

Bi- $\bar{\mathbb{Q}}$ -structures on Hermitian symmetric spaces and quadratic relations between CM periods

Ziyang Gao, Emmanuel Ullmo, and Andrei Yafaev

Abstract. In this paper, we introduce the notion of a bi- $\bar{\mathbb{Q}}$ -structure on the tangent space at a CM point on a locally Hermitian symmetric domain. We prove that this bi- $\bar{\mathbb{Q}}$ -structure decomposes into the direct sum of 1-dimensional bi- $\bar{\mathbb{Q}}$ -subspaces, and make this decomposition explicit for the moduli space of abelian varieties \mathbb{A}_g .

We propose an *analytic subspace conjecture*, which is the analogue of the Wüstholtz's analytic subgroup theorem in this context. We show that this conjecture, applied to \mathbb{A}_g , implies that all quadratic $\bar{\mathbb{Q}}$ -relations among the holomorphic periods of CM abelian varieties arise from elementary ones.

1. Introduction

1.1. Prologue and motivation

In all this paper, $\bar{\mathbb{Q}}$ will denote the algebraic closure of \mathbb{Q} in \mathbb{C} . The uniformizing map $u: \mathbb{C} \rightarrow \mathbb{C}^*, z \mapsto e^{2\pi iz}$, is a map between algebraic varieties defined over $\bar{\mathbb{Q}}$. This map is transcendental, and the Gelfond–Schneider theorem asserts the following: if $\alpha \in \bar{\mathbb{Q}}$ and $u(\alpha) \in \bar{\mathbb{Q}}^*$, then $\alpha \in \mathbb{Q}$. On the other hand, if we consider the usual exponential $\exp: \mathbb{C} \rightarrow \mathbb{C}^*, z \mapsto e^z$, the situation is different. In this case, if $\alpha \in \bar{\mathbb{Q}}$ and $e^\alpha \in \bar{\mathbb{Q}}^*$, then $\alpha = 0$ by a theorem of Lindemann.

We can look at this from a different viewpoint, which we will further develop and adapt in this paper. Consider the natural $\bar{\mathbb{Q}}$ -structure on $\mathbb{C}^* = \mathbb{G}_m(\mathbb{C})$ given by $\mathbb{G}_{m,\mathbb{C}} = \mathbb{G}_{m,\bar{\mathbb{Q}}} \otimes \mathbb{C}$, and the usual exponential map $\exp: \mathbb{C} \rightarrow \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$. Endow \mathbb{C} with two $\bar{\mathbb{Q}}$ -structures: one given by $W_1 := 2\pi i \bar{\mathbb{Q}} \subseteq \mathbb{C}$, the other given by $W_2 := \bar{\mathbb{Q}} \subseteq \mathbb{C}$. An $\alpha \in \mathbb{C}$ is called bi- $\bar{\mathbb{Q}}$ for W_j if $\alpha \in W_j$ and $e^\alpha \in \bar{\mathbb{Q}}^*$. From the previous paragraph, $\alpha \in \mathbb{C}$ is bi- $\bar{\mathbb{Q}}$ for W_1 (resp. W_2) if and only if $\alpha \in \mathbb{Q}$ (resp. $\alpha = 0$). These two $\bar{\mathbb{Q}}$ -structures on \mathbb{C} are related by $2\pi i$ which is a transcendental number and is a period of \mathbb{G}_m . We will call (W_1, W_2) a *bi- $\bar{\mathbb{Q}}$ -structure on \mathbb{C} with period $2\pi i$* .

The situation of bi- $\bar{\mathbb{Q}}$ -structure also occurs in the case of complex CM abelian varieties A . Let $g = \dim A$. By definition if

$$A \simeq A_1^{n_1} \times \cdots \times A_r^{n_r}$$

Mathematics Subject Classification 2020: 11J81 (primary); 11G15, 14K22, 14G35 (secondary).

Keywords: bi- $\bar{\mathbb{Q}}$ -structures, Shimura varieties, CM periods, transcendence, quadratic relations, hyperbolic analytic subspace conjecture.

is the isotypic decomposition of A , then each A_i is a simple CM abelian variety. If g_i denotes the dimension of A_i , then $E_i := \text{End}(A_i) \otimes \mathbb{Q}$ is a CM field of dimension $2g_i$. In this situation $\text{End}(A) \otimes \mathbb{Q} = M_{n_1}(E_1) \times \cdots \times M_{n_r}(E_r)$ is a CM-algebra.

In Section 2.3, we will see that $T_0 A$ is naturally endowed with a bi- $\bar{\mathbb{Q}}$ -structure. Moreover, we show in Proposition 2.9 that this bi- $\bar{\mathbb{Q}}$ -structure is *split*, which means

$$T_0 A = \bigoplus_{j=1}^g \mathbb{C}(\theta_j) \quad (1.1)$$

where each $\mathbb{C}(\theta_j)$ is a 1-dimensional complex vector subspace endowed with the induced bi- $\bar{\mathbb{Q}}$ -structure. The complex number θ_j measures the difference between the two $\bar{\mathbb{Q}}$ -structures and is defined up to multiplication by some element in $\bar{\mathbb{Q}}^*$, and they are precisely the holomorphic periods of A defined as follows. For each 1-form $\omega \in H_{\text{dR}}^1(A)$, $\int_\gamma \omega$ is independent of the choice of $\gamma \in H_1(A, \mathbb{Z})$ up to $\bar{\mathbb{Q}}$ and is non-zero for some $\gamma \in H_1(A, \mathbb{Z})$ by Shimura [17, Rem. 3.4]. Moreover, if $\int_\gamma \omega \neq 0$ and $\int_{\gamma'} \omega \neq 0$, then there exists $\theta \in \bar{\mathbb{Q}}^*$ such that $\int_\gamma \omega = \theta \int_{\gamma'} \omega$. Note that Shimura [17, Rem. 3.4] has the hypothesis that $\text{End}(A) \otimes \mathbb{Q}$ has a totally real subfield of dimension $\dim(A)$, but we can apply his result to every isotypical factors of the above decomposition of A .

Convention 1.1. Let $\{\omega_1, \dots, \omega_g\}$ be an $\text{End}(A) \otimes \mathbb{Q}$ -eigenbasis of Ω_A (holomorphic 1-forms). We can choose $\gamma_j \in \Gamma$ such that $\theta_j := \int_{\gamma_j} \omega_j \neq 0$ for each $j \in \{1, \dots, g\}$. We call $\theta_1, \dots, \theta_g$ the *holomorphic periods* of A ; they are well defined up to $\bar{\mathbb{Q}}^*$.

A very general conjecture in transcendental number theory is Grothendieck's period conjecture and its generalizations by André and Konsevich–Zagier, which predicts the polynomials relations between the periods. Wüstholz's famous analytic subgroup theorem (WAST) [21] gives a complete answer of the linear relations of this conjecture for linear algebraic groups and more generally for 1-motives; it includes most known results in transcendental number theory. We refer to [1, 13, 19] for relevant discussions. For a CM abelian variety A and its holomorphic periods defined above, WAST implies: the θ_j 's are transcendental numbers, and that they are $\bar{\mathbb{Q}}$ -linearly independent if A has no square factors.¹

However, almost nothing is known beyond the linear relations (except if the question can be reduced to linear case, for example the transcendence of $\theta_j \theta_{j'}/\pi$). For example, little is known for quadratic relations among the θ_j 's.

In a subsequent paper by the first- and second-named authors [12], we will present an example of non-trivial quadratic relation which takes the following simple form:

$$\theta_j \theta_{j'} - c_{jj'kk'} \theta_k \theta_{k'} = 0 \quad \text{with } \{j, j'\} \neq \{k, k'\} \text{ and } c_{jj'kk'} \in \bar{\mathbb{Q}}^*. \quad (1.2)$$

¹By the Poincaré irreducibility theorem, A is isogenous to a product of simple CM abelian varieties. We say that A has no square factors if these simple CM abelian varieties are all distinct.

In this example, A is simple and has dimension 2^5 . Based on this observation, we make the following definition.

Definition 1.2. A quadratic relation among $\theta_1, \dots, \theta_g$ is called *elementary* if it is of the form (1.2).

Now, a natural question is the following.

Question 1.3. *Is it true that any non-trivial $\bar{\mathbb{Q}}$ -quadratic relation among the θ_j 's, or equivalently any $\bar{\mathbb{Q}}$ -linear relation among the $\theta_j \theta_{j'}$'s, is a $\bar{\mathbb{Q}}$ -linear combination of elementary ones?*

Moreover, if the Mumford–Tate group of A is a maximal torus, is it true that

$$\dim_{\bar{\mathbb{Q}}} \sum_{1 \leq j \leq j' \leq g} \bar{\mathbb{Q}} \theta_j \theta_{j'} = \frac{g(g+1)}{2} ? \quad (1.3)$$

The aim of this article is to propose a possible framework to study this question. We will introduce a natural bi- $\bar{\mathbb{Q}}$ -structure on the tangent space at a CM point on a Shimura variety, prove that it decomposes into the direct sum of 1-dimensional bi- $\bar{\mathbb{Q}}$ -subspaces $\mathbb{C}(\alpha_j)$ (and hence yields complex numbers α_j which measure the differences between the two $\bar{\mathbb{Q}}$ -structures on each $\mathbb{C}(\alpha_j)$), and compute the α_j 's in the case of the moduli space of abelian varieties \mathbb{A}_g . We then propose an *analytic subspace conjecture*, which is the analogue of WAST in this context, and show that this conjecture when applied to \mathbb{A}_g gives an affirmative answer to Question 1.3.

1.2. Bi- $\bar{\mathbb{Q}}$ -structure associated with Shimura varieties

For Shimura varieties, we use Deligne's notations [9, 10]. A quick summary which suffices for our use can be found in Section 4.1.

Let (\mathbf{G}, X) be a connected Shimura datum. Here \mathbf{G} is a reductive group defined over \mathbb{Q} and X is a Hermitian symmetric domain. Let Γ be an arithmetic subgroup of $\mathbf{G}(\mathbb{Q})_+$ which acts on X . The Baily–Borel theorem [4] asserts that $S := \Gamma \backslash X$ has a unique structure of quasi-projective complex algebraic variety. Write

$$u: X \longrightarrow S$$

for the uniformizing map. Moreover, the general theory of Shimura varieties asserts that S admits a unique model over $\bar{\mathbb{Q}}$ and we write $S_{\bar{\mathbb{Q}}}$ for this model.

Let $[o] \in S(\bar{\mathbb{Q}})$ and $o \in X$ such that $u(o) = [o]$. The holomorphic tangent space $T_o X$ is a finite-dimensional complex vector space. Assume furthermore that o is special point (or CM point). We endow $T_o X$ with two $\bar{\mathbb{Q}}$ -structures, the *arithmetic $\bar{\mathbb{Q}}$ -structure* and the *geometric $\bar{\mathbb{Q}}$ -structure* as follows.

1.2.1. The arithmetic $\bar{\mathbb{Q}}$ -structure on $T_o X$. The algebraic tangent space $T_{[o]} S_{\bar{\mathbb{Q}}}$ is a finite-dimensional $\bar{\mathbb{Q}}$ -vector space and we have a \mathbb{C} -isomorphism

$$du: T_o X \longrightarrow T_{[o]} S_{\bar{\mathbb{Q}}} \otimes \mathbb{C}.$$

Then $(du)^{-1}(T_{[o]}S_{\bar{\mathbb{Q}}})$ defines a $\bar{\mathbb{Q}}$ -structure on $T_o X$. We call it the *arithmetic $\bar{\mathbb{Q}}$ -structure*; see Section 4.2 for more details.

1.2.2. The geometric $\bar{\mathbb{Q}}$ -structure on $T_o X$. The complex dual X^\vee of X is a projective algebraic variety. If a faithful rational representation ρ of \mathbf{G} on a finite-dimensional \mathbb{Q} -vector space V , then X^\vee is naturally a subvariety of a flag variety defined over $\bar{\mathbb{Q}}$. Write $X_{\bar{\mathbb{Q}}}^\vee$ for this $\bar{\mathbb{Q}}$ -structure on X^\vee . The natural inclusion $\iota: X \rightarrow X^\vee$ is a holomorphic map and X is an open subset of X^\vee in the usual topology. Since o is a special point, $\iota(o)$ is a $\bar{\mathbb{Q}}$ -point of X^\vee . So $T_{\iota(o)}X^\vee$ is a $\bar{\mathbb{Q}}$ -vector space of dimension $\dim X$. We have an isomorphism

$$d\iota: T_o X \longrightarrow T_{\iota(o)}X_{\bar{\mathbb{Q}}}^\vee \otimes \mathbb{C}$$

and $(d\iota)^{-1}(T_{\iota(o)}X_{\bar{\mathbb{Q}}}^\vee)$ defines a $\bar{\mathbb{Q}}$ -structure on $T_o X$. We call it the *geometric $\bar{\mathbb{Q}}$ -structure*; see Section 4.3 for more details.

The first result of this work is the following statement.

Theorem 1.4 (Theorem 5.1). *The bi- $\bar{\mathbb{Q}}$ -structure on $T_o X$ defined above is split, i.e., we have a decomposition*

$$T_o X = \bigoplus_{i=1}^{\dim X} \mathbb{C}(\alpha_j),$$

where each $\mathbb{C}(\alpha_j)$ is a 1-dimensional complex vector space endowed with the restriction of the bi- $\bar{\mathbb{Q}}$ -structures on $T_o X$ and $\alpha_j \in \mathbb{C}/\bar{\mathbb{Q}}^*$ compares the two $\bar{\mathbb{Q}}$ -structures on $\mathbb{C}(\alpha_j)$.

We can be much more precise when $S = \mathbb{A}_g = \mathrm{Sp}(2g, \mathbb{Z}) \backslash \mathfrak{H}_g$ is the moduli space of principally polarized abelian varieties of dimension g .

Theorem 1.5 (Theorem 7.1). *Let $[o]$ be a CM point of $\mathbb{A}_g(\bar{\mathbb{Q}})$ corresponding to a CM abelian variety A with holomorphic periods $\theta_1, \dots, \theta_g$. Let $o \in \mathfrak{H}_g$ mapping to $[o]$. Then*

$$T_o \mathfrak{H}_g = \bigoplus_{1 \leq j \leq j' \leq g} \mathbb{C} \left(\frac{\theta_j \theta_{j'}}{\pi} \right). \quad (1.4)$$

Moreover, we prove that the decompositions (1.4) and (1.1) are compatible with the Kodaira–Spencer map; see Theorem 7.1'. Better, we show that each $\mathbb{C} \left(\frac{\theta_j \theta_{j'}}{\pi} \right)$ arises naturally as a *root* space; see Theorem 7.4 and Corollary 7.5 for the precise statements.

The following conjecture of Lang was proved by Cohen–Wolfart [8]. As an application of this framework of bi- $\bar{\mathbb{Q}}$ -structures and of Theorem 1.5, we give a new and simpler proof.

Proposition 1.6 (Proposition 8.1). *Let C be a connected Shimura variety of dimension 1 of genus > 1 . Suppose that the universal holomorphic covering map*

$$\varphi: E_\rho := \{z \in \mathbb{C} : |z| < \rho\} \longrightarrow C^{\mathrm{an}}$$

is normalized in such a way that $\varphi(0) \in C(\bar{\mathbb{Q}})$ is a CM point and that $\varphi'(0) \in \bar{\mathbb{Q}}$. Then ρ is a transcendental number.

1.3. The hyperbolic analytic subspace conjecture

Another main objective of this article is to formulate a *hyperbolic analytic subspace conjecture*, which is the analogue of WAST in this context, and to derive some of its immediate consequences. We start by recalling Wüstholz's famous analytic subgroup theorem.

Theorem 1.7 (WAST, [21]). *Let G be a commutative algebraic group over $\bar{\mathbb{Q}}$. Let $\mathfrak{g} = \text{Lie } G = T_0 G$ viewed as a $\bar{\mathbb{Q}}$ -vector space, and consider the exponential map*

$$\exp_G: \mathfrak{g}_{\mathbb{C}} \longrightarrow G(\mathbb{C}).$$

Let $z \in \mathfrak{g}_{\mathbb{C}}$ be such that $\exp_G(z) \in G(\bar{\mathbb{Q}})$. Let V be a sub-vector space of the $\bar{\mathbb{Q}}$ -vector subspace of \mathfrak{g} with $z \in V \otimes \mathbb{C}$. Then there exists a connected algebraic subgroup H of G defined over $\bar{\mathbb{Q}}$ which contains $\exp_G(z)$ such that $V \supseteq T_0 H$.

In the context of Shimura varieties, the tangent space $T_0 G$ corresponds to the arithmetic $\bar{\mathbb{Q}}$ -structure on $T_o X$ defined in Section 1.2.1 and the natural $\bar{\mathbb{Q}}$ -structure on the uniformizing space gives rise to the geometric $\bar{\mathbb{Q}}$ -structure on $T_o X$ defined in Section 1.2.2. Hence the analogue of WAST in this context, which we propose, is the following conjecture.

Conjecture 1.8 (HASC; Conjecture 10.1). *Let $S = \Gamma \backslash X$ be a connected Shimura variety. Let o be a special point of X . Let $\mathcal{D} \subseteq T_o X$ be the Harish-Chandra realization of X as a bounded symmetric domain centered at o . Let $u: \mathcal{D} \rightarrow S$ be the uniformizing map. Let $z \in \mathcal{D}$ be such that $[z] := u(z) \in S(\bar{\mathbb{Q}})$. Let V be an arithmetic $\bar{\mathbb{Q}}$ -subspace of $T_o X$. Then $V \supseteq V'$ for some bi- $\bar{\mathbb{Q}}$ -subspace $V' \subseteq T_o X$ which contains z .*

If moreover $\text{MT}(o)$ is a maximal torus, we can take $V' = T_{[o]} S'_{\bar{\mathbb{Q}}}$ for some Shimura subvariety S' of S which contains $[z]$ and $[o]$.

An equivalent but more precise statement of this conjecture is presented as Conjecture 10.1. Observe that if $u(z) \neq u(o)$, then in the conclusion of the conjecture we have $\dim V' > 0$ and $\dim S' > 0$. Note that the statement of the conjecture depends on the choice of a special point of S (up to Hecke operations).

It is known that WAST implies that there exists no non-trivial $\bar{\mathbb{Q}}$ -linear relations between the holomorphic periods $\theta_1, \dots, \theta_g$ of a CM abelian variety A of dimension g if A has no square factors. In the same way we show that our hyperbolic counterpart HASC implies the desired result for quadratic relations. More precisely, we prove the following.

Proposition 1.9 (Proposition 10.2). *Assume that Conjecture 1.8 holds true for $S = \mathbb{A}_g$ and the point $[o] \in \mathbb{A}_g(\bar{\mathbb{Q}})$ parametrizing A . Then Question 1.3 has an affirmative answer for A .*

We think that a proof of the Conjecture 1.8 could be quite a challenge as the classical version of WASP for commutative algebraic groups is a major achievement. Our hope is that it could be still more tractable than Grothendieck's period conjecture.

1.4. Organization of the article

In Section 2, we define split bi- $\bar{\mathbb{Q}}$ -structures and discuss the cases of algebraic tori Section 2.2 and CM abelian varieties Section 2.3. We also reformulate WAST in these two cases in this language. In Section 3, we explain why WAST applied to A^\natural , the universal vector extension of A , yields the transcendence of $\theta_j \theta_j^\vee / \pi$ with the θ_j 's the holomorphic periods of a CM abelian variety A . We also restate Question 1.3 as a motivating question for the current article.

After this prologue, we start to build up our framework and relate it to Question 1.3 in Sections 4–10.

The bi- $\bar{\mathbb{Q}}$ -structure on the tangent space of a Shimura variety at a special point is defined in Section 4, and is proved to be split in Section 5. In Section 6, we discuss on the family version of the de Rham–Betti comparison and the Kodaira–Spencer map, which will be used to compute the periods of the split bi- $\bar{\mathbb{Q}}$ -structure for \mathbb{A}_g in Section 7. In Section 8, we prove the conjecture of Lang (Proposition 1.6) and compute the periods for Hilbert modular varieties.

Then after some preliminary discussion on CM points in Section 9, we propose in Section 10 our hyperbolic analytic subspace conjecture (Conjecture 1.8), and explains how this conjecture applied to \mathbb{A}_g gives a positive answer to Question 1.3.

The last section Section 11 explains some consequences of Grothendieck's period conjecture on the algebraic relations between π and holomorphic periods of CM abelian varieties. This section is rather independent of the rest of the article.

2. Bi- $\bar{\mathbb{Q}}$ -spaces and preliminary examples

The goal of this section is to define the notion of a bi- $\bar{\mathbb{Q}}$ -structure on a complex vector space and to discuss examples given by algebraic tori and CM abelian varieties.

2.1. Bi- $\bar{\mathbb{Q}}$ -structures on a complex vector space

Let V be a complex vector space of finite dimension n . A $\bar{\mathbb{Q}}$ -structure on V is a $\bar{\mathbb{Q}}$ -vector space W generated by a basis $\mathcal{B} = \{e_1, \dots, e_n\}$ of V . If W is $\bar{\mathbb{Q}}$ -structure on V , then $W \otimes_{\bar{\mathbb{Q}}} \mathbb{C} \simeq V$. A complex vector subspace V' of V is said to be *rational for the $\bar{\mathbb{Q}}$ -structure W* if V' admits a basis consisting of elements in W . In general if V' is a vector subspace of V then $V' \cap W$ is a $\bar{\mathbb{Q}}$ -vector subspace of W and

$$\dim_{\bar{\mathbb{Q}}}(V' \cap W) \leq \dim_{\mathbb{C}}(V')$$

with equality if and only if V' is $\bar{\mathbb{Q}}$ -rational for the $\bar{\mathbb{Q}}$ -structure W .

Definition 2.1. Let V be a complex vector space. A *bi- $\bar{\mathbb{Q}}$ -structure* (W_1, W_2) on V is the data given by two $\bar{\mathbb{Q}}$ -structures W_1 and W_2 on V . A complex vector subspace V' of V is said to be bi- $\bar{\mathbb{Q}}$ -rational (or simply bi- $\bar{\mathbb{Q}}$) if V' is rational for both W_1 and W_2 .

If V' is a bi- $\bar{\mathbb{Q}}$ -subspace of dimension 1, then V' determines a complex number $\theta = \theta(V', W_1, W_2)$, well defined up to multiplication by an element of $\bar{\mathbb{Q}}^*$, in the following way. Let $e_1 \in W_1$ and $e_2 \in W_2$ be some bases of V' defining the corresponding $\bar{\mathbb{Q}}$ -structures. Then $e_2 = \theta e_1$ for some $\theta \in \mathbb{C}$. Such a θ will be called a *period* of V' for (W_1, W_2) .

This terminology of *periods* is motivated by the examples of the exponential function and of the map uniformizing CM abelian varieties. We will explain this in later sections.

To ease notation, we use the following convention.

Notation 2.2. Let z_1 and z_2 be complex numbers. We will write $z_1 \simeq z_2$ if there exists $\alpha \in \bar{\mathbb{Q}}^*$ such that $z_2 = \alpha z_1$.

Definition 2.3. A bi- $\bar{\mathbb{Q}}$ -structure (W_1, W_2) on V is said to be *split* if V is a direct sum of bi- $\bar{\mathbb{Q}}$ -subspaces of dimension 1. In this situation

$$V = \bigoplus_j \mathbb{C} e_j = \bigoplus_j \mathbb{C} f_j$$

for a $\bar{\mathbb{Q}}$ -basis $\{e_1, \dots, e_n\}$ of W_1 and a $\bar{\mathbb{Q}}$ -basis $\{f_1, \dots, f_n\}$ of W_2 such that for each j , there exists $\theta_j \in \mathbb{C}^*$ with $f_j = \theta_j e_j$. We say that $\{\theta_1, \dots, \theta_n\}$ is a *complete set of periods* of V for (W_1, W_2) .

Let (W_1, W_2) be a split bi- $\bar{\mathbb{Q}}$ -structure on V and let $\theta_1, \dots, \theta_n$ be the associated periods. These periods can be regrouped in the following way: Set (J_1, \dots, J_r) to be the partition of $\{1, \dots, n\}$ such that $\theta_j \simeq \theta_{j'}$ if and only if $j, j' \in J_s$ for some $s \in \{1, \dots, r\}$. For each J_s , denote by θ_{J_s} the period θ_j for any $j \in J_s$; it is well defined up to multiplication by a number in $\bar{\mathbb{Q}}^*$.

Definition 2.4. Let $s \in \{1, \dots, r\}$. The sub-vector space

$$V_s := \bigoplus_{j \in J_s} \mathbb{C} e_j = \bigoplus_{j \in J_s} \mathbb{C} f_j$$

of V is called the isotypic subspace of V associated to the period θ_{J_s} .

The following simple proposition fully describes the bi- $\bar{\mathbb{Q}}$ -subspaces of V .

Proposition 2.5. *The following holds true:*

- (i) *There is a decomposition $V = \bigoplus_s V_s$ into a direct sum of isotypic subspaces.*
- (ii) *Let V' be a subspace of some isotypic subspace V_s . Then V' is rational for W_1 if and only if it is rational for W_2 . Thus V' is bi- $\bar{\mathbb{Q}}$ if and only if it is rational for one of two $\bar{\mathbb{Q}}$ -structures.*
- (iii) *Any bi- $\bar{\mathbb{Q}}$ -sub-space V' of V has a decomposition in isotypic components $V' = \bigoplus V'_s$, where $V'_s := V_s \cap V'$ is a bi- $\bar{\mathbb{Q}}$ -subspace of V_s .*

An equivalent way of stating the proposition is to consider the category $\mathcal{C}_{\bar{\mathbb{Q}}, \bar{\mathbb{Q}}}^{\text{split}}$ of finite-dimensional complex vector spaces endowed with a split bi- $\bar{\mathbb{Q}}$ -structure. The morphism

in $\mathcal{C}_{\bar{\mathbb{Q}}, \bar{\mathbb{Q}}}^{\text{split}}$ are the linear maps respecting the two $\bar{\mathbb{Q}}$ -structures. Then $\mathcal{C}_{\bar{\mathbb{Q}}, \bar{\mathbb{Q}}}^{\text{split}}$ is a semi-simple category and the simple objects are the one-dimensional vector spaces $(\mathbb{C}; \theta)$ with period $\theta \in \mathbb{C}^*/\bar{\mathbb{Q}}^*$. For any n -tuples of non-zero complex numbers $(\theta_1, \dots, \theta_n)$ (modulo $\bar{\mathbb{Q}}^*$), denote by $(\mathbb{C}^n; \theta_1, \dots, \theta_n)$ the object in $\mathcal{C}_{\bar{\mathbb{Q}}, \bar{\mathbb{Q}}}^{\text{split}}$ of dimension n with periods $(\theta_1, \dots, \theta_n)$. The following proposition summarizes the two extreme cases.

Proposition 2.6. *Consider an object $(\mathbb{C}^n; \theta_1, \dots, \theta_n)$ in $\mathcal{C}_{\bar{\mathbb{Q}}, \bar{\mathbb{Q}}}^{\text{split}}$.*

- (i) *If $\theta_1 \simeq \theta_2 \simeq \dots \simeq \theta_n$, then any complex vector subspace V' of \mathbb{C}^n which is $\bar{\mathbb{Q}}$ -rational for one of the two $\bar{\mathbb{Q}}$ -structures is bi- $\bar{\mathbb{Q}}$.*
- (ii) *If $\theta_j \theta_{j'}^{-1} \notin \bar{\mathbb{Q}}^*$ for all $j \neq j'$, then there are only finitely many bi- $\bar{\mathbb{Q}}$ -subspaces of $(\mathbb{C}^n; \theta_1, \dots, \theta_n)$.*

In case (i), \mathbb{C}^n is isotypic. In case (ii), the isotypic subspaces are all of dimension 1, which we denote by V_1, \dots, V_n . Then the bi- $\bar{\mathbb{Q}}$ -subspaces are precisely the $\bigoplus_{j \in J} V_j$'s, where J runs over all subsets of $\{1, \dots, n\}$. The proofs of Propositions 2.5 and 2.6 are simple linear algebra exercises and are left to the reader.

