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On the Krein–Milman theorem for the space of sofic representations

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Abstract. Denote by $\mathrm{Sof}(G)$ the space of sofic representations of a countable group G. This space is known by a result of the second author to have a convex-like structure. We show that, in this space, minimal faces are extreme points. We then construct uncountably many extreme points for $\mathrm{Sof}(\mathbb{F}_r)$ and show that there exists a decreasing chain of closed faces with empty intersection. Finally, we construct a strange-looking sofic representation in $\mathrm{Sof}(\mathbb{F}_r)$ that we believe to be outside of the closure of the convex hull of extreme points.

The starting point of our discussion is the paper [4]. Nate Brown considered the space of all morphisms from a fixed finite von Neumann algebra N to the ultrapower of the hyperfinite factor R, up to unitary equivalence, denoted by $\mathrm{Hom}(N, R^\omega)$. He then constructed a convex structure on this space and studied its extreme points. For a sofic group G, the space $\mathrm{Sof}(G, P^\omega)$ is constructed analogously. Similar properties hold for extreme points, as shown in [15, 16].

Over the years, these spaces have been studied by different authors, possibly using other frameworks. In [14], it is shown that $\mathrm{Hom}(N, R^\omega)$ consists of one point if and only if N is amenable. Elek and Szabó showed the same thing for $\mathrm{Sof}(G, P^\omega)$ in [11]. Capraro and Fritz [6] showed that these spaces can be embedded in an abstractly constructed Banach space, while Atkinson [3] studied finite-dimensional faces of $\mathrm{Hom}(N, R^\omega)$. In [16], the second author showed that there are groups $H \subset G$ such that the restriction $R : \mathrm{Sof}(G, P^\omega) \to \mathrm{Sof}(H, P^\omega)$ is not surjective. This is an obstruction to soficity that hints at the existence of non-sofic groups. For example, if we find two groups G_1, G_2, with a common subgroup H, such that $R_1 : \mathrm{Sof}(G_1, P^\omega) \to \mathrm{Sof}(H, P^\omega)$ and $R_2 : \mathrm{Sof}(G_2, P^\omega) \to \mathrm{Sof}(H, P^\omega)$ have disjoint images, then $G_1 *_H G_2$ is not sofic. Let us state the main question to be investigated in this work.

Question. Do $\mathrm{Hom}(N, R^\omega)$ and $\mathrm{Sof}(G, P^\omega)$ satisfy a Krein–Milman result?

Apart from the original work of Brown and Capraro [4, 5], this problem has been tackled before by Chirvasitu in [9]. The author hints at an argument in favour of a positive

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answer (see discussion before Proposition 2.10 of [9]). Let us recall the Krein–Milman theorem.

Theorem (Krein–Milman). *Let X be a locally convex topological vector space, and let K be a compact convex subset of X. Then K is the closed convex hull of its extreme points.*

$\mathrm{Hom}(N, R^{\omega})$ and $\mathrm{Sof}(G, P^{\omega})$ are subsets in a Banach space (that is always locally convex). However, in [4, Appendix A], Ozawa showed that they are never compact (unless N or G is amenable per Jung and Elek–Szabó's results). For general non-compact convex sets, the Krein–Milman theorem fails.

In the proof of the theorem, compactness is used in two places: to show that minimal faces are points and to deduce that a decreasing chain of closed faces has nontrivial intersection. In Section 2, we show that minimal faces of $\mathrm{Sof}(G, P^{\omega})$ are points, as before, even though $\mathrm{Sof}(G, P^{\omega})$ is not compact. We then construct a decreasing chain of closed faces, with trivial intersection, provided that we have infinite extreme points, in Section 3. In Sections 4 and 5, we show that $\mathrm{Sof}(\mathbb{F}_r, P^{\omega})$ has uncountably many extreme points. Finally, we construct a sofic representation of the free group, with trivial commutant and uncountably many cuts, in Section 6.

1. Introduction

For a matrix $x \in M_n$, we define its normalised trace as $\mathrm{Tr}(x) = 1/n \sum_i x(i,i)$ and its trace norm $\|x\|_2 = \sqrt{\mathrm{Tr}(x^*x)} = \sqrt{1/n \sum_{i,j} |x(i,j)|^2}$. Throughout the paper, we denote by $P_n \subset M_n$ the group of permutation matrices and $D_n \subset M_n$ the maximal abelian subalgebra of diagonal matrices. The group P_n is isomorphic to $\mathrm{Sym}(n)$, the symmetric group on a set of n elements. For simplicity, we call elements in P_n permutations, instead of permutation matrices. For $p \in P_n$, $\mathrm{Tr}(p) = 1 - 1/n \cdot |\mathrm{Fix}(p)| = 1 - d_H(p, \mathrm{Id})$, where d_H is the normalised Hamming distance: for $p,q \in \mathrm{Sym}(n)$, $d_H(p,q) = 1/n \cdot |\{i : p(i) \neq q(i)\}|$. On the algebra D_n, $\mathrm{Tr} : D_n \to \mathbb{C}$ is acting as an integral and D_n is isomorphic to $L^{\infty}(\{1, \ldots, n\})$ endowed with the normalised cardinal measure.

Fix now ω to be a free ultrafilter on $\mathbb{N}$, and let $(n_k)_k$ denote a sequence of natural numbers, $\lim_k n_k = \infty$. The ultraproduct $\Pi_{k\to\omega} P_{n_k}$ is called *the universal sofic group*. It was introduced by Elek and Szabó in [10]. They showed that a countable group is sofic if and only if it is a subgroup of $\Pi_{k\to\omega} P_{n_k}$.

The ultraproduct $(\Pi_{k\to\omega} D_{n_k}, \mathrm{Tr})$ yields an abelian finite von Neumann algebra. As such, there exists a probability space (X_ω, μ_ω) such that $(\Pi_{k\to\omega} D_{n_k}, \mathrm{Tr}) \simeq L^{\infty}(X_\omega, \mu_\omega)$. We can construct (X_ω, μ_ω) as an ultraproduct of finite probability spaces, that is, a Loeb construction. It comes with a measurable map called *the standard part* $\mathrm{St} : X_\omega \to [0,1]$. This map induces a canonical embedding $\mathrm{St}^* : L^{\infty}([0,1], \mu) \to L^{\infty}(X_\omega, \mu_\omega)$, where μ is the Lebesgue measure. For more details on the Loeb space, standard part and also the action of $\Pi_{k\to\omega} P_{n_k}$ on $(\Pi_{k\to\omega} D_{n_k}, \mathrm{Tr})$, check [8, Section 1].

We assume familiarity with the space of sofic representations for a group G, denoted by $\mathrm{Sof}(G, P^\omega)$, and its convex-like structure. We refer the reader to [15, Sections 2.1 and 2.2]. We use the same notations.

Proposition 1.1 ([15, Proposition 2.2]). *Let Θ_i, $i = 1, \dots, n$, be sofic representations of a group G and $\lambda_i \in [0,1]$ be such that $\sum_{i=1}^n \lambda_i = 1$. Then there exists a well-defined element of* $\mathrm{Sof}(G, P^\omega)$ *denoted by $\sum_{i=1}^n \lambda_i [\Theta_i]$ such that the axioms of convex-like structures are observed.*

The reverse operation of taking convex combinations is cutting with a commuting projection. Here we recall the construction of a cut sofic representation, as it is central to the paper.

Definition 1.2. Let $\Theta : G \to \Pi_{k\to\omega} P_{n_k}$ be a sofic representation and $p \in \Pi_{k\to\omega} D_{n_k r_k}$ be a non-zero projection commuting with $\Theta \otimes \mathrm{Id}_{r_k}$. Let m_k be natural numbers such that $\mathrm{Tr}(p) = \lim_{k\to\omega} m_k/(n_k r_k)$. Denote by Θ_p the map: $p(\Theta \otimes \mathrm{Id}_{r_k}) : G \to \Pi_{k\to\omega} P_{m_k}$. The sofic representation Θ_p depends on the numbers m_k, but its class in $\mathrm{Sof}(G, P^\omega)$ does not.

We call such a $p \in \Pi_{k\to\omega} D_{n_k r_k}$ a *cutting projection of* Θ, and we denote the set of such projections by $\mathcal{C}(\Theta)$. Note that $0 < \mathrm{Tr}(p) \leq 1$ for any p a cutting projection.

Observation 1.3. For any sequence $\{s_k\}_k$, we have $[\Theta_p] = [\Theta_{p\otimes \mathrm{Id}_{s_k}}]$ and $\mathrm{Tr}(p) = \mathrm{Tr}(p \otimes \mathrm{Id}_{s_k})$.

Observation 1.4 ([15, Observation 2.5]). If $[\Theta] = \lambda[\Psi_1] + (1-\lambda)[\Psi_2]$, $\lambda > 0$, then there exists p a cutting projection with $\mathrm{Tr}(p) = \lambda$ such that $[\Theta_p] = [\Psi_1]$.

Observation 1.5. If p and q are two cutting projections in the same sequence of dimensions, then $[(\Theta_p)_q] = [\Theta_{pq}]$.

