Thematic Working Group on the Teaching and Learning of Calculus, TWG25

ERME column regularly presented by Frode Rønning and Andreas Stylianides

In this issue presented by the group leader Laura Branchetti

CERME Thematic Working Groups

We continue the initiative of introducing the CERME Thematic Working Groups, which we began in the September 2017 issue, focusing on ways in which European research in the field of mathematics education may be of interest or relevance to people working in pure and applied mathematics. We aim to disseminate developments in mathematics education research discussed at CERMEs and enrich the ERME community with new participants, who may benefit from hearing about research methods and findings and contribute to future CERMEs.

Introduction of CERME's Thematic Working Group 25 – Teaching and Learning of Calculus

This paper reports on the current trends in research related to the teaching and learning of calculus within Thematic Working Group 25 (TWG25) at CERME14, as well as the development of research in this field, particularly in the context of interactions between mathematics and other disciplines. We explore the challenges related to conceptual understanding, formalization, institutional constraints, and interdisciplinary applications. We highlight innovative task designs, technological tools, and implications for teaching practices in diverse educational contexts.

1 Teaching and learning of calculus across grades and disciplines

The Thematic Working Group 25 (TWG25) is a new group proposed to create a new community that investigates the teaching and learning of calculus at secondary and tertiary levels and at the transition between secondary school and university. Calculus plays a key role in mathematics education, both for its interesting features *per se*—difficulties in learning calculus and infinitesimal processes have been investigated since the 1970s—and for its transversal role in society. Due to its powerful concepts, tools and theories developed to deal with graphs, variations, trends, and accumulation processes, calculus is relevant for many other disciplines and professional fields (the so-called *non-mathematicians*). It is taught in most countries at secondary school, college and university first-year level. The undeniable value and flexibility of calculus in modeling processes and in the conceptualization and formalization of multiple real-world problems do not translate into recognition of its value by many teachers and students. Addressing the issue of teaching calculus to most students in several different contexts is challenging. Indeed, its intrinsic complexity—semiotic,[1] conceptual, and argumentative aspects—combined with lecturers' habit to formalize from the very beginning its notions, usually leads students to experience several difficulties [3, 18]. The difference between the conceptual and semiotic aspects of key notions like continuous function, derivative, integral, limit, and the formal and theoretical evolution of their definitions and properties is often stressed using two different terms—calculus and analysis—to refer to different mathematical practices and the related didactic issues, as highlighted by Bergé [2]. Separating calculus and analysis sharply and definitively is not easy and can be misleading, but a distinction is worthwhile. Transitioning from calculus to analysis, the students are asked to move from *doing* to *justifying* and *structuring.* Bergé [2] provided interesting examples of differences that appear in syllabi of courses addressed to undergraduate students. Calculus courses emphasize operational, representational and algorithmic tasks; analysis courses focus on theoretical, formal, axiomatic, and proof-oriented tasks. For instance, in calculus courses, continuity, derivative and integral are introduced via geometric intuition, computing with examples, graphs, and applications. Students work with tasks like "find where f is increasing," "compute $\int f(x)\,dx$ from a to b." In analysis, continuity and differentiability are explored in much greater generality: rigorous (ε, δ)-definitions, proofs of theorems, and often the role of completeness (or compactness) is used.

[1] *Semiotics*—the study of signs and meaning-making—has become a major focus in mathematics education over the past three decades. Research has shown that mathematical thinking and learning are mediated through symbols, gestures, diagrams, and language. Influential works highlighted the central role of semiotic processes in constructing and communicating mathematical meaning [14].

2 The main motivations to propose a new group on calculus

There are at least two motivations for making the effort to create a new community for the teaching and learning of calculus. First of all, it is necessary to provide new research—empirical, practice-based, and practice-oriented, and theoretical—to grasp the main variables affecting the conceptualization process, in particular the phase of representation and interpretation. This phase is necessary to ground the formal knowledge on strong and appropriate personal meanings. Unfortunately, as stressed in the pioneering work developed by David Tall and colleagues since the 1980s (see, for instance, [15–17]), often undergraduate students fail in constructing fruitful connection between their mental images, emerged after their learning experiences at the secondary level, and the formal definitions that are proposed suddenly at the undergraduate level. Tall and Vinner [17] introduced the terms *concept image* and *concept definition* to refer to these two aspects of the conceptualization, showing that many students struggle with the cognitive conflicts between them. Moreover, such conflicts often hinder their learning processes and cause ruptures between the manipulation of symbols and reasoning. Such undesired effects are partially due to promoting the learning of analysis without a meaning-making process and conceptual regulation processes behind, as documented by several researchers (e.g., [9]; for a systematic review, [5]).

The second motivation is that pre-calculus reasoning and notions and other calculus-related advanced mathematical practices (multivariable calculus, modeling processes based on rate of change and accumulation) have specificities that usually are lost while moving to the formal world of analysis and cannot be disregarded as informal knowledge and processed to be refined or formalized as soon as possible pursuing mainly the formal rigor [5, 13].