We close this subsection by defining the *Tate twist* in $\mathcal{C}_{\bar{\mathbb{Q}}, \bar{\mathbb{Q}}}^{\text{split}}$.

Definition 2.7. The *Tate twist* in $\mathcal{C}_{\bar{\mathbb{Q}}, \bar{\mathbb{Q}}}^{\text{split}}$, denoted by $\mathfrak{L}(1)$, is defined to be $(\mathbb{C}; 2\pi i)$.

As we will see in the next subsection, the Tate twist arises naturally from the bi- $\bar{\mathbb{Q}}$ -structure associated with algebraic tori.

2.2. Example: Algebraic tori

We look at two concrete examples of bi- $\bar{\mathbb{Q}}$ -structures on a complex space in the current and next subsections. The current one is in line with Proposition 2.6(i). The next one is more general, and in some cases in line with Proposition 2.6(ii).

Consider the algebraic torus $\mathbb{G}_{\text{m}, \bar{\mathbb{Q}}}^n$ defined over $\bar{\mathbb{Q}}$ and the uniformization in the category of complex spaces

$$u: \mathbb{C}^n \longrightarrow \mathbb{G}_{\text{m}}^n(\mathbb{C}) = (\mathbb{C}^*)^n, \quad (z_1, \dots, z_n) \longmapsto (e^{2\pi i z_1}, \dots, e^{2\pi i z_n}).$$

We can endow \mathbb{C}^n with two $\bar{\mathbb{Q}}$ -structures. The *geometric $\bar{\mathbb{Q}}$ -structure* on \mathbb{C}^n is the one induced by the natural inclusion $\bar{\mathbb{Q}} \subseteq \mathbb{C}$ (i.e., $\mathbb{C}^n = \bar{\mathbb{Q}}^n \otimes \mathbb{C}$), and the *arithmetic $\bar{\mathbb{Q}}$ -structure* on \mathbb{C}^n is the one induced by $\mathbb{C}^n \simeq \text{Lie } \mathbb{G}_{\text{m}, \bar{\mathbb{Q}}}^n \otimes \mathbb{C}$. A natural $\bar{\mathbb{Q}}$ -basis for the geometric structure is the canonical basis (e_1, \dots, e_n) of \mathbb{C}^n . Then $(2i\pi e_1, \dots, 2i\pi e_n)$ is a $\bar{\mathbb{Q}}$ -basis of the arithmetic structure. Thus they define a bi- $\bar{\mathbb{Q}}$ -structure on \mathbb{C}^n . This bi- $\bar{\mathbb{Q}}$ -structure is split and all the periods are $2\pi i$. This easily follows from the fact that the derivative of u defines the isomorphism

$$\psi := du = \text{diag}(2\pi i, \dots, 2\pi i): \mathbb{C}^n = \bar{\mathbb{Q}}^n \otimes \mathbb{C} \longrightarrow \text{Lie } \mathbb{G}_{\text{m}, \bar{\mathbb{Q}}}^n \otimes \mathbb{C} \simeq \mathbb{C}^n.$$

Hence this bi- $\bar{\mathbb{Q}}$ -structure on \mathbb{C}^n associated with $\mathbb{G}_{\text{m}, \bar{\mathbb{Q}}}^n$ is the n -copy of the Tate twist $\mathfrak{L}(1)^{\oplus n}$.

As a consequence, a subspace V' of \mathbb{C}^n is rational for the geometric $\bar{\mathbb{Q}}$ -structure if and only if it is rational for the arithmetic $\bar{\mathbb{Q}}$ -structure.

On the other hand, the arithmetic and the geometric $\bar{\mathbb{Q}}$ -structures on \mathbb{C} are different. Indeed, let $x \in \mathbb{G}_m^n(\bar{\mathbb{Q}})$, and fix some $\tilde{x} \in u^{-1}(x)$. By Gelfond–Schneider, \tilde{x} is a $\bar{\mathbb{Q}}$ -point in \mathbb{C}^n with respect to the geometric $\bar{\mathbb{Q}}$ -structure if and only if x is a torsion point in $\mathbb{G}_m^n(\bar{\mathbb{Q}})$. On the other hand by Lindemann, \tilde{x} is a $\bar{\mathbb{Q}}$ -point in \mathbb{C}^n with respect to the arithmetic $\bar{\mathbb{Q}}$ -structure if and only if $\tilde{x} = 0$.

We close this subsection by a reformulation of WAST, which in this case has its roots in Baker's theorem on $\bar{\mathbb{Q}}$ -linear independence of \mathbb{Q} -linearly independent logarithms of algebraic numbers. The theorem concerns the base change of the exponential map

$$\exp: \text{Lie } \mathbb{G}_{m,\bar{\mathbb{Q}}}^n \longrightarrow \mathbb{G}_{m,\bar{\mathbb{Q}}}^n$$

to \mathbb{C} , which is precisely $u \circ \psi^{-1}$.

Theorem 2.8 (Wüstholz). *Let $z \in \mathbb{C}^n$ such that $u(z) \in \mathbb{G}_m^n(\bar{\mathbb{Q}})$. Let V be a $\bar{\mathbb{Q}}$ -sub-vector space of $\text{Lie } \mathbb{G}_{m,\bar{\mathbb{Q}}}^n$ such that $\psi(z) \in V \otimes \mathbb{C}$. Then $V \supseteq \text{Lie } H$ for some algebraic subgroup H of $\mathbb{G}_{m,\bar{\mathbb{Q}}}^n$ which contains $u(z)$.*

2.3. Example: CM abelian varieties

In this subsection, we take a first look at CM abelian varieties defined over $\bar{\mathbb{Q}}$ and explain how its uniformizing space can be endowed with a bi- $\bar{\mathbb{Q}}$ -structure which, under some mild assumptions, fits into Proposition 2.6(ii). More detailed discussions on CM abelian varieties will be given in later sections.

Let A be a complex CM abelian variety of dimension g . Then A is the extension to \mathbb{C} of an abelian variety $A_{\bar{\mathbb{Q}}}$ defined over $\bar{\mathbb{Q}}$. The universal covering of A^{an} is \mathbb{C}^g . We endow $\mathbb{C}^g = \text{Lie}(A_{\bar{\mathbb{Q}}}) \otimes \mathbb{C}$ with a bi- $\bar{\mathbb{Q}}$ -structure:

- The *arithmetic $\bar{\mathbb{Q}}$ -structure* on \mathbb{C}^g is defined to be the one given by

$$\mathbb{C}^g \simeq \text{Lie } A_{\bar{\mathbb{Q}}} \otimes \mathbb{C}.$$

- The *geometric $\bar{\mathbb{Q}}$ -structure* on \mathbb{C}^g is slightly more complicated. It is known that there exists a matrix $\tau \in \text{Mat}_{g \times g}(\bar{\mathbb{Q}})$ such that $A(\mathbb{C}) \simeq \mathbb{C}^g / (\mathbb{Z}^g + \tau \mathbb{Z}^g)$. Hence,

$$\bar{\mathbb{Q}} \cdot (\mathbb{Z}^g + \tau \mathbb{Z}^g) \subseteq \mathbb{C}^g$$

gives a $\bar{\mathbb{Q}}$ -structure on \mathbb{C}^g , and we call it the *geometric $\bar{\mathbb{Q}}$ -structure*.

Let $\theta_1, \dots, \theta_g$ be the holomorphic periods of A as defined in Convention 1.1.

Proposition 2.9. *The bi- $\bar{\mathbb{Q}}$ -structure on $\mathbb{C}^g \simeq \text{Lie}(A_{\bar{\mathbb{Q}}}) \otimes \mathbb{C}$ defined above is split, and is isomorphic to $(\mathbb{C}^g; \theta_1, \dots, \theta_g)$.*

Proof. We start with the case where A is simple. Then $\text{End}^0(A) := \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a CM field E of degree $2g$. Let Φ be the CM type of A .

For the arithmetic $\bar{\mathbb{Q}}$ -structure, E acts on $\text{Lie } A_{\mathbb{C}}$ through the embeddings $\sigma: E \rightarrow \mathbb{C}$ belonging to Φ . For each such σ , its eigenspace $(\text{Lie } A_{\mathbb{C}})_{\sigma}$ has dimension 1. Moreover, as σ factors through $\bar{\mathbb{Q}} \subseteq \mathbb{C}$, its eigenspace $(\text{Lie } A_{\mathbb{C}})_{\sigma}$ descends to $\bar{\mathbb{Q}}$, i.e., there exists $e_{\sigma} \in \text{Lie } A_{\bar{\mathbb{Q}}}$ such that the eigenspace for σ is $\mathbb{C}e_{\sigma}$. Now the action of E on $\text{Lie } A_{\mathbb{C}}$ induces

$$\mathbb{C}^g = \text{Lie } A_{\mathbb{C}} = \bigoplus_{\sigma \in \Phi} \mathbb{C}e_{\sigma}.$$

For the geometric $\bar{\mathbb{Q}}$ -structure, denote for simplicity by $\Lambda := \mathbb{Z}^g + \tau\mathbb{Z}^g$. Then the action of $\text{End}(A)$ on $A(\mathbb{C}) \simeq \mathbb{C}^g / \Lambda$ induces an action of $\text{End}(A)$ on $\Lambda \subseteq \mathbb{C}^g$. Thus tensoring \mathbb{Q} , we obtain an action of E on $\mathbb{Q}\Lambda \subseteq \mathbb{C}^g$ and hence on $\bar{\mathbb{Q}}\Lambda = \bar{\mathbb{Q}}^g \subseteq \mathbb{C}^g$. If we identify \mathbb{C}^g with $\prod_{\sigma \in \Phi} \mathbb{C}$, then Λ equals $\Phi(R_E)$ for some order R_E in \mathcal{O}_E under

$$\Phi: E \otimes_{\mathbb{Q}} \mathbb{R} \simeq \prod_{\sigma \in \Phi} \mathbb{C}.$$

In particular, the action of E on \mathbb{C}^g is via the embeddings $\sigma: E \rightarrow \bar{\mathbb{Q}}$ belonging to Φ . Hence in the previous paragraph, we have $\mathbb{C}^g = \bigoplus_{\sigma \in \Phi} \mathbb{C}f_{\sigma}$ with $f_{\sigma} \in \bar{\mathbb{Q}}\Lambda$ an eigenvector for σ .

Now that we have two actions of E on \mathbb{C}^g via the embeddings $\sigma: E \rightarrow \bar{\mathbb{Q}}$, the eigenspaces for σ coincide. Hence $\mathbb{C}e_{\sigma} = \mathbb{C}f_{\sigma}$. Hence $f_{\sigma} = \theta_{\sigma}e_{\sigma}$ for some $\theta_{\sigma} \in \mathbb{C}^*$. So in the case where A is simple, the bi- $\bar{\mathbb{Q}}$ -structure on \mathbb{C}^g associated with A is split.

For a general CM abelian variety A of dimension g defined over $\bar{\mathbb{Q}}$, A is isogenous to the product of simple CM abelian varieties. We use Φ to denote the (disjoint) union of all the CM types of these simple CM abelian varieties. Then we can also find $\{e_{\sigma}\}_{\sigma \in \Phi}$ a $\bar{\mathbb{Q}}$ -basis of the arithmetic $\bar{\mathbb{Q}}$ -structure on \mathbb{C}^g and $\{f_{\sigma}\}_{\sigma \in \Phi}$ a $\bar{\mathbb{Q}}$ -basis of the geometric $\bar{\mathbb{Q}}$ -structure on \mathbb{C}^g . Then $f_{\sigma} = \theta_{\sigma}e_{\sigma}$ for some $\theta_{\sigma} \in \mathbb{C}^*$. Notice that θ_{σ} is uniquely determined by σ up to multiplication by a element in $\bar{\mathbb{Q}}^*$. Hence the bi- $\bar{\mathbb{Q}}$ -structure on \mathbb{C}^g associated with A is split.

It remains to show that the θ_{σ} 's are precisely the holomorphic periods of A .

It is known that Ω_A (the $\bar{\mathbb{Q}}$ -vector space consisting of invariant holomorphic 1-forms) is the dual of $\text{Lie } A_{\bar{\mathbb{Q}}}$. Set $\omega_{\sigma} := e_{\sigma}^{\vee}$ for each $\sigma \in \Phi$. Then $\{\omega_{\sigma}\}_{\sigma \in \Phi}$ is an $\text{End}^0(A)$ -eigenbasis of Ω_A . Now each $f_{\sigma} \in \bar{\mathbb{Q}}\Lambda$ can be written as $f_{\sigma} = \sum_{j=1}^{2g} a_{\sigma,j} \gamma_j$ for some $a_{\sigma,j} \in \bar{\mathbb{Q}}$ and $\gamma_j \in \Lambda$. Thus

$$1 = e_{\sigma}(\omega_{\sigma}) = \theta_{\sigma}^{-1} \cdot f_{\sigma}(\omega_{\sigma}) = \theta_{\sigma}^{-1} \cdot \sum_{j=1}^{2g} a_{\sigma,j} \int_{\gamma_j} \omega_{\sigma} \simeq \theta_{\sigma}^{-1} \cdot \sum_{j=1}^{2g} \int_{\gamma_j} \omega_{\sigma}.$$

But up to $\bar{\mathbb{Q}}^*$, $\int_{\gamma_j} \omega_{\sigma}$ is independent of the choice of γ by Shimura [17, Rem. 3.4] (see the discussion before Convention 1.1). So $\theta_{\sigma} \simeq \int_{\gamma_j} \omega_{\sigma}$. We are done. \blacksquare

The following result is a consequence of Wüstholz's analytic subgroup theorem applied to $\mathbb{G}_{\text{a}} \times A$ and to A , respectively; see [19, Prop. 1.2, Thm. 1.4].

Proposition 2.10 (Wüstholz). *The following statements are true:*

- (i) *The periods θ_j are transcendental.*
- (ii) *Assume A has no square factors up to isogeny. Then $\dim_{\bar{\mathbb{Q}}} \Sigma_{j=1}^g \bar{\mathbb{Q}}\theta_j = g$.*

On the other hand, isogenous abelian varieties have the same periods up to $\bar{\mathbb{Q}}^*$. From part (ii) of this proposition, we can easily deduce that the bi- $\bar{\mathbb{Q}}$ -structure on \mathbb{C}^g defined at the beginning of this subsection satisfies the properties of the part (ii) of Proposition 2.6, if A has no square factors. This holds true for example when A is simple.

We close this subsection by a reformulation of Wüstholtz's analytic subgroup theorem for this case. Consider the uniformization

$$u: \mathbb{C}^g \longrightarrow A(\mathbb{C}) = \mathbb{C}^g / \Lambda.$$

It can be shown that: for any $x \in \mathbb{C}^g$ which is $\bar{\mathbb{Q}}$ in the geometric $\bar{\mathbb{Q}}$ -structure, we have $u(x) \in A(\bar{\mathbb{Q}})$ if and only if $u(x)$ is a torsion point. We are thus viewing \mathbb{C}^g as $\bar{\mathbb{Q}}\Lambda \otimes \mathbb{C}$.

The map u induces an isomorphism

$$\psi := du: \mathbb{C}^g \simeq \bar{\mathbb{Q}}\Lambda \otimes \mathbb{C} \longrightarrow \text{Lie } A_{\bar{\mathbb{Q}}} \otimes \mathbb{C} \simeq \mathbb{C}^g,$$

which, under the bases $\{f_\sigma\}_{\sigma \in \Phi}$ and $\{e_\sigma\}_{\sigma \in \Phi}$ from the proof of Proposition 2.9, is the diagonal matrix $\text{diag}(\theta_\sigma)_{\sigma \in \Phi}$.

The base change of the exponential map

$$\exp: \text{Lie } A \longrightarrow A$$

to \mathbb{C} is $u \circ \psi^{-1}$. WAST in this case can be rephrased to be as the following.

Theorem 2.11 (Wüstholtz). *Let $z \in \mathbb{C}^g$ be such that $u(z) \in A(\bar{\mathbb{Q}})$. Let V be a sub-vector space of $\text{Lie } A_{\bar{\mathbb{Q}}}$ such that $\psi(z) \in V \otimes \mathbb{C}$. Then $V \supseteq \text{Lie } B$ for some abelian subvariety B of A which contains $u(z)$.*

3. A first discussion on products of periods of CM abelian varieties

In Section 2.3, or more precisely Proposition 2.10, we have seen the transcendence and the linear relations between the holomorphic periods of CM abelian varieties, both as consequences of WAST. In this section, we attempt to understand and ask questions about the quadratic relations.

Let A be a CM abelian variety of dimension g defined over $\bar{\mathbb{Q}}$. Let $\theta_1, \dots, \theta_g$ be its holomorphic periods as defined in Convention 1.1. Then up to $\bar{\mathbb{Q}}^*$, each θ_j is $\int_\gamma \omega_j$ for an $\text{End}(A) \otimes \mathbb{Q}$ -eigenbasis $\{\omega_1, \dots, \omega_g\}$ of Ω_A (with $\gamma \in H_1(A, \mathbb{Z})$).

WAST, when applied to A^\sharp (the universal vector extension of A) implies that the quotient of a holomorphic period by an anti-holomorphic period is transcendental. In the case of CM abelian varieties, this yields a transcendence result on the product of two holomorphic periods. This result is well known to experts, but we will include its proof in the current paper for the convenience of readers.

Corollary 3.1. *Each $\theta_j \theta_{j'} / \pi$ is a transcendental number for $j, j' \in \{1, \dots, g\}$.*

Before proceeding to its proof, at this stage a natural question to ask is what are the possible non-trivial $\bar{\mathbb{Q}}$ -linear relations among $\theta_j \theta_{j'}$ for all $1 \leq j \leq j' \leq g$, or equivalently what are the possible non-trivial $\bar{\mathbb{Q}}$ -quadratic relations between the θ_j 's. Unlike the transcendence of $\theta_j \theta_{j'}/\pi$, this question is still widely open; see Question 1.3 for the prediction. It turns out that this question is closely related to bi- $\bar{\mathbb{Q}}$ -structures arising from Shimura varieties, which we will discuss from Section 4.

In the rest of this section we prove Corollary 3.1. It follows immediately from the following proposition.

Proposition 3.2. *Assume that A has no square factor up to isogeny. Then*

$$\dim_{\bar{\mathbb{Q}}} \left(\bar{\mathbb{Q}} 2\pi i + \bar{\mathbb{Q}} \theta_1 + \cdots + \bar{\mathbb{Q}} \theta_g + \bar{\mathbb{Q}} \frac{2\pi i}{\theta_1} + \cdots + \bar{\mathbb{Q}} \frac{2\pi i}{\theta_g} \right) = 2g + 1. \quad (3.1)$$

Proof of Proposition 3.2 implying Corollary 3.1. Using equation (3.1) we conclude that θ_j and $2\pi i/\theta_{j'}$ are linearly independent over $\bar{\mathbb{Q}}$ for each $j, j' \in \{1, \dots, g\}$. Dividing both numbers by $\pi/\theta_{j'}$, we get that $\theta_j \theta_{j'}/\pi$ and $2i$ are linearly independent over $\bar{\mathbb{Q}}$. But $2i \in \bar{\mathbb{Q}}$. Hence $\theta_j \theta_{j'}/\pi$ is transcendental. \blacksquare

Now let us prove Proposition 3.2.

Proof of Proposition 3.2. We may assume $A = A_1 \times \cdots \times A_n$ is the product of simple CM abelian varieties which are 2-by-2 non-isogenous. We start with some preparation.

The key idea is to apply Wüstholz's analytic subgroup theorem to A^\natural , the universal vector extension of A . It fits into the short exact sequence

$$0 \longrightarrow \Omega_{A^\natural} \longrightarrow A^\natural \longrightarrow A \longrightarrow 0.$$

The natural projection $A^\natural \rightarrow A$ induces canonical isomorphisms

$$H_{\text{dR}}^1(A^\natural) = H_{\text{dR}}^1(A), \quad H^1(A^\natural, \mathbb{Z}) = H^1(A, \mathbb{Z}).$$

As a complex variety, $A^\natural(\mathbb{C}) = H_1(A^\natural, \mathbb{C})/H_1(A^\natural, \mathbb{Z})$. The Lie algebra $\text{Lie } A^\natural$ is $H_{\text{dR}}^1(A^\natural)^\vee$.

Let η_j be the complex conjugation of ω_j . Then $\{\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g\}$ is a $\bar{\mathbb{Q}}$ -basis of $H_{\text{dR}}^1(A^\natural) = H_{\text{dR}}^1(A)$. Take any basis $\{\gamma_1^\vee, \dots, \gamma_{2g}^\vee\}$ of $H^1(A^\natural, \mathbb{Z}) = H^1(A, \mathbb{Z})$. Then the Hodge-de Rham comparison $H_{\text{dR}}^1(A^\natural) \xrightarrow{\sim} H^1(A^\natural, \mathbb{C})$ under these $\bar{\mathbb{Q}}$ -bases is

$$\beta = \left[\left(\int_{\gamma_k} \omega_j \right)_{1 \leq k \leq 2g, 1 \leq j \leq g} \quad \left(\int_{\gamma_k} \eta_j \right)_{1 \leq k \leq 2g, 1 \leq j \leq g} \right].$$

For each $j \in \{1, \dots, g\}$, we have $\int_{\gamma_k} \omega_j \simeq \theta_j$. Since $\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g$ are eigenforms for the CM action, the reciprocity law for the differential forms of the 1st and the 2nd kinds implies that $\int_{\gamma_k} \eta_j \simeq 2\pi i/\theta_j$; see [5, p. 36, equation (3)] for more details. Hence this $2g \times 2g$ -matrix β is similar to the diagonal matrix $\text{diag}(\theta_1, \dots, \theta_g, 2\pi i/\theta_1, \dots, 2\pi i/\theta_g)$ over $\bar{\mathbb{Q}}$. For more details about this computation we refer to the last sentence of Remark 6.5 and above.

Denote for simplicity by $g_j := \dim A_j$, and $\theta_1^{(j)}, \dots, \theta_{g_j}^{(j)}$ the holomorphic periods of A_j . Write $\exp_j: \text{Lie } A_j^\natural \rightarrow A_j^\natural(\mathbb{C})$ for the exponential map, which is the composite of $\text{Lie } A_j^\natural = H_{\text{dR}}^1(A_j^\natural)^\vee \simeq H_1(A_j^\natural, \mathbb{C})$ and the quotient $H_1(A_j^\natural, \mathbb{C}) \rightarrow H_1(A_j^\natural, \mathbb{C})/H_1(A_j^\natural, \mathbb{Z})$. By the conclusion of the last paragraph, under suitable $\bar{\mathbb{Q}}$ -bases, the isomorphism

$$H_{\text{dR}}^1(A_j^\natural)^\vee \simeq H_1(A_j^\natural, \mathbb{C})$$

is represented by the diagonal matrix $\text{diag}(\theta_1^{(j)}, \dots, \theta_{g_j}^{(j)}, 2\pi i/\theta_1^{(j)}, \dots, 2\pi i/\theta_{g_j}^{(j)})$. Thus there exist $\lambda_1^{(j)}, \dots, \lambda_{g_j}^{(j)}, \mu_1^{(j)}, \dots, \mu_{g_j}^{(j)} \in \bar{\mathbb{Q}}^*$ such that $\exp_j(\mathbf{x}^{(j)}) = 0$ for the point $\mathbf{x}^{(j)} := (\lambda_1^{(j)}\theta_1^{(j)}, \dots, \lambda_{g_j}^{(j)}\theta_{g_j}^{(j)}, \mu_1^{(j)} \cdot 2\pi i/\theta_1^{(j)}, \dots, \mu_{g_j}^{(j)} \cdot 2\pi i/\theta_{g_j}^{(j)}) \in \text{Lie } A_j^\natural$.

Set $\mathbf{x} := (2\pi i, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}) \in \text{Lie}(\mathbb{G}_m \times A^\natural)$ and

$$\exp = (e^\cdot, \exp_1, \dots, \exp_n): \text{Lie}(\mathbb{G}_m \times A^\natural) \longrightarrow \mathbb{G}_m \times A^\natural.$$

Then $\exp(\mathbf{x})$ is the origin of $\mathbb{G}_m \times A^\natural$.

Let $\ell \gg 1$ be a prime number such that $\exp_j(\mathbf{x}^{(j)}/\ell)$ is a non-zero torsion point in A_j^\natural for each j . Let V be the smallest $\bar{\mathbb{Q}}$ -subspace of $\text{Lie}(\mathbb{G}_m \times A^\natural)$ such that $\mathbf{x}/\ell \in V_{\mathbb{C}}$. Then WAST applied to $\exp: \text{Lie}(\mathbb{G}_m \times A^\natural) \rightarrow \mathbb{G}_m \times A^\natural$, and the minimality assumption made on V , ensure that $V = \text{Lie } H$ for some (algebraic subgroup) H of $\mathbb{G}_m \times A^\natural$. Since $\exp(\mathbf{x}/\ell)$ is non-zero, we have $\dim H > 0$.

Write $\pi: \mathbb{G}_m \times A^\natural \rightarrow \mathbb{G}_m \times A$. As $\pi(H)$ is an irreducible subgroup of $\mathbb{G}_m \times A$ and A is the product of 2-by-2 non-isogenous simple abelian varieties, we have that $\pi(H) = \mathbb{G}_m \times A_1 \times \dots \times A_m \times \{0\}$ up to reordering.

We claim that $m = n$ and $\text{Lie}(H) = \text{Lie}(\mathbb{G}_m \times A^\natural)$. Indeed, set $B := A_1 \times \dots \times A_m$ and $B' := A_{m+1} \times \dots \times A_n$. Then $A = B \times B'$. Since $\mathbb{G}_m \times B \times \{0\} = \pi(H)$, we have $\mathbb{G}_m \times B^\natural \times \{0\} \subseteq H$. Hence the group H equals $\mathbb{G}_m \times B^\natural \times V'$ for some vector subgroup V' of B' . So $\exp(\mathbf{x}/\ell) \in (\mathbb{G}_m \times B^\natural \times V')(\bar{\mathbb{Q}}) \subseteq (\mathbb{G}_m \times A^\natural)(\bar{\mathbb{Q}})$. But $\exp(\mathbf{x}/\ell)$ is a torsion point, so $\exp(\mathbf{x}/\ell) \in (\mathbb{G}_m \times B^\natural)(\bar{\mathbb{Q}}) \times \{0\}$. By minimality of V , we then have $\pi(H) = \mathbb{G}_m \times B^\natural \times \{0\}$ and $V = \text{Lie}(\mathbb{G}_m \times B^\natural) \times \{0\} = \text{Lie } \mathbb{G}_m \times \text{Lie } A_1^\natural \times \dots \times \text{Lie } A_m^\natural \times \{0\}$. If $m < n$, then $\mathbf{x}^{(m+1)}/\ell$ is not 0, and hence

$$\mathbf{x}/\ell = (2\pi i, \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)})/\ell \notin \text{Lie } \mathbb{G}_m \times \text{Lie } A_1^\natural \times \dots \times \text{Lie } A_m^\natural \times \{0\} = V.$$

This contradicts the choice of V . So we must have $m = n$. Therefore $B = \mathbb{G}_m \times A$ and $V = \text{Lie}(\mathbb{G}_m \times A^\natural)$.