This last observation is just an algebraic consequence of the definition. However there is a discussion here to be made. When we write $a = \Pi_{k\to\omega} a_k \in \Pi_{k\to\omega} P_{n_k}$, we think of the permutations $a_k \in P_{n_k}$ as acting on the set $\{1, \dots, n_k\}$. But, for $\Theta_p(g) = p(\Theta(g) \otimes \mathrm{Id}_{r_k})$, $(\Theta_p(g))_k$ is a permutation on the support of p_k. As a projection in $D_{n_k r_k}$, p_k is a projection on some subset of $\{1, \dots, n_k r_k\}$. It is on these subsets that $\Theta_p(g)$ is acting as an ultraproduct of permutations. Having this in mind, now $(\Theta_p)_q$ makes sense.

1.1. The smallest face of a sofic representation

The goal of this subsection is to provide a characterisation of the smallest face of $\mathrm{Sof}(G, P^\omega)$ containing an arbitrary point $[\Theta]$. This is an adaptation of results in [3] to the convex structure on sofic embeddings.

Definition 1.6. We denote by $F_{[\Theta]}$ the set $\{[\Theta_p] : p$ a cutting projection of $\Theta\}$.

We now prove that $F_{[\Theta]}$ is indeed the smallest face containing $[\Theta]$. First, we present some helpful lemmas.

Lemma 1.7. *If p, q are disjoint cutting projections in the same sequence of dimensions, then*

$$[\Theta_{p+q}] = \frac{\mathrm{Tr}(p)}{\mathrm{Tr}(p+q)}[\Theta_p] + \frac{\mathrm{Tr}(q)}{\mathrm{Tr}(p+q)}[\Theta_q].$$

Proof. Let $p, q \in \Pi_{k\to\omega} D_{n_k r_k}$. Then $\Theta_{p+q} = (\Theta \otimes \mathrm{Id}_{r_k})_p \oplus (\Theta \otimes \mathrm{Id}_{r_k})_q$. As Θ_{p+q} is a direct sum, by the definition of the convex structure, its class is a convex combination of its summands. Thus $[\Theta_{p+q}]$ is a convex combination of $[\Theta_p]$ and $[\Theta_q]$. The coefficients of this convex combination are given by how much space $[\Theta_p]$ and $[\Theta_q]$ occupy in the direct sum. A close inspection of the dimensions of the permutations involved yields the stated result. ∎

The following result is important, as it shows that $F_{[\Theta]}$ is convex, a first requirement of being a face.

Lemma 1.8 ([3, analogue of Proposition 3.2]). *Let $\Theta : G \to \Pi_{k\to\omega} P_{n_k}$ be a sofic representation, p, q be two cutting projections and $\lambda \in [0, 1]$. Then there exists s a cutting projection such that*

$$\lambda[\Theta_p] + (1-\lambda)[\Theta_q] = [\Theta_s].$$

Proof. We can assume that p and q are in the same sequence of dimensions, that is, $p, q \in \Pi_{k\to\omega} D_{n_k r_k}$. Let $t \in \Pi_{k\to\omega} D_{n_k}$ be any projection such that

$$\mathrm{Tr}(t) = \frac{\lambda \mathrm{Tr}(q)}{\lambda \mathrm{Tr}(q) + (1-\lambda)\mathrm{Tr}(p)}.$$

This value is chosen such that $\lambda = \mathrm{Tr}(p)\mathrm{Tr}(t)/[\mathrm{Tr}(p)\mathrm{Tr}(t) + \mathrm{Tr}(q)(1 - \mathrm{Tr}(t))]$. Construct $s \in \Pi_{k\to\omega} D_{n_k^2 r_k}$ by

$$s = p \otimes t + q \otimes (1-t).$$

It is easy to see that s is a cutting projection. Moreover,

$$\Theta_s = \Theta_{p\otimes t} \oplus \Theta_{q\otimes(1-t)} = (\Theta \otimes \mathrm{Id}_{r_k})_p \otimes (\mathrm{Id}_{n_k})_t \oplus (\Theta \otimes \mathrm{Id}_{r_k})_q \otimes (\mathrm{Id}_{n_k})_{1-t}.$$

As in the previous lemma, $[\Theta_s]$ is a convex combination between $[\Theta_p]$ and $[\Theta_q]$. The value of $\mathrm{Tr}(t)$ is chosen such that this yields the required combination, that is, $[\Theta_s] = \lambda[\Theta_p] + (1-\lambda)[\Theta_q]$. ∎

Proposition 1.9 ([3, analogue of Proposition 3.3]). *The set $F_{[\Theta]}$ is the smallest face containing* $[\Theta]$.

Proof. We first show that $F_{[\Theta]}$ is indeed a face. By the previous lemma, $F_{[\Theta]}$ is convex. Now let $\lambda[\Psi_1] + (1-\lambda)[\Psi_2]$ be an element of $F_{[\Theta]}$. So there is a cutting projection p such that $[\Theta_p] = \lambda[\Psi_1] + (1-\lambda)[\Psi_2]$. By Observation 1.4, there exists q a cutting projection such that $[(\Theta_p)_q] = [\Psi_1]$, and by Observation 1.5, $[(\Theta_p)_q] = [\Theta_{pq}]$ (amplifying

p and q to the same sequence of dimensions if needed). Thus $[\Psi_1] = [\Theta_{pq}] \in F_{[\Theta]}$. By similar arguments, we have $[\Psi_2] \in F_{[\Theta]}$, and therefore, $F_{[\Theta]}$ is a face.

Now let F be a face containing $[\Theta]$, and let p be a cutting projection. As a particular case of Lemma 1.7, we get $[\Theta] = \mathrm{Tr}(p)[\Theta_p] + (1 - \mathrm{Tr}(p))[\Theta_{1-p}]$. It follows that F contains $[\Theta_p]$, so it contains $F_{[\Theta]}$. ∎

Theorem 1.10. *The set $F_{[\theta]}$ is closed.*

Proof. This follows from a diagonal argument constructing a sofic representation from a sequence of sofic representations. Let $\{[\theta_i]\}_i$ be a Cauchy sequence of elements in $F_{[\theta]}$, $\theta_i = \Pi_{k\to\omega}\theta_i^k$.

We now describe what a diagonal argument means, in general, when an ultrafilter is present. We will extensively use diagonal arguments throughout the paper. We want to construct sets $F_i \in \omega$, $F_i \supseteq F_{i+1}$, $\bigcap_i F_i = \emptyset$ such that $\Psi = \Pi_{k\to\omega}\psi^k$ is a sofic representation with the required properties, where $\psi^k = \theta_i^k$ for $k \in F_i \setminus F_{i+1}$.

More detailed, let ε_i be a decreasing sequence converging to 0. For each i, construct a set $F_i \in \omega$ such that θ_i^k satisfies some ε_i-inequalities related to our goals for each $k \in F_i$. The set F_i represents the "good" part of θ_i, that is, the set of indices k coming from θ_i that can be used for the construction of the limit sofic representation. Then, inductively replace F_i with $F_i \cap F_{i-1}$, in order to ensure the $F_i \supseteq F_{i+1}$ condition. Note that $F_i \cap F_{i-1}$ is still an element of ω, per axioms of the ultrafilter. Finally, replace F_i with $F_i \cap \{i, i+1, \ldots\}$ to get $\bigcap_i F_i = \emptyset$. Again, $\{i, i+1, \ldots\} \in \omega$, as ω is a free ultrafilter. The last step is to construct $\psi^k = \theta_i^k$ for $k \in F_i \setminus F_{i+1}$ and $\Psi = \Pi_{k\to\omega}\psi^k$.

In our case, we want Ψ to be the limit of the sequence $\{[\theta_i]\}_i$ and $\Psi \in F_{[\theta]}$. Choosing a subsequence, we can assume that $d([\theta_i], [\theta_{i+1}]) < 1/4^i$. Also, for each i, $[\theta_i] = [\theta_{p_i}]$ for some $p_i \in \mathcal{C}(\theta)$. Let $p_i = \Pi_{k\to\omega}p_i^k$. Finally, let $G = \bigcup_i S_i$, with $S_i \subset S_{i+1}$, S_i finite. Now choose $F_i \in \omega$ such that for each $k \in F_i$ we have:

(1) $d(\theta_i^k, \theta_{i+1}^k) < 2 \cdot 1/4^i$;

(2) $\|p_i^k(\theta(g) \otimes \mathrm{Id}), (\theta(g) \otimes \mathrm{Id})p_i^k\| < 1/4^i$ for $g \in S_i$.

Modify the sets $\{F_i\}_i$ to get a decreasing sequence with empty intersection, as described in the previous paragraph. Construct $\psi_k = \theta_i^k$ for $k \in F_i \setminus F_{i+1}$ and $\Psi = \Pi_{k\to\omega}\psi^k$, as planned. Note that $\Psi = \theta_p$, where $p = \Pi_{k\to\omega}p^k$ and $p^k = p_i^k$ for $k \in F_i \setminus F_{i+1}$.

By the first requirement, $d([\Psi], [\theta_i]) < \sum_{j\geq i} 2 \cdot 1/4^j = 1/4^i \cdot 6/4 \to_i 0$. The second requirement ensures that $p \in \mathcal{C}(\theta)$, so $[\Psi] \in F_{[\theta]}$. ∎

1.2. Infinite convex combinations

In this paper, we also use infinite convex combinations. They are constructed similarly to [15, Section 2.2] plus a diagonal argument.