Several contributions to calculus had been proposed in other groups (in particular, TWG14 on University Mathematics Education), but some specificities were not addressed in a systematic and pertinent way. The main motivation for proposing a thematic group like TWG25 is precisely to clarify the specificities of the challenges related to teaching of calculus and to explore ways to support meaning-making processes. Taking this topic outside the university mathematics group opens a space to discuss it alongside high school calculus, thus bringing a richer perspective on pre-calculus and on the transition from secondary to tertiary.

The group inherits the work developed by researchers who worked on this topic for several years, including the calculus and analysis working groups attending the INDRUM conference [12, 18] and the calculus conferences (CalcConf1 in 2019 in Norway, CalcConf2 in 2023 in Norway, and CalcConf3 in 2025 in Italy) and new fresh perspectives by researchers of several countries with different theoretical perspectives and goals.

3 State of the art about the teaching and learning of calculus

Recent literature stressed that these issues, that mainly regard the epistemic and cognitive level, are intertwined with institutional, linguistic and affective variables that make the challenge even harder to address. The main results are summarized in a chapter of the first handbook about the INDRUM community (International Network on Didactics of University Mathematics), resuming the current challenging issues university mathematics education is facing [10]—in particular, [18]—and in the systematic review [4]. At the institutional level, two important phenomena have been stressed by researchers that have a strong impact on the teaching process and on the analysis of learning outcomes.

Artigue [1] emphasized that, among the various theoretical perspectives on investigating students' difficulties in mathematics during the transition from secondary school to university—as well as on innovations in university-level mathematics teaching—the institutional dimension has been less frequently addressed in the past. However, over the last two decades, there has been growing attention to the analysis of this aspect. In particular, the research community engaged in the research program developed by Chevallard since the late 1990s—adopting an anthropological perspective on the investigation of mathematical practices within institutional contexts (the anthropological theory of didactic, or ATD [6])—has made significant contributions to the characterization of mathematical practices at the secondary and tertiary levels across various fields (mathematics, engineering, physics, economics, etc.). A significant contribution is highlighting the mismatches between teaching and learning activities carried out in different contexts that can be a major source of difficulties for students and teachers. Among many contributions in this direction, the work by Gueudet et al. [11] is particularly important, since the authors highlighted relevant features of such a mismatch at the transition towards university and the related implications for research. A different use of ATD is the analysis of praxeologies across different tertiary institutions, in particular stressing the difference between the praxeologies that characterize the world of pure mathematics and the ones developed by other professional or scientific communities.

Biza et al. [3] presented a relevant issue that relates to the intention itself of teaching calculus to *non-mathematicians* in courses of the first year and on the assessment practices. Calculus is one of the main causes of dropout. Unfortunately, calculus is often intended, and used, as a filtering tool to test if students are good enough to attend the university courses [4]. Regardless of the features of mathematical practices related to a specific field, the students' interests and needs, calculus is taught in a "standard" way, without any attempt to fit the students' needs, or to take care of those linguistic or affective needs of the students. The timely problem of inclusion and diversity in calculus teaching and learning, which deserves particular attention in courses for *non-mathematicians* [8], is still underexplored.

4 Overview of TWG25: research foci and theoretical perspectives

In this contribution, we summarize how the papers presented in TWG25 contributed to addressing these challenges, combining different theoretical lenses and addressing several of the aspects highlighted in the literature, and the new issues that emerged during the discussion groups that opened up new avenues for research. The group brought together a broad range of research contributions focused on the learning and teaching of calculus, bridging foundational secondary-level concepts and advanced topics in tertiary education. The authors addressed the development of students' understanding of key calculus ideas and examined instructional practices, theoretical frameworks, and methodologies that inform this field, building on research traditions spanning decades while responding to current curricular, technological, and interdisciplinary demands. Several contributors emphasized the importance of early conceptual readiness, focusing on variables, covariation, functional thinking, and graphical reasoning. These ideas form the cognitive substrate for understanding the rate of change and accumulation—central themes in calculus. Some papers deepened core topics such as the derivative as rate of change, the integral as accumulation, differentials, and the continuity and integrability of functions.

Among the diverse contributions to TWG25, we can identify three macro categories: students' conceptualization of properties of real numbers relevant to calculus, single variable calculus, and multivariable calculus.

A first area of inquiry concerns students' conceptualization of real numbers, particularly their understanding of the convergence of infinite series and the role of partial sums. Persistent challenges emerge here, including tensions between formal definitions and intuitive conceptions. Another crucial but often overlooked theme is the density of the rational numbers within the reals, which plays an essential role in grasping continuity. Classroom activities such as those involving continued fractions, together with analyses of student responses, highlight the variety and dynamism of students' reasoning processes.

