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Laplacian vanishing theorem for a quantized singular Liouville equation

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Abstract. In this article we establish a vanishing theorem for a singular Liouville equation with a quantized singular source. If a blowup sequence tends to infinity near the quantized singular source and the blowup solutions violate the spherical Harnack inequality around the singular source (non-simple blowups), the Laplacian of the coefficient function must tend to zero. This seems to be the first second order estimates for a Liouville equation with a quantized source and non-simple blowups. This result as well as the key ideas of the proof will be useful for various applications.

Keywords: Laplace, vanishing, non-simple blowup, Liouville equation, mean field equation, blowup solutions, singular source, Harnack inequality, classification theorem, linearized equation.

1. Introduction

This is the third article in our series studying blowup solutions of

$$\Delta u + |x|^{2N}\mathrm{H}(x)e^{u} = 0 \tag{1.1}$$

in a neighborhood of the origin in $\mathbb{R}^2$. Here H is a positive smooth function and N is a positive integer. Since the analysis is local in nature we focus the discussion on a neighborhood of the origin: Let $\mathfrak{u}_k$ be a sequence of solutions of

$$\Delta \mathfrak{u}_k(x) + |x|^{2N}\mathrm{H}_k(x)e^{\mathfrak{u}_k} = 0 \quad \text{in } B_\tau \tag{1.2}$$

for some $\tau > 0$ independent of k, where B_τ is the ball centered at the origin with radius τ. In addition we postulate the usual assumptions on $\mathfrak{u}_k$ and H_k: For a positive constant C

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independent of k, the following holds:

$$\begin{cases} \|\mathrm{H}_k\|_{C^3(\bar{B}_\tau)} \le C, \quad 1/C \le \mathrm{H}_k(x) \le C, \quad x \in \bar{B}_\tau, \\ \int_{B_\tau} \mathrm{H}_k e^{\mathfrak{u}_k} \le C, \\ |\mathfrak{u}_k(x) - \mathfrak{u}_k(y)| \le C, \quad \forall x, y \in \partial B_\tau, \end{cases} \tag{1.3}$$

and since we study the asymptotic behavior of blowup solutions around the singular source, we assume that there is no blowup point except at the origin:

$$\max_{K \subset\subset B_\tau \setminus \{0\}} \mathfrak{u}_k \le C(K). \tag{1.4}$$

If a sequence $\{u^k\}_{k=1}^\infty$ of solutions of (1.1) satisfies

$$\lim_{k\to\infty} u^k(x_k) = \infty \quad \text{for some } \bar{x} \in B_\tau \text{ and } x_k \to \bar{x},$$

we say $\{u^k\}$ is a sequence of *bubbling solutions* or *blowup solutions*, and $\bar{x}$ is called a *blowup point*. The question we consider in this work is: when 0 is the only blowup point in a neighborhood of the origin, what vanishing theorems will the coefficient functions H_k satisfy?

One indispensable assumption is that the blowup solutions violate the spherical Harnack inequality around the origin:

$$\max_{x \in B_\tau} [\mathfrak{u}_k(x) + 2(1+N)\log|x|] \to \infty \quad \text{as } k \to \infty. \tag{1.5}$$

It is also mentioned in the literature (see [20,26]) that 0 is then called a *non-simple* blowup point. The main result of this article is the following.

Theorem 1.1. *Let $\{\mathfrak{u}_k\}$ be a sequence of solutions of* (1.2) *such that* (1.3)–(1.4) *hold and the spherical Harnack inequality is violated as in* (1.5)*. Then along a subsequence,*

$$\lim_{k\to\infty} \Delta(\log \mathrm{H}_k)(0) = 0.$$

Theorem 1.1 is a continuation of our previous result in [27]:

Theorem A. *Let $\{\mathfrak{u}_k\}$ be a sequence of solutions of* (1.2) *such that* (1.3)–(1.5) *hold. Then along a subsequence,*

$$\lim_{k\to\infty} \nabla(\log \mathrm{H}_k + \phi_k)(0) = 0$$

where ϕ_k is defined as

$$\begin{cases} \Delta\phi_k(x) = 0 \quad \text{in } B_\tau, \\ \phi_k(x) = \mathfrak{u}_k(x) - \dfrac{1}{2\pi\tau}\displaystyle\int_{\partial B_\tau} \mathfrak{u}_k \, dS, \quad x \in \partial B_\tau. \end{cases} \tag{1.6}$$

Equation (1.1) comes from its equivalent form

$$\Delta v + \mathrm{H}e^v = 4\pi N\delta_0$$

by using a logarithmic function to eliminate the Dirac mass on the right hand side. Since the strength of the Dirac mass is a multiple of 4π, this type of singularity is called "quantized". Equations with a quantized singular source are ubiquitous in the literature. In particular, the following mean field equation defined on a Riemann surface (M, g):

$$\Delta_g u + \rho\left(\frac{h(x)e^{u(x)}}{\int_M he^u} - 1\right) = 4\pi \sum_{t=1}^{M} \alpha_t(\delta_{p_t} - 1), \tag{1.7}$$

represents a conformal metric with prescribed conic singularities (see [15,24,25]), where h is a positive smooth function, $\rho > 0$ is a constant and the volume of M is assumed to be 1 for convenience, and the $\alpha_j > -1$ are constants as well. If the singular source is quantized, the equation is profoundly linked to algebraic geometry, integrable systems, number theory and complex Monge–Ampère equations (see [12]). In physics the main equation (1.1) reveals key features of mean field limits of point vortices in the Euler flow [8,9] and models in Chern–Simons–Higgs theory [19] and electroweak theory [2], etc.

