On reflection maps from the *n*-space to the *n* + 1-space

Milena Barbosa Gama and Otoniel Nogueira da Silva

Abstract. We consider some problems about a reflected graph map germ f from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^{n+1}, 0)$. A reflected graph map is a particular case of a reflection map, which is defined using an embedding of $\mathbb{C}^n$ in $\mathbb{C}^p$ and then applying the action of a reflection group G on $\mathbb{C}^p$. We present a description of the presentation matrix of $f_* \mathcal{O}_n$ as an $\mathcal{O}_{n+1}$-module via f in terms of the action of the associated reflection group G. We also give a description for a defining equation of the image of f in terms of the action of G. Finally, we provide an upper (and also a lower) bound for the multiplicity of the image of f and some applications.

1. Introduction

In this work, we consider some problems on reflection maps from $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$. Reflection maps have recently emerged in the literature and have proven to be a very interesting subject for singularity theory, with interesting properties and challenging problems to be explored. These kind of maps were introduced by Peñafort-Sanchis in [21], where several interesting problems are considered, such as Lê's conjecture, normal crossings and $\mathcal{A}$-finite determinacy. Reflection maps were also used to produce the first known counter-example of Ruas's conjecture ([23] see also [26]). In order to explain what a reflection map is, let us first introduce some ingredients and notation.

Let $\mathrm{GL}(\mathbb{C}^p)$ be the group of all invertible linear maps from $\mathbb{C}^p$ to $\mathbb{C}^p$ (the general linear group), and $U(\mathbb{C}^p)$ the group of unitary automorphisms of $\mathbb{C}^p$. A *reflection* in $\mathbb{C}^p$ is a linear map $g \colon \mathbb{C}^p \to \mathbb{C}^p$ which is unitary, has finite order (as an element of $\mathrm{GL}(\mathbb{C}^p)$) and the set of points fixed by the action of g has dimension $p - 1$. A finite subgroup G of $U(\mathbb{C}^p)$ is a (unitary) reflection group if it is generated by reflections. The cyclic group $\mathbb{Z}_d$ and the dihedral D_{2m} are typical examples of reflection groups (as long as they are considered with convenient representations in $\mathrm{GL}(\mathbb{C}^p)$, see Remark 2.8). There is a vast theory regarding reflection groups, and we will only explain in this work what we need to get our results. A general reference for the topic is [10] (see also [8]) where one can find a detailed description of reflection groups and also the classification of irreducible reflection groups obtained by Shephard–Todd in 1954 (see [25]).

In few words, a *reflection map* $f: \mathbb{C}^n \to \mathbb{C}^p$ is defined by Peñafort-Sanchis in [21] simply as the composition of the orbit map $w: \mathbb{C}^p \to \mathbb{C}^p$ of the group G (see Definition 9.2 in [10]) with an embedding $\rho: \mathbb{C}^n \hookrightarrow \mathbb{C}^p$, i.e., $f = w \circ \rho$.

We remark that according to Peñafort-Sanchis, the formulation of a precise definition of a reflection map is attributed to Fernández de Bobadilla. A reflected graph is a particular case of a reflection map. These kind of maps were also introduced by Peñafort-Sanchis in [21], and play a very important role in the theory of reflection maps. For instance, Peñafort-Sanchis shows that with a certain hypothesis, any reflection map germ is $\mathcal{A}$-equivalent to a reflected graph map germ (see Proposition 3.2 in [21]). Let $h: \mathbb{C}^p \to \mathbb{C}^r$ be any holomorphic map and G a reflection group acting on $\mathbb{C}^p$. In few words, we can say that a reflected graph map $(w, h): \mathbb{C}^p \to \mathbb{C}^{p+r}$ is the reflection map obtained by taking the embedding of h given by $x \mapsto (x, h(x))$, and letting G act on $\mathbb{C}^p \times \mathbb{C}^r$, trivially on the second factor.

The objective of this work is to study some problems about reflected graph map germs from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^{n+1}, 0)$, where h is a map germ from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}, 0)$. Before we describe the problems we will deal with, let us look at some historical cases.

The simplest class of examples of reflected graph maps are the fold maps. For instance, the C_5-singularity of Mond's list (see p. 378 in [15]) defined by $f(x, y) = (x, y^2, xy^3 - x^5 y)$ (see Figure 1). Historically, fold maps are the first examples of reflection maps that we find in the literature. The first people to study the subject were Bruce in [2] and Mond in [14]. A fold map germ $f: (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is a reflected graph map germ where $\rho: (\mathbb{C}^2, 0) \hookrightarrow (\mathbb{C}^3, 0)$ is defined by $\rho(x, y) = (x, y, h(x, y))$ and $w: (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ is the orbit map $w(X, Y, Z) = (X, Y^2, Z)$ of the group $\mathbb{Z}_2 \times G_{\text{id}}$, where G_{id} is trivial group.
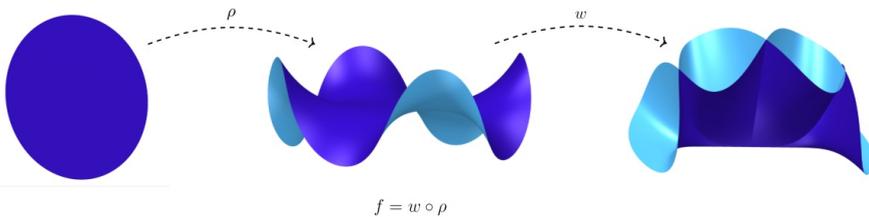

$$f = w \circ \rho$$

Figure 1. The C_5 singularity viewed as a reflection map.

Later, in 2008, Marar and Nuño-Ballesteros introduced in [11] the double fold maps. These maps are similar to folds maps, but for this class, w is the orbit map of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $f = w \circ \rho$ takes the form $f(x, y) = (x^2, y^2, h(x, y))$. A detailed study of double fold maps is given in [20]. A typical example of a double fold map germ is the one given by $f(x, y) = (x^2, y^2, x^3 + y^3 + xy)$ (see Figure 2).

Note that the image of f above is a singular surface in $\mathbb{C}^3$. If (X, Y, Z) denotes a system of coordinates in $\mathbb{C}^3$, then a defining equation of the image of f is

(1.1)
$$X^2 Y^2 - 2XYZ^2 + Z^4 - 2X^4 Y - 2XY^4 - 8X^2 Y^2 Z$$
$$- 2X^3 Z^2 - 2Y^3 Z^2 + X^6 - 2X^3 Y^3 + Y^6 = 0.$$
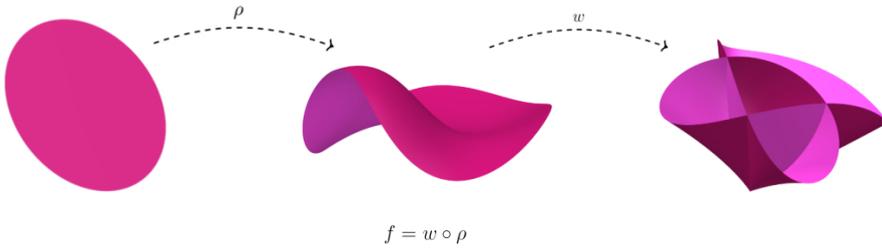
$$f = w \circ \rho$$

Figure 2. The reflection map $f(x, y) = (x^2, y^2, x^3 + y^3 + xy)$ (real points).

We remark that determining a defining equation of the image of a map germ in general is not an easy task. In this way, let G be a reflection group acting in $\mathbb{C}^n \times \mathbb{C}$, trivially on the second factor, through an orbit map $w = (w_1, \ldots, w_n, Z)$. Let $h \colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function. Consider a reflected graph map germ $f \colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$, $f(x) = (w_1(x), \ldots, w_n(x), h(x))$, where $x = (x_1, x_2, \ldots, x_n)$. We have that the action of G on $\mathbb{C}^n \times \mathbb{C}$ induces an action on the ring of analytic functions $\mathcal{O}_{n+1}$ (which is isomorphic to $\mathbb{C}\{X_1, X_2, \ldots, X_n, Z\}$, the ring of convergent series in a neighborhood of the origin). Thus a natural question in this setting is:

Question 1. *How can we describe a defining equation of the image of a reflected graph map germ $f \colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ in terms of the action of G on h?*

When $n = 2$, a defining equation of the image of a reflection map is described in [3] in the case of the cyclic groups $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4$ and the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. In Chapter 4 of [26], the case of the groups $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_5$ and $\mathbb{Z}_4$ is considered. However, none of these works describe a defining equation of the image in terms of the action of G on h.

Also in the case where $n = 2$, Marar and Nuño-Ballesteros present a defining equation of the image of f in the case of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ (see the proof of Proposition 3.1 in [11]). The technique they use is to consider the fact that one can see the pushforward $f_* \mathcal{O}_2$ as an $\mathcal{O}_3$-module, via composition with f. In the sequence, using the Mond–Pellikaan algorithm (see [18]), they determine a presentation matrix of $f_* \mathcal{O}_2$ and then they find a defining equation of the image as the 0-Fitting ideal of $f_* \mathcal{O}_2$, which is simply the determinant of the presentation matrix of $f_* \mathcal{O}_2$. For instance, a presentation matrix of $f = (x^2, y^2, x^3 + y^3 + xy)$ (with relation to the basis $1, x, y, xy$) is given by

$$M = \begin{bmatrix} -Z & X & Y & 1 \\ X^2 & -Z & X & Y \\ Y^2 & Y & -Z & X \\ XY & Y^2 & X^2 & -Z \end{bmatrix}.$$

Note that the defining equation of $f = (x^2, y^2, x^3 + y^3 + xy)$ described in (1.1) is precisely the determinant of the matrix M above. Thus a natural question in this setting is the following.

Question 2. *Let $f(x) = (w(x), h(x))$ be a reflected graph map germ as in Question 1. How can we describe the presentation matrix of the pushforward $f_* \mathcal{O}_n$ in terms of the action of G on h?*

Note that finding the presentation matrix of $f_*\mathcal{O}_n$ may not be an easy task. For the computations, one can use the software SINGULAR [6] and the implementation of Mond–Pellikaan's algorithm given by Hernandes, Miranda, and Peñafort-Sanchis in [7]. However, depending on the complexity of f, even with the help of a regular computer, the calculations can take days and in some cases cannot be completed due to lack of computer memory. This is the case, for example, of maps whose coordinate functions have high multiplicities.

In this work, we answer both questions. For Question 2, in a few words, we describe the presentation matrix of $f_*\mathcal{O}_n$ as a product of three matrices, whose entries depend on the action of G on h (see Theorem 3.3). We note that one importance consequence of obtaining a presentation matrix of $f_*\mathcal{O}_n$ is the fact that the Fitting ideals give a convenient analytic structure not only for the image of f, but also for the image $M_k(f)$ of the (source) multiple points $D^k(f)$ of f (see [18]).

We consider a reflection group G of order d and a reflected graph map $f(\boldsymbol{x}) = (w(\boldsymbol{x}), h(\boldsymbol{x}))$ as in Question 1. We present the following result (see Theorem 4.2).

Theorem 1.1. *A defining equation of the image of* $f(\boldsymbol{x}) = (w(\boldsymbol{x}), h(\boldsymbol{x}))$ (*given by the* 0-*Fitting ideal of the presentation matrix of* $f_*\mathcal{O}_n$) *is given by the following alternating sum*:

$$
(1.2) \quad
\begin{aligned}
F(X_1, &\ldots, X_n, Z) \\
&= Z^d - Q_{d-1}Z^{d-1} + Q_{d-2}Z^{d-2} + \cdots + (-1)^{d-1}Q_1 Z + (-1)^d Q_0.
\end{aligned}
$$

The description of Q_{d-k} in Theorem 1.1 is given in terms of a G-invariant polynomial $q_{d-k}(x_1, \ldots, x_n)$, which can be seen as a symmetric polynomial in the variables $h_1, h_2, \ldots, h_d$, where h_i denotes the action of an element g_i of G on h (see Lemma 4.1). We remark that another way to obtain the defining equation of the image of a reflection map is presented in [1] using sections of the orbit map, which is a different technique from the one we use in this work.

As a corollary of Theorem 1.1, we obtain an upper bound for the multiplicity of f, where f is a reflected graph map germ. More precisely, write

$$
F = F_m + F_{m+1} + \cdots + F_k + \cdots,
$$

where each F_k is a homogeneous polynomial of degree k and $F_m \neq 0$. The integer m is called the multiplicity of $\mathbf{V}(F)$ in 0 and is denoted by $m(\mathbf{V}(F), 0)$. Note that by the defining equation (1.2) in Theorem 1.1, we obtain as a corollary that $m(f(\mathbb{C}^n), 0) \leq d$, where d is the order of the reflection group G. Now, consider a reflection map germ (not necessarily a reflected graph one) $f: (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$. A natural question is:

Question 3. *For a fixed* (*representation of*) *the group* G, *consider a reflection map germ* $f: (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$. *How big can the multiplicity of the image of* f *be? In other words, is there an upper bound for the multiplicity of the image of a reflection map?*

In this work, we give an answer to Question 3. An important set of invariants of a reflection group G acting on $\mathbb{C}^{n+1}$ is the set of degrees of G, denoted by $\{d_1, d_2, \ldots, d_{n+1}\}$ (see Proposition 3.25 in [10]). Ordering the degrees of G in the form $d_1 \leq d_2 \leq \cdots \leq d_{n+1}$, we show (see Theorem 5.4) that

$$
d_1 d_2 \cdots d_n \leq m(f(\mathbb{C}^n), 0) \leq d_2 d_3 \cdots d_{n+1}.
$$

Finally, as an application of our results, in the case of reflected graph map germs, we present another way to obtain a defining equation of the double point hypersurface of f different from the one obtained by in Borges Zampiva, Peñafort-Sanchis, Oréfice Okamoto and Tomazella in [1].

In order to illustrate our results, we introduce in Section 6.2 a new class of map germs called "*2m-dihedral map germs*". We show the presentation matrix of the pushforward $f_* \mathcal{O}_2$ and a defining equation of the image of a 6-dihedral map germ in an explicit way (see Proposition 6.7). In Theorem 3.4 of [11], Marar and Nuño-Ballesteros showed that there are no finitely determined quasihomogeneous (with distinct weights) double fold map germs, i.e., f is a reflection graph map germ with reflection group $\mathbb{Z}_2 \times \mathbb{Z}_2$. We finish this work presenting an extension of this result to the group $\mathbb{Z}_r \times \mathbb{Z}_s$, with $r, s \geq 2$ (see Lemma 6.9).

2. Preliminaries

Throughout the text, we assume that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ is a finite holomorphic map germ, unless otherwise stated. Moreover, $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $(X, Z) = (X_1, \ldots, X_n, Z)$ are used to denote systems of coordinates in $\mathbb{C}^n$ (source) and $\mathbb{C}^{n+1}$ (target), respectively. We also use the standard notation of singularity theory as the reader can find in [17].

2.1. Reflection groups and reflection maps

Consider $\mathrm{GL}(\mathbb{C}^p)$, the group of all invertible linear transformations of $\mathbb{C}^p$. Let Id be the identity element of $\mathrm{GL}(\mathbb{C}^p)$. A linear representation of a group G with representation space $\mathbb{C}^p$ is a homomorphism $\psi : G \to \mathrm{GL}(\mathbb{C}^p)$. If $\psi : G \to \mathrm{GL}(\mathbb{C}^p)$ is a representation, we say that G acts on $\mathbb{C}^p$ and we call $\mathbb{C}^p$ a G-module [24]. The action of $g \in G$ on $v \in \mathbb{C}^p$ is defined by $g v := \psi(g) v$, and we usually omit ψ but denote $g v$ by $g_\bullet v$. For $g \in \mathrm{GL}(\mathbb{C}^p)$, we have $\mathrm{Fix}\, g := \{v \in \mathbb{C}^p \mid g_\bullet v = v\}$.

Definition 2.1. A reflection on $\mathbb{C}^p$ is a linear map $g : \mathbb{C}^p \to \mathbb{C}^p$ satisfying

  (i) g is unitary,

  (ii) g has finite order,

  (iii) $\dim \mathrm{Fix}\, g = p - 1$.

