), z) \mid (x, z) \in \widehat{\Omega}\}$$

for some domain $\widehat{\Omega} \subset \mathbb{R}^2_+$, and both u_1, u_2 satisfy

$$z^2 Q(u) + \widehat{B}(x, z, u, Du) = 0$$

for $\widehat{B}(x, z, u, Du) = 2(xu_x - u)(1 + u_x^2 + u_z^2)$ and Q as in (3.30). Once again, we obtain from Theorem 2.22 in [19] that Σ_1 and Σ_2 coincide in a neighbourhood of p.

At this point, we have shown that if Σ_1 and Σ_2 are tangent at a point p, they must coincide in neighbourhoods which are either horizontal or vertical graphs for Σ_1 and Σ_2. The proof for the case where Σ_1 and Σ_2 are complete and connected now follows from covering Σ_1 and Σ_2 with such (overlapping) neighbourhoods. ∎

Remark 3.24. The tangency principle for translators contrasts with the one for constant mean curvature surfaces (see, for instance, Theorem 3.2.4 in [14]): two distinct geodesic spheres in $\mathbb{R}^3$ with the same mean curvature can be tangent to each other without violating the tangency principle. In the setting of translators, the tangency principle does not require any assumptions on the orientation of Σ_1 and Σ_2 because, from (3.1), if Σ_1 and Σ_2 are translators to MCF which are tangent at a point p, then necessarily their mean curvature vectors $\mathbf{H}_1$ and $\mathbf{H}_2$ must agree at p, which defines a coinciding, *standard* (local) orientation for both Σ_1 and Σ_2.

On the remainder of the section we will apply the tangency principle to establish some classification results concerning translators in $\mathbb{H}^3$. First, we show that properly immersed translators in $\mathbb{H}^3$ are never cylindrically bounded, and next we prove that any horoconvex translator which is complete or transversal to the z-axis is necessarily an open set of a horizontal horosphere.

Recall that a circular cone in $\mathbb{R}^3_+ := \mathbb{R}^2 \times (0, +\infty)$ with vertex at $p \in \mathbb{R}^2$ and axis $\gamma_p := \{p\} \times (0, +\infty)$ constitutes a *cylinder* $\mathcal{C}$ in $\mathbb{H}^3$, that is, the set of points of $\mathbb{H}^3$ at a fixed distance to the vertical geodesic γ_p. The convex side of $\mathcal{C}$ is the component of $\mathbb{H}^3 - \mathcal{C}$ which contains γ_p.

Theorem 3.25. *There is no properly immersed translator to MCF in $\mathbb{H}^3$ which is contained in the convex side of a cylinder with vertex at $p = (0,0)$. In particular, there is no closed (i.e., compact without boundary) translator to MCF in $\mathbb{H}^3$.*

Proof. Suppose, by contradiction, that there exists a properly immersed translator Σ to MCF in $\mathbb{H}^3$ which is contained in the convex side Ω of a cylinder $\mathscr{C}$ with vertex at $p = (0,0)$. Clearly, the property of being a translator is invariant by the translations $\Gamma_t(p) := e^t p$, $t \in \mathbb{R}$. Therefore, we can assume without loss of generality that Σ intersects the horosphere $\mathcal{H}$ of height 1.

Under the above conditions, since r^+ is uniformly bounded, we have from item (v) of Theorem 3.10 that there exists $R > 0$ such that, for any $r > R$, the translating catenoid Σ_r of the family $\mathcal{C}$ is disjoint from $\mathscr{C}$, and so from Σ. On the other hand, for a sufficiently small $r > 0$, Σ_r and Σ have nonempty intersection. Taking into account the asymptotic behaviour of Σ_r, together with the hypothesis that Σ is contained in Ω, as r decreases from R to zero, a standard argument shows that there will be a first value r_* such that Σ_{r_*} is the element of $\mathcal{C}$ that first establishes a contact with Σ at a point $p \in \Sigma \cap \Sigma_{r_*}$, as in Figure 6. Then Σ and Σ_{r_*} are tangent at p with Σ on one side of Σ_r, and the tangency principle applies to show that $\Sigma = \Sigma_{r_*}$, which is a contradiction, since Σ is contained in Ω and Σ_{r_*} is not. ∎

An oriented surface Σ of $\mathbb{H}^3$ is called *horoconvex* if its principal curvatures k_1 and k_2 satisfy $k_i \geq 1$, $i = 1, 2$. It is well known that every point p of a horoconvex surface $\Sigma \subset \mathbb{H}^3$ is locally supported by a horosphere $\mathscr{H}$, meaning that $\mathscr{H}$ is tangent to Σ at p, and there exists a neighbourhood of p in Σ which lies in the horoball bounded by $\mathscr{H}$ (see, e.g., Lemma 1 in [4]). Since the tangency principle immediately implies that no translator can attain a local maximum or local minimum for its height function, we obtain that *any horoconvex translator whose tangent plane is horizontal at one of its points is necessarily an open set of a horizontal horosphere*. With these considerations, we are in position to establish the following uniqueness result.

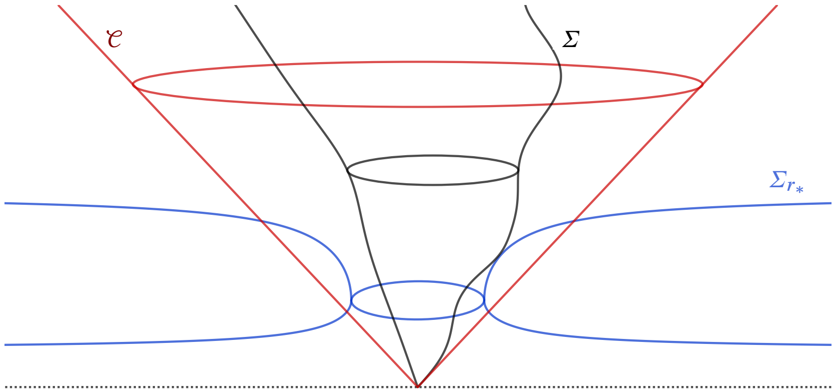

Figure 6. In the proof of Theorem 3.25, there exists a smallest $r_* > 0$ such that Σ_{r_*} intersects Σ tangentially, with Σ in one of the two regions of $\mathbb{H}^3$ defined by Σ_{r_*}.