So far the non-simple bubbling situation has been observed in Liouville equations [6,20], Liouville systems [17,18,28] and fourth order equations [1]. The main theorem in this article should impact the study of these equations as well as some well known open questions in Monge–Ampère equations [26].

When compared with Theorem A, Theorem 1.1 is clearly more challenging. In fact, the proof of Theorem A is a special case of one step of the proof of Theorem 1.1. However, their major difference is in applications. Theorem 1.1 is significantly more influential for many reasons: First the main motivation to study equation (1.1) is for equations or systems defined on manifolds. Usually blowup analysis near a singular point needs to reflect the curvature at the blowup point. In this respect Theorem 1.1 is directly related to the Gauss curvature at the blowup point. Second, the harmonic function in Theorem A causes inconvenience in applications since it is generally hard to identify what the harmonic function is. On the other hand, Theorem 1.1 is only involved with the Laplacian of the coefficient function. This may lead to substantial advances in applications: In many degree counting problems a major difficulty is bubble-coalition, which means bubbling disks may collide into one point. The formation of bubbling disks tending to one point is accurately represented by (1.1). Theorem 1.1 and its proof may be useful to simplify blowup pictures. Third, the proof of Theorem 1.1 is also important for proving uniqueness of bubbling solutions, and the results for Liouville equations with quantized singular sources are inspirational for many equations and systems with similar singular poles. Before our series of works most of the study of singular equations or systems focused on non-quantized singular situations. However, it is the "quantized situations" that manifest profound connections to different fields of mathematics and physics. Theorem 1.1 may be a starting point of multiple directions of exciting adventures.

As a first application of Theorem 1.1 we present an advancement for the mean field equation (1.7). Let Λ be defined as

$$\Lambda = \left\{8\pi k + \sum_{j\in A} 8\pi(1+\alpha_j);\ k \in \mathbb{N}\cup\{0\},\ A \subset \{1,\dots,M\}\right\}$$

where $\mathbb{N} = \{1,2,\dots\}$. By the work of Bartolucci–Tarantello [4,5], Chen–Lin [12] etc., an a priori estimate holds if $\rho \notin \Lambda$. In other words, if u^k is a sequence of blowup solutions with parameters ρ^k, the limit of ρ^k is in Λ. Our second main theorem is the following.

Theorem 1.2. *Let u_k be a sequence of blowup solutions of (1.7) with parameters $\rho^k \to \rho \in \Lambda$, where h is a positive smooth function, and $\alpha_1,\dots,\alpha_M > -1$ are constants. If at each quantized blowup point p we have*

$$\Delta \log h(p) - 2K(p) - 4\pi \sum_{t=1}^{M} \alpha_t + \rho \neq 0, \tag{1.8}$$

where $K(p)$ is the Gauss curvature at p, then all blowup points of u_k are simple.

The organization of the article is as follows. In Section 2 we cite preliminary results related to the proof of the main theorem. Then in Section 3 we approximate the blowup solutions by a family of global solutions that agree with the blowup solutions at one local maximum point. This is crucial for our argument. Then we derive some intermediate estimates as preparation for a more precise analysis. In Section 4 we prove first order estimates that cover the main result in [27]. This section proves a stronger result than in [27] and provides more details. Finally, in Section 5 we take advantage of the first order estimates and complete the proof of Theorem 1.1. The proof of Theorem 1.2 is given in Section 6. The final section is an appendix that contains certain computations needed in the proof of Theorem 1.1.

Notation. We will use $B(x_0, r)$ to denote a ball centered at x_0 with radius r. If x_0 is the origin we use B_r. Also C represents a positive constant that may change from place to place.

2. Preliminary discussions

In the first stage of the proof of Theorem 1.1 we set up some notations and cite some preliminary results. Set

$$u_k(x) = \mathrm{u}_k(x) - \phi_k(x), \tag{2.1}$$

$$h_k(x) = \mathrm{H}_k(x)e^{\phi_k(x)}. \tag{2.2}$$

Then the equation for u_k is

$$\Delta u_k(x) + |x|^{2N} h_k(x) e^{u_k} = 0 \quad \text{in } B_\tau. \tag{2.3}$$

Without loss of generality we assume

$$\lim_{k\to\infty} h_k(0) = 1. \tag{2.4}$$

Obviously (1.5) is equivalent to

$$\max_{x\in B_\tau}[u_k(x) + 2(1+N)\log|x|] \to \infty \quad \text{as } k \to \infty, \tag{2.5}$$

It is well known [6,20] that u_k exhibits a non-simple blowup profile. It is established in [6,20] that there are $N+1$ local maximum points of u_k: $p_0^k,\dots,p_N^k$, and they are evenly distributed on $\mathbb{S}^1$ after scaling according to their magnitude: if along a subsequence,

$$\lim_{k\to\infty} p_0^k/|p_0^k| = e^{i\theta_0},$$

then

$$\lim_{k\to\infty} \frac{p_l^k}{|p_0^k|} = e^{i(\theta_0+\frac{2\pi l}{N+1})}, \quad l = 1,\dots,N.$$

For many reasons it is convenient to denote

$$\delta_k = |p_0^k| \quad \text{and} \quad \mu_k = u_k(p_0^k) + 2(1+N)\log\delta_k. \tag{2.6}$$

Also we use

$$\varepsilon_k = e^{-\frac{1}{2}\mu_k}$$

as the scaling factor most of the time. Since p_l^k's are evenly distributed around ∂B_{δ_k}, standard results for Liouville equations around a regular blowup point can be applied to get $u_k(p_l^k) = u_k(p_0^k) + o(1)$. Also, (1.5) gives $\mu_k \to \infty$. The interested readers may look into [6,20] for more detailed information.