Let $U(\mathbb{C}^p)$ be the group of unitary automorphisms of $\mathbb{C}^p$. A subgroup G of $U(\mathbb{C}^p)$ is said to be a reflection group if it is generated by reflections. If g is a reflection, the subspace $\mathrm{Fix}\, g$ is a hyperplane, called the *reflecting hyperplane* of g.

We note that the action of G on $\mathbb{C}^p$ induces an action of G on a holomorphic function, as we will see in the next definition.

Definition 2.2. Let $g \in \mathrm{GL}(\mathbb{C}^p)$ and let $P \in \mathcal{O}_p$ be a holomorphic function. We define the action of g on P by

$$(g_\bullet P)(v) := P(g^{-1}(v)), \quad \text{for all } v \in \mathbb{C}^p.$$

We note that for any $g \in G$, the action of g on $P, Q \in \mathcal{O}_p$ has the following property: $g_\bullet(PQ) = g_\bullet(P) g_\bullet(Q)$. We say that $P \in S$ is G-invariant if $g_\bullet P = P$ for all $g \in G$.
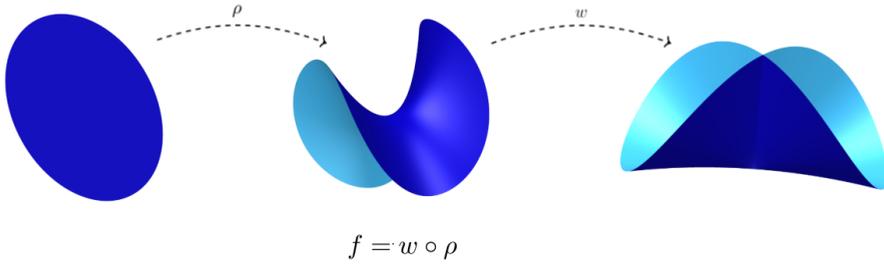
$$f = w \circ \rho$$

Figure 3. Illustration of a reflection map.

The *algebra of invariants of* G is the algebra of G-invariant holomorphic functions

$$\mathcal{O}_p^G = \{P \in \mathcal{O}_p \mid g_\bullet P = P \text{ for all } g \in G\}.$$

The following lemma can be found for instance in Lemma 3.17 of [10]. It gives us a way to obtain the action of a reflection of G on an element of $\mathcal{O}_p$.

Lemma 2.3. *Let g be a reflection in* $\mathrm{GL}(\mathbb{C}^p)$, *and let* $\mathrm{Fix}\, g$ *be is its reflecting hyperplane, with* $\mathrm{Fix}\, g = \mathrm{Ker}\, L_g$. *Then for all $P \in \mathcal{O}_p$, there exists $Q \in \mathcal{O}_p$ such that*

$$g_\bullet P = P + L_g Q.$$

It follows from Lemma 2.3 that if g is a reflection, then $(P - g_\bullet P)/L_g = -Q \in \mathcal{O}_p$. So we can consider the operator $\Delta_g \colon \mathcal{O}_p \to \mathcal{O}_p$ given by

$$\Delta_g(P) = \frac{P - g_\bullet P}{L_g},$$

where g is a reflection of G. This operator is also known as the *Demazure operator*.

Consider now the operator $\mathrm{Av} \colon \mathcal{O}_p \to \mathcal{O}_p^G$ given by

$$\mathrm{Av}(P) := \frac{1}{|G|} \Big(\sum_{g \in G} g_\bullet P \Big).$$

This operator is also known as the *Reynolds operator*. It is clear from the definition that $\mathrm{Av}(P) \in \mathcal{O}_p^G$ and that $\mathrm{Av}(P)$ is either 0 or has the same degree as P. Moreover, for $P \in \mathcal{O}_p^G$, we have $\mathrm{Av}(P) = P$ and therefore $\mathrm{Av}^2 = \mathrm{Av}$. Thus Av is a projection of $\mathcal{O}_p$ onto $\mathcal{O}_p^G$. In fact, a somewhat stronger statement is true, namely that for $P \in \mathcal{O}_p^G$ and $Q \in S$, we have $\mathrm{Av}(PQ) = P\,\mathrm{Av}(Q)$, so that Av is a $\mathcal{O}_p^G$-module homomorphism.

The orbit map w of a group G acting on $\mathbb{C}^p$ determines a way of "folding" $\mathbb{C}^p$, gluing an orbit of G to a point [21]. Figure 3 illustrates a geometrical idea of the notion of a reflection map.

Since G acts on $\mathbb{C}^p$, then by Shephard–Todd's theorem (Theorem 5.1 in [25], see also Theorem 3.20 in [10]), we have that the algebra of G-invariant holomorphic functions $\mathcal{O}_p^G$ can be generated by p homogeneous holomorphic functions.

Definition 2.4. The orbit map of a reflection group G is a map $w: \mathbb{C}^p \to \mathbb{C}^p$ whose coordinate functions are homogeneous polynomials $w_1, w_2, \ldots, w_p$ in S that generate J. The set of degrees $d_1, \ldots, d_p$ of G are the degrees of $w_1, w_2, \ldots, w_p$, respectively.

The map w is unique up to invertible polynomial transformations in the target. Furthermore, it is well known that the degrees of G do not depend on the choice of the set of generators $w_1, \ldots, w_p$ of $\mathcal{O}_p^G$. Since we work with objects which are invariant under such transformations, the choice of w does not matter. This justifies our calling w the "orbit map" of G.

Remark 2.5. From Theorem 3.20 in [10], we know that $\mathcal{O}_p^G \simeq \mathbb{C}\{w_1, \ldots, w_p\}$, which means that for all $g \in G$ and all $P \in \mathbb{C}\{w_1, \ldots, w_p\}$, we have $g_\bullet P = P$.

Example 2.6. Consider the generators of the dihedral group D_{2m} in $\mathrm{GL}(\mathbb{C}^2)$,

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} e^{2\pi i/m} & 0 \\ 0 & e^{-2\pi i/m} \end{bmatrix}.$$

The orbit map $w: \mathbb{C}^2 \to \mathbb{C}^2$ of D_{2m} is given by

$$(x, y) \mapsto (xy, \; x^m + y^m).$$

One of the most important results about the orbit map is Noether's theorem [19], which allows us to conclude that the set $w^{-1}(w(v))$ is the orbit of G in v, i.e., the set

$$Gv := \{g_\bullet v \mid g \in G\}.$$

Theorem 2.7 (Noether). *For any $v \in \mathbb{C}^p$, $w^{-1}(w(v)) = Gv$.*

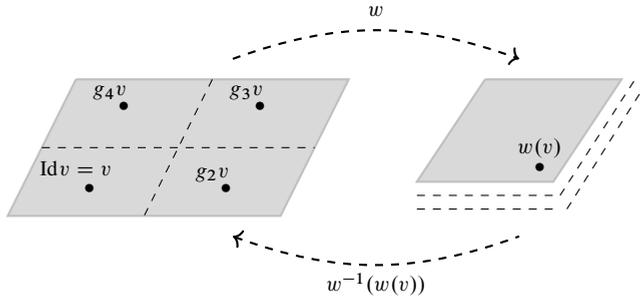Considering $G = \mathbb{Z}_4$, Figure 4 illustrates Noether's theorem.

Figure 4. Orbit map of a reflection group.

Remark 2.8. It is important to mention that whether a group is a reflection group or not depends on its representation in $\mathrm{GL}(\mathbb{C}^p)$. For example, the cyclic group $\mathbb{Z}_d$ generated by the matrix

$$K = \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i/d} \end{bmatrix}$$

is a reflection group, since K is a reflection. On the other hand, the group generated by the matrix

$$K' = \begin{bmatrix} e^{2\pi i/d} & 0 \\ 0 & (e^{2\pi i/d})^k \end{bmatrix},$$

with $\gcd(k, d) = 1$, is isomorphic to $\mathbb{Z}_d$.

Suppose that $\langle K' \rangle \simeq \mathbb{Z}_d$ is a reflection group. By Section 8 of [25] (see also Theorem 4.14 in [10]), we have that the number of reflections in $\mathbb{Z}_d$ is $d - 1$ (obviously the identity matrix is not a reflection). Therefore, every element $g \in \langle K' \rangle$ with $g \neq \text{Id}$ is a reflection. Note that $\text{Fix } K' = 0$ (which has dimension 0). By Definition 2.1, K' is not a reflection, a contradiction. Therefore, $\langle K' \rangle$ cannot be generated by reflections.

Let G be a reflection group acting on $\mathbb{C}^p$. Let $\text{Fix } g_1, \ldots, \text{Fix } g_l$ be all the (distinct) reflecting hyperplanes of G (where g_i is a reflection of G). Let $\text{ord}(g_i)$ be the order of g_i and let $L_{g_1}, \ldots, L_{g_\ell}$ be linear forms such that $\text{Fix } g_i := \text{Ker } L_{g_i}$. An important relation between the orbit map and the defining equations of the reflecting hyperplanes of G is given by the following proposition, which can be found for instance in Theorem 9.8 of [25]. We shall denote the Jacobian matrix of the orbit map w by $\text{jac}(w)$.

Proposition 2.9. *For some non-zero constant c, we have that*

$$\det(\text{jac}(w)) = c \prod_{i=1}^{\ell} L_{g_i}^{\text{ord}(g_i)-1}.$$

Definition 2.10. Let G be a reflection group acting on $\mathbb{C}^p$ and let $h: \mathbb{C}^p \to \mathbb{C}^r$ be any holomorphic map.

(a) A G-reflection map $f: \mathbb{C}^n \to \mathbb{C}^p$ is a map given by the composition of an embedding $\rho: \mathbb{C}^n \hookrightarrow \mathbb{C}^p$ with the orbit map $w: \mathbb{C}^p \to \mathbb{C}^p$ of G, i.e., $f = w \circ \rho$.

(b) The G-reflected graph of h is the map $(w, h): \mathbb{C}^p \to \mathbb{C}^{p+r}$, given by $f(x) = (w(x), h(x))$.

The G-reflected graph $f = (w, h)$ is the G-reflection map obtained by taking the graph embedding h, given by $x \mapsto (x, h(x))$, and letting G act on $\mathbb{C}^p \times \mathbb{C}^r$, trivially on the second factor (a trivial extension of the action to $\mathbb{C}^{p+r}$). To simplify, we will omit G and simply say reflection map, when G is clear in the context. Throughout this work, we will only consider G-reflected graphs in the case where $r = 1$. For a G-reflection map, we will consider only the case where $p = n + 1$.

3. The presentation matrix of reflection maps

In this section, we will consider a reflection group G acting on $\mathbb{C}^n$ and a holomorphic function $h: \mathbb{C}^n \to \mathbb{C}$. We consider an extension of the action of G on $\mathbb{C}^n \times \mathbb{C}$, simply by making G act trivially on the second factor; in this sense we also consider an orbit map $w = (w_1, \ldots, w_n, Z)$. We will study a reflection graph from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^{n+1}, 0)$ given by $f = (w_1, \ldots, w_n, h)$. In this context, we will provide an answer to Question 2 in the introduction, i.e., we will present a presentation matrix of the pushforward $f_* \mathcal{O}_n$

as an $\mathcal{O}_{n+1}$-module, via composition with f, where f is a reflected graph map germ. We note that our way to construct the presentation matrix depends only on the action of G on h.

Before we do this, we introduce some notation. Let $f\colon(\mathbb{C}^n,0)\to(\mathbb{C}^{n+1},0)$ be a finite analytic map germ, let $r_1,r_2,\ldots,r_\ell$ be elements in $\mathcal{O}_n$, and denote the maximal ideal of $\mathcal{O}_{n+1}$ by $\mathbf{m}$. Note that $\mathcal{O}_n$ is a finite $\mathcal{O}_{n+1}$-module via f and if the classes of $r_1,\ldots,r_\ell$ in $\mathcal{O}_n/f^*\mathbf{m}$ generate it as a vector space over $\mathbb{C}=\mathcal{O}_{n+1}/\mathbf{m}$, then $r_1,\ldots,r_\ell$ generate $\mathcal{O}_n$ as $\mathcal{O}_{n+1}$-module via f (see, e.g., Corollaries D.1 and D.2 in [17]). It is clear that we can take $r_1=1$, and we shall do so from now on. A presentation of $\mathcal{O}_n$ over $\mathcal{O}_{n+1}$ is an exact sequence

$$(3.1) \qquad \mathcal{O}_{n+1}^r \xrightarrow{\;\lambda\;} \mathcal{O}_{n+1}^s \xrightarrow{\;\psi\;} \mathcal{O}_n \longrightarrow 0$$

of $\mathcal{O}_{n+1}$-modules. It follows by Lemma 2.1 in [18] that $r=s=\ell$ in (3.1). We say that λ in (3.1) is the *presentation matrix* of $f_*\mathcal{O}_n$. Mond and Pellikaan also give an algorithm to construct a presentation (for details, see Section 2.2 of [18]).

Let $G=\{g_1,\ldots,g_d\}$ be a reflection group of order d acting on $\mathbb{C}^n$. We will adopt the notation $\boldsymbol{w}=(w_1,w_2,\ldots,w_n)$ for the orbit map of G, $(\boldsymbol{X},Z)=(X_1,\ldots,X_n,Z)$ for the coordinates of $\mathbb{C}^{n+1}$ (target), and $\boldsymbol{x}=(x_1,x_2,\ldots,x_n)$ for the coordinates of $\mathbb{C}^n$ (source). Throughout this work, the bold notation (e.g., $\boldsymbol{w}$, $\boldsymbol{x}$ and $\boldsymbol{X}$) will always be used to represent an n-tuple. Recall that, according to Definition 2.1, an element $g\in G$ is a *reflection* on $\mathbb{C}^n$ if g is unitary, has finite order, and satisfies $\dim\operatorname{Fix}g=n-1$. We denote by $\mathcal{R}$ the set of all reflections of G.

Recall that a reflected graph map germ $f\colon(\mathbb{C}^n,0)\to(\mathbb{C}^{n+1},0)$, $f(\boldsymbol{x})=(w(\boldsymbol{x}),h(\boldsymbol{x}))$, is obtained as the composition $f=w\circ\rho$, where ρ is the embedding $(\boldsymbol{x},h(\boldsymbol{x}))$ and w is the orbit map of G. Denote by $\mathcal{Y}:=\rho(\mathbb{C}^n)$ and $\mathcal{X}:=w(\mathcal{Y})$. Note that $\mathcal{X}=f(\mathbb{C}^n)=\mathbf{V}(F(\boldsymbol{X},Z))$, where $F(\boldsymbol{X},Z)$ is a defining equation of the image of f which is obtained as the determinant of the presentation matrix of $f_*(\mathcal{O}_n)$. It follows that a defining equation of $(\mathcal{Y},0)$ is $Z-h(X_1,\ldots,X_n)=0$. By Noether's theorem (see Theorem 2.7), we have that

$$w^{-1}(w(\mathcal{Y}))=\bigcup_{i=1}^{d}g_i{\scriptstyle\bullet}\mathcal{Y}.$$

The pre-image of $\mathcal{X}$ by w consists of the orbit of $\mathcal{Y}$ under the action of G. We will show in the following lemma that a defining equation of $w^{-1}(w(\mathcal{Y}))$ is given by

$$(3.2) \qquad (Z-g_1{\scriptstyle\bullet}h(\boldsymbol{X}))\cdot(Z-g_2{\scriptstyle\bullet}h(\boldsymbol{X}))\cdots(Z-g_d{\scriptstyle\bullet}h(\boldsymbol{X}))=0.$$

The following lemma will be a key tool to prove Theorem 4.2, where we will provide a defining equation of the image of f. We will also show that the defining equation of $w^{-1}(w(\mathcal{Y}))$, described in (3.2), coincides with the pullback of the defining equation $F(\boldsymbol{X},Z)$ of $(\mathcal{X},0)$ by $\boldsymbol{w}$, which will be denoted by $F(w_1,\ldots,w_n,Z)=F(\boldsymbol{w},Z)$.