Theorem 3.26. *Let Σ be a horoconvex translator in $\mathbb{H}^3$ and assume that one of the following occurs*:

(i) *Σ is complete,*

(ii) *Σ intersects the z-axis transversely.*

Then Σ is an open set of a horizontal horosphere.

Proof. Let Σ be as stated. By the comments above, in order to prove that Σ is an open set of a horizontal horosphere, it suffices to show that the tangent plane of Σ is horizontal at one of its points.

First, assume that item (i) holds, so Theorem 3.25 implies that Σ is noncompact. The main results in [4] show that horospheres are the unique complete and noncompact horoconvex surfaces of $\mathbb{H}^3$, so Σ is a horosphere. In particular, there is a point $p \in \Sigma$, where $T_p\Sigma$ is horizontal, so Σ is a horizontal horosphere.

Assume now that Σ intersects the z-axis transversally at a point. Then there is an open neighbourhood U of Σ, which is a graph of a smooth function u over an open ball $\Omega \subset \mathbb{R}^2$ centred at the origin $(0,0)$. Since Σ is a translator, u satisfies (see the proof of the tangency principle),

$$u^2 Q(u) + B(x,y,u,Du) = 0, \tag{3.32}$$

where Q is defined by (3.30) and $B(x,y,u,Du) = 2(1+u_x^2+u_y^2)(xu_x+yu_y)$. In addition, it follows from the horoconvexity of Σ that its mean curvature $\overline{H}$ with respect to the induced Euclidean metric cannot change sign, being either everywhere nonpositive or everywhere nonnegative, which, when orienting U with respect to the upwards pointing normal, correspond to the respective cases $Q(u) \geq 0$ or $Q(u) \leq 0$.

First, assume that $Q(u) \geq 0$. Then it follows from (3.32) that

$$xu_x(x,y) + yu_y(x,y) \leq 0 \quad \text{for all } (x,y) \in \Omega. \tag{3.33}$$

Let $\theta \in [0, \pi]$ be given and consider $u_\theta(t) = u(t\cos(\theta), t\sin(\theta))$, the restriction of u to the line through the origin $L_\theta = \{(t\cos(\theta), t\sin(\theta)) \mid t \in \mathbb{R}\} \cap \Omega$. Then (3.33) implies that

$$tu'_\theta(t) = t\cos(\theta)u_x(t\cos(\theta), t\sin(\theta)) + t\sin(\theta)u_y(t\cos(\theta), t\sin(\theta)) \leq 0.$$

In particular, $u'_\theta(t) \geq 0$ when $t < 0$ and $u'_\theta(t) \leq 0$ when $t > 0$, therefore $t = 0$ is a point where u_θ assumes a local maximal value. Since θ is any given value in the compact set $[0, \pi]$, this implies that the tangent plane of Σ at $p = (0, 0, u(0, 0))$ is horizontal, as we wished to prove.

The case when $Q(u) \leq 0$ (i.e., when the mean curvature vector of Σ points downwards) can be treated analogously. ∎

Remark 3.27. The horoconvexity hypothesis in Theorem 3.26 is necessary to the conclusion, since translating catenoids and grim reapers are complete, and any grim reaper is transversal to the z-axis. Furthermore, as we proved in Theorem 3.10 (iv), horizontal horospheres are limits of hyperbolic translating catenoids, so that they constitute the analogues of the bowl soliton of $\mathbb{R}^3$ (cf. [3]). From this point of view, Theorem 3.26 relates to certain uniqueness results for the bowl soliton, such as the celebrated theorem by Wang [21], which asserts that the bowl soliton is the only convex entire translator of $\mathbb{R}^3$.

4. Rotators to MCF in $\mathbb{H}^3$

Let us consider now the one-parameter group $\mathscr{G} \subset \mathrm{Iso}(\mathbb{H}^3)$ of rotations Γ_t of $\mathbb{H}^3 = (\mathbb{R}^3_+, ds^2)$ about the z-axis. Considering the decomposition $\mathbb{R}^3_+ = \mathbb{R}^2 \times (0, +\infty)$, we have that

$$\Gamma_t = \begin{bmatrix} e^{tJ} & \\ & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

In this setting, an initial condition of a $\mathscr{G}$-soliton will be called a *rotating soliton* or simply a *rotator*. The (horizontal) Killing field associated to $\mathscr{G}$ is $\xi(p) = J\pi(p)$, $p \in \mathbb{H}^3$, where π denotes the projection over $\{(0, 0, 1)\}^\perp \subset \mathbb{R}^3$, i.e., $\pi(x, y, z) = x\partial_x + y\partial_y$. Hence, a surface Σ of hyperbolic space $\mathbb{H}^3$ is a rotator to MCF if and only if

$$H(p) = \langle J\pi(p), \eta(p)\rangle \quad \text{for all } p \in \Sigma. \tag{4.1}$$

Clearly, a minimal surface of $\mathbb{H}^3$ is a (stationary) rotator if and only if it is invariant by rotations about the z-axis. These minimal surfaces were classified in [6], and they include, of course, the one-parameter family of totally geodesic planes which intersect the z-axis orthogonally, as well as the hyperbolic catenoid obtained by Mori [17]. On the other hand, it is clear that no horosphere is a rotator in $\mathbb{H}^3$. These facts, together with the considerations of Remark 3.5, yield the following theorem.