Finally, we shall use E to denote a frequently appearing error term of the size $O(\delta_k^2) + O(\mu_k e^{-\mu_k})$.

3. Approximating bubbling solutions by global solutions

We write $p_0^k = \delta_k e^{i\theta_k}$ and define

$$v_k(y) = u_k(\delta_k y e^{i\theta_k}) + 2(N+1)\log\delta_k, \quad |y| < \tau\delta_k^{-1}. \tag{3.1}$$

If we write out each component, (3.1) is

$$v_k(y_1,y_2) = u_k\big(\delta_k(y_1\cos\theta_k - y_2\sin\theta_k), \delta_k(y_1\sin\theta_k + y_2\cos\theta_k)\big) \\ + 2(1+N)\log\delta_k.$$

It is standard to verify that v_k solves

$$\Delta v_k(y) + |y|^{2N}\mathfrak{h}_k(\delta_k y)e^{v_k(y)} = 0, \quad |y| < \tau/\delta_k, \tag{3.2}$$

where

$$\mathfrak{h}_k(x) = h_k(xe^{i\theta_k}), \quad |x| < \tau. \tag{3.3}$$

Thus the image of p_0^k after scaling is $Q_1^k = e_1 = (1,0)$. Let $Q_1^k, \dots, Q_N^k$ be the images of p_i^k $(i = 1, \dots, N)$ after scaling:

$$Q_l^k = \frac{p_l^k}{\delta_k} e^{-i\theta_k}, \quad l = 1, \dots, N.$$

It was established by Kuo–Lin [20] and independently by Bartolucci–Tarantello [6] that

$$\lim_{k\to\infty} Q_l^k = \lim_{k\to\infty} p_l^k/\delta_k = e^{\frac{2l\pi i}{N+1}}, \quad l = 0, \dots, N. \tag{3.4}$$

Then it was proved in our previous work [26, (3.13)] that

$$Q_l^k - e^{\frac{2\pi l i}{N+1}} = O(\mu_k e^{-\mu_k}) + O(|\nabla \log \mathfrak{h}_k(0)|\delta_k).$$

Using the rate of $\nabla \mathfrak{h}_k(0)$ in [26] we have

$$Q_l^k - e^{\frac{2\pi l i}{N+1}} = O(\mu_k e^{-\mu_k}) + O(\delta_k^2). \tag{3.5}$$

Choosing $3\varepsilon > 0$ small and independent of k, we can make disks centered at Q_l^k with radius 3ε (denoted as $B(Q_l^k, 3\varepsilon)$) mutually disjoint. Let

$$\mu_k = \max_{B(Q_0^k,\varepsilon)} v_k. \tag{3.6}$$

Since Q_l^k are evenly distributed around ∂B_1, it is easy to use standard estimates for single Liouville equations [11, 16, 30] to obtain

$$\max_{B(Q_l^k,\varepsilon)} v_k = \mu_k + o(1), \quad l = 1, \dots, N.$$

Let

$$V_k(x) = \log \frac{e^{\mu_k}}{\left(1 + \frac{e^{\mu_k}\mathfrak{h}_k(\delta_k e_1)}{8(1+N)^2}|y^{N+1} - e_1|^2\right)^2}. \tag{3.7}$$

Clearly V_k is a solution of

$$\Delta V_k + \mathfrak{h}_k(\delta_k e_1)|y|^{2N} e^{V_k} = 0 \quad \text{in } \mathbb{R}^2, \quad V_k(e_1) = \mu_k. \tag{3.8}$$

This expression is based on the classification theorem of Prajapat–Tarantello [23].

The estimate of $v_k(x) - V_k(x)$ is important for the main theorem of this article. For convenience we use

$$\beta_l = \frac{2\pi l}{N+1}, \quad \text{so} \quad e_1 = e^{i\beta_0} = Q_0^k, \quad e^{i\beta_l} = Q_l^k + E \quad \text{for } l = 1, \dots, N.$$

4. Vanishing of the first derivatives

Our first goal is to prove the following vanishing rate for $\nabla \mathfrak{h}_k(0)$:

Theorem 4.1.

$$\nabla(\log \mathfrak{h}_k)(0) = O(\delta_k \mu_k). \tag{4.1}$$

Proof. Note that we have proved in [26] that

$$\nabla(\log \mathfrak{h}_k)(0) = O(\delta_k^{-1}\mu_k e^{-\mu_k}) + O(\delta_k).$$

If $\delta_k \geq C\varepsilon_k$, there is nothing to prove. So we assume that

$$\delta_k = o(\varepsilon_k). \tag{4.2}$$

By way of contradiction we assume that

$$|\nabla \mathfrak{h}_k(0)|/(\delta_k\mu_k) \to \infty. \tag{4.3}$$

Another observation is that based on (3.5) we have

$$\varepsilon_k^{-1}|Q_l^k - e^{i\beta_l}| \leq C\varepsilon_k^{\varepsilon}, \quad l = 0, \dots, N,$$

for some small $\varepsilon > 0$. Thus ξ_k tends to U after scaling. We need this fact in our argument.