Lemma 3.1. *With the notation above, let $f\colon(\mathbb{C}^n,0)\to(\mathbb{C}^{n+1},0)$, $f(\boldsymbol{x})=(w(\boldsymbol{x}),h(\boldsymbol{x}))$ be a reflected graph map germ, then $w^{-1}(w(\mathcal{Y}))=V(F(\boldsymbol{w},Z))$. Furthermore,*

$$(3.3) \qquad F(\boldsymbol{w},Z)=\prod_{k=1}^{d}(Z-g_k{\scriptstyle\bullet}(h(\boldsymbol{x}))).$$

Proof. Note that $(\mathcal{X}, 0) \subset (\mathbb{C}^{n+1}, 0)$ is defined by a single equation (the determinant of the presentation matrix $\lambda[X, Z]$ of $f_*\mathcal{O}_n$ as a $\mathcal{O}_{n+1}$-module via f, denoted by $F(X, Z)$). Since $F(X, Z)$ is a defining equation of $\mathcal{X}$, it follows that the pullback $F(\boldsymbol{w}, Z)$ of $F(X, Z)$ by w is a defining equation of $w^{-1}(w(\mathcal{Y}))$. Let us show that $F(\boldsymbol{w}, Z)$ can be factored as in (3.3).

By Corollary 3.29 in [10], we have that

$$\dim_\mathbb{C} \frac{\mathbb{C}\{x_1, x_2, \ldots, x_n\}}{\langle w_1, w_2, \ldots, w_n \rangle} = |G| = d.$$

Let $1 = r_1, r_2, \ldots, r_d$ be a basis of $\mathbb{C}\{\boldsymbol{x}\}/\langle \boldsymbol{w} \rangle$ as a $\mathbb{C}$-vector space. By Lemma 3.28 in [10], there are $p_1, p_2, \ldots, p_d$ in $\mathbb{C}\{\boldsymbol{x}\}$ such that

$$h := r_1 p_1(w) + r_2 p_2(w) + \cdots + r_d p_d(w),$$

Note that $p_i(\boldsymbol{w}) = p_i(w_1, \ldots, w_n)$ are uniquely determined by h and the orbit map w. Now we will follow Mond–Pellikaan's algorithm (see Section 2.2 of [18]) to construct a presentation matrix $\lambda[X, Z]$ of $f_*\mathcal{O}_n$ as an $\mathcal{O}_{n+1}$-module via f. Let us first consider the matrix

$$\alpha[X] = \begin{bmatrix} \alpha_{1,1} & \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,d} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \cdots & \alpha_{2,d} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \cdots & \alpha_{3,d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{d,1} & \alpha_{d,2} & \alpha_{d,3} & \cdots & \alpha_{d,d} \end{bmatrix}_{d \times d} ,$$

where the $\alpha_{i,j}(X)$ are in $\mathbb{C}\{X\}$ and satisfy the relation

$$(3.4) \qquad\qquad r_i \cdot h = \sum_{j=1}^{d} (\alpha_{i,j}(\boldsymbol{w})) \cdot r_j, \quad \text{for } i = 1, \ldots, d.$$

Again by Lemma 3.28 in [10], we note that the $\alpha_{i,j}$ are uniquely determined by h and w. Thus, the presentation matrix of $f_*\mathcal{O}_n$ is

$$\lambda[X, Z] = \alpha[X] - ZId = \begin{bmatrix} \alpha_{1,1}-Z & \alpha_{1,2} & \alpha_{1,3} & \cdots & \alpha_{1,d} \\ \alpha_{2,1} & \alpha_{2,2}-Z & \alpha_{2,3} & \cdots & \alpha_{2,d} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3}-Z & \cdots & \alpha_{3,d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{d,1} & \alpha_{d,2} & \alpha_{d,3} & \cdots & \alpha_{d,d}-Z \end{bmatrix} .$$

Having defined the presentation matrix $\lambda[X, Z]$ according to Mond–Pellikaan's algorithm, now we would like to factorize $F(\boldsymbol{w}, Z)$ as in (3.3). Substituting X for $\boldsymbol{w}$ in $\lambda[X, Z]$ and $\alpha[X]$, we obtain the matrices $\lambda[\boldsymbol{w}, Z]$ and $\alpha[\boldsymbol{w}]$, i.e., $\lambda[\boldsymbol{w}, Z]$ (respectively, $\alpha[\boldsymbol{w}]$) is the pullback of $\lambda[X, Z]$ (respectively, $\alpha[X]$) by w. Clearly, the determinant of $\lambda[\boldsymbol{w}, Z]$ is equal to $F(\boldsymbol{w}, Z)$.

Let $\mathbb{K} := \mathrm{Frac}(\mathbb{C}\{x\})$ and consider $\alpha[w]$ as a matrix with entries in $\mathbb{K}$. Note that $\lambda[w, Z] = (\alpha[w] - Z\,\mathrm{Id})$, where Id is the identity matrix. Therefore $F(w, Z)$ is precisely the characteristic polynomial of $\alpha[w]$. Let us find the eigenvalues of $\alpha[w]$. Consider $\mathrm{Id} = g_1, g_2, \ldots, g_d$ the elements of the reflection group G. For all $i = 1, \ldots, d$, we obtain from equation (3.4) that

$$(3.5) \qquad \alpha_{i,1}(w) \cdot r_1 + \cdots + \alpha_{i,d}(w) \cdot r_d = h \cdot r_i.$$

Since each $\alpha_{i,j}(w) = \alpha_{i,j}(w_1, \ldots, w_n)$ is in $\mathbb{C}\{w_1, \ldots, w_p\}$, by Remark 2.5 we have that $\alpha_{i,j}(w)$ is invariant under the action of G for all $i, j \in \{1, \ldots, n\}$. Thus, applying the action of $g_k \in G$ in (3.5), we obtain that

$$\alpha_{i,1}(w) \cdot g_{k\bullet}(r_1) + \cdots + \alpha_{i,d}(w) \cdot g_{k\bullet}(r_d) = g_{k\bullet}(h) \cdot g_{k\bullet}(r_i),$$

from which we conclude that

$$\begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \cdots & \alpha_{1,d} \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \cdots & \alpha_{2,d} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \cdots & \alpha_{3,d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{d,1} & \alpha_{d,2} & \alpha_{d,3} & \cdots & \alpha_{d,d} \end{bmatrix} \cdot \begin{bmatrix} g_{k\bullet}(r_1) \\ g_{k\bullet}(r_2) \\ g_{k\bullet}(r_3) \\ \vdots \\ g_{k\bullet}(r_d) \end{bmatrix} = g_{k\bullet}(h) \begin{bmatrix} g_{k\bullet}(r_1) \\ g_{k\bullet}(r_2) \\ g_{k\bullet}(r_3) \\ \vdots \\ g_{k\bullet}(r_d) \end{bmatrix}.$$

In this way, for all $k = 1, \ldots, d$, we have that $(g_{k\bullet}(r_1), g_{k\bullet}(r_2), \ldots, g_{k\bullet}(r_d))$ is an eigenvector of $\alpha[w]$ with respective eigenvalue $g_{k\bullet}(h)$. Consider the matrix E whose columns are the eigenvectors $(g_{k\bullet}(r_1), g_{k\bullet}(r_2), \ldots, g_{k\bullet}(r_d))$, i.e.,

$$E = \begin{bmatrix} g_{1\bullet}(r_1) & g_{2\bullet}(r_1) & \cdots & g_{d\bullet}(r_1) \\ g_{1\bullet}(r_2) & g_{2\bullet}(r_2) & \cdots & g_{d\bullet}(r_2) \\ \vdots & \vdots & \ddots & \vdots \\ g_{1\bullet}(r_{d-1}) & g_{2\bullet}(r_{d-1}) & \cdots & g_{d\bullet}(r_{d-1}) \\ g_{1\bullet}(r_d) & g_{2\bullet}(r_d) & \cdots & g_{d\bullet}(r_d) \end{bmatrix}.$$

It follows, by Gutkin's theorem (see Theorem 10.13 in [10], where the matrix $A_{i,j}$ in Definition 10.6 of [10] is the matrix E in our setting), that the determinant of E is non-zero. Hence, the set of these eigenvectors are linearly independent. Therefore,

$$(3.6) \qquad \det \lambda[w, Z] := F(w, Z) = \prod_{k=1}^{d}(Z - g_{k\bullet}(h(x))).$$

This completes the proof. ∎

Remark 3.2. (a) In the sequel, given a matrix $M = (m_{i,j}(w_1, \ldots, w_n))$, where each $m_{i,j}$ is in $\mathbb{C}[w_1, \ldots, w_n]$ (i.e., $m_{i,j}$ is G-invariant), we denote by $w^*(M)$ the matrix obtained by exchanging w_i for X_i in $m_{i,j}$, i.e.,

$$w^*(M) = (m_{i,j}(X_1, \ldots, X_n)).$$

(b) Let $\lambda[\mathbf{X}, Z]$ be the presentation matrix of $f_* \mathcal{O}_n$ as a $\mathcal{O}_{n+1}$-module and let $\lambda[\boldsymbol{w}, Z]$ be as in Lemma 3.1. With the notation above, consider the matrices E and A below:

$$
E = \begin{bmatrix}
g_{1\bullet}(r_1) & g_{2\bullet}(r_1) & \cdots & g_{d\bullet}(r_1) \\
g_{1\bullet}(r_2) & g_{2\bullet}(r_2) & \cdots & g_{d\bullet}(r_2) \\
\vdots & \vdots & \ddots & \vdots \\
g_{1\bullet}(r_{d-1}) & g_{2\bullet}(r_{d-1}) & \cdots & g_{d\bullet}(r_{d-1}) \\
g_{1\bullet}(r_d) & g_{2\bullet}(r_d) & \cdots & g_{d\bullet}(r_d)
\end{bmatrix},
$$

$$
A = \begin{bmatrix}
g_{1\bullet}(h) & 0 & 0 & \cdots & 0 \\
0 & g_{2\bullet}(h) & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & g_{d\bullet}(h)
\end{bmatrix}.
$$

As we mentioned in the proof of Lemma 3.1, E is the eigenvector matrix of $\lambda[\boldsymbol{w}, 0]$ and the elements that appear on the diagonal of the matrix A are the eigenvalues of $\lambda[\boldsymbol{w}, 0]$. This motivates the following theorem.

Theorem 3.3. *Let* $f \colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$, $f(\boldsymbol{x}) = (w(\boldsymbol{x}), h(\boldsymbol{x}))$, *be a reflected graph map germ. Consider the matrices* E *and* A *as above and set* $A_Z := (A - Z\,\mathrm{Id})$. *The presentation matrix of* $f_* \mathcal{O}_n$ *as a* $\mathcal{O}_{n+1}$-*module* $\lambda[\mathbf{X}, Z]$ *is expressed as the product of the matrices* E, A *and* E^{-1}, *i.e.*

$$
\lambda[\mathbf{X}, Z] = w^*(E \cdot A_Z \cdot E^{-1}).
$$

Proof. From the proof of Lemma 3.1, we have that $E^{-1}\lambda[\boldsymbol{w}, 0]E = A$. In other words, the matrix E diagonalizes the matrix $\lambda[\boldsymbol{w}, 0]$. Thus,

$$
\lambda[\boldsymbol{w}, Z] = EAE^{-1} - Z\,\mathrm{Id} = E(A - Z\,\mathrm{Id})\,E^{-1} = E \cdot A_Z \cdot E^{-1}.
$$

Now, the result follows from the fact that $w^*(\lambda[\boldsymbol{w}, Z]) = \lambda[\mathbf{X}, Z]$. ∎

Definition 3.4. The quotient $\mathcal{O}_n/(w_1, \ldots, w_n)$ is called the *coinvariant algebra* of the reflection group G (see Section 5 in Chapter 3 of [10]).

Another important space in the theory of reflection groups is the space $\mathcal{H}$ of G-harmonic analytic functions (see Definition 9.35 in [10]). For the purposes of this work, we highlight the fact that $\mathcal{H}$ is isomorphic (as a G-module) to the coinvariant algebra of G. Thus, in what follows, we can identify $\mathcal{H}$ with $\mathcal{O}_n/(w_1, \ldots, w_n)$ as G-modules.

In the proof of Lemma 3.1, we obtained that the determinant of the matrix E is not zero. Actually, we can say more about that determinant. First, let us establish some notation. Consider a reflection group G of order d acting on $\mathbb{C}^n \times \mathbb{C}$ (trivially on the second factor). Denote by $\mathrm{Fix}\,g_1, \ldots, \mathrm{Fix}\,g_l$ the (distinct) reflecting hyperplanes of G. Let L_{g_i} be a linear form from $\mathbb{C}^{n+1}$ to $\mathbb{C}$ such that $\mathrm{Fix}\,g_i = \mathbf{V}(L_{g_i})$ for all i. Let g_i be a generator of the cyclic group which fixes $\mathrm{Fix}\,g_i$ and denote the order of g_i by $\mathrm{ord}(g_i)$. The following proposition gives us an expression for the determinant of E.

Proposition 3.5. *Let G be a finite reflection group and let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be a reflected graph map germ. Then for some non-zero constant c, we have*

$$\det(E) = c \prod_{i=1}^{\ell} L_{g_i}{}^{|G|(\mathrm{ord}(g_i)-1)/2}.$$

Proof. Consider the coinvariant algebra $\mathcal{O}_n/(w_1, \ldots, w_n)$ of G. By Corollary 3.29 in [10], we have that

$$\mathcal{O}_n/(w_1, \ldots, w_n) \simeq \mathbb{C}^d,$$

as $\mathbb{C}$-vector spaces, where $d := |G|$. Take a basis of the coinvariant algebra $\{r_1, r_2, \ldots, r_d\}$ with $r_1 = 1$ as usually. By Proposition 3.2 in [10], we can see $M := \mathbb{C}^d$ as a G-module, where the representation of G on $\mathbb{C}^d \simeq \mathcal{O}_n/(w_1, \ldots, w_n)$ is the regular representation (see Section 1.2 of [24] for the definition of a regular representation of a finite group). Consider the canonical basis

$$\{e_{g_1}, e_{g_2}, \ldots, e_{g_d}\}$$

of $\mathbb{C}^d$ indexed by the elements of G. The dual $M^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ has dual basis $\{y_1, \ldots, y_d\}$, where $y_j := e^{g_j}$ is the linear functional from $\mathbb{C}^d$ to $\mathbb{C}$ defined by $y_j(e_{g_k}) = \delta_{jk}$ (the Kronecker symbol). By Corollary 9.37 in [10], we have that $\mathcal{H}$ is isomorphic to the coinvariant algebra of G (as G-modules). Furthermore, there is a canonical isomorphism

$$(\mathcal{H} \otimes M^*)^G \cong \mathrm{Hom}_{\mathbb{C}G}(M, \mathcal{H})$$

sending $\sum_{\ell} A_{\ell} \otimes \phi_{\ell}$ to the linear operator $v \mapsto \sum_{\ell} \phi_{\ell}(v) A_{\ell}$ (see Lemma 10.2 (ii) in [10]). Now construct explicitly d morphisms $\varphi_k \in \mathrm{Hom}_{\mathbb{C}G}(M, \mathcal{H})$ (one for each $k = 1, \ldots, d$) as follows:

$$\varphi_k : M \to \mathcal{H}, \quad \varphi_k(e_g) := g \cdot r_k.$$

Note that for $g, g' \in G$, we have that

$$\varphi_k(g' \cdot e_g) = \varphi_k(e_{g'g}) = (g'g) \cdot r_k = g' \cdot (g \cdot r_k) = g' \cdot \varphi_k(e_g).$$

Thus $\varphi_k \in \mathrm{Hom}_{\mathbb{C}G}(M, \mathcal{H})$. Therefore, $\varphi_1, \ldots, \varphi_d$ span the space $\mathrm{Hom}_{\mathbb{C}G}(M, \mathcal{H})$. Passing back to the tensor side, the image corresponds to an element $u_k \in (\mathcal{H} \otimes M^*)^G$:

$$u_k = \sum_{j=1}^{d} A_{kj} \otimes y_j, \quad \text{where } A_{kj} \in \mathcal{H}.$$

By definition of the correspondence,

$$A_{kj} = \varphi_k(e_{g_j}) = g_j \cdot r_k.$$

So we obtain that

(3.7) $$A_{kj} = g_j \cdot r_k, \quad 1 \leq k, j \leq d.$$

Thus, in this context, the matrix E in Remark 3.2 (b) coincides exactly with the matrix A_{kj} in (3.7) (see also Definition 10.6 in [10]). Recall that $\mathscr{A}$ is the set of hyperplanes $\mathrm{Fix}\, g$,

where g is a reflection in G. Gutkin's theorem (Theorem 10.13 in [10]) states that there exists a constant $c \neq 0$ such that

$$\det(E) = c \cdot \prod_{\mathrm{Fix}g \in \mathcal{A}} L_g^{C(\mathrm{Fix}g, M)}.$$

Here, $C(\mathrm{Fix}g, M)$ is determined by the restriction of M to the cyclic stabilizer $G_{\mathrm{Fix}g}$. For the regular representation, one finds that

$$C(\mathrm{Fix}g, M) = |G| \cdot \left(\frac{\mathrm{ord}(g) - 1}{2} \right)$$

(see for instance the proof of Theorem 10.13 in [10]). This completes the proof. ∎

Corollary 3.6. *Let* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$, $f(x) = (w(x), h(x))$, *be a reflected graph map germ. Then*

$$\lambda[X, Z] = w^* \left(\frac{1}{\prod_{i=1}^{\ell} L_{H_i}^{|G|(e_i - 1)/2}} E \cdot A_Z \cdot \mathrm{Adj}(E) \right),$$

where $\mathrm{Adj}(E)$ *denotes the adjoint matrix of* E (*the transpose of the cofactor matrix of* E).