Theorem 4.1. *The only rotators of constant mean curvature in $\mathbb{H}^3$ are the minimal surfaces of revolution with axis $\{(0, 0)\} \times (0, +\infty)$.*

We shall seek for rotators in $\mathbb{H}^3$ in the class of *helicoidal surfaces*, which are described as follows. Choose a smooth curve with trace contained in the horosphere $\mathcal{H} := \mathbb{R}^2 \times \{1\}$ of height 1:

$$s \in \mathbb{R} \mapsto (\alpha(s), 1) \in \mathcal{H},$$

where $\alpha\colon \mathbb{R} \to \mathbb{R}^2$ is a regular curve parametrized by arc length. Given a constant $h > 0$, we call a parametrized surface $\Sigma = X(\mathbb{R}^2) \subset \mathbb{H}^3$ a *helicoidal surface* generated by α with *pitch* h if the parametrization $X\colon \mathbb{R}^2 \to \mathbb{H}^3$ writes as

$$X(u,v) = e^{hv}(e^{vJ}\alpha(u), 1), \quad (u,v) \in \mathbb{R}^2. \tag{4.2}$$

Considering a parametrization of α by arc length, $\alpha(s) = (u(s), v(s), 0)$, $s \in \mathbb{R}$, and writing

$$T(s) = u'(s)\partial_x + v'(s)\,\partial_y \quad \text{and} \quad N(s) = -v'(s)\partial_x + u'(s)\,\partial_y,$$

the curvature of α is given by

$$k(s) = \langle \alpha''(s), N(s)\rangle_e = -u''(s)v'(s) + v''(s)u'(s),$$

where $\langle\,,\rangle_e$ stands for the Euclidean metric of $\mathbb{R}^2$. Furthermore, by the well-known Frenet–Serret equations, one has

$$T' = kN, \quad N' = -kT.$$

In this setting, if we define the functions

$$\tau := \langle \alpha, T\rangle_e \quad \text{and} \quad \mu := \langle \alpha, N\rangle_e, \tag{4.3}$$

we get from a direct computation that

$$\bar\eta = \varrho(e^{vJ}N, -(\tau + h\mu)/h), \quad \varrho := h(h^2 + (\tau + h\mu)^2)^{-1/2}, \tag{4.4}$$

is an Euclidean unit normal to the helicoidal surface Σ, and that its Euclidean mean curvature in this orientation is

$$\bar H = e^{-hv}\varrho\,\frac{k((h^2+1)r^2 + h^2) - (h\tau - \mu)}{2(h^2 + (\tau + h\mu)^2)},$$

where $r^2 := \tau^2 + \mu^2$. From this equality and (2.1), we have that the hyperbolic mean curvature H of Σ is

$$H = \frac{\varrho}{h}\Big(h\,\frac{k((h^2+1)r^2 + h^2) - (h\tau - \mu)}{2(h^2 + (\tau + h\mu)^2)} - (\tau + h\mu)\Big). \tag{4.5}$$

These considerations yield the following existence result, which brings Theorem 3.1 in [9] to $\mathbb{H}^3$.

Theorem 4.2. *For any smooth function $\Psi\colon \mathbb{R}^2 \to \mathbb{R}$ and any constant $h > 0$, there exists a one-parameter family of complete helicoidal surfaces of pitch h in $\mathbb{H}^3$ each of them with mean curvature function H satisfying*

$$H(X(u,v)) = \Psi(\tau(u), \mu(u)),$$

where X is the parametrization given in (4.2) *and τ and μ are as in* (4.3).

Proof. Considering equality (4.5) for the given function $H = H(\tau, \mu)$ and solving for k, we have that $k = k(\tau, \mu)$ is a smooth function of $(\tau, \mu) \in \mathbb{R}^2$. However, by Lemma 3.2

in [9], there exists a one-parameter family of plane curves $\alpha\colon \mathbb{R} \to \mathbb{R}^2$, each of them with curvature k. Therefore, for such an α, and for a given $h > 0$, the helicoidal surface of $\mathbb{H}^3$ with pitch h whose generating curve is α has mean curvature function $H = H(\tau, \mu)$, as we wished to prove. ∎

Now, we verify the conditions under which a helicoidal surface $\Sigma = X(\mathbb{R}^2)$ of $\mathbb{H}^3$ is a rotator to MCF. By (4.2)–(4.4),

$$\begin{aligned}\langle J\pi(X), \eta(X)\rangle &= \langle J(e^{hv}e^{vJ}\alpha), e^{hv}\,\bar\eta(X)\rangle = \langle Je^{vJ}\alpha, \bar\eta(X)\rangle_e\\ &= \langle e^{vJ}J\alpha, \varrho e^{vJ}N\rangle_e = \varrho\langle J\alpha, N\rangle_e\\ &= -\varrho\langle\alpha, JN\rangle_e = \varrho\langle\alpha, T\rangle_e = \varrho\tau,\end{aligned}$$

which, together with (4.1), implies the following result.

Lemma 4.3. *A helicoidal surface $\Sigma = X(\mathbb{R}^2)$ of pitch $h > 0$ parametrized as in (4.2) is a rotator to MCF in $\mathbb{H}^3$ if and only if its mean curvature function $H = H(\tau, \mu)$ satisfies*

$$H = \frac{h\tau}{\sqrt{h^2 + (h\mu + \tau)^2}}\cdot \tag{4.6}$$

In what follows, we prove the main result of this section, which provides the existence of complete rotators in $\mathbb{H}^3$ by means of helicoidal surfaces, and completely describe the topology of the corresponding generating curves (see Figure 7).