Under the assumption (4.2) we cite Proposition 3.1 of [27]:

Let $l = 0, \dots, N$ and δ be so small that $B(e^{i\beta_l}, \delta) \cap B(e^{i\beta_s}, \delta) = \emptyset$ for $l \neq s$. In each $B(e^{i\beta_l}, \delta)$,

$$|v_k(x) - V_k(x)| \leq \begin{cases} C\mu_k e^{-\mu_k/2}, & |x - e^{i\beta_l}| \leq Ce^{-\mu_k/2}, \\ C\dfrac{\mu_k e^{-\mu_k}}{|x - e^{i\beta_l}|} + O(\mu_k^2 e^{-\mu_k}), & Ce^{-\mu_k/2} \leq |x - e^{i\beta_l}| \leq \delta. \end{cases} \tag{4.4}$$

Remark 4.1. We only need a rescaled version of the proposition above:

$$|v_k(e^{i\beta_l} + \varepsilon_k y) - V_k(e^{i\beta_l} + \varepsilon_k y)| \leq C\varepsilon_k^{\varepsilon}(1 + |y|)^{-1}, \quad 0 < |y| < \tau\varepsilon_k^{-1}, \tag{4.5}$$

for some small constants $\varepsilon, \tau > 0$ both independent of k.

One major step in the proof of Theorem 4.1 is the following estimate:

Proposition 4.1. *Let $w_k = v_k - V_k$. Then*

$$|w_k(y)| \leq C\tilde{\delta}_k, \quad y \in \Omega_k := B(0, \tau\delta_k^{-1}),$$

where $\tilde{\delta}_k = |\nabla \mathfrak{h}_k(0)|\delta_k + \delta_k^2\mu_k$.

Proof. Obviously we can assume that $|\nabla \mathfrak{h}_k(0)|\delta_k > 2\delta_k^2\mu_k$ because otherwise there is nothing to prove. Now we recall that the equation for v_k is (3.2), and v_k is a constant on $\partial B(0, \tau\delta_k^{-1})$. Moreover, $v_k(e_1) = \mu_k$. Recall that V_k defined in (3.7) satisfies

$$\Delta V_k + \mathfrak{h}_k(\delta_k e_1)|y|^{2N}e^{V_k} = 0 \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |y|^{2N}e^{V_k} < \infty,$$

V_k has its local maximums at $e^{i\beta_l}$ for $l = 0, \dots, N$, and $V_k(e_1) = \mu_k$. For $|y| \sim \delta_k^{-1}$,

$$V_k(y) = -\mu_k - 4(N+1)\log \delta_k^{-1} + C + O(\delta_k^{N+1}).$$

Let $\Omega_k = B(0, \tau\delta_k^{-1})$. We shall derive a precise, pointwise estimate of w_k on $B_3 \setminus \bigcup_{l=1}^N B(Q_l^k, \tau)$ where $\tau > 0$ is a small number independent of k. Here we note that among $N + 1$ local maximum points, we already have e_1 as a common local maximum point for both v_k and V_k, and we shall prove that w_k is very small in B_3 if we exclude all bubbling disks except the one around e_1. Before we carry out more specific computations, we emphasize the importance of

$$w_k(e_1) = |\nabla w_k(e_1)| = 0. \tag{4.6}$$

Now we write the equation for w_k as

$$\Delta w_k + \mathfrak{h}_k(\delta_k y)|y|^{2N} e^{\xi_k} w_k = (\mathfrak{h}_k(\delta_k e_1) - \mathfrak{h}_k(\delta_k y))|y|^{2N} e^{V_k} \tag{4.7}$$

in Ω_k, where ξ_k is obtained from the mean value theorem:

$$e^{\xi_k(x)} = \begin{cases} \dfrac{e^{v_k(x)} - e^{V_k(x)}}{v_k(x) - V_k(x)} & \text{if } v_k(x) \neq V_k(x), \\ e^{V_k(x)} & \text{if } v_k(x) = V_k(x). \end{cases}$$

An equivalent form is

$$e^{\xi_k(x)} = \int_0^1 \frac{d}{dt} e^{t v_k(x) + (1-t) V_k(x)}\, dt = e^{V_k(x)} \Big(1 + \frac{1}{2} w_k(x) + O(w_k(x)^2)\Big). \tag{4.8}$$

For convenience we write the equation for w_k as

$$\Delta w_k + \mathfrak{h}_k(\delta_k y)|y|^{2N} e^{\xi_k} w_k = \delta_k \nabla \mathfrak{h}_k(\delta_k e_1) \cdot (e_1 - y)|y|^{2N} e^{V_k} + E_1, \tag{4.9}$$

where

$$E_1 = O(\delta_k^2)|y - e_1|^2 |y|^{2N} e^{V_k}, \quad y \in \Omega_k.$$

Note that the oscillation of w_k on $\partial\Omega_k$ is $O(\delta_k^{N+1})$, which all comes from the oscillation of V_k.

Let $M_k = \max_{y \in \bar{\Omega}_k} |w_k(y)|$. We shall get a contradiction by assuming $M_k/\tilde{\delta}_k \to \infty$. This assumption implies

$$M_k/(\delta_k^2 \mu_k) \to \infty. \tag{4.10}$$

Set

$$\tilde{w}_k(y) = w_k(y)/M_k, \quad y \in \Omega_k.$$

Clearly $\max_{y\in\Omega_k} |\tilde{w}_k(y)| = 1$. The equation for $\tilde{w}_k$ is

$$\Delta \tilde{w}_k(y) + |y|^{2N} \mathfrak{h}_k(\delta_k e_1) e^{\xi_k} \tilde{w}_k(y) = a_k \cdot (e_1 - y)|y|^{2N} e^{V_k} + \tilde{E}_1 \tag{4.11}$$

in Ω_k, where $a_k = \delta_k \nabla \mathfrak{h}_k(0)/M_k \to 0$ and

$$\tilde{E}_1 = o(1)|y - e_1|^2 |y|^{2N} e^{V_k}, \quad y \in \Omega_k. \tag{4.12}$$