Proposition 1.11. *Let* $\{\Theta_i\}_{i\in\mathbb{N}^*}$ *be a sequence of sofic representations of a countable group* G*, and let* $\lambda_i \in [0,1]$ *be such that* $\sum_{i\in\mathbb{N}} \lambda_i = 1$*. Then there exists a sofic representation* $\Psi : G \to \Pi_{k\to\omega} P_{m_k}$ *such that for each* $i \in \mathbb{N}$ *with* $\lambda_i > 0$*, there exists a cutting projection* $p_i \in \Pi_{k\to\omega} D_{m_k}$ *such that* $\mathrm{Tr}(p_i) = \lambda_i$*,* $\sum_i p_i = \mathrm{Id}$ *and* $[\Psi_{p_i}] = [\Theta_i]$*.*

1.3. Order relation on $\mathcal{C}(\Theta)$

There is a partial relation for cutting projections. Firstly, we consider a projection equivalent to any of its amplifications, that is, $p \simeq p \otimes \mathrm{Id}_{s_k}$.

Definition 1.12. Let Θ be some sofic representation, and let $p, q \in \mathcal{C}(\Theta)$. Then $p \leq q$ if they have amplifications p_1, q_1 to the same sequences of dimensions such that $p_1 q_1 = p_1$.

Let $\Theta : G \to \Pi_{k\to\omega} P_{n_k}$. Sometimes, it is useful to view a cutting projection of Θ as an element of $\Pi_{k\to\omega} D_{n_k}$ instead of using amplifications. For a finite von Neumann algebra (N, Tr), we let N_+^1 denote the set of strictly positive contractions in N given by

$$N_+^1 = \{x \in N : 0 < x \leq 1\}.$$

We also need a notation for the *commutant*, a central tool of the article.

Notation 1.13. For a map $\Theta : G \to \Pi_{k\to\omega} P_{n_k}$, we denote by Θ' the commutant of the image of Θ in $\Pi_{k\to\omega} M_{n_k}$, that is,

$$\Theta' = \{x \in \Pi_{k\to\omega} M_{n_k} : xa = ax \ \forall a \in \Theta(G)\}.$$

Proposition 1.14. *For a sofic representation* $\Theta : G \to \Pi_{k\to\omega} P_{n_k}$*, we have* $\mathcal{C}(\Theta) \simeq (\Pi_{k\to\omega} D_{n_k})_+^1 \cap \Theta'$*.*

Proof. Let $p \in \Pi_{k\to\omega} D_{n_k r_k}$ be a non-zero projection commuting with $\Theta \otimes \mathrm{Id}_{r_k}$. Then $p = \Pi_{k\to\omega} p_k$, where p_k is a projection in $D_{n_k r_k}$. As such $p_k = \bigoplus_{i=1}^{r_k} p_k^i$, with $p_k^i \in D_{n_k}$. Define $f_k = 1/r_k \sum_i p_k^i$ and $f = \Pi_{k\to\omega} f_k$. Then $\mathrm{Tr}(p) = \mathrm{Tr}(f)$ and $0 < f \leq \mathrm{Id}$. Also p commuting with Θ implies f commutes with Θ. Denote by $\Lambda : \mathcal{C}(\Theta) \to (\Pi_{k\to\omega} D_{n_k})_+^1 \cap \Theta'$ the map constructed in this paragraph.

For the reverse, if $f \in (\Pi_{k\to\omega} D_{n_k})_+^1 \cap \Theta'$ is a projection, then $f \in \mathcal{C}(\Theta)$ and $\Lambda(f) = f$. If $f = 1/r \sum_{i=1}^r p_i$, where each p_i is a projection in $(\Pi_{k\to\omega} D_{n_k})_+^1 \cap \Theta'$, then $p = \bigoplus_i p_i \in \Pi_{k\to\omega} D_{n_k}$ is in $\mathcal{C}(\theta)$ and $\Lambda(p) = f$.

Now choose an arbitrary $f \in (\Pi_{k\to\omega} D_{n_k})_+^1 \cap \Theta'$. Fix $r \in \mathbb{N}^*$. Let $c_i : \mathbb{R} \to \mathbb{R}$ be the characteristic function of the set $[i/r, \infty)$. By functional calculus, $c_i(f) \in \Theta'$, and $c_i(f)$ is a projection. By the previous paragraph, $1/r \sum_{i=1}^r c_i(f)$ is in the image of Λ. Moreover, $\left\| f - 1/r \sum_{i=1}^r c_i(f) \right\|_2 < 1/r$. The proof can be finished by a diagonal argument. ∎

Observation 1.15. For $p, q \in \mathcal{C}(\Theta)$, we have $p \leq q$ if and only if $f_p \leq f_q$, where f_p, f_q are the associated elements in $\Pi_{k\to\omega} D_{n_k}$.

1.4. Direct integrals and ergodic decompositions

In general, when discussing convex structures and group actions, direct integrals and ergodic decompositions are important tools. The main difficulty in our context is that both the Loeb space and $\mathrm{Sof}(G, P^\omega)$ are non-separable (for G sofic and non-amenable).

Let $f : [0,1] \to \mathrm{Sof}(G, P^\omega)$ be a measurable function ($\mathrm{Sof}(G, P^\omega)$ is a metric space, so it has a measurable structure). As $\mathrm{Sof}(G, P^\omega)$ is non-separable, we do not have tools like Luzin's theorem in order to approximate f with continuous functions. We can define $\int f(\lambda)d\lambda$ if f is continuous, as a limit in $\mathrm{Sof}(G, P^\omega)$ of finite convex combinations of elements in $\{f(\lambda) : \lambda \in [0,1]\}$. However, we do not know what representations we get by using cutting projections. In particular, we do not know if for almost all $\lambda \in [0,1]$ there exists a cutting projection p such that $\left[\left(\int f(\lambda)d\lambda\right)_p\right] = [f(\lambda)]$.

Let $\{[\Theta_n] : n \in \mathbb{N}\}$ be a sequence convergent to $[\Theta]$ in $\mathrm{Sof}(G, P^\omega)$. Understanding the connection between $F_{[\Theta]}$ and $\{F_{[\Theta_n]} : n \in \mathbb{N}\}$ is an important problem in the context of sofic approximations.

Let $\Theta : G \to \Pi_{k\to\omega} P_{n_k}$ be a sofic representation. Then we have a p.m.p. action on the Loeb space, $G \curvearrowright (X_\omega, \mu_\omega)$. As X_ω is not separable, we cannot consider the ergodic decomposition of this action. Moreover, if Θ_1, Θ_2 are two sofic representations with measurable equivalent actions on X_ω, we cannot deduce that $[\Theta_1] = [\Theta_2]$. For example, if two sofic representations are automorphically conjugated, that is, conjugated by an automorphism of $\Pi_{k\to\omega} P_{n_k}$, then they have measurable equivalent actions. This is a consequence of [13, Lemma 3.3]. At the moment, automorphically conjugated is not known to imply conjugation by an element of $\Pi_{k\to\omega} P_{n_k}$.

2. Minimal faces

The goal of this section is to prove that minimal faces are points, and thus extreme points. For a compact convex subset, this is an easy consequence of the Hahn–Banach theorem and part of the proof of the Krein–Milman theorem. Here we do not have compactness, so we deduce this result by other means.

Proposition 2.1. *If $\Psi \in F_{[\Theta]}$, there exists a maximal cutting projection s of $[\Theta]$ such that $[\Psi] = [\Theta_s]$.*

Proof. First of all, when we say *maximal* projection, we consider a projection to be equivalent to any of its amplifications. So if p, q are two projections, then $p \le q$ if they have amplifications p_1, q_1 to the same sequences of dimensions such that $p_1 q_1 = p_1$.

Let

$$A = \{p : p \text{ cutting projection}, [\Theta_p] = [\Psi]\}$$

endowed with the order relation specified above. Let P be a totally ordered subset of A. We want to prove that P has an upper bound in A. The important thing to notice here is that for $p, q \in P$, $\mathrm{Tr}(p) \le \mathrm{Tr}(q)$ implies $p \le q$. This comes from the fact that P is totally ordered. If $\sup\{\mathrm{Tr}(p) : p \in P\}$ is attained in P, then that element would be maximal in P, and we are done. Otherwise, choose $q_i \in P$, $i \in \mathbb{N}$, such that $q_i < q_{i+1}$ for $i \in \mathbb{N}$ and

$$\sup\{\mathrm{Tr}(q_i) : i \in \mathbb{N}\} = \sup\{\mathrm{Tr}(p) : p \in P\}.$$

The goal here is to reduce P to a countable subset. Indeed, for each $p \in P$, there is $i \in \mathbb{N}$ such that $p < q_i$. All we have to do is construct an upper bound in A of the sequence $\{q_i\}_{i\in\mathbb{N}}$.