Assume there exists $(c, a_1, \dots, a_g, b_1, \dots, b_g) \in \bar{\mathbb{Q}}^{2g+1}$ such that

$$c2\pi i + \sum a_k \theta_k + \sum b_l \frac{2\pi i}{\theta_l} = 0.$$

Set

$$W := \{(x_0, x_1, \dots, x_g, x_{g+1}, \dots, x_{2g}) \in \text{Lie}(\mathbb{G}_m \times A^\natural) : cx_0 + \sum a_k x_k + \sum b_l x_{g+l} = 0\}.$$

Then W is defined over $\bar{\mathbb{Q}}$. It contains the point \mathbf{x}/ℓ . So $V \subseteq W$ by definition of V . But then $\text{Lie}(\mathbb{G}_m \times A^\natural) \subseteq W$, and hence $W = \text{Lie}(\mathbb{G}_m \times A^\natural)$ and

$$c = a_1 = \cdots = a_g = b_1 = \cdots = b_g = 0.$$

This finishes the proof. \blacksquare

4. Bi- $\bar{\mathbb{Q}}$ -spaces arising from Shimura varieties: Definition

From this section, we will build up a framework to relate Conjecture 1.3 to split bi- $\bar{\mathbb{Q}}$ -structures arising from Shimura varieties. The plan is as follows. The current Section 4 defines the bi- $\bar{\mathbb{Q}}$ -structure in question, and Section 5 proves that this bi- $\bar{\mathbb{Q}}$ -structure is split. In Section 7 we compute the periods of this split bi- $\bar{\mathbb{Q}}$ -structure for \mathbb{A}_g , the moduli space of principally polarized abelian varieties; the computation uses the discussion on the family version of the de Rham–Betti comparison and the Kodaira–Spencer map in Section 6. Then in Section 10, we propose the analogue of Wüstholz’s analytic subgroup theorem for Shimura varieties, and explains how this conjecture applied to \mathbb{A}_g implies Conjecture 1.3.

Let (\mathbf{G}, X) be a connected Shimura datum, and fix a special point $o \in X$; for definitions see Section 4.1. It is known that X is a Hermitian symmetric domain. The goals of this section are to endow $T_o X$ with a natural bi- $\bar{\mathbb{Q}}$ -structure. More precisely, we will endow $T_o X$ with

- the *arithmetic $\bar{\mathbb{Q}}$ -structure* in Section 4.2,
- the *geometric $\bar{\mathbb{Q}}$ -structure* in Section 4.3.

We also discuss in Section 4.4 on the $\bar{\mathbb{Q}}$ -structure on the Harish–Chandra (bounded) realization of X . This discussion is important when one relates the bi- $\bar{\mathbb{Q}}$ -space arising from Shimura varieties to transcendence theory.

4.1. Shimura data and Shimura varieties

We give a quick summary of Deligne’s language of Shimura varieties. References are [9, 10, 15].

A *connected Shimura datum* is a pair (\mathbf{G}, X) where \mathbf{G} is a reductive algebraic group over \mathbb{Q} and X is the $\mathbf{G}(\mathbb{R})^+$ -conjugacy class of a morphism

$$x: \mathbb{S} \longrightarrow \mathbf{G}_{\mathbb{R}}$$

satisfying Deligne’s conditions (SV1)–(SV3). It is known that X is a Hermitian symmetric space, and each Hermitian symmetric space arises in this way.

Let $\Gamma \subseteq \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup. The Baily–Borel theorem [4] asserts that $S := \Gamma \backslash X$ has a natural structure of quasi-projective complex algebraic variety. Such an S is called a *connected Shimura variety associated with (\mathbf{G}, X)* .

Let $u: X \rightarrow S^{\text{an}}$ denote the uniformization.

Consider $x \in X$. It gives rise to a morphism $x: \mathbb{S} \rightarrow \mathbf{G}_{\mathbb{R}}$. The *Mumford–Tate group* of x , denoted by $\mathrm{MT}(x)$, is defined to be the smallest \mathbb{Q} -subgroup \mathbf{H} of \mathbf{G} such that $x(\mathbb{S}) \subseteq \mathbf{H}_{\mathbb{R}}$.

A point $x \in X$ is said to be *special* if $\mathrm{MT}(x)$ is a torus. A point in $S(\mathbb{C})$ is said to be *special* if it is the image of a special point in X under the uniformization u . In the particular case $S = \mathbb{A}_g$, the moduli space of principally polarized abelian varieties, special points of S are precisely the points parametrizing CM abelian varieties.

4.2. Arithmetic $\bar{\mathbb{Q}}$ -structure on $T_o X$

By general theory of Shimura varieties (see also Faltings), the Shimura variety S has a canonical model over $\bar{\mathbb{Q}}$ which we denote by $S_{\bar{\mathbb{Q}}}$. Moreover, the special point $[o] := u(o)$ is in $S_{\bar{\mathbb{Q}}}(\bar{\mathbb{Q}})$. The differential of the uniformization $u: X \rightarrow S$ at o is then a \mathbb{C} -linear map

$$\psi: T_o X \longrightarrow T_{[o]} S.$$

Now $T_{[o]} S$ has a natural $\bar{\mathbb{Q}}$ -structure $T_{[o]} S = T_{[o]} S_{\bar{\mathbb{Q}}} \otimes_{\bar{\mathbb{Q}}} \mathbb{C}$. Thus ψ defines a $\bar{\mathbb{Q}}$ -structure on $T_o X$ as follows: For a $\bar{\mathbb{Q}}$ -basis $\{e_1, \dots, e_N\}$ of $T_{[o]} S_{\bar{\mathbb{Q}}}$, the set $\{\psi^{-1}(e_1), \dots, \psi^{-1}(e_N)\}$ is a basis of $T_o X$, and $\bigoplus_{j=1}^N \bar{\mathbb{Q}} \psi^{-1}(e_j)$ defines a $\bar{\mathbb{Q}}$ -structure on $T_o X$. It is called the *arithmetic $\bar{\mathbb{Q}}$ -structure* on $T_o X$.

The arithmetic $\bar{\mathbb{Q}}$ -structure on $T_o X$ can be characterized as follows: A subspace W of $T_o X$ is rational for the arithmetic $\bar{\mathbb{Q}}$ -structure if and only if $\dim_{\bar{\mathbb{Q}}} (\psi(W) \cap T_{[o]} S_{\bar{\mathbb{Q}}}) = \dim_{\mathbb{C}} \psi(W)$ (i.e., $\psi(W)$ is rational for $T_{[o]} S_{\bar{\mathbb{Q}}}$).

Using knowledge on Hecke correspondences, it is not hard to check that the arithmetic- $\bar{\mathbb{Q}}$ structure on $T_o X$ does not depend on the choice of the arithmetic subgroup $\Gamma \subseteq \mathbf{G}(\mathbb{Q})$.

4.3. Geometric $\bar{\mathbb{Q}}$ -structure on $T_o X$

In this subsection, we define a natural $\bar{\mathbb{Q}}$ -structure on X (denoted by $X_{\bar{\mathbb{Q}}}$), via the Hodge theoretic interpretation of the Borel embedding theorem given by Deligne, for which o lies in $X_{\bar{\mathbb{Q}}}(\bar{\mathbb{Q}})$. This then endows $T_o X$ with a $\bar{\mathbb{Q}}$ -structure $T_o X_{\bar{\mathbb{Q}}} \otimes_{\bar{\mathbb{Q}}} \mathbb{C}$ on $T_o X$, called the *geometric $\bar{\mathbb{Q}}$ -structure*.

Let $x: \mathbb{S} \rightarrow \mathbf{G}_{\mathbb{R}}$ be an element of X . Let $x_{\mathbb{C}}: \mathbb{G}_{m, \mathbb{C}} \times \mathbb{G}_{m, \mathbb{C}} \rightarrow \mathbf{G}_{\mathbb{C}}$ be its extension to \mathbb{C} , and denote the associated character by

$$\mu_x: \mathbb{G}_{m, \mathbb{C}} \longrightarrow \mathbf{G}_{\mathbb{C}}, \quad z \longmapsto x_{\mathbb{C}}(z, 1).$$

For each rational representation (\mathbf{V}, ρ) of \mathbf{G} and any $x \in X$, we have a Hodge structure $(V_{\mathbb{R}}, \rho \circ x)$ and an associated Hodge filtration $F(x)$ on $V_{\mathbb{C}}$ induced by the cocharacter μ_x . There exists a flag variety $\mathrm{Fl}(X, V)$ defined over \mathbb{C} such that each $F(x)$ is parametrized by a point in $\mathrm{Fl}(X, V)(\mathbb{C})$. Moreover, this flag variety has a natural model over $\bar{\mathbb{Q}}$, denoted by $\mathrm{Fl}(X, V)_{\bar{\mathbb{Q}}}$, such that the $\bar{\mathbb{Q}}$ -points correspond to filtrations of $V_{\bar{\mathbb{Q}}}$ by $\bar{\mathbb{Q}}$ -subvector spaces. Since $o \in X$ is a special point, $F(o)$ is a filtration by $\bar{\mathbb{Q}}$ -spaces; see [20, Prop. 3.7]. So we have $\mathrm{Fl}(X, V)_{\bar{\mathbb{Q}}} \simeq \mathrm{GL}(V_{\bar{\mathbb{Q}}})/\mathrm{Stab}(F(o))$.

The center $Z(\mathbf{G})(\mathbb{R})$ acts trivially on X . If the kernel of the representation $\rho: \mathbf{G} \rightarrow \mathrm{GL}(\mathbf{V})$ is contained in $Z(\mathbf{G})$, then the map

$$X \longrightarrow \mathrm{Fl}(X, V), \quad x \longmapsto F(x) \quad (4.1)$$

is injective. It factors through the $\mathbf{G}(\mathbb{C})$ -conjugacy class of μ_o in $\mathrm{hom}(\mathbb{S}_{\mathbb{C}}, \mathbf{G}_{\mathbb{C}})$, which by the first part of the Borel embedding theorem (Theorem 4.1) can be realized as the compact dual X^{\vee} of X . Now $\mathbf{G}_{\mathbb{R}} \subseteq \mathbf{G}_{\mathbb{C}} \xrightarrow{\rho} \mathrm{GL}(V_{\mathbb{C}})$ induces injective maps

$$\begin{aligned} X = \frac{\mathbf{G}(\mathbb{R})^+}{\mathrm{Stab}_{\mathbf{G}(\mathbb{R})^+}(o)} &\longrightarrow X^{\vee} = \frac{\mathbf{G}(\mathbb{C})}{\mathrm{Stab}_{\mathbf{G}(\mathbb{C})}(\mu_o)} \longrightarrow \mathrm{Fl}(X, V) = \frac{\mathrm{GL}(V_{\mathbb{C}})}{\mathrm{Stab}_{\mathrm{GL}(V_{\mathbb{C}})}(F(o))} \\ x \longmapsto &\mu_x \longmapsto F(x). \end{aligned}$$

Borel's embedding theorem (Theorem 4.1) asserts that the first map realizes X as an open subset (in the usual topology) of X^{\vee} . The second map makes X^{\vee} into a projective complex algebraic subvariety of $\mathrm{Fl}(X, V)$, which furthermore descends to $\bar{\mathbb{Q}}$ since μ_o descends to $\bar{\mathbb{Q}}$. In other words, there exists a subvariety $X_{\bar{\mathbb{Q}}}^{\vee}$ of $\mathrm{Fl}(X, V)_{\bar{\mathbb{Q}}}$, defined over $\bar{\mathbb{Q}}$, such that $X^{\vee} = X_{\bar{\mathbb{Q}}}^{\vee} \otimes_{\bar{\mathbb{Q}}} \mathbb{C}$.

Now we are ready to define the $\bar{\mathbb{Q}}$ -structure on X by setting $X(\bar{\mathbb{Q}}) := X \cap X_{\bar{\mathbb{Q}}}^{\vee}(\bar{\mathbb{Q}}) = X \cap \mathrm{Fl}(X, V)_{\bar{\mathbb{Q}}}(\bar{\mathbb{Q}})$. We have seen that $o \in X(\bar{\mathbb{Q}})$. Hence we obtain the *geometric* $\bar{\mathbb{Q}}$ -structure on $T_o X$ as explained at the beginning of this section; it equals $T_o X_{\bar{\mathbb{Q}}}^{\vee}$.

We finish this subsection with the following remark. Notice that $X_{\bar{\mathbb{Q}}}^{\vee}$ does not depend on the choice of the rational representation (\mathbf{V}, ρ) (with $\ker(\rho) \subseteq Z(\mathbf{G})$). In particular, since \mathbf{G} is a reductive group, we can take (\mathbf{V}, ρ) to be $\mathbf{V} = \mathrm{Lie} \mathbf{G}^{\mathrm{ad}}$ and

$$\rho: \mathbf{G} \longrightarrow \mathbf{G}^{\mathrm{ad}} = \mathbf{G}/Z(\mathbf{G}) \longrightarrow \mathrm{GL}(\mathrm{Lie} \mathbf{G}^{\mathrm{ad}}) \quad (4.2)$$

where the second morphism is the adjoint representation.

4.4. Bounded realization of X

To make the correct analogue of Wüstholtz's analytic subgroup theorem in the Shimura setting, it is important to work with the *Harish–Chandra realization* of X which includes two aspects:

- (i) identifies $T_o X$ with a certain abelian sub-Lie algebra \mathfrak{m}^+ of $\mathfrak{g}_{\mathbb{C}} := \mathrm{Lie}(\mathbf{G}(\mathbb{C}))$;
- (ii) realizes X as a bounded symmetric domain \mathcal{D} in $\mathfrak{m}^+ \simeq \mathbb{C}^N$ such that o becomes the origin.

Here $N = \dim S = \dim X$. We will recall both aspects in the current subsection, with Lemma 4.2 for (i) and Theorem 4.3 for (ii). Notice that this gives an inclusion of X in $T_o X$. For the purpose of our paper, it is important to understand how the various $\bar{\mathbb{Q}}$ -structures are related under the Harish–Chandra realization. More precisely, we will show that the $\bar{\mathbb{Q}}$ -structure on X and the geometric $\bar{\mathbb{Q}}$ -structure on $T_o X$, both given in the previous subsection, are compatible under this inclusion; see the paragraph below Proposition 4.4 for more details.

For our connected Shimura datum (G, X) , we fix the following notation. Denote by $G := \mathbf{G}^{\text{der}}(\mathbb{R})^+$, $G_{\mathbb{R}} := \mathbf{G}_{\mathbb{R}}^{\text{der}}$ and $G_{\mathbb{C}} = \mathbf{G}_{\mathbb{C}}^{\text{der}}$. The Lie algebras are denoted by using fractur letters: denote by $\text{Lie}(\mathbf{G}^{\text{der}}) = \mathfrak{g}$, $\text{Lie}(G) = \mathfrak{g}_{\mathbb{R}}$ and $\text{Lie}(G(\mathbb{C})) = \mathfrak{g}_{\mathbb{C}}$.

The underlying space X is a G -orbit because the center of $\mathbf{G}(\mathbb{R})$ acts trivially on X .

Our point $o \in X$ gives rise to a morphism $o: \mathbb{S} \rightarrow \mathbf{G}_{\mathbb{R}}$. By (SV2) in the definition of Shimura data, $o(\sqrt{-1})$ defines a Cartan involution θ on $\mathfrak{g}_{\mathbb{R}}$. Write $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{m}$ for the associated Cartan decomposition, with \mathfrak{k} the eigenspace of 1 and \mathfrak{m} the eigenspace of -1 . Then $G = K_{\infty}M$ with $K_{\infty} = \exp(\mathfrak{k})$ a maximal compact subgroup of G and $M = \exp(\mathfrak{m})$. Moreover, $X = G/K_{\infty}$ as a Riemannian symmetric space. Denote by $T_{\mathbb{R}}(X)$ the real tangent space of X at o . Then $T_{\mathbb{R}}(X) = \mathfrak{m}$.

The Cartan involution θ extends to $\mathfrak{g}_{\mathbb{C}}$ and we have a corresponding Cartan decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{m}_{\mathbb{C}}$. Let $\mathfrak{g}_c := \mathfrak{k} \oplus \sqrt{-1}\mathfrak{m} \subseteq \mathfrak{g}_{\mathbb{C}}$. Since G is semi-simple of non-compact type, $G_c := \exp(\mathfrak{g}_c)$ is a compact Lie group and $X^{\vee} := G_c/K_{\infty}$ is the compact dual of the Riemannian symmetric space $X = G/K_{\infty}$.

The complex structure on X is given by an endomorphism J of $T_{\mathbb{R}}(X)$ such that $J^2 = -\text{Id}$. We have a decomposition

$$T_{\mathbb{R}}(X) \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}(X) \oplus T^{0,1}(X)$$

where J acts by multiplication by $\sqrt{-1}$ on $T^{1,0}(X)$ and by $-\sqrt{-1}$ on $T^{0,1}(X)$. In this description $T^{1,0}(X)$ is identified with the holomorphic tangent space at o and there is a real isomorphism $T_{\mathbb{R}}(X) \simeq T^{1,0}(X)$. We have $\mathfrak{m}_{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$ where J acts by multiplication by $\sqrt{-1}$ on \mathfrak{m}^+ and by $-\sqrt{-1}$ on \mathfrak{m}^- . It is not hard to check that \mathfrak{m}^+ and \mathfrak{m}^- are abelian sub-Lie algebras, i.e., $[\mathfrak{m}^+, \mathfrak{m}^+] = 0$ and $[\mathfrak{m}^-, \mathfrak{m}^-] = 0$.

Let $M^+ = \exp(\mathfrak{m}^+)$, $M^- = \exp(\mathfrak{m}^-)$, $K_{\mathbb{C}} = \exp(\mathfrak{k}_{\mathbb{C}})$ and $P_{\mathbb{C}} = \exp(\mathfrak{k}_{\mathbb{C}} + \mathfrak{m}^-) = K_{\mathbb{C}}M_{\mathbb{C}}^-$. Notice that $P_{\mathbb{C}}$ is a subgroup of $G_{\mathbb{C}}$ because $\mathfrak{p} := \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{m}^-$ is a complex sub-Lie algebra.

Theorem 4.1 (Borel embedding theorem). *The embedding $G_c \rightarrow G(\mathbb{C})$ induces a biholomorphism $X^{\vee} = G_c/K_{\infty} \rightarrow G(\mathbb{C})/P(\mathbb{C})$. The embedding $G \rightarrow G(\mathbb{C})$ induces an open embedding*

$$X = G/K_{\infty} \rightarrow G(\mathbb{C})/P_{\mathbb{C}} \simeq X^{\vee},$$

realizing X as a open subset (in the usual topology) of its compact dual X^{\vee} .

Let $T_o X$ be the holomorphic tangent space of X at o . It is canonically isomorphic to $T_o X^{\vee}$ under the Borel embedding theorem, which is furthermore canonically isomorphic to $\mathfrak{g}_{\mathbb{C}}/(\mathfrak{k}_{\mathbb{C}} + \mathfrak{m}^-) = \mathfrak{m}^+$. Hence we have:

Lemma 4.2. *Under the identification $X = G/K_{\infty}$ and the Borel embedding theorem, we have $T_o X = T_o X^{\vee} = \mathfrak{m}^+$.*

The Harish–Chandra embedding theorem states the following.

Theorem 4.3 (Harish–Chandra). *The map*

$$F: M^+ \times K_{\mathbb{C}} \times M^- \rightarrow G_{\mathbb{C}}, \quad (m^+, k, m^-) \mapsto m^+ k m^-$$

is a biholomorphism of $M^+ \times K_{\mathbb{C}} \times M^-$ onto an open subset of $G(\mathbb{C})$ containing G . As a consequence the map

$$\eta: \mathfrak{m}^+ \longrightarrow G(\mathbb{C})/P_{\mathbb{C}} = X^\vee, \quad m^+ \longmapsto \exp(m^+)P_{\mathbb{C}} \quad (4.3)$$

is a biholomorphism onto a dense open subset of X^\vee containing X . Furthermore, $\mathcal{D} := \eta^{-1}(X)$ is a bounded symmetric domain in $\mathfrak{m}^+ \simeq \mathbb{C}^N$ and $\eta^{-1}(o) = 0$.

Up to now in this subsection, we have *not* used any assumption on the chosen point $o \in X$. From now on, recall that $o \in X(\bar{\mathbb{Q}})$.

We start with understanding the geometric $\bar{\mathbb{Q}}$ -structure on $T_o X$ given in Section 4.3 from the Lie algebra point of view. Notice that the geometric $\bar{\mathbb{Q}}$ -structure on $T_o X$ and the identification of \mathfrak{m}^+ with $T_o X$ from Lemma 4.2 together induce a natural $\bar{\mathbb{Q}}$ -structure on \mathfrak{m}^+ , which we denote by $\mathfrak{m}_{\bar{\mathbb{Q}}, \text{geom}}^+$.

Proposition 4.4. *We have:*

- (i) $\mathfrak{m}_{\bar{\mathbb{Q}}, \text{geom}}^+ = \mathfrak{m}^+ \cap \mathfrak{g}_{\bar{\mathbb{Q}}}$.
- (ii) *The map η from (4.3) descends to a polynomial morphism $\mathfrak{m}_{\bar{\mathbb{Q}}, \text{geom}}^+ \rightarrow X^\vee(\bar{\mathbb{Q}})$ defined over $\bar{\mathbb{Q}}$.*

Define the $\bar{\mathbb{Q}}$ -structure on \mathcal{D} by setting $\mathcal{D}(\bar{\mathbb{Q}}) := \mathfrak{m}_{\bar{\mathbb{Q}}, \text{geom}}^+ \cap \mathcal{D}$. Then by Proposition 4.4, η induces a bijection between $\mathcal{D}(\bar{\mathbb{Q}})$ and $X(\bar{\mathbb{Q}})$.

Proof of Proposition 4.4. From the discussion above Lemma 4.2, the identification $\mathfrak{m}^+ = T_o X$ is given by $\mathfrak{m}^+ = \mathfrak{g}_{\mathbb{C}}/(\mathfrak{k}_{\mathbb{C}} + \mathfrak{m}^-)$. Since $o \in X(\bar{\mathbb{Q}})$, in the Cartan decomposition $\mathfrak{k}_{\mathbb{C}}$, \mathfrak{m}^+ and \mathfrak{m}^- descend to $\bar{\mathbb{Q}}$. Hence

$$\mathfrak{m}_{\bar{\mathbb{Q}}, \text{geom}}^+ = \mathfrak{g}_{\bar{\mathbb{Q}}}/((\mathfrak{k}_{\mathbb{C}} + \mathfrak{m}^-) \cap \mathfrak{g}_{\bar{\mathbb{Q}}}) = \mathfrak{m}^+ \cap \mathfrak{g}_{\bar{\mathbb{Q}}}.$$

This proves part (i).

As \mathfrak{m}^+ is an abelian Lie algebra, the exponential map on \mathfrak{m}^+ is given by a polynomial expression with coefficient in $\bar{\mathbb{Q}}$. Thus part (ii) holds true. \blacksquare

5. Bi- $\bar{\mathbb{Q}}$ -spaces arising from Shimura varieties: Proof of the splitting

Let (G, X) be a connected Shimura datum, and fix a special point $o \in X$. We have endowed $T_o X$ with a bi- $\bar{\mathbb{Q}}$ -structure in Section 4, i.e., the *arithmetic $\bar{\mathbb{Q}}$ -structure* in Section 4.2 and the *geometric $\bar{\mathbb{Q}}$ -structure* in Section 4.3. The goal of this section is to prove the following.

Theorem 5.1. *The bi- $\bar{\mathbb{Q}}$ -structure on $T_o X$ thus defined is split.*

Moreover, we will explain how the splitting of this bi- $\bar{\mathbb{Q}}$ -structure on $T_o X$ is obtained; see Theorem 5.1'.

By Theorem 5.1 and the general theory from Section 2, we have an isomorphism of bi- $\bar{\mathbb{Q}}$ -structure

$$T_o(X) \simeq (\mathbb{C}^N; \alpha_1, \dots, \alpha_N) \quad \text{with } \alpha_i \in \mathbb{C}^*/\bar{\mathbb{Q}}^*.$$

If $S = \mathbb{A}_g$, then o corresponds to a CM abelian variety A_o and $\text{Lie}(A_o) \simeq (\mathbb{C}^g; \theta_1, \dots, \theta_g)$ with $\alpha_i \in \mathbb{C}^* / \bar{\mathbb{Q}}^*$ by the results of Section 2.3. We will describe relations between the α_i and the θ_i using deformation theory in the next sections.

5.1. Notation

We retain the notation from Section 4.4 that $\mathfrak{g} := \text{Lie}(\mathbf{G}^{\text{der}})$. The natural morphism $\mathbf{G}^{\text{der}} \subseteq \mathbf{G} \rightarrow \mathbf{G}^{\text{ad}} = \mathbf{G}/Z(\mathbf{G})$ induces $\mathfrak{g} = \text{Lie}(\mathbf{G}^{\text{ad}})$.

The point $o \in X$ defines a Cartan involution θ of $\mathfrak{g}_{\mathbb{R}}$ by (SV2) of Deligne's definition of Shimura data. Write $\theta_{\mathbb{C}}$ for its extension to $\mathfrak{g}_{\mathbb{C}}$. We have the following Cartan decompositions

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{m}, \quad \mathfrak{g}_c = \mathfrak{k} + \sqrt{-1}\mathfrak{m} \quad \text{and} \quad \mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}},$$

with $\mathfrak{m}_{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$. See Section 4.4 for the notation.

Since o is a special point, its Mumford–Tate group $\text{MT}(o)$ is a torus. Let \mathbf{T} be a maximal torus of \mathbf{G} which contains $\text{MT}(o)$. Write \mathbf{T}^{ad} for its image under the morphism $\mathbf{G} \rightarrow \mathbf{G}^{\text{ad}}$. Then \mathbf{T}^{ad} is a maximal torus of \mathbf{G}^{ad} , and hence $\mathfrak{h}_{\mathbb{R}} := \text{Lie} \mathbf{T}^{\text{ad}}(\mathbb{R})$ is a real Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$. Moreover, $\mathfrak{h}_{\mathbb{R}} \subseteq \mathfrak{k}$ since \mathfrak{k} is the eigenspace of 1 for the Cartan involution.

5.2. Decomposition of the Geometric $\bar{\mathbb{Q}}$ -structure on $T_o X$

In this subsection, we will show that the $T_o X_{\bar{\mathbb{Q}}}$, i.e., the geometric $\bar{\mathbb{Q}}$ -structure on $T_o X$, can be decomposed into the direct sum of 1-dimensional $\bar{\mathbb{Q}}$ -sub-vector spaces, each being an eigenspace for the action of a maximal torus in \mathbf{G} which contains $\text{MT}(o)$.

5.2.1. Statement of the result. Let \mathfrak{z} be the center of \mathfrak{k} . Then $\mathfrak{z} \subseteq \mathfrak{h}_{\mathbb{R}}$ since $\mathfrak{h}_{\mathbb{R}}$ is abelian. The following result can be found in [16, p. 54, Prop. 1].

Lemma 5.2. *There exists $z \in \mathfrak{z}$ such that the J -operator on \mathfrak{m} , which induces the complex structure on \mathfrak{m} , is defined by $J.v = [z, v]$ for any $v \in \mathfrak{m}$.*

Proposition 4.4 (i) gives the geometric $\bar{\mathbb{Q}}$ -structure on $T_o X$ a Lie algebra point of view, i.e., it is naturally identified with $\mathfrak{m}_{\bar{\mathbb{Q}}, \text{geom}}^+ = \mathfrak{m}^+ \cap \mathfrak{g}_{\bar{\mathbb{Q}}}$ via Lemma 4.2.

Proposition 5.3. *$\mathfrak{m}_{\bar{\mathbb{Q}}, \text{geom}}^+$ can be decomposed into the direct sum of 1-dimensional $\bar{\mathbb{Q}}$ -spaces.*

In fact, we will prove a more precise version, Proposition 5.5, which explains how this decomposition is constructed. This construction is important.

5.2.2. Complex multiplication and root space decomposition. Let $\mathfrak{h}_{\mathbb{R}}^\vee$ be the dual space of $\mathfrak{h}_{\mathbb{R}}$. Let $\Phi \subseteq \mathfrak{h}_{\mathbb{C}}^\vee$ be the set of roots of $\mathfrak{h}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$. Then we have the root decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \left(\bigoplus_{\phi \in \Phi} \mathfrak{g}_{\mathbb{C}}^\phi \right), \quad (5.1)$$

where $\mathfrak{g}_{\mathbb{C}}^\phi = \{x \in \mathfrak{g}_{\mathbb{C}} : [h, x] = \phi(h)x \text{ for all } h \in \mathfrak{h}_{\mathbb{C}}\}$ has dimension 1. For each root $\phi \in \Phi$, let f_ϕ be a generator of the root space $\mathfrak{g}_{\mathbb{C}}^\phi$.