Also on the boundary, since $M_k/\tilde{\delta}_k \to \infty$, we have

$$\tilde{w}_k = C + o(1/\mu_k) \quad \text{on } \partial\Omega_k. \tag{4.13}$$

By [27, Proposition 3.1],

$$\xi_k(e_1 + \varepsilon_k z) = V_k(e_1 + \varepsilon_k z) + O(\varepsilon_k^{\varepsilon})(1 + |z|)^{-1}. \tag{4.14}$$

Since V_k is not exactly symmetric around e_1, we shall replace the rescaled version of V_k around e_1 by a radial function. Let U_k be solutions of

$$\Delta U_k + \mathfrak{h}_k(\delta_k e_1)e^{U_k} = 0 \quad \text{in } \mathbb{R}^2, \quad U_k(0) = \max_{\mathbb{R}^2} U_k = 0. \tag{4.15}$$

By the classification theorem of Caffarelli–Gidas–Spruck [7] we have

$$U_k(z) = \log \frac{1}{\left(1 + \frac{\mathfrak{h}_k(\delta_k e_1)}{8}|z|^2\right)^2}$$

and standard refined estimates yield (see [11, 16, 30])

$$V_k(e_1 + \varepsilon_k z) + 2\log \varepsilon_k = U_k(z) + O(\varepsilon_k)|z| + O(\mu_k^2 \varepsilon_k^2). \tag{4.16}$$

Also we observe that

$$\log|e_1 + \varepsilon_k z| = O(\varepsilon_k)|z|. \tag{4.17}$$

Thus, the combination of (4.14), (4.16) and (4.17) gives

$$\begin{aligned} &2N\log|e_1 + \varepsilon_k z| + \xi_k(e_1 + \varepsilon_k z) + 2\log\varepsilon_k - U_k(z) \\ &\qquad = O(\varepsilon_k^{\varepsilon})(1 + |z|), \quad 0 \le |z| < \delta_0 \varepsilon_k^{-1}, \end{aligned} \tag{4.18}$$

for a small $\varepsilon > 0$ independent of k. Since we shall use the rescaled version, based on (4.18) we write

$$\varepsilon_k^2 |e_1 + \varepsilon_k z|^{2N} e^{\xi_k(e_1 + \varepsilon_k z)} = e^{U_k(z)} + O(\varepsilon_k^{\varepsilon})(1 + |z|)^{-3}. \tag{4.19}$$

Here we note that the estimate in (4.18) is not optimal.

In the following we reduce the proof of Proposition 4.1 to a few estimates. First we prove the following.

Lemma 4.1. *For $\delta > 0$ small and independent of k,*

$$\tilde{w}_k(y) = o(1), \quad \nabla\tilde{w}_k = o(1) \quad \text{in } B(e_1, \delta) \setminus B(e_1, \delta/8), \tag{4.20}$$

where $B(e_1, 3\delta)$ does not include other blowup points.

Proof. If (4.20) is not true, without loss of generality $\tilde{w}_k \to c > 0$. This is based on the fact that $\tilde{w}_k$ tends to a global harmonic function with removable singularity, that is, to a constant. Here we assume $c > 0$ but the argument for $c < 0$ is the same. Let

$$W_k(z) = \tilde{w}_k(e_1 + \varepsilon_k z), \quad \varepsilon_k = e^{-\frac{1}{2}\mu_k}, \tag{4.21}$$

and let W denote the limit of W_k. Then

$$\Delta W + e^U W = 0 \quad \text{in } \mathbb{R}^2, \quad |W| \le 1,$$

where U is a solution of $\Delta U + e^U = 0$ in $\mathbb{R}^2$ with $\int_{\mathbb{R}^2} e^U < \infty$. Since 0 is the local maximum of U,

$$U(z) = \log \frac{1}{\left(1 + \frac{1}{8}|z|^2\right)^2}.$$

Here we further claim that $W \equiv 0$ in $\mathbb{R}^2$ because $W(0) = |\nabla W(0)| = 0$, a fact well known based on the classification of the kernel of the linearized operator. Going back to W_k, we have

$$W_k(z) = o(1), \quad |z| \le R_k \text{ for some } R_k \to \infty.$$

Based on the expression of $\tilde{w}_k$, (4.16) and (4.19), we write the equation for W_k as

$$\Delta W_k(z) + \mathfrak{h}_k(\delta_k e_1) e^{U_k(z)} W_k(z) = E_2^k \tag{4.22}$$

for $|z| < \delta_0 \varepsilon_k^{-1}$, where a crude estimate of the error term E_2^k is

$$E_2^k(z) = o(1)\varepsilon_k^{\varepsilon}(1 + |z|)^{-3}.$$

Let

$$g_0^k(r) = \frac{1}{2\pi}\int_0^{2\pi} W_k(r,\theta)\, d\theta. \tag{4.23}$$

Then clearly $g_0^k(r) \to c > 0$ for $r \sim \varepsilon_k^{-1}$. The equation for g_0^k is

$$\begin{aligned}
&\frac{d^2}{dr^2} g_0^k(r) + \frac{1}{r}\frac{d}{dr} g_0^k(r) + \mathfrak{h}_k(\delta_k e_1) e^{U_k(r)} g_0^k(r) = \tilde{E}_0^k(r),\\
&g_0^k(0) = \frac{d}{dr} g_0^k(0) = 0,
\end{aligned}$$

where $\tilde{E}_0^k(r)$ has the same upper bound as $E_2^k(r)$:

$$|\tilde{E}_0^k(r)| \le o(1)\varepsilon_k^{\varepsilon}(1 + r)^{-3}.$$

For the homogeneous equation, the two fundamental solutions are known: g_{01}, g_{02}, where

$$g_{01} = \frac{1 - c_1 r^2}{1 + c_1 r^2}, \quad c_1 = \frac{\mathfrak{h}_k(\delta_k e_1)}{8}.$$