Let $q_i = \Pi_{k\to\omega} q_i^k$. We know that $(q_i \otimes \mathrm{Id})(q_{i+1} \otimes \mathrm{Id}) = q_i \otimes \mathrm{Id}$. It is a rather involved diagonal argument here to show that there exists p a cutting projection $q_i < p$ for any i and $\mathrm{Tr}(p) = \sup\{\mathrm{Tr}(q_i) : i \in \mathbb{N}\}$. Alternatively, we can simply take the supremum of the sequence $\{q_i\}_{i\in\mathbb{N}}$, when these projections are embedded in a direct limit of amplifications, like the map Ψ from [16, Notation 2.2].

We only need to show that $[\Theta_p] = [\Psi]$. As $q_i < p$, by Lemma 1.7 and Observation 1.5, we have

$$[\Theta_p] = \frac{\mathrm{Tr}(q_i)}{\mathrm{Tr}(p)}[\Theta_{q_i}] + \frac{\mathrm{Tr}(p) - \mathrm{Tr}(q_i)}{\mathrm{Tr}(p)}[\Theta_{p-q_i}].$$

As $\lim_{i\to\infty} \mathrm{Tr}(q_i) = \mathrm{Tr}(p)$, the axioms of the convex structure imply that

$$\lim_{i\to\infty} d([\Theta_p], [\Theta_{q_i}]) = 0.$$

But $[\Theta_{q_i}] = [\Psi]$, as $q_i \in A$. We conclude that $[\Theta_p] = [\Psi]$. The proposition follows now from Zorn's lemma. ∎

Theorem 2.2. *The minimal faces are exactly the extreme points.*

Proof. Let F be a minimal face, and let $\Theta \in F$. As F is minimal, $F = F_{[\Theta]}$. Consider $[\Psi] \in F_{[\Theta]}$. By the previous proposition, there exists a maximal projection q such that $[\Theta_q] = [\Psi]$. We will show that $q = 1$. Assume that $q < 1$. We know that $[\Theta_{(1-q)}] \in F$. Thus $F_{[\Theta_{(1-q)}]} \subset F$, and since F is minimal, we have $F = F_{[\Theta_{(1-q)}]}$. As $[\Psi] \in F_{[\Theta_{(1-q)}]}$, there exists a non-zero projection p such that $[\Psi] = [(\Theta_{(1-q)})_p]$. Then, by Observation 1.5, $[\Psi] = [\Theta_{(1-q)p}]$, and by Lemma 1.7, $[\Psi] = [\Theta_{q+(1-q)p}]$. This contradicts the maximality of q. So any element in F is equal to $[\Theta]$. ∎

Observation 2.3. The proof works in the same way for Brown's convex structure on $\mathrm{Hom}(N, R^\omega)$, where N is a type II_1 factor, and R is the hyperfinite factor.

Open Problem 2.4. Does this result follow directly from Nate Brown's axioms for any convex-like structure?

3. Decreasing chain of faces

In this section, we construct a decreasing chain of faces in $\mathrm{Sof}(G, P^\omega)$ with trivial intersection, provided that $\mathrm{Sof}(G, P^\omega)$ has an infinite number of extreme points. Recall that $[\Theta] \in \mathrm{Sof}(G, P^\omega)$ is an extreme point if and only if $[\Theta] = [\Theta_p]$ for any cutting projection p [15, Lemma 2.12].

Proposition 3.1. *Let $\{\Theta_i\}_{i\in\mathbb{N}^*}$ be a sequence of different extreme sofic representations of a countable group G. Then there exists a sofic representation Ψ such that:*

(1) *for each $i \in \mathbb{N}^*$, there exists $p_i \in \mathcal{C}(\Psi)$ such that $[\Psi_{p_i}] = [\Theta_i]$;*

(2) *for each $p \in \mathcal{C}(\Psi)$, there exist $q \in \mathcal{C}(\Psi)$, $q \leq p$ and $i \in \mathbb{N}$ such that $[\Psi_q] = [\Theta_i]$.*

Proof. Let $(\lambda_i)_{i\in\mathbb{N}^*} \subset (0,1)$ be a sequence such that $\sum_i \lambda_i = 1$. Use Proposition 1.11 to construct a sofic representation $\Psi : G \to \Pi_{k\to\omega} P_{n_k}$ with the given properties. Then point (1) is automatic and does not require Θ_i to be extreme points.

Now suppose $p \in \mathcal{C}(\Psi)$, $p \in \Pi_{k\to\omega} D_{n_k r_k}$. Then there exists i such that $\mathrm{Tr}(p(p_i \otimes \mathrm{Id}_{r_k})) \neq 0$. Let $q = p(p_i \otimes \mathrm{Id}_{r_k})$. Then $q \leq p$ and $q \leq p_i$. As $[\Psi_{p_i}] = [\Theta_i]$ is an extreme point, it follows that $[\Psi_q] = [\Theta_i]$. ∎

Theorem 3.2. *Let $\{\Theta_i\}_{i\in\mathbb{N}^*}$ be a sequence of different extreme sofic representations of a countable group G. Then there exists a decreasing chain of faces in* $\mathrm{Sof}(G, P^\omega)$ *with trivial intersection.*

Proof. Let $[\Psi]$ be the sofic representation constructed above. By construction, we have cutting projections $(p_i)_{i\in\mathbb{N}^*}$ such that $[\Psi_{p_i}] = [\Theta_i]$ and $\sum_i p_i = \mathrm{Id}$. As $[\Theta_i]$ are different extreme points, $p_i \in \mathcal{C}(\Psi)$ is maximal with the property $[\Psi_{p_i}] = [\Theta_i]$.

Let $q_j = \sum_{i\geq j} p_i$ and define the face $F_j = F_{[\Psi_{q_j}]}$. We show that $\bigcap_j F_j = \emptyset$. If $i < j$, then $[\Theta_i] \notin F_j$; otherwise, p_i would not be maximal. Assume that $\bigcap_j F_j$ is not empty. So there exists $p \in \mathcal{C}(\Psi)$ such that $[\Psi_p] \in \bigcap_j F_j$. By the second property of the above proposition, there exist $q \leq p$ and $i \in \mathbb{N}^*$ such that $[\Psi_q] = [\Theta_i]$. But $q \leq p$ implies $[\Psi_q] \in \bigcap_j F_j$, so $[\Theta_i] \in \bigcap_j F_j$, a contradiction. ∎

Of course, this chain of decreasing faces does not help towards settling Krein–Milman for $\mathrm{Sof}(G, P^\omega)$. We actually know for sure that the faces constructed in the proof of Theorem 3.2 are in the closure of the convex hull of extreme points. We later construct a sofic representation that is a candidate for an element outside of the convex hull.

For a sofic, non-amenable group G, there are two possibilities: either $\mathrm{Sof}(G, P^\omega)$ has a finite number of extreme points, a case in which Krein–Milman does not hold, as $\mathrm{Sof}(G, P^\omega)$ is not separable; or $\mathrm{Sof}(G, P^\omega)$ has an infinite number of extreme points, in which case, by the above theorem, it has a decreasing chain of faces with trivial intersection. In the next section, we prove that the space $\mathrm{Sof}(\mathbb{F}_r, P^\omega)$ has uncountably many extreme points.

4. Extreme points in $\mathrm{Sof}(\mathbb{F}_2, P^\omega)$

In this section, we construct uncountably many extreme points in the space $\mathrm{Sof}(\mathbb{F}_2, P^\omega)$. Each of these points will be an *expander*, so let us first recall the definition.

Definition 4.1. Let $\lambda > 0$ and $p_1, \ldots, p_s \in P_n$. We say that $(p_1, \ldots, p_s)$ is a λ-*expander* if for any projection $e \in D_n$ with $0 < \mathrm{Tr}(e) \leq 1/2$ we have $\lambda \mathrm{Tr}(e) < \sum_{i=1}^{s} d_H(e, p_i e p_i^*)$.

Definition 4.2. A sofic representation $\Theta : G \to \Pi_{k\to\omega} P_{n_k}$ is called an *expander* or λ-*expander* if there exist $g_1, \ldots, g_s \in G$ and $p_k^i \in P_{n_k}$ such that $\Theta(g_i) = \Pi_{k\to\omega} p_k^i$ for $i = 1, \ldots, s$ and $(p_k^1, \ldots, p_k^s)$ is a λ-expander for each $k \in \mathbb{N}$.

From now on, a or a_n denotes the matrix in P_n associated to the cycle permutation $i \to i + 1 \pmod n$. We also fix $\lambda = 0.1$ for the rest of this section. The value is chosen such that $2\lambda < 1/e$, as needed in the proof of Proposition 4.11. Also this value allows us to import some results from [16]. For example, taking two random elements in P_n yields a λ-expander with great probability. For n-cycles, this was already shown.

Proposition 4.3 ([16, Proposition 5.11]). *There exists a constant μ such that for at least $(1 - \mu/n) \cdot (n-1)!$ n-cycles c, the following holds: for any projection $p \in D_n$ with $0 < \mathrm{Tr}(p) \leq 1/2$, we have $\lambda \mathrm{Tr}(p) < d_H(p, apa^*) + d_H(p, cpc^*)$.*

This time we work with arbitrary permutations, not just cycles. The above proposition still holds true, with virtually the same proof. If we track down the proof of Theorem 5.11 from [16], we see that the key ingredient was [12, Theorem 1.2]. Theorem 1.1 of the same article yields the following.