A second major theme focuses on the notions of the derivative and integrals, explored through graphical and dynamic representations, technology-enhanced environments, and modeling tasks. Findings emphasize how different forms of representation shape students' evolving understanding and engagement with concepts of variation and rate of change. Research has also pointed to the importance of connecting procedural fluency with a deeper conceptual awareness, especially regarding continuity and accumulation processes.

Another line of investigation examines the design of qualitative and graphical tasks. These activities have shown potential to foster flexible mathematical reasoning and to move learners beyond narrow algorithmic strategies. In this regard, extra-mathematical contexts appear particularly valuable: connecting calculus concepts to real-world situations allows students to anchor abstract notions in meaningful experiences, thereby strengthening their understanding of rates of change.

Teacher knowledge and classroom practices have also received attention. Studies reveal how pedagogical choices and epistemological perspectives strongly influence the way the derivative is taught, and how inconsistencies in the use of notation can hinder students' conceptual clarity. Some approaches propose rethinking traditional topics such as inverse proportion or substitution in integration, underlining the role of foundational reasoning structures in supporting students' learning trajectories.

Finally, research on multivariable calculus has highlighted the difficulties students face when dealing with double and triple integrals, multivariable limits, and functions of several variables. Digital tools and dynamic visualizations, such as interactive applets, have been shown to provide important scaffolds, supporting spatial reasoning and bridging the gap between symbolic manipulation and geometric intuition.

The contributions showcased a wide range of theoretical perspectives, including APOS theory, commognition, ATD, concept image and concept definition, and semiotic approaches. While this diversity enriched the dialogue, it also raised questions about coherence and integration. Participants debated whether future work should aim for theoretical unification or respect and encourage pluralism. Methodologies varied from thematic and content analysis to grounded theory and discourse analysis [7]. Participants emphasized the importance of aligning research methods with theoretical assumptions. Analysis of students' "scratch work" was highlighted as a valuable yet underutilized method for gaining insight into student thinking. Several recurring cross-cutting themes emerged, with particular attention to:

1. the tension between conceptual understanding and formal definitions;
2. the role of visual and embodied reasoning in fostering insight;
3. the influence of institutional and cultural contexts on curriculum;
4. the need to bridge the research-practice divide;
5. the foundational role of pre-calculus knowledge.

During the discussion session, we explored several important research questions and considerations that warrant further investigation. Examples include: How should various institutional conditions (e.g., cultural factors, prior knowledge, and experiences of both STEM and non-STEM students) be considered in the teaching and learning of calculus and multivariable calculus? How can we enhance accessibility to calculus education while considering subject-specific norms? Do task design principles in calculus differ from those in other areas of mathematics? If so, what are the key design principles unique to calculus? For instance, what types of contextual tasks are most effective for teaching and learning calculus? Even within calculus, do these principles vary across topics—such as derivatives, limits, and integrals—or do they remain consistent?

5 The learning and teaching of calculus across disciplines

More recently, an explicitly interdisciplinary focus has gained attention due to the increasing awareness that the meanings and uses of calculus are strongly shaped by the disciplinary contexts in which it is applied. In this perspective, two conferences have been organized that focus on "The Learning and Teaching of Calculus Across Disciplines" (CalcConf2 in Bergen in 2023 and CalcConf3 in Milan in 2025).

Several contributions in the proceedings show how the teaching and learning of calculus cannot be separated from the practices of the sciences that rely on it. Several contributions addressed covariational reasoning as a cornerstone of students' understanding of the derivative and accumulation. Others explored how basic mental models of the integral may support comprehension in thermodynamics and chemistry, or how conservation laws in physics can be interpreted through the fundamental theorem of calculus. A number of papers investigated the disciplinary specificity of calculus, for instance in economics, biology, chemistry and engineering, pointing out how meanings of differentials, partial derivatives or multivariable limits vary according to context.

The contributions also highlighted the increasing role of technology and innovation. Studies reported the use of eye-tracking to analyze reasoning with simulations, the integration of computational thinking in calculus teaching, or the affordances of digital platforms and AI tools, such as MAPLE Learn and large language models (LLMs), to create authentic problem situations. More creative approaches were presented as well, including the design of a calculus card game and the use of history-based artifacts as mediators of conceptual understanding.

Altogether, these contributions suggest that research on the teaching and learning of calculus is today both diversified and coherent: diversified, because of the variety of contexts, approaches and methods represented; coherent, because of the shared recognition that calculus learning should be meaningful, conceptually grounded, and responsive to the epistemological needs of the sciences.

From this perspective, the results and innovation proposed in CalcConf2 and CalcConf3 complement earlier discussions promoted in other international contexts, and reinforce the idea that future work should increasingly move toward concrete didactic proposals and collaborative interdisciplinary designs. This appears to be a promising way forward to ensure that calculus teaching supports students not only in mastering procedures, but also in developing the conceptual and modeling resources needed for their scientific and professional trajectories. Future research should also consider the future potential of AI and LLMs in analyzing student responses and designing adaptive tasks. Cross-institutional collaboration and attention to diverse educational contexts will be key to addressing the evolving challenges in calculus education.

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