Doc. Math. (Online first)
DOI 10.4171/DM/1054

Bessel models for representations of GSp$(4, q)$

Jonathan Cohen

Abstract. We compute the Bessel models of every irreducible representation of the finite group GSp$(4, q)$.

1. Introduction

For F a local or global field, there has been significant work on Bessel models for GSp$(4, F)$ and their applications to Siegel modular forms and automorphic representations; see [3–6] for some examples. In this article, we consider the analogous situation over finite fields, and our main result is the computation of the Bessel models for all irreducible representations of GSp$(4, q)$; see Theorem 4.1. One of our primary motivations is the determination of the (nonsplit) Bessel models for the depth zero supercuspidals of p-adic GSp$(4, F)$, which we will carry out in a forthcoming paper. We also remark that there are some notable differences between the p-adic and finite field cases. For example, it is shown in [5] that all split Bessel models are unique (when they exist) over the p-adics, while a consequence of our computations is that the split Bessel models over finite fields can have multiplicity 2; see Corollary 4.2 below. On the other hand, over both p-adic and finite fields, the generic representations admit every split Bessel model and (if $q > 3$ in the finite field case) this characterizes genericity.

We now briefly outline the contents of this article. In Section 2 we introduce the requisite notation, in Section 3 we carry out a technical preliminary computation, and in Section 4 we prove the main result.

2. Notations

Let ζ_n denote a complex primitive nth root of unity and $\delta_{i,j}$ denote the Kronecker delta for i, j varying over some set. Let q be a power of a prime p and $\mathbb{F}_q$ denote the field of order q. If q is even then the group homomorphism $\mathbb{F}_q \to \mathbb{F}_q$ given by $x \mapsto x^2 + x$ is two-to-one. Let $\mathbb{F}_q^\circ$ be its image and $\varepsilon\colon \mathbb{F}_q \to \{\pm 1\}$ be the homomorphism with kernel $\mathbb{F}_q^\circ$. Let ξ be a fixed element of $\mathbb{F}_q$ that is not in $\mathbb{F}_q^\circ$; so $\varepsilon(t^2 + t) = 1$ and $\varepsilon(t^2 + t + \xi) = -1$ for all $t \in \mathbb{F}_q$. If q is odd let ξ be a fixed nonsquare in $\mathbb{F}_q^\times$.

Mathematics Subject Classification 2020: 20C33.
Keywords: GSp(4), Bessel models, finite groups.

Fix the symplectic form

$$J = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix} \tag{2.1}$$

and define

$$G := \mathrm{GSp}(4,q) = \left\{ g \in \mathrm{GL}(4,q) \mid {}^t g J g = \mu(g) J \text{ for some } \mu(g) \in \mathbb{F}_q^\times \right\}. \tag{2.2}$$

The kernel of the multiplier homomorphism $\mu\colon \mathrm{GSp}(4,q) \to \mathbb{F}_q^\times$ is the symplectic group $\mathrm{Sp}(4,q)$. The center Z of $\mathrm{GSp}(4,q)$ consists of scalar invertible matrices. If q is even then $G = \mathrm{Sp}(4,q) \times Z$. We further define the Siegel parabolic subgroup

$$P = G \cap \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{bmatrix} \tag{2.3}$$

and its unipotent radical N, given by

$$N = \left\{ \begin{bmatrix} 1 & & y & z \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \mid x, y, z \in \mathbb{F}_q \right\}. \tag{2.4}$$

If $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}(2,q)$ then let $A' := \frac{1}{ad-bc} \left[\begin{smallmatrix} a & -b \\ -c & d \end{smallmatrix}\right]$ so $\left[\begin{smallmatrix} A & \\ & uA' \end{smallmatrix}\right] \in P$ with multiplier $u \in \mathbb{F}_q^\times$.

Let ψ be a nontrivial character of $\mathbb{F}_q$. For $a, b, c \in \mathbb{F}_q$, let $\psi_{a,b,c}$ be the character of N given by

$$\psi_{a,b,c}\left(\begin{bmatrix} 1 & & y & z \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) = \psi(ax + by + cz) \tag{2.5}$$

for $x, y, z \in \mathbb{F}_q$. We denote by $\mathbb{C}_{a,b,c}$ the one-dimensional representation of N given by the character $\psi_{a,b,c}$. We also define the group

$$T = \left\{ \begin{bmatrix} r-bs & -as & & \\ cs & r & & \\ & & r-bs & as \\ & & -cs & r \end{bmatrix} \mid r, s \in \mathbb{F}_q,\ r^2 - brs + s^2 ac \in \mathbb{F}_q^\times \right\}. \tag{2.6}$$

The reader may readily check that T normalizes N and preserves the form $ax + by + cz$. Let $\chi\colon T \to \mathbb{C}^\times$ be a character. The map $tn \mapsto \chi(t)\psi_{a,b,c}(n)$ defines a character of the *Bessel subgroup*

$$R := TN, \tag{2.7}$$

which we denote by $\chi \otimes \psi_{a,b,c}$. Let $\mathbb{C}_{\chi,a,b,c}$ be the one-dimensional representation of R given by the character $\chi \otimes \psi_{a,b,c}$.

The irreducible characters of G have been determined in [7] (if q is odd) and [2] (if q is even). Some of the mistakes in [2, Table IV-2] have been corrected in [1, Table 7]; we remark that the values for θ_1 and θ_2 there are transposed.

3. Fourier coefficients

The main result of this paper is a computation of $\dim \mathrm{Hom}_R(V_\sigma, \mathbb{C}_{\chi,a,b,c})$ for all irreducible representations (σ, V_σ) of G, when $b^2 - 4ac \neq 0$. To carry this out it is convenient to first prove a preliminary result.

Proposition 3.1. *The dimensions of the spaces* $\mathrm{Hom}_N(V_\sigma, \mathbb{C}_{a,b,c})$ *for all irreducible representations* (σ, V_σ) *of* $\mathrm{GSp}(4, q)$ *are as given in Table 1 (if q is odd) and Table 3 (if q is even).*

Proof. By character theory we have

$$\dim \mathrm{Hom}_N(V_\sigma, \mathbb{C}_{a,b,c}) = \frac{1}{q^3} \sum_{x,y,z\in\mathbb{F}_q} \psi(-ax - by - cz)\theta\left(\begin{bmatrix} 1 & & y & z \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix}\right),$$

where θ is the trace character of σ. For 2×2 matrices X, Y over $\mathbb{F}_q$, write $X \sim Y$ if there exist $A \in \mathrm{GL}(2, \mathbb{F}_q)$ and $u \in \mathbb{F}_q^\times$ with $uAX(A')^{-1} = Y$.