Proposition 4.4. *There exists a constant μ such that for at least $(1 - \mu/n) \cdot n!$ permutations c, the following holds: for any projection $p \in D_n$ with $0 < \mathrm{Tr}(p) \leq 1/2$, we have $\lambda \mathrm{Tr}(p) < d_H(p, apa^*) + d_H(p, cpc^*)$.*

The same transition from cycles to arbitrary permutations has to be applied to [16, Theorem 5.20].

Proposition 4.5 ([16, Theorem 5.20]). *For any $\varepsilon > 0$ and nontrivial $\gamma \in \mathbb{F}_2$, there exists n_0 such that for any $n > n_0$ for at least $(1-\varepsilon)[(n-1)!]$ n-cycles $c \in P_n$, we have $d_H(\gamma(a, c), 1_n) > 1 - \varepsilon$, where $\gamma(a, c)$ is the element in P_n obtained by replacing the two generators of γ by a and c, respectively.*

Again, replacing Theorem 1.2 of [12] with Theorem 1.1 of the same article, we get the following.

Proposition 4.6. *For any $\varepsilon > 0$ and nontrivial $\gamma \in \mathbb{F}_2$, there exists n_0 such that for any $n > n_0$ for at least $(1-\varepsilon) \cdot n!$ elements $c \in P_n$, we have $d_H(\gamma(a, c), 1_n) > 1 - \varepsilon$.*

In our argument, we need two more results from [16].

Proposition 4.7 ([16, Proposition 5.12]). *Let $\varepsilon > 0$ and $x \in P_n$. Then the number of permutations $y \in P_n$ such that $d_H(x, y) < \varepsilon$ is less than $n^{\lfloor n\varepsilon \rfloor}$.*

Proposition 4.8 ([16, Proposition 5.13]). *Let $\varepsilon > 0$. The number of permutations $y \in P_n$ such that $d_H(ay, ya) < \varepsilon$ is less than $n^{\lfloor n\varepsilon \rfloor + 1}$.*

The goal of this section is to construct uncountably many, extreme sofic representations of $\mathbb{F}_2$. The first step is to construct uncountably many sofic representations that are not conjugated. We do this with the help of the following distance.

Definition 4.9. Given two collections of permutations in P_n, $(x_1, x_2, \dots, x_k)$ and $(y_1, y_2, \dots, y_k)$, define the distance

$$d_S((x_1, x_2, \dots, x_k), (y_1, y_2, \dots, y_k)) = \min_{p \in P_n} \sum_{i=1}^{k} d_H(x_i, p y_i p^*).$$

We now prove a first result estimating the number of permutations $c \in P_n$ with certain properties.

Proposition 4.10. *Suppose $b \in P_n$. The number of permutations $c \in P_n$ such that $d_S((a, b), (a, c)) < \lambda$ is less than $n^{2\lfloor n\lambda \rfloor + 1}$.*

Proof. Let $c \in P_n$ be such that $d_S((a, b), (a, c)) < \lambda$. Then there exists $p \in P_n$ such that $d_H(a, pap^*) < \lambda$ and $d_H(b, pcp^*) < \lambda$. Let $A = \{q^* b q : q \in P_n, d_H(qa, aq) < \lambda\}$. It follows that $d_H(c, A) < \lambda$.

By Proposition 4.8, the cardinality of A is less than $n^{\lfloor n\lambda \rfloor + 1}$. As $d_H(c, A) < \lambda$, by Proposition 4.7, c is in a set of cardinality at most $n^{\lfloor n\lambda \rfloor + 1} \cdot n^{\lfloor n\lambda \rfloor} = n^{2\lfloor n\lambda \rfloor + 1}$. ∎

Now we are ready to prove the main technical result that will make the construction of uncountably many points in $\mathrm{Sof}(\mathbb{F}_2, P^\omega)$ possible.

Proposition 4.11. *Let $k \in \mathbb{N}^*$. There exist $n_k \in \mathbb{N}$ and k permutations $c_1, \dots, c_k \in P_{n_k}$ such that:*

1. *(a, c_i) is a λ-expander for each $i = 1, \dots, k$;*
2. *$d_S((a, c_i), (a, c_j)) > \lambda$ for all $i \neq j$;*
3. *$\mathrm{Tr}(\gamma(a, c_i)) < 1/k$ for all $i = 1, \dots, k$ and $\gamma \in B_k(\mathbb{F}_2)$.*

Proof. Recall that $B_k(\mathbb{F}_2)$ is the ball of radius k around the origin in the Cayley graph of $\mathbb{F}_2$. We cannot impose condition (3) on all elements of $\mathbb{F}_2$, so we do it for larger and larger finite subsets. This is the standard procedure in the theory of sofic groups.

Fix $\varepsilon > 0$. By Propositions 4.4 and 4.6, applied to each $\gamma \in B_k(\mathbb{F}_2)$, we get that for large enough n, for at least $(1 - \varepsilon) \cdot n!$ permutations $c \in P_n$, conditions (1) and (3) are satisfied.

We must now choose $c_1, \dots, c_k$ in this set such that $d_S((a, c_i), (a, c_j)) > \lambda$. Here is where Proposition 4.10 comes in. Choosing a permutation $c \in P_n$ will exclude at most $n^{2\lfloor n\lambda \rfloor + 1}$ other permutations. For large enough n, we have $k \cdot n^{2\lfloor n\lambda \rfloor + 1} < (1 - \varepsilon) \cdot n!$. As such, we can choose $c_1, \dots, c_k$ with the required properties. ∎

Theorem 4.12. *For each $i \in [0, 1]$, there exists a sofic representation $\Theta_i : \mathbb{F}_2 \to \Pi_{k\to\omega} P_{n_k}$ such that each one is an expander, and Θ_i, Θ_j are not conjugated for any $i \neq j$.*

Proof. Use the previous proposition to construct a sequence $(n_k)_k$, and sets $A_k = \{c_k^1, \ldots, c_k^k\} \subset P_{n_k}$, with the specified properties.

In order to construct our sofic representations, we need an uncountable family $\mathcal{F}$ of infinite subsets of $\mathbb{N}$ such that for $F_1, F_2 \in \mathcal{F}$, $F_1 \neq F_2$, we have $F_1 \cap F_2$ is finite. An example was constructed in [7, Remark 3.3]. For $t \in [1/10, 1)$, construct $F_t = \{\lfloor 10^k t \rfloor : k \in \mathbb{N}^*\}$. It is easy to see that $\mathcal{F} = \{F_t : t \in [1/10, 1)\}$ has the required property.

For $F \in \mathcal{F}$, construct a function $f : \mathbb{N} \to \mathbb{N}$, $f_F(k) = \max\{i \in F : i \leq k\}$. Let x_1, x_2 be the two generators of $\mathbb{F}_2$. For $F \in \mathcal{F}$, define a sofic morphism $\Theta_F : \mathbb{F}_2 \to \Pi_{k\to\omega} P_{n_k}$ by $\Theta_F(x_1) = \Pi_{k\to\omega} a_{n_k}$ and $\Theta_F(x_2) = \Pi_{k\to\omega} c_k^{f_F(k)}$. The third condition in Proposition 4.11 ensures that Θ_F is a sofic representation of $\mathbb{F}_2$. The first condition implies that Θ_F is an expander. The second condition, together with the properties of the family $\mathcal{F}$, guarantees that $\{\Theta_F : F \in \mathcal{F}\}$ are not conjugated. ∎

We have not specified until now, but an expander easily implies being an extreme point.

Proposition 4.13 ([1, Proposition 3.13]). *An expander sofic representation is an extreme point.*

We constructed uncountably many extreme sofic representations, in the same sequence of dimensions, that are not conjugated. All that we have to do is to show that these sofic representations do not have amplifications that are conjugated; otherwise, they still yield the same element in $\mathrm{Sof}(G, P^\omega)$. Getting rid of an amplification is no easy feat, and the problem deserves its own section.

5. Representations with conjugated amplifications

Open Problem 5.1. Let $\Theta_1, \Theta_2 : G \to \Pi_{k\to\omega} P_{n_k}$ be two sofic representations such that $[\Theta_1] = [\Theta_2]$, that is, there exists a sequence $(r_k)_k \in \mathbb{N}^*$ and $p \in \Pi_{k\to\omega} P_{n_k r_k}$ such that $p(\Theta_1 \otimes \mathrm{Id}_{r_k})p^* = \Theta_2 \otimes \mathrm{Id}_{r_k}$. Then there exists $q \in \Pi_{k\to\omega} P_{n_k}$ such that $q\Theta_1 q^* = \Theta_2$.

We suspect that this last statement is true. Let us see some particular examples where this property holds. First of all, a similar statement holds for unitaries.

Theorem 5.2 ([2, Theorem 6.4]). *Let $\Theta_1, \Theta_2 : G \to \Pi_{k\to\omega} \mathcal{U}(n_k)$ be two morphisms such that there exists a sequence $(r_k)_k$ and $u \in \Pi_{k\to\omega} \mathcal{U}(n_k r_k)$ such that $u(\Theta_1 \otimes \mathrm{Id}_{r_k})u^{-1} = \Theta_2 \otimes \mathrm{Id}_{r_k}$. Then there exists $v \in \Pi_{k\to\omega} \mathcal{U}(n_k)$ such that $v\Theta_1 v^{-1} = \Theta_2$.*

Atkinson uses embeddings in type II_1 factors, as opposed to matrix algebras, as we do here. However, the proof and the difficulty of the problem are the same.