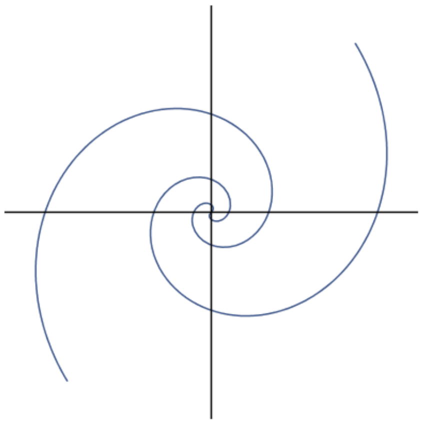

Figure 7. The generating curve of a helicoidal rotator in $\mathbb{H}^3$.

Theorem 4.4. *For any $h > 0$, there exists a one-parameter family of complete rotators to MCF in $\mathbb{H}^3$ whose elements are all helicoidal surfaces of pitch h. For each such surface, the trace of the generating curve $\alpha\colon \mathbb{R} \to \mathbb{R}^2$ consists of two unbounded properly embedded arms centred at the point of α which is closest to the origin $o \in \mathbb{R}^2$, with each arm spiraling around o.*

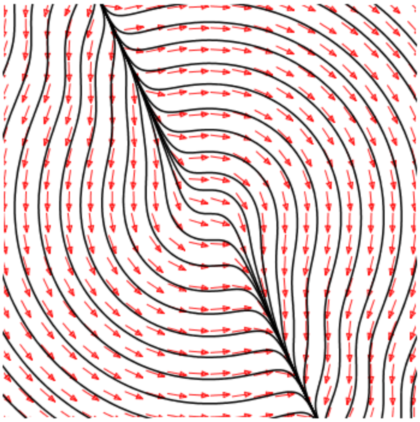

Figure 8. Phase portrait of system (4.9) for $h = 1$.

Proof. The existence part of the statement follows directly from Lemma 4.3 and Theorem 4.2. So, it remains to prove that the generating curve α of any such helicoidal surface has the asserted geometric properties.

Keeping the above notation, we first observe that, from equalities (4.5) and (4.6), the curvature k of α satisfies

$$k = \frac{2(h^2 + (\tau + h\mu)^2)((h+1)\tau + h\mu) + h(h\tau - \mu)}{h((h^2+1)r^2 + h^2)}. \tag{4.7}$$

Also, from (4.3) and the Frenet–Serret equations, one has

$$\tau' = 1 + k\mu \quad \text{and} \quad \mu' = -k\tau, \tag{4.8}$$

which, together with (4.7), yields the ODE system (see Figure 8)

$$\begin{cases} \tau' = 1 + \dfrac{2(h^2 + (\tau + h\mu)^2)((h+1)\tau\mu + h\mu^2) + h^2\tau\mu - h\mu^2}{h((h^2+1)r^2 + h^2)}, \\ \mu' = -\dfrac{2(h^2 + (\tau + h\mu)^2)((h+1)\tau^2 + h\tau\mu) + h^2\tau^2 - h\tau\mu}{h((h^2+1)r^2 + h^2)}. \end{cases} \tag{4.9}$$

Now, we establish the properties of $\alpha = \tau T + \mu N$ through the following claims.

Claim 4.5. The ODE system (4.9) has no constant solutions, and all solutions are defined on $\mathbb{R}$.

Proof of Claim 4.5. Assume, by contradiction, that $\psi(s) = (\tau_0, \mu_0)$, $s \in \mathbb{R}$, is a constant solution. Since $\tau' = \mu' = 0$, we have from (4.8) that $k_0 := k(\tau_0, \mu_0)$ satisfies $k_0\mu_0 = -1$ and $k_0\tau_0 = 0$, which yields $\tau_0 = 0$ and $\mu_0 \neq 0$. However, from the first equation in (4.9), one has

$$\tau' = 1 + \frac{2h(1+\mu_0^2)\mu_0^2 - \mu_0^2}{(h^2+1)\mu_0^2 + h^2} = \frac{h^2(\mu_0^2+1) + 2h(1+\mu_0^2)\mu_0^2}{(h^2+1)\mu_0^2 + h^2} > 0,$$

which is a contradiction. Therefore, (4.9) has no constant solutions. From this fact, and since $k = k(\tau, \mu)$ is defined on $\mathbb{R}^2$, we conclude that any solution of (4.9) is defined on $\mathbb{R}$. ∎

Claim 4.6. Suppose that any integral curve $\psi(s) := (\tau(s), \mu(s))$ of (4.9) satisfies that the limit $\lim_{s\to+\infty} \tau(s)$ (respectively, $\lim_{s\to+\infty} \mu(s)$) exists. Then $\lim_{s\to-\infty} \tau(s)$ (respectively, $\lim_{s\to-\infty} \mu(s)$) also exists. Furthermore, if there exists some $L \in [-\infty, +\infty]$ with the property that for any integral curve,

$$\lim_{s\to+\infty} \tau(s) = L \quad \big(\text{respectively, } \lim_{s\to+\infty} \mu(s) = L\big),$$

then it also holds that any integral curve satisfies

$$\lim_{s\to-\infty} \tau(s) = -L \quad \big(\text{respectively, } \lim_{s\to-\infty} \mu(s) = -L\big).$$

Proof of Claim 4.6. Let $\psi(s) := (\tau(s), \mu(s))$ be an integral curve of the system (4.9). Then it is easily checked that $\overline{\psi}(s) := -\psi(-s)$ is also an integral curve of that system. Setting $\overline{\psi} = (\bar{\tau}, \bar{\mu})$, we have that $\bar{\tau}(s) = -\tau(-s)$ and $\bar{\mu}(s) = -\mu(-s)$. By hypothesis, $\lim_{s\to+\infty} \bar{\tau}(s)$ exists and the first part of the claim follows from observing that $\lim_{s\to-\infty} \mu(s) = -\lim_{s\to+\infty} \bar{\mu}(s)$. The remainder of the proof is argued analogously and will be omitted. ∎

Claim 4.7. The function τ has precisely one zero s_0 and τ is negative in $(-\infty, s_0)$ and positive in $(s_0, +\infty)$. As a consequence, the function $r^2 = \tau^2 + \mu^2$ has a global minimum and satisfies $\lim_{s\to\pm\infty} r^2 = +\infty$.