We say that a root ϕ is *compact* (resp. *non-compact*) if $f_\phi \in \mathfrak{k}_\mathbb{C}$ (resp. $f_\phi \in \mathfrak{m}_\mathbb{C}$). Let Φ_K be the set of compact roots and Φ_M be the set of non-compact roots. We claim that $\Phi = \Phi_K \cup \Phi_M$. Indeed, since θ is an involution fixing $\mathfrak{k}_\mathbb{C}$ and $\mathfrak{h}_\mathbb{C} \subseteq \mathfrak{k}_\mathbb{C}$, we have $[h, \theta(f_\phi)] = \phi(h)\theta(f_\phi)$. Hence $\theta(f_\phi) \in \mathfrak{g}_\mathbb{C}^\phi$, and therefore $\theta(f_\phi) = \lambda f_\phi$ for some $\lambda \in \mathbb{C}^*$. If $f_\phi = a_K f_{\phi, K} + a_M f_{\phi, M}$ under the Cartan decomposition with $a_K, a_M \in \mathbb{C}$, then $\theta(f_\phi) = a_K f_{\phi, K} - a_M f_{\phi, M}$. Then either $a_K = 0$ or $a_M = 0$, and hence we conclude that f_ϕ is in $\mathfrak{k}_\mathbb{C}$ or in $\mathfrak{m}_\mathbb{C}$.

Lemma 5.4. *There exists a choice Φ^+ of positive roots, with $\Phi = \Phi^+ \sqcup -\Phi^+$, satisfying the following property. For $\Phi_M^+ := \Phi_M \cap \Phi^+$, we have*

$$\mathfrak{m}^+ = \bigoplus_{\phi \in \Phi_M^+} \mathfrak{g}_\mathbb{C}^\phi. \quad (5.2)$$

Before proving this lemma, let us explain how Φ^+ is constructed. Each $\phi \in \Phi$ takes real values on $\sqrt{-1}\mathfrak{h}_\mathbb{R}$, so Φ is a root system in $\sqrt{-1}\mathfrak{h}_\mathbb{R}^\vee$. Since $\mathfrak{g}_\mathbb{C}$ is semi-simple, Φ spans $\sqrt{-1}\mathfrak{h}_\mathbb{R}^\vee$. From now on, we identify $\sqrt{-1}\mathfrak{h}_\mathbb{R}^\vee$ with $\sqrt{-1}\mathfrak{h}_\mathbb{R}$ using the Killing form on $\mathfrak{g}_\mathbb{R}$.² Now consider the Weyl chambers in $\sqrt{-1}\mathfrak{h}_\mathbb{R}$ with respect to the root system Φ . Take a Weyl chamber C such that $-\sqrt{-1}z \in \bar{C}$, with z from Lemma 5.2. For each γ in the interior of C , the set $\Phi^+ := \{\phi \in \Phi : \phi(\gamma) > 0\}$ does not depend on the choice of γ . By theory of root systems, we have $\Phi = \Phi^+ \sqcup -\Phi^+$. We will show that this is our desired Φ^+ .

Proof. We start with the following observation: For any non-compact root ϕ , either $f_\phi \in \mathfrak{m}^+$ or $f_\phi \in \mathfrak{m}^-$. Indeed by Lemma 5.2, $Jf_\phi = [z, f_\phi]$ for some $z \in \mathfrak{z}$. Since $\mathfrak{h}_\mathbb{C} \subseteq \mathfrak{k}_\mathbb{C}$, we have $[h, z] = 0$ for all $h \in \mathfrak{h}_\mathbb{C}$. Hence for each $h \in \mathfrak{h}_\mathbb{C}$, we have $[h, Jf_\phi] = [h, [z, f_\phi]] = [z, [h, f_\phi]]$, with the last equality induced by the Jacobi identity, and further equals

$$[z, \phi(h)f_\phi] = \phi(h)[z, f_\phi] = \phi(h)Jf_\phi.$$

Hence $Jf_\phi \in \mathfrak{g}_\mathbb{C}^\phi$. So Jf_ϕ is a scalar of f_ϕ . Now that $f_\phi \in \mathfrak{m}_\mathbb{C} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$, we have $f_\phi = a^+m^+ + a^-m^-$ for some $m^+ \in \mathfrak{m}^+$, $m^- \in \mathfrak{m}^-$ and $a^+, a^- \in \mathbb{C}$. As \mathfrak{m}^+ is the J -eigenspace associated with $\sqrt{-1}$ and \mathfrak{m}^- is the J -eigenspace associated with $-\sqrt{-1}$, we then have $Jf_\phi = \sqrt{-1}a^+m^+ - \sqrt{-1}a^-m^-$. But Jf_ϕ is a scalar of f_ϕ . So either $a^+ = 0$ or $a^- = 0$. Hence we are done for the claim.

This shows that $\mathfrak{m}^+ = \bigoplus_{\phi \in \Phi_0} \mathfrak{g}_\mathbb{C}^\phi$ for a subset $\Phi_0 \subseteq \Phi_M$. It remains to show $\Phi_0 = \Phi_M^+$.

Assume $\phi \in \Phi_0$, i.e., $f_\phi \in \mathfrak{m}^+$. Then

$$[-\sqrt{-1}z, f_\phi] = -\sqrt{-1}Jf_\phi = f_\phi.$$

Thus $\phi(-\sqrt{-1}z) = 1 > 0$. So $\phi \in \Phi^+$ by construction. This proves $\Phi_0 \subseteq \Phi^+ \cap \Phi_M = \Phi_M^+$.

²This Killing form is positive definite when restricted to $\sqrt{-1}\mathfrak{h}_\mathbb{R}$, and hence induces $\sqrt{-1}\mathfrak{h}_\mathbb{R}^\vee \simeq \sqrt{-1}\mathfrak{h}_\mathbb{R}$.

Assume $\phi \in \Phi_M^+$. Then $\phi(-\sqrt{-1}z) \geq 0$ by construction. If $\phi \notin \Phi_0$, then

$$[-\sqrt{-1}z, f_\phi] = -\sqrt{-1}Jf_\phi = -f_\phi$$

and hence $\phi(-\sqrt{-1}z) = -1 < 0$, contradiction. This proves $\Phi_M^+ \subseteq \Phi_0$.

Now we are done. \blacksquare

5.2.3. Decomposition over $\bar{\mathbb{Q}}$. Another way to see the root decomposition (5.1) is by the adjoint representation restricted to the maximal torus \mathbf{T} and the character group $X^*(\mathbf{T})$. More precisely, the morphism $\mathbf{G} \rightarrow \mathbf{G}^{\text{ad}}$ induces an inclusion $X^*(\mathbf{T}^{\text{ad}}) \subseteq X^*(\mathbf{T})$. In fact, $Z(\mathbf{G}) < \mathbf{T}$ and $\mathbf{T}^{\text{ad}} = \mathbf{T}/Z(\mathbf{G})$, and hence

$$X^*(\mathbf{T}^{\text{ad}}) = \{\chi \in X^*(\mathbf{T}) : \chi|_{Z(\mathbf{G})} \equiv 1\}. \quad (5.3)$$

The adjoint representation of \mathbf{G}^{ad} on \mathfrak{g} induces

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \left(\bigoplus_{\chi \in \Phi(\mathbf{T}_{\mathbb{C}}, \mathbf{G}_{\mathbb{C}})} \mathfrak{g}_{\mathbb{C}}^{\chi} \right) \quad (5.4)$$

where

$$\Phi(\mathbf{T}_{\mathbb{C}}, \mathbf{G}_{\mathbb{C}}) \subseteq X^*(\mathbf{T}_{\mathbb{C}}^{\text{ad}}) \subseteq X^*(\mathbf{T}_{\mathbb{C}}) \text{ and } \mathfrak{g}_{\mathbb{C}}^{\chi} = \{x \in \mathfrak{g}_{\mathbb{C}} : \text{Ad}(t)(x) = \chi(t)x \text{ for all } t \in \mathbf{T}(\mathbb{C})\}$$

has dimension 1.

By general theory of root decomposition for semi-simple groups, the sets of spaces $\{\mathfrak{g}_{\mathbb{C}}^{\phi}\}_{\phi \in \Phi}$ and $\{\mathfrak{g}_{\mathbb{C}}^{\chi}\}_{\chi \in \Phi(\mathbf{T}_{\mathbb{C}}, \mathbf{G}_{\mathbb{C}})}$ coincide. More precisely, there exists a bijection $\Phi \simeq \Phi(\mathbf{T}_{\mathbb{C}}, \mathbf{G}_{\mathbb{C}})$, $\phi \mapsto \chi_{\phi}$, such that $\mathfrak{g}_{\mathbb{C}}^{\phi} = \mathfrak{g}_{\mathbb{C}}^{\chi_{\phi}}$. Both \mathbf{T} and \mathbf{G} are \mathbb{Q} -groups, so there is a natural isomorphism $X^*(\mathbf{T}_{\bar{\mathbb{Q}}}) = X^*(\mathbf{T}_{\mathbb{C}})$. Thus each $\mathfrak{g}_{\mathbb{C}}^{\chi}$ is naturally defined over $\bar{\mathbb{Q}}$; more precisely $\mathfrak{g}_{\mathbb{C}}^{\chi} = (\mathfrak{g}_{\mathbb{C}}^{\chi} \cap \mathfrak{g}_{\bar{\mathbb{Q}}}) \otimes_{\bar{\mathbb{Q}}} \mathbb{C}$.

Let Φ_M^+ be from Lemma 5.4. By abuse of notation, we still use Φ_M^+ to denote the image of $\Phi_M^+ \subseteq \Phi$ under $\Phi \simeq \Phi(\mathbf{T}_{\mathbb{C}}, \mathbf{G}_{\mathbb{C}}) \subseteq X^*(\mathbf{T}_{\mathbb{C}}) = X^*(\mathbf{T}_{\bar{\mathbb{Q}}})$.

Proposition 5.5. Denote by $\mathfrak{m}_{\bar{\mathbb{Q}}, \text{geom}}^{+, \chi} := \mathfrak{g}_{\mathbb{C}}^{\chi} \cap \mathfrak{g}_{\bar{\mathbb{Q}}}$ for each $\chi \in \Phi_M^+$. Then

$$\mathfrak{m}_{\bar{\mathbb{Q}}, \text{geom}}^+ = \mathfrak{m}^+ \cap \mathfrak{g}_{\bar{\mathbb{Q}}} = \bigoplus_{\chi \in \Phi_M^+} \mathfrak{m}_{\bar{\mathbb{Q}}, \text{geom}}^{+, \chi}, \quad (5.5)$$

with $\dim_{\bar{\mathbb{Q}}} \mathfrak{m}_{\bar{\mathbb{Q}}, \text{geom}}^{+, \chi} = 1$ for each $\chi \in \Phi_M^+$ (equivalently, $\mathfrak{m}_{\bar{\mathbb{Q}}, \text{geom}}^{+, \chi} \otimes \mathbb{C} = \mathfrak{g}_{\mathbb{C}}^{\chi}$).

This proposition is a more precise version of Proposition 5.3.

Proof of Proposition 5.5. The second equality follows immediately from Lemma 5.4, and we have seen $\dim_{\bar{\mathbb{Q}}} \mathfrak{g}_{\bar{\mathbb{Q}}}^{\chi} = 1$ above. To see the first equality, recall that $\mathfrak{m}_{\bar{\mathbb{Q}}, \text{geom}}^+ = T_0 X_{\bar{\mathbb{Q}}}^{\vee}$. The $\bar{\mathbb{Q}}$ -structure $X_{\bar{\mathbb{Q}}}^{\vee}$ is given by the flag variety associated with the representation (4.2), which is precisely the representation giving the root decomposition (5.4). Hence we are done. \blacksquare

5.3. Decomposition of the arithmetic $\bar{\mathbb{Q}}$ -structure on $T_o X$

In this subsection, we turn to the decomposition of the arithmetic $\bar{\mathbb{Q}}$ -structure on $T_o X$ into the direct sum of 1-dimensional $\bar{\mathbb{Q}}$ -space.

We retain the notation from Section 4.1 and 4.2. Now S is a connected Shimura variety associated with the connected Shimura datum (\mathbf{G}, X) , and S has a canonical model $S_{\bar{\mathbb{Q}}}$ over $\bar{\mathbb{Q}}$. As a complex variety $S^{\text{an}} = \Gamma \backslash X$ for some arithmetic subgroup $\Gamma \subseteq \mathbf{G}(\mathbb{Q})$.

From the uniformization $u: X \rightarrow S$ and the special point $o \in X$, we get a $\bar{\mathbb{Q}}$ -point $[o] := u(o) \in S_{\bar{\mathbb{Q}}}(\bar{\mathbb{Q}})$. The arithmetic $\bar{\mathbb{Q}}$ -structure on $T_o X$ is the one obtained from $\psi^{-1}(T_{[o]} S_{\bar{\mathbb{Q}}})$, where ψ is the differential $du: T_o X \rightarrow T_{[o]} S$.

The point $o \in X$ gives rise to a Cartan decomposition $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{m}$, and we have $\mathfrak{m}_{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$; see Section 4.4. By Lemma 4.2, \mathfrak{m}^+ can be identified with $T_o X$. Hence the arithmetic $\bar{\mathbb{Q}}$ -structure on $T_o X$ defined above yields a $\bar{\mathbb{Q}}$ -structure on \mathfrak{m}^+ , which we denote by $\mathfrak{m}_{\bar{\mathbb{Q}}, \text{arith}}^+$.

Let \mathbf{T} be a maximal torus in \mathbf{G} which contains $\text{MT}(o)$ such that $\text{Lie } \mathbf{T}(\mathbb{R})$ is contained in \mathfrak{k} modulo $\text{Lie } Z(\mathbf{G})(\mathbb{R})$. In particular, $\mathbf{T}(\mathbb{R})o = o$. Then any $t \in \mathbf{T}(\mathbb{Q})$ induces the following commutative diagram, and $o \in X$ is mapped

$$\begin{array}{ccccc}
 X & \xleftarrow{\text{id}} & X & \xrightarrow{t \cdot} & X \\
 u \downarrow & & u_t \downarrow & & \downarrow u \\
 S & \xleftarrow{[\text{id}]} & S_t := (\Gamma \cap t^{-1}\Gamma t) \backslash X & \xrightarrow{[t \cdot]} & S
 \end{array}
 \quad
 \begin{array}{ccccc}
 o & \xleftarrow{\quad} & o & \xrightarrow{\quad} & o \\
 \downarrow & & \downarrow & & \downarrow \\
 [o] & \xleftarrow{\quad} & [o_t] := u_t(o) & \xrightarrow{\quad} & [o]
 \end{array} \tag{5.6}$$

where u_t is the quotient. The theory of Shimura varieties asserts that every morphism in the bottom line is algebraic, and is furthermore defined over $\bar{\mathbb{Q}}$. The morphism $[\text{id}]$ induces $T_{[o]} S_{t, \bar{\mathbb{Q}}} = T_{[o]} S_{\bar{\mathbb{Q}}}$. Thus the differential of $[t \cdot]$ induces a $\bar{\mathbb{Q}}$ -linear map

$$\rho_t: T_{[o]} S_{\bar{\mathbb{Q}}} \longrightarrow T_{[o]} S_{\bar{\mathbb{Q}}}.$$

By abuse of notation, we also use ρ_t to denote its base change to \mathbb{C} .

Proposition 5.6. *Under the identification of $T_o X$ with \mathfrak{m}^+ from Lemma 4.2, we have*

$$\rho_t = \psi \circ \text{Ad}(t) \circ \psi^{-1}. \tag{5.7}$$

Proof. Differentiating (5.6), we get that $\psi^{-1} \circ \rho_t \circ \psi: T_o X \rightarrow T_o X$ is $d(t \cdot)_o$.

Use the notation from Lemma 4.2 and above. Recall our choice of \mathbf{T} that $\text{Lie } \mathbf{T}(\mathbb{R})$ is contained in \mathfrak{k} modulo $\text{Lie } Z(\mathbf{G})(\mathbb{R})$. Since $[\mathfrak{k}, \mathfrak{m}] \subseteq \mathfrak{m}$ and $[\mathfrak{k}_{\mathbb{C}}, \mathfrak{m}^+] \subseteq \mathfrak{m}^+$, we have $\text{Ad}(\mathbf{T}(\mathbb{R}))(\mathfrak{m}) \subseteq \mathfrak{m}$ and $\text{Ad}(\mathbf{T}(\mathbb{C}))(\mathfrak{m}^+) \subseteq \mathfrak{m}^+$.

For $X = G/K_{\infty}$ as a Riemannian symmetric space, the real tangent space of X at o is \mathfrak{m} . The map $t \cdot X \rightarrow X$ is induced by $G \rightarrow G, g \mapsto tgt^{-1}$, and hence the differential of $t \cdot$ at o is induced by $\text{Ad}(t): \mathfrak{m} \rightarrow \mathfrak{m}$. Passing from \mathbb{R} to \mathbb{C} and considering the decomposition of $\mathfrak{m}_{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$, we can conclude. \blacksquare

Let Φ_M^+ be the subset of $X^*(\mathbf{T}_{\bar{\mathbb{Q}}})$ defined as above Proposition 5.5. Then $\mathfrak{m}^+ = \bigoplus_{\chi \in \Phi_M^+} \mathfrak{g}_{\mathbb{C}}^{\chi}$, with $\mathfrak{g}_{\mathbb{C}}^{\chi} = \{x \in \mathfrak{g}_{\mathbb{C}} : \text{Ad}(t)(x) = \chi(t)x \text{ for all } t \in \mathbf{T}(\mathbb{C})\}$ of dimension 1.

Proposition 5.7. *For each $\chi \in \Phi_M^+$,*

$$\mathfrak{m}_{\bar{\mathbb{Q}}, \text{arith}}^{+, \chi} := \mathfrak{m}_{\bar{\mathbb{Q}}, \text{arith}}^+ \cap \mathfrak{g}_{\mathbb{C}}^\chi$$

is a $\bar{\mathbb{Q}}$ -space of dimension 1 (equivalently, $\mathfrak{m}_{\bar{\mathbb{Q}}, \text{arith}}^{+, \chi} \otimes \mathbb{C} = \mathfrak{g}_{\mathbb{C}}^\chi$).

As a consequence, $\mathfrak{m}^+ = \bigoplus_{\chi \in \Phi_M^+} \mathfrak{m}_{\bar{\mathbb{Q}}, \text{arith}}^{+, \chi}$.

Proof. By Proposition 5.6 and the definition of the arithmetic $\bar{\mathbb{Q}}$ -structure $\mathfrak{m}_{\bar{\mathbb{Q}}, \text{arith}}^+$ on \mathfrak{m}^+ , $\mathbf{T}(\mathbb{Q})$ acts on $\mathfrak{m}_{\bar{\mathbb{Q}}, \text{arith}}^+$ via $t \cdot x = \text{Ad}(t)(x)$. For each $\chi \in \Phi_M^+ \subseteq X^*(\mathbf{T}_{\bar{\mathbb{Q}}})$, the weight space associated with χ (defined as $\{x \in \mathfrak{m}_{\bar{\mathbb{Q}}, \text{arith}}^+ : \text{Ad}(t)(x) = \chi(t)x \text{ for all } t \in \mathbf{T}(\bar{\mathbb{Q}})\}$) is precisely $\mathfrak{m}_{\bar{\mathbb{Q}}, \text{arith}}^{+, \chi}$. Extend this notation and write $\mathfrak{m}_{\bar{\mathbb{Q}}, \text{arith}}^{+, \chi}$ for the weight space associated with χ for each $\chi \in X^*(\mathbf{T}_{\bar{\mathbb{Q}}})$.

The general theory of algebraic tori and their character groups then asserts that

$$\mathfrak{m}_{\bar{\mathbb{Q}}, \text{arith}}^+ = \bigoplus_{\chi \in X^*(\mathbf{T}_{\bar{\mathbb{Q}}})} \mathfrak{m}_{\bar{\mathbb{Q}}, \text{arith}}^{+, \chi}.$$

Tensoring \mathbb{C} on both sides, we get

$$\mathfrak{m}^+ = \bigoplus_{\chi \in \Phi_M^+} (\mathfrak{m}_{\bar{\mathbb{Q}}, \text{arith}}^{+, \chi} \otimes \mathbb{C}) \oplus \bigoplus_{\chi \notin \Phi_M^+} (\mathfrak{m}_{\bar{\mathbb{Q}}, \text{arith}}^{+, \chi} \otimes \mathbb{C}).$$

By Lemma 5.4, $\mathfrak{m}^+ = \bigoplus_{\chi \in \Phi_M^+} \mathfrak{g}_{\mathbb{C}}^\chi$. Thus the conclusion follows because

$$\mathfrak{m}_{\bar{\mathbb{Q}}, \text{arith}}^{+, \chi} \otimes \mathbb{C} \subseteq \mathfrak{g}_{\mathbb{C}}^\chi \quad \text{for each } \chi \in X^*(\mathbf{T}_{\bar{\mathbb{Q}}}). \quad \blacksquare$$

5.4. Conclusion for Theorem 5.1

By Propositions 5.5 and 5.7, the bi- $\bar{\mathbb{Q}}$ -structure on $T_o X = \mathfrak{m}^+$ is split. This establishes Theorem 5.1. In fact, these two propositions yield a more precise statement as follows.

Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{m}$ be the Cartan decomposition of $\mathfrak{g}_{\mathbb{R}} := \text{Lie } \mathbf{G}^{\text{der}}(\mathbb{R})$ associated with the special point $o \in X$ by (SV2). Let \mathbf{T} be a maximal torus in \mathbf{G} . By Propositions 5.5 and 5.7, we have the following.

Theorem 5.1'. *Endow $T_o X$ with the bi- $\bar{\mathbb{Q}}$ -structure from Sections 4.2 and 4.3. Then $\mathbf{T}(\mathbb{Q})$ acts on $T_o X$ via bi- $\bar{\mathbb{Q}}$ -automorphisms and the splitting under this action gives the bi- $\bar{\mathbb{Q}}$ -splitting of $T_o X$.*

6. De Rham–Betti comparison in family and the Kodaira–Spencer map

In this section, we recall the comparison of the relative de Rham cohomology and the Betti cohomology for families of abelian varieties. These are useful in the computation of the periods of the Siegel modular varieties in the next section Section 7.

Let $k \subseteq \mathbb{C}$ be an algebraically closed field. Let S be a smooth irreducible variety and let $f: \mathcal{A} \rightarrow S$ be an abelian scheme of relative dimension g , all defined over k . Assume that \mathcal{A} carries a principal polarization, i.e., an isomorphism of abelian schemes $\lambda: \mathcal{A} \xrightarrow{\sim} \mathcal{A}^\dagger$ where \mathcal{A}^\dagger is the dual abelian scheme of \mathcal{A}/S .

Let $u: \tilde{S} \rightarrow S^{\text{an}}$ be the uniformization in the category of complex analytic spaces. Set

$$\begin{array}{ccc} \mathcal{A}_{\tilde{S}} := \tilde{S} \times_S \mathcal{A} & \longrightarrow & \mathcal{A} \\ \tilde{f} \downarrow \lrcorner & & \downarrow f \\ \tilde{S} & \xrightarrow{u} & S. \end{array} \quad (6.1)$$

Then $\mathcal{A}_{\tilde{S}}/\tilde{S}$ is a family of abelian varieties which carries a principal polarization.

For each $s \in S(\mathbb{C})$ (resp. $\tilde{s} \in \tilde{S}$), denote by $\mathcal{A}_s := f^{-1}(s)$ (resp. $\mathcal{A}_{\tilde{s}} := \tilde{f}^{-1}(\tilde{s})$). Notice that for each $\tilde{s} \in \tilde{S}$, $\mathcal{A}_{\tilde{s}}$ can be canonically identified with $\mathcal{A}_{u(\tilde{s})}$.

We have a canonical short exact sequence

$$0 \longrightarrow f^* \Omega_S^1 \longrightarrow \Omega_{\mathcal{A}}^1 \longrightarrow \Omega_{\mathcal{A}/S}^1 \longrightarrow 0. \quad (6.2)$$

It is known that the de Rham complex $\Omega_{\mathcal{A}/S}^\bullet = (\mathcal{O}_{\mathcal{A}} \rightarrow \Omega_{\mathcal{A}/S}^1 \rightarrow \Omega_{\mathcal{A}/S}^2 \rightarrow \dots)$ is a resolution of $f^{-1}\mathcal{O}_S$.

6.1. De Rham cohomology and symplectic basis

The relative de Rham bundle on S is defined as follows. Let $\mathcal{H}_{\text{dR}}^1(\mathcal{A}/S) := R^1 f_* \Omega_{\mathcal{A}/S}^\bullet$; it is a locally free sheaf of rank $2g$, which we view as a vector bundle of rank $2g$ over S . We have a subbundle $\Omega_{\mathcal{A}} := f_* \Omega_{\mathcal{A}/S}^1$ of $\mathcal{H}_{\text{dR}}^1(\mathcal{A}/S) \rightarrow S$ which has rank g ; over $s \in S(k)$, the fiber $\Omega_{\mathcal{A},s}$ is precisely $\Omega_{\mathcal{A}_s}$ and hence consists of invariant holomorphic differentials on \mathcal{A}_s .

Set

$$\mathcal{H}_{\text{dR}}^1(\mathcal{A}_{\tilde{S}}/\tilde{S}) := u^* \mathcal{H}_{\text{dR}}^1(\mathcal{A}/S) \quad \text{and} \quad \Omega_{\mathcal{A}_{\tilde{S}}} := u^* \Omega_{\mathcal{A}}. \quad (6.3)$$

Restricted to each $\tilde{s} \in \tilde{S}$, the inclusion $\Omega_{\mathcal{A}_{\tilde{s}}} \subseteq \mathcal{H}_{\text{dR}}^1(\mathcal{A}_{\tilde{S}}/\tilde{S})$ becomes $\Omega_{\mathcal{A}_{\tilde{s}}} \subseteq H_{\text{dR}}^1(\mathcal{A}_{\tilde{s}})$.

The advantage of using $\Omega_{\mathcal{A}_{\tilde{s}}} \subseteq \mathcal{H}_{\text{dR}}^1(\mathcal{A}_{\tilde{S}}/\tilde{S})$ instead of working directly on S is that we can take global basis. More precisely, we have the following.

Construction 6.1. There exists a *global* basis $\{\omega_1, \dots, \omega_g\}$ of $\Omega_{\mathcal{A}_{\tilde{s}}}$,³ which can be completed into a global basis $\{\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g\}$ of $\mathcal{H}_{\text{dR}}^1(\mathcal{A}_{\tilde{S}}/\tilde{S})$, with $\eta_j(\tilde{s})$ being the complex conjugate of $\omega_j(\tilde{s})$ for each $\tilde{s} \in \tilde{S}$.

Moreover, if $\tilde{s} \in \tilde{S}$ satisfies that $\mathcal{A}_{\tilde{s}}$ is a CM abelian variety, then $E := \text{End}(\mathcal{A}_{\tilde{s}}) \otimes \mathbb{Q}$ acts on $\Omega_{\mathcal{A}_{\tilde{s}}}$ and we can make a choice such that each $\omega_i(\tilde{s})$ is an eigenvector for this action.

³Namely, the ω_j 's are sections of $\Omega_{\mathcal{A}_{\tilde{s}}} \rightarrow \tilde{S}$ such that $\sum_{j=1}^g \mathbb{C} \omega_j(\tilde{s}) = \Omega_{\mathcal{A}_{\tilde{s}}}$ for each $\tilde{s} \in \tilde{S}$.

Proof. The existence of $\{\omega_1, \dots, \omega_g\}$ holds true since \tilde{S} is simply-connected. If $\mathcal{A}_{\tilde{S}}$ is a CM abelian variety, then by looking at the CM type, we can make a choice such that each $\omega_i(\tilde{s})$ is an eigenvector for the action of E on $\Omega_{\mathcal{A}_{\tilde{s}}}$.

