J. Fractal Geom. (Online first)
DOI 10.4171/JFG/172

The distribution of the largest digit for parabolic Iterated Function Systems of the interval

Hiroki Takahasi

Abstract. We investigate the distribution of the largest digit for a wide class of infinite parabolic Iterated Function Systems (IFSs) of the unit interval. Due to the recurrence to parabolic (neutral) fixed points, the dimension analysis of these systems becomes more delicate than that of uniformly contracting IFSs. We show that the Hausdorff dimensions of level sets associated with the largest digits are constantly equal to the Hausdorff dimension of the limit set of the IFS. This result is an analogue of Wu and Xu's theorem [Math. Proc. Cambridge Philos. Soc. 146 (2009), 207–212] on the regular continued fraction. Examples of application of our result include the backward (aka minus, or negative) continued fractions, even-integer continued fractions, and go beyond. Our main tool is a dimension theory for non-uniformly expanding Bernoulli interval maps with infinitely many branches.

1. Introduction

Any irrational number x in $(0,1)$ has a unique infinite expansion of the form

$$x = \frac{1|}{|a_1(x)} + \frac{1|}{|a_2(x)} + \frac{1|}{|a_3(x)} + \cdots,$$

called *the regular continued fraction*, where each digit $a_n(x)$ belongs to the set $\mathbb{N}$ of positive integers. For a typical irrational $x \in (0,1)$ in the sense of the Lebesgue measure, the sequence $\{a_n(x)\}_{n\in\mathbb{N}}$ is unbounded and consists of mostly small digits, punctuated by occasional larger ones, see Hensley [10]. It is clear that the occurrence of large digits influences statistical properties of digit sequences, see, e.g., [5] for details. Therefore, it is important to investigate the growth of the largest digit $[L_n(x) = \max\{a_1(x),\dots,a_n(x)\}$ as $n \to \infty$. A pioneering result in this direction is due to Galambos [7] who proved that

$$\lim_{n\to\infty} \mu_{\mathrm{G}}\left\{x \in (0,1) \setminus \mathbb{Q} : \frac{L_n(x)}{n} < \frac{y}{\log 2}\right\} = e^{-\frac{1}{y}} \quad \text{for } y > 0,$$

Mathematics Subject Classification 2020: 11A55 (primary); 11K50, 37D25, 37E05 (secondary).

Keywords: Iterated Function System (IFS), continued fraction, dimension theory.

where μ_{G} denotes the Gauss measure $\frac{1}{\log 2}\frac{dx}{1+x}$. Later in [8] he proved that

$$\lim_{n\to\infty}\frac{\log L_n(x)}{\log n}=1 \text{ for Lebesgue a.e. } x\in(0,1)\setminus\mathbb{Q}.$$

Philipp [19] considered the number

$$\ell(x)=\liminf_{n\to\infty}\frac{L_n(x)\log\log n}{n},$$

and proved that $\ell(x)=1/\log 2$ for Lebesgue a.e. $x\in(0,1)\setminus\mathbb{Q}$, solving Erdős' conjecture apart from the value. Okano [18] proved that for any $\alpha\in(0,\infty)$ there exists $x\in(0,1)\setminus\mathbb{Q}$ such that $\ell(x)=\alpha$. Wu and Xu [25, Theorem 1.1] significantly strengthened Okano's result, proving that for any $\alpha\in[0,\infty)$ the set

$$\left\{x\in(0,1)\setminus\mathbb{Q}\colon \lim_{n\to\infty}\frac{L_n(x)\log\log n}{n}=\alpha\right\}$$

is of Hausdorff dimension 1. Chang and Chen [4] proved results analogous to that of Wu and Xu replacing $L_n(x)$ or the norming function $n/\log\log n$.

All these interesting developments have taken place exclusively for the regular continued fraction. The aim of this paper is to investigate growth rates of the largest digits for a wide class of infinite parabolic Iterated Function Systems. As a corollary we obtain an analogue of Wu and Xu's theorem for other continued fractions with totally different Lebesgue typical behaviors.

Let X be a compact interval with positive Euclidean diameter. Let I be a subset of $\mathbb{N}$ with $\#I\geq 2$, and let $\phi_i\colon X\to X$ $(i\in I)$ be C^1 maps. The collection $\Phi=\{\phi_i\}_{i\in I}$ is called *an Iterated Function System (IFS)* on X. It is called *an infinite (resp. finite)* IFS if I is an infinite (resp. finite) set. We say an IFS Φ satisfies *the open set condition* if for all distinct indices $i,j\in I$,

$$\phi_i(\mathrm{int}X)\cap\phi_j(\mathrm{int}X)=\emptyset.$$

Let $\Phi=\{\phi_i\}_{i\in I}$ be an IFS on X. For $\omega=(\omega_1,\omega_2,\ldots)\in I^{\mathbb{N}}$ and $n\in\mathbb{N}$, we set

$$\phi_{\omega_1\cdots\omega_n}=\phi_{\omega_1}\circ\cdots\circ\phi_{\omega_n}.$$

If the set $\bigcap_{n=1}^{\infty}\phi_{\omega_1\cdots\omega_n}(X)$ is a singleton for any $\omega\in I^{\mathbb{N}}$, we define *an address map* $\Pi\colon I^{\mathbb{N}}\to X$ by

$$\Pi(\omega)\in\bigcap_{n=1}^{\infty}\phi_{\omega_1\cdots\omega_n}(X),$$

and *the limit set* of Φ by

$$\Lambda(\Phi)=\Pi(I^{\mathbb{N}}).$$

Since the address map may not be injective, we introduce the set

$$\Lambda_*(\Phi) = \left\{x \in \Lambda(\Phi) \colon \#\Pi^{-1}(x) = 1\right\}.$$

Since X is an interval, if the open set condition holds then $\Lambda(\Phi) \setminus \Lambda_*(\Phi)$ is countable and of Hausdorff dimension zero. For each $x \in \Lambda_*(\Phi)$, there is a unique sequence $\{a_n(x)\}_{n=1}^{\infty} \in I^{\mathbb{N}}$ that satisfies $x = \Pi(\{a_n(x)\}_{n=1}^{\infty})$. Note that

$$x = \lim_{n\to\infty} \phi_{a_1(x)} \circ \cdots \circ \phi_{a_n(x)}(y) \ \text{ for all } y \in X.$$

If Φ is an infinite IFS, then for each $\alpha \in [0, \infty]$, define a level set

$$L(\alpha) = \left\{x \in \Lambda_*(\Phi) \colon \lim_{n\to\infty} \frac{\max\{a_1(x), \dots, a_n(x)\} \log\log n}{n} = \alpha\right\}.$$

Moreover, if there exist constants $c > 0$, $d > 1$ such that

$$\min_{x\in X} |\phi_i'(x)| \geq \frac{c}{i^d} \ \text{ for every } i \in I,$$

then we say Φ is *d-decaying*.

An IFS Φ on X is called *parabolic* if the open set condition holds, and the following two conditions hold:

(A1) (Non-uniform contraction) $|\phi_i'(x)| < 1$ everywhere except for finitely many pairs (i, x_i), $i \in I$, for which x_i is the unique fixed point of ϕ_i and $|\phi_i'(x_i)| = 1$. Such pairs and indices i are called *parabolic*.