Proof of Claim 4.7. First, observe that the equalities (4.8) yield

$$(r^2)' = 2(\tau\tau' + \mu\mu') = 2(\tau(1 + k\mu) + \mu(-k\tau)) = 2\tau,$$

which implies that the zeroes of τ are the critical points of r^2. Also, as seen in the first part of the proof of Claim 4.5, if $\tau(s_0) = 0$ for some s_0, then $\tau'(s_0) > 0$, which gives that τ has at most one zero s_0, in which case τ is negative in $(-\infty, s_0)$ and positive in $(s_0, +\infty)$.

Next, we argue by contradiction and assume that τ has no zeroes. We will also assume that $\tau > 0$ on $\mathbb{R}$, since the complementary case $\tau < 0$ can be treated analogously. Under this assumption, the function r^2 is strictly increasing. So, there exists $\delta \geq 0$ such that

$$\lim_{s\to-\infty} r^2(s) = \delta.$$

In particular, since $\tau = (r^2)'/2$, we also have that

$$\lim_{s\to-\infty} \tau(s) = 0, \tag{4.10}$$

which implies that $\mu^2 \to \delta$ as $s \to -\infty$. However, the first equality in (4.9) yields that $\lim_{s\to-\infty} \tau'(s) > 0$, which contradicts (4.10), proving that τ has exactly one zero and also that r^2 has only one critical point. Consequently, both the limits of r^2 as $s \to \pm\infty$ exist in $[0, +\infty]$.

In order to complete the proof of the claim, just note that if either $\lim_{s\to-\infty} r^2 = \delta$ or $\lim_{s\to+\infty} r^2 = \delta$ for some $\delta > 0$, the same arguments as before lead to a contradiction, thus $\lim_{s\to\pm\infty} r^2(s) = +\infty$. ∎

Claim 4.8. The limits of τ and μ as $s \to \pm\infty$ exist (possibly being infinite).

Proof of Claim 4.8. First, we show that k has at most one zero in $\mathbb{R}$. To this end, assume that $k(s_0) = 0$ for some $s_0 \in \mathbb{R}$. We have from (4.7) that, at s_0,

$$2(h^2 + (\tau + h\mu)^2)((h+1)\tau + h\mu) + h(h\tau - \mu) = 0. \tag{4.11}$$

Also, by (4.8), $\tau'(s_0) = 1$ and $\mu'(s_0) = 0$. This, together with (4.11), gives that, at s_0,

$$k' = \frac{2(h+1)(h^2 + (\tau + h\mu)^2) + 4((h+1)\tau + h\mu)(\tau + h\mu) + h^2}{h((h^2+1)r^2 + h^2)}. \tag{4.12}$$

If $\tau(s_0)\,\mu(s_0) \geq 0$, we have from (4.12) that $k'(s_0) > 0$. Assume then $\tau(s_0)\,\mu(s_0) < 0$ and notice that, by (4.11), one has

$$\operatorname{sign}((h+1)\tau(s_0) + h\mu(s_0)) = \operatorname{sign}(\mu(s_0) - h\tau(s_0)). \tag{4.13}$$

If $\tau(s_0) < 0 < \mu(s_0)$, then both signs in (4.13) are positive. In addition,

$$\tau(s_0) + h\mu(s_0) = (h+1)\tau(s_0) + h\mu(s_0) - h\tau(s_0) > 0,$$

and then (4.12) yields $k'(s_0) > 0$. Analogously, $\mu(s_0) < 0 < \tau(s_0)$ implies $k'(s_0) > 0$.

It follows from the above that k has at most one zero $s_0 \in \mathbb{R}$ and, if so, k is negative in $(-\infty, s_0)$ and positive in $(s_0, +\infty)$. Since, by Claim 4.7, τ has exactly one zero, we have that $\mu' = -k\tau$ has at most two zeroes, which implies that μ has at most two critical points. In particular, the limits $\lim_{s\to\pm\infty} \mu(s)$ exist.

To finish the proof of the claim, let us assume, by contradiction, that the limit of τ as $s \to +\infty$ does not exist. In this case, for some $\tau_0 > 0$, there exists a strictly increasing sequence $(s_n)_{n\in\mathbb{N}}$ diverging to $+\infty$ and such that (see Figure 9)

$$\tau(s_n) = \tau_0 \quad \text{and} \quad \tau'(s_n)\tau'(s_{n+1}) < 0 \quad \text{for all } n \in \mathbb{N}.$$

Claim 4.7 implies that $\lim r^2(s_n) = +\infty$. Then we must have $\lim \mu^2(s_n) = +\infty$. In this case, our previous arguments show that either $\lim \mu(s_n) = +\infty$ or $\lim \mu(s_n) = -\infty$. In any case, we have from (4.7) that

$$\lim_{n\to+\infty} (k(s_n)\,\mu(s_n)) = \lim_{n\to+\infty} \frac{2h^3\mu(s_n)^4}{h(h^2+1)\,\mu(s_n)^2} = +\infty.$$

In particular, for any sufficiently large $n \in \mathbb{N}$, $\tau'(s_n) = 1 + k(s_n)\,\mu(s_n) > 0$. This, however, contradicts the fact that $(\tau'(s_n))_{n\in\mathbb{N}}$ is an alternating sequence. So, $\lim_{s\to+\infty} \tau(s)$ exists. Since (τ, μ) is an arbitrary integral curve of (4.9), Claim 4.6 implies that $\lim_{s\to-\infty} \tau(s)$ also exists, thereby finishing the proof of the claim. ■

Claim 4.9. We have $\lim_{s\to\pm\infty} \tau(s) = \pm\infty$ and $\lim_{s\to\pm\infty} \mu(s) = \mp\infty$.