By the standard reduction of order process, $g_{02}(r) = O(\log r)$ for $r > 1$. Then it is easy to find, assuming $|W_k(z)| \le 1$, that

$$\begin{aligned}
|g_0(r)| &\le C|g_{01}(r)| \int_0^r s|\tilde{E}_0^k(s) g_{02}(s)|\, ds + C|g_{02}(r)| \int_0^r s|g_{01}(s)\tilde{E}_0^k(s)|\, ds\\
&\le C\varepsilon_k^{\varepsilon} \log(2 + r), \quad 0 < r < \delta_0 \varepsilon_k^{-1}.
\end{aligned}$$

Clearly this is a contradiction to (4.23). We have proved $c = 0$, which means $\tilde{w}_k = o(1)$ in $B(e_1, \delta_0) \setminus B(e_1, \delta_0/8)$. Then the equation for $\tilde{w}_k$ and the standard Harnack inequality yield $\nabla \tilde{w}_k = o(1)$ in the same region. Lemma 4.1 is established. ∎

The second estimate is a more precise description of $\tilde{w}_k$ around e_1:

Lemma 4.2. *For any given $\sigma \in (0,1)$ there exists $C > 0$ such that*

$$|\tilde{w}_k(e_1 + \varepsilon_k z)| \le C\varepsilon_k^\sigma (1 + |z|)^\sigma, \quad 0 < |z| < \tau\varepsilon_k^{-1}, \tag{4.24}$$

for some $\tau > 0$.

Remark 4.2. Lemma 4.2 is an intermediate estimate for $\tilde{w}_k$. Eventually we shall improve (4.24) to an error with leading term $o(\varepsilon_k)$.

Proof of Lemma 4.2. Let W_k be defined as in (4.21). In order to obtain a better estimate we need to write the equation of W_k more precisely than (4.22):

$$\Delta W_k + \mathfrak{h}_k(\delta_k e_1) e^{\Theta_k} W_k = E_3^k(z), \quad z \in \Omega_{W_k}, \tag{4.25}$$

where Θ_k is defined by

$$e^{\Theta_k(z)} = |e_1 + \varepsilon_k z|^{2N} e^{\xi_k(e_1 + \varepsilon_k z) + 2\log \varepsilon_k},$$

$\Omega_{W_k} = B(0, \tau\varepsilon_k^{-1})$ and $E_3^k(z)$ satisfies

$$E_3^k(z) = O(\varepsilon_k)(1 + |z|)^{-3}, \quad z \in \Omega_{W_k}.$$

Here we observe that by Lemma 4.1, $W_k = o(1)$ on $\partial\Omega_{W_k}$. Let

$$\Lambda_k = \max_{z \in \Omega_{W_k}} \frac{|W_k(z)|}{\varepsilon_k^\sigma (1 + |z|)^\sigma}.$$

If (4.24) does not hold, then $\Lambda_k \to \infty$ and we use z_k to denote where Λ_k is attained. Note that because of the smallness of W_k on $\partial\Omega_{W_k}$, z_k is an interior point. Letting

$$g_k(z) = \frac{W_k(z)}{\Lambda_k (1 + |z_k|)^\sigma \varepsilon_k^\sigma}, \quad z \in \Omega_{W_k},$$

we see immediately that

$$|g_k(z)| = \frac{|W_k(z)|}{\varepsilon_k^\sigma \Lambda_k (1 + |z|)^\sigma} \cdot \frac{(1 + |z|)^\sigma}{(1 + |z_k|)^\sigma} \le \frac{(1 + |z|)^\sigma}{(1 + |z_k|)^\sigma}. \tag{4.26}$$

Note that σ can be as close to 1 as needed. The equation for g_k is

$$\Delta g_k(z) + \mathfrak{h}_k(\delta_k e_1) e^{\Theta_k} g_k = o(\varepsilon_k^{1-\sigma}) \frac{(1 + |z|)^{-3}}{(1 + |z_k|)^\sigma} \quad \text{in } \Omega_{W_k}.$$

Then we can obtain a contradiction to $|g_k(z_k)| = 1$ as follows: If $\lim_{k\to\infty} z_k = P \in \mathbb{R}^2$, this is not possible because the fact that $g_k(0) = |\nabla g_k(0)| = 0$ and the sublinear growth of g_k in (4.26) implies that $g_k \to 0$ over any compact subset of $\mathbb{R}^2$ (see [11, 30]). So we have $|z_k| \to \infty$. But this would lead to a contradiction again by using the Green's representation of g_k:

$$\begin{aligned}&\pm 1 = g_k(z_k) = g_k(z_k) - g_k(0)\\ &= \int_{\Omega_{k,1}} (G_k(z_k,\eta) - G_k(0,\eta))\Big(\mathfrak{h}_k(\delta_k e_1)e^{\Theta_k} g_k(\eta) + o(\varepsilon_k^{1-\sigma})\frac{(1+|\eta|)^{-3}}{(1+|z_k|)^{\sigma}}\Big)\, d\eta + o(1).\end{aligned} \tag{4.27}$$

where $G_k(y,\eta)$ is the Green's function on Ω_{W_k}, and $o(1)$ in the equation above comes from the smallness of W_k on $\partial\Omega_{W_k}$. Let $L_k = \tau\varepsilon_k^{-1}$. Then the expression of G_k is

$$G_k(y,\eta) = -\frac{1}{2\pi}\log|y-\eta| + \frac{1}{2\pi}\log\Big(\frac{|\eta|}{L_k}\Big|\frac{L_k^2\eta}{|\eta|^2} - y\Big|\Big),$$

and

$$G_k(z_k,\eta) - G_k(0,\eta) = -\frac{1}{2\pi}\log|z_k-\eta| + \frac{1}{2\pi}\log\Big|\frac{z_k}{|z_k|} - \frac{\eta z_k}{L_k^2}\Big| + \frac{1}{2\pi}\log|\eta|.$$