Example 3.7. Consider $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and a double fold map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ given by

$$f(x, y) = (x^2, y^2, h(x, y)).$$

We can write

$$h(x, y) = xp_1 + yp_2 + xyp_3, \quad \text{where } p_i = p_i(x^2, y^2).$$

Note that $r_1 = 1, r_2 = x, r_3 = y$ and $r_4 = xy$ generates $\mathbb{C}[x, y]/(x^2, y^2)$ as a $\mathbb{C}$-complex vector space. The matrices E, $\mathrm{Adj}(E)$ and A_Z are

$$E = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x & -x & x & -x \\ y & y & -y & -y \\ xy & -xy & -xy & xy \end{bmatrix}, \quad \mathrm{Adj}(E) = \begin{bmatrix} 4x^2y^2 & 4xy^2 & 4x^2y & 4xy \\ 4x^2y^2 & -4xy^2 & 4x^2y & -4xy \\ 4x^2y^2 & 4xy^2 & -4x^2y & -4xy \\ 4x^2y^2 & -4xy^2 & -4x^2y & 4xy \end{bmatrix},$$

and

$$A_Z = \begin{bmatrix} h_1 - Z & 0 & 0 & 0 \\ 0 & h_2 - Z & 0 & 0 \\ 0 & 0 & h_3 - Z & 0 \\ 0 & 0 & 0 & h_4 - Z \end{bmatrix},$$

where

$$h_1 = xp_1 + yp_2 + xyp_3,$$
$$h_2 = -xp_1 + yp_2 - xyp_3,$$
$$h_3 = xp_1 - yp_2 - xyp_3,$$
$$h_4 = -xp_1 - yp_2 + xyp_3.$$

The group $\mathbb{Z}_2 \times \mathbb{Z}_2$ has two reflecting hyperplanes given by $L_{g_1} = \mathbf{V}(x)$ and $L_{g_2} = \mathbf{V}(y)$. All reflections of G have order 2, therefore $e_1 = e_2 = 2$. By Proposition 3.5, we obtain that $\det(E) = cx^2 y^2$ for some constant c. Actually, $\det(E) = 16x^2 y^2$.

Making the product $E \cdot A_Z \cdot \mathrm{Adj}(E)$ and multiplying the entries of the resulting matrix by $1/16x^2 y^2$, we obtain the following matrix:

$$\lambda[w, Z] = \begin{bmatrix} -Z & p_1 & p_2 & p_3 \\ x^2 p_1 & -Z & x^2 p_3 & p_2 \\ y^2 p_2 & y^2 p_3 & -Z & p_1 \\ x^2 y^2 p_3 & y^2 p_2 & x^2 p_1 & -Z \end{bmatrix}.$$

Now, making the change of $x^2 \mapsto X$, $y^2 \mapsto Y$ in $\lambda[w, Z]$, we obtain the presentation matrix

$$\lambda[X, Y, Z] = \begin{bmatrix} -Z & p_1 & p_2 & p_3 \\ Xp_1 & -Z & Xp_3 & p_2 \\ Yp_2 & Yp_3 & -Z & p_1 \\ XYp_3 & Yp_2 & Xp_1 & -Z \end{bmatrix},$$

which is exactly the same matrix $\lambda[X, Y, Z]$ that appears in the proof of Proposition 3.1 in [11].

4. The image of a reflected graph map

In this section, we provide an answer to Question 1 from the introduction. Specifically, we give a defining equation of the image of a reflected graph map germ f from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^{n+1}, 0)$ in terms of the action of the reflection group G on h (the last coordinate function of f). Let us start with an example to illustrate our answer to Question 1.

Consider the reflection group $G = \mathbb{Z}_4$ acting on $\mathbb{C}^2 \times \mathbb{C}$ (trivially on the second factor) with orbit map $w = (x, y^4, z)$. Let $f: (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a reflected graph map germ, given by

$$f(x, y) = (x, y^4, yp_1 + y^2 p_2 + y^3 p_3),$$

where $p_i = p_i(x, y^4)$ for $i = 1, 2, 3$. Note that $r_1 = 1$, $r_2 = y$, $r_3 = y^2$ and $r_4 = y^3$ generate $\mathbb{C}[x, y]/(x, y^4)$ as a complex vector space. Applying Theorem 3.3, we obtain the presentation matrix of $f_*(\mathcal{O}_2)$ as a $\mathcal{O}_3$-module via f given by

$$\lambda[X, Y, Z] = \begin{bmatrix} -Z & p_1 & p_2 & p_3 \\ Yp_3 & -Z & p_1 & p_2 \\ Yp_2 & Yp_3 & -Z & p_1 \\ Yp_1 & Yp_2 & Yp_3 & -Z \end{bmatrix}.$$

Therefore, calculating the determinant of $\lambda[X, Y, Z]$, a defining equation of the image of f is given by

$$F(X, Y, Z) = Z^4 + Q_2 Z^2 - Q_1 Z + Q_0,$$

where

$$Q_2 = -4Yp_1p_3 - 2p_2^2,$$
$$Q_1 = 4Yp_1^2p_2 + 4Y^2p_2p_3^2,$$
$$Q_0 = Y^2(p_2^4 - 4p_1p_2^2p_3 + 2p_1^2p_3^2) - Y^3p_3^4 - Yp_1^4.$$

Now set $q_i(x, y) := w_*(Q_i) = Q_i(x, y^4)$. Thus we obtain that

$$q_2(x, y) = -4y^4p_1p_3 - 2p_2^2,$$
$$q_1(x, y) = 4y^4p_1^2p_2 + 4y^8p_2p_3^2,$$
$$q_0(x, y) = y^8(p_2^4 - 4p_1p_2^2p_3 + 2p_1^2p_3^2) - y^{12}p_3^4 - y^4p_1^4,$$

where now we have $p_i = p_i(x, y^4)$.

A curious fact is that we can express each q_i in terms of symmetric polynomials in the "variables" h_i; more precisely,

$$q_2(x, y) = h_1h_2 + h_1h_3 + h_1h_4 + h_2h_3 + h_3h_4,$$
$$q_1(x, y) = h_1h_2h_3 + h_1h_2h_4 + h_1h_3h_4 + h_2h_3h_4,$$
$$q_0(x, y) = h_1h_2h_3h_4,$$

where

$$h_i = g_{i_\bullet}(xp_1 + y^2p_2 + y^3p_3).$$

More precisely,

$$h_1 = yp_1 + y^2p_2 + y^3p_3,$$
$$h_2 = -yp_1 + y^2p_2 - y^3p_3,$$
$$h_3 = iyp_1 - y^2p_2 - iy^3p_3,$$
$$h_4 = -iyp_1 - y^2p_2 + iy^3p_3.$$

Note that, in particular, the polynomials q_i are invariant under the action of G. We will show in this section that this same behaviour happens for any reflected graph map germ.

Lemma 4.1. *Let $G = \{g_1 = \mathrm{Id}, g_2, \ldots, g_d\}$ be a reflection group and let h be a function in $\mathbb{C}\{x\}$. Define h_i by $g_{i_\bullet}h$, that is, h_i is the action of g_i on h. Then for all $k \in \{1, 2, \ldots, d\}$, the polynomial*

$$q_{d-k} := \sum_{i_1 < i_2 < \cdots < i_d} h_{i_1}h_{i_2} \cdots h_{i_k}$$

is invariant under the action of G. In particular, $q_{d-k} \in \mathbb{C}[w_1, \ldots, w_d]$.

Proof. Consider the ring $A := \mathbb{C}[h_1, h_2, \ldots, h_d]$ and let $\mathrm{Sym}(A)$ be the set of symmetric polynomials in the "variables" $h_1, h_2, \ldots, h_d$. Note that by definition, $q_{d-k} \in \mathrm{Sym}(A)$ for each k. For all $l \geq 1$, consider the power sum symmetric polynomial $m_l = h_1^l + h_2^l + \cdots + h_d^l$. A well-known result states that any symmetric polynomial in $h_1, \ldots, h_d$ can be

expressed as a polynomial expression with rational coefficients in the power sum symmetric polynomials $m_1, \ldots, m_d$ (see Theorem 6.1 in [9]). In other words, the $m_1, m_2, \ldots, m_d$ are the generators of the algebra of symmetric polynomials $\text{Sym}(A)$, that is,

$$\text{Sym}(A) = \mathbb{C}[m_1, \ldots, m_d].$$

Observe that, since $q_{d-k} \in \text{Sym}(A)$, there is $Q_{d-k} \in \mathbb{C}[m_1, m_2, \ldots, m_d]$ such that $q_{d-k} = Q_{d-k}(m_1, m_2, \ldots, m_d)$. From the fact that the Reynolds operator is a projection, it follows that each m_i is invariant under the action of G. Therefore, q_{d-k} is also invariant under the action of G. ∎

Now, since each q_{d-k} is invariant under the action of G, i.e., we can think that q_{d-k} is in $\mathbb{C}[w_1, \ldots, w_d]$. Thus, we can define $w^*(q_{d-k}(w_1, \ldots, w_d)) =: Q_{d-k}(X_1, \ldots, X_d)$, i.e., we can change the "variable" w_i by the (target) variable X_i in q_{d-k}. We are now able to present a defining equation of the image of a reflected graph map germ.

Theorem 4.2. *Let G be a reflection group of order d. Let $f = (\mathbf{w}, h)$ be a reflected graph map germ from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^{n+1}, 0)$. The image of f is given as the zero set of F (which is the determinant of the presentation matrix of $f_* \mathcal{O}_n$), where F is described as the following alternating sum:*

$$F(X, Z) = Z^d - Q_{d-1} Z^{d-1} + Q_{d-2} Z^{d-2} + \cdots + (-1)^{d-1} Q_1 Z + (-1)^d Q_0,$$

where $Q_{d-k} = q_{d-k}(X)$, and q_{d-k} is described in Lemma 4.1.

Proof. Consider the image of f given by the zero set of $F(X, Z)$, where F denotes the determinant of the presentation matrix $\lambda[X, Z]$ of $f_* \mathcal{O}_n$ as a $\mathcal{O}_{n+1}$-module via f. Consider the pullback $F(\mathbf{w}, Z)$ of $F(X, Z)$ by w. It follows by Lemma 3.1 that

$$(4.1) \qquad F(\mathbf{w}, Z) = \prod_{k=1}^{d} (Z - g_{k \bullet}(h(\mathbf{x}))).$$

Considering $h_k := g_{k \bullet} h$ as in Lemma 4.1, we have that

$$(4.2) \qquad F(\mathbf{w}, Z) = (Z - h_1)(Z - h_2) \cdots (Z - h_d).$$

Expanding the expression (4.2), it follows from Lemma 4.1 that

$$F(\mathbf{w}, Z) = Z^d - q_{d-1} Z^{d-1} + q_{d-2} Z^{d-2} + \cdots + (-1)^{d-1} q_1 Z + (-1)^d q_0.$$

Since the polynomials q_{d-k} are invariant, we can consider

$$w^*(q_{d-k}(w_1, \ldots, w_d)) := Q_{d-k}(X_1, \ldots, X_d),$$

and therefore,

$$F(X, Z) = Z^d - Q_{d-1} Z^{d-1} + Q_{d-2} Z^{d-2} + \cdots + (-1)^{d-1} Q_1 Z + (-1)^d Q_0,$$

as desired. ∎

5. The multiplicity of a reflection map germ

In this section, we will present an upper (and also a lower) bound for the multiplicity of the image of a generically 1-to-1 reflection map germ f from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^{n+1}, 0)$, in general, not necessarily a reflected graph map. To state our result, we begin by recalling the notion of multiplicity.

Consider a germ of analytic function $F : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ reduced at the origin with $F \neq 0$. Let $(\mathbf{V}(F), 0)$ be the germ of the zero set of F at the origin. Write

$$F = F_m + F_{m+1} + \cdots + F_k + \cdots ,$$

where each F_k is a homogeneous polynomial of degree k and $F_m \neq 0$. The integer m is called the multiplicity of $\mathbf{V}(F)$ in 0 and is denoted by $m(\mathbf{V}(F), 0)$. Clearly, the multiplicity of $m(\mathbf{V}(F), 0)$ is greater than or equal to 1. An important property of the multiplicity is that $m(\mathbf{V}(F), 0) = 1$ if and only if $(\mathbf{V}(F), 0)$ is non-singular.

Once we have the notion of multiplicity in hand, let us return to the Question 3 from the introduction. Consider a reflection map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$. If $f = (\boldsymbol{w}, h)$ is a singular generically 1-to-1 reflected graph map germ, then by Theorem 4.2 we have that

$$F(\boldsymbol{X}, Z) = Z^d - Q_{d-1} Z^{d-1} + Q_{d-2} Z^{d-2} + \cdots + (-1)^{d-1} Q_1 Z + (-1)^d Q_0$$

is a defining equation of the image of f, i.e., $f(\mathbb{C}^n) = \mathbf{V}(F)$.

Since f is generically 1-to-1, we have that F is reduced (see Proposition 3.1 in [18]). As a consequence, we obtain that

(5.1) $2 \leq m(f(\mathbb{C}^n), 0) \leq d = |G|.$

However, if f is a reflection map germ (not necessarily a reflected graph map germ one), it is not clear how big the multiplicity of the image can be. In Theorem 5.4, we provide an upper bound (and also a lower bound) that generalizes the one given in (5.1). Before stating the theorem, we introduce an auxiliary lemma.

Remark 5.1. Let $(X, 0)$ and $(Y, 0)$ be germs of irreducible analytic sets. Let $f : (X, 0) \to (Y, 0)$ be a finite surjective analytic map germ. We denote the degree of a map f by $\deg(f)$. Roughly speaking, the degree of f is the number of pre-images of a generic value in the image of f. The precise definition is given, for instance, in Definition D.2 of [17]. An important fact about the concept of the degree of a map is its multiplicative property. Suppose $g : (Y, 0) \to (W, 0)$ is a finite surjective analytic map germ, where $(W, 0)$ is an irreducible analytic set. Then $\deg(g \circ f) = \deg(g) \cdot \deg(f)$.

Suppose that $(X, 0)$ is a germ of dimension n and that $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$. Now let $l_1, l_2, \ldots, l_n : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be generic (reduced) linear forms. Let $\pi : (X, 0) \to (\mathbb{C}^n, 0)$, $\pi = (l_1, \ldots, l_n)$, be the restriction to X of the (generic) linear projection π from $\mathbb{C}^{n+1}$ to $\mathbb{C}^n$. Here, "*generic linear projection*" means that $\mathrm{Ker}(\pi) := \pi^{-1}(0)$ is a generic line in $\mathbb{C}^{n+1}$ such that $\mathrm{Ker}(\pi) \cap X = \{0\}$.

For a generic x close enough to 0, $\pi^{-1}(x)$ is a subspace parallel to $\mathrm{Ker}(\pi)$ which intersects X in a finite number of points; this number is precisely $m(X, 0)$ (see for instance Section D.3 of [17]). In other words, the multiplicity can be seen as the local intersection

number at 0 of X with a generic line in $\mathbb{C}^{n+1}$. We note that this local intersection number is independent of the choice of the generic line (see Section D.3 of [17]).