Proof of Claim 4.9. By Claim 4.8, all the limits above exist and, arguing by contradiction, we first treat the case when $\lim_{s\to+\infty} \mu(s) = L \in \mathbb{R}$. Under this assumption, we have from Claims 4.7 and 4.8 that $\lim_{s\to+\infty} \tau(s) = +\infty$. Then, it follows from the second equation in (4.9) that $\lim_{s\to+\infty} \mu'(s) = -\infty$, which contradicts the assumed fact $L \in \mathbb{R}$.

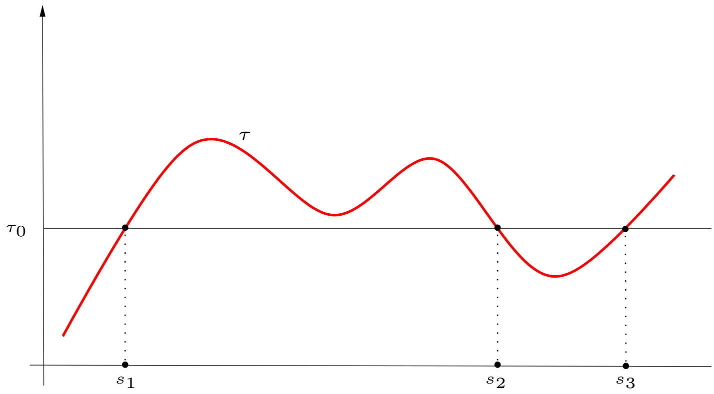

Figure 9. Graph of τ.

Suppose now that $\lim_{s\to+\infty}\mu(s)=+\infty$. From this assumption and Claim 4.7, we have that $h^2+(\tau(s)+h\mu(s))^2>1/2$ for all sufficiently large $s>0$. Applying this last inequality to (4.7) yields

$$k(s)>\frac{(h+1)\tau(s)+h\mu(s)+h(h\tau(s)-\mu(s))}{h((h^2+1)r^2+h^2)}=\frac{(h^2+h+1)\tau(s)}{h((h^2+1)r^2+h^2)}>0.$$

However, for such values of s, $\mu'(s)=-k(s)\tau(s)<0$, which is a contradiction. Therefore, $\lim_{s\to+\infty}\mu(s)=-\infty$. Since (τ,μ) is an arbitrary integral curve, Claim 4.6 applies to show that $\lim_{s\to-\infty}\mu(s)=+\infty$.

To finish the proof, assume, by contradiction, that $\lim_{s\to+\infty}\tau(s)$ is finite. Then, since $\lim_{s\to+\infty}\mu(s)=-\infty$, we have from (4.7) that $\lim_{s\to+\infty}k(s)=-\infty$. From this, we have $\lim_{s\to+\infty}\tau'(s)=\lim_{s\to+\infty}(1+k(s)\mu(s))=+\infty$, which is a contradiction. Therefore, Claim 4.7 gives that $\lim_{s\to+\infty}\tau(s)=+\infty$. Once again, $\lim_{s\to-\infty}\tau(s)=-\infty$ follows from Claim 4.6. ■

Claim 4.10. The function $\nu:=-\tau/\mu$ is bounded outside of a compact interval.

Proof of Claim 4.10. It follows from Claim 4.9 that ν is well defined and positive at all points outside of a compact interval of $\mathbb{R}$. Assume, by contradiction, that there exists a sequence $(s_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ which diverges to infinity, and such that $\lim\nu(s_n)=+\infty$, i.e., $\lim(-\mu(s_n)/\tau(s_n))=0$.

From (4.7), one has

$$k\tau=\frac{2(h^2+(\tau+h\mu)^2)(h+1+h\mu/\tau)+h^2-h\mu/\tau}{h((h^2+1)(1+\mu^2/\tau^2)+h^2/\tau^2)}.$$

Hence, after passing to a subsequence, we can assume that $k(s_n)\tau(s_n)$ is positive and bounded away from zero for all $n\in\mathbb{N}$. However,

$$+\infty=\lim\nu(s_n)=\lim\frac{\tau(s_n)}{-\mu(s_n)}=\lim\frac{\tau'(s_n)}{-\mu'(s_n)}=\lim\left(\frac{1}{k(s_n)\tau(s_n)}+\frac{\mu(s_n)}{\tau(s_n)}\right)<+\infty,$$

which is a contradiction.

Analogously, we derive a contradiction by assuming that there exists $s_n \to -\infty$ satisfying $\lim \nu(s_n) = +\infty$. This proves Claim 4.10. ∎

In what follows, we shall denote by $\omega = \omega(s)$ the angle function of α, that is,

$$\alpha = r(\cos\omega, \sin\omega).$$

It then follows from (4.8) that the equality

$$T = \frac{\tau}{r^2}\alpha + \omega' J\alpha \tag{4.14}$$

holds at all points where $r \neq 0$.

Claim 4.11. We have $\omega(s) \to +\infty$ as $s \to \pm\infty$.