Using this expression in (4.27) and an elementary computation shows that the right hand side of (4.27) is $o(1)$, a contradiction to $|g_k(z_k)| = 1$. Lemma 4.2 is established. ∎

The smallness of $\tilde{w}_k$ around e_1 can be used to obtain the following third key estimate:

Lemma 4.3.

$$\tilde{w}_k = o(1) \quad in\ B(e^{i\beta_l},\tau),\ l = 1,\dots,N. \tag{4.28}$$

Proof. We abuse the notation W_k by defining

$$W_k(z) = \tilde{w}_k(e^{i\beta_l} + \varepsilon_k z), \quad z \in \Omega_{k,l} := B(0,\tau\varepsilon_k^{-1}).$$

Here we point out that based on (3.5) and (4.2) we have $\varepsilon_k^{-1}|Q_l^k - e^{i\beta_l}| \to 0$. So the scaling around $e^{i\beta_l}$ or Q_l^k does not affect the limit function. We have

$$\varepsilon_k^2|e^{i\beta_l} + \varepsilon_k z|^{2N}\mathfrak{h}_k(\delta_k e_1)e^{\xi_k(e^{i\beta_l}+\varepsilon_k z)} \to e^{U(z)}$$

where $U(z)$ is a solution of

$$\Delta U + e^U = 0 \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^U < \infty.$$

Here we recall that $\lim_{k\to\infty}\mathfrak{h}_k(\delta_k e_1) = 1$. Since W_k converges to a solution of the linearized equation

$$\Delta W + e^U W = 0 \quad \text{in } \mathbb{R}^2,$$

W can be written as a linear combination of three functions:

$$W(x) = c_0\phi_0 + c_1\phi_1 + c_2\phi_2,$$

where

$$\phi_0 = \frac{1-\frac{1}{8}|x|^2}{1+\frac{1}{8}|x|^2}, \quad \phi_1 = \frac{x_1}{1+\frac{1}{8}|x|^2}, \quad \phi_2 = \frac{x_2}{1+\frac{1}{8}|x|^2}.$$

It remains to prove $c_0 = c_1 = c_2 = 0$. First we prove $c_0 = 0$.

Step 1: $c_0 = 0$. First we write the equation for W_k in a convenient form. Since

$$|e^{i\beta_l} + \varepsilon_k z|^{2N}\mathfrak{h}_k(\delta_k e_1) = \mathfrak{h}_k(\delta_k e_1) + O(\varepsilon_k z),$$
$$\varepsilon_k^2 e^{\xi_k(e^{i\beta_l}+\varepsilon_k z)} = e^{U_k(z)} + O(\varepsilon_k^\varepsilon)(1+|z|)^{-3},$$

based on (4.11) we write the equation for W_k as

$$\Delta W_k(z) + \mathfrak{h}_k(\delta_k e_1)e^{U_k}W_k = E_l^k(z), \tag{4.29}$$

where

$$E_l^k(z) = O(\varepsilon_k^\varepsilon)(1+|z|)^{-3} \quad \text{in } \Omega_{k,l}.$$

In order to prove $c_0 = 0$, the key is to control the derivative of

$$W_0^k(r) = \frac{1}{2\pi r}\int_{\partial B_r} W_k(re^{i\theta})\,dS, \quad 0 < r < \tau\varepsilon_k^{-1}.$$

To do so, we use the radial solution $\phi_0^k(r)$ of

$$\Delta\phi_0^k + \mathfrak{h}_k(\delta_k e_1)e^{U_k}\phi_0^k = 0 \quad \text{in } \mathbb{R}^2.$$

When $k\to\infty$, $\phi_0^k \to c_0\phi_0$. Thus using the equations for ϕ_0^k and W_k, we have

$$\int_{\partial B_r}(\partial_\nu W_k\phi_0^k - \partial_\nu\phi_0^k W_k) = o(\varepsilon_k^\varepsilon). \tag{4.30}$$

Thus from (4.30) we have

$$\frac{d}{dr}W_0^k(r) = \frac{1}{2\pi r}\int_{\partial B_r}\partial_\nu W_k = o(\varepsilon_k^\varepsilon)/r + O(1/r^3), \quad 1 < r < \tau\varepsilon_k^{-1}. \tag{4.31}$$

Since we know that

$$W_0^k(\tau\varepsilon_k^{-1}) = o(1),$$

by the fundamental theorem of calculus we have

$$W_0^k(r) = W_0^k(\tau\varepsilon_k^{-1}) + \int_{\tau\varepsilon_k^{-1}}^r\left(\frac{o(\varepsilon_k^\varepsilon)}{s} + O(s^{-3})\right)ds = O(1/r^2) + O\left(\varepsilon_k^\varepsilon\log\frac{1}{\varepsilon_k}\right)$$

for $r\ge 1$. Thus $c_0 = 0$ because $W_0^k(r)\to c_0\phi_0$, which means that when r is large, $W_0^k(r) = -c_0 + O(1/r^2)$.