(A2) (Bounded distortion) There exists a constant $C \geq 1$ such that for all $\omega \in I^{\mathbb{N}}$ and $n \in \mathbb{N}_{\geq 2}$ such that ω_n is not a parabolic index, or else $\omega_{n-1} \neq \omega_n$,

$$|\phi_{\omega_1\cdots\omega_n}'(x)| \leq C|\phi_{\omega_1\cdots\omega_n}'(y)| \ \text{ for all } x, y \in X.$$

For results on a dimension theory of finite or infinite parabolic IFSs, see [9, 11, 14–17, 24] for example. Here, our interest is in infinite parabolic IFSs. A prime example is given by the backward (aka minus, or negative) continued fraction, see Section 4 for more details. Our definition of parabolic IFS is simpler than that in [16, Section 8] since we only work on IFSs on a compact interval.

Let $\dim_{\mathrm{H}}$ denote the Hausdorff dimension on $[0, 1]$. Our main result is stated as follows.

Theorem 1.1. *Let $d > 1$, and let $\Phi = \{\phi_i\}_{i\in I}$ be a d-decaying parabolic IFS on $[0, 1]$ that satisfies (B1) (B2). For any $\alpha \in [0, \infty]$ we have*

$$\dim_{\mathrm{H}} L(\alpha) = \dim_{\mathrm{H}} \Lambda(\Phi).$$

Conditions (B1), (B2) are given in Section 2.1. For any infinite parabolic IFS Φ on [0, 1], the set $\bigcap_{n=1}^{\infty} \phi_{\omega_1\cdots\omega_n}([0,1])$ is a singleton for any $\omega \in I^{\mathbb{N}}$ (see Lemma 2.2). Hence, the limit set $\Lambda(\Phi)$ is defined. A key observation is that the Hausdorff dimension of the set

$$\{x \in \Lambda_*(\Phi)\colon a_n(x) \text{ is not parabolic for every } n \in \mathbb{N}\}$$

becomes strictly smaller than that of $\Lambda(\Phi)$ when Φ has a parabolic index. This means that we cannot exclude parabolic indices from consideration to establish the equality in Theorem 1.1.

To prove Theorem 1.1, it suffices to show the inequality $\dim_{\mathrm{H}} L(\alpha) \geq \dim_{\mathrm{H}} \Lambda(\Phi)$. To this end, we first extract from Φ a family of finite IFSs whose limit sets have Hausdorff dimension arbitrarily close to $\dim_{\mathrm{H}} \Lambda(\Phi)$. We then construct a subset of $L(\alpha)$ by inserting large digits into each limit set without substantially losing Hausdorff dimension. This two-step construction has been inspired by the above-mentioned work of Wu and Xu [25]. However, if Φ has parabolic indices then the construction of a family of finite IFSs and the selection of positions of digits to be inserted have to be done carefully, in order to avoid the influence of parabolic indices on distortion estimates. We assume (B1) (B2) to facilitate these issues. These two conditions are translations from the paper [13] by Jaerisch and the author. For more details, see Section 2.1.

The rest of this paper is organized as follows. In Section 2 we provide preliminary materials, including the definitions of (B1) (B2) and the construction of a family of finite IFSs. In Section 3 we complete the proof of Theorem 1.1. In Section 4 we give a verifiable sufficient condition for (B1) (B2), and provide examples of parabolic IFSs satisfying them.

2. Preliminaries

This section provides preliminary materials. In Section 2.1 we introduce conditions (B1) (B2). In Section 2.2 we translate a few results in [13] into the language of IFS. In Section 2.3 we construct a family of finite IFSs with large limit sets, from a given parabolic IFS satisfying (B1) (B2).

2.1. Saturation and mild distortion

Let $\Phi = \{\phi_i\}_{i\in I}$ be a parabolic IFS. For each $n \in \mathbb{N}$ let I^n denote the set of words from I with word length n. Let $\mathcal{M}$ denote the set of shift invariant ergodic Borel probability measures on the Cartesian product topological space $I^{\mathbb{N}}$. For each $\nu \in \mathcal{M}$, define *the Lyapunov exponent* of ν by

$$\chi(\nu) = -\int \log|\phi'_{\omega_1}(\Pi(\omega))| d\nu(\omega) \in [0,\infty],$$

and set

$$\mathcal{M}(\Phi) = \{\nu \in \mathcal{M} \colon \chi(\nu) < \infty\}.$$

For a Borel probability measure μ on $[0,1]$, define

$$\dim(\mu) = \inf\{\dim_{\mathrm{H}} A \colon A \subset [0,1],\ \mu(A) = 1\}.$$

(B1) (Saturation) $\dim_{\mathrm{H}} \Lambda(\Phi) = \sup\{\dim(\nu \circ \Pi^{-1}) \colon \nu \in \mathcal{M}(\Phi)\}$.

For each $n \in \mathbb{N}$ we define

$$D_n(\Phi) = \sup_{\omega_1\cdots\omega_n \in I^n} \max_{x,y\in[0,1]} \log \frac{\phi'_{\omega_1\cdots\omega_n}(x)}{\phi'_{\omega_1\cdots\omega_n}(y)}.$$

(B2) (Mild distortion) $D_1(\Phi) < \infty$ and $D_n(\Phi) = o(n)$.

The open set condition allows us to convert an IFS Φ on $[0,1]$ to iterations of a Bernoulli map on $[0,1]$, see Section 2.2 for the definition. Each neutral fixed point of this map corresponds to a parabolic index of Φ. This correspondence allows us to import results from paper [13] on a dimension theory of non-uniformly expanding one-dimensional Markov maps with infinitely many branches. Conditions (B1) (B2) are translations from [13] into the language of IFS. A verifiable sufficient condition for (B1) (B2) is given in Proposition 4.1.

Remark 2.1. The regular continued fraction is generated by the 2-decaying Iterated Function System $\Phi = \{\phi_i\}_{i\in\mathbb{N}}$ on $[0,1]$ given by $\phi_i(x) = 1/(x+i)$, which is a parabolic IFS without parabolic indices with

$$\Lambda(\Phi) = \Lambda_*(\Phi) = (0,1) \setminus \mathbb{Q} : x = \lim_{n\to\infty} \phi_{a_1(x)} \cdots \phi_{a_n(x)}(0)$$

for all $x \in \Lambda(\Phi)$. Condition (B1) holds because $\dim(\mu_{\mathrm{G}}) = 1$. Using the uniform contraction $\sup_{\omega\in\mathbb{N}^2} \max_{x\in[0,1]} |\phi'_\omega(x)| < 1$ and so-called Rényi's condition, one can verify $\sup_{n\in\mathbb{N}} D_n(\Phi) < \infty$ which is stronger than (B2). As a result, one can recover [25, Theorem 1.1] by applying Theorem 1.1 to this IFS Φ.

2.2. Entropy, dimension, decay of fundamental intervals

We say $f\colon \Delta \to [0,1]$ is *a Markov map* if the following three conditions hold:

(M0) there exist a subset I of $\mathbb{N}$ and a family $\{\Delta_i\}_{i\in I}$ of pairwise disjoint non-empty open intervals in $(0,1)$ such that $\Delta = \bigcup_{i\in I} \Delta_i$;

(M1) for each $i \in I$, the restriction $f|_{\Delta_i}$ extends to a C^1 diffeomorphism from the closure of Δ_i onto its image;

(M2) $f(\Delta_i) \cap \Delta_j \neq \emptyset$ for $i, j \in I$ implies $\Delta_j \subset f(\Delta_j)$.