The set $\{\eta_1, \dots, \eta_g\}$ can be constructed as follows. Apply (6.1) and (6.3) to the dual abelian scheme $\mathcal{A}^t \rightarrow S$. Then we obtain a family of abelian varieties $\mathcal{A}_{\tilde{S}}^t = \tilde{S} \times_S \mathcal{A}^t \rightarrow \tilde{S}$ and a vector bundle $\Omega_{\mathcal{A}_{\tilde{S}}^t} \rightarrow \tilde{S}$. The principal polarization $\lambda: \mathcal{A} \simeq \mathcal{A}^t$ induces an isomorphism $\lambda^*: \Omega_{\mathcal{A}_{\tilde{S}}^t} \simeq \Omega_{\mathcal{A}_{\tilde{S}}}$ of vector bundles on \tilde{S} , and hence we obtain a global basis $\{\lambda^*(\omega_1), \dots, \lambda^*(\omega_g)\}$ of $\Omega_{\mathcal{A}_{\tilde{S}}^t}$. For the exact sequence

$$0 \longrightarrow \Omega_{\mathcal{A}_{\tilde{S}}} \longrightarrow \mathcal{H}_{\text{dR}}^1(\mathcal{A}_{\tilde{S}}/\tilde{S}) \longrightarrow \text{Lie}(\mathcal{A}_{\tilde{S}}^t/\tilde{S}) = (\Omega_{\mathcal{A}_{\tilde{S}}^t})^\vee \longrightarrow 0, \quad (6.4)$$

$\{2\pi i \lambda^*(\omega_1)^\vee, \dots, 2\pi i \lambda^*(\omega_g)^\vee\}$ lifts to a set of global sections $\{\eta_1, \dots, \eta_g\}$ of $\mathcal{H}_{\text{dR}}^1(\mathcal{A}_{\tilde{S}}/\tilde{S})$. For each $s \in \tilde{S}$, use ρ to denote the complex conjugation on $H_{\text{dR}}^1(\mathcal{A}_{\tilde{s}})$. Then $\eta_j(\tilde{s}) = \rho(\omega_j(\tilde{s}))$ because $\rho(\Omega_{\mathcal{A}_{\tilde{s}}}) \xrightarrow{\sim} \text{Lie}(\mathcal{A}_{\tilde{s}}^t)$, $\rho(\omega) \mapsto 2\pi i(\lambda^*\omega)^\vee$. We are done. ■

Remark 6.2. If $s \in S(k)$ and $\tilde{s} \in \tilde{S}$ lies above s (i.e., $u(\tilde{s}) = s$), then the short exact sequence (6.4) restricted over \tilde{s} becomes $0 \rightarrow \Omega_{\mathcal{A}_s} \rightarrow H_{\text{dR}}^1(\mathcal{A}_s) \rightarrow \text{Lie}(\mathcal{A}_s^t) \rightarrow 0$, which is defined over k .

6.2. Betti (co)homology and symplectic basis

Let $\underline{\mathbb{Z}}_{\mathcal{A}}$ be the locally constant sheaf of \mathbb{Z} on \mathcal{A} . Write $R_1 f_* \underline{\mathbb{Z}}_{\mathcal{A}} := (R^1 f_* \underline{\mathbb{Z}}_{\mathcal{A}})^\vee$. Then $R_1 f_* \underline{\mathbb{Z}}_{\mathcal{A}}$ is a local system on S such that $(R_1 f_* \underline{\mathbb{Z}}_{\mathcal{A}})_s = H_1(\mathcal{A}_s, \mathbb{Z})$ for each $s \in S(k)$. Use $\mathbb{V}(\mathcal{A}/S)$ to denote the vector bundle over S associated with $R_1 f_* \underline{\mathbb{Z}}_{\mathcal{A}} \otimes_{\mathbb{Z}} \mathcal{O}_S$, i.e., $\mathbb{V}(\mathcal{A}/S) = \underline{\text{Spec}}_S(\text{Sym}(R_1 f_* \underline{\mathbb{Z}}_{\mathcal{A}} \otimes_{\mathbb{Z}} \mathcal{O}_S))$.

Set

$$R_1 \tilde{f}_* \underline{\mathbb{Z}} := u^*(R_1 f_* \underline{\mathbb{Z}}_{\mathcal{A}}) \quad \text{and} \quad \mathbb{V}(\mathcal{A}_{\tilde{S}}/\tilde{S}) := u^* \mathbb{V}(\mathcal{A}/S). \quad (6.5)$$

Then $R_1 \tilde{f}_* \underline{\mathbb{Z}}$ is a local system on \tilde{S} with $(R_1 \tilde{f}_* \underline{\mathbb{Z}})_{\tilde{s}} = H_1(\mathcal{A}_{\tilde{s}}, \mathbb{Z})$ for each $\tilde{s} \in \tilde{S}$, and $\mathbb{V}(\mathcal{A}_{\tilde{S}}/\tilde{S}) \rightarrow \tilde{S}$ is a vector bundle whose sheaf of sections is $R_1 \tilde{f}_* \underline{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_{\tilde{S}}$.

Since \tilde{S} is simply-connected, we can take a set of *global* sections $\{\gamma_1, \dots, \gamma_{2g}\}$ of $R_1 \tilde{f}_* \underline{\mathbb{Z}}$ which is a global basis of $\mathbb{V}(\mathcal{A}_{\tilde{S}}/\tilde{S}) \rightarrow \tilde{S}$, i.e., we have the following \tilde{S} -isomorphism

$$\mathbb{C}^{2g} \times \tilde{S} \xrightarrow{\sim} \mathbb{V}(\mathcal{A}_{\tilde{S}}/\tilde{S}), \quad ((k_1, \dots, k_{2g}), \tilde{s}) \mapsto \sum_{j=1}^{2g} k_j \gamma_j(\tilde{s}). \quad (6.6)$$

The basis $\{\gamma_1, \dots, \gamma_{2g}\}$ can be furthermore chosen to be *symplectic* in the following sense. The principal polarization on $\mathcal{A}_{\tilde{S}}$ endows, for each $\tilde{s} \in \tilde{S}$, $(R_1 \tilde{f}_* \underline{\mathbb{Z}})_{\tilde{s}} = H_1(\mathcal{A}_{\tilde{s}}, \mathbb{Z})$ with a symplectic form $\Psi_{\tilde{s}}$, which furthermore induces a pairing on \mathbb{Z}^{2g} via

$$\mathbb{Z}^{2g} \xrightarrow{\sim} H_1(\mathcal{A}_{\tilde{s}}, \mathbb{Z}), \quad (k_1, \dots, k_{2g}) \mapsto \sum_{j=1}^{2g} k_j \gamma_j(\tilde{s}).$$

We say that $\{\gamma_1, \dots, \gamma_{2g}\}$ is *symplectic* if this induced pairing on \mathbb{Z}^{2g} is $\begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix}$ for each $\tilde{s} \in \tilde{S}$.

Finally, notice that the natural differential map $\mathcal{O}_{\tilde{S}} \rightarrow \Omega_{\tilde{S}}^1$ gives a natural map

$$R_1 \tilde{f}_* \mathbb{Z} \otimes \mathcal{O}_{\tilde{S}} \longrightarrow R_1 \tilde{f}_* \mathbb{Z} \otimes \Omega_{\tilde{S}}^1,$$

which in turn becomes a connection on $\mathbb{V}(\mathcal{A}_{\tilde{S}}/\tilde{S})^\vee$. Denote this connection by d .

6.3. Gauss–Manin connection

Taking exterior powers, the canonical short exact sequence (6.2) yields a decreasing filtration $\Omega_{\mathcal{A}}^\bullet = \text{Fil}^0 \Omega_{\mathcal{A}}^\bullet \supseteq \text{Fil}^1 \Omega_{\mathcal{A}}^\bullet \supseteq \dots$ on $\Omega_{\mathcal{A}}^\bullet$ such that $\text{Fil}^r \Omega_{\mathcal{A}}^\bullet / \text{Fil}^{r+1} \Omega_{\mathcal{A}}^\bullet = f^* \Omega_S^r \otimes_{\mathcal{O}_{\mathcal{A}}} \Omega_{\mathcal{A}/S}^{\bullet-r}$. Thus from $0 \rightarrow \text{Fil}^1 / \text{Fil}^2 \rightarrow \text{Fil}^0 / \text{Fil}^2 \rightarrow \text{Fil}^0 / \text{Fil}^1 \rightarrow 0$, we obtain

$$0 \longrightarrow f^* \Omega_S^1 \otimes_{\mathcal{O}_{\mathcal{A}}} \Omega_{\mathcal{A}/S}^{\bullet-1} \longrightarrow \Omega_{\mathcal{A}}^\bullet / \text{Fil}^2 \longrightarrow \Omega_{\mathcal{A}/S}^\bullet \longrightarrow 0.$$

Thus we have a connection map

$$R^1 f_* \Omega_{\mathcal{A}/S}^\bullet \longrightarrow R^2 f_*(f^* \Omega_S^1 \otimes_{\mathcal{O}_{\mathcal{A}}} \Omega_{\mathcal{A}/S}^{\bullet-1}) = \Omega_S^1 \otimes R^1 f_* \Omega_{\mathcal{A}/S}^\bullet.$$

Recall the definition of the relative de Rham cohomology $\mathcal{H}_{\text{dR}}^1(\mathcal{A}/S) = R^1 f_* \Omega_{\mathcal{A}/S}^\bullet$. Then the co-boundary map above is precisely the *Gauss–Manin connection*

$$\nabla_{\text{GM}}: \mathcal{H}_{\text{dR}}^1(\mathcal{A}/S) \longrightarrow \mathcal{H}_{\text{dR}}^1(\mathcal{A}/S) \otimes \Omega_S^1.$$

By abuse of notation, we also use ∇_{GM} to denote the Gauss–Manin connection

$$\mathcal{H}_{\text{dR}}^1(\mathcal{A}_{\tilde{S}}/\tilde{S}) \longrightarrow \mathcal{H}_{\text{dR}}^1(\mathcal{A}_{\tilde{S}}/\tilde{S}) \otimes \Omega_{\tilde{S}}^1.$$

6.4. Comparison of de Rham and Betti cohomologies

By the theory of vector bundles with connections, there is an isomorphism of vector bundles over \tilde{S}

$$\beta: (\mathcal{H}_{\text{dR}}^1(\mathcal{A}_{\tilde{S}}/\tilde{S}), \nabla_{\text{GM}}) \xrightarrow{\sim} (\mathbb{V}(\mathcal{A}_{\tilde{S}}/\tilde{S})^\vee, d). \quad (6.7)$$

Set $\Omega_1(\tilde{s}) = [\int_{\gamma_l} \omega_j(\tilde{s})]_{1 \leq j, l \leq g}$, $\Omega_2(\tilde{s}) = [\int_{\gamma_{g+l}} \omega_j(\tilde{s})]_{1 \leq j, l \leq g}$, $N_1(\tilde{s}) = [\int_{\gamma_l} \eta_j(\tilde{s})]_{1 \leq j, l \leq g}$, and $N_2(\tilde{s}) = [\int_{\gamma_{g+l}} \eta_j(\tilde{s})]_{1 \leq j, l \leq g}$. Under the global basis $\{\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g\}$ of $\mathcal{H}_{\text{dR}}^1(\mathcal{A}_{\tilde{S}}/\tilde{S})$ and the global basis $\{\gamma_1^\vee, \dots, \gamma_{2g}^\vee\}$ of $\mathbb{V}(\mathcal{A}_{\tilde{S}}/\tilde{S})^\vee$, the isomorphism β is represented by the matrix

$$\beta_{\tilde{s}} := \begin{bmatrix} \Omega_1(\tilde{s}) & N_1(\tilde{s}) \\ \Omega_2(\tilde{s}) & N_2(\tilde{s}) \end{bmatrix} \quad (6.8)$$

over each $\tilde{s} \in \tilde{S}$, i.e., the following diagram (of \tilde{S} -morphisms) commutes

$$\begin{array}{ccc} \mathcal{H}_{\text{dR}}^1(\mathcal{A}_{\tilde{S}}/\tilde{S}) & \xrightarrow{\sim} & \mathbb{V}(\mathcal{A}_{\tilde{S}}/\tilde{S})^\vee \\ \uparrow \simeq & & \uparrow \simeq \\ \mathbb{C}^{2g} \times \tilde{S} & \xrightarrow[\substack{(\mathbf{v}, \tilde{s}) \mapsto (\beta_{\tilde{s}} \mathbf{v}, \tilde{s})}]{} & \mathbb{C}^{2g} \times \tilde{S} \end{array} \quad (6.9)$$

where the left isomorphism is by sending

$$((l_1, \dots, l_g, l_{g+1}, \dots, l_{2g}), \tilde{s}) \mapsto \sum_{j=1}^g l_j \omega_j(\tilde{s}) + \sum_{j=1}^g l_{g+j} \eta_j(\tilde{s}),$$

and the right isomorphism is defined by sending $((k_1, \dots, k_{2g}), \tilde{s}) \mapsto \sum_{j=1}^{2g} k_j \gamma_j^\vee(\tilde{s})$.

We take a closer look at this comparison. Let $\tilde{s} \in \tilde{S}$. Set $\tau_{\tilde{s}} := \Omega_2(\tilde{s})\Omega_1(\tilde{s})^{-1}$. Under the basis $\{\gamma_1^\vee(\tilde{s}), \dots, \gamma_{2g}^\vee(\tilde{s})\}$ of $\mathbb{V}(\mathcal{A}_{\tilde{S}}/\tilde{S})_{\tilde{s}}^\vee = H^1(\mathcal{A}_{\tilde{s}}, \mathbb{C})$, we have

$$\beta_{\tilde{s}}(\Omega_{\mathcal{A}_{\tilde{s}}}) = \{(\Omega_1(\tilde{s})\mathbf{x}, \Omega_2(\tilde{s})\mathbf{x}) : \mathbf{x} \in \mathbb{C}^g\}.$$

It is known that $\Omega_1(\tilde{s})$ is invertible for each $\tilde{s} \in \tilde{S}$. Hence

$$\beta_{\tilde{s}}(\Omega_{\mathcal{A}_{\tilde{s}}}) = \{(\mathbf{x}, \Omega_2(\tilde{s})\Omega_1(\tilde{s})^{-1}\mathbf{x}) : \mathbf{x} \in \mathbb{C}^g\} = \{(\mathbf{x}, \tau_{\tilde{s}} \cdot \mathbf{x}) : \mathbf{x} \in \mathbb{C}^g\}. \quad (6.10)$$

More precisely, $\beta_{\tilde{s}}(\Omega_{\mathcal{A}_{\tilde{s}}})$ is the subspace of $\bigoplus_{j=1}^{2g} \mathbb{C} \gamma_j^\vee(\tilde{s})$ consisting of the vectors of the form $[\gamma_1^\vee(\tilde{s}) \ \dots \ \gamma_g^\vee(\tilde{s})] \mathbf{x} + [\gamma_{g+1}^\vee(\tilde{s}) \ \dots \ \gamma_{2g}^\vee(\tilde{s})] \tau_{\tilde{s}} \cdot \mathbf{x}$, with \mathbf{x} running over all (column) vectors in \mathbb{C}^g . Taking \mathbf{x} to be the vector with 1 on the j -th entry and 0 elsewhere with j running over $\{1, \dots, g\}$, we obtain a basis of $\beta_{\tilde{s}}(\Omega_{\mathcal{A}_{\tilde{s}}})$. Notice that this basis is precisely the columns vectors of $[\gamma_1^\vee(\tilde{s}) \ \dots \ \gamma_g^\vee(\tilde{s})] + [\gamma_{g+1}^\vee(\tilde{s}) \ \dots \ \gamma_{2g}^\vee(\tilde{s})] \tau_{\tilde{s}}$. The comparison $\beta_{\tilde{s}}$ induces the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega_{\mathcal{A}_{\tilde{s}}} & \longrightarrow & H_{\text{dR}}^1(\mathcal{A}_{\tilde{s}}) & \longrightarrow & \text{Lie}(\mathcal{A}_{\tilde{s}}^t) = \Omega_{\mathcal{A}_{\tilde{s}}^t}^\vee \longrightarrow 0 \\ & & \downarrow & & \downarrow \beta_{\tilde{s}} & & \downarrow \bar{\beta}_{\tilde{s}} \\ 0 & \longrightarrow & \beta_{\tilde{s}}(\Omega_{\mathcal{A}_{\tilde{s}}}) & \longrightarrow & H^1(\mathcal{A}_{\tilde{s}}, \mathbb{C}) & \longrightarrow & H^1(\mathcal{A}_{\tilde{s}}, \mathbb{C})/\beta_{\tilde{s}}(\Omega_{\mathcal{A}_{\tilde{s}}}) \longrightarrow 0. \end{array} \quad (6.11)$$

For each $j \in \{1, \dots, 2g\}$, write $\bar{\gamma}_j^\vee(\tilde{s})$ for the image of $\gamma_j^\vee(\tilde{s})$ under quotient

$$H^1(\mathcal{A}_{\tilde{s}}, \mathbb{C}) \longrightarrow H^1(\mathcal{A}_{\tilde{s}}, \mathbb{C})/\beta_{\tilde{s}}(\Omega_{\mathcal{A}_{\tilde{s}}}).$$

Then $\{\bar{\gamma}_{g+1}^\vee(\tilde{s}), \dots, \bar{\gamma}_{2g}^\vee(\tilde{s})\}$ is a basis of $H^1(\mathcal{A}_{\tilde{s}}, \mathbb{C})/\beta_{\tilde{s}}(\Omega_{\mathcal{A}_{\tilde{s}}})$.

The following lemma follows from a direct computation.

Lemma 6.3. *We have*

- (i) *Under the basis $\{\omega_1(\tilde{s}), \dots, \omega_g(\tilde{s})\}$ of $\Omega_{\mathcal{A}_{\tilde{s}}}$ and the (natural choice of) basis of $\beta_{\tilde{s}}(\Omega_{\mathcal{A}_{\tilde{s}}})$ below (6.10), the matrix for $\beta_{\tilde{s}}|_{\Omega_{\mathcal{A}_{\tilde{s}}}}$ is $\Omega_1(\tilde{s})$.*
- (ii) *Under the basis $\{\bar{\eta}_1(\tilde{s}), \dots, \bar{\eta}_g(\tilde{s})\}$ of $\text{Lie}(\mathcal{A}_{\tilde{s}}^t)$ and the basis $\{\bar{\gamma}_{g+1}^\vee(\tilde{s}), \dots, \bar{\gamma}_{2g}^\vee(\tilde{s})\}$ of $H^1(\mathcal{A}_{\tilde{s}}, \mathbb{C})/\beta_{\tilde{s}}(\Omega_{\mathcal{A}_{\tilde{s}}})$, the matrix for $\bar{\beta}_{\tilde{s}}$ is $N_2(\tilde{s}) - \Omega_2(\tilde{s})\Omega_1(\tilde{s})^{-1}N_1(\tilde{s}) = N_2(\tilde{s}) - \tau_{\tilde{s}}N_1(\tilde{s})$.*

Recall from Construction 6.1 that $\bar{\eta}_j$ is $(\lambda^* \omega_j)^\vee$, where $\lambda: \mathcal{A}_{\tilde{S}} \xrightarrow{\sim} \mathcal{A}_{\tilde{S}}^t$ is the principal polarization.

Proof. We only prove part (i). Part (ii) follows from a similar computation.

From (6.8), we have

$$\begin{aligned}
& [\beta_{\tilde{s}}(\omega_1(\tilde{s})) \quad \cdots \quad \beta_{\tilde{s}}(\omega_g(\tilde{s}))] \\
&= [\gamma_1^{\vee}(\tilde{s}) \quad \cdots \quad \gamma_{2g}^{\vee}(\tilde{s})] \cdot \begin{bmatrix} \Omega_1(\tilde{s}) \\ \Omega_2(\tilde{s}) \end{bmatrix} \\
&= [\gamma_1^{\vee}(\tilde{s}) \quad \cdots \quad \gamma_g^{\vee}(\tilde{s})] \Omega_1(\tilde{s}) + [\gamma_{g+1}^{\vee}(\tilde{s}) \quad \cdots \quad \gamma_{2g}^{\vee}(\tilde{s})] \Omega_2(\tilde{s}) \\
&= [\gamma_1^{\vee}(\tilde{s}) \quad \cdots \quad \gamma_g^{\vee}(\tilde{s})] \Omega_1(\tilde{s}) + [\gamma_{g+1}^{\vee}(\tilde{s}) \quad \cdots \quad \gamma_{2g}^{\vee}(\tilde{s})] \tau_{\tilde{s}} \Omega_1(\tilde{s}) \\
&= ([\gamma_1^{\vee}(\tilde{s}) \quad \cdots \quad \gamma_g^{\vee}(\tilde{s})] + [\gamma_{g+1}^{\vee}(\tilde{s}) \quad \cdots \quad \gamma_{2g}^{\vee}(\tilde{s})] \tau_{\tilde{s}}) \cdot \Omega_1(\tilde{s}).
\end{aligned}$$

We are done. \blacksquare

6.5. Over CM fibers

In this subsection, we assume $k = \bar{\mathbb{Q}}$. Fix a point $\tilde{s} \in \tilde{S}$ such that $\mathcal{A}_{\tilde{s}}$ is a CM abelian variety. Set $E := \text{End}(\mathcal{A}_{\tilde{s}}) \otimes \mathbb{Q}$.

All CM abelian varieties over \mathbb{C} are defined over $\bar{\mathbb{Q}}$. So $s := u(\tilde{s}) \in S(\bar{\mathbb{Q}})$. Moreover, it is known that $\tau_{\tilde{s}} \in \text{Mat}_{g \times g}(\bar{\mathbb{Q}})$. The basis $\{\omega_1(\tilde{s}), \dots, \omega_g(\tilde{s}), \eta_1(\tilde{s}), \dots, \eta_g(\tilde{s})\}$ is by choice a $\bar{\mathbb{Q}}$ -eigenbasis of $H_{\text{dR}}^1(\mathcal{A}_{\tilde{s}})$ for the action of E , and $\{\gamma_1^{\vee}(\tilde{s}), \dots, \gamma_{2g}^{\vee}(\tilde{s})\}$ is a $\bar{\mathbb{Q}}$ -basis of $H^1(\mathcal{A}_{\tilde{s}}, \bar{\mathbb{Q}})$.

Let $\theta_1, \dots, \theta_g$ be the holomorphic periods of $\mathcal{A}_{\tilde{s}}$.

Since $\tau_{\tilde{s}} \in \text{Mat}_{g \times g}(\bar{\mathbb{Q}})$, the subspace

$$\beta_{\tilde{s}}(\Omega_{\mathcal{A}_{\tilde{s}}}) = \{(\mathbf{x}, \tau_{\tilde{s}} \mathbf{x}) : \mathbf{x} \in \mathbb{C}^g\}$$

of $H^1(\mathcal{A}_{\tilde{s}}, \mathbb{C}) = \bigoplus \mathbb{C} \gamma_j^{\vee}(\tilde{s})$ is rational for the $\bar{\mathbb{Q}}$ -structure given by $H^1(\mathcal{A}_{\tilde{s}}, \bar{\mathbb{Q}}) \otimes \mathbb{C}$. Thus $H^1(\mathcal{A}_{\tilde{s}}, \bar{\mathbb{Q}})$ gives a $\bar{\mathbb{Q}}$ -structure on $H^1(\mathcal{A}_{\tilde{s}}, \mathbb{C})/\beta_{\tilde{s}}(\Omega_{\mathcal{A}_{\tilde{s}}})$, for which $\{\bar{\gamma}_{g+1}^{\vee}(\tilde{s}), \dots, \bar{\gamma}_{2g}^{\vee}(\tilde{s})\}$ is a $\bar{\mathbb{Q}}$ -basis. Moreover, the action of E on $H^1(\mathcal{A}_{\tilde{s}}, \bar{\mathbb{Q}})$ induces an action of E on $H^1(\mathcal{A}_{\tilde{s}}, \mathbb{C})/\beta_{\tilde{s}}(\Omega_{\mathcal{A}_{\tilde{s}}})$ which preserves its $\bar{\mathbb{Q}}$ -structure. Therefore we can find a $\bar{\mathbb{Q}}$ -eigenbasis $\{\mathbf{f}_1, \dots, \mathbf{f}_g\}$ of $H^1(\mathcal{A}_{\tilde{s}}, \mathbb{C})/\beta_{\tilde{s}}(\Omega_{\mathcal{A}_{\tilde{s}}})$ with respect to the action of E .

Lemma 6.4. *Under the $\bar{\mathbb{Q}}$ -basis $\{\bar{\eta}_1(\tilde{s}), \dots, \bar{\eta}_g(\tilde{s})\}$ of $\text{Lie}(\mathcal{A}_{\tilde{s}}^1)$ and a choice of the $\bar{\mathbb{Q}}$ -eigenbasis $\{\mathbf{f}_1, \dots, \mathbf{f}_g\}$ of $H^1(\mathcal{A}_{\tilde{s}}, \mathbb{C})/\beta_{\tilde{s}}(\Omega_{\mathcal{A}_{\tilde{s}}})$ as above, the matrix for $\bar{\beta}_{\tilde{s}}$ is $\text{diag}(\pi \theta_1^{-1}, \dots, \pi \theta_g^{-1})$.*

Proof. Since both bases are eigenbases for the actions of E and the linear map $\bar{\beta}_{\tilde{s}}$ is E -equivariant (because $\beta_{\tilde{s}}$ is E -equivariant), the matrix for $\bar{\beta}_{\tilde{s}}$ under these bases is diagonal up to reordering $\mathbf{f}_1, \dots, \mathbf{f}_g$. Call this matrix D .

Let us look at the matrix $N_2(\tilde{s}) - \tau_{\tilde{s}} N_1(\tilde{s})$. By our choice of the basis, $\eta_j(\tilde{s})$ is the complex conjugate of $\omega_j(\tilde{s})$ for each $j \in \{1, \dots, g\}$. Since $\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g$ are eigenforms for the CM action, the reciprocity law for the differential forms of the 1st and the 2nd spaces implies that $\int_{\gamma} \eta_j(\tilde{s}) \simeq 2\pi i / \theta_j$ for any $\gamma \in H_1(\mathcal{A}_{\tilde{s}}, \mathbb{Z})$; see [5, p. 36, equation (3)] for more details. So both $N_1(\tilde{s})$ and $N_2(\tilde{s})$ are in $\text{Mat}_{g \times g}(\bar{\mathbb{Q}}) \cdot \text{diag}(\pi \theta_1^{-1}, \dots, \pi \theta_g^{-1})$. So

$N_2(\tilde{s}) - \tau_{\tilde{s}} N_1(\tilde{s})$ is in $\text{Mat}_{g \times g}(\bar{\mathbb{Q}}) \cdot \text{diag}(\pi\theta_1^{-1}, \dots, \pi\theta_g^{-1})$ since $\tau_{\tilde{s}} \in \text{Mat}_{g \times g}(\bar{\mathbb{Q}})$, i.e.,

$$N_2(\tilde{s}) - \tau_{\tilde{s}} N_1(\tilde{s}) = M \cdot \text{diag}(\pi\theta_1^{-1}, \dots, \pi\theta_g^{-1})$$

for some $M \in \text{Mat}_{g \times g}(\bar{\mathbb{Q}})$. Notice that M is an invertible matrix, since $\bar{\beta}_{\tilde{s}}$ is an isomorphism of \mathbb{C} -vector spaces.

Set $[v_1(\tilde{s}) \ \dots \ v_g(\tilde{s})] := [\bar{\gamma}_{g+1}^\vee(\tilde{s}) \ \dots \ \bar{\gamma}_{2g}^\vee(\tilde{s})] M$. By Lemma 6.3 (ii), under the basis $\{\bar{\eta}_1(\tilde{s}), \dots, \bar{\eta}_g(\tilde{s})\}$ of $\text{Lie}(\mathcal{A}_{\tilde{s}})$ and the basis $\{v_1(\tilde{s}), \dots, v_g(\tilde{s})\}$ of $H^1(\mathcal{A}_{\tilde{s}}, \mathbb{C})/\beta_{\tilde{s}}(\Omega_{\mathcal{A}_{\tilde{s}}})$, the matrix for $\bar{\beta}_{\tilde{s}}$ is $M \cdot \text{diag}(\pi\theta_1^{-1}, \dots, \pi\theta_g^{-1})$.