Lemma 5.2. *Consider a reflection map germ $f: (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$, $f = w \circ \rho$. As in Remark 5.1, let $\pi = (l_1, \ldots, l_n)$ be a generic linear projection from $(\mathbb{C}^{n+1}, 0)$ to $(\mathbb{C}^n, 0)$. Consider the image of f with the Fitting structure, i.e., $(f(\mathbb{C}^n), 0) = (V(\mathcal{F}_0(f_* \mathcal{O}_n)), 0)$ and let $L: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a reduced (non-singular) analytic function such that $(\rho(\mathbb{C}^n), 0) = (V(L), 0)$. If f is generically 1-to-1, then*

$$m(f(\mathbb{C}^n), 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+1}}{\langle L, l_1 \circ w, \ldots, l_n \circ w \rangle}.$$

Proof. As in Section 3, set

$$(f(\mathbb{C}^n), 0) = (X, 0) \quad \text{and} \quad (\rho(\mathbb{C}^n), 0) = (Y, 0).$$

Let $\pi: (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^n, 0)$ be a generic linear projection,

$$\pi(x, x_{n+1}) = (l_1(x, x_{n+1}), \ldots, l_n(x, x_{n+1})).$$

By Remark 5.1, we obtain that

$$\deg(\pi \circ w_{|y}) = \deg(w_{|y}) \cdot \deg(\pi_{|x}) \quad \text{and} \quad m(f(\mathbb{C}^n), 0) = \deg(\pi_{|x}).$$

Furthermore,

$$1 = \deg(f) = \deg(\rho \circ w_{|y}) = \deg(\rho) \cdot \deg(w_{|y}) = \deg(w_{|y}).$$

Hence,

$$\deg(\pi \circ w_{|y}) = \deg(\pi_{|x}).$$

Therefore,

$$m(f(\mathbb{C}^n)_{\text{red}}) = \deg(\pi \circ w_{|y}).$$

Finally, we have that

$$\deg(\pi \circ w_{|y}) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{y,0}}{\langle l_1(x), \ldots, l_n(w) \rangle} = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+1}}{\langle L, l_1 \circ w, \ldots, l_n \circ w \rangle},$$

which concludes the proof. ∎

Let $(Y_1, 0), \ldots, (Y_{n+1}, 0)$ be germs of hypersurfaces in $(\mathbb{C}^{n+1}, 0)$. Let $L_1, \ldots, L_{n+1}$ be germs of analytic functions from $(\mathbb{C}^{n+1}, 0)$ to $(\mathbb{C}, 0)$ such that $(Y_i, 0) = (V(L_i), 0)$. We will denote the intersection multiplicity of $(Y_1, 0), \ldots, (Y_{n+1}, 0)$ at 0 by $i(Y_1, \ldots, Y_{n+1})$. If the intersection $(Y_1, 0) \cap \cdots \cap (Y_{n+1}, 0)$ is just the origin, then the intersection multiplicity is a finite number and can be calculated as

$$i(Y_1, \ldots, Y_{n+1}) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+1}}{\langle L_1, L_2, \ldots, L_{n+1} \rangle}.$$

Remark 5.3. We remark that the intersection multiplicity of hypersurfaces is greater than or equal to the product of the multiplicities of each hypersurface, i.e.,

$$(5.2) \qquad i(\mathcal{Y}_1, \ldots, \mathcal{Y}_{n+1}) \geq m(\mathcal{Y}_1, 0) \cdot m(\mathcal{Y}_2, 0) \cdots m(\mathcal{Y}_{n+1}, 0),$$

with equality if and only if the intersection $(\mathcal{Y}_1, 0) \cap \cdots \cap (\mathcal{Y}_{n+1}, 0)$ is transversal, i.e., the intersection of tangent cones of $(\mathcal{Y}_j, 0)$ is also just the origin 0 in $\mathbb{C}^{n+1}$. See Chapter 7 of [5] and p. 151 in [4] for details on the intersection multiplicity of hypersurfaces.

Theorem 5.4. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$, $f = w \circ \rho$, be a generically 1-to-1 reflection map germ, and let G be a reflection group acting on $\mathbb{C}^{n+1}$. Let $d_1 \leq d_2 \leq \cdots \leq d_{n+1}$ be the degrees of G. Then*

$$(5.3) \qquad d_1 d_2 \cdots d_n \leq m(f(\mathbb{C}^n), 0) \leq d_2 d_3 \cdots d_{n+1} \leq |G|.$$

Proof. Denote by $(X) = (X_1, \ldots, X_{n+1})$ the coordinates of $\mathbb{C}^{n+1}$ (the target of f).

Let $L : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a (reduced) non-singular analytic function such that $(\rho(\mathbb{C}^n), 0) = (\mathbf{V}(L), 0)$. We have $\rho : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ and $w : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$, with $w = (w_1, \ldots, w_{n+1})$. In order to avoid confusion with the notation, let $(x) = (x_1, \ldots, x_n)$ be the coordinates of $\mathbb{C}^n$ (the source of ρ), and let $(y) = (y_1, \ldots, y_{n+1})$ be the coordinates of $\mathbb{C}^{n+1}$ (the source of w, equivalently in the target of ρ). Let $\pi : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^n, 0)$ be a generic linear projection,

$$\pi(X) = (l_1(X), \ldots, l_n(X)),$$

in the sense of Remark 5.1, where for $i = 1, \ldots, n$,

$$l_i(X) := b_{i,1} X_1 + b_{i,2} X_2 + \cdots + b_{i,n+1} X_{n+1},$$

with $b_{i,j} \in \mathbb{C}$, and

$$\mathrm{Ker}(\pi) \cap f(\mathbb{C}^n) = \{0\}.$$

Thus,

$$l_i \circ w = \sum_{j=1}^{n+1} b_{i,j} w_j.$$

By the genericity of π, we can assume that all the $b_{i,j}$ are non-zero complex constants. By Lemma 5.2, we have that

$$(5.4) \quad m(f(\mathbb{C}^n), 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+1}}{\left\langle L, \; \sum_{j=1}^{n+1} b_{1,j} w_j, \; \sum_{j=1}^{n+1} b_{2,j} w_j, \ldots, \; \sum_{j=1}^{n+1} b_{n,j} w_j \right\rangle}.$$

After equivalences in the quotient ring in (5.4) and by the genericity of π, we can find non-zero complex constants $\tilde{b}_{i,j}$ such that

$$m(f(\mathbb{C}^n), 0)$$

$$(5.5) \quad = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+1}}{\left\langle L, \; \sum_{j=1}^{n+1} \tilde{b}_{1,j} w_j, \; \sum_{j=2}^{n+1} \tilde{b}_{2,j} w_j, \; \sum_{j=3}^{n+1} \tilde{b}_{3,j} w_j, \ldots, \; \sum_{j=n}^{n+1} \tilde{b}_{n,j} w_j \right\rangle}.$$

Note that the right-hand side of (5.5) can be viewed as an intersection multiplicity of $n + 1$ hypersurfaces in $\mathbb{C}^{n+1}$. Using the notation $\mathcal{Y} = V(L)$, we obtain that

$$(5.6) \quad m(f(\mathbb{C}^n), 0) = i\left(\mathcal{Y}, \mathbf{V}\left(\sum_{j=1}^{n+1} \widetilde{b}_{1,j}\, w_j\right), \mathbf{V}\left(\sum_{j=2}^{n+1} \widetilde{b}_{2,j}\, w_j\right), \dots, \mathbf{V}\left(\sum_{j=n}^{n+1} \widetilde{b}_{n,j}\, w_j\right)\right).$$

Note that $d_i = m(\mathbf{V}(w_i), 0)$ by definition and $m(\mathcal{Y}, 0) = 1$. By (5.2) in Remark 5.3 and (5.6), we conclude that

$$m(f(\mathbb{C}^n), 0) \geq d_1 d_2 \cdots d_n.$$

Now, we need to obtain an upper bound for the multiplicity of the image of f. Since w if finite, then $\mathcal{O}_{n+1}/\langle w_1(y), \dots, w_{n+1}(y)\rangle < \infty$. Therefore, if y_i is a factor of w_j, then y_i is not a factor of w_s for all $s \neq j$. After a change of coordinates in $(\mathbb{C}^{n+1}, 0)$, we can assume without loss of generality that y_{n+1} does not divide $w_2, \dots, w_{n+1}$. Since ρ is an embedding, we can write ρ as

$$\rho(\mathbf{x}) = (\hat{l}_1(\mathbf{x}) + g_1(\mathbf{x}), \ \hat{l}_2(\mathbf{x}) + g_2(\mathbf{x}), \dots, \hat{l}_{n+1}(\mathbf{x}) + g_{n+1}(\mathbf{x})),$$

where $g_i \in \mathbf{m}^2$ for all $i = 1, \dots, n + 1$, $\mathbf{m}$ denotes the maximal ideal of $\mathbb{C}\{\mathbf{x}\}$, and

$$\hat{l}_i(\mathbf{x}) = a_{i,1}\, x_1 + \cdots + a_{i,n}\, x_n.$$

Note that the $n + 1$ vectors $v_j := (a_{1,j}, \dots, a_{n+1,j})$ in $\mathbb{C}^{n+1}$ (with $j = 1, \dots, n + 1$) generate the tangent space of $(\mathcal{Y}, 0)$ (actually, only n vectors are needed). Up to a change of coordinates in $\mathbb{C}^{n+1}$, we can assume that

$$(5.7) \qquad \rho(\mathbf{x}) = (x_1 + g_1(\mathbf{x}), \ x_2 + g_2(\mathbf{x}), \dots, \ x_n + g_n(\mathbf{x}), \ \hat{l}_{n+1}(\mathbf{x}) + g_{n+1}(\mathbf{x})),$$

and therefore,

$$
\begin{aligned}
(5.8) \qquad f(\mathbf{x}) = & (w_1(x_1 + g_1(\mathbf{x}), \dots, \hat{l}_{n+1}(\mathbf{x}) + g_{n+1}(\mathbf{x})), \\
& \dots, \ w_{n+1}(x_1 + g_1(\mathbf{x}), \dots, \hat{l}_{n+1}(\mathbf{x}) + g_{n+1}(\mathbf{x}))).
\end{aligned}
$$

Note that in (5.7), y_{n+1} (the last coordinate of $\mathbb{C}^{n+1}$) is not affected by the last coordinate change. Therefore, we can still assume that y_{n+1} does not divide $w_2, \dots, w_{n+1}$. Let $\pi_1 \colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^n, 0)$ be the linear projection defined by

$$\pi_1(X_1, X_2, \dots, X_{n+1}) = (X_2, \dots, X_{n+1}).$$

Statement. *The map germ* $(\pi_1 \circ f) \colon (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ *is finite.*

By (5.8) and Theorem D.5 in [17], it is sufficient to show that
(5.9)
$$\frac{\mathbb{C}\{x_1, \dots, x_n\}}{\langle w_2(x_1 + g_1(\mathbf{x}), \dots, \hat{l}_{n+1}(\mathbf{x}) + g_{n+1}(\mathbf{x})), \dots, w_{n+1}(x_1 + g_1(\mathbf{x}), \dots, \hat{l}_{n+1}(\mathbf{x}) + g_{n+1}(\mathbf{x}))\rangle}$$

is a finite-dimensional vector space over $\mathbb{C}$.

Note that the quotient ring in (5.9) is isomorphic to the following quotient ring:

$$(5.10) \quad \frac{\mathbb{C}\{x_1, \ldots, x_n, u\}}{\langle \hat{l}_{n+1}(\boldsymbol{x}) + g_{n+1}(\boldsymbol{x}) - u, w_2(x_1, \ldots, x_n, u) + \cdots, \ldots, w_{n+1}(x_1, \ldots, x_n, u) + \cdots \rangle}$$

where the dots appearing in $w_i(x_1, x_2, \ldots, x_n, u) + \cdots$ consists of terms of degree greater than the degree of w_i. In other words, $w_i(x_1, x_2, \ldots, x_n, u) + \cdots$ is just the expansion of $w_i(x_1 + g_1(\boldsymbol{x}), \ldots, x_n + g_n(\boldsymbol{x}), u)$, and u is a new variable.

Since the ring $\mathcal{O}_{n+1}/\langle w_2(\boldsymbol{y}), \ldots, w_{n+1}(\boldsymbol{y}) \rangle$ is unmixed (it is a Cohen–Macaulay ring), we obtain that the zero-divisors of

$$R := \frac{\mathbb{C}\{x_1, \ldots, x_n, u\}}{\langle w_2(x_1, \ldots, x_n, u), \ldots, w_{n+1}(x_1, \ldots, x_n, u) \rangle}$$

are precisely the elements in the union of the minimal primes of R.

Since y_{n+1} does not divide $w_i(y_1, \ldots, y_n, y_{n+1})$ for all $i \neq 1$, then u does not divide $w_i(x_1, \ldots, x_n, u)$ for all $i \neq 1$. Thus u cannot be contained in any minimal prime of R. It follows that u is not a zero divisor of R. By a suitable version of Krull's principal ideal theorem, we obtain that

$$\frac{R}{\langle u \rangle} \simeq \frac{\mathbb{C}\{x_1, \ldots, x_n, u\}}{\langle u, w_2(x_1, \ldots, x_n, u), \ldots, w_{n+1}(x_1, \ldots, x_n, u) \rangle}$$

has dimension zero, hence it is a finite-dimensional vector space over $\mathbb{C}$. This implies that the intersection in (5.10) (and also in (5.9)) is transversal. By Remark 5.3, we conclude that the dimension as a $\mathbb{C}$-vector space of the quotient ring in (5.9) is precisely the product $d_2 \cdot d_3 \cdots d_{n+1}$, which proves the statement.

To conclude the proof, in the Mond–Pellikaan algorithm (see Section 2.2 of [18]), consider the projection π_1 above. By the statement, we have that $\pi_1 \circ f$ is finite. Thus the presentation matrix λ of $f_*(\mathcal{O}_n)$ with respect to π_1 has the following form:

$$\lambda[X_1, \ldots, X_{n+1}] = \begin{bmatrix} \alpha_{1,1} - X_1 & \alpha_{1,2} & \alpha_{1,3} & \cdots & \alpha_{1,d'} \\ \alpha_{2,1} & \alpha_{2,2} - X_1 & \alpha_{2,3} & \cdots & \alpha_{2,d'} \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} - X_1 & \cdots & \alpha_{3,d'} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \alpha_{d',1} & \alpha_{d',2} & \alpha_{d',3} & \cdots & \alpha_{d',d'} - X_1 \end{bmatrix},$$

where $\alpha_{i,j} \in \mathbb{C}\{X_2, \ldots, X_{n+1}\}$ and $d' = d_2 \cdot d_3 \cdots d_{n+1}$.

As a consequence, we obtain that the term $X_1^{d'}$ appears in the defining equation of $f(\mathbb{C}^n)$ given by the determinant of λ. Therefore,

$$m(f(\mathbb{C}^n), 0) \leq d_2 \cdots d_{n+1}.$$

The last inequality in (5.3) follows from the fact that $d_1 \cdot d_2 \cdots d_{n+1} = |G|$, since G is a reflection group ([25], see also Theorem 4.19 in [10]). ∎

Remark 5.5. Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ be a reflected graph map germ with respect to a reflection group G acting on $\mathbb{C}^{n+1} = \mathbb{C}^n \times \mathbb{C}$, as in Section 4. Since the group G acts trivially on the second factor, the orbit map $w\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ is $w(\mathbf{y}) = (w_1(\mathbf{y}), \dots, w_n(\mathbf{y}), y_{n+1})$. In other words, if $d_1, \dots, d_{n+1}$ are the degrees of G, then $d_{n+1} = 1$. After a reordering of the remaining degrees of G, we can assume that $d_1 \leq d_2 \leq \cdots \leq d_n$. By Theorem 5.4, we obtain that

$$d_1 d_2 \cdots d_{n-1} \leq m(f(\mathbb{C}^n), 0) \leq d_1 \cdots d_n = d = |G|.$$

When working with a reflected graph map, to avoid unnecessary notation, we can think of G as acting nontrivially only on $\mathbb{C}^n$ (the first factor of $\mathbb{C}^n \times \mathbb{C}$, and not on all $\mathbb{C}^{n+1}$). In this case, we can consider only $d_1, \dots, d_n$ as "non-trivial degrees" of G and ignore the degree $d_{n+1} = 1$. Whenever there is no danger of confusing the notation, we will use this simplification.