We say $f\colon \Delta \to [0,1]$ is *a Bernoulli map* if (M0) (M1), and the following holds:

(M3) for every $i \in I$, $f(\Delta_i) = (0,1)$.

Given a parabolic IFS $\Phi = \{\phi_i\}_{i\in I}$ on $[0,1]$, we can associate *a Bernoulli map* $f\colon \bigcup_{i\in I} \phi_i((0,1)) \to (0,1)$ by

$$f|_{\phi_i((0,1))} = \phi_i^{-1}|_{\phi_i((0,1))}.$$

The parabolic indices of Φ correspond to neutral fixed points of f, $\Lambda(\Phi)$ contains $\bigcap_{n=0}^{\infty} f^{-n}(\Delta)$, and $\Lambda(\Phi) \setminus \bigcap_{n=0}^{\infty} f^{-n}(\Delta)$ is a countable set. For each $\nu \in \mathcal{M}$, the measure $\nu \circ \Pi^{-1}$ is f-invariant. Let $h(\nu)$ denote the measure-theoretic entropy of $\nu \circ \Pi^{-1}$ with respect to f. We say ν is *ergodic* if $\nu \circ \Pi^{-1}$ is ergodic with respect to f, and is *expanding* if $\chi(\nu) > 0$. If $\nu \in \mathcal{M}(\Phi)$ is ergodic and expanding, then

$$\dim(\nu \circ \Pi^{-1}) = \frac{h(\nu)}{\chi(\nu)}, \tag{2.1}$$

see, e.g., [13] and [16, Section 4.4] for details.

Let $\Phi = \{\phi_i\}_{i\in I}$ be a parabolic IFS on $[0,1]$. For each $n \in \mathbb{N}$ and each element $\omega = (\omega_1, \dots, \omega_n) \in I^n$, we call the closed interval

$$J(\omega) = J(\omega_1, \dots, \omega_n) = \phi_{\omega_1\cdots\omega_n}([0,1])$$

a fundamental interval of order n. For convenience, let us call $[0,1]$ a fundamental interval of order 0. Fundamental intervals of the same order are either disjoint, coincide or intersect only at their boundary points.

We say Φ has *decay of fundamental intervals* if $\bigcap_{n=1}^{\infty} \phi_{\omega_1\cdots\omega_n}([0,1])$ is a singleton for any $\omega \in I^{\mathbb{N}}$. We say Φ has *uniform decay of fundamental intervals* if

$$\lim_{n\to\infty} \sup_{\omega\in I^n} |J(\omega)| = 0.$$

Lemma 2.2. *Every parabolic IFS Φ has decay of fundamental intervals. If moreover $D_1(\Phi) < \infty$, then Φ has uniform decay of fundamental intervals.*

Proof. This follows from [13, Proposition 3.1] applied to the Bernoulli map associated with the IFS Φ in the lemma. For each $n \in \mathbb{N}$, fundamental intervals of order n are the closures of the n-cylinders in the language of [13]. ∎

Lemma 2.3. *Let $\Phi = \{\phi_i\}_{i\in I}$ be a parabolic IFS satisfying (B1) (B2). Let $\nu \in \mathcal{M}(\Phi)$ be ergodic and expanding. For any $\varepsilon > 0$ there exists $p_0 \geq 2$ such that for every integer $p \geq p_0$ there exists a finite subset V_p of I^p such that*

$$\left|\frac{1}{p}\log \# V_p - h(\nu)\right| < \varepsilon \ \textit{and} \ \max_{\omega\in V_p}\max_{x\in[0,1]}\left|\frac{1}{p}\log|\phi'_\omega(x)|^{-1} - \chi(\nu)\right| < \varepsilon.$$

Proof. This follows from the proof of [13, Lemma 3.5] applied to the Bernoulli map associated with the IFS Φ in the lemma. ∎

2.3. Construction of finite IFSs with large limit sets

The next proposition provides a family of finite IFSs without parabolic indices whose limit sets approximate that of the original parabolic IFS in terms of Hausdorff dimension.

Proposition 2.4. *Let $\Phi = \{\phi_i\}_{i\in I}$ be a parabolic IFS on $[0,1]$ that satisfies (B1) (B2). For any $\varepsilon > 0$ there exist $p_0 \geq 2$ such that for every integer $p \geq p_0$ there exist a constant $\gamma > 0$, a non-empty finite subset W_p of I^p with the following properties:*

(a) *for all distinct words $\omega, \eta \in W_p$, $J(\omega) \cap J(\eta) = \emptyset$;*

(b) *for any $\omega = \omega_1 \cdots \omega_p \in W_p$, ω_p is not a parabolic index;*

(c) *for any $\omega \in W_p$, $\max_{x\in[0,1]} |\phi'_\omega(x)| < e^{-\gamma p}$;*

(d) *for the finite IFS $\Phi_p = \{\phi_\omega\}_{\omega\in W_p}$ on $[0,1]$ without parabolic indices,*

$$\dim_{\mathrm{H}} \Lambda(\Phi_p) > \dim_{\mathrm{H}} \Lambda(\Phi) - \varepsilon.$$

Proof. Fix an index $a \in I$ that is not parabolic. By (B1) and (2.1), for any $\varepsilon > 0$ there exists an ergodic expanding measure $\nu \in \mathcal{M}(\Phi)$ such that

$$\frac{h(\nu)}{\chi(\nu)} > \dim_{\mathrm{H}} \Lambda(\Phi) - \frac{\varepsilon}{2}. \tag{2.2}$$

Let $\delta \in (0, h(\nu))$. By Lemma 2.3, for all sufficiently large integer $p \geq 2$ there exists a non-empty finite subset V_{p-1} of I^{p-1} such that

$$\left|\frac{1}{p-1}\log \# V_{p-1} - h(\nu)\right| < \frac{\delta}{2}, \tag{2.3}$$

and for every $\omega \in V_{p-1}$,

$$\max_{x\in[0,1]}\left|\frac{1}{p-1}\log|\phi'_\omega(x)|^{-1} - \chi(\nu)\right| < \frac{\delta}{2}. \tag{2.4}$$

The fundamental intervals in the collection $\{J(\omega_1,\ldots,\omega_{p-1},a) \colon \omega_1\cdots\omega_{p-1} \in V_{p-1}\}$ may not be pairwise disjoint, intersecting each other at their endpoints. We remove from this collection every other interval with respect to the natural order in $[0,1]$. What is left is the collection of pairwise disjoint fundamental intervals of order p containing at least $\lfloor(\#V_{p-1})/2\rfloor$ elements. Define W_p to be the collection of words in I^p that correspond to these remaining intervals. Clearly (a) (b) hold. For all sufficiently large p, the estimates in (2.3) and (2.4) remain intact. Indeed, we have

$$\left|\frac{1}{p}\log\#W_p - h(\nu)\right| \leq \frac{1}{p(p-1)}\log\#W_p + \frac{1}{p-1}\left|\log\#W_{p-1} - \log\#V_{p-1}\right| + \left|\frac{1}{p-1}\log\#V_{p-1} - h(\nu)\right| < \delta, \tag{2.5}$$

and for every $\omega \in W_p$,

$$\max_{x\in[0,1]}\left|\frac{1}{p}\log|\phi_\omega'(x)|^{-1} - \chi(\nu)\right| \leq \frac{1}{p(p-1)}\max_{x\in[0,1]}\log|\phi_\omega'(x)|^{-1} + \max_{x\in[0,1]}\left|\frac{1}{p-1}\log|\phi_\omega'(x)|^{-1} - \chi(\nu)\right| < \delta. \tag{2.6}$$

Item (c) follows from (2.6) by setting $\gamma = \chi(\nu) - \delta > 0$.