Proof of Claim 4.11. Considering (4.14) and the equality $(r^2)' = 2\tau$, we have that

$$r' = \frac{\tau}{r} \quad \text{and} \quad \omega' = -\frac{\mu}{r^2}\cdot$$

So, given a differentiable function $\varphi = \varphi(r)$, $r \in (0, +\infty)$, one has

$$\frac{d\varphi}{d\omega} = \frac{d\varphi}{dr}\frac{dr}{ds}\frac{ds}{d\omega} = -r\varphi'(r)\,\frac{\tau}{\mu}\cdot \tag{4.15}$$

Now, define $\varphi(r) = \log(\log r)$. Then $\varphi(r) \to +\infty$ as $r \to +\infty$ and

$$r\varphi'(r) = \frac{1}{\log r} \to 0 \quad \text{as } r \to +\infty. \tag{4.16}$$

Since, by Claim 4.10, $-\tau/\mu$ is bounded outside of a compact interval, it follows from Claim 4.7 and (4.15)–(4.16) that $d\varphi/d\omega \to 0$ as $s \to \pm\infty$. Consequently, $d\omega/d\varphi \to +\infty$ as $s \to \pm\infty$, which proves Claim 4.11. ∎

It follows from the above that the trace of α has one point p_0 closest to the origin o (Claim 4.7), and consists of two properly embedded arms centred at p_0 (Claim 4.9) which proceed to infinity by spiraling around o (Claim 4.11). This finishes our proof. ∎

Let $\Sigma = X(\mathbb{R}^2)$ be a helicoidal surface of pitch h in $\mathbb{H}^3$ as given in (4.2). Consider the subgroup $\mathscr{G} = \{\Gamma_t \mid t \in \mathbb{R}\} \subset \mathrm{Iso}(\mathbb{H}^3)$ of (downward) hyperbolic translations of constant speed h, i.e., $\Gamma_t(p) = e^{-ht}p$, and notice that the Killing field on $\mathbb{H}^3$ determined by $\mathscr{G}$ is $\xi(p) = -hp$. Now, recall that the unit normal to Σ is $\eta = e^{hv}\bar\eta$, with $\bar\eta$ as in (4.4). From this, we have

$$\langle \xi(X), \eta\rangle = -h\varrho\Big(\mu - \frac{\tau + h\mu}{h}\Big) = \varrho\tau,$$

so that Σ is a $\mathscr{G}$-soliton if and only if its mean curvature function is given by

$$H = \varrho\tau = \frac{h\tau}{\sqrt{h^2 + (h\mu + \tau)^2}}\cdot$$

From this last equality and Lemma 4.3, we have the following result.

Proposition 4.12. *Let $\Sigma = X(\mathbb{R}^2)$ be a helicoidal surface of pitch h in $\mathbb{H}^3$. Then the following assertions are equivalent*:

(i) *Σ is a rotator to MCF.*

(ii) *Σ is a translator to MCF with respect to the Killing field $\xi(p) = -hp$.*

5. The classification of minimal translators

In this section, we prove Theorem 3.2, which is a classification of complete, properly immersed minimal surfaces of $\mathbb{H}^3$ invariant under the 1-parameter group $\{\Gamma_t\}_{t\in\mathbb{R}}$ of hyperbolic isometries of $\mathbb{H}^3$ defined (in the half-space model) by

$$(x, y, z) \in \mathbb{R}^3_+ \mapsto \Gamma_t(x, y, z) = (e^t x, e^t y, e^t z).$$

We let $\overline{\mathbb{H}}^3$ be the topological space given by the compactification of $\mathbb{H}^3$ with respect to the so-called *cone topology* (as defined in [7]) and let $S^2(\infty)$ denote the asymptotic boundary of $\mathbb{H}^3$. In the upper half-space model of $\mathbb{H}^3$, $S^2(\infty)$ is identified with the one point compactification of $\mathbb{R}^2 = \{z = 0\}$:

$$S^2(\infty) = \mathbb{R}^2 \cup \{\infty\}.$$

Along the proof, given a surface $\Sigma \subset \mathbb{H}^3$, we will write $\partial_\infty\Sigma$ for the asymptotic boundary of Σ, that is, $\partial_\infty\Sigma := \overline{\Sigma} \cap S^2(\infty)$, where $\overline{\Sigma}$ is the closure of Σ in $\overline{\mathbb{H}}^3$.

Proof of Theorem 3.2. Consider α a curve in the horosphere $\mathcal{H} := \{z = 1\}$ and assume that $\Sigma = \{e^t\alpha \mid t \in \mathbb{R}\}$ is a complete, properly immersed minimal surface invariant under the action of $\{\Gamma_t\}_{t\in\mathbb{R}}$ and generated by α. For the remainder of the proof, we will assume that α is parametrized by arc length over a maximal interval I.

Claim 5.1. For $\theta \in [0, \pi)$, let $L_\theta = \{(r\cos(\theta), r\sin(\theta), 1) \mid r \in \mathbb{R}\}$. If α intersects L_θ in two (or more) points, then $\alpha = L_\theta$.

Proof. After a rotation, it suffices to prove the claim for $\theta = 0$. Assume that there are two distinct points $p_1 = (r_1, 0, 1)$, $p_2 = (r_2, 0, 1) \in \alpha$ and, arguing by contradiction, assume that $\alpha \neq L_0$.

Consider the compact arc a that $\{p_1, p_2\}$ bounds in α. Since $\alpha \neq L_0$, there exists a point $\widehat{p}$ in the interior of a where the second coordinate function y has a local maximum or a local minimum, and we may rotate α once again to assume it is a local maximum, attained at $(\widehat{x}, \widehat{y}, 1)$ with $\widehat{y} > 0$. Let P be the tilted plane of $\mathbb{H}^3$ that contains the line $\{(r, \widehat{y}, 1) \mid r \in \mathbb{R}\}$ and whose asymptotic boundary contains $(0, 0, 0)$. Then P is an equidistant surface to the totally geodesic plane $\{y = 0\}$ and its mean curvature vector points upwards. Since Σ locally stays in the mean convex side of P and intersects P tangentially along the line $\{e^t\widehat{p} \mid t \in \mathbb{R}\}$, we obtain a contradiction with the mean curvature comparison principle. ■