6. Some applications and examples

In this section, we discuss present some applications of our results. In the first part, we provide an alternative proof of a result of (see Theorem 5.2 in [1]) about a defining equation of the double point space of a reflected graph map germ. In the second part, we will introduce the notion of a dihedral map germ and we will apply our results to describe the presentation matrix and a defining equation of the image of maps of this kind. Finally, in the last part, we extend a result given by Marar and Nuño-Ballesteros (see Theorem 3.4 in [11]) about the non-existence of quasihomogeneous (with distinct weights) finitely determined reflection map germs, for some reflection groups.

6.1. Double point spaces

When we study a finite map f from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^p, 0)$ with $n \leq p$, the multiple point spaces of f play an important role in the study of its geometry.

In the particular case where $p = n + 1$ and f is a reflection map, a natural question is if we can describe a defining equation of the hypersurface of double points $D(f)$ in the source in terms of the action of G in some sense.

Recently, Borges Zampiva, Peñafort-Sanchis, Oréfice Okamoto and Tomazella provided an answer to this question in a more general context. They give a defining equation of $D(f)$ which depends only on the action of G in an appropriate way (see [1]). As our first application, we will use Lemma 3.1 to offer an alternative proof of the formula given in the case of reflected graph map germs. For the convenience of the reader, we will recall the notion of the double point space of a map germ. We follow [15] to describe the double point set of a map germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, with $n < p$.

Let $f\colon U \to \mathbb{C}^p$ be a homomorphic map, where $U \subset \mathbb{C}^n$ is an open subset and $n < p$. Now let us define the set of double point space of f, denoted by $D^2(f)$. Denote the diagonals of $\mathbb{C}^n \times \mathbb{C}^n$ and $\mathbb{C}^p \times \mathbb{C}^p$ by Δ_n and Δ_p, and denote the ideal schemes of defining Δ_n and Δ_p by $\mathcal{I}_n$ and $\mathcal{I}_p$. Write points of $\mathbb{C}^n \times \mathbb{C}^n$ as (x, x'). It is clear that for each $i = 1, \dots, p$,

$$f_i(x) - f_i(x') \in \mathcal{I}_n,$$

so there exist $\alpha_{i,j}$, $1 \leq i \leq n$, $1 \leq j \leq n+1$, such that

$$f_i(x) - f_i(x') = \sum \alpha_{i,j}(x - x')(x_j - x'_j).$$

If $f(x) = f(x')$ and $x \neq x'$, then clearly every $n \times n$ minor of the matrix $\alpha = [\alpha_{i,j}]$ must vanish at (x, x'). Now denote by $\mathcal{R}_n(\alpha)$ the ideal in $\mathcal{O}_{\mathbb{C}^{2n}}$ generated by the $n \times n$ minors of α.

Definition 6.1. Let $f : U \to \mathbb{C}^p$ be as above. Then the double point space of f is the complex space

$$D^2(f) = \mathbf{V}((f \times f)^* \mathcal{I}_p + \mathcal{R}_n(\alpha)).$$

Definition 6.2. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a finite map germ, where $n \leq p$.

(a) Let us denote by I_p and $R_n(\alpha)$ the stalks at 0 of the sheaves $\mathcal{I}_p$ and $\mathcal{R}_n(\alpha)$. Taking a representative of f, we define the double point space of the map germ f as the analytic space

$$D^2(f) := \mathbf{V}((f \times f)^* I_p + R_n(\alpha)).$$

(b) The mapping $\pi : (D^2(f), 0) \to (\mathbb{C}^n, 0)$, given by $(x, x') \mapsto x$, is finite. The source double point set $D(f)$ is defined as the image of π with the analytic structure given by the 0-Fitting ideal of $\pi_* \mathcal{O}_{D^2(f)}$, that is,

$$D(f) := \mathbf{V}(\mathcal{F}_0(\pi_* \mathcal{O}_{*D^2(f)})).$$

Now we will present an alternative proof for Theorem 5.2 in [1] in the case of a reflected graph map germ. We recall that $\mathcal{R}$ denotes the set of all reflections of G and that Δ_{g_k} denotes the Demazure operator (see Section 2.1).

Proposition 6.3 ([1]). *Let* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$, $f(\mathbf{x}) = (w(\mathbf{x}), h(\mathbf{x}))$ *be a reflected graph map germ. Then*

$$D(f) = V\left(\left(\prod_{g_k \in \mathcal{R}} \Delta_{g_k}(h)\right)\left(\prod_{g_k \notin \mathcal{R}, \, g_k \neq \mathrm{Id}} (h - g_{k\bullet}h)\right)\right).$$

Proof. Denote by $g_1 = \mathrm{Id}, g_2, \ldots, g_d$ the elements of G. By Theorem 4.2, we obtain that a defining equation of the determinant of the image of f is given by

(6.1) $F(X, Z) = Z^d - Q_{d-1} Z^{d-1} + Q_{d-2} Z^{d-2} + \cdots + (-1)^{d-1} Q_1 Z + (-1)^d Q_0.$

Note that replacing X by w in (6.1) and taking the partial derivative with respect to the variable Z is equivalent to taking the partial derivative of (6.1) with respect to the variable Z and then replacing X with w, i.e., the order of execution of these operations does not matter. By Lemma 3.1, we have that

$$F(\mathbf{w}, Z) = \prod_{k=1}^{d} (Z - g_{k\bullet}(h(\mathbf{x}))).$$

Therefore,

$$\frac{\partial F(\boldsymbol{w}, Z)}{\partial Z} = \frac{\partial(Z - g_{1\bullet}h(\boldsymbol{x}))}{\partial Z}\Big[\prod_{k=2}^{d}(Z - g_{k\bullet}(h(\boldsymbol{x})))\Big]$$

$$+ (Z - g_{1\bullet}h(\boldsymbol{x}))\frac{\partial\big[\prod_{k=2}^{d}(Z - g_{k\bullet}(h(\boldsymbol{x})))\big]}{\partial Z}.$$

Note that $g_{1\bullet}h(\boldsymbol{x}) = h(\boldsymbol{x})$, and so, replacing Z by h, we obtain that

$$\frac{\partial F(\boldsymbol{w}, h)}{\partial Z} = (1 - h(\boldsymbol{x}))\Big[\prod_{k=2}^{d}(h(\boldsymbol{x}) - g_{k\bullet}(h(\boldsymbol{x})))\Big].$$

By [22] (see also Proposition 2.5 in [3]), we obtain that

$$(D(f), 0) = \Big(\mathbf{V}\Big(\frac{\prod_{k=2}^{d}(h(\boldsymbol{x}) - g_{k\bullet}h(\boldsymbol{x}))}{\det(\mathrm{jac}(\boldsymbol{w}))}\Big), 0\Big).$$

Consider the hyperplanes $\mathrm{Fix}\, g$, where g runs through the reflections of G, and call these the reflecting hyperplanes of G. For each i, let $\mathrm{ord}(g_i)$ be the order of the cyclic group $G_{\mathrm{Fix}\, g_i}$ fixing $\mathrm{Fix}\, g_i$ pointwise, and let $L_{g_1}, L_{g_2}, \dots, L_{g_l}$ be linear forms such that $\mathrm{Fix}\, g_i = \mathrm{Ker}\, L_{g_i}$. We have that

$$\frac{\prod_{g_k \in \mathcal{R}}(h(\boldsymbol{x}) - g_{k\bullet}h(\boldsymbol{x}))}{\det(\mathrm{jac}(\boldsymbol{w}))} = \frac{\prod_{g_k \in \mathcal{R}}(h(\boldsymbol{x}) - g_{k\bullet}h(\boldsymbol{x}))}{c\prod_{i=1}^{\ell}L_{g_i}^{\mathrm{ord}(g_i)-1}} = c\prod_{g_k \in \mathcal{R}}\Delta_{g_k}(h),$$

where $\mathcal{R}$ denotes the set of all reflections of G (see for instance Lemma 9.7 in [10]). Therefore,

$$D(f) = \mathbf{V}\Big(\Big(\prod_{g_k \in \mathcal{R}}\Delta_{g_k}(h)\Big)\Big(\prod_{g_k \notin \mathcal{R}, g_k \neq \mathrm{Id}}(h - g_{k\bullet}h)\Big)\Big),$$

which concludes the proof. ∎

Example 6.4. Consider $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $f(x, y) = (x^2, y^2, h(x, y))$, where

$$h(x, y) = xp_1(x^2, y^2) + yp_2(x^2, y^2) + xyp_3(x^2, y^2).$$

Note that the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ can be represented in $\mathrm{GL}_2(\mathbb{C})$ by the following matrices

$$\mathrm{Id} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad g_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad g_3 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, we obtain that

$$h(x, y) - g_{1\bullet}h(x, y) = 2(xp_1 + yp_2),$$
$$h(x, y) - g_{2\bullet}h(x, y) = 2y(p_2 + xp_3),$$
$$(x, y) - g_{3\bullet}h(x, y) = 2x(p_1 + yp_3).$$

Note that $\det(\mathrm{jac}(x^2, y^2)) = 4xy$. Thus, from Proposition 6.3, we obtain that

$$(6.2) \qquad D(f) = \mathbf{V}((yp_3 + p_1)(xp_3 + p_2)(xp_1 + yp_2)).$$

We remark that the defining equation that appears in (6.2) for $D(f)$ is exactly the same presented in Proposition 3.1 of [11].

Corollary 6.5. *With the notation used in Proposition 6.3, let $f = (\boldsymbol{w}, h)$ be a reflected graph map germ from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^{n+1}, 0)$. Suppose that h is regular, say*

$$h(\boldsymbol{x}) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n + R(\boldsymbol{x}),$$

with $m(R(\boldsymbol{x}), 0) > 1$ or $R = 0$. Let H be the hyperplane defined by $H := \mathbf{V}(\sum_{i=1}^{n} a_i x_i)$. Suppose that for every reflection $g \in G$, H is not the reflecting hyperplane of g (equivalently, $H \cap (V(\mathrm{jac}(w))) \neq H$ as a set). Then,

$$D(f) = V\Big(\prod_{g_k \notin \mathcal{R}, \, g_k \neq \mathrm{Id}} (h(\boldsymbol{x}) - g_k \bullet h(\boldsymbol{x})) \Big).$$

Proof. By Proposition 6.3 we have that

$$D(f) = \mathbf{V}\Big(\Big(\prod_{g_k \in \mathcal{R}} \Delta_{g_k}(h(\boldsymbol{x})) \Big) \Big(\prod_{g_k \notin \mathcal{R}, \, g_k \neq Id} (h(\boldsymbol{x})) - g_k \bullet h(\boldsymbol{x})) \Big) \Big).$$

We just need to show that $\Delta_g(h(\boldsymbol{x}))$ is an invertible element in the ring of analytic series $\mathbb{C}\{\boldsymbol{x}\}$ for all $g \in \mathcal{R}$. If $g \in \mathcal{R}$ and $\mathrm{Fix}\, g$ is its reflecting hyperplane, with $\mathrm{Fix}\, g = \mathrm{Ker}\, L_g$, then by the definition of the Demazure operator and Lemma 2.3, we obtain that

$$(6.3) \qquad \Delta_g(h(\boldsymbol{x})) = \frac{h(\boldsymbol{x}) - g_\bullet h(\boldsymbol{x})}{L_g} = Q(\boldsymbol{x}),$$

for some $Q(\boldsymbol{x}) \in \mathbb{C}\{\boldsymbol{x}\}$.

The action of G on $\mathbb{C}\{\boldsymbol{x}\}$ has the property that if $P \in \mathbb{C}\{\boldsymbol{x}\}$ is homogeneous of degree d, then $g_\bullet P$ is also homogeneous of degree d (see Section 2 in Chapter 3 of [10]). Also, recall that

$$g_\bullet(P_1 + P_2) = g_\bullet(P_1) + g_\bullet(P_2) \quad \text{for all } P_1, P_2 \in \mathbb{C}\{\boldsymbol{x}\}.$$

By hypothesis, H is not a hyperplane of g. Therefore,

$$g_\bullet(a_1 x_1 + a_2 x_2 + \cdots + a_n x_n) \neq a_1 x_1 + a_2 x_2 + \cdots + a_n x_n,$$

i.e., the defining equation of H is not invariant under the action of g. In this way, we can write

$$h(\boldsymbol{x}) - g_\bullet h(\boldsymbol{x}) = b_1 x_1 + b_2 x_2 + \cdots + b_n x_n + Q'(\boldsymbol{x}),$$

where either $m(Q'(\boldsymbol{x}), 0) > 1$ or $Q' = 0$. Also, $b_i \in \mathbb{C}$ for all i, with at least one b_i being non-zero.

By (6.3), we have that L_g (which is a linear form) divides $h(\boldsymbol{x}) - g_\bullet h(\boldsymbol{x})$. This implies that $Q(0) \neq 0$, and so $Q(\boldsymbol{x})$ must necessarily be an invertible element in $\mathbb{C}\{\boldsymbol{x}\}$. ∎

Example 6.6. Let $f \colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a reflected graph map germ, given by

$$f(x, y) = (x^2, y^2, x - 3y + y^3).$$

Using the notation in Example 6.4, by Corollary 6.5 we obtain that

$$D(f) = \mathbf{V}(h(x, y) - g_{1\bullet}h(x, y))) = \mathbf{V}(x - 3y + y^3).$$

Note that in this case we do not need to consider all elements of G to obtain a defining equation of $D(f)$, only those that are not reflections (excluding of course the identity of the group).

6.2. Dihedral map germs

Inspired by the work of Marar and Nuño-Ballesteros in [11], where they introduce the double fold map germs, in this section we will introduce the "*dihedral map germs*". Consider the generators of the dihedral group D_{2m} in $\mathrm{GL}(\mathbb{C}^2)$ given by in Example 2.6,

$$R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{m-1} \end{bmatrix},$$

where $\zeta = e^{2\pi i/m}$.

Note that with this representation D_{2m} is a reflection group. By Proposition 2.22 in [10], we have that the polynomials $w_1 = x^m + y^m$ and $w_2 = xy$ are algebraically independent and they are invariant by the action of D_{2m} (in [10], D_{2m} is identified as being the group $G(m, m, 2)$ in the Shephard and Todd classification). Therefore, the orbit map for the group D_{2m} acting on $\mathbb{C}^2$ is $w(x, y) = (x^m + y^m, \ xy)$.

Thus, a the reflection map $f \colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is a *2m-dihedral map germ*, or simply "*a dihedral map germ*", if f is a reflected graph map germ in the following form:

$$f(x, y) = (x^m + y^m, \ xy, \ h(x, y)).$$

Note that $r_1 = 1, r_2 = x, r_3 = x^2, \ldots, r_m = x^{m-1}$ and $r_{m+k} = y^k$, with $k \in \{1, \ldots, m\}$, is a basis of $\mathcal{O}_2/(x^m + y^m, xy)$ as a $\mathbb{C}$-vector space. By taking coordinate changes and using the Malgrange preparation theorem, we can write h in the form

$$h(x, y) = \sum_{j=1}^{m-1} x^j p_j + \sum_{k=1}^{m} y^k p_{m-1+k},$$

where $p_i = p_i(x^m + y^m, xy)$ and we suppose that $h(0, 0) = 0$. As an example, we will provide in the next result a study of a 6-dihedral map germ, i.e., using the D_6 group. We consider the following representation in $\mathrm{GL}(\mathbb{C}^2)$ of the dihedral group D_6:

$$g_1 = \mathrm{Id} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad g_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad g_3 = \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{bmatrix},$$

$$g_4 = \begin{bmatrix} \zeta^2 & 0 \\ 0 & \zeta \end{bmatrix}, \quad g_5 = \begin{bmatrix} 0 & \zeta \\ \zeta^2 & 0 \end{bmatrix}, \quad g_6 = \begin{bmatrix} 0 & \zeta^2 \\ \zeta & 0 \end{bmatrix},$$

where $\zeta = e^{2\pi i/3}$, so $\zeta^2 + \zeta + 1 = 0$.