For convenience, we set $I^0 = \{0\}$, and set ϕ_0 to be the identity map on $[0,1]$. Let $\nu_{\max}$ denote the measure of maximal entropy for the restriction of the p-iteration of the associated Bernoulli map to $\Lambda(\Phi_p)$. The measure $\tilde{\nu} = (1/p)\sum_{i=0}^{p-1}\sum_{\omega\in I^i}\nu_{\max}\circ\phi_\omega$ belongs to $\mathcal{M}(\Phi)$, ergodic, expanding and so $\dim(\tilde{\nu}) = h(\tilde{\nu})/\chi(\tilde{\nu})$ by (2.1). Moreover, we have

$$h(\tilde{\nu}) = \frac{1}{p}\log\#W_p \quad\text{and}\quad \chi(\tilde{\nu}) \leq \frac{1}{p}\max_{\omega\in W_p}\max_{x\in[0,1]}|\phi_\omega'(x)|^{-1}. \tag{2.7}$$

Combining (2.5), (2.6) and (2.7) yields

$$\dim(\tilde{\nu}) = \frac{h(\tilde{\nu})}{\chi(\tilde{\nu})} > \frac{h(\nu)-\delta}{\chi(\nu)+\delta}.$$

Since the set $\bigcup_{i=0}^{p-1}\bigcup_{\omega\in I^i}\phi_\omega(\Lambda(\Phi_p))$ has full $\tilde{\nu}$-measure, we have

$$\dim_{\mathrm{H}}\left(\bigcup_{i=0}^{p-1}\bigcup_{\omega\in I^i}\phi_\omega(\Lambda(\Phi_p))\right) \geq \dim(\tilde{\nu}).$$

Combining these two inequalities yields

$$\dim_{\mathrm{H}}\Lambda(\Phi_p) = \dim_{\mathrm{H}}\left(\bigcup_{i=0}^{p-1}\bigcup_{\omega\in I^i}\phi_\omega(\Lambda(\Phi_p))\right) > \frac{h(\nu)-\delta}{\chi(\nu)+\delta}. \tag{2.8}$$

Since $\delta > 0$ is arbitrary, combining (2.2), (2.8) and then reducing δ if necessary we obtain the desired inequality in (d). ■

3. Proof of the main result

In this section we complete the proof of Theorem 1.1.

3.1. Initial setup

Let $d > 1$, and let $\Phi = \{\phi_i\}_{i\in I}$ be a d-decaying parabolic IFS on $[0, 1]$ that satisfies (B1) (B2). To prove Theorem 1.1 it suffices to show $\dim_{\mathrm{H}} L(\alpha) \geq \dim_{\mathrm{H}} \Lambda(\Phi)$ for any $\alpha \in [0, \infty]$. Let $\varepsilon > 0$. Let $p \geq 2$ be a sufficiently large integer such that

$$(k + p + 1)^d + 2p + 1 < (k + p + 2)^d \quad \text{for every } k \in \mathbb{N} \cup \{0\}. \tag{3.1}$$

Let W_p be a non-empty finite subset of I^p, and let $\gamma > 0$ be a constant for which the conclusions of Proposition 2.4 hold. Let $\Phi_p = \{\phi_\omega\}_{\omega\in W_p}$ be a finite IFS on $[0, 1]$ given by Proposition 2.4. Clearly the limit set of Φ_p is contained in $L(0)$. Proposition 2.4(d) gives

$$\dim_{\mathrm{H}} L(0) \geq \dim_{\mathrm{H}} \Lambda(\Phi_p) > \dim_{\mathrm{H}} \Lambda(\Phi) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we obtain $\dim_{\mathrm{H}} L(0) \geq \dim_{\mathrm{H}} \Lambda(\Phi)$ as required. In what follows, we only consider the case $\alpha \in (0, \infty)$. The case $\alpha = \infty$ is covered by a slight modification of the following argument.

3.2. Construction of a subset of the level set

We construct a subset of the level set $L(\alpha)$ with large Hausdorff dimension by inserting digits into the limit set of Φ_p. In order to select positions of these digits, we define sequences $\{n(k)\}_{k=0}^{\infty}$, $\{m(k)\}_{k=0}^{\infty}$ of non-negative integers inductively as follows. Start with $n(0) = 0$. Given $n(k)$ for $k \geq 0$ such that $n(k) + p < (k + p + 1)^d$, define

$$n(k + 1) = n(k) + m(k)p + 1, \tag{3.2}$$

where $m(k) \geq 2$ is the positive integer satisfying

$$n(k) + \big(m(k) - 1\big)p \leq (k + p + 1)^d < n(k) + m(k)p.$$

By (3.1) and (3.2) we have

$$n(k + 1) + p = n(k) + m(k)p + p + 1 \leq (k + p + 1)^d + 2p + 1 < (k + p + 2)^d,$$

which proves the assumption of induction. The above definition implies

$$2 \leq n(k) - (k + p)^d \leq p \quad \text{for every } k \geq 1. \tag{3.3}$$

From (3.3) we obtain

$$\lim_{k\to\infty}\frac{k}{n(k)}=0. \tag{3.4}$$

Define a function $\tau\colon[2,\infty)\to(0,\infty)$ by $\tau(x)=x/\log\log x$. Notice that

$$\lim_{k\to\infty}\frac{\tau(k^d)}{\tau((k+1)^d)}=1.$$

We have $\lim_{x\to\infty}\tau'(x)=0$. Applying the mean value theorem to τ and then using (3.3), we get

$$\lim_{k\to\infty}\frac{\tau(n(k))-\tau(k^d)}{\tau((k+1)^d)}=0.$$

These two equalities shows $\lim_{k\to\infty}\tau(n(k))/\tau((k+1)^d)=1$, and

$$\lim_{k\to\infty}\frac{\tau(n(k))}{\tau(n(k+1))}=1. \tag{3.5}$$

Let $y\in\Lambda(\Phi_p)$. Then $\{a_n(y)\}_{n\in\mathbb{N}}$ is a concatenation of elements of W_p. We insert the digits $a_{n(k)}=\lfloor\alpha\tau(n(k))\rfloor$, $k\in\mathbb{N}$ into $\{a_n(y)\}_{n\in\mathbb{N}}$ to define a new sequence

$$\ldots,a_{(m(0)+\cdots+m(k-1))p-1}(y),a_{(m(0)+\cdots+m(k-1))p}(y),\boxed{a_{n(k)}},$$
$$a_{(m(0)+\cdots+m(k-1))p+1}(y),\ldots,a_{(m(0)+\cdots+m(k))p-1}(y),a_{(m(0)+\cdots+m(k))p}(y),$$
$$\boxed{a_{n(k+1)}},a_{(m(0)+\cdots+m(k))p+1}(y),\ldots$$

Let $x(y)$ denote the point in $(0,1)\setminus\mathbb{Q}$ whose regular continued fraction expansion is given by this new sequence. Let $L_p(\alpha)$ denote the collection of these points:

$$L_p(\alpha)=\big\{x(y)\in(0,1)\setminus\mathbb{Q}\colon y\in\Lambda(\Phi_p)\big\}.$$

Lemma 3.1. *We have* $L_p(\alpha)\subset L(\alpha)$.