Now we have two $\bar{\mathbb{Q}}$ -bases $\{\mathbf{f}_1, \dots, \mathbf{f}_g\}$ and $\{v_1(\tilde{s}), \dots, v_g(\tilde{s})\}$ of $H^1(\mathcal{A}_{\tilde{s}}, \mathbb{C})/\beta_{\tilde{s}}(\Omega_{\mathcal{A}_{\tilde{s}}})$. Let $T \in \text{GL}_g(\bar{\mathbb{Q}})$ be the transition matrix. Then $D = (TM) \cdot \text{diag}(\pi\theta_1^{-1}, \dots, \pi\theta_g^{-1})$. So TM is diagonal because both D and $\text{diag}(\pi\theta_1^{-1}, \dots, \pi\theta_g^{-1})$ are diagonal. Thus we can conclude by replacing each \mathbf{f}_j by a suitable $\bar{\mathbb{Q}}$ -multiple. ■

Remark 6.5. Similar to the proof of Lemma 6.4 and using Lemma 6.3 (i), we can prove the following assertion. Under the basis $\{\omega_1(\tilde{s}), \dots, \omega_g(\tilde{s})\}$ of $\Omega_{\mathcal{A}_{\tilde{s}}}$ and a $\bar{\mathbb{Q}}$ -eigenbasis $\{\mathbf{e}_1, \dots, \mathbf{e}_g\}$ of $\beta_{\tilde{s}}(\Omega_{\mathcal{A}_{\tilde{s}}})$, the matrix for $\beta_{\tilde{s}}|_{\Omega_{\mathcal{A}_{\tilde{s}}}}$ is $\text{diag}(\theta_1, \dots, \theta_g)$.

Therefore the matrix for $\beta_{\tilde{s}}$ is $\text{diag}(\theta_1, \dots, \theta_g, \pi\theta_1^{-1}, \dots, \pi\theta_g^{-1})$ under suitable $\bar{\mathbb{Q}}$ -bases of $H_{\text{dR}}^1(\mathcal{A}_{\tilde{s}})$ and $H^1(\mathcal{A}_{\tilde{s}}, \mathbb{C})$.

6.6. Kodaira–Spencer map

Consider the maps defined over the field k

$$\begin{aligned} \Omega_{\mathcal{A}} &\subseteq \mathcal{H}_{\text{dR}}^1(\mathcal{A}/S) \xrightarrow{\nabla_{\text{GM}}} \mathcal{H}_{\text{dR}}^1(\mathcal{A}/S) \otimes_{\mathcal{O}_S} \Omega_S^1 \longrightarrow \mathcal{H}_{\text{dR}}^1(\mathcal{A}/S)/\Omega_{\mathcal{A}} \otimes_{\mathcal{O}_S} \Omega_S^1 \\ &\simeq \text{Lie}(\mathcal{A}^\dagger/S) \otimes_{\mathcal{O}_S} \Omega_S^1. \end{aligned} \quad (6.12)$$

Thus for the tangent bundle $TS = (\Omega_S^1)^\vee$ we have

$$TS \longrightarrow \text{Lie}(\mathcal{A}^\dagger/S) \otimes_{\mathcal{O}_S} \text{Lie}(\mathcal{A}/S). \quad (6.13)$$

Using the principal polarization $\lambda: \mathcal{A} \simeq \mathcal{A}^\dagger$, the right-hand side is isomorphic (over \mathcal{O}_S) to $\text{Lie}(\mathcal{A}/S) \otimes_{\mathcal{O}_S} \text{Lie}(\mathcal{A}/S) = \text{Sym}_{\mathcal{O}_S}^2 \text{Lie}(\mathcal{A}/S) \oplus \text{Alt}_{\mathcal{O}_S}^2 \text{Lie}(\mathcal{A}/S)$. We introduce the following (non-standard) notation:

$$S^2 \text{Lie}(\mathcal{A}/S) := (\lambda_*, 1)(\text{Sym}_{\mathcal{O}_S}^2 \text{Lie}(\mathcal{A}/S)) \subseteq \text{Lie}(\mathcal{A}^\dagger/S) \otimes_{\mathcal{O}_S} \text{Lie}(\mathcal{A}/S). \quad (6.14)$$

It is known that the image of (6.13) lies in $S^2 \text{Lie}(\mathcal{A}/S)$, i.e., we have

$$\text{KS}: TS \longrightarrow S^2 \text{Lie}(\mathcal{A}/S) \subseteq \text{Lie}(\mathcal{A}^\dagger/S) \otimes_{\mathcal{O}_S} \text{Lie}(\mathcal{A}/S). \quad (6.15)$$

Let $s \in S(k)$. Then (6.15) restricted over s becomes

$$\text{KS}_s: T_s S \longrightarrow S^2 \text{Lie}(\mathcal{A}_s) \subseteq \text{Lie} \mathcal{A}_s^\dagger \otimes \text{Lie} \mathcal{A}_s. \quad (6.16)$$

We close this section by the following discussion, which will be used in the computation of the periods of the Siegel modular varieties.

For the universal covering $u: \tilde{S} \rightarrow S$, the sequence of maps (6.12) exists with \mathcal{A}/S replaced by $\mathcal{A}_{\tilde{S}}/\tilde{S}$. Then combined with the isomorphism $\beta: (\mathcal{H}_{\text{dR}}^1(\mathcal{A}_{\tilde{S}}/\tilde{S}), \nabla_{\text{GM}}) \xrightarrow{\sim} (\mathbb{V}(\mathcal{A}_{\tilde{S}}/\tilde{S})^\vee, \text{d})$ from (6.7), we obtain

$$\begin{array}{ccccccc} \Omega_{\mathcal{A}_{\tilde{S}}} & \hookrightarrow & \mathcal{H}_{\text{dR}}^1(\mathcal{A}_{\tilde{S}}/\tilde{S}) & \xrightarrow{\nabla_{\text{GM}}} & \mathcal{H}_{\text{dR}}^1(\mathcal{A}_{\tilde{S}}/\tilde{S}) \otimes \Omega_{\tilde{S}}^1 & \longrightarrow & \mathcal{H}_{\text{dR}}^1(\mathcal{A}_{\tilde{S}}/\tilde{S}) / \Omega_{\mathcal{A}_{\tilde{S}}} \otimes \Omega_{\tilde{S}}^1 = \text{Lie}(\mathcal{A}_{\tilde{S}}^t) \otimes \Omega_{\tilde{S}}^1 \\ \downarrow & & \beta \downarrow & & \downarrow \beta \otimes \text{id} & & \downarrow \bar{\beta} \otimes \text{id} \\ \beta(\Omega_{\mathcal{A}_{\tilde{S}}}) & \hookrightarrow & \mathbb{V}(\mathcal{A}_{\tilde{S}}/\tilde{S})^\vee & \xrightarrow{\text{d}} & \mathbb{V}(\mathcal{A}_{\tilde{S}}/\tilde{S})^\vee \otimes \Omega_{\tilde{S}}^1 & \longrightarrow & \mathbb{V}(\mathcal{A}_{\tilde{S}}/\tilde{S})^\vee / \beta(\Omega_{\mathcal{A}_{\tilde{S}}}) \otimes \Omega_{\tilde{S}}^1, \end{array} \quad (6.17)$$

from which we get the commutative diagram

$$\begin{array}{ccc} T\tilde{S} & \xrightarrow{\text{KS} \circ \text{d}u} & \text{Lie}(\mathcal{A}_{\tilde{S}}^t) \otimes \text{Lie}(\mathcal{A}_{\tilde{S}}) \\ \downarrow & & \downarrow \bar{\beta} \otimes \beta^\vee \\ T\tilde{S} & \longrightarrow & \frac{\mathbb{V}(\mathcal{A}_{\tilde{S}}/\tilde{S})^\vee}{\beta(\Omega_{\mathcal{A}_{\tilde{S}}})} \otimes \beta(\Omega_{\mathcal{A}_{\tilde{S}}})^\vee. \end{array} \quad (6.18)$$

Take $\tilde{s} \in u^{-1}(s)$. Then (6.18) yields, after the natural identification $\mathcal{A}_{\tilde{s}} = \mathcal{A}_s$ under u ,

$$\begin{array}{ccc} T_s S & \xrightarrow{\text{KS}_s} & \text{Lie}(\mathcal{A}_{\tilde{s}}^t) \otimes \text{Lie}(\mathcal{A}_{\tilde{s}}) \\ \uparrow \simeq \text{d}u & & \downarrow \bar{\beta}_{\tilde{s}} \otimes \beta_{\tilde{s}}^\vee \\ T_{\tilde{s}} \tilde{S} & \longrightarrow & \frac{H^1(\mathcal{A}_{\tilde{s}}, \mathbb{C})}{\beta_{\tilde{s}}(\Omega_{\mathcal{A}_{\tilde{s}}})} \otimes \beta_{\tilde{s}}(\Omega_{\mathcal{A}_{\tilde{s}}})^\vee. \end{array} \quad (6.19)$$

7. Periods of the Siegel modular variety

In this section, we turn to the Siegel case. Let \mathbb{A}_g be the moduli space of principally polarized abelian varieties of dimension g . The associated connected Shimura datum is $(\text{GSp}_{2g}, \mathfrak{H}_g)$, where \mathfrak{H}_g is the Siegel upper half space $\{\tau \in \text{Mat}_{g \times g}(\mathbb{C}) : \tau = \tau^\top, \text{Im } \tau > 0\}$.

Let $o \in \mathfrak{H}_g$ be a CM point. The goal of this section is to compute the periods of the (split) bi- $\bar{\mathbb{Q}}$ -structure on $T_o \mathfrak{H}_g$ defined in Section 4, or more precisely in Sections 4.2 and 4.3.

The abelian variety $A_o := \mathbb{C}^g / (\mathbb{Z}^g + o\mathbb{Z}^g)$ is a CM abelian variety. Let $\theta_1, \dots, \theta_g$ be its holomorphic periods as defined in Convention 1.1.

Theorem 7.1. *The periods of the (split) bi- $\bar{\mathbb{Q}}$ -structure on $T_o \mathfrak{H}_g$ defined in Section 4 are $\theta_j \theta_{j'} / \pi$ with $1 \leq j \leq j' \leq g$.*

Our proof of Theorem 7.1 uses the Kodaira–Spencer map and does not use the root space decompositions discussed in Sections 5.2 and 5.3. At the end of this section, we will show in Section 7.4 that the computation with the Kodaira–Spencer map is compatible with the root space decomposition.

We have the following more precise formulation of Theorem 7.1 in view of the language of split bi- $\bar{\mathbb{Q}}$ -structures introduced in the current paper. By dimension comparison, the Kodaira–Spencer map (6.16) applied to the universal abelian variety $\mathfrak{A}_g \rightarrow \mathbb{A}_g$ at the point $[o] \in \mathbb{A}_g(\bar{\mathbb{Q}})$ gives an isomorphism $\text{KS}_{[o]}: T_{[o]}\mathbb{A}_g \xrightarrow{\sim} S^2 \text{Lie}(A_o)$ over $\bar{\mathbb{Q}}$. Thus for the uniformization $u: \mathfrak{H}_g \rightarrow \mathbb{A}_g$, the composite

$$\text{KS}_o: T_o \mathfrak{H}_g \xrightarrow{\text{d}u} T_{[o]}\mathbb{A}_g \xrightarrow[\text{KS}_{[o]}]{\sim} S^2 \text{Lie } A_o$$

is an isomorphism of \mathbb{C} -vector spaces. The split bi- $\bar{\mathbb{Q}}$ -structure on $\text{Lie } A_o \simeq \mathbb{C}^g$ given by Section 2.3 induces a natural split bi- $\bar{\mathbb{Q}}$ -structure on $S^2 \text{Lie } A_o$.

Theorem 7.1'. *KS_o induces an isomorphism of bi- $\bar{\mathbb{Q}}$ -structures between $T_o \mathfrak{H}_g \otimes \mathfrak{L}(1)$ and $S^2 \text{Lie } A_o$, where $\mathfrak{L}(1)$ is the Tate twist from Definition 2.7.*

7.1. Universal abelian variety and eigen-symplectic bases

By abuse of notation, denote by \mathbb{A}_g be the fine moduli space of principally polarized abelian varieties of dimension g with level- ℓ -structure for some $\ell \geq 3$. Then there exists a universal abelian variety

$$f: \mathfrak{A}_g \longrightarrow \mathbb{A}_g,$$

which is an abelian scheme of relative dimension g defined over $\bar{\mathbb{Q}}$ and is endowed with a principal polarization $\lambda: \mathfrak{A}_g \xrightarrow{\sim} \mathfrak{A}_g^\dagger$; here $\mathfrak{A}_g^\dagger \rightarrow \mathbb{A}_g$ is the dual abelian scheme.

For each $\tau \in \mathfrak{H}_g$, denote by A_τ the abelian variety parametrized by $[\tau]$. If $[\tau] = [\tau'] \in \mathbb{A}_g(k)$ (for an algebraically closed field $k \subseteq \mathbb{C}$), then A_τ and $A_{\tau'}$ are k -isomorphic as polarized abelian varieties. All discussions in Section 6 apply to $k = \bar{\mathbb{Q}}$, $(\pi: \mathcal{A} \rightarrow S) = (f: \mathfrak{A}_g \rightarrow \mathbb{A}_g)$ and $(u: \tilde{S} \rightarrow S^{\text{an}}) = (u: \mathfrak{H}_g \rightarrow \mathbb{A}_g^{\text{an}})$. In particular, one has

- a symplectic global basis $\{\omega_1, \dots, \omega_g, \eta_1, \dots, \eta_g\}$ of $\mathcal{H}_{\text{dR}}^1(\mathcal{A}_{\mathfrak{H}_g}/\mathfrak{H}_g) \rightarrow \mathfrak{H}_g$ as in Construction 6.1, such that when evaluated at o we get precisely the $\omega_j(o)$'s as above Theorem 7.1;
- a symplectic global basis $\{\gamma_1, \dots, \gamma_{2g}\}$ as in (6.6) of the vector bundle $\mathbb{V}(\mathcal{A}_{\mathfrak{H}_g}/\mathfrak{H}_g)$ over \mathfrak{H}_g (whose fiber over each $\tau \in \mathfrak{H}_g$ equals $H_1(A_\tau, \mathbb{C})$).

7.2. Realization of \mathfrak{H}_g

Let

$$(\Lambda, \Psi) := \left(\mathbb{Z}^{2g}, 2\pi i \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix} \right),$$

where Ψ is viewed as a symplectic form on $\Lambda_{\mathbb{C}}^\vee$. Moreover, let E act on $\Lambda_{\mathbb{Q}}^\vee$ via the isomorphism $\Lambda_{\mathbb{Q}}^\vee \simeq H^1(A_o, \mathbb{Q})$.

Let \mathfrak{H}_g^\vee be the parametrizing space of Lagrangians in $\Lambda_{\mathbb{C}}^\vee$, i.e.,

$$\mathfrak{H}_g^\vee = \{[W \subseteq \Lambda_{\mathbb{C}}^\vee] : \dim_{\mathbb{C}} W = g, \Psi(W, W) = 0\}.$$

Then \mathfrak{H}_g^\vee has a natural $\bar{\mathbb{Q}}$ -structure, for which a point $[W \subseteq \Lambda_{\mathbb{C}}^\vee] \in \mathfrak{H}_g^\vee$ is a $\bar{\mathbb{Q}}$ -point if and only if W descends to $\bar{\mathbb{Q}}$, i.e., there exists a $\bar{\mathbb{Q}}$ -subspace $V \subseteq \Lambda_{\bar{\mathbb{Q}}}^\vee$ such that $W = V \otimes_{\bar{\mathbb{Q}}} \mathbb{C}$.

Construction 7.2. The map (4.1) for the Siegel case (in particular, $X = \mathfrak{H}_g$) factors through an injective map $\iota: \mathfrak{H}_g \rightarrow \mathfrak{H}_g^\vee$ with the following property: For any $\tau \in \mathfrak{H}_g$, the entries of τ are in $\bar{\mathbb{Q}}$ if and only if $\iota(\tau)$ is a $\bar{\mathbb{Q}}$ -point.⁴

This construction of ι is important. It implies that the $\bar{\mathbb{Q}}$ -structure on \mathfrak{H}_g constructed in Section 4.3, denoted by $\mathfrak{H}_{g,\bar{\mathbb{Q}}}$, is equivalent to the following simple construction: \mathfrak{H}_g is a semi-algebraic open subset of $\{\tau \in \text{Mat}_{g \times g}(\mathbb{C}) : \tau = \tau^\top\} = \{\tau \in \text{Mat}_{g \times g}(\bar{\mathbb{Q}}) : \tau = \tau^\top\} \otimes_{\bar{\mathbb{Q}}} \mathbb{C}$, and a point $\tau \in \mathfrak{H}_g$ is a $\bar{\mathbb{Q}}$ -point if all entries of τ are in $\bar{\mathbb{Q}}$.

Proof. The vector bundle of Betti cohomology in family $\mathbb{V}(\mathcal{A}_{\mathfrak{H}_g}/\mathfrak{H}_g)^\vee$ over \mathfrak{H}_g can be trivialized by the dual basis $\{\gamma_1^\vee, \dots, \gamma_{2g}^\vee\}$ in a similar way as (6.6)

$$\Lambda_{\mathbb{C}}^\vee \times \mathfrak{H}_g \xrightarrow{\sim} \mathbb{V}(\mathcal{A}_{\mathfrak{H}_g}/\mathfrak{H}_g)^\vee, \quad ((k_1, \dots, k_{2g}), \tau) \mapsto \sum_{j=1}^{2g} k_j \gamma_j^\vee(\tau). \quad (7.1)$$

We work under this identification. Then the comparison between de Rham and Betti cohomologies (6.7) (applied to the current situation) becomes

$$\beta: \mathcal{H}_{\text{dR}}^1(\mathcal{A}_{\mathfrak{H}_g}/\mathfrak{H}_g) \xrightarrow{\sim} \Lambda_{\mathbb{C}}^\vee \times \mathfrak{H}_g.$$

For each $\tau \in \mathfrak{H}_g$, set $\beta_\tau: H_{\text{dR}}^1(A_\tau) \xrightarrow{\sim} \Lambda_{\mathbb{C}}^\vee$ to be the restriction of this comparison over τ .

We claim that $\beta_\tau(\Omega_{A_\tau})$ is a Lagrangian. By (6.10), we have

$$\beta_\tau(\Omega_{A_\tau}) = \{(\mathbf{x}, \tau \mathbf{x}) : \mathbf{x} \in \mathbb{C}^g\} \quad (7.2)$$

under the basis $\{\gamma_1^\vee(\tau), \dots, \gamma_{2g}^\vee(\tau)\}$ of $\Lambda_{\mathbb{C}}^\vee$. Since $\dim \beta_\tau(\Omega_{A_\tau}) = g$, it suffices to prove $\Psi(\beta_\tau(\Omega_{A_\tau}), \beta_\tau(\Omega_{A_\tau})) = 0$. This follows from the direct computation $(\mathbf{x}, \tau \mathbf{x}) \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix} (\mathbf{x}, \tau \mathbf{x})^\top = 0$. Hence we are done.

Now we are ready to define

$$\iota: \mathfrak{H}_g \longrightarrow \mathfrak{H}_g^\vee, \quad \tau \mapsto [\beta_\tau(\Omega_{A_\tau}) \subseteq \Lambda_{\mathbb{C}}^\vee]. \quad (7.3)$$

It is clear that (4.1) factors through ι and that ι is injective.

By (7.2), τ has entries in $\bar{\mathbb{Q}}$ if and only if $\beta_\tau(\Omega_{A_\tau})$ is defined over $\bar{\mathbb{Q}}$. Hence the entries of τ are in $\bar{\mathbb{Q}}$ if and only if $\iota(\tau)$ is a $\bar{\mathbb{Q}}$ -point. We are done. ■

7.3. Proof of Theorems 7.1 and 7.1'

From

$$\begin{array}{ccc} \mathfrak{H}_g & \xrightarrow{\iota} & \mathfrak{H}_g^\vee, \\ u \downarrow & & \downarrow \\ \mathbb{A}_g & & [o] \end{array} \quad \begin{array}{ccc} o & \longmapsto & \iota(o) = [\beta_o(\Omega_{A_o}) \subseteq \Lambda_{\mathbb{C}}^\vee] \end{array}$$

⁴In fact, \mathfrak{H}_g^\vee is the compact dual of \mathfrak{H}_g , with ι the map in the Borel embedding theorem.

we get

$$\begin{array}{ccc} T_o \mathfrak{H}_g & \xrightarrow{dt} & T_{\iota(o)} \mathfrak{H}_g^\vee \\ \downarrow du & & \\ T_{[o]} \mathbb{A}_g. & & \end{array}$$

The Kodaira–Spencer map (6.16) applied to $\mathfrak{A}_g \rightarrow \mathbb{A}_g$ at the point $[o] \in \mathbb{A}_g(\bar{\mathbb{Q}})$ gives $\text{KS}_{[o]}: T_{[o]} \mathbb{A}_g \rightarrow S^2 \text{Lie}(A_o)$. In this particular case (of moduli space), this map is injective; moreover, both sides have the same dimension $g(g+1)/2$. Hence $\text{KS}_{[o]}$ is an isomorphism of $\bar{\mathbb{Q}}$ -vector spaces. To sum it up, we have

$$\text{KS}_{[o]}: T_{[o]} \mathbb{A}_g \simeq S^2 \text{Lie}(A_o) \subseteq \text{Lie}(A_o^t) \otimes \text{Lie}(A_o) \quad (7.4)$$

and $\text{KS}_{[o]}$ is defined over $\bar{\mathbb{Q}}$.

By knowledge on the tangent space of Grassmannian, we have

$$T_{\iota(o)} \mathfrak{H}_g^\vee \subseteq \text{Hom}(\beta_o(\Omega_{A_o}), \Lambda_{\mathbb{C}}^\vee / \beta_o(\Omega_{A_o})) = \frac{\Lambda_{\mathbb{C}}^\vee}{\beta_o(\Omega_{A_o})} \otimes \beta_o(\Omega_{A_o})^\vee. \quad (7.5)$$

Recall from (6.11) the map $\beta_o: H_{\text{dR}}^1(\mathcal{A}_o) \rightarrow H^1(\mathcal{A}_o, \mathbb{C}) = \Lambda_{\mathbb{C}}^\vee$ and the induced maps

$$\bar{\beta}_o: \text{Lie}(A_o^t) \longrightarrow \Lambda_{\mathbb{C}}^\vee / \beta_o(\Omega_{A_o}), \quad \beta_o^\vee: \beta_o(\Omega_{A_o})^\vee \longrightarrow \text{Lie } A_o = \Omega_{A_o}^\vee. \quad (7.6)$$

Apply (6.19) to $(\tilde{S}, \tilde{s}) = (\mathfrak{H}_g, o)$, and notice that the bottom arrow in this case is precisely $du: T_o \mathfrak{H}_g \rightarrow T_{\iota(o)} \mathfrak{H}_g^\vee$ composed with (7.5). Thus $du \circ (dt)^{-1}$ is the restriction of $\bar{\beta}_o^{-1} \otimes \beta_o^\vee$.

By Lemma 6.4, $\bar{\beta}_o^{-1}$ is diagonalizable under $\bar{\mathbb{Q}}$ -bases to be $\text{diag}(\theta_1/\pi, \dots, \theta_g/\pi)$. By Remark 6.5, β_o^\vee is diagonalizable under $\bar{\mathbb{Q}}$ -bases to be $\text{diag}(\theta_1, \dots, \theta_g)$. Hence under suitable $\bar{\mathbb{Q}}$ -bases of $T_{\iota(o)} \mathfrak{H}_g^\vee$ and $T_{[o]} \mathbb{A}_g$, the matrix of the linear map $du \circ (dt)^{-1}$ is $\text{diag}(\theta_j \theta_{j'}/\pi)_{1 \leq j \leq j' \leq g}$.

This yields Theorem 7.1, because the geometric $\bar{\mathbb{Q}}$ -structure on $T_o \mathfrak{H}_g$ is by definition given by the $\bar{\mathbb{Q}}$ -structure on $T_{\iota(o)} \mathfrak{H}_g^\vee$.

We can do better. The computation above and (6.19) applied to $(\tilde{S}, \tilde{s}) = (\mathfrak{H}_g, o)$ yield the following assertion: $\text{KS}_{[o]}$ maps a 1-dimensional bi- $\bar{\mathbb{Q}}$ -subspace of $T_{[o]} \mathbb{A}_g$ with period $\theta_j \theta_{j'}/\pi$ to a 1-dimensional bi- $\bar{\mathbb{Q}}$ -subspace of $S^2 \text{Lie}(A_o)$ with period $\theta_j \theta_{j'}$. This proves Theorem 7.1'. We are done. ■

Remark 7.3. By general knowledge on symplectic vector spaces, for the Lagrangian $\beta_o(\Omega_{A_o})$ of $\Lambda_{\mathbb{C}}^\vee$ we have a decomposition $\Lambda_{\mathbb{C}}^\vee = \beta_o(\Omega_{A_o}) \oplus \beta_o(\Omega_{A_o})^\vee$ given as follows: take a $\bar{\mathbb{Q}}$ -basis $\{x_1, \dots, x_g\}$ of $\beta_o(\Omega_{A_o})$, and then one has a dual $\bar{\mathbb{Q}}$ -basis for a complement defined by $\Psi(x_j, y_l) = 2\pi i \delta_{jl}$, and this complement is canonically isomorphic over \mathbb{C} to $\beta_o(\Omega_{A_o})^\vee$. Over $\bar{\mathbb{Q}}$, this decomposition then becomes $\Lambda_{\bar{\mathbb{Q}}}^\vee = \beta_o(\Omega_{A_o}) \oplus 2\pi i \beta_o(\Omega_{A_o})^\vee$. Hence we have a canonical isomorphism

$$2\pi i \beta_o(\Omega_{A_o})^\vee = \Lambda_{\mathbb{C}}^\vee / \beta_o(\Omega_{A_o}) \quad (7.7)$$

defined over $\bar{\mathbb{Q}}$.

We close this subsection with the following (slightly more precise) formulation of Theorem 7.1, which follows immediately from the computation above. Denote by $\mathfrak{H}_{g,\bar{\mathbb{Q}}}$ the $\bar{\mathbb{Q}}$ -structure on \mathfrak{H}_g explained below Construction 7.2. Consider the tangent of the uniformization u at the CM point o

$$\psi := du: T_o \mathfrak{H}_g \longrightarrow T_{[o]} \mathbb{A}_g.$$

Theorem 7.1''. *Under a suitable $\bar{\mathbb{Q}}$ -basis of $T_o \mathfrak{H}_g = T_o \mathfrak{H}_{g,\bar{\mathbb{Q}}} \otimes_{\bar{\mathbb{Q}}} \mathbb{C}$ and a suitable $\bar{\mathbb{Q}}$ -basis of $T_{[o]} \mathbb{A}_g = T_{[o]} \mathbb{A}_{g,\bar{\mathbb{Q}}} \otimes_{\bar{\mathbb{Q}}} \mathbb{C}$, the matrix of ψ is $\text{diag}(\theta_j \theta_{j'}/\pi)_{1 \leq j \leq j' \leq g}$.*

7.4. In relation to the root space decomposition

Let \mathbf{T} be a maximal torus which contains $\text{MT}(o)$. The last sentence of Remark 6.5 yields a \mathbb{Z} -basis $\{\varepsilon_0^*, \varepsilon_1^*, \dots, \varepsilon_g^*\}$ of $X^*(\mathbf{T})$ such that $\varepsilon_j^*(\beta_o) = \theta_j$ for each $j \in \{1, \dots, g\}$ and ε_0^* is a character of \mathbb{G}_m . The root system $\Phi(\mathbf{T}, \text{GSp}_{2g}) \subseteq X^*(\mathbf{T})$ defined in [6, §13.18] is $\{\varepsilon_j^* + \varepsilon_{j'}^* : 1 \leq j \leq j' \leq g\} \cup \{\varepsilon_j^* - \varepsilon_{j'}^* : 1 \leq j < j' \leq g\}$. The set Φ_M^+ defined above Proposition 5.5 is $\{\varepsilon_j^* + \varepsilon_{j'}^* : 1 \leq j \leq j' \leq g\}$.