The first case we can solve for permutations is the one of a bounded sequence $(r_k)_k$.

Proposition 5.3. *Let $\Theta_1, \Theta_2 : \mathbb{F}_2 \to \Pi_{k\to\omega} P_{n_k}$ be two sofic representations such that there exists an $r \in \mathbb{N}^*$ and $p \in \Pi_{k\to\omega} P_{n_k r}$ such that $p(\Theta_1 \otimes \mathrm{Id}_r)p^* = \Theta_2 \otimes \mathrm{Id}_r$. Then there exists $q \in \Pi_{k\to\omega} P_{n_k}$ such that $q\Theta_1 q^* = \Theta_2$.*

The proof is just a consequence of results in [17]. We do not get into all the details here, as we do not need this result, and it will take some time and space to familiarise the reader with the concepts of that article. In short, an element $p \in \Pi_{k\to\omega} P_{n_k r}$ will generate a DSE of multiplicity r on the Loeb space $\Pi_{k\to\omega} D_{n_k}$. By [17, Theorem 3.8], we can construct $q_\varepsilon \in \Pi_{k\to\omega} P_{n_k}$ such that $d(q_\varepsilon \Theta_1 q_\varepsilon^*, \Theta_2) < \varepsilon$. The proof can be finished with a diagonal argument.

We prove the result in the case of expander sofic representations in order to finish the results in the previous section. The proof is based on the results of [16, Section 5.1]. We first introduce some tools.

Definition 5.4. For $x, y \in M_n(\mathbb{C})$, define *the Hamming distance* on matrices as

$$d_H(x, y) = \frac{1}{n}|\{i : \exists j \ x(i, j) \neq y(i, j)\}|.$$

The formula counts the number of rows that are different in x and y. If $x, y \in P_n$, then this is the usual Hamming distance on a symmetric group. We shall use this extended metric mainly on "pieces of permutations".

Definition 5.5. A matrix $q \in M_n(\mathbb{C})$ is called a *piece of permutation* if q has only 0 and 1 entries and at most one entry of 1 on each row and each column. Alternatively, $q = pa$ where $p \in P_n$, and $a \in D_n$ is a projection.

Proposition 5.6. *Let $x, y \in M_n(\mathbb{C})$ and $p \in P_n$. We still have bi-invariance*

$$d_H(x, y) = d_H(px, py) = d_H(xp, yp).$$

If p is a piece of permutation, then

$$d_H(px, py) \leq d_H(x, y) \quad \textit{and} \quad d_H(xp, yp) \leq d_H(x, y).$$

The main technical part of the proof is contained in the next lemma.

Proposition 5.7. *Let $\varepsilon > 0$, $n, r \in \mathbb{N}^*$ and $x_1, \ldots, x_k, y_1, \ldots, y_k \in P_n$ be such that $\{y_1, \ldots, y_k\}$ is a λ-expander. Assume that there exists a permutation $u \in P_{nr}$ such that $d_H(u(x_t \otimes 1_r), (y_t \otimes 1_r)u) < \varepsilon$ for all $t = 1, \ldots, k$. Accordingly, there is $v \in P_n$ such that $d_H(vx_t, y_t v) < 20k^2\varepsilon/\lambda$ for all $t = 1, \ldots, k$.*

Proof. The key point here is that the term $20k^2\varepsilon/\lambda$ does not contain r. Otherwise, the problem would be solved using Proposition 5.3.

By bi-invariance, we also have $d_H((x_t \otimes 1_r)u^*, u^*(y_t \otimes 1_r)) < \varepsilon$ for all $t = 1, \dots, k$. As $M_{nr}(\mathbb{C}) \simeq M_r(\mathbb{C}) \otimes M_n(\mathbb{C})$, elements in $M_{nr}(\mathbb{C})$ can be viewed as functions from $\{1, 2, \dots, r\}^2$ to $M_n(\mathbb{C})$. Then $[u(x_t \otimes 1_r)](i, j) = u(i, j)x_t$ and $[(y_t \otimes 1_r)y](i, j) = y_t u(i, j)$. Also $u(i, j)$ is a piece of permutation for any i, j. If $A, B \in P_{nr}$, then

$$2d_H(A, B) \geq \frac{1}{r} \sum_{i,j=1}^{r} d_H(A(i, j), B(i, j)).$$

Let $\varepsilon_{i,j}^t = d_H(u(i, j)x_t, y_t u(i, j))$ and $\delta_{i,j}^t = d_H(x_t u^*(i, j), u^*(i, j)y_t)$. Then, for all $t = 1, \dots, k$, we have

$$\frac{1}{r} \sum_{i,j=1}^{r} \varepsilon_{i,j}^t \leq 2d_H(u(x_t \otimes 1_r), (y_t \otimes 1_r)u) < 2\varepsilon,$$
$$\frac{1}{r} \sum_{i,j=1}^{r} \delta_{i,j}^t \leq 2d_H((x_t \otimes 1_r)u^*, u^*(y_t \otimes 1_r)) < 2\varepsilon.$$

From these inequalities, we can deduce the existence of an $i \in \{1, 2, \dots, r\}$ such that

$$\sum_{j=1}^{r} \varepsilon_{i,j}^t < 4k\varepsilon, \quad \sum_{j=1}^{r} \delta_{j,i}^t < 4k\varepsilon$$

for all $t = 1, \dots, k$. From now on, i is fixed with this property. For $t = 1, \dots, k$, we have

$$\begin{aligned} d_H(&u(i, j)u^*(j, i)y_t, y_t u(i, j)u^*(j, i)) \\ &\leq d_H(u(i, j)u^*(j, i)y_t, u(i, j)x_t u^*(j, i)) \\ &\quad + d_H(u(i, j)x_t u^*(j, i), y_t u(i, j)u^*(j, i)) \\ &\leq d_H(u^*(j, i)y_t, x_t u^*(j, i)) + d_H(u(i, j)x_t, y_t u(i, j)) = \delta_{j,i}^t + \varepsilon_{i,j}^t. \end{aligned}$$

For $j = 1, \dots, r$, let $p_j = u(i, j)u^*(j, i)$. As $u(i, j)$ is a piece of permutation, p_j is a projection in D_n. Moreover, $\sum_{j=1}^{r} p_j = \mathrm{Id}_n$, so these are disjoint projections. Also

$$d_H(p_j, y_t p_j y_t^*) = d_H(p_j y_t, y_t p_j) \leq \delta_{j,i}^t + \varepsilon_{i,j}^t.$$

For $S \subset \{1, 2, \dots, r\}$, define $p_S = \sum_{j \in S} p_j$. Notice that $d_H(p_S, y_t p_S y_t^*) \leq \sum_{j \in S} d_H(p_j, y_t p_j y_t^*)$, and by the above inequalities, we get, for any subset S and for any t,

$$d_H(p_S, y_t p_S y_t^*) \leq \sum_{j \in S} (\delta_{j,i}^t + \varepsilon_{i,j}^t) < 8k\varepsilon.$$

By the expander hypothesis, we have $\lambda \mathrm{Tr}(p_S) \leq \sum_{t=1}^{k} d_H(p_S, y_t p_S y_t^*)$. As such, if $\mathrm{Tr}(p_S) \leq 1/2$, then actually $\mathrm{Tr}(p_S) \leq 8k^2\varepsilon/\lambda$. As $\sum_{i=1}^{r} p_j = \mathrm{Id}_n$, it is easy to see that there exists j such that $\mathrm{Tr}(p_j) \geq 1/2$. Let $S = \{1, 2, \ldots, r\} \setminus j$. Then $\mathrm{Tr}(p_S) < 8k^2\varepsilon/\lambda$, so $\mathrm{Tr}(p_j) \geq 1 - 8k^2\varepsilon/\lambda$. This means that $u(i, j)$ is "almost" a permutation matrix.

Let $v \in P_n$ be such that $d_H(v, u(i, j)) < 8k^2\varepsilon/\lambda$. For all $t = 1, \ldots, r$, we get

$$\begin{aligned} d_H(vx_t, y_t v) &\leq d_H(vx_t, u(i, j)x_t) + d_H(u(i, j)x_t, y_t u(i, j)) \\ &\quad + d_H(y_t u(i, j), y_t v) \\ &< 8k^2 \frac{\varepsilon}{\lambda} + 4k\varepsilon + 8k^2 \frac{\varepsilon}{\lambda} < 20k^2 \frac{\varepsilon}{\lambda}. \end{aligned}$$

■

Theorem 5.8. *Let $\Theta_1, \Theta_2 : \mathbb{F}_2 \to \Pi_{k\to\omega} P_{n_k}$ be two sofic representations such that $[\Theta_1] = [\Theta_2]$, that is, there exists a sequence $(r_k)_k \in \mathbb{N}^*$ and $p \in \Pi_{k\to\omega} P_{n_k r_k}$ such that $p(\Theta_1 \otimes \mathrm{Id}_{r_k})p^* = \Theta_2 \otimes \mathrm{Id}_{r_k}$. Assume that Θ_1 is an expander. Then there exists $q \in \Pi_{k\to\omega} P_{n_k}$ such that $q\Theta_1 q^* = \Theta_2$.*

Proof. Let $\Theta_1(\gamma_i) = \Pi_{k\to\omega} x_i^k$, $i = 1, 2$ be such that (x_1^k, x_2^k) is a λ-expander. Let permutations $y_i^k \in P_{n_k}$ such that $\Theta_2(\gamma_i) = \Pi_{k\to\omega} y_i^k$. Also choose p_k such that $p = \Pi_{k\to\omega} p_k$. Apply the previous proposition to $y_1^k, y_2^k, x_1^k, x_2^k$ and p_k in order to get $q_k \in P_{n_k}$ with the stated property. Then $\Pi_{k\to\omega} q_k$ conjugates Θ_1 to Θ_2. ■

We managed to prove this result because an expander representation cannot be cut into pieces. As such, inside the conjugating $p \in P_{nr}$, we found our $q \in P_n$ as one of the r^2 squares.