Proof. Recall that $\Phi=\{\phi_i\}_{i\in I}$ is the d-decaying parabolic IFS on $[0,1]$. Let M denote the maximum of the finite subset of I that consists of integers appearing in some element of W_p. Fix a sufficiently large integer $k_0\geq 1$ such that $\alpha\tau(n(k_0))\geq M$. Let $x\in L_p(\alpha)$. For any $n\geq n(k_0)$ there exists $k\geq k_0$ such that

$$n(k)\leq n<n(k+1).$$

Then

$$\max\big\{a_1(x),\ldots,a_n(x)\big\}=\max\big\{M,a_{n(k)}(x)\big\}=\lfloor\alpha\tau(n(k))\rfloor.$$

Since τ is monotone increasing, we have

$$\frac{\alpha\tau(n(k))-1}{\tau(n(k+1))}\leq\frac{\lfloor\alpha\tau(n(k))\rfloor}{\tau(n(k+1))}\leq\frac{\max\{a_1(x),\ldots,a_n(x)\}}{\tau(n)}\leq\frac{\lfloor\alpha\tau(n(k))\rfloor}{\tau(n(k))}\leq\alpha.$$

As $n \to \infty$, we have $k \to \infty$, and from (3.5) we obtain

$$\lim_{n\to\infty} \frac{\max\{a_1(x), \ldots, a_n(x)\}}{\tau(n)} = \alpha,$$

namely $x \in L(\alpha)$ as required. ∎

3.3. Completing the proof of Theorem 1.1

The map $y \in \Lambda(\Phi_p) \mapsto x(y) \in L_p(\alpha)$ is bijective. Let $h_\varepsilon\colon L_p(\alpha) \to \Lambda(\Phi_p)$ denote the inverse of this map that eliminates all the inserted digits $a_{n(k)}$, $k \in \mathbb{N}$.

Proposition 3.2. *The map h_ε is Hölder continuous with exponent $1/(1+\varepsilon)$.*

We finish the proof of Theorem 1.1 subject to Proposition 3.2. The next lemma is standard in dimension theory.

Lemma 3.3 ([6, Proposition 3.3]). *Let $\Lambda \subset [0,1]$ and let $h\colon \Lambda \to [0,1]$ be Hölder continuous with exponent $\beta \in (0,1]$. Then*

$$\dim_{\mathrm{H}} \Lambda \geq \beta \cdot \dim_{\mathrm{H}} h(\Lambda).$$

Proposition 3.2 allows us to apply Lemma 3.3 to the map h_ε. Further, Lemma 3.1, Lemma 3.3 and Proposition 2.4(d) altogether yield

$$\dim_{\mathrm{H}} L(\alpha) \geq \dim_{\mathrm{H}} L_p(\alpha) \geq \frac{1}{1+\varepsilon} \dim_{\mathrm{H}} \Lambda(\Phi_p) \geq \frac{1}{1+\varepsilon}\big(\dim_{\mathrm{H}} \Lambda(\Phi) - \varepsilon\big).$$

Since $\varepsilon > 0$ is arbitrary, we obtain $\dim_{\mathrm{H}} L(\alpha) \geq \dim_{\mathrm{H}} \Lambda(\Phi)$ as required in Theorem 1.1.

3.4. The first technical lemma

For a proof of Proposition 3.2 we need two technical lemmas. The first one compares diameters of two fundamental intervals of different orders associated with $L_p(\alpha)$, one obtained from the other eliminating all the inserted digits. To give a precise statement we need some definitions.

For each $k \in \mathbb{N}$, let A_k denote the set of $a_1 \cdots a_{n(k+1)} \in I^{n(k+1)}$ for which the following two conditions hold for every $0 \leq j \leq k$:

- $a_{n(j)+1} \cdots a_{n(j+1)-1} \in I^{n(j+1)-n(j)-1}$ is a concatenation of elements of W_p;
- $a_{n(j+1)} = \lfloor \alpha\tau(n(j+1)) \rfloor$.

We set

$$B_k = \big\{n(k) + mp\colon m = 0, 1, \ldots, m(k)\big\}.$$

Notice that

$$L_p(\alpha) = \bigcap_{k=0}^{\infty} \bigcup_{a_1\cdots a_{n(k+1)}\in A_k} J(a_1,\ldots,a_{n(k+1)}).$$

For each $n \in \mathbb{N}$ and $a_1 \cdots a_n \in I^n$, let $\overline{a_1 \cdots a_n}$ denote the word from I obtained by eliminating from $a_1 \cdots a_n$ the digits $a_{n(1)}, a_{n(2)}, \ldots, a_{n(k)}$, $n(k) \le n < n(k+1)$. Set

$$\overline{J}(a_1,\ldots,a_n) = \phi_{\overline{a_1\cdots a_n}}([0,1]).$$

Let $|\cdot|$ denote the Euclidean diameter of a set in [0, 1].

Lemma 3.4. *There exists $k_1 \in \mathbb{N}$ with the property that, for any integer $k \ge k_1$, any $a_1 \cdots a_{n(k+1)} \in A_k$ and any $n \in B_k$ we have*

$$|J(a_1,\ldots,a_n)| \ge |\overline{J}(a_1,\ldots,a_n)|^{1+\varepsilon}.$$

Proof. Take $\delta > 0$ such that

$$\delta \log C \le \frac{\varepsilon\gamma(1-\delta)}{4}, \tag{3.6}$$

where $C \ge 1$ is the constant in (A2). Let $k \in \mathbb{N}$, $a_1 \cdots a_{n(k+1)} \in A_k$ and let $n \in B_k$. We have

$$n(k) \le n < n(k+1).$$

In view of (3.4), we assume k is sufficiently large so that

$$k+1 \le \delta n(k). \tag{3.7}$$

On the one hand, by the mean value theorem there exists $x_0 \in [0, 1]$ such that

$$|J(a_1,\ldots,a_n)| = |\phi'_{a_1\cdots a_n}(x_0)|.$$

For $x \in [0, 1]$ we set

$$R(x) = \begin{cases} |(\phi_{a_{n(k)+1}} \circ \cdots \circ \phi_{a_n})'(x_0)| & \text{if } n(k) < n, \\ 1 & \text{if } n(k) = n. \end{cases}$$

The chain rule gives

$$\begin{aligned} |J(a_1,\ldots,a_n)| = &\prod_{j=1}^{k} |(\phi_{a_{n(j-1)+1}} \circ \cdots \circ \phi_{a_{n(j)-1}})'(\phi_{a_{n(j)}\cdots a_n}(x_0))| \\ &\times \prod_{j=1}^{k} |\phi'_{a_{n(j)}}(\phi_{a_{n(j)+1}} \circ \cdots \circ \phi_{a_n}(x_0))| \times R(x_0). \end{aligned} \tag{3.8}$$