Step 2: $c_1 = c_2 = 0$. We first observe that Lemma 4.3 follows from this. Indeed, once we have proved $c_1 = c_2 = c_0 = 0$ around each $e^{i\beta_l}$, it is easy to use the maximum principle to prove $\tilde{w}_k = o(1)$ in B_3 using $\tilde{w}_k = o(1)$ on ∂B_3 and the Green's representation of $\tilde{w}_k$. The smallness of $\tilde{w}_k$ immediately implies $\tilde{w}_k = o(1)$ in B_R for any fixed $R \gg 1$. Outside B_R, a crude estimate of v_k is

$$v_k(y) \le -\mu_k - 4(N+1)\log|y| + C, \quad 3 < |y| < \tau\delta_k^{-1}.$$

Using this and the Green's representation of w_k we can first observe that the oscillation of w_k on each ∂B_r is $o(1)$ ($R < r < \tau\delta_k^{-1}/2$) and then by the Green's representation of $\tilde{w}_k$ and fast decay rate of e^{V_k} we obtain $\tilde{w}_k = o(1)$ in $\overline{B(0, \tau\delta_k^{-1})}$, contradicting $\max|\tilde{w}_k| = 1$.

There are $N+1$ local maximum points, one of them being e_1. Correspondingly, there are $N+1$ global solutions $V_{l,k}$ that approximate v_k accurately near Q_l^k for $l = 0, \dots, N$. Note that $Q_0^k = e_1$. For $V_{l,k}$ the expression is

$$V_{l,k} = \log \frac{e^{\mu_l^k}}{\left(1 + \frac{e^{\mu_l^k}}{D_l^k}|y^{N+1} - (e_1 + p_l^k)|^2\right)^2}, \quad l = 0, \dots, N,$$

where $p_l^k = E$ and

$$D_l^k = 8(N+1)^2/\mathfrak{h}_k(\delta_k Q_l^k). \tag{4.32}$$

The equation that $V_{l,k}$ satisfies is

$$\Delta V_{l,k} + |y|^{2N}\mathfrak{h}_k(\delta_k Q_l^k)e^{V_{l,k}} = 0 \quad \text{in } \mathbb{R}^2.$$

Since v_k and $V_{l,k}$ have the same common local maximum at Q_l^k, it is easy to see that

$$Q_l^k = e^{i\beta_l} + \frac{p_l^k e^{i\beta_l}}{N+1} + O(|p_l^k|^2), \quad \beta_l = \frac{2l\pi}{N+2}. \tag{4.33}$$

Let $M_{l,k}$ be the maximum of $|v_k - V_{l,k}|$. We claim that all these $M_{l,k}$ are comparable:

$$M_{l,k} \sim M_{s,k}, \quad \forall s \ne l. \tag{4.34}$$

The proof of (4.34) is as follows: We use $L_{s,l}$ to denote the limit of $(v_k - V_{l,k})/M_{l,k}$ around Q_s^k:

$$\frac{(v_k - V_{l,k})(Q_s^k + \varepsilon_k z)}{M_{l,k}} = L_{s,l} + o(1), \quad |z| \le \tau\varepsilon_k^{-1},$$

where

$$L_{s,l} = c_{1,s,l}\frac{z_1}{1 + \frac{1}{8}|z|^2} + c_{2,s,l}\frac{z_2}{1 + \frac{1}{8}|z|^2}, \quad L_{l,l} = 0, \quad s, l = 0, \dots, N.$$

If all $c_{1,s,l}$ and $c_{2,s,l}$ are zero for a fixed l, we can obtain a contradiction just as at the beginning of Step 2. So at least one of them is not zero. For each $s \ne l$, by Lemma 4.2 we have

$$v_k(Q_s^k + \varepsilon_k z) - V_{s,k}(Q_s^k + \varepsilon_k z) = O(\varepsilon_k^\sigma)(1 + |z|)^\sigma M_{s,k}, \quad |z| < \tau\varepsilon_k^{-1}. \tag{4.35}$$

Let $M_k = \max_i M_{i,k}$ $(k = 0, \dots, N)$ and we suppose $M_k = M_{l,k}$. Then to determine $L_{s,l}$ we see that

$$\frac{v_k(Q_s^k + \varepsilon_k z) - V_{l,k}(Q_s^k + \varepsilon_k z)}{M_k} = o(\varepsilon_k^\sigma)(1 + |z|)^\sigma + \frac{V_{s,k}(Q_s^k + \varepsilon_k z) - V_{l,k}(Q_s^k + \varepsilon_k z)}{M_k}.$$

This expression says that $L_{s,l}$ is mainly determined by the difference of two global solutions $V_{s,k}$ and $V_{l,k}$. In order to obtain a contradiction to our assumption we will decompose the difference into several terms. The main idea in this part of the reasoning is that "first order terms" tell us what the kernel functions should be; then the "second order terms" tell us where the pathology is.

We write

$$V_{s,k}(y) - V_{l,k}(y) = \mu_s^k - \mu_l^k + 2A - A^2 + O(|A|^3)$$

where

$$A(y) = \frac{\frac{e^{\mu_l^k}}{D_l^k}|y^{N+1} - e_1 - p_l^k|^2 - \frac{e^{\mu_s^k}}{D_s^k}|y^{N+1} - e_1 - p_s^k|^2}{1 + \frac{e^{\mu_s^k}}{D_s^k}|y^{N+1} - e_1 - p_s^k|^2}.$$

Here for convenience we abuse the notation ε_k by assuming $\varepsilon_k = e^{-\mu_s^k/2}$. Note that $\varepsilon_k = e^{-\mu_t^k/2}$ for some t, but it does not matter which t it is. From the formula for A we claim that

$$V_{s,k}(Q_s^k + \varepsilon_k z) - V_{l,k}(Q_s^k + \varepsilon_k z) = \phi_1 + \phi_2 + \phi_3 + \phi_4 + \mathfrak{R}, \tag{4.36}$