Combining Theorem 4.12, Proposition 4.13 and Theorem 5.8, we now have uncountably many extreme sofic representations of $\mathbb{F}_2$. The result easily extends to $\mathbb{F}_r$.

6. A sofic representation with many cuts and trivial commutant

In this section, we construct a sofic representation of $\mathbb{F}_2$ that has trivial commutant in $\Pi_{k\to\omega} P_{n_k}$ but has uncountably many non-conjugated cuts. This suggests that this sofic representation has uncountably many, non-isomorphic cuts that are extreme points.

6.1. Trivial commutant

Sofic representations with trivial commutant in permutations were first constructed by Arzhantseva and the second author in [1, Theorem 3.2]. For the scope of the present paper, we have to rework the proof in order to get a better convergence for the counting argument used. In [1], it is shown that two permutations in $\mathrm{Sym}(n)$ randomly chosen have trivial commutant with respect to the Hamming distance, with probability converging to 1 as $n \to \infty$. The takeaway from this subsection is that the rate of this convergence is exponential. Recall that $a \in P_n$ is the matrix corresponding to the maximal cycle $i \to i + 1$. We already used the estimation of the number of permutations commuting with such a cycle.

Proposition 6.1 ([16, Proposition 5.13]). *Let $\varepsilon > 0$. The number of permutations $y \in P_n$ such that $d_H(ay, ya) < \varepsilon$ is less than $n^{\lfloor n\varepsilon \rfloor + 1}$.*

This time we also need an estimate for the number of permutations commuting with an arbitrary element.

Proposition 6.2 ([1, Proposition 3.5]). *Let $b \in P_n$ be such that $d_H(b, 1_n) > 11\delta$. The number of permutations $c \in P_n$ such that $d_H(bc, cb) < \delta$ is less than $n!/n^{4n\delta}$ for large enough n.*

Proof. Define $C = \{c \in P_n : d_H(bc, cb) < \delta\}$. Choose $c \in C$. Consider the following subsets of $\{1, \dots, n\}$: $A_c = \{i : bc(i) = cb(i)\}$ and $B = \{i : b(i) \neq i\}$. Then $|A_c| > (1-\delta)n$ and $|B| > 11\delta n$. It follows that $|A_c \cap B| > (11\delta - \delta) \cdot n = 10\delta \cdot n$.

Let $i \in A_c \cap B$. Then $c(b(i)) = bc(i)$ and $b(i) \neq i$. Hence, once the value of $c(i)$ is fixed, the value of c on $b(i)$ must be $bc(i)$. Unfortunately, the set A_c depends on c. This makes the counting argument a little more involved.

Let us recall how to count the number of permutations $p \in P_n$: $p(1)$ can take any of the n values in the set $\{1, \dots, n\}$; $p(2)$ can take any of the remaining $n-1$ values and so on. Hence, the cardinality of P_n is $n!$. We adapt this argument to count the number of permutations c with the required properties. Without loss of generality, we can assume that $B = \{1, 2, \dots, |B|\}$. As before $c(1)$ can take n values. If $1 \in A_c$, information that we do not have at the moment, then $c(b(1))$ is also set. Thus the following value of c to be decided ($c(2)$ if $b(1) \neq 2$, and $c(3)$ otherwise) has only $n-2$ options. If $1 \notin A_c$, we continue our enumeration of elements in B till $|B|$. In the worst scenario, the first $\delta \cdot n$ elements of B will not be in A_c. After this, all remaining elements of B are also bound to be in A_c.

Thus, denoting by $t = \lfloor \delta n \rfloor$ and $s = \lfloor (11\delta - \delta)n/2 \rfloor = \lfloor 5\delta \cdot n \rfloor$, our estimation for the maximal number of elements in C is

$$\underbrace{n(n-1)\cdots(n-t+1)}_{t \text{ terms}}\underbrace{(n-t)(n-t-2)\cdots(n-t-2s+2)}_{s \text{ terms}}(n-t-2s) \cdot (n-t-2s-1)\cdots 1.$$

Hence,

$$|C| < \frac{n!}{(n-t-2s+1)^s} < \frac{n!}{[(1-11\delta)n]^{5\delta n - 1}}.$$

We only need to show that $[(1-11\delta)n]^{5\delta n-1} > n^{4n\delta}$. Using the logarithm, this is equivalent to

$$(5\delta n - 1)\ln[(1-11\delta)n] > 4n\delta \cdot \ln n.$$

We factor the two terms and compute the limit via L'Hospital's rule:

$$\lim_{n\to\infty} \frac{(5\delta n - 1)\ln[(1-11\delta)n]}{4n\delta \cdot \ln n} = \lim_n \frac{5\delta \cdot \ln[(1-11\delta)n] + (5\delta n - 1)\cdot 1/n}{4\delta \cdot \ln n + 4n\delta \cdot 1/n}$$
$$= \lim_n \frac{5\delta \cdot \ln[(1-11\delta)n]}{4\delta \cdot \ln n} = \frac{5}{4} > 1.$$ ∎

We continue our counting argument by introducing two sets of n-cycles with specific properties. Given $\delta > 0$, we define

$$L_n^\delta = \{c \in P_n : \nexists b \in P_n \text{ with } d_H(b, 1_n) > 11\delta, d_H(ab, ba) < \delta \text{ and } d_H(cb, bc) < \delta\}.$$

Proposition 6.3 ([1, Proposition 3.6]). *For a fixed* $\delta > 0$ *and large enough* $n \in \mathbb{N}$,

$$\text{Card } L_n^\delta > (1 - n^{-2\delta n}) \cdot (n!).$$

Proof. According to Proposition 4.8, there are at most $n^{n\delta+1}$ permutations $b \in P_n$ such that $d_H(ab, ba) < \delta$. By Proposition 6.2, for each of those permutations b with $d_H(b, 1_n) > 11\delta$, there are at most $n! \cdot n^{-4n\delta}$ cycles c such that $d_H(cb, bc) < \delta$. All in all, the complement of L_n^δ has a cardinality less than $n^{n\delta+1} \cdot n! \cdot n^{-4n\delta} < n^{-2\delta n} \cdot n!$. The conclusion hence follows. ∎

The set L_n^δ cannot be used directly to construct the required sofic representation. This is because $d_H(b, 1_n)$ is in some sense a moving target, while in the definition of L_n^δ it is supposed to be fixed. This is why we introduce the following set:

$$K_n^\delta = \{c \in P_n : \forall b \in P_n,\ d_H(b, 1_n) \leq 22 \cdot \max\{d_H(ab, ba), d_H(bc, cb), \delta\}.\}$$

Proposition 6.4 ([1, Proposition 3.7]). *For a fixed* $1 > \delta > 0$ *and a large enough* $n \in \mathbb{N}$,

$$\text{Card } K_n^\delta > \left(1 - \frac{1}{n^{\delta n}}\right)[n!]$$

Proof. The proof is almost over when we notice that $K_n^\delta \supseteq L_n^\delta \cap L_n^{2\delta} \cap \cdots \cap L_n^{2^k\delta}$, where k is minimal with the property that $2^{k+2}\delta > 1$. So let $c \in L_n^\delta \cap L_n^{2\delta} \cap \cdots \cap L_n^{2^k\delta}$ and take $b \in P_n$. Denote by $\lambda = \max\{d_H(ab, ba), d_H(bc, cb)\}$. If $\lambda < \delta$, then as $c \in L_n^\delta$, $d_H(b, 1_n) \leq 11\delta$.

Assume that $\lambda \geq \delta$. Then, there exists $i > 0$ such that $2^{i-1}\delta \leq \lambda < 2^i\delta$. If $i \leq k$, then $c \in L_n^{2^i\delta}$, so $d_H(b, 1_n) \leq 11 \cdot 2^i\delta \leq 22\lambda$. If $i > k$, then $8\lambda > 1$. This proves $c \in K_n^\delta$.