By the mean value theorem and the chain rule, there exists $y_0 \in [0,1]$ such that

$$|\overline{J}(a_1,\dots,a_n)| = \prod_{j=1}^{k} |(\phi_{a_{n(j-1)+1}} \circ \cdots \circ \phi_{a_{n(j)-1}})'(\phi_{\overline{a_{n(j)}\cdots a_n}}(y_0))| \times R(y_0). \quad (3.9)$$

From Proposition 2.4(b), $a_{n(j)-1}$ is not a parabolic index for every $1 \le j \le k$. By (A2) we have

$$\frac{|(\phi_{a_{n(j-1)+1}} \circ \cdots \circ \phi_{a_{n(j)-1}})'(\phi_{a_{n(j)}\cdots a_n}(x_0))|}{|(\phi_{a_{n(j-1)+1}} \circ \cdots \circ \phi_{a_{n(j)-1}})'(\phi_{\overline{a_{n(j)}\cdots a_n}}(y_0))|} \ge C^{-1}.$$

Since $k \ge k_1$ and the number of parabolic indices is finite by (A1), if k_1 is sufficiently large then $a_{n(k)}$ is not a parabolic index. Since $n \in B_k$, a_n is not a parabolic index and (A2) gives

$$\frac{R(x_0)}{R(y_0)} \ge C^{-1}.$$

Combining (3.8), (3.9) and then applying the above two distortion estimates, we get

$$\begin{aligned}\frac{|J(a_1,\dots,a_n)|}{|\overline{J}(a_1,\dots,a_n)|} &\ge C^{-k-1} \prod_{j=1}^{k} |\phi'_{a_{n(j)}}(\phi_{a_{n(j)+1}} \circ \cdots \circ \phi_{a_n}(x_0))| \\ &\ge C^{-k-1} \prod_{j=1}^{k} \min_{x\in[0,1]} |\phi'_{n(j)}(x)|. \end{aligned} \quad (3.10)$$

We estimate the two factors in the last expression separately. By $k+1 \le \delta n(k) \le \delta n$ from (3.7), and then by (3.6) we have

$$C^{-k-1} \ge C^{-\delta n} \ge e^{-\varepsilon\gamma(n-\delta n)/4}.$$

Since $n \ge n(k)$ and Φ is d-decaying, (3.3) implies $k \le 2n^{\frac{1}{d}}$, and

$$\prod_{j=1}^{k} \min_{x\in[0,1]} |\phi'_{n(j)}(x)| \ge \left(\frac{c}{n^d}\right)^{2n^{\frac{1}{d}}} \ge e^{-\varepsilon\gamma(n-\delta n)/4},$$

provided k is sufficiently large. Plugging these two estimates into (3.10) yields

$$\frac{|J(a_1,\dots,a_n)|}{|\overline{J}(a_1,\dots,a_n)|} \ge e^{-\varepsilon\gamma(n-\delta n)/2} \ge e^{-\varepsilon\gamma(n-k)/2}. \quad (3.11)$$

On the other hand, since $\overline{a_1\cdots a_n}$ contains a concatenation of $\lfloor (n-k)/p \rfloor$ elements of W_p counted with multiplicity, Proposition 2.4(c) yields

$$|\overline{J}(a_1,\dots,a_n)|^{\varepsilon} \le e^{-\varepsilon\gamma(n-k)/2}, \quad (3.12)$$

provided k is sufficiently large. From equations (3.11) and (3.12) we obtain the desired inequality. ■

3.5. The second technical lemma

Before proceeding further, we introduce a positive constant

$$K = \min_{\substack{\omega,\eta\in W_p \\ \omega\neq\eta}} \min\bigl\{|x-y|: x\in J(\omega),\ y\in J(\eta)\bigr\}. \tag{3.13}$$

Fundamental intervals are closed subintervals of [0, 1], and all fundamental intervals of order p corresponding to words in W_p are pairwise disjoint by Proposition 2.4(a). Hence $K > 0$ holds.

The second technical lemma for the proof of Proposition 3.2 gives a lower bound on the distance between two nearby points in $L_p(\alpha)$ in terms of the diameter of some fundamental interval. For a pair (x, y) of distinct points in $L_p(\alpha)$, let $s(x, y)$ denote the maximal integer $n \geq 0$ for which there exists a fundamental interval of order n that contains x and y. By Lemma 2.2, $s(x, y)$ is well defined.

Lemma 3.5. *For any pair (x, y) of distinct points in $L_p(\alpha)$ satisfying $s(x, y) \geq p$, there exist integers $k \geq 0$, $m \geq 0$ such that*

$$n(k) + mp \leq s(x, y),\ n(k) + mp \in B_k$$

and

$$|J(a_1(x), \dots, a_{n(k)+mp}(x))| \leq CK^{-1}|x - y|.$$

Proof. There exists $k \geq 0$ such that $n(k) \leq s(x, y) \leq n(k+1) - 1$. By the definition of $s(x, y)$ and $a_{n(k+1)}(x) = a_{n(k+1)}(y)$, the second inequality is actually strict

$$n(k) \leq s(x, y) < n(k+1) - 1.$$

From the definition (3.2), there exists $m \geq 0$ such that

$$n(k) + mp \leq s(x, y) < n(k) + (m+1)p \leq n(k+1) - 1. \tag{3.14}$$

Hence $n(k) + mp \in B_k$ holds. Since $s(x, y) \geq p$ and $n(0) = 0$, $k = 0$ implies $m \geq 1$.

The first and second inequalities in (3.14) together imply that

$$\begin{aligned} &a_i(x) = a_i(y) \ \text{ for } i = 1, \dots, n(k) + mp \ \text{ and} \\ &a_i(x) \neq a_i(y) \ \text{ for some } i \in \bigl\{n(k) + mp + 1, \dots, n(k) + (m+1)p\bigr\}. \end{aligned}$$

Set

$$\begin{aligned} a_i = a_i(x) \ \text{ for } i = 1, \dots, n(k) + mp,\ \phi = \phi_{a_1\cdots a_{n(k)+mp}} \ \text{ and} \\ \omega(z) = a_{n(k)+mp+1}(z) \cdots a_{n(k)+(m+1)p}(z) \in I^p \ \text{ for } z = x, y. \end{aligned}$$