The orbit map for the group D_6 acting on $\mathbb{C}^2$ is $w(x, y) = (x^3 + y^3, xy)$. Thus, a reflection map $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ for the dihedral group D_6 is given by $f(x, y) = (x^3 + y^3, xy, h(x, y))$, where $h(x, y) = xp_1 + x^2 p_2 + yp_3 + y^2 p_4 + y^3 p_5$. Note that

$$h_1 := g_1 \bullet h = h,$$
$$h_2 := g_2 \bullet h = yp_1 + y^2 p_2 + xp_3 + x^2 p_4 + x^3 p_5,$$
$$h_3 := g_3 \bullet h = \zeta xp_1 + \zeta^2 x^2 p_2 + \zeta^2 yp_3 + \zeta y^2 p_4 + y^3 p_5,$$
$$h_4 := g_4 \bullet h = \zeta^2 xp_1 + \zeta x^2 p_2 + \zeta yp_3 + \zeta^2 y^2 p_4 + y^3 p_5,$$
$$h_5 := g_5 \bullet h = \zeta yp_1 + \zeta^2 y^2 p_2 + \zeta^2 xp_3 + \zeta x^2 p_4 + x^3 p_5,$$
$$h_6 := g_6 \bullet h = \zeta^2 yp_1 + \zeta y^2 p_2 + \zeta xp_3 + \zeta^2 x^2 p_4 + x^3 p_5.$$

Recall from Lemma 4.1 that we can define

$$q_5(xy, x^3 + y^3) := h_1 + h_2 + h_3 + h_4 + h_5 + h_6,$$
$$q_4(xy, x^3 + y^3) := h_1 h_2 + h_1 h_3 + h_1 h_4 + \cdots + h_5 h_6,$$
$$q_3(xy, x^3 + y^3) := h_1 h_2 h_3 + h_1 h_2 h_4 + \cdots + h_4 h_5 h_6,$$
$$q_2(xy, x^3 + y^3) := h_1 h_2 h_3 h_4 + h_1 h_2 h_3 h_5 + \cdots + h_3 h_4 h_5 h_6,$$
$$q_1(xy, x^3 + y^3) := h_1 h_2 h_3 h_4 h_5 + h_1 h_2 h_3 h_4 h_6 + \cdots + h_2 h_3 h_4 h_5 h_6,$$
$$q_0(xy, x^3 + y^3) := h_1 h_2 h_3 h_4 h_5 h_6.$$

Proposition 6.7. *Let* $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ *be a 6-dihedral map germ and write it in the form*

$$f(x, y) = (x^3 + y^3, \ xy, \ xp_1 + x^2 p_2 + yp_3 + y^2 p_4 + y^3 p_5),$$

where $p_i = p_i(x^3 + y^3, xy)$. *Then:*

(a) *The presentation matrix* $\lambda[X, Y, Z]$ *of* $f_* \mathcal{O}_2$ *as an* $\mathcal{O}_3$-*module via* f *is*

$$\lambda = \begin{bmatrix} -Z & p_1 & p_2 & p_3 & p_4 & p_5 \\ p_2 X + p_3 Y & -Z & p_1 & p_4 Y & p_5 Y & -p_2 \\ p_1 X + p_4 Y^2 & p_2 X + p_3 Y & -Z & p_5 Y^2 & -p_2 Y & -p_1 \\ p_1 Y & p_2 Y & -p_5 Y & p_5 X - Z & p_3 & p_4 \\ p_2 Y^2 & -p_5 Y^2 & -p_4 Y & p_4 X + p_1 Y & p_5 X - Z & p_3 \\ -p_5 Y^3 & -p_4 Y^2 & -p_3 Y & p_3 X + p_2 Y^2 & p_4 X + p_1 Y & p_5 X - Z \end{bmatrix},$$

where $p_i = p_i(X, Y)$.

(b) *A defining equation of the image of* f *is*

$$F(X, Y, Z) = Z^6 - Q_5 Z^5 + Q_4 Z^4 - Q_3 Z^3 + Q_2 Z^2 - Q_1 Z + Q_0,$$

where

$$Q_{6-k}(X, Y) := w^*(q_{6-k}(x^3 + y^3, xy)),$$

i.e., we simply change the "variables" $x^3 + y^3$ *and* xy *by the (target) variables* X *and* Y *in* q_{6-k} *described just above this proposition.*

(c) *A defining equation of the double point curve $D(f)$ of f is*

$$D(f) = V(\eta_1 \eta_2 \eta_3),$$

where

$$\eta_1 = -p_1 + p_3 - p_2 x + p_4 x - p_2 y + p_4 y + p_5 x^2 + p_5 xy + p_5 y^2,$$

$$\begin{aligned}\eta_2 = {}& p_1^2 + p_1 p_3 + p_3^2 + 2p_1 p_2 x + p_2 p_3 x + p_1 p_4 x - p_3 p_4 x - p_1 p_2 y + p_2 p_3 y \\ & + p_1 p_4 y + 2p_3 p_4 y + p_2^2 x^2 + p_2 p_4 x^2 + p_4^2 x^2 - 2p_1 p_5 x^2 - p_3 p_5 x^2 - p_2^2 xy \\ & - p_2 p_4 xy - p_4^2 xy + p_1 p_5 xy - p_3 p_5 xy + p_2^2 y^2 + p_2 p_4 y^2 + p_4^2 y^2 + p_1 p_5 y^2 \\ & + 2p_3 p_5 y^2 - 2p_2 p_5 x^3 - p_4 p_5 x^3 + 2p_2 p_5 x^2 y + p_4 p_5 x^2 y - p_2 p_5 xy^2 \\ & - 2p_4 p_5 xy^2 + p_2 p_5 y^3 + 2p_4 p_5 y^3 + p_5^2 x^4 - p_5^2 x^3 y - p_5^2 xy^3 + p_5^2 y^4,\end{aligned}$$

$$\begin{aligned}\eta_3 = {}& p_1^2 x^2 + p_1 p_3 xy + p_3^2 y^2 + p_1 p_2 x^3 + 2p_2 p_3 x^2 y + 2p_1 p_4 xy^2 + p_3 p_4 y^3 \\ & + p_2^2 x^4 + p_2 p_4 x^2 y^2 + p_4^2 y^4.\end{aligned}$$

(d) *The multiplicity of the image of f satisfies*

$$2 \le m(f(\mathbb{C}^2, 0)) \le 6.$$

Proof. (a) Consider the basis $r_1 = 1, r_2 = x, r_3 = x^2, r_4 = y, r_5 = y^2$ and $r_6 = y^3$ for $\mathcal{O}_2/(x^3 + y^3, xy)$. Using the same notation as in Remark 3.2, we obtain that

$$E = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ x & y & \zeta x & \zeta^2 x & \zeta y & \zeta^2 y \\ x^2 & y^2 & \zeta^2 x^2 & \zeta x^2 & \zeta^2 y^2 & \zeta y^2 \\ y & x & \zeta^2 y & \zeta y & \zeta^2 x & \zeta x \\ y^2 & x^2 & \zeta y^2 & \zeta^2 y^2 & \zeta x^2 & \zeta^2 x^2 \\ y^3 & x^3 & y^3 & y^3 & x^3 & x^3 \end{bmatrix}$$

and

$$A_Z = \begin{bmatrix} h_1 - Z & 0 & 0 & 0 & 0 & 0 \\ 0 & h_2 - Z & 0 & 0 & 0 & 0 \\ 0 & 0 & h_3 - Z & 0 & 0 & 0 \\ 0 & 0 & 0 & h_4 - Z & 0 & 0 \\ 0 & 0 & 0 & 0 & h_5 - Z & 0 \\ 0 & 0 & 0 & 0 & 0 & h_6 - Z \end{bmatrix}.$$

The reflections of D_6 are g_2, g_5 and g_6, and their respective reflecting hyperplanes are

$$L_{H_1} = x - y, \quad L_{H_2} = x - \zeta y \quad \text{and} \quad L_{H_3} = x - \zeta^2 y.$$

Each reflection of D_6 has order 2. By Proposition 3.5, we obtain that $\det(E) = 27(x^3 - y^3)^3$. The adjoint matrix of E is

$$\text{Adj}(E) = 9(x^3 - y^3)^2 \begin{bmatrix} x^3 & x^2 & x & -y^2 & -y & -1 \\ -y^3 & -y^2 & -y & x^2 & x & 1 \\ x^3 & \zeta^2 x^2 & \zeta x & -\zeta y^2 & -\zeta^2 y & -1 \\ x^3 & \zeta x^2 & \zeta^2 x & -\zeta^2 y^2 & -\zeta y & -1 \\ -y^3 & -\zeta^2 y^2 & -\zeta y & \zeta x^2 & \zeta^2 x & 1 \\ -y^3 & -\zeta y^2 & -\zeta^2 y & \zeta^2 x^2 & \zeta x & 1 \end{bmatrix}.$$

Applying Corollary 3.6, we obtain that a presentation matrix of $f_*(\mathcal{O}_2)$ as a $\mathcal{O}_3$-module via f is given by

$$\lambda[X,Y,Z] = \begin{bmatrix} -Z & p_1 & p_2 & p_3 & p_4 & p_5 \\ p_2X+p_3Y & -Z & p_1 & p_4Y & p_5Y & -p_2 \\ p_1X+p_4Y^2 & p_2X+p_3Y & -Z & p_5Y^2 & -p_2Y & -p_1 \\ p_1Y & p_2Y & -p_5Y & p_5X-Z & p_3 & p_4 \\ p_2Y^2 & -p_5Y^2 & -p_4Y & p_4X+p_1Y & p_5X-Z & p_3 \\ -p_5Y^3 & -p_4Y^2 & -p_3Y & p_3X+p_2Y^2 & p_4X+p_1Y & p_5X-Z \end{bmatrix},$$

as desired.

(b) It follows by Theorem 4.2.

To prove (c), we can apply Proposition 6.3 to obtain

$$D(f) = \mathbf{V}\left(\frac{(h-g_{2\bullet}h)(h-g_{3\bullet}h)(h-g_{4\bullet}h)(h-g_{5\bullet}h)(h-g_{6\bullet}h)}{3(x-y)(x-\zeta y)(x-\zeta^2 y)}\right).$$

Note that

$$h - g_{2\bullet}h = (y-x)(-p_1+p_3+p_2(x+y)+p_4(x+y)-p_5(x^2+xy+y^2)),$$
$$h - g_{3\bullet}h = p_1(x-\zeta^2 x)+p_2(x^2-\zeta x^2)+p_3(y-\zeta y)+p_4(y^2-\zeta^2 y^2),$$
$$h - g_{4\bullet}h = p_1(x-\zeta x)+p_2(x^2-\zeta^2 x^2)+p_3(y-\zeta^2 y)+p_4(y^2-\zeta y^2),$$
$$h - g_{5\bullet}h = p_1(x-\zeta y)+p_2(x^2-\zeta^2 y^2)+p_3(y-\zeta^2 x)+p_4(y^2-\zeta x^2)+p_5(y^3-x^3),$$
$$h - g_{6\bullet}h = p_1(x-\zeta^2 y)+p_2(x^2-\zeta y^2)+p_3(y-\zeta x)+p_4(y^2-\zeta^2 x^2)+p_5(y^3-x^3).$$

Therefore, a calculation shows that $D(f) = \mathbf{V}(f_1 f_2 f_3)$.

The proof of (d) follows by Theorem 5.4 and Remark 5.5. ∎

Remark 6.8. A straightforward (but tedious) calculation can be done to present the $Q_i's$ coefficients in Proposition 6.7(b) explicitly. For instance,

$$q_5 = h_1+h_2+h_3+h_4+h_5+h_6 = 3p_5(x^3+y^3),$$
$$q_4 = -6xyp_1p_3 - 3p_1p_2(x^3+y^3) - 3p_3p_4(x^3+y^3) - 6x^2y^2p_2p_4$$
$$+ 3p_5^2(x^3+y^3)^2 + 3p_5^2 x^3 y^3.$$

Therefore,

$$Q_5(X,Y) = 3Xp_5$$

and

$$Q_4(X,Y) = -6Yp_1p_3 - 3Xp_1p_2 - 3Xp_3p_4 - 6Y^2p_2p_4 + 3X^2p_5^2 + 3Y^3p_5^2.$$

In the same way, we obtain from q_3, q_2, q_1 and q_0 that

$$Q_3(X,Y) = -X(-p_1^3 - p_3^3 - Xp_2^3 - Xp_4^3 + 9Xp_1p_2p_5 + 3Xp_3p_4p_5 - X^2p_5^3)$$
$$- XY(-3p_2^2 p_3 - 3p_1 p_4^2 + 12p_1p_3p_5 + 12Y^2p_2p_4p_5 - 6Y^2p_5^3)$$
$$- Y^2(-6p_2p_3^2 - 6p_1^2 p_4 + 2Yp_2^3 + 2Yp_4^3 - 12Yp_1p_2p_5 + 12Yp_3p_4p_5),$$

$$Q_2(X, Y) = 9XYp_1^2 p_2 p_3 + 9Y^2 p_1^2 p_3^2 + 9X^2 p_1 p_2 p_3 p_4 + 9XYp_1 p_3^2 p_4 + 3X^2 p_1^3 p_5$$
$$+ 9Y^3 p_1^2 p_2^2 + 9XY^2 p_1 p_2^2 p_4 + 9XY^2 p_2 p_3 p_4^2 + 9Y^3 p_3^2 p_4^2 - 6Y^3 p_1^3 p_5$$
$$+ 3X^3 p_2^2 p_5 + 9X^2 Y p_2^2 p_3 p_5 + 9XY^2 p_2 p_3^2 p_5 + 6Y^3 p_3^3 p_5 + 9XY^2 p_1^2 p_4 p_5$$
$$- 9X^3 p_1 p_2 p_5^2 - 9X^2 Y p_1 p_3 p_5^2 + 9Y^4 p_2^2 p_4^2 - 9XY^3 p_2^3 p_5 - 18Y^4 p_2^2 p_3 p_5$$
$$+ 18Y^4 p_1 p_4^2 p_5 + 3XY^3 p_4^3 p_5 + 18XY^3 p_1 p_2 p_5^2 - 9X^2 Y^2 p_2 p_4 p_5^2$$
$$- 18XY^3 p_3 p_4 p_5^2 + 3X^2 Y^3 p_5^4 + 3Y^6 p_5^4,$$

$$Q_1(X, Y) = -\big[3XYp_1^4 p_3 + 3X^2 p_1 p_2 p_3^3 + 3XYp_1 p_3^4 + 3X^2 p_1^3 p_3 p_4 + 6Y^3 p_1^4 p_2$$
$$+ 3X^2 Y p_1 p_2^3 p_3 + 18XY^2 p_1 p_2^2 p_3^2 + 12Y^3 p_1 p_2 p_3^3 + 12XY^2 p_1^3 p_2 p_4$$
$$+ 12Y^3 p_1^3 p_3 p_4 + 3X^3 p_2^3 p_3 p_4 + 9X^2 Y p_2^2 p_3^2 p_4 + 12XY^2 p_2 p_3^3 p_4$$
$$+ 6Y^3 p_3^4 p_4 + 9X^2 Y p_1^2 p_2 p_4^2 + 18XY^2 p_1^2 p_3 p_4^2 + 3X^3 p_1 p_2 p_4^3$$
$$+ 3X^2 Y p_1 p_3 p_4^3 - 9X^2 Y p_1^2 p_2 p_3 p_5 - 9XY^2 p_1^2 p_3^2 p_5 - 9X^3 p_1 p_2 p_3 p_4 p_5$$
$$- 9X^2 Y p_1 p_3^2 p_4 p_5 - 3X^3 p_1^3 p_5^2 + 3XY^3 p_1 p_2^4 + 12Y^4 p_1 p_2^3 p_3$$
$$+ 3X^2 Y^2 p_2^4 p_4 + 12Y^4 p_1 p_3 p_4^3 + 3X^2 Y^2 p_2 p_4^4 + 3XY^3 p_3 p_4^4$$
$$- 9XY^3 p_1^2 p_2^2 p_5 - 9X^2 Y^2 p_1 p_2^2 p_4 p_5 - 9X^2 Y^2 p_2 p_3 p_4^2 p_5 - 9XY^3 p_3^2 p_4^2 p_5$$
$$+ 9XY^3 p_1^3 p_5^2 - 3X^4 p_2^3 p_5^2 - 9X^3 Y p_2^2 p_3 p_5^2 - 9X^2 Y^2 p_2 p_3^2 p_5^2$$
$$- 3XY^3 p_3^3 p_5^2 - 9X^2 Y^2 p_1^2 p_4 p_5^2 + 3X^4 p_1 p_2 p_5^3 + 3X^3 Y p_1 p_3 p_5^3$$
$$- 6Y^5 p_2^4 p_4 - 6Y^5 p_2 p_4^4 - 9XY^4 p_2^2 p_4^2 p_5 + 12X^2 Y^3 p_2^3 p_5^2$$
$$+ 27XY^4 p_2^2 p_3 p_5^2 - 9XY^4 p_1 p_4^2 p_5^2 - 3X^2 Y^3 p_1 p_2 p_5^3 + 3X^3 Y^2 p_2 p_4 p_5^3$$
$$+ 3X^2 Y^3 p_3 p_4 p_5^3 - 6Y^6 p_2^3 p_5^2 - 6Y^6 p_4^3 p_5^2 - 12Y^6 p_1 p_2 p_5^3$$
$$+ 12Y^6 p_3 p_4 p_5^3 - 3XY^6 p_5^5 + 18Y^5 p_2 p_3^2 p_5^2 + 18Y^5 p_1^2 p_4 p_5^2 \big],$$