Theorem 7.4. *The 1-dimensional bi- $\bar{\mathbb{Q}}$ subspace of $T_o \mathfrak{H}_g$ associated with $\varepsilon_j^* + \varepsilon_{j'}^*$ has period $\theta_j \theta_{j'}/\pi$.*

Proof. All vector spaces in the proof are \mathbb{C} -spaces unless otherwise stated. We start by defining an action of $\mathbf{T}(\mathbb{Q})$ on $\text{Lie } A_o = \Omega_{A_o}^\vee$. By (7.2), we have $\beta_o(\Omega_{A_o}) = \{(ox, \mathbf{x}) : \mathbf{x} \in \mathbb{C}^g\}$ under the basis $\{\gamma_{g+1}^\vee(o), \dots, \gamma_{2g}^\vee(o), \gamma_1^\vee(o), \dots, \gamma_g^\vee(o)\}$ of $H^1(A_o, \mathbb{C})$. The action of $\text{GSp}_{2g}(\mathbb{R})^+$ on \mathfrak{H}_g is defined by $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \tau = (A\tau + B)(C\tau + D)^{-1}$. By the discussion at the end of Section 5.1, $\mathbf{T}(\mathbb{R})o = o$. So for each $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbf{T}(\mathbb{R})$, we have $(Ao + B)(Co + D)^{-1} = o$, and hence

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} o\mathbf{x} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} (Ao + B)\mathbf{x} \\ (Co + D)\mathbf{x} \end{bmatrix} = \begin{bmatrix} oy \\ \mathbf{y} \end{bmatrix} \in \beta_o(\Omega_{A_o})$$

with $\mathbf{y} = (Co + D)\mathbf{x}$. This defines an action of $\mathbf{T}(\mathbb{Q})$ on $\text{Lie } A_o$, and hence an action of $\mathbf{T}(\mathbb{Q})$ on $S^2 \text{Lie } A_o$. Notice that the Kodaira–Spencer map KS_o is $\mathbf{T}(\mathbb{Q})$ -equivariant because every morphism on the bottom line of (6.17), when restricted to o , is $\mathbf{T}(\mathbb{Q})$ -equivariant. Thus the 1-dimensional bi- $\bar{\mathbb{Q}}$ subspace of $T_o \mathfrak{H}_g$ associated with $\varepsilon_j^* + \varepsilon_{j'}^*$ is the eigenspace of $\varepsilon_j^* + \varepsilon_{j'}^*$ for the action of $\mathbf{T}(\mathbb{Q})$ on $S^2 \text{Lie } A_o$.

Recall the bi- $\bar{\mathbb{Q}}$ structure on $\text{Lie } A_o$ defined in Section 2.3. By the first part of Remark 6.5 and the choice of ε_j^* , the eigenspace of ε_j^* for the action of $\mathbf{T}(\mathbb{Q})$ on $\text{Lie } A_o$ is the 1-dimensional bi- $\bar{\mathbb{Q}}$ subspace of $\text{Lie } A_o$ associated with θ_j . Hence we are done by the conclusion of the previous paragraph. ■

Theorem 7.4 yields the following immediate corollary. By a *root space*, we mean a 1-dimensional bi- $\bar{\mathbb{Q}}$ -subspace of $T_o \mathfrak{H}_g$ associated with some $\varepsilon_j^* + \varepsilon_{j'}^*$.

Corollary 7.5. *The following statements are equivalent:*

- (i) $\frac{\theta_j \theta_{j'}}{\theta_k \theta_{k'}} \notin \bar{\mathbb{Q}}^*$ for all distinct pairs $\{j, j'\}$ and $\{k, k'\}$;

- (ii) each 1-dimensional bi- $\bar{\mathbb{Q}}$ -subspace of $T_o\mathfrak{H}_g$ is a root space;
- (iii) each bi- $\bar{\mathbb{Q}}$ -subspace of $T_o\mathfrak{H}_g$ is the direct sum of root spaces.

In particular, if there exists a special subvariety (i.e., a connected sub-Shimura variety) S of \mathbb{A}_g passing through $[o]$ such that $T_{[o]}S$ is not the direct sum of root spaces of $T_{[o]}\mathbb{A}_g$, then the holomorphic periods of A_o satisfy some elementary non-trivial quadratic relation as defined in Definition 1.2. We shall elaborate this phenomenon in a subsequent work [12, §9] by the first- and second-named authors.

8. Shimura curves and Hilbert modular varieties

Let S be a connected Shimura variety associated with a connected Shimura datum (\mathbf{G}, X) . Let $S_{\mathbf{H}}$ be a connected Shimura subvariety associated with $(\mathbf{H}, X_{\mathbf{H}})$. We have seen that X can be endowed with a split bi- $\bar{\mathbb{Q}}$ -structure in Section 4, and the restriction of this split bi- $\bar{\mathbb{Q}}$ -structure on $X_{\mathbf{H}}$ gives precisely the bi- $\bar{\mathbb{Q}}$ -structure defined for $X_{\mathbf{H}}$ in Section 4. In particular, each period of $X_{\mathbf{H}}$ is a period of X by Proposition 2.5 (iii).

In this section, we see two aspects of this discussion. The first is about Shimura curves, for which we use Corollary 3.1 (which is a consequence of WAST) and our Theorem 7.1 to give a new proof of a conjecture of Lang. The second is to compute the periods of Hilbert modular varieties.

8.1. Shimura curves

Lang [14] raised the following transcendence question in uniformization theory. Let C be a smooth projective algebraic curve of genus > 1 defined over $\bar{\mathbb{Q}}$. Suppose that the universal holomorphic covering map

$$\varphi: E_{\rho} := \{z \in \mathbb{C} : |z| < \rho\} \longrightarrow C^{\text{an}} \quad (8.1)$$

is normalized in such a way that $\varphi(0) \in C(\bar{\mathbb{Q}})$ and that $\varphi'(0) \in \bar{\mathbb{Q}}$. Is the covering radius ρ then a transcendental number?

This question was answered affirmatively by Cohen and Wolfart [8] when C is a Shimura variety of dimension 1 and $\varphi(0)$ is a CM point. Apart from Wüstholz's result on transcendental numbers, their proof relies on some hard computation of Shimura [18, Thm. 1.2, Thm. 7.1, Thm. 7.6].

We hereby give a new and easier proof using the framework of this paper which does not use Shimura's computation. More precisely we use the bi- $\bar{\mathbb{Q}}$ -structure on the Hermitian symmetric space and our Theorem 7.1, in addition to the consequence of Wüstholz presented in our paper (Corollary 3.1).

Proposition 8.1. *The question of Lang mentioned in the paragraph of (8.1) has an affirmative answer if C is a Shimura variety of dimension 1 and $\varphi(0)$ is a CM point.*

Proof. By general theory of Shimura varieties, C is of abelian type, i.e., there exists a connected Shimura subvariety C' of \mathbb{A}_g (for some $g \geq 1$) of dimension 1 together with a finite map $C' \rightarrow C$. Moreover, it is known that this finite map is defined over $\bar{\mathbb{Q}}$, and hence it suffices to prove the result with C replaced by C' .

Now let us explain how the uniformization φ is obtained from the Shimura setting. Take $o \in \mathfrak{H}_g$ which maps to $\varphi(0)$ under the uniformization $\mathfrak{H}_g \rightarrow \mathbb{A}_g$. Let A_o be the CM abelian variety parametrized by $\varphi(0)$, and let $\theta_1, \dots, \theta_g$ be its holomorphic periods.

Write \mathcal{D}_g for the bounded realization of \mathfrak{H}_g based on o .

As a sub-Shimura variety of \mathbb{A}_g of dimension 1, $C = \Gamma \backslash \mathcal{D}$ where $\mathcal{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the Poincaré unit disk and Γ is a subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$. The Shimura subdatum with which C is associated can be recovered as follows. Let \mathbf{H}' be the neutral component of the Zariski closure of Γ in GSp_{2g} , then it is known that $\mathcal{D} = \mathbf{H}'(\mathbb{R})^+ o$. Now let $\mathbf{H} = \mathbb{G}_m \cdot \mathbf{H}' < \mathrm{GSp}_{2g}$. Then $\mathcal{D} = \mathbf{H}(\mathbb{R})^+ o \subseteq \mathrm{GSp}_{2g}(\mathbb{R})^+ o = \mathcal{D}_g$, and thus we have the sub-Shimura datum $(\mathbf{H}, \mathcal{D})$ of $(\mathrm{GSp}_{2g}, \mathcal{D}_g)$. Now C is associated with $(\mathbf{H}, \mathcal{D})$, and we have a uniformization $u_C: \mathcal{D} \rightarrow C = \Gamma \backslash \mathcal{D}$.

Now φ from (8.1) is the composite of $\rho^{-1}: E_\rho \rightarrow \mathcal{D}$ and u_C . Thus $\varphi'(0) = \rho^{-1} u'_C(0)$. Since $\varphi'(0) \in \bar{\mathbb{Q}}$ by assumption, we have that $\rho = u'_C(0)$ up to $\bar{\mathbb{Q}}^*$. Hence it suffices to prove that $u'_C(0)$ is a transcendental number.

By Theorem 7.1 (and the formulation Theorem 7.1'') and Proposition 2.5(iii), we have $u'_C(0) = \theta_j \theta_{j'} / \pi$ for some j and j' . Thus $u'_C(0)$ is a transcendental number by Corollary 3.1. We are done. ■

8.2. Hilbert modular varieties

Let F be a totally real field with $[F : \mathbb{Q}] = g$. Let $\mathbf{G} := \mathbb{G}_m \cdot \mathrm{Res}_{F/\mathbb{Q}} \mathrm{SL}_2$. Then $\mathbf{G}^{\mathrm{der}}(\mathbb{Q}) = \mathrm{SL}_2(F)$, and $\mathbf{G}^{\mathrm{der}}(\mathbb{R}) = \prod_{\sigma: F \rightarrow \mathbb{R}} \mathrm{SL}_2(\mathbb{R})$. For the Siegel upper half plane $\mathfrak{H} = \{\tau \in \mathbb{C} : \mathrm{Im} \tau > 0\}$, $G := \mathbf{G}(\mathbb{R})^+$ acts on \mathfrak{H}^g componentwise. The pair $(\mathbf{G}, \mathfrak{H}^g)$ is a connected Shimura datum. For each congruence subgroup Γ of $\mathbf{G}(\mathbb{Q})$, the connected Shimura variety $\Gamma \backslash \mathfrak{H}^g$ is, by Baily–Borel, the analytification of an algebraic variety, which furthermore is defined over $\bar{\mathbb{Q}}$. This Shimura variety, which we denote by \mathcal{H}_F , is called the *Hilbert modular variety* defined by \mathbf{G} (or, more simply, associated with F). Then \mathcal{H}_F is a Shimura subvariety of \mathbb{A}_g .

Let \mathcal{O}_F be the ring of integers of F . An abelian scheme $\mathcal{A} \rightarrow S$ of relative dimension g is said to *have RM by F* if there exists an injective ring homomorphism $\iota: R_F \hookrightarrow \mathrm{End}_S(\mathcal{A})$ (for some order R_F in \mathcal{O}_F) such that $\mathrm{Lie}(\mathcal{A}/S)$ is a locally free $R_F \otimes \mathcal{O}_S$ -module. It is known that the Hilbert modular variety \mathcal{H}_F parametrizes abelian varieties with RM by F and some extra structures.

The goal of this section is to prove the following theorem. Let $[o] \in \mathcal{H}_F(\bar{\mathbb{Q}})$ be a CM point. Let A_o be the CM variety parametrized by $[o]$, and let $\theta_1, \dots, \theta_g$ be the holomorphic periods of A_o as defined in the introduction (Convention 1.1).

Let $o \in \mathfrak{H}^g$ be a point such that $o \mapsto [o]$. The natural inclusion $\mathfrak{H}^g \subseteq \mathfrak{H}_g$ induces $T_o \mathfrak{H}^g \subseteq T_o \mathfrak{H}_g$. Thus the split bi- $\bar{\mathbb{Q}}$ -structure on $T_o \mathfrak{H}_g$ defined in Section 4 gives a split bi- $\bar{\mathbb{Q}}$ -structure on $T_o \mathfrak{H}^g$.

Theorem 8.2. *Under suitable $\bar{\mathbb{Q}}$ -bases of*

$$T_o \mathfrak{H}^g = T_o \mathfrak{H}_{\bar{\mathbb{Q}}}^g \otimes_{\bar{\mathbb{Q}}} \mathbb{C} \quad \text{and} \quad T_{[o]} \mathcal{H}_F = T_{[o]} \mathcal{H}_{F, \bar{\mathbb{Q}}} \otimes_{\bar{\mathbb{Q}}} \mathbb{C},$$

the matrix of the linear map du is $\frac{1}{\pi} \operatorname{diag}(\theta_j^2)_{j \in \{1, \dots, g\}}$. In particular, the periods of the split bi- $\bar{\mathbb{Q}}$ -structure on $T_o \mathfrak{H}^g$ are θ_j^2/π for $j \in \{1, \dots, g\}$.

Proof. We write for simplicity \mathcal{H} for \mathcal{H}_F .

The following holds true by the modular interpretation of \mathcal{H} above. For a suitable Γ , the Hilbert modular variety \mathcal{H} is a fine moduli space with the following property: the universal family $\mathcal{A} \rightarrow \mathcal{H}$ satisfies that $\operatorname{Lie}(\mathcal{A}/\mathcal{H})$ is a locally free $R_F \otimes \mathcal{O}_{\mathcal{H}}$ -module. Hence $\operatorname{Lie}(A_o)$ is a free R_F -module. But $\operatorname{Lie}(A_o)$ has dimension g and R_F is a \mathbb{Z} -module of rank g , so $\operatorname{Lie}(A_o)$ has rank 1 as an R_F -module. We write $\operatorname{Lie}(A_o) = R_F \mathbf{f}$ with \mathbf{f} a $\bar{\mathbb{Q}}$ -element of $\operatorname{Lie}(A_o)$.

As $\mathcal{H} \subseteq \mathbb{A}_g$, the Kodaira–Spencer map (7.4) gives, in combination with $A_o^t \simeq A_o$,

$$T_{[o]} \mathcal{H} \subseteq T_{[o]} \mathbb{A}_g \xrightarrow{\operatorname{KS}_{[o]}} \operatorname{Lie}(A_o) \otimes \operatorname{Lie}(A_o).$$

The R_F -action on each fiber of $\operatorname{Lie}(\mathcal{A}/\mathcal{H})$ (and of $\operatorname{Lie}(\mathcal{A}^t/\mathcal{H})$) then implies $\operatorname{KS}_{[o]}(T_{[o]} \mathcal{H}) \subseteq \operatorname{Lie}(A_o) \otimes_{R_F} \operatorname{Lie}(A_o)$. By comparing dimensions of both sides, we have equality. Hence we have the isomorphism defined over $\bar{\mathbb{Q}}$

$$\operatorname{KS}_{[o]}: T_{[o]} \mathcal{H} \xrightarrow{\sim} \operatorname{Lie}(A_o) \otimes_{R_F} \operatorname{Lie}(A_o).$$

Similarly, by using (7.5) and taking into consideration of (7.7) we get

$$T_o \mathfrak{H}^g = 2\pi i \beta_o(\Omega_{A_o})^\vee \otimes_{R_F} \beta_o(\Omega_{A_o})^\vee.$$

Similarly to $\operatorname{Lie}(A_o)$, $\beta_o(\Omega_{A_o})^\vee$ is a free R_F -module of rank 1. Write $\beta_o(\Omega_{A_o})^\vee = R_F \mathbf{e}$ with \mathbf{e} a $\bar{\mathbb{Q}}$ -element of $\beta_o(\Omega_{A_o})^\vee$. The following diagram commutes, with all horizontal maps defined over $\bar{\mathbb{Q}}$:

$$\begin{array}{ccccccc} T_o \mathfrak{H}^g & \xrightarrow{=} & 2\pi i \beta_o(\Omega_{A_o})^\vee \otimes_{R_F} \beta_o(\Omega_{A_o})^\vee & \xrightarrow[\sim]{2\pi i f_1 \mathbf{e} \otimes f_2 \mathbf{e} \mapsto 2\pi i f_1 f_2 \mathbf{e}} & 2\pi i \beta_o(\Omega_{A_o})^\vee & & \\ \downarrow du & & \downarrow \frac{1}{2\pi i} \beta_o^\vee \otimes \beta_o^\vee & & & & \downarrow \frac{1}{2\pi i} (\beta_o^\vee)^2 \\ T_{[o]} \mathcal{H} & \xrightarrow[\sim]{\operatorname{KS}_{[o]}} & \operatorname{Lie}(A_o) \otimes_{R_F} \operatorname{Lie}(A_o) & \xrightarrow[\sim]{f_1 \mathbf{f} \otimes f_2 \mathbf{f} \mapsto f_1 f_2 \mathbf{f}} & \operatorname{Lie}(A_o) & & \end{array}$$

Thus the conclusion follows from the first part of Remark 6.5. ■

9. Some general discussion on CM points

In this section, we gather some preliminary results regarding the Mumford–Tate group of CM abelian varieties and on CM points in \mathbb{A}_g . They will be used in later discussions.

9.1. Mumford–Tate group of CM abelian varieties

Let A be a CM abelian variety, and let $\mathbf{T}_o := \mathrm{MT}(A)$. Let $E := \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$.

If A has no square factors, then $E = E_1 \times \cdots \times E_k$ is a product of CM fields. In this case, there exists an element $\iota \in E$ such that $\bar{\iota} = -\iota$. Then E can be endowed with the \mathbb{Q} -symplectic form

$$\langle x, y \rangle := \mathrm{Tr}_{E/\mathbb{Q}}(\bar{x}\iota y).$$

This makes $(E, \langle \cdot, \cdot \rangle) \simeq (\mathbb{Q}^{2g}, \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix})$ into a symplectic space. Set GU_E to be the subgroup of GSp_{2g} generated by $\mathbb{G}_m = Z(\mathrm{GSp}_{2g})$ and

$$\mathrm{U}_E := \{x \in \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_m : x\bar{x} = 1\}.$$

Lemma 9.1. *Assume A has no square factors. Then GU_E is a maximal torus of GSp_{2g} and equals $Z_{\mathrm{GSp}_{2g}}(\mathbf{T}_o)$.*

Proof. For each $j \in \{1, \dots, k\}$, let F_j be the largest totally real subfield of E_j . Set $F := F_1 \times \cdots \times F_k$. Then

$$\mathrm{U}_E = \ker(\mathrm{Nm}: \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{m,E} \longrightarrow \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{m,F}).$$

Hence U_E is a torus. Moreover, $\sum_{j=1}^k [F_j : \mathbb{Q}] = g$. So $\dim \mathrm{U}_E = g$.

Therefore GU_E is a torus of dimension $g + 1$. But $\mathrm{rk} \mathrm{GSp}_{2g} = g + 1$. So GU_E is a maximal torus of GSp_{2g} .

Now let us prove $\mathrm{GU}_E = Z_{\mathrm{GSp}_{2g}}(\mathbf{T}_o)$. Since $\mathbf{T}_o < \mathrm{GU}_E$ and GU_E is a torus, we have $\mathrm{GU}_E < Z_{\mathrm{GSp}_{2g}}(\mathbf{T}_o)$. Hence it suffices to prove that $Z_{\mathrm{GSp}_{2g}}(\mathbf{T}_o)$ is a torus.

Write $V = H_1(A, \mathbb{Q})$, which is a \mathbb{Q} -Hodge structure of type $(-1, 0) + (0, -1)$ and whose Mumford–Tate group is \mathbf{T}_o . Then for any $\alpha \in \mathrm{End}(V_{\mathbb{Q}})$, we have $\alpha \in \mathrm{End}_{\mathbb{Q}\text{-HS}}(V)$ if and only if α commutes with all elements in $\mathbf{T}_o(\mathbb{Q})$. Thus $\mathrm{End}_{\mathbb{Q}\text{-HS}}(V) = \mathrm{Lie} Z_{\mathrm{GSp}_{2g}}(\mathbf{T}_o)$. Observe that $\mathrm{End}_{\mathbb{Q}\text{-HS}}(V)$ is abelian because A has no square factors. So $Z_{\mathrm{GSp}_{2g}}(\mathbf{T}_o)$ is a torus. We are done. \blacksquare

9.2. Weyl points

Let $[o] \in \mathbb{A}_g(\bar{\mathbb{Q}})$ be a CM point, and let A_o be the associated CM abelian variety. Assume that A_o is simple. Then $E := \mathrm{End}^0(A_o)$ is a CM field. Let F be the maximal totally real subfield of E , then $[F : \mathbb{Q}] = g$ and $[E : F] = 2$.

Write E^c and F^c for the Galois closures of E and F in $\bar{\mathbb{Q}}$. Then $\mathrm{Gal}(F^c/\mathbb{Q}) < \mathfrak{S}_g$ and $\mathrm{Gal}(E^c/\mathbb{Q}) < (\mathbb{Z}/2\mathbb{Z})^g \rtimes \mathfrak{S}_g$.

The point $[o]$ is called a *Weyl point* (or *Galois generic*) if $\mathrm{Gal}(E^c/\mathbb{Q}) = (\mathbb{Z}/2\mathbb{Z})^g \rtimes \mathfrak{S}_g$.

Proposition 9.2. *For each Weyl point $[o] \in \mathbb{A}_g(\bar{\mathbb{Q}})$, there are precisely 3 special subvarieties of \mathbb{A}_g which pass through $[o]$: $\{[o]\}$, \mathbb{A}_g , and the Hilbert modular variety defined by $\mathbb{G}_m \cdot \mathrm{Res}_{F/\mathbb{Q}} \mathrm{SL}_2$.⁵*

⁵For readers who are not familiar with this terminology, we refer to Section 8.2 for the definition.

Proof. The result is clearly true if $g = 1$. From now on, assume $g \geq 2$.

Let S be a special subvariety of \mathbb{A}_g , which passes through $[o]$, associated with the connected Shimura datum $(\mathbf{H}, X_{\mathbf{H}})$. Assume $S \neq \{[o]\}$.

Let $\mathbf{T}_o := \mathrm{MT}([o])$. Then $\mathbf{T}_o < \mathbf{H}$, \mathbf{T}_o is a maximal torus of GSp_{2g} , and the Galois closure of the splitting field of \mathbf{T}_o is E^c . There exists a maximal torus \mathbf{T}'_o of Sp_{2g} such that $\mathbf{T}_o = \mathbb{G}_m \cdot \mathbf{T}'_o$, where $\mathbb{G}_m = Z(\mathrm{GSp}_{2g})$. In particular, $\dim \mathbf{T}'_o = g$.

We claim that $\mathbf{T}'_o \subseteq \mathbf{H}^{\mathrm{der}}$. Indeed, $\mathbf{T}'_o \cap \mathbf{H}^{\mathrm{der}}$ is a non-trivial subtorus of \mathbf{T}'_o , and hence $\mathbf{T}'_o = (\mathbf{T}'_o \cap \mathbf{H}^{\mathrm{der}})\mathbf{T}''_o$ for some subtorus \mathbf{T}''_o of \mathbf{T}'_o . Let $r_1 := \dim(\mathbf{T}'_o \cap \mathbf{H}^{\mathrm{der}})$ and $r_2 := \dim \mathbf{T}''_o$. Let L_1^c (resp. L_2^c) be the Galois closure of the splitting field of $\mathbf{T}'_o \cap \mathbf{H}^{\mathrm{der}}$ (resp. of \mathbf{T}''_o). Then $\mathrm{Gal}(L_i^c/\mathbb{Q}) < (\mathbb{Z}/2\mathbb{Z})^{r_i} \rtimes \mathfrak{S}_{r_i}$ for $i \in \{1, 2\}$. Since E^c is the Galois closure of the splitting field of \mathbf{T}'_o , we then have that

$$\mathrm{Gal}(E^c/\mathbb{Q}) < \prod_{i=1}^2 (\mathbb{Z}/2\mathbb{Z})^{r_i} \rtimes \mathfrak{S}_{r_i}.$$

But $\mathrm{Gal}(E^c/\mathbb{Q}) = (\mathbb{Z}/2\mathbb{Z})^g \rtimes \mathfrak{S}_g$ since $[o]$ is Weyl. Notice that $r_1 + r_2 = g$ and $r_1 > 0$. So $r_1 = g$ and $r_2 = 0$. Therefore $\mathbf{T}'_o = \mathbf{T}'_o \cap \mathbf{H}^{\mathrm{der}}$, and hence $\mathbf{T}'_o < \mathbf{H}^{\mathrm{der}}$.

The upshot is that \mathbf{T}'_o is a maximal torus of $\mathbf{H}^{\mathrm{der}}$. Hence $\mathrm{rk} \mathbf{H}^{\mathrm{der}} = g$. For the root system of $(\mathbf{T}'_o, \mathbf{H}^{\mathrm{der}})$, the Weyl group of $\mathbf{H}^{\mathrm{der}}$ is then $(\mathbb{Z}/2\mathbb{Z})^g \rtimes \mathfrak{S}_g$.

We claim that $\mathbf{H}^{\mathrm{der}}$ is a \mathbb{Q} -simple algebraic group. Assume not, then we can get a contradiction to $\mathrm{Gal}(E^c/\mathbb{Q}) = (\mathbb{Z}/2\mathbb{Z})^g \rtimes \mathfrak{S}_g$ with a similar argument as for $\mathbf{T}'_o < \mathbf{H}^{\mathrm{der}}$.

The upshot is that $\mathbf{H}^{\mathrm{der}} = \mathrm{Res}_{F'/\mathbb{Q}} \tilde{\mathbf{H}}$ for some F' totally real and $\tilde{\mathbf{H}}$ simply-connected. Thus

$$\mathbf{H}_{\mathbb{C}} = \bigoplus_{\sigma: F' \rightarrow \mathbb{R}} \tilde{\mathbf{H}}_{\sigma, \mathbb{C}}.$$

For $r := [F' : \mathbb{Q}]$, we then have $r \cdot \mathrm{rk}(\mathbf{H}_{\sigma, \mathbb{C}}) = g$.

Let Φ be the root system of $\mathbf{H}_{\mathbb{C}}$, and let Φ_{σ} be the root system of $\mathbf{H}_{\sigma, \mathbb{C}}$. Then $\Phi = \bigsqcup_{\sigma: F' \rightarrow \mathbb{R}} \Phi_{\sigma}$. Then

$$\mathrm{Aut}(\Phi) = \left(\prod_{\sigma} \mathrm{Aut}(\Phi_{\sigma}) \right) \rtimes \mathfrak{S}_r.$$

By the theory of root systems, $\mathrm{Aut}(\Phi_{\sigma}) = W_{\sigma} \rtimes \mathrm{Aut}(\Delta_{\sigma})$, where W_{σ} is the Weyl group of $H_{\sigma, \mathbb{C}}$, and $\mathrm{Aut}(\Delta_{\sigma})$ is the transformation group of the Dynkin diagram Δ_{σ} associated with $\mathbf{H}_{\sigma, \mathbb{C}}$. Then W_{σ} is a Weyl group of type A_r, B_r, C_r , or D_r with r dividing g . We may therefore have $W_{\sigma} = \mathfrak{S}_{r+1}$, $W_{\sigma} = (\mathbb{Z}/2\mathbb{Z})^r \rtimes \mathfrak{S}_r$ or $W_{\sigma} = (\mathbb{Z}/2\mathbb{Z})^{r-1} \rtimes \mathfrak{S}_r$. Moreover, $\mathrm{Aut}(\Delta_{\sigma})$ equals 1, $\mathbb{Z}/2\mathbb{Z}$, or \mathfrak{S}_3 (in which case $\mathbf{H}_{\sigma, \mathbb{C}}$ is of type D_3).

Assume $g \geq 4$. Notice that $\mathrm{Aut}(\Phi)$ contains the Weyl group of $\mathbf{H}^{\mathrm{der}}$, which is $(\mathbb{Z}/2\mathbb{Z})^g \rtimes \mathfrak{S}_g$. So we must have $r = 1$ or $r = g$. The same holds true for $g = 2$ and $g = 3$ because $r|g$.

If $r = 1$, then $F' = \mathbb{Q}$ and $\mathbf{H}^{\mathrm{der}}$ is a simple-connected simple group of rank g . Then it is of type A_g, B_g, C_g , or D_g . Since the Weyl group of $\mathbf{H}^{\mathrm{der}}$ is $(\mathbb{Z}/2\mathbb{Z})^g \rtimes \mathfrak{S}_g$ and $\mathbf{H}^{\mathrm{der}}$ is a subgroup of Sp_{2g} , we conclude that $\mathbf{H}^{\mathrm{der}} = \mathrm{Sp}_{2g}$. And hence $S = \mathbb{A}_g$.

If $r = g$, then $[F' : \mathbb{Q}] = g \geq 2$ and $\text{rk } \tilde{\mathbf{H}} = 1$. So $\tilde{\mathbf{H}}$ is a simple simply-connected group of type A_1 . Therefore $\tilde{\mathbf{H}}_{\bar{\mathbb{Q}}}$ is either SL_2 or PSL_2 . As $\mathbf{H}^{\text{der}} = \text{Res}_{F'/\mathbb{Q}} \tilde{\mathbf{H}}$ is a subgroup of Sp_{2g} , we have $\tilde{\mathbf{H}}_{\bar{\mathbb{Q}}} = \text{SL}_2$. Hence $\tilde{\mathbf{H}}$ is a form of SL_2 over F' . Hence $\tilde{\mathbf{H}}$ is either $\text{SL}_{2,F'}$ or a quaternion algebra over F' which is non-split at some real place σ of F' . We exclude the second case as follows. Recall that $\text{rk } \mathbf{H}^{\text{der}} = g$. Since $\text{rk } \mathbf{H} \leq \text{rk } \text{GSp}_{2g} = g + 1$, we then have $\mathbf{H} = \mathbb{G}_m \cdot \mathbf{H}^{\text{der}} = \mathbb{G}_m \cdot \text{Res}_{F'/\mathbb{Q}} \tilde{\mathbf{H}}$. Thus $\tilde{\mathbf{H}}$ is split at some real place σ' of F' by (SV3) of the definition of Shimura data. Write V for the \mathbb{Q} -vector space of dimension $2g$ on which GSp_{2g} naturally acts. If $\tilde{\mathbf{H}}$ is a quaternion over F' which is non-split at σ , then the action of $\mathbf{H}_{\sigma, \mathbb{R}}$ on $V_{\sigma, \mathbb{R}}$ has weight 0, while the action of $\mathbf{H}_{\sigma', \mathbb{R}}$ on $V_{\sigma', \mathbb{R}}$ has weight 1. This is impossible by definition of Shimura data. Hence $\tilde{\mathbf{H}} = \text{SL}_{2,F'}$. Thus S is the Hilbert modular variety defined by the totally real field F' of degree g . By the modular interpretation at the beginning of Section 8.2, an order of $\mathcal{O}_{F'}$ is thus contained in the endomorphism ring of A_o . This obliges $F' = F$. Hence we are done. ■

Corollary 9.3. *Let $[o]$ be a CM point in $\mathbb{A}_g(\bar{\mathbb{Q}})$. Set*

$$\begin{aligned} \Sigma := \{[z] \in \mathbb{A}_g(\bar{\mathbb{Q}}) : [z] \text{ is CM, the only special subvariety of} \\ \mathbb{A}_g \text{ passing through } [o] \text{ and } [z] \text{ is } \mathbb{A}_g\}. \end{aligned}$$

Then Σ is dense in \mathbb{A}_g^{an} in the usual topology.

Proof. Let \mathcal{W} be the set of all Weyl points in \mathbb{A}_g . It is known that \mathcal{W} is dense in \mathbb{A}_g^{an} in the usual topology; see [7, Prop. 2.1].

Consider $\text{End}^0(A_o)$. There are only finitely many totally real fields of degree g which are contained in $\text{End}^0(A_o)$. For each such totally real field F' , one can associate a Hilbert modular variety $\mathcal{H}_{F'}$ obtained from $\mathbb{G}_m \cdot \text{Res}_{F'/\mathbb{Q}} \text{SL}_2$. Then $\mathcal{W} \setminus \bigcup_{F'} \mathcal{H}_{F'}(\bar{\mathbb{Q}})$ is dense in \mathbb{A}_g^{an} in the usual topology.

It remains to show that $\mathcal{W} \setminus \bigcup_{F'} \mathcal{H}_{F'}(\bar{\mathbb{Q}})$ is contained in Σ . Indeed, for any $[z] \in \mathcal{W} \setminus \Sigma$ with F the maximal totally real subfield of $\text{End}^0(A_z)$, Proposition 9.2 implies that $[o]$ is contained in the Hilbert modular variety defined by $\mathbb{G}_m \cdot \text{Res}_{F/\mathbb{Q}} \text{SL}_2$, and hence F is a subfield of $\text{End}^0(A_o)$ and has degree g . So we are done. ■

10. Analytic subspace conjecture and its consequence on transcendence

Let S be a connected Shimura variety associated with the Shimura datum (\mathbf{G}, X) , and let $u: X \rightarrow S^{\text{an}}$ be the uniformization. Endow X with the $\bar{\mathbb{Q}}$ -structure as in Section 4.3. Fix a special point $[o] \in S(\bar{\mathbb{Q}})$ and some $o \in u^{-1}([o])$. Then o is a $\bar{\mathbb{Q}}$ -point of X . The goals of this section are to formulate an analogue of Wüstholz's analytic subgroup theorem for the holomorphic tangent space $T_o(X)$, which we call the *analytic subspace conjecture*, and to show how this conjecture gives an affirmative answer to Question 1.3.

10.1. Analytic subspace conjecture

Inspired by the reformulations of WAST for tori (Theorem 2.8) and for CM abelian varieties (Theorem 2.11), we make the following conjecture, which is precisely the analogous statement for Shimura varieties.

Let $T_o X$ and $T_{[o]} S$ be the tangent space defined over \mathbb{C} . Set

$$\psi := du: T_o X \longrightarrow T_{[o]} S.$$

Then the Harish-Chandra realization of X makes X into a bounded symmetric domain $\mathcal{D} \subseteq T_o X$ centered at o . By abuse of notation, we also write $u: \mathcal{D} \rightarrow S$.

Recall that $T_o X$ has a bi- $\bar{\mathbb{Q}}$ -structure defined in Section 4.

Conjecture 10.1. *Let $z \in \mathcal{D}$ be such that $[z] := u(z) \in S(\bar{\mathbb{Q}})$. Let V be a sub-vector space of $T_{[o]} S_{\bar{\mathbb{Q}}}$ with $\psi(z) \in V \otimes \mathbb{C}$. Then $V \supseteq \psi(V')$ for some bi- $\bar{\mathbb{Q}}$ -subspace $V' \subseteq T_o X$ which contains z .*

If moreover $MT(o)$ is a maximal torus, then we can take $V' = T_{[o]} S'_{\bar{\mathbb{Q}}}$ for some Shimura subvariety S' of S which contains $[z]$ and $[o]$.

The linear map ψ was computed for the Siegel case, i.e., $S = \mathbb{A}_g$, $(G, X) = (\mathrm{GSp}_{2g}, \mathfrak{H}_g)$ and $\mathcal{D} = \mathcal{D}_g$. Indeed by Theorem 7.1'', under suitable $\bar{\mathbb{Q}}$ -bases, the map ψ is the diagonal matrix $\frac{1}{\pi} \mathrm{diag}(\theta_j \theta_{j'})_{1 \leq j \leq j' \leq g}$, where the θ_j 's are the holomorphic periods of the CM abelian variety A_o parametrized by $[o]$. Note that in this case, if $MT(o)$ is a maximal torus, then A_o has no square factors.

10.2. Consequence on quadratic relations of holomorphic periods

Let A be a CM abelian variety of dimension g defined over $\bar{\mathbb{Q}}$ with no square factors. Let $\theta_1, \dots, \theta_g$ be its holomorphic periods as defined in Convention 1.1.

Proposition 10.2. *Assume that Conjecture 10.1 holds true for $S = \mathbb{A}_g$, the point $[o] \in \mathbb{A}_g(\bar{\mathbb{Q}})$ parametrizing A , and any V of dimension $\dim S - 1$. Then Question 1.3 has an affirmative answer for A .*

Proof. Each θ_j is well defined up to $\bar{\mathbb{Q}}^*$. By abuse of notation, we write $\theta_j \in \mathbb{C}$ for a representative. It is known by WAST that $\theta_j \notin \bar{\mathbb{Q}}$ for each $j \in \{1, \dots, g\}$.

For the uniformization $u: \mathfrak{H}_g \rightarrow \mathbb{A}_g$, take $o \in u^{-1}([o])$ and the derivative of u at o

$$\psi: T_o \mathfrak{H}_g \longrightarrow T_{[o]} \mathbb{A}_g.$$

We start the proof by fixing $\bar{\mathbb{Q}}$ -bases of $T_o \mathfrak{H}_g$ and of $T_{[o]} \mathbb{A}_g$. We have seen in Theorem 7.1'' that the matrix for ψ is $\mathrm{diag}(\theta_j \theta_{j'} / \pi)_{1 \leq j \leq j' \leq g}$ under suitable $\bar{\mathbb{Q}}$ -bases $\{f_{jj'}\}_{1 \leq j \leq j' \leq g}$ of $T_o \mathfrak{H}_{g, \bar{\mathbb{Q}}}$ and $\{e_{jj'}\}_{1 \leq j \leq j' \leq g}$ of $T_{[o]} \mathbb{A}_{g, \bar{\mathbb{Q}}}$. Moreover, by Theorem 7.4, these $\bar{\mathbb{Q}}$ -bases can be chosen such that each $\mathbb{C} f_{jj'} = \mathbb{C} e_{jj'}$ is a root space.

Write \mathcal{D}_g for the bounded realization of \mathfrak{H}_g based at o . Then $o \in \mathcal{D}_{g, \bar{\mathbb{Q}}} \subseteq T_o \mathfrak{H}_{g, \bar{\mathbb{Q}}}$ and o is the origin. For each $z \in \mathcal{D}_g$, write $[z]$ for the image of z under $\mathcal{D}_g \simeq \mathfrak{H}_g \rightarrow \mathbb{A}_g$.

Claim. There exists a $\bar{\mathbb{Q}}$ -basis $\{f_{jj'}\}_{1 \leq j \leq j' \leq g}$ of $T_o \mathfrak{H}_{g, \bar{\mathbb{Q}}}$ as above such the point $z := \sum f_{jj'}$ satisfies that $[z] \in \mathbb{A}_g(\bar{\mathbb{Q}})$.

Let us prove this claim. Set

$$\tilde{\Sigma} := \{z \in \mathcal{D}_g : z \text{ is CM, the only special subvariety of } \mathbb{A}_g \text{ passing through } [o] \text{ and } [z] \text{ is } \mathbb{A}_g\}.$$

Then $\tilde{\Sigma}$ is dense in \mathcal{D}_g in the usual topology by Corollary 9.3. Since $\tilde{\Sigma}$ is dense in \mathcal{D}_g , there exists $z = \sum k_{jj'} f_{jj'} \in \tilde{\Sigma}$ with $k_{jj'} \neq 0$ for each j, j' . Moreover, as each CM point is $\bar{\mathbb{Q}}$ in $T_o \mathfrak{H}_g$, we have that $k_{jj'} \in \bar{\mathbb{Q}}$. Thus $z = \sum k_{jj'} f_{jj'} \in \tilde{\Sigma}$ with each $k_{jj'} \in \bar{\mathbb{Q}}^*$. Now the claim holds true by replacing each $f_{jj'}$ by $k_{jj'} f_{jj'}$.

Finally, set

$$e_{jj'} := \frac{\psi(f_{jj'})}{\theta_j \theta_{j'}/\pi}.$$

By Theorem 7.1'' $\{e_{jj'}\}_{1 \leq j \leq j' \leq g}$ is a $\bar{\mathbb{Q}}$ -eigenbasis of $T_{[o]} \mathbb{A}_{g, \bar{\mathbb{Q}}}$.

Now assume

$$\sum_{1 \leq j \leq j' \leq g} c_{jj'} \theta_j \theta_{j'} = 0 \quad (10.1)$$

with $c_{jj'} \in \bar{\mathbb{Q}}$.

Define

$$V := \left\{ \sum a_{jj'} e_{jj'} \in T_{[o]} \mathbb{A}_{g, \bar{\mathbb{Q}}} : \sum c_{jj'} a_{jj'} = 0 \right\};$$

it is a $\bar{\mathbb{Q}}$ -subspace of $T_{[o]} \mathbb{A}_{g, \bar{\mathbb{Q}}}$ because each $c_{jj'} \in \bar{\mathbb{Q}}$. A direct computation shows that $\psi(z) = \psi(\sum f_{jj'}) = \frac{1}{\pi} \sum \theta_j \theta_{j'} e_{jj'}$. So $\psi(z) \in V \otimes \mathbb{C}$ by (10.1).

Hence we can apply Conjecture 10.1 to the point $z \in \mathcal{D}_g$ and V . So there exists a bi- $\bar{\mathbb{Q}}$ -subspace V' of $T_o \mathfrak{H}_{g, \bar{\mathbb{Q}}}$ containing $z = \sum f_{jj'}$ such that $\psi(V') \subset V$.

If $\text{MT}(o) = \text{MT}(A)$ is a maximal torus, then the ‘‘moreover’’ part of Conjecture 10.1 furthermore implies that we can take $V' = T_{[o]} S'_{\bar{\mathbb{Q}}}$ for some special subvariety S' of \mathbb{A}_g . But then by choice of z , we have $S' = \mathbb{A}_g$. So $V = T_{[o]} \mathbb{A}_{g, \bar{\mathbb{Q}}}$ and hence $c_{jj'} = 0$ for all j, j' .

In general, there exists an equivalence relation among the pairs in $\{1, \dots, g\}^2$ defined as follows: $(j, j') \sim (k, k')$ if and only if $\theta_j \theta_{j'} \simeq \theta_k \theta_{k'}$. Let $\{1, \dots, g\}^2 = \coprod_s J_s$ be the partition into equivalence classes, and for each s let us fix a pair $(j_s, j'_s) \in J_s$. Then (10.1) can be rewritten such that $c_{jj'} = 0$ if $(j, j') \neq (j_s, j'_s)$ for any s . Then our goal is to prove that $c_{j_s j'_s} = 0$ for each s . With this choice

$$V := \left\{ \sum a_{jj'} e_{jj'} \in T_{[o]} \mathbb{A}_{g, \bar{\mathbb{Q}}} : \sum_s c_{j_s j'_s} a_{j_s j'_s} = 0 \right\}.$$

By Proposition 2.5, the bi- $\bar{\mathbb{Q}}$ -subspace V' equals $\bigoplus_s V'_s$ for some $V'_s \subseteq \bigoplus_{(j, j') \in J_s} \mathbb{C} f_{jj'}$. Since $z = \sum f_{jj'} \in V'$, we have $\mathbb{C} \cdot \sum_{(j, j') \in J_s} f_{jj'} \subseteq V'_s$ for each s . Therefore, the subspace $\psi(V') \supset V$ contains the point $e_{j_s j'_s} + \sum_{(j, j') \in J_s \setminus \{(j_s, j'_s)\}} \frac{\theta_j \theta_{j'}}{\theta_{j_s} \theta_{j'_s}} e_{jj'}$ for each s . But from the last paragraph we assumed $c_{jj'} = 0$ for any $(j, j') \in J_s \setminus \{(j_s, j'_s)\}$. So $c_{j_s j'_s} = 0$ by definition of V . Hence we are done. ■

11. Grothendieck's period conjecture and consequences on CM periods

This section is rather independent of the rest of the paper. We recall the basic versions of Grothendieck's period conjecture, and explain some consequences on the algebraic relations among π and holomorphic periods of CM abelian varieties. We show that all non-trivial such algebraic relations are generated by a fixed set of monomials (Proposition 11.4).

11.1. Statement of the conjecture

Let X be a smooth projective irreducible variety defined over $\bar{\mathbb{Q}}$. Let

$$\beta: H_{\text{dR}}^*(X) \otimes_{\bar{\mathbb{Q}}} \mathbb{C} \xrightarrow{\sim} H^*(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

be the de Rham–Betti comparison. Under suitable bases, β is a matrix in $\text{GL}_N(\mathbb{C})$. Write $(\beta_{jj'})_{1 \leq j, j' \leq N}$ for this matrix.

For each $n \in \mathbb{N}$, each algebraic cycle in X^n induces $\bar{\mathbb{Q}}$ -polynomial relations, all homogenous of degree n , for the $\beta_{jj'}$'s. Indeed, for $n = 1$, an algebraic cycle of X of codimension d gives an element in $H_{\text{dR}}^{2d}(X) \cap \beta^{-1}(H^{2d}(X, \mathbb{Q}))$. Hence there exists a vector $\mathbf{v} = [v_1 \ \dots \ v_N]^T \in \bar{\mathbb{Q}}^N$ such that $\beta\mathbf{v} \in \mathbb{Q}^N$. Writing it out, we obtain N linear relations over $\bar{\mathbb{Q}}$ among the $\beta_{jj'}$'s. For general n , the assertion follows from Künneth's Formula.

Notice that among such relations, some of them are products of polynomials of lower degrees. This notably happens for elements in $\text{End}(X)$, which are cycles in X^2 but give linear relations. Indeed, for each $\alpha \in \text{End}(X)$, we have the following commutative diagram

$$\begin{array}{ccc} H_{\text{dR}}^*(X) \otimes_{\bar{\mathbb{Q}}} \mathbb{C} & \xrightarrow{\beta} & H^*(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \\ \alpha^* \downarrow & & \downarrow \alpha^* \\ H_{\text{dR}}^*(X) \otimes_{\bar{\mathbb{Q}}} \mathbb{C} & \xrightarrow{\beta} & H^*(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}. \end{array}$$

Under suitable bases, the α^* on the left is an $N \times N$ -matrix with entries in $\bar{\mathbb{Q}}$, and the α^* on the right is an $N \times N$ -matrix with entries in \mathbb{Q} . Thus this commutative diagram gives linear relations over $\bar{\mathbb{Q}}$ among the $\beta_{jj'}$'s.

Conjecture 11.1 (Grothendieck, strong version). *Consider the $\bar{\mathbb{Q}}$ -variety GL_N . Set I to be the ideal of the subvariety $\beta^{\bar{\mathbb{Q}}\text{-Zar}}$.⁶ Then I is generated by the polynomial relations obtained from all algebraic cycles in all powers of X .*

Let us take a closer look at this conjecture. Write $I = \bigoplus_{j \geq 1} I_j$ the homogenous decomposition. A consequence of this conjecture for I_1 is that I_1 is generated by relations

⁶Thus β is the generic point of this subvariety.

obtained from elements in $\text{End}(X)$. In case of commutative algebraic groups, this consequence can be deduced from Wüstholz's analytic subgroup theorem. Little is known for $j \geq 2$.

There is also a weaker version of this conjecture, sometimes also known as Grothendieck's period conjecture. Let G_{mod} be the motivic Galois group defined by Nori [11] and Ayoub [3]. In case of the first cohomology for abelian varieties, G_{mod} is precisely the Mumford–Tate group by André [2].

Conjecture 11.2 (Grothendieck, weak version). $\text{trdeg}_{\bar{\mathbb{Q}}}(\beta_{jj'})_{1 \leq j, j' \leq N} = \dim G_{\text{mod}}$.

The strong version implies the weak version. Conversely, the weak version implies the strong version if the universal period torsor is connected. We refer to [1, 7.5.2.2. Prop] for more details.

11.2. Consequences on holomorphic CM periods

Let A be a CM abelian variety over $\bar{\mathbb{Q}}$ with no square factors. Let $\theta_1, \dots, \theta_g$ be its holomorphic periods.

Let $\mathbf{T}_0 := \text{MT}(A)$. Then \mathbf{T}_0 is a subtorus of GSp_{2g} ; it contains \mathbb{G}_m , the center of GSp_{2g} . Thus the multiplier $\text{GSp}_{2g} \rightarrow \mathbb{G}_m$, $\alpha \mapsto \det(\alpha)^{1/g}$, is non-trivial when restricted to \mathbf{T}_0 . Hence there is a non-constant homomorphism $X^*(\mathbb{G}_m) \rightarrow X^*(\mathbf{T}_0)$. Let ε_0^* be the image of 1 of this homomorphism.

By Lemma 9.1, $\mathbf{T} := Z_{\text{GSp}_{2g}}(\mathbf{T}_0)$ is a maximal torus of GSp_{2g} . The last sentence of Remark 6.5 yields a \mathbb{Z} -basis $\{\varepsilon_0^*, \varepsilon_1^*, \dots, \varepsilon_g^*\}$ of $X^*(\mathbf{T})$ satisfying the following property: For de Rham–Hodge comparison $\beta \in \text{GSp}_{2g}(\mathbb{C})$, we have $\varepsilon_j^*(\beta) = \theta_j$ for each $j \in \{1, \dots, g\}$.

Proposition 11.3. *Assume the weak version of Grothendieck's period conjecture for A . Then each $\theta_j \theta_{j'}$ is a transcendental number.*

Proof. Consider the root system $\Phi(\mathbf{T}, \text{GSp}_{2g}) \subseteq X^*(\mathbf{T})$ defined in [6, §13.18]; it contains $2\varepsilon_j^*$, $\varepsilon_j^* + \varepsilon_{j'}^*$ and $\varepsilon_j^* - \varepsilon_{j'}^*$, for all $1 \leq j < j' \leq g$.

As $Z_{\text{GSp}_{2g}}(\mathbf{T}_0)$ is solvable, \mathbf{T}_0 is a semi-regular torus by [6, §13.1, Prop.]. So $\mathbf{T}_0 \not\subseteq \ker \chi$ for any $\chi \in \Phi(\mathbf{T}, \text{GSp}_{2g})$ by [6, §13.2, Prop.]. In particular,

$$(\varepsilon_j^* + \varepsilon_{j'}^*)|_{\mathbf{T}_0} \not\equiv 0 \quad \text{for all } 1 \leq j < j' \leq g. \quad (11.1)$$

By our choice of ε_j^* , we have $(\varepsilon_j^* + \varepsilon_{j'}^*)(\beta) = \theta_j \theta_{j'}$ for each $1 \leq j < j' \leq g$.

Now comes the step where we assume the weak version of Grothendieck's period conjecture for A . Then β is the generic point of \mathbf{T}_0 . Hence (11.1) implies $(\varepsilon_j^* + \varepsilon_{j'}^*)(\beta) \notin \bar{\mathbb{Q}}$ for all $1 \leq j < j' \leq g$. We are done. ■

Proposition 11.4. *Assume the weak version of Grothendieck's period conjecture for A . Then there exist monomials over $\bar{\mathbb{Q}}$ in π and the θ_j 's (with $j \in \{1, \dots, g\}$) such that each algebraic relations over $\bar{\mathbb{Q}}$ among these numbers are generated by these monomials.*

Proof. Set $l := \dim \mathbf{T} - \dim \mathbf{T}_0$.

The inclusion $\mathbf{T}_0 < \mathbf{T}$ induces a group homomorphism $X^*(\mathbf{T}) \rightarrow X^*(\mathbf{T}_0)$, which is furthermore surjective. The kernel of this homomorphism, denoted by N , is a free- \mathbb{Z} -module of rank l . Let $\{a_{j0}\varepsilon_0^* + \dots + a_{jg}\varepsilon_g^* : j = 1, \dots, l\}$ be a set of generators of N . Then

$$\mathbf{T}_0 = \{t \in \mathbf{T} : \prod_{k=0}^g \varepsilon_k^*(t)^{a_{jk}} = 1 \text{ for all } j = 1, \dots, l\}. \quad (11.2)$$

Denote by (t_0, t_1, \dots, t_g) the coordinates of $\mathbb{A}_{\bar{\mathbb{Q}}}^{g+1}$. The image of $\mathbf{T}_{\bar{\mathbb{Q}}} \xrightarrow{(\varepsilon_0^*, \dots, \varepsilon_g^*)} \mathbb{G}_{m, \bar{\mathbb{Q}}}^{g+1} \subseteq \mathbb{A}_{\bar{\mathbb{Q}}}^{g+1}$ is an open $\bar{\mathbb{Q}}$ -subvariety. For each $j \in \{1, \dots, l\}$, set $P_j := \prod_{k=0}^g t_k^{a_{jk}} - 1$.

By (11.2), as a $\bar{\mathbb{Q}}$ -subvariety we have

$$\mathbf{T}_0 = \mathbf{T} \cap Z(P_1, \dots, P_l).$$

As $l = \dim \mathbf{T} - \dim \mathbf{T}_0$ and $\mathbf{T}_{\bar{\mathbb{Q}}}$ is open in $\mathbb{P}_{\bar{\mathbb{Q}}}^{g+1}$, by Hilbert Nullstellensatz, each polynomial $P \in \bar{\mathbb{Q}}[t_0, \dots, t_g]$ which vanishes identically on \mathbf{T}_0 is in the radical of the ideal (P_1, \dots, P_l) , which is precisely (P_1, \dots, P_l) since the kernel of $X^*(\mathbf{T}) \rightarrow X^*(\mathbf{T}_0)$ is torsion free.

Now let us study the algebraic relations over $\bar{\mathbb{Q}}$ among $2\pi i, \theta_1, \dots, \theta_g$. Assume $P \in \bar{\mathbb{Q}}[t_0, \dots, t_g]$ is a polynomial such that $P(2\pi i, \theta_1, \dots, \theta_g) = 0$. Since $\varepsilon_0^*(\beta) = 2\pi i$ and $\varepsilon_j^*(\beta) = \theta_j$ for each $j \in \{1, \dots, g\}$, we have $P(\varepsilon_0^*(\beta), \varepsilon_1^*(\beta), \dots, \varepsilon_g^*(\beta)) = 0$.

Now comes the step where we assume the weak version of Grothendieck's period conjecture for A . Then β is the generic point of \mathbf{T}_0 . Thus from the last paragraph we have $P|_{\mathbf{T}_0} \equiv 0$. Hence P is in the ideal (P_1, \dots, P_l) . Therefore each algebraic relation among $\pi, \theta_1, \dots, \theta_g$ is generated by the l -monomials

$$(2\pi i)^{a_{j0}} \theta_1^{a_{j1}} \dots \theta_g^{a_{jg}} \in \bar{\mathbb{Q}}, \quad j \in \{1, \dots, l\}.$$

We are done. ■

This proposition yields the following immediate corollary.

Corollary 11.5. *The weak version of Grothendieck's period conjecture for A gives an affirmative answer to Question 1.3 for A .*

Acknowledgments. Both ZG and AY would like to thank the IHES for its hospitality during the preparation of this paper. Both ZG and EU would like to thank the MSRI for its hospitality during the work. We would like to thank Yves André and Daniel Bertrand for various discussions on transcendental number theory, especially on Shimura relations, Wüstholz's analytic subgroup theorem and Grothendieck's period conjecture. We would like to thank Martin Orr for discussion on problems related to isogeny and Hecke orbits. We would like to thank Gregorio Baldi for pointing out the reference [8] and for his comments on a previous version of the preprint. Finally, we would like to thank the reviewer for their constructive comments and careful reading of the manuscript, which helped us to improve the presentation of the paper.

Funding. Ziyang Gao has received fundings from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement n° 945714).

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Communicated by Otmar Venjakob

Received 11 April 2024; revised 15 September 2025.

Ziyang Gao

Department of Mathematics, University of California, Los Angeles, 520 Portola Plaza, Box 951555, Los Angeles, CA 90095, USA; ziyang.gao@math.ucla.edu

Emmanuel Ullmo

Laboratoire Alexander Grothendieck, IHES, Université Paris-Saclay, 35 route de Chartres, 91440 Bures sur Yvette, France; ullmo@ihes.fr

Andrei Yafaev

Department of Mathematics, University College London, Gower street, London WC1E 6BT, UK; yafaev@math.ucl.ac.uk