We have $\phi^{-1}(x) \in J(\omega(x))$, $\phi^{-1}(y) \in J(\omega(y))$ and $\omega(x) \neq \omega(y)$. By the definition of $L_p(\alpha)$, both $\omega(x)$ and $\omega(y)$ belong to W_p. By (A2) and the definition of K in (3.13), we have

$$\frac{|J(a_1,\dots,a_{n(k)+mp})|}{|x-y|} \le C\frac{|\phi^{-1}(J(a_1,\dots,a_{n(k)+mp}))|}{|\phi^{-1}(x)-\phi^{-1}(y)|} \le CK^{-1},$$

as required. ∎

3.6. Proof of Proposition 3.2

We set

$$K_1 = \min_{\substack{x,y\in L_p(\alpha)\\ x\neq y}} \{|x-y| \colon s(x,y) \le n(k_1)-1\},$$

where k_1 is the positive integer in Lemma 3.4. Since $K > 0$ we have $K_1 > 0$. Let (x, y) be a pair of distinct points in $L_p(\alpha)$. If $s(x,y) \le n(k_1)-1$ then $|x-y| \ge K_1$, and so

$$|h_\varepsilon(x) - h_\varepsilon(y)| \le 1 \le K_1^{-1}|x-y|. \tag{3.15}$$

If $s(x,y) \ge n(k_1) \ge p$, then let $k \ge 0$, $m \ge 0$ be the integers for which the conclusion of Lemma 3.5 holds. Since $x, y \in J(a_1(x),\dots,a_{n(k)+mp}(x))$, $h_\varepsilon(x)$, $h_\varepsilon(y)$ are contained in the interval $\overline{J}(a_1(x),\dots,a_{n(k)+mp}(x))$, and thus

$$\begin{aligned}|h_\varepsilon(x)-h_\varepsilon(y)| &\le |\overline{J}(a_1(x),\dots,a_{n(k)+mp}(x))| \\ &\le |J(a_1(x),\dots,a_{n(k)+mp}(x))|^{\frac{1}{1+\varepsilon}} \le \left(CK^{-1}|x-y|\right)^{\frac{1}{1+\varepsilon}}. \end{aligned} \tag{3.16}$$

To deduce the second inequality, we have used Lemma 3.4. By (3.15) and (3.16), h_ε is Hölder continuous with exponent $1/(1+\varepsilon)$ as required. ∎

4. Verification of (B1) (B2) and examples

In this last section we give a verifiable sufficient condition for (B1) (B2), and provide examples of 2-decaying parabolic IFSs satisfying them.

Proposition 4.1. *Let $\Phi = \{\phi_i\}_{i\in I}$ be an infinite parabolic IFS such that ϕ_i is C^2 for each $i \in I$, and*

$$\sup_{i\in I}\max_{x\in[0,1]} |(\log|\phi_i'(x)|)'| < \infty. \tag{4.1}$$

Then (B1) (B2) hold.

Proof. Let $f\colon \bigcup_{i\in I}\Delta_i \to [0,1]$ be the Bernoulli map associated with the IFS Φ: $\Delta_i = \phi_i((0,1))$ and $f_i = f|_{\Delta_i}$ for each $i \in I$. We have $\phi_i \circ f_i(x) = x$ for all $x \in \Delta_i$. Differentiating this equality twice and rearranging the result gives

$$\frac{f_i''(x)}{(f_i'(x))^2} = -\frac{\phi_i''(f_i(x))}{\phi_i'(f_i(x))}.$$

Since $i \in I$ and $x \in \Delta_i$ are arbitrary, this equality and (4.1) together imply that f satisfies Rényi's condition

$$\sup_{i\in I}\sup_{x\in\Delta_i}\frac{|f_i''(x)|}{|f_i'(x)|^2} < \infty.$$

Condition (4.1) also implies $D_1(\Phi) < \infty$. By Lemma 2.2, Φ has uniform decay of fundamental intervals. It follows from [13, Lemma 5.1] that Φ satisfies (B2).

Since $\Lambda(\Phi) \setminus \bigcap_{n=0}^{\infty} f^{-n}(\Delta)$ is a countable set, to verify (B1) it suffices to check (i), (ii), (iii), (iv) in [13, Proposition 5.2] for f. We have already shown that f satisfies Rényi's condition. By (A1), (M3) in [13] holds and (i) in [13, Proposition 5.2] is vacuous. If we consider the first return map to the domain $\bigcup\{\Delta_i\colon i \in I, \text{not parabolic}\}$, then all (ii), (iii), (iv) in [13, Proposition 5.2] hold. ∎

Example 1 (Backward continued fraction). Any irrational number x in $(0,1)$ has a unique expansion of the form

$$x = 1 - \frac{1|}{|a_1(x)} - \frac{1|}{|a_2(x)} - \frac{1|}{|a_3(x)} - \cdots,$$

where $a_n(x) \in \mathbb{N}_{\geq 2}$ for every $n \in \mathbb{N}$, called *the backward continued fraction*, see [12, 20]. The typical behavior of digits in this expansion in the sense of the Lebesgue measure on $(0,1) \setminus \mathbb{Q}$ is totally different from that of the regular continued fraction. For example, each integer $k \in \mathbb{N}$ typically appears with positive definite asymptotic frequency $\frac{1}{\log 2}\log\frac{(k+1)^2}{k(k+2)}$ in the latter, while in the former, any digit other than 2 appears with asymptotic frequency zero. For more details on typical behaviors of digits in the backward continued fraction, see [1–3, 12, 23], for example.

The backward continued fraction is generated by the 2-decaying parabolic IFS $\Phi = \{\phi_i\}_{i\in\mathbb{N}_{\geq 2}}$ given by $\phi_i(x) = 1 - 1/(x + i - 1)$. The index 2 is the only parabolic index: $\phi_2(0) = 0$ and $\phi_2'(0) = 1$. We have $\Lambda(\Phi) = \{0\} \cup ((0,1) \setminus \mathbb{Q})$, and

$$x = \lim_{n\to\infty}\phi_{a_1(x)}\cdots\phi_{a_n(x)}(0)$$

for all $x \in \Lambda(\Phi)$. A direct calculation shows (A1). Condition (A2) follows from (4.1) and the standard bounded distortion lemma near neutral fixed points as in the lemma below. Conditions (B1) (B2) follow from Proposition 4.1.

Lemma 4.2 (Proof of [13, Lemma 5.3]). *Let $f\colon [0,1) \to \mathbb{R}$ be a C^2 map satisfying $f(0)=0$, $f'(0)=1$ and $f'(x)>1$ for all $x \in (0,1)$. There exists a constant $C>0$ such that for every $n \in \mathbb{N}$ and all $x, y \in J_{n-1}$,*

$$\log \frac{(f^n)'(x)}{(f^n)'(y)} \le C|f^n(x)-f^n(y)| \sum_{i=0}^{n-1} \frac{|J_i|}{|J_0|},$$

where $q_0=1$, $f(q_{i+1})=q_i$ and $J_i=[q_{i+1},q_i)$ for $i=0,\ldots,n-1$.

Example 2 (Even-integer continued fraction). Any irrational number x in $(0,1)$ has a unique continued fraction expansion of the form

$$x = \frac{1|}{|b_1} + \frac{\varepsilon_1|}{|b_2} + \frac{\varepsilon_2|}{|b_3} + \cdots,$$

where b_n is a positive even integer and $\varepsilon_n \in \{1,-1\}$ for all $n \in \mathbb{N}$, called *the even-integer continued fraction* [21, 22]. This expansion is generated by the 2-decaying parabolic IFS $\Psi = \{\psi_i\}_{i\in\mathbb{N}}$ given by $\psi_i(x) = 1/(i-x)$ for i even and

$$\psi_i(x) = 1/(x+i+1)$$

for i odd. The index 2 is the only parabolic index: $\psi_2(1)=1$ and $\psi_2'(1)=1$. We have $\Lambda(\Psi) = \{1\} \cup ((0,1)\setminus\mathbb{Q})$, and for all $x \in \Lambda(\Psi)$,

$$x = \lim_{n\to\infty} \psi_{a_1(x)} \cdots \psi_{a_n(x)}(0)$$

and

$$x = \frac{1|}{|b_1(x)} + \frac{\varepsilon_1(x)|}{|b_2(x)} + \frac{\varepsilon_2(x)|}{|b_3(x)} + \cdots,$$

where

$$\big(b_n(x), \varepsilon_n(x)\big) = \big(a_n(x), -1\big) \text{ if } a_n(x) \text{ is even}$$

and

$$\big(b_n(x), \varepsilon_n(x)\big) = \big(a_n(x)+1, 1\big) \text{ if } a_n(x) \text{ is odd.}$$

In this expansion, the typical behavior of digits in the sense of the Lebesgue measure on $(0,1)\setminus\mathbb{Q}$ is similar to that of the backward continued fraction. A direct calculation shows (A1). By the same reasoning as for the backward continued fraction, one can check (A2). Conditions (B1) (B2) follow from Proposition 4.1.