and

$$Q_0(X, Y) = X^2 p_1^3 p_3^3 + Y^3 p_1^6 + 3XY^2 p_1^3 p_2 p_3^2 - 2Y^3 p_1^3 p_3^3 + X^3 p_2^3 p_3^3 + 3X^2 Y p_2^2 p_3^4$$
$$+ 3XY^2 p_2 p_3^5 + Y^3 p_3^6 + 3XY^2 p_1^5 p_4 + 3XY^2 p_1^2 p_3^3 p_4 + 3X^2 Y p_1^4 p_4^2$$
$$+ X^3 p_1^3 p_4^3 - 3X^2 Y p_1^4 p_3 p_5 - 3X^3 p_1^3 p_3 p_4 p_5 + XY^3 p_1^3 p_2^3 + 6Y^4 p_1^3 p_2^2 p_3$$
$$+ 3X^2 Y^2 p_2^4 p_3^2 + 6XY^3 p_2^3 p_3^3 + 3Y^4 p_2^2 p_3^4 + 3X^2 Y^2 p_1^2 p_2^3 p_4$$
$$+ 9XY^3 p_1^2 p_2^2 p_3 p_4 + 18Y^4 p_1^2 p_2 p_3^2 p_4 + 3Y^4 p_1^4 p_4^2 + 3X^3 Y p_1 p_2^3 p_4^2$$
$$+ 9X^2 Y^2 p_1 p_2^2 p_3 p_4^2 + 9XY^3 p_1 p_2 p_3^2 p_4^2 + 6Y^4 p_1 p_3^3 p_4^2 + 6XY^3 p_1^3 p_4^3$$
$$+ X^4 p_2^3 p_4^3 + 3X^3 Y p_2^2 p_3 p_4^3 + 3X^2 Y^2 p_2 p_3^2 p_4^3 + XY^3 p_3^3 p_4^3$$
$$+ 3X^2 Y^2 p_1^2 p_4^4 - 3XY^3 p_1^4 p_2 p_5 + 6Y^4 p_1^4 p_3 p_5 - 3X^3 Y p_1 p_2^3 p_3 p_5$$
$$- 9X^2 Y^2 p_1 p_2^2 p_3^2 p_5 - 12XY^3 p_1 p_2 p_3^3 p_5 - 6Y^4 p_1 p_3^4 p_5 - 3X^2 Y^2 p_1^3 p_2 p_4 p_5$$
$$- 3X^4 p_2^3 p_3 p_4 p_5 - 9X^3 Y p_2^2 p_3^2 p_4 p_5 - 9X^2 Y^2 p_2 p_3^3 p_4 p_5 - 3XY^3 p_3^4 p_4 p_5$$
$$- 9X^2 Y^2 p_1^2 p_3 p_4^2 p_5 + X^4 p_1^3 p_5^3 + 3XY^4 p_2^5 p_3 + 3Y^5 p_2^4 p_3^2$$
$$- 6Y^5 p_1^2 p_2^3 p_4 - 9XY^4 p_1 p_2^3 p_4^2 - 18Y^5 p_1 p_2^2 p_3 p_4^2 - 4X^2 Y^3 p_2^3 p_4^3$$

$$- 9XY^4 p_2^2 p_3 p_4^3 - 6Y^5 p_2 p_3^2 p_4^3 + 3Y^5 p_1^2 p_4^4 + 3XY^4 p_1 p_4^5$$

$$- 3X^2Y^3 p_1 p_2^4 p_5 - 12Y^5 p_1^3 p_2 p_4 p_5 - 3X^3Y^2 p_2^4 p_4 p_5 + 3X^2Y^3 p_2^3 p_3 p_4 p_5$$

$$+ 18XY^4 p_2^2 p_3^2 p_4 p_5 + 12Y^5 p_2 p_3^3 p_4 p_5 - 18XY^4 p_1^2 p_2 p_4^2 p_5$$

$$- 3X^2Y^3 p_1 p_2 p_4^3 p_5 - 12XY^4 p_1 p_3 p_4^3 p_5 + 9XY^4 p_1^2 p_2 p_3 p_5^2 + 9Y^5 p_1^2 p_3^2 p_5^2$$

$$+ 9X^2Y^3 p_1 p_2 p_3 p_4 p_5^2 + 9XY^4 p_1 p_3^2 p_4 p_5^2 - 4X^2Y^3 p_1^3 p_5^2 + X^5 p_2^3 p_5^2$$

$$+ 3X^4Y p_2^2 p_3 p_5^3 + 3X^3Y^2 p_2 p_3^2 p_5^3 + X^2Y^3 p_3^3 p_5^3 + 3X^3Y^2 p_1^2 p_4 p_5^3$$

$$+ Y^6 p_2^6 + 2Y^6 p_2^3 p_4^3 + Y^6 p_4^6 + 6Y^6 p_1 p_2^4 p_5 + 9XY^5 p_2^4 p_4 p_5$$

$$+ 12Y^6 p_2^3 p_3 p_4 p_5 - 12Y^6 p_1 p_2 p_4^3 p_5 - 3XY^5 p_2 p_4^4 p_5 - 6Y^6 p_3 p_4^4 p_5$$

$$+ 9Y^6 p_1^2 p_2^2 p_5^2 + 9XY^5 p_1 p_2^2 p_4 p_5^2 + 9XY^5 p_2 p_3 p_4^2 p_5^2 + 9Y^6 p_3^2 p_4^2 p_5^2$$

$$+ 2Y^6 p_1^3 p_5^3 - 5X^3Y^3 p_2^3 p_5^3 - 12X^2Y^4 p_2^2 p_3 p_5^3 - 9XY^5 p_2 p_3^2 p_5^3$$

$$- 2Y^6 p_3^3 p_5^3 - 9XY^5 p_1^2 p_4 p_5^3 + 3X^2Y^4 p_1 p_4^2 p_5^3 - 3X^3Y^3 p_1 p_2 p_5^4$$

$$- 3X^2Y^4 p_1 p_3 p_5^4 + 9Y^7 p_2^2 p_4^2 p_5^2 + 5XY^6 p_2^3 p_5^3 + 6Y^7 p_2^2 p_3 p_5^3$$

$$- 6Y^7 p_1 p_4^2 p_5^3 + XY^6 p_4^3 p_5^3 + 9XY^6 p_1 p_2 p_5^4 + 6Y^7 p_1 p_3 p_5^4$$

$$- 3X^2Y^5 p_2 p_4 p_5^4 - 3XY^6 p_3 p_4 p_5^4 + 6Y^8 p_2 p_4 p_5^4 + Y^9 p_5^6.$$

6.3. Quasihomogeneous reflection maps

Another application of our results is about quasihomogeneous map germs. A polynomial $p(x_1, \ldots, x_n)$ is *quasihomogeneous* if there are positive coprime integers $b_1, \ldots, b_n$, and an integer d such that

$$p(k^{b_1} x_1, \ldots, k^{b_n} x_x) = k^d p(x_1, \ldots, x_n).$$

The number b_i is called the weight of the variable x_i and d is called the weighted degree of p. In this case, we say p is of type $(d; b_1, \ldots, b_n)$. This definition extends to polynomial map germs $f: (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ by just requiring each coordinate function f_i to be quasihomogeneous of type $(d_i; b_1, \ldots, b_n)$, for fixed weights $b_1, \ldots, b_n$. In particular, for a quasihomogeneous map germ $f: (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, we say that it is quasihomogeneous of type $(d_1, d_2, d_3; b_1, b_2)$.

Marar and Nuño-Ballesteros studied in [11] the case where $f: (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is a corank 2 quasihomogeneous and finitely determined map germ. They say that the mere existence of this kind of maps is quite a surprise. Indeed, the three adjectives create tremendous restrictions and examples seem hard to find. They showed that if $f = (x^2, y^2, h(x, y))$ is finitely determined, then f is in fact homogeneous, i.e., $b_1 = b_2 = 1$ (see Theorem 3.4 in [11]). In particular, there is no finitely determined quasihomogeneous double fold map germ with distinct weights. On the other hand, examples of finitely determined homogeneous (where $b_1 = b_2 = 1$) reflected graph map germs exist, see for instance Example 3.6 in [11] and Example 16 in [21]. Thus, we can consider the following question.

Question 4. *Is there any corank 2 reflected graph map germ $f = (w_1, w_2, h)$ from $(\mathbb{C}^2, 0)$ to $(\mathbb{C}^3, 0)$ such that f is finitely determined and quasihomogeneous with distinct weights?*

Since the coordinate functions w_1 and w_2 to the orbit map $w = (w_1, w_2)$ of G are always homogeneous (see Chapter 9 of [10]), we will restrict ourselves to studying this question only for the group $\mathbb{Z}_r \times \mathbb{Z}_s$, where the orbit map $w = (x^r, x^s)$ of $\mathbb{Z}_r \times \mathbb{Z}_s$ is a quasihomogeneous map. If $f(x, y) = (x^r, y^s, h(x, y))$, then the corank 2 hypothesis implies that $r, s \geq 2$ and $h \in \mathbf{m}^2$, where $\mathbf{m}$ denotes the maximal ideal of $\mathcal{O}_2$. The following lemma can be viewed as an extension of Theorem 3.4 in [11] for the reflection group $\mathbb{Z}_r \times \mathbb{Z}_s$. We will consider $\mathbb{Z}_r \times \mathbb{Z}_s$ as a subgroup of $\mathrm{GL}_2(\mathbb{C})$ generated by the reflections

$$R' = \begin{bmatrix} \theta & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad S' = \begin{bmatrix} 1 & 0 \\ 0 & \xi \end{bmatrix},$$

where θ (respectively, ξ) is a primitive $\mathbf{r}$-th (respectively, $\mathbf{s}$-th) root of unity.

Denote the elements of $\mathbb{Z}_r \times \mathbb{Z}_s$ by $g_{i,j}$, where $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, s\}$, with $g_{r,s} = \mathrm{Id}$, the identity matrix. Note that after an eventual reordering of the indices i and j, we have that

$$g_{i,j} {}_\bullet h(x, y) = h(\theta^i x, \xi^j y).$$

Lemma 6.9. *Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a reflected graph map germ given by $f(x, y) = (x^r, y^s, h(x, y))$, with $r, s \geq 2$ and $h \in \mathbf{m}^2$. If f is quasihomogeneous and finitely determined, then f is homogeneous.*

Proof. The case where $r, s = 2$ was considered in Theorem 3.4 of [11]. Therefore, we can suppose that $(r, s) \neq (2, 2)$. Denote the weight of x by a and the weight of y by b. By hypothesis, we have that f is quasihomogeneous, therefore we can write h in the form

$$(6.4) \quad h(x, y) = x^\alpha y^\beta (c_k (x^b)^k + c_{k-1}(x^b)^{k-1} y^a + \cdots + c_1 x^b (y^a)^{k-1} + c_0 (y^a)^k)$$

for some non-negative integers α, β and k, where $c_0, \ldots, c_k \in \mathbb{C}$ and $c_0, c_k \neq 0$. We will show that $\alpha, \beta = 0$. Since $\det(\mathrm{jac}(w)) = rs x^{r-1} y^{s-1}$, then by Proposition 6.3, we obtain that

$$(6.5) \quad D(f) = \mathbf{V}\left(\frac{\prod_{(i,j) \neq (r,s)} (h - g_{i,j} {}_\bullet h)}{x^{r-1} y^{s-1}} \right).$$

Suppose that $\beta \neq 0$. Then the restriction of f to $\mathbf{V}(y)$ is r-to-1 (see Lemma 6.1 in [23]). Since f is finitely determined, it follows that $1 \leq r \leq 2$. If we suppose that $\alpha \neq 0$, we obtain with a similar argument that $1 \leq s \leq 2$. Hence, if $r, s \geq 3$, then $\alpha = \beta = 0$. Let us consider the following remaining cases:

Case a.1: $r = 2$ and $s \geq 3$.

Since $s \geq 3$, we have that $\alpha = 0$. Suppose that $\beta \geq 1$. Then by (6.4) and (6.5), we obtain that

$$D(f) = \mathbf{V}(y^{\beta(2s-1)-(s-1)} \lambda_1(x, y))$$

for some $\lambda_1(x, y)$ in $\mathcal{O}_2$. Now, note that

$$\beta(2s - 1) - (s - 1) = \beta s + \beta(s - 1) - (s - 1) \geq 3.$$

Therefore, $D(f)$ is not reduced, thus it follows by Corollary 3.5 in [12] that f is not finitely determined, a contradiction. Hence, we obtain that $\beta = 0$.

Case a.2: $s = 2$ and $r \geq 3$.

The proof of this case is similar to the one given to show Case a.1.

Finally, we will show that the weights of f are equal, i.e., $a = b = 1$. Since $\alpha = \beta = 0$, then by (6.4) and (6.5), we obtain that

$$D(f) = \mathbf{V}(x^{b(r-1)-(r-1)} y^{a(s-1)-(s-1)} \lambda_2(x, y))$$

for some $\lambda_2(x, y)$ in $\mathcal{O}_2$. Therefore, since f is finitely determined, it follows that

(6.6) $$0 \leq (r - 1)(b - 1) \leq 1 \quad \text{and} \quad 0 \leq (s - 1)(a - 1) \leq 1.$$

From (6.6), we obtain that if $r \geq 3$, then $b = 1$. On the other hand, if $s \geq 3$, then $a = 1$. Hence, if $r, s \geq 3$, then $a = b = 1$. Let us consider the following remaining cases.

Case b.1: $r = 2$ and $s \geq 3$.

Note that in this case,

$$f = (f_1, f_2, f_3) = (x^2, y^s, c_k(x^b)^k + \cdots + c_0(y^a)^k),$$

with $c_0, c_k \neq 0$. From (6.6), we obtain that $a = 1$ and $1 \leq b \leq 2$. Suppose that $b = 2$. Then the weighted degree of f_1, f_2 and f_3 are $2, 2s$ and $2k$. By Proposition 1.4 in [16], we obtain that the number $C(f)$ of cross-caps of f is given by

(6.7) $$C(f) = \frac{1}{2} ((2s - 1)(2k - 2) + (2s - 1)),$$

which is not an integer number, since the numerator of (6.7) is an odd integer number. In particular, this implies that f is not finitely determined, a contradiction. Therefore, we obtain that $a = b = 1$.

Case b.2: $s = 2$ and $r \geq 3$.

The proof of this case is similar to the one given to show Case b.1. ∎

We note that for the group $\mathbb{Z}_1 \times \mathbb{Z}_d$, where $\mathbb{Z}_1$ denotes the trivial group, it is not hard to find quasihomogeneous finitely determined map germs which are not homogeneous. For instance,

$$f(x, y) = (x, y^4, y^6 + xy)$$

is an example of a corank 1 finitely determined map germ which is quasihomogeneous of type $(5, 4, 6; 1, 5)$.

Remark 6.10. All figures used in this work were created by the authors using the software Surfer [27].

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Milena Barbosa Gama
Department of Mathematics, Universidade Federal da Paraíba
Cidade Universitária, Campus I, PB 58051-900 João Pessoa, Brazil;
milena.gama@academico.ufpb.br, gamamilena32@gmail.com

Otoniel Nogueira da Silva
Department of Mathematics, Universidade Federal da Paraíba
Cidade Universitária, Campus I, PB 58051-900 João Pessoa, Brazil;
otoniel.silva@academico.ufpb.br, otoniel@ufu.br