By Proposition 6.3 and using De Morgan's formula $\bigcap_{j=0}^k L_n^{2^j\delta} = \overline{\bigcup_{j=0}^k \overline{L_n^{2^j\delta}}}$, we obtain that

$$|L_n^\delta \cap L_n^{2\delta} \cap \cdots \cap L_n^{2^k\delta}| > \left(1 - \frac{k+1}{n^{2\delta n}}\right)[n!].$$

As k is fixed, depending only on δ, for large enough n, $|K_n^\delta| > (1 - 1/n^{\delta n})[n!]$. ∎

6.2. Permutations with small Coxeter length

In the previous subsection, we saw that most permutations are in the set K_n^δ that is needed to ensure a trivial commutant in permutations. Here we show that sufficiently many permutations can be cut into many pieces. The main tool is the *Coxeter length*.

We first describe what it means for a sofic representation to have many cuts. If an element $p \in \Pi_{k\to\omega} P_{n_k}$ commutes with all $\Pi_{k\to\omega} D_{n_k}$, then p has to be the identity. This is because $\Pi_{k\to\omega} D_{n_k}$ is a MASA in $\Pi_{k\to\omega} M_{n_k}$. Thus we only ask for commutativity on a separable subalgebra. Let (X,μ) be the unit interval endowed with the Lebesgue measure. There is a canonical measure-preserving map, called *the standard part*, from the Loeb space to (X,μ). This induces an embedding $\mathrm{St}^*(L^\infty(X,\mu)) \subset \Pi_{k\to\omega} D_{n_k}$, so $\mathrm{St}^*(L^\infty(X,\mu))$ is a canonical separable von Neumann subalgebra in $\Pi_{k\to\omega} D_{n_k}$. For more information, the reader should check [8, Section 1]. We do not need these details; we only need to understand when an element of $\Pi_{k\to\omega} P_{n_k}$ commutes with $\mathrm{St}^*(L^\infty(X,\mu))$. Here is where the Coxeter length comes in.

Definition 6.5. For $p \in P_n$, the *Coxeter length* is defined as

$$\ell_C(p) = \frac{2}{n(n-1)}\mathrm{Card}\{i < j : p(i) > p(j)\}.$$

This is the normalised number of inversions of the permutation p.

Example 6.6. For the n-cycle, $\ell_C(a_n) = 2/n(n-1)\cdot(n-1) = 2/n$. If $p(i) = n+1-i$, then $\ell_C(p) = 1$, and this is the maximum value.

The Coxeter function can be defined on elements of the universal sofic group as an ultralimit. The only problem is that in this ultralimit, it becomes just a semi-length. For details, consult [8, Section 4]. For our discussion, the Coxeter semi-length is relevant because of the following.

Proposition 6.7. *An element $p \in \Pi_{k\to\omega} P_{n_k}$ commutes with $\mathrm{St}^*(L^\infty(X,\mu))$ if and only if $\ell_C(p) = 0$.*

This is mostly [8, Proposition 4.4], but one also has to check the definitions from Section 2 of the same article. Going back to our problem, we show that there are "enough" permutations with small Coxeter length.

Define the following set:

$$T_n^\delta = \Big\{ p \in P_n : \ell_C(p) < 2\delta \text{ and } d_H(w(a_n,p),\mathrm{Id}_n) > 1-\delta \\ \text{for every } w \neq 1_{\mathbb{F}_2} \text{ of length at most } \frac{1}{\delta} \Big\}.$$

A permutation p is in T_n^δ if it has small Coxeter length, and together with a_n, it provides a δ-sofic approximation of $\mathbb{F}_2$ on a ball of radius $\frac{1}{\delta}$.

Proposition 6.8. *For a fixed $1 > \delta > 0$ and a large enough $n \in \mathbb{N}$,* $\mathrm{Card}\, T_n^\delta > \delta^n \cdot n!$.

Proof. Let $k = \lfloor 1/\delta \rfloor$ and assume that $n = k \cdot m$. It might not be the case that $k \mid n$, but k is fixed, so the error will not affect our computations for large enough n. We consider only elements that permute the first m points, the next m points and so on. Any such permutation has total displacement less than 2δ:

$$\ell_C(p) \le \frac{2}{n(n-1)} \cdot \frac{m(m-1)}{2} \cdot k = \frac{m-1}{mk-1} < \frac{1}{k} \le 2\delta.$$

We construct permutations in T_n^δ by choosing k permutations $q \in P_m$ such that for each one of them $d_H(w(a_m, q), \mathrm{Id}_m) > 1 - \delta$ for every $w \neq 1_{\mathbb{F}_2}$ of length at most $1/\delta$. According to Proposition 4.6, and as the ball of radius $1/\delta$ in $\mathbb{F}_2$ is finite, for any $\varepsilon > 0$, for large enough m, there are at least $(1-\varepsilon) \cdot m!$ permutations in P_m satisfying this property.

We proved that Card $T_n^\delta > [(1-\varepsilon) \cdot m!]^k$. We only need to prove that this value is larger than $\delta^n \cdot n!$, for large enough n. Recall Stirling's approximation: $\sqrt{2\pi} \cdot (n/e)^n \cdot n \le n! \le e \cdot (n/e)^n \cdot n$. Setting $c = \sqrt{2\pi}$, we have

$$[(1-\varepsilon)(m!)]^k \ge c^k (1-\varepsilon)^k \cdot \left(\frac{m}{e}\right)^{mk} \cdot m^k = c^k (1-\varepsilon)^k \left(\frac{1}{k}\right)^n \left(\frac{n}{e}\right)^n \cdot m^k$$
$$> c^k (1-\varepsilon)^k \delta^n \left(\frac{n}{e}\right)^n \cdot n \ge c^k \frac{(1-\varepsilon)^k}{e} \cdot \delta^n \cdot n!$$

We can choose $\varepsilon > 0$ small enough to guarantee that $c^k(1-\varepsilon)^k/e > 1$. ∎

Putting everything together, we get the following.

Proposition 6.9. *For any $\delta > 0$, for large enough n, $K_n^\delta \cap T_n^\delta \neq \emptyset$.*

Proof. We need to show that in the limit, $\delta^n \cdot n! > n^{-\delta n} \cdot n!$. This is equivalent to $\delta \cdot n^\delta > 1$. ∎

6.3. A sofic representation away from extreme points

Theorem 6.10. *There exists a sofic representation $\Theta : \mathbb{F}_2 \to \Pi_{k\to\omega} P_{n_k}$ such that $\Theta(\mathbb{F}_2)' \cap \Pi_{k\to\omega} P_{n_k} = \{\mathrm{Id}\}$ and $\mathrm{St}^*(L^\infty(X,\mu)) \subset \Theta(\mathbb{F}_2)'$.*

Proof. Let $(\delta_k)_k \in \mathbb{R}_+^*$ be a decreasing sequence converging to 0. By Proposition 6.9, applied to each δ_k, we construct a sequence of permutations $p_k \in P_{n_k}$ such that $p_k \in K_{n_k}^\delta \cap T_{n_k}^\delta$. Set $\Theta(x_1) = \Pi_{k\to\omega} a_{n_k}$ and $\Theta(x_2) = \Pi_{k\to\omega} p_k$. Then, by Example 6.6, $\ell_C(\Theta(x_1)) = \lim_{k\to\omega} \frac{2}{n_k} = 0$ and $\ell_C(\Theta(x_2)) = \lim_{k\to\omega} \ell_C(p_k) = 0$. This shows that $\mathrm{St}^*(L^\infty(X,\mu)) \subset \Theta(\mathbb{F}_2)'$.

Suppose $b = \Pi_{k\to\omega} b_k \in \Pi_{k\to\omega} P_{n_k}$ is in the commutant of Θ. This means that $\lim_{k\to\omega} d_H(a_{n_k} b_k, b_k a_{n_k}) = 0$ and $\lim_{k\to\omega} d_H(p_k b_k, b_k p_k) = 0$. Let $\varepsilon > 0$ and take $F \in \omega$ such that for all $k \in F$, $\delta_k < \varepsilon$ and $d_H(a_{n_k} b_k, b_k a_{n_k}) < \varepsilon$, $d_H(p_k b_k, b_k p_k) < \varepsilon$. As $p_k \in K_{n_k}^{\delta_k}$, we get

$$d_H(b_k, \mathrm{Id}_{n_k}) \le 22 \cdot \max\{d_H(a_{n_k} b_k, b_k a_{n_k}), d_H(p_k b_k, b_k p_k), \delta_k\} < 22\varepsilon.$$

As such, $d_H(b, \mathrm{Id}) \leq 22\varepsilon$. As $\varepsilon > 0$ is arbitrary, $b = \mathrm{Id}$. ∎

The reason that we believe Θ is not in the closure of the convex hull of extreme points in $\mathrm{Sof}(\mathbb{F}_2, P^\omega)$ is that for each A, a measurable subset of the unit interval, we can define Θ_A a cut of Θ. The fact that Θ has no commutant in $\Pi_{k\to\omega} P_{n_k}$ suggests that these cuts are different as elements in $\mathrm{Sof}(\mathbb{F}_2, P^\omega)$. A first step in proving this statement is Open Problem 5.1. We end with another open problem.

Open Problem 6.11. Let $\Theta : G \to \Pi_{k\to\omega} P_{n_k}$ be a sofic representation in the closure of the convex hull of extreme points in $\mathrm{Sof}(G, P^\omega)$. Then there are at most countably many different extreme sofic representations that are obtained as cuts of Θ.

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