Example 3. From a given infinite parabolic IFS, one can define a new one by removing countably many non-parabolic indices. For example, for any proper infinite subset I of $\mathbb{N}_{\ge 2}$ (resp. of $\mathbb{N}$) containing 2, the IFS $\{\phi_i\}_{i\in I}$ from Example 1 (resp. the IFS $\{\psi_i\}_{i\in I}$ from Example 2) is a 2-decaying parabolic IFS. Conditions (B1) (B2) follow from Proposition 4.1.

Acknowledgments. I thank the referee for its careful reading of the manuscript and giving valuable comments and suggestions. I thank Johannes Jaerisch and Yuto Nakajima for fruitful discussions.

Funding. This research was partially supported by the JSPS KAKENHI 25K21999, Grant-in-Aid for Challenging Research (Exploratory).

References

[1] J. Aaronson, Random f-expansions. *Ann. Probab.* **14** (1986), no. 3, 1037–1057 Zbl 0658.60050 MR 0841603

[2] J. Aaronson and H. Nakada, Trimmed sums for non-negative, mixing stationary processes. *Stochastic Process. Appl.* **104** (2003), no. 2, 173–192 Zbl 1075.60511 MR 1961618

[3] C. Bonanno and T. I. Schindler, Almost sure limit theorems with applications to non-regular continued fraction algorithms. *Stochastic Process. Appl.* **183** (2025), article no. 104573 Zbl 1567.37009 MR 4857830

[4] J. Chang and H. Chen, Slow increasing functions and the largest partial quotients in continued fraction expansions. *Math. Proc. Cambridge Philos. Soc.* **164** (2018), no. 1, 1–14 Zbl 1378.11080 MR 3733236

[5] H. G. Diamond and J. D. Vaaler, Estimates for partial sums of continued fraction partial quotients. *Pacific J. Math.* **122** (1986), no. 1, 73–82 Zbl 0589.10056 MR 0825224

[6] K. Falconer, *Fractal geometry: Mathematical foundations and applications.* Third edition, John Wiley & Sons, Ltd., Chichester, 2014 Zbl 1285.28011 MR 3236784

[7] J. Galambos, The distribution of the largest coefficient in continued fraction expansions. *Quart. J. Math. Oxford Ser. (2)* **23** (1972), no. 2, 147–151 Zbl 0234.10041 MR 0299576

[8] J. Galambos, An iterated logarithm type theorem for the largest coefficient in continued fractions. *Acta Arith.* **25** (1973/74), 359–364 Zbl 0255.10036 MR 0344212

[9] K. Gelfert and M. Rams, The Lyapunov spectrum of some parabolic systems. *Ergodic Theory Dynam. Systems* **29** (2009), no. 3, 919–940 Zbl 1180.37051 MR 2505322

[10] D. Hensley, The statistics of the continued fraction digit sum. *Pacific J. Math.* **192** (2000), no. 1, 103–120 Zbl 1015.11038 MR 1741027

[11] G. Iommi, Multifractal analysis of the Lyapunov exponent for the backward continued fraction map. *Ergodic Theory Dynam. Systems* **30** (2010), no. 1, 211–232 Zbl 1184.37038 MR 2586352

[12] M. Iosifescu and C. Kraaikamp, *Metrical theory of continued fractions*. Mathematics and its Applications 547, Kluwer Academic Publishers, Dordrecht, 2002 Zbl 1122.11047 MR 1960327

[13] J. Jaerisch and H. Takahasi, Mixed multifractal spectra of Birkhoff averages for non-uniformly expanding one-dimensional Markov maps with countably many branches. *Adv. Math.* **385** (2021), article no. 107778 Zbl 1483.11162 MR 4256139

[14] A. Johansson, T. M. Jordan, A. Öberg, and M. Pollicott, Multifractal analysis of non-uniformly hyperbolic systems. *Israel J. Math.* **177** (2010), 125–144 Zbl 1214.37029 MR 2684415

[15] R. D. Mauldin and M. Urbański, Parabolic iterated function systems. *Ergodic Theory Dynam. Systems* **20** (2000), no. 5, 1423–1447 Zbl 0982.37045 MR 1786722

[16] R. D. Mauldin and M. Urbański, *Graph directed Markov systems: Geometry and dynamics of limit sets*. Cambridge Tracts in Math. 148, Cambridge University Press, Cambridge, 2003 Zbl 1033.37025 MR 2003772

[17] K. Nakaishi, Multifractal formalism for some parabolic maps. *Ergodic Theory Dynam. Systems* **20** (2000), no. 3, 843–857 Zbl 0956.37004 MR 1764931

[18] T. Okano, Explicit continued fractions with expected partial quotient growth. *Proc. Amer. Math. Soc.* **130** (2002), no. 6, 1603–1605 Zbl 1028.11003 MR 1887004

[19] W. Philipp, A conjecture of Erdös on continued fractions. *Acta Arith.* **28** (1975/76), no. 4, 379–386 Zbl 0332.10033 MR 0387226

[20] A. Rényi, On algorithms for the generation of real numbers. *Magyar Tud. Akad. Mat. Fiz. Oszt. Közl.* **7** (1957), 265–293 Zbl 0161.05102 MR 0097645

[21] F. Schweiger, Continued fractions with odd and even partial quotients. *Arbeit. Math. Inst. Salzburg* **1982** (1982), 59–70 Zbl 0506.10038

[22] F. Schweiger, On the approximation by continued fractions with odd and even partial quotients. *Arbeit. Math. Inst. Salzburg* **1984** (1984), 105–114

[23] H. Takahasi, Large deviation principle for the backward continued fraction expansion. *Stochastic Process. Appl.* **144** (2022), 153–172 Zbl 1486.11096 MR 4347489

[24] M. Urbański, Parabolic Cantor sets. *Fund. Math.* **151** (1996), no. 3, 241–277 Zbl 0895.58036 MR 1424576

[25] J. Wu and J. Xu, The distribution of the largest digit in continued fraction expansions. *Math. Proc. Cambridge Philos. Soc.* **146** (2009), no. 1, 207–212 Zbl 1162.11042 MR 2461878

Received 30 November 2024; revised 15 August 2025.

Hiroki Takahasi
Keio Institute of Pure and Applied Sciences (KiPAS), Department of Mathematics, Keio University, 3-14-1 Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan; hiroki@math.keio.ac.jp