A straightforward consequence of Claim 5.1 is that the curve α must be properly embedded, $I = \mathbb{R}$ and, if $(0, 0, 1) \in \alpha$, Σ is a vertical plane. Thus, assuming that Σ is not a vertical plane, we may parametrize α as

$$\alpha(s) = (r(s)\cos(\theta(s)), r(s)\sin(\theta(s)), 1), \tag{5.1}$$

where s is the arc length of α and $r(s) > 0$ for all $s \in \mathbb{R}$. Claim 5.1 implies that, after a rotation in $\mathbb{H}^3$ (and possibly reparametrizing on the opposite orientation) the function θ must satisfy $\theta'(s) \geq 0$ and

$$\lim_{s\to-\infty} \theta(s) = 0, \qquad \lim_{s\to+\infty} \theta(s) = \theta_+ > 0. \tag{5.2}$$

In fact, $\theta_+ \in (0, \pi]$. Indeed, arguing by contradiction, assume that $\theta_+ > \pi$ and choose $\theta^* \in (\pi, \theta_+)$. Then Claim 5.1 implies that α intersects $L = L_{\theta^+ - \pi}$ at most in one point, so the fact that $(0, 0, 1) \notin \alpha$ implies that either $\theta(s) \in (0, \theta^*)$ for all $s \in I$ or $\theta(s) \in (\theta^* - \pi, \theta_+)$ for all $s \in I$, both in contradiction with (5.2).

For the next claim, the union of two half-lines in $\mathbb{R}^2$ issuing from a point p and making an oriented angle $\theta \in (0, 2\pi)$ will be called a θ-*hinge* with *vertex* p.

Claim 5.2. $\partial_\infty \Sigma \cap \mathbb{R}^2$ is a θ_+-hinge with vertex at the origin $\mathbf{0} := (0, 0, 0)$.

Proof. Using the notation of (5.1), we may parametrize Σ as

$$\Sigma = \{(e^t r(s)\cos(\theta(s)),\, e^t r(s)\sin(\theta(s)),\, e^t) \mid t, s \in \mathbb{R}\}.$$

Our next argument is to show that $\ell_0 \cup \ell_{\theta_+} \cup \{\mathbf{0}\} \subset \partial_\infty \Sigma$, where, for $\theta \in [0, 2\pi)$, we let $\ell_\theta = \{(r\cos(\theta), r\sin(\theta), 0) \mid r > 0\}$.

Note that $\lim_{s\to\pm\infty} r(s) = +\infty$, as α is properly embedded and noncompact. Consider a divergent sequence $(s_n)_{n\in\mathbb{N}} \subset \mathbb{R}$, so $\lim_{n\to\infty} r(s_n) = +\infty$. Take $r > 0$ and let, for $n \in \mathbb{N}$, $t_n = \log(r/r(s_n))$ and

$$p_n = e^{t_n}\alpha(s_n) = \left(r\cos(\theta(s_n)), r\sin(\theta(s_n)), \frac{r}{r(s_n)}\right) \in \Sigma.$$

Since $r > 0$ and $r(s_n) \to +\infty$, $\lim_{n\to\infty} r/r(s_n) = 0$. Furthermore, it follows from (5.2) that $\lim_{n\to\infty} \theta(s_n) = \theta_+$, thus

$$\lim_{n\to\infty} p_n = (r\cos(\theta_+), r\sin(\theta_+), 0) \in \partial_\infty \Sigma.$$

Since r is arbitrary, this gives $\ell_{\theta_+} \subset \partial_\infty \Sigma$. In a similar manner, we can prove that $\{\mathbf{0}\} \cup \ell_0 \subset \partial_\infty \Sigma$.

Next, we prove that $\ell_0 \cup \ell_{\theta_+} \cup \{\mathbf{0}\} \supset (\partial_\infty \Sigma \cap \mathbb{R}^2)$. Choose $\overline{p} \in \partial_\infty \Sigma \cap \mathbb{R}^2$ and, assuming that $\overline{p} \neq \mathbf{0}$, write $\overline{p} = (r\cos(\theta), r\sin(\theta), 0)$ for some $r > 0$ and $\theta \in [0, 2\pi)$. Let $(p_n)_{n\in\mathbb{N}}$ be a sequence in Σ such that $p_n \to \overline{p}$, so there exist uniquely defined $s_n, t_n \in \mathbb{R}$ such that

$$p_n = e^{t_n}\alpha(s_n) = (e^{t_n} r(s_n)\cos(\theta(s_n)),\, e^{t_n} r(s_n)\sin(\theta(s_n)),\, e^{t_n}).$$

The fact that $p_n \to \overline{p}$ implies that $e^{t_n} \to 0$. Moreover, $\lim_{n\to\infty} e^{t_n} r(s_n) = r$, thus $r(s_n) \to +\infty$, and it follows that either $s_n \to +\infty$ (in which case $\theta(s_n) \to \theta_+$) or $s_n \to -\infty$ (and $\theta(s_n) \to 0$). In both situations we obtain $\overline{p} \in \ell_0 \cup \ell_{\theta_+}$, which proves the claim. ∎

At this point, we have shown that a properly immersed minimal surface $\Sigma \subset \mathbb{H}^3$ which is invariant under the group $\{\Gamma_t\}_{t\in\mathbb{R}}$ is in fact properly embedded and its asymptotic boundary $\partial_\infty \Sigma \cap \mathbb{R}^2$ is a θ_+-hinge with vertex at $\mathbf{0}$. The existence and uniqueness of such surfaces was proven in [8] (unpublished) and also presented in Proposition A.1 of [20], which finishes the proof of Theorem 3.2. ∎

The theory of translators in $\mathbb{R}^3$ is well developed, having many classification results under certain geometric constraints, see Chapter 13 of [18]. On the other hand, to our knowledge, there are no classification results for translators in $\mathbb{H}^3$ apart the ones we obtained here. In light of the most known results on construction and classification of translators in $\mathbb{R}^3$ (e.g., Corollary 13.48, Theorem 13.53, and Theorem 13.67 in [18]), a natural question arises: Can translators of $\mathbb{H}^3$ be constructed and classified under additional geometric conditions such as being a graph, or mean convex?

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