Suppose that q is odd. Then, for $X = \left[\begin{smallmatrix} y & z \\ x & y \end{smallmatrix}\right]$,

$$\begin{array}{ll} \text{if } X \neq 0 \text{ and } \det(X) = 0, & \text{then } X \sim \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right], \\ \text{if } \det(X) \in \mathbb{F}_q^{\times 2}, & \text{then } X \sim \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \\ \text{if } \det(X) \in \xi\mathbb{F}_q^{\times 2}, & \text{then } X \sim \left[\begin{smallmatrix} 0 & -\xi \\ 1 & 0 \end{smallmatrix}\right]. \end{array}$$

Here, ξ is a fixed nonsquare in $\mathbb{F}_q^\times$. Hence

$$\begin{aligned} &\dim \mathrm{Hom}_N(V_\sigma, \mathbb{C}_{a,b,c}) \\ &= \frac{1}{q^3}\Biggl(\theta\left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) + \sum_{\substack{x,y,z\in\mathbb{F}_q \\ y^2 - xz = 0 \\ (x,y,z)\neq(0,0,0)}} \psi(-ax - by - cz)\cdot\theta\left(\begin{bmatrix} 1 & & & 1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) \\ &\qquad + \sum_{\substack{x,y,z\in\mathbb{F}_q \\ y^2 - xz \in \mathbb{F}_q^{\times 2}}} \psi(-ax - by - cz)\cdot\theta\left(\begin{bmatrix} 1 & & 1 & \\ & 1 & & 1 \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) \\ &\qquad + \sum_{\substack{x,y,z\in\mathbb{F}_q \\ y^2 - xz \in \xi\mathbb{F}_q^{\times 2}}} \psi(-ax - by - cz)\cdot\theta\left(\begin{bmatrix} 1 & & & -\xi \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{bmatrix}\right)\Biggr). \end{aligned}$$

The four matrices in this equation represent the conjugacy classes denoted by A_0, A_1, A_{21} and A_{22} in [7]. Hence the assertion follows from [7, Tables 5–11] and Lemma 3.2 below.

Now suppose that q is even. Then, for $X = \left[\begin{smallmatrix} y & z \\ x & y \end{smallmatrix}\right]$,

$$\begin{array}{ll} \text{if } X \neq 0 \text{ and } \det(X) = 0, & \text{then } X \sim \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right], \\ \text{if } \det(X) \in \mathbb{F}_q^\times \text{ and } x = z = 0, & \text{then } X \sim \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \\ \text{if } \det(X) \in \mathbb{F}_q^\times \text{ and } (x,z) \neq (0,0), & \text{then } X \sim \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]. \end{array}$$

σ	$\dim V_\sigma$	rank 0	rank 1	rank 2 square	rank 2 nonsquare	cusp	gen
$X_1(\lambda,\mu,\nu)$	$(q+1)^2(q^2+1)$	$4(q+1)$	$q+3$	$q+3$	$q+1$		$\bullet$
$X_2(\Lambda,\nu)$	q^4-1	$2(q-1)$	$q-1$	$q+1$	$q-1$		$\bullet$
$X_3(\Lambda,\nu)$	q^4-1	0	$q+1$	$q-1$	$q+1$		$\bullet$
$X_4(\Theta)$	$(q^2-1)^2$	0	$q-1$	$q-1$	$q+1$	$\bullet$	$\bullet$
$X_5(\Lambda,\omega)$	$(q-1)^2(q^2+1)$	0	$q-1$	$q-1$	$q-3$	$\bullet$	$\bullet$
$\chi_1(\lambda,\nu)$	$(q+1)(q^2+1)$	$2(q+1)$	1	2	0		
$\chi_2(\lambda,\nu)$	$q(q+1)(q^2+1)$	$2(q+1)$	$q+2$	$q+1$	$q+1$		$\bullet$
$\chi_3(\lambda,\nu)$	$(q+1)(q^2+1)$	$q+3$	2	1	1		
$\chi_4(\lambda,\nu)$	$q(q+1)(q^2+1)$	$3q+1$	$q+1$	$q+2$	q		$\bullet$
$\chi_5(\omega,\nu)$	$(q-1)(q^2+1)$	$q-1$	0	1	1		
$\chi_6(\omega,\nu)$	$q(q-1)(q^2+1)$	$q-1$	$q-1$	q	$q-2$		$\bullet$
$\chi_7(\Lambda)$	$(q-1)(q^2+1)$	0	1	0	2		
$\chi_8(\Lambda)$	$q(q-1)(q^2+1)$	0	q	$q-1$	$q-1$		$\bullet$
$\tau_1(\nu)$	q^2+1	2	1	0	0		
$\tau_2(\nu)$	$q(q^2+1)$	$q+1$	1	1	1		
$\tau_3(\nu)$	$q^2(q^2+1)$	$2q$	q	$q+1$	$q-1$		$\bullet$
$\tau_4(\lambda')$	q^2-1	0	1	0	0		
$\tau_5(\lambda')$	$q^2(q^2-1)$	0	q	$q-1$	$q+1$		$\bullet$
$\theta_1(\nu)$	$\frac{1}{2}q(q+1)^2$	$q+1$	1	1	0		
$\theta_2(\nu)$	$\frac{1}{2}q(q-1)^2$	0	0	0	1	$\bullet$	
$\theta_3(\nu)$	$\frac{1}{2}q(q^2+1)$	1	1	0	1		
$\theta_4(\nu)$	$\frac{1}{2}q(q^2+1)$	q	0	1	0		
$\theta_5(\nu)$	q^4	q	q	q	q		$\bullet$
$\theta_0(\nu)$	1	1	0	0	0		

Table 1. The irreducible characters of $\mathrm{GSp}(4,q)$ for q odd, as listed in [7]. If (σ, V_σ) is the representation in the first column, then the "rank 0" column shows $\dim \mathrm{Hom}_N(V_\sigma, \mathbb{C}_{0,0,0})$. The "rank 1" column shows $\dim \mathrm{Hom}_N(V_\sigma, \mathbb{C}_{a,b,c})$ if $b^2-4ac=0$ and $(a,b,c)\neq(0,0,0)$. The "rank 2 square" column shows $\dim \mathrm{Hom}_N(V_\sigma, \mathbb{C}_{a,b,c})$ if $b^2-4ac \in \mathbb{F}_q^{\times 2}$. The "rank 2 nonsquare" column shows $\dim \mathrm{Hom}_N(V_\sigma, \mathbb{C}_{a,b,c})$ if $b^2-4ac \in \xi\mathbb{F}_q^{\times 2}$. The last two columns indicate the cuspidal and generic representations.

Hence

$$
\begin{aligned}
&\dim \operatorname{Hom}_N(V_\sigma, \mathbb{C}_{a,b,c}) \\
&= \frac{1}{q^3}\Bigg(\theta\left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) + \sum_{\substack{x,y,z\in\mathbb{F}_q \\ y^2-xz=0 \\ (x,y,z)\neq(0,0,0)}} \psi(-ax-by-cz)\cdot\theta\left(\begin{bmatrix} 1 & & & 1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) \\
&\qquad + \sum_{\substack{x,y,z\in\mathbb{F}_q \\ y^2-xz\neq0 \\ (x,z)=(0,0)}} \psi(-ax-by-cz)\cdot\theta\left(\begin{bmatrix} 1 & & 1 & \\ & 1 & & 1 \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) \\
&\qquad + \sum_{\substack{x,y,z\in\mathbb{F}_q \\ y^2-xz\neq0 \\ (x,z)\neq(0,0)}} \psi(-ax-by-cz)\cdot\theta\left(\begin{bmatrix} 1 & & 1 & 1 \\ & 1 & & 1 \\ & & 1 & \\ & & & 1 \end{bmatrix}\right)\Bigg).
\end{aligned}
$$

The four matrices in this equation represent the conjugacy classes denoted by A_1, A_2, A_{31} and A_{32} in [2]. Hence the assertion follows from [2, Table IV-2] and Lemma 3.3 below. ∎

Lemma 3.2. *Let q be odd and $a, b, c \in \mathbb{F}_q$. Then*

(i)
$$
\sum_{\substack{x,y,z\in\mathbb{F}_q \\ y^2-xz=0 \\ (x,y,z)\neq(0,0,0)}} \psi(-ax-by-cz) = \begin{cases} q^2-1 & \text{if } a=b=c=0, \\ -1 & \text{if } b^2-4ac=0,\ (a,c)\neq(0,0), \\ q-1 & \text{if } b^2-4ac\in\mathbb{F}_q^{\times2}, \\ -q-1 & \text{if } b^2-4ac\in\xi\mathbb{F}_q^{\times2}. \end{cases}
$$

(ii)
$$
\sum_{\substack{x,y,z\in\mathbb{F}_q \\ y^2-xz\in\mathbb{F}_q^{\times2}}} \psi(-ax-by-cz) = \begin{cases} \dfrac{q(q^2-1)}{2} & \text{if } a=b=c=0, \\ \dfrac{q(q-1)}{2} & \text{if } b^2-4ac=0,\ (a,c)\neq(0,0), \\ -q & \text{if } b^2-4ac\in\mathbb{F}_q^{\times2}, \\ 0 & \text{if } b^2-4ac\in\xi\mathbb{F}_q^{\times2}. \end{cases}
$$

(iii)
$$
\sum_{\substack{x,y,z\in\mathbb{F}_q \\ y^2-xz\in\xi\mathbb{F}_q^{\times2}}} \psi(-ax-by-cz) = \begin{cases} \dfrac{q(q-1)^2}{2} & \text{if } a=b=c=0, \\ -\dfrac{q(q-1)}{2} & \text{if } b^2-4ac=0,\ (a,c)\neq(0,0), \\ 0 & \text{if } b^2-4ac\in\mathbb{F}_q^{\times2}, \\ q & \text{if } b^2-4ac\in\xi\mathbb{F}_q^{\times2}. \end{cases}
$$

Proof. Since

$$
\sum_{x,y,z\in\mathbb{F}_q} \psi(-ax-by-cz) = \begin{cases} q^3 & \text{if } a=b=c=0, \\ 0 & \text{otherwise}, \end{cases}
$$

(iii) will follow once we prove (i) and (ii).

(i) We calculate

$$\sum_{\substack{x,y,z\in\mathbb{F}_q\\ y^2-xz=0\\ (x,y,z)\neq(0,0,0)}} \psi(-ax-by-cz) = \sum_{\substack{x\in\mathbb{F}_q^\times\\ y\in\mathbb{F}_q}} \psi\big(-x(a+by+cy^2)\big) + \sum_{z\in\mathbb{F}_q^\times} \psi(-cz).$$

If $c=0$, then we see easily that

$$\sum_{\substack{x,y,z\in\mathbb{F}_q\\ y^2-xz=0\\ (x,y,z)\neq(0,0,0)}} \psi(-ax-by-cz) = \begin{cases} q-1 & \text{if } b\neq 0,\\ -1 & \text{if } a\neq 0,\ b=0,\\ q^2-1 & \text{if } a=b=0. \end{cases}$$

Suppose that $c\neq 0$. Then

$$\begin{aligned}\sum_{\substack{x,y,z\in\mathbb{F}_q\\ y^2-xz=0\\ (x,y,z)\neq(0,0,0)}} \psi(-ax-by-cz) &= \sum_{\substack{x\in\mathbb{F}_q^\times\\ y\in\mathbb{F}_q}} \psi\big(-x(a+by+cy^2)\big)-1\\ &= \begin{cases} 1\cdot(q-1)+(q-1)(-1)-1 & \text{if } b^2-4ac=0,\\ 2\cdot(q-1)+(q-2)(-1)-1 & \text{if } b^2-4ac\in\mathbb{F}_q^{\times 2},\\ 0\cdot(q-1)+q(-1)-1 & \text{if } b^2-4ac\in\xi\mathbb{F}_q^{\times 2}, \end{cases}\\ &= \begin{cases} -1 & \text{if } b^2-4ac=0,\\ q-1 & \text{if } b^2-4ac\in\mathbb{F}_q^{\times 2},\\ -q-1 & \text{if } b^2-4ac\in\xi\mathbb{F}_q^{\times 2}. \end{cases}\end{aligned}$$

This concludes the proof of (i).

(ii) We calculate

$$\sum_{\substack{x,y,z\in\mathbb{F}_q\\ y^2-xz\in\mathbb{F}_q^{\times 2}}} \psi(-ax-by-cz) = \frac{1}{2}\sum_{u\in\mathbb{F}_q^\times}\sum_{\substack{x,y,z\in\mathbb{F}_q\\ y^2-xz=1}} \psi\big(u(ax+by+cz)\big).$$

The $x=0$ terms contribute

$$\frac{1}{2}\sum_{u\in\mathbb{F}_q^\times}\sum_{z\in\mathbb{F}_q} \psi(cz)\big(\psi(ub)+\psi(-ub)\big) = q\delta_{c,0}(-1+q\delta_{b,0}).$$

The $x\neq 0$ terms contribute

$$\begin{aligned}&\frac{1}{2}\sum_{u,x\in\mathbb{F}_q^\times}\sum_{y\in\mathbb{F}_q} \psi\left(u\left(ax+by+c\left(\frac{y^2-1}{x}\right)\right)\right)\\ &\qquad= \frac{1}{2}\sum_{u,x\in\mathbb{F}_q^\times}\sum_{y\in\mathbb{F}_q} \psi\big(u(ax^2+byx+cy^2-c)\big).\end{aligned}$$

First suppose that $c = 0$. Then

$$\frac{1}{2}\sum_{u,x\in\mathbb{F}_q^\times}\sum_{y\in\mathbb{F}_q}\psi\big(u(ax^2+byx+cy^2-c)\big) = \frac{1}{2}\sum_{u,x\in\mathbb{F}_q^\times}\sum_{y\in\mathbb{F}_q}\psi(ua+by)$$
$$= \frac{q(q-1)}{2}\delta_{b,0}(-1+q\delta_{a,0}).$$

This confirms the formulas when $c = 0$. From now on suppose that $c \neq 0$. Then

$$\frac{1}{2}\sum_{u,x\in\mathbb{F}_q^\times}\sum_{y\in\mathbb{F}_q}\psi\big(u(ax^2+byx+cy^2-c)\big)$$
$$= \frac{1}{2}\sum_{u,x\in\mathbb{F}_q^\times}\sum_{y\in\mathbb{F}_q}\psi\big(u\big(y^2-x^2(b^2-4ac)-1\big)\big). \tag{3.1}$$

Suppose that $b^2-4ac = 0$. Then

$$(3.1) = \frac{(q-1)}{2}\sum_{\substack{u\in\mathbb{F}_q^\times\\ y\in\mathbb{F}_q}}\psi\big(u(y^2-1)\big) = \frac{(q-1)}{2}\big(2(q-1)-(q-2)\big) = \frac{q(q-1)}{2}.$$

Next suppose that $b^2-4ac \in \xi\mathbb{F}_q^{\times 2}$. Then using the fact that there are $q+1$ solutions $(x,y)\in\mathbb{F}_q\times\mathbb{F}_q$ to the equation $y^2-x^2\xi = 1$, we obtain

$$2*(3.1) = \sum_{\substack{u\in\mathbb{F}_q^\times\\ x,y\in\mathbb{F}_q}}\psi\big(-u(y^2-x^2\xi-1)\big) - \sum_{\substack{u\in\mathbb{F}_q^\times\\ y\in\mathbb{F}_q}}\psi\big(-u(y^2-1)\big)$$
$$= (q-1)(q+1)-\big(q^2-(q+1)\big)-\big(2(q-1)-(q-2)\big) = 0.$$

Finally, suppose that $b^2-4ac \in \mathbb{F}_q^{\times 2}$. Then

$$(3.1) = \frac{1}{2}\sum_{\substack{u,x\in\mathbb{F}_q^\times\\ y\in\mathbb{F}_q}}\psi\big(u(y^2-x^2-1)\big) = \frac{1}{2}\sum_{\substack{u,x\in\mathbb{F}_q^\times\\ y\in\mathbb{F}_q}}\psi\big(u(y^2+2yx-1)\big)$$
$$= \frac{1}{2}\sum_{u,x,y\in\mathbb{F}_q^\times}\psi\big(u(y^2+x-1)\big) + \frac{1}{2}\sum_{u,x\in\mathbb{F}_q^\times}\psi(-u)$$
$$= \sum_{u,x\in\mathbb{F}_q^\times}\psi(ux) + \frac{(q-3)}{2}\sum_{u,x\in\mathbb{F}_q^\times}\psi\big(u(x+1)\big) - \frac{q-1}{2}$$
$$= -(q-1) + \frac{q-3}{2}\big(q-1-(q-2)\big) - \frac{q-1}{2} = -q.$$

This concludes the proof. ∎

Lemma 3.3. *Let q be even and $a, b, c \in \mathbb{F}_q$. Then*

(i)
$$\sum_{\substack{x,y,z\in\mathbb{F}_q\\ y^2-xz=0\\ (x,y,z)\neq(0,0,0)}} \psi(-ax-by-cz) = \begin{cases} q^2-1 & \text{if } a=b=c=0,\\ -1 & \text{if } b=0,\ (a,c)\neq(0,0),\\ q-1 & \text{if } b\neq 0,\ \varepsilon(acb^{-2})=1,\\ -q-1 & \text{if } b\neq 0,\ \varepsilon(acb^{-2})=-1.\end{cases}$$

(ii)
$$\sum_{\substack{x,y,z\in\mathbb{F}_q\\ y^2-xz\neq0\\ (x,z)=(0,0)}} \psi(-ax-by-cz) = \begin{cases} q-1 & \text{if } a=b=c=0,\\ q-1 & \text{if } b=0,\ (a,c)\neq(0,0),\\ -1 & \text{if } b\neq 0,\ \varepsilon(acb^{-2})=1,\\ -1 & \text{if } b\neq 0,\ \varepsilon(acb^{-2})=-1.\end{cases}$$

(iii)
$$\sum_{\substack{x,y,z\in\mathbb{F}_q\\ y^2-xz\neq0\\ (x,z)\neq(0,0)}} \psi(-ax-by-cz) = \begin{cases} (q-1)(q^2-1) & \text{if } a=b=c=0,\\ -q+1 & \text{if } b=0,\ (a,c)\neq(0,0),\\ -q+1 & \text{if } b\neq 0,\ \varepsilon(acb^{-2})=1,\\ q+1 & \text{if } b\neq 0,\ \varepsilon(acb^{-2})=-1.\end{cases}$$

Proof. Since

$$\sum_{x,y,z\in\mathbb{F}_q} \psi(-ax-by-cz) = \begin{cases} q^3 & \text{if } a=b=c=0,\\ 0 & \text{otherwise,}\end{cases}$$

(iii) will follow once we prove (i) and (ii). Note that (ii) is an easy exercise. To prove (i), we calculate

$$\begin{aligned}\sum_{\substack{x,y,z\in\mathbb{F}_q\\ y^2-xz=0\\ (x,y,z)\neq(0,0,0)}} \psi(-ax-by-cz) &= \sum_{\substack{x\in\mathbb{F}_q^\times\\ y\in\mathbb{F}_q}} \psi\Big(-ax-by-c\frac{y^2}{x}\Big) + \sum_{z\in\mathbb{F}_q^\times}\psi(-cz)\\ &= \sum_{\substack{x\in\mathbb{F}_q^\times\\ y\in\mathbb{F}_q}} \psi\big(-x(a+by+cy^2)\big) + \sum_{z\in\mathbb{F}_q^\times}\psi(-cz).\end{aligned}$$

If $c=0$, then we see easily that

$$\sum_{\substack{x,y,z\in\mathbb{F}_q\\ y^2-xz=0\\ (x,y,z)\neq(0,0,0)}} \psi(-ax-by-cz) = \begin{cases} q-1 & \text{if } b\neq 0,\\ -1 & \text{if } a\neq 0,\ b=0,\\ q^2-1 & \text{if } a=b=0.\end{cases}$$

Suppose that $c\neq 0$. Then

$$\sum_{\substack{x,y,z\in\mathbb{F}_q\\ y^2-xz=0\\ (x,y,z)\neq(0,0,0)}} \psi(-ax-by-cz) = \sum_{\substack{x\in\mathbb{F}_q^\times\\ y\in\mathbb{F}_q}} \psi\big(-x(a+by+cy^2)\big) - 1.$$

If $a = b = 0$, this is easily seen to be -1. Suppose that $b = 0$ and $a \neq 0$. Then $a + by + cy^2 = 0$ for exactly one value of y, so that

$$\sum_{\substack{x,y,z\in\mathbb{F}_q\\ y^2-xz=0\\ (x,y,z)\neq(0,0,0)}} \psi(-ax - by - cz) = 1(q-1) + (q-1)(-1) - 1 = -1.$$

Suppose that $b \neq 0$ (and still $c \neq 0$). Then

$$a + by + cy^2 = 0 \iff bcy + (cy)^2 = ca \iff cb^{-1}y + (cb^{-1}y)^2 = cab^{-2}.$$

If $\varepsilon(cab^{-2}) = 1$, then this equation has exactly two solutions, otherwise none. Hence

$$\sum_{\substack{x,y,z\in\mathbb{F}_q\\ y^2-xz=0\\ (x,y,z)\neq(0,0,0)}} \psi(-ax-by-cz) = \begin{cases} 2\cdot(q-1) + (q-2)(-1) - 1 & \text{if } \varepsilon(cab^{-2})=1,\\ 0\cdot(q-1) + q(-1) - 1 & \text{if } \varepsilon(cab^{-2})=-1 \end{cases}$$

$$= \begin{cases} q-1 & \text{if } \varepsilon(cab^{-2}) = 1,\\ -q-1 & \text{if } \varepsilon(cab^{-2}) = -1. \end{cases}$$

This concludes the proof of (i). ■

4. Bessel models

We now state and prove the main result of this paper.

Theorem 4.1. *If $b^2 - 4ac \neq 0$ then the dimensions of the spaces $\mathrm{Hom}_R(V_\sigma, \mathbb{C}_{\chi,a,b,c})$ for all irreducible representations (σ, V_σ) of $\mathrm{GSp}(4,q)$ are as given in Table 2 if q is odd and in Table 4 if q is even.*

Proof. Let ω be the central character and θ the trace character of σ. By character theory,

$$\begin{aligned}\dim \mathrm{Hom}_R(V_\sigma, \mathbb{C}_{\chi,a,b,c}) &= \frac{1}{|T|q^3} \sum_{\substack{t\in T\\ x,y,z\in\mathbb{F}_q}} \chi(t)^{-1}\psi(-ax-by-cz)\theta\left(t\begin{bmatrix}1&&y&z\\&1&x&y\\&&1&\\&&&1\end{bmatrix}\right)\\ &= S_1 + S_2,\end{aligned}$$

where

$$S_1 = \frac{1}{|T|q^3}\sum_{t\in Z}\sum_{x,y,z\in\mathbb{F}_q} \chi(t)^{-1}\psi(-ax-by-cz)\theta\left(t\begin{bmatrix}1&&y&z\\&1&x&y\\&&1&\\&&&1\end{bmatrix}\right),$$

$$S_2 = \frac{1}{|T|q^3}\sum_{t\in T\setminus Z}\sum_{x,y,z\in\mathbb{F}_q} \chi(t)^{-1}\psi(-ax-by-cz)\theta\left(t\begin{bmatrix}1&&y&z\\&1&x&y\\&&1&\\&&&1\end{bmatrix}\right).$$

σ	square	nonsquare
$X_1(\lambda, \mu, \nu)$	$1 + (\chi_+, \nu + \nu\mu + \nu\lambda + \nu\mu\lambda)$	1
$X_2(\Lambda, \nu)$	$1 + (\chi_+, \nu + \nu\Lambda\|_{\mathbb{F}_q^\times})$	$1 - \big(\chi, \Lambda(\nu \circ N) + \Lambda^q(\nu \circ N)\big)$
$X_3(\Lambda, \nu)$	1	1
$X_4(\Theta)$	1	1
$X_5(\Lambda, \omega)$	1	$1 - (\chi, \Lambda + \Lambda^q + \Lambda\widetilde{\omega} + \Lambda^q\widetilde{\omega}^q)$
$\chi_1(\lambda, \nu)$	$(\chi_+, \nu + \nu\lambda)$	0
$\chi_2(\lambda, \nu)$	$1 + (\chi_+, \nu + \nu\lambda)$	1
$\chi_3(\lambda, \nu)$	$(\chi_+, \nu\lambda)$	$\big(\chi, (\lambda\nu) \circ N\big)$
$\chi_4(\lambda, \nu)$	$1 + (\chi_+, \nu + \nu\lambda + \nu\lambda^2)$	$1 - \big(\chi, (\lambda\nu) \circ N\big)$
$\chi_5(\omega, \nu)$	(χ_+, ν)	$(\chi, \nu \circ N)$
$\chi_6(\omega, \nu)$	$1 + (\chi_+, \nu)$	$1 - (\chi, \nu \circ N + \widetilde{\omega}\nu \circ N + \widetilde{\omega}^q\nu \circ N)$
$\chi_7(\Lambda)$	0	$(\chi, \Lambda + \Lambda^q)$
$\chi_8(\Lambda)$	1	$1 - (\chi, \Lambda + \Lambda^q)$
$\tau_1(\nu)$	0	0
$\tau_2(\nu)$	(χ_+, ν)	$(\chi, \nu \circ N)$
$\tau_3(\nu)$	$1 + (\chi_+, \nu + \nu\alpha_0)$	$1 - \big(\chi, \nu \circ N + (\nu\alpha_0) \circ N\big)$
$\tau_4(\lambda')$	0	0
$\tau_5(\lambda')$	1	1
$\theta_1(\nu)$	(χ_+, ν)	0
$\theta_2(\nu)$	0	$(\chi, \nu \circ N)$
$\theta_3(\nu)$	0	$(\chi, \nu \circ N)$
$\theta_4(\nu)$	(χ_+, ν)	0
$\theta_5(\nu)$	$1 + (\chi_+, \nu)$	$1 - (\chi, \nu \circ N)$
$\theta_0(\nu)$	0	0

Table 2. The irreducible characters of $\mathrm{GSp}(4, q)$ for q odd, as listed in [7]. If (σ, V_σ) is the representation in the first column, the remainder shows $\dim \mathrm{Hom}_R(V_\sigma, \mathbb{C}_{\chi,a,b,c})$ assuming $\chi|_Z = \omega_\sigma$. In the square column, in which $b^2 - 4ac \in \mathbb{F}_q^{\times 2}$, we write $\chi_+ = \chi|_{T_+}$ and use the canonical isomorphism $T_+ \cong \mathbb{F}_q^\times$ to identify χ_+ with a character of $\mathbb{F}_q^\times$. In the square column the inner product is for characters of $\mathbb{F}_q^\times$ while in the nonsplit column it is for $\mathbb{F}_{q^2}^\times$. By $N\colon \mathbb{F}_{q^2}^\times \to \mathbb{F}_q^\times$ we denote the norm map.

Clearly if $\chi|_Z \neq \omega$ then $S_1 = S_2 = 0$, so assume that $\chi|_Z = \omega$. Then

$$S_1 = \frac{q-1}{|T|q^3} \sum_{x,y,z \in \mathbb{F}_q} \psi(-ax - by - cz)\theta\left(\begin{bmatrix} 1 & & y & z \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix}\right).$$

Up to the factor of $(q-1)/|T|$, this was computed in the previous section.

σ	$\dim V_\sigma$	$b=0$ $(a,c)=(0,0)$	$b=0$ $(a,c)\neq(0,0)$	$b\neq 0$ $\varepsilon(\frac{ac}{b^2})=1$	$b\neq 0$ $\varepsilon(\frac{ac}{b^2})=-1$	cusp	gen
θ_0	1	1	0	0	0		
θ_1	$\frac{1}{2}q(q+1)^2$	$q+1$	1	1	0		
θ_2	$\frac{1}{2}q(q^2+1)$	1	1	0	1		
θ_3	$\frac{1}{2}q(q^2+1)$	q	0	1	0		
θ_4	q^4	q	q	q	q		•
θ_5	$\frac{1}{2}q(q-1)^2$	0	0	0	1	•	
$\chi_1(k,l)$	$(q+1)^2(q^2+1)$	$4(q+1)$	$q+3$	$q+3$	$q+1$		•
$\chi_2(k)$	q^4-1	$2(q-1)$	$q-1$	$q+1$	$q-1$		•
$\chi_3(k,l)$	q^4-1	0	$q+1$	$q-1$	$q+1$		•
$\chi_4(k,l)$	$(q-1)^2(q^2+1)$	0	$q-1$	$q-1$	$q-3$	•	•
$\chi_5(k)$	$(q^2-1)^2$	0	$q-1$	$q-1$	$q+1$	•	•
$\chi_6(k)$	$(q+1)(q^2+1)$	$q+3$	2	1	1		
$\chi_7(k)$	$(q+1)(q^2+1)$	$2(q+1)$	1	2	0		
$\chi_8(k)$	$(q-1)(q^2+1)$	$q-1$	0	1	1		
$\chi_9(k)$	$(q-1)(q^2+1)$	0	1	0	2		
$\chi_{10}(k)$	$q(q+1)(q^2+1)$	$3q+1$	$q+1$	$q+2$	q		•
$\chi_{11}(k)$	$q(q+1)(q^2+1)$	$2(q+1)$	$q+2$	$q+1$	$q+1$		•
$\chi_{12}(k)$	$q(q-1)(q^2+1)$	$q-1$	$q-1$	q	$q-2$		•
$\chi_{13}(k)$	$q(q-1)(q^2+1)$	0	q	$q-1$	$q-1$		•

Table 3. The irreducible characters of GSp(4, q) for q even. Shown are the irreducible characters of Sp(4, q) as determined in [2]. The irreducible characters of GSp(4, q) follow from GSp(4, q) $= \mathbb{F}_q^\times \times$ Sp(4, q). If (σ, V_σ) is the representation in the first column, then columns 3, 4, 5, 6 show $\dim \mathrm{Hom}_N(V_\sigma, \mathbb{C}_{a,b,c})$ under the indicated conditions. The last columns indicate the cuspidal and generic representations.

Suppose first that q is odd. Evidently $T \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$ if b^2-4ac is a square and $T \cong \mathbb{F}_{q^2}^\times$ otherwise. Let

$$g = \begin{bmatrix} r-sb/2 & -sa & & \\ sc & r+sb/2 & & \\ & & r-sb/2 & sa \\ & & -sc & r+sb/2 \end{bmatrix} \begin{bmatrix} 1 & & y & z \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \in TN.$$

The eigenvalues of g are $\alpha_\pm = \alpha_\pm(g) = r \pm s(b^2/4 - ac)^{1/2}$. If $s=0$, the conjugacy class of g was determined in the previous section. Suppose $s \neq 0$. If b^2-4ac is a square then

$$g \text{ is in a conjugacy class of type } \begin{cases} D_0(\alpha_+,\alpha_-) & \text{if } ax+by+cz=0, \\ D_1(\alpha_+,\alpha_-) & \text{if } ax+by+cz\neq 0, \end{cases}$$

σ	$\varepsilon(acb^{-2}) = 1$	$\varepsilon(acb^{-2}) = -1$
θ_0	0	0
θ_1	$\delta_{j,0}$	0
θ_2	0	$\delta_{j,0}$
θ_3	$\delta_{j,0}$	0
θ_4	$1+\delta_{j,0}$	$1-\delta_{j,0}$
θ_5	0	$\delta_{j,0}$
$\chi_1(k,l)$	$1+\delta_{j,k\pm l}+\delta_{j,-k\pm l}$	1
$\chi_2(k)$	$1+\delta_{j,\pm k}$	$1-\delta_{j,\pm k}$
$\chi_3(k,l)$	1	1
$\chi_4(k,l)$	1	$1-\delta_{j,k\pm l}-\delta_{j,-k\pm l}$
$\chi_5(k)$	1	1
$\chi_6(k)$	$\delta_{j,0}$	$\delta_{j,0}$
$\chi_7(k)$	$\delta_{j,\pm k}$	0
$\chi_8(k)$	$\delta_{j,0}$	$\delta_{j,0}$
$\chi_9(k)$	0	$\delta_{j,\pm k}$
$\chi_{10}(k)$	$1+\delta_{j,0}+\delta_{j,\pm k}$	$1-\delta_{j,0}$
$\chi_{11}(k)$	$1+\delta_{j,\pm k}$	1
$\chi_{12}(k)$	$1+\delta_{j,0}$	$1-\delta_{j,0}-\delta_{j,\pm 2k}$
$\chi_{13}(k)$	1	$1-\delta_{j,\pm k}$

Table 4. The irreducible characters of $\mathrm{GSp}(4,q)$ for q even. Shown are the irreducible characters of $\mathrm{Sp}(4,q)$ as determined in [2]. The irreducible characters of $\mathrm{GSp}(4,q)$ follow from $\mathrm{GSp}(4,q) = \mathbb{F}_q^\times \times \mathrm{Sp}(4,q)$. If (σ, V_σ) is the representation in the first column, the next columns give $\dim \mathrm{Hom}_R(V_\sigma, \mathbb{C}_{\chi,a,b,c})$ assuming $\chi|_Z = \omega_\sigma$. If $\varepsilon(acb^{-2}) = 1$ then we identify $T \cap \mathrm{Sp}(4,q)$ with $\langle\gamma\rangle$, define the index j by $\chi(\gamma) = \zeta_{q-1}^j$, and $\delta_{y,z}$ denotes the Kronecker delta over $\mathbb{Z}/(q-1)\mathbb{Z}$. If $\varepsilon(acb^{-2}) = -1$ then we identify $T \cap \mathrm{Sp}(4,q)$ with $\langle\eta\rangle$, define the index j by $\chi(\gamma) = \zeta_{q+1}^j$, and $\delta_{y,z}$ denotes the Kronecker delta over $\mathbb{Z}/(q+1)\mathbb{Z}$.

while if $b^2 - 4ac$ is a nonsquare then

$$g \text{ is in a conjugacy class of type } \begin{cases} F_0(\alpha_+) & \text{if } ax+by+cz = 0, \\ F_1(\alpha_+) & \text{if } ax+by+cz \neq 0. \end{cases}$$

Here we are using the notation of [7].

Suppose $b^2 - 4ac$ is a square, so $|T| = (q-1)^2$. Define the subgroups $T_\pm = \{t \in T \mid \alpha_\mp(t) = 1\}$ and write $\chi_\pm := \chi|_{T_\pm}$. Clearly $T = T_+T_- = T_\pm Z \cong T_+ \times T_- \cong Z \times T_\pm$ and $T_\pm \cong \mathbb{F}_q^\times$ via $t \mapsto \alpha_\pm(t)$. The reader can check that if $t \in T_\pm$ then $\alpha_\pm(t) = 2r - 1$.

Let t_d denote the unique element of T_+ with eigenvalues d and 1. From the above we have

$$\begin{aligned}
S_2 &= \frac{1}{q^3(q-1)^2}\sum_{t\in T\setminus Z}\Bigg(\sum_{\substack{x,y,z\in\mathbb{F}_q\\ ax+by+zc=0}}\chi(t)^{-1}\psi(-ax-by-cz)\theta\big(D_0\big(\alpha_+(t),\alpha_-(t)\big)\big)\\
&\qquad\qquad + \sum_{\substack{x,y,z\in\mathbb{F}_q\\ ax+by+zc\neq 0}}\chi(t)^{-1}\psi(-ax-by-cz)\theta\big(D_1\big(\alpha_+(t),\alpha_-(t)\big)\big)\Bigg)\\
&= \frac{1}{q(q-1)^2}\sum_{t\in T\setminus Z}\chi(t)^{-1}\big[\theta\big(D_0\big(\alpha_+(t),\alpha_-(t)\big)\big)-\theta\big(D_1\big(\alpha_+(t),\alpha_-(t)\big)\big)\big]\\
&= \frac{1}{q(q-1)^2}\sum_{t\in T\setminus Z}\chi\left(\frac{1}{\alpha_-(t)}t\right)^{-1}\big[\theta\big(D_0\big(\alpha_+(t)/\alpha_-(t),1\big)\big)-\theta\big(D_1\big(\alpha_+(t)/\alpha_-(t),1\big)\big)\big]\\
&= \frac{1}{q(q-1)}\sum_{1\neq t\in T_+}\chi(t)^{-1}\big[\theta\big(D_0\big(\alpha_+(t),1\big)\big)-\theta\big(D_1\big(\alpha_+(t),1\big)\big)\big]\\
&= \frac{1}{q(q-1)}\sum_{r\in\mathbb{F}_q\setminus\{1/2,1\}}\chi(t_{2r-1})^{-1}\big[\theta\big(D_0(2r-1,1)\big)-\theta\big(D_1(2r-1,1)\big)\big]\\
&= \frac{1}{q(q-1)}\sum_{1\neq d\in\mathbb{F}_q^\times}\chi_+(d)^{-1}\big[\theta\big(D_0(d,1)\big)-\theta\big(D_1(d,1)\big)\big].
\end{aligned}$$

In the last line we identify d and t_d via the isomorphism $T_+\cong\mathbb{F}_q^\times$. The value of S_2 can now readily be computed from the character table in [7].

Now suppose b^2-4ac is not a square, so $|T|=q^2-1$. For $t\in T\setminus Z$, let α_t,α_t^q denote the eigenvalues of t. By the above considerations,

$$\begin{aligned}
S_2 &= \frac{1}{q^3(q^2-1)}\sum_{t\in T\setminus Z}\Bigg(\sum_{\substack{x,y,z\in\mathbb{F}_q\\ ax+by+zc=0}}\chi(t)^{-1}\psi(-ax-by-cz)\theta\big(F_0(\alpha_t)\big)\\
&\qquad\qquad + \sum_{\substack{x,y,z\in\mathbb{F}_q\\ ax+by+zc\neq 0}}\chi(t)^{-1}\psi(-ax-by-cz)\theta\big(F_1(\alpha_t)\big)\Bigg)\\
&= \frac{1}{q^3(q^2-1)}\sum_{t\in T\setminus Z}\chi(t)^{-1}\Big(q^2\theta\big(F_0(\alpha_t)\big)\\
&\quad+\Big(\sum_{x,y,z\in\mathbb{F}_q}\psi(-ax-by-cz)-\sum_{\substack{x,y,z\in\mathbb{F}_q\\ ax+by+zc=0}}\psi(-ax-by-cz)\Big)\theta\big(F_1(\alpha_t)\big)\Big)\\
&= \frac{1}{q(q^2-1)}\sum_{t\in T\setminus Z}\chi(t)^{-1}\big(\theta\big(F_0(\alpha_t)\big)-\theta\big(F_1(\alpha_t)\big)\big).
\end{aligned}$$

The value of S_2 can now readily be computed from the character tables in [7].

Now suppose q is even, so $b \neq 0$. Following [2], we fix elements γ and η of $\mathbb{F}_{q^2}$ of orders $q-1$ and $q+1$, respectively. Evidently $T \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$ if $\varepsilon(acb^{-2}) = 1$ and $T \cong \mathbb{F}_{q^2}^\times$ otherwise. Let

$$g = \begin{bmatrix} r+bs & as & & \\ sc & r & & \\ & & r+bs & as \\ & & cs & rs \end{bmatrix} \begin{bmatrix} 1 & & y & z \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix} \in TN. \tag{4.1}$$

Note $\mu(g)^{-1/2} g \in \mathrm{Sp}(4,q)$ and if $s=0$ then the conjugacy class of $\mu(g)^{-1/2}g = r^{-1/2}g$ was found in the previous section. So assume $s \neq 0$. If $\varepsilon(acb^{-2}) = 1$ then the eigenvalues of g are $\gamma^{\pm i}$ for some i and

$$\mu(g)^{-1/2} g \text{ is in a conjugacy class of type } \begin{cases} C_2(i) & \text{if } ax+by+cz=0, \\ D_2(i) & \text{if } ax+by+cz \neq 0, \end{cases} \tag{4.2}$$

while if $\varepsilon(acb^{-2}) = -1$ then the eigenvalues of g are $\eta^{\pm i}$ for some i and

$$\mu(g)^{-1/2} g \text{ is in a conjugacy class of type } \begin{cases} C_4(i) & \text{if } ax+by+cz=0, \\ D_4(i) & \text{if } ax+by+cz \neq 0. \end{cases} \tag{4.3}$$

Here we are using the notation of [2]. We identify $T \cap \mathrm{Sp}(4,q)$ with $\mathbb{F}_q^\times = \langle \gamma \rangle$ if $\varepsilon(acb^{-2}) = 1$ and with $\langle \eta \rangle$ if $\varepsilon(acb^{-2}) = -1$. Arguing in a similar manner to the odd q case, we obtain

$$\begin{aligned} S_2 &= \frac{1}{q^3|T|} \sum_{t \in T \setminus Z} \sum_{x,y,z \in \mathbb{F}_q} \chi(t)^{-1} \omega\big(\mu(t) I_4\big)^{1/2} \psi(-ax-by-cz) \theta\left(\mu(t)^{-1/2} t \begin{bmatrix} 1 & & y & z \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) \\ &= \frac{q-1}{q^3|T|} \sum_{t \in (T\setminus Z)\cap \mathrm{Sp}(4,q)} \sum_{x,y,z \in \mathbb{F}_q} \chi(t)^{-1} \psi(-ax-by-cz) \theta\left(t \begin{bmatrix} 1 & & y & z \\ & 1 & x & y \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) \\ &= \begin{cases} \frac{1}{q(q-1)} \sum_{i=1}^{q-2} \chi^{-1}(\gamma^i)\big[\theta\big(C_2(i)\big) - \theta\big(D_2(i)\big)\big] & \text{if } \varepsilon(acb^{-2}) = 1, \\ \frac{1}{q(q+1)} \sum_{i=1}^{q} \chi^{-1}(\eta^i)\big[\theta\big(C_4(i)\big) - \theta\big(D_4(i)\big)\big] & \text{if } \varepsilon(acb^{-2}) = -1. \end{cases} \end{aligned}$$

The value of S_2 can now readily be found from the character tables in [2]. ■

Corollary 4.2. *Let (σ, V_σ) be an irreducible representation of* $\mathrm{GSp}(4,q)$*, and $a,b,c \in \mathbb{F}_q$ with $b^2 - 4ac \neq 0$. Let $R = TN$ be the associated Bessel subgroup and $(\chi \otimes \psi_{a,b,c}, \mathbb{C}_{\chi,a,b,c})$ a one-dimensional representation of R.*

(a) *Suppose $T \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$. If σ is generic then*

$$1 \leq \dim \mathrm{Hom}_R(V_\sigma, \mathbb{C}_{\chi,a,b,c}) \leq 2$$

and if σ is nongeneric then

$$\dim \mathrm{Hom}_R(V_\sigma, \mathbb{C}_{\chi,a,b,c}) \leq 1.$$

In addition, if σ is nongeneric then $\dim \mathrm{Hom}_R(V_\sigma, \mathbb{C}_{\chi,a,b,c}) = 1$ *for at most one χ, except for $\sigma = \chi_7(k)$ (when q is even) and $\sigma = \chi_1(\lambda,\nu)$ (when q is odd), in which case there are two such χ.*

(b) *If $T \cong \mathbb{F}_{q^2}^\times$ then*

$$\dim \mathrm{Hom}_R(V_\sigma, \mathbb{C}_{\chi,a,b,c}) \leq 1.$$

In addition, if σ is nongeneric then $\dim \mathrm{Hom}_R(V_\sigma, \mathbb{C}_{\chi,a,b,c}) = 1$ *for at most one χ, except for $\sigma = \chi_9(k)$ (when q is even) and $\sigma = \chi_7(\Lambda)$ (when q is odd) in which case there are two such χ.*

Acknowledgments. The author thanks Ralf Schmidt for many useful conversations during the writing of this article.

References

[1] V. Dabbaghian-Abdoly, Characters of some finite groups of Lie type with a restriction containing a linear character once. *J. Algebra* **309** (2007), no. 2, 543–558 Zbl 1123.20009 MR 2303192

[2] H. Enomoto, The characters of the finite symplectic group $\mathrm{Sp}(4,q)$, $q = 2^f$. *Osaka Math. J.* **9** (1972), 75–94 Zbl 0254.20005 MR 0302750

[3] A. Pitale, Steinberg representation of GSp(4): Bessel models and integral representation of L-functions. *Pacific J. Math.* **250** (2011), no. 2, 365–406 Zbl 1264.11038 MR 2794606

[4] D. Prasad and R. Takloo-Bighash, Bessel models for GSp(4). *J. Reine Angew. Math.* **655** (2011), 189–243 Zbl 1228.11070 MR 2806111

[5] B. Roberts and R. Schmidt, Some results on Bessel functionals for GSp(4). *Doc. Math.* **21** (2016), 467–553 Zbl 1417.11088 MR 3522249

[6] R. Schulze-Pillot, Bessel functionals and Siegel modular forms. *J. Théor. Nombres Bordeaux* **7** (1995), no. 1, 15–20 Zbl 0848.11020 MR 1413563

[7] K. Shinoda, The characters of the finite conformal symplectic group, $\mathrm{CSp}(4,q)$. *Comm. Algebra* **10** (1982), no. 13, 1369–1419 Zbl 0492.20024 MR 0662708

Communicated by Jessica Fintzen

Received 17 October 2024; revised 15 May 2025.

Jonathan Cohen
Department of Mathematics, University of North Texas, 1155 Union Circle #311430, Denton, TX 76203-5017, USA; jonathan.cohen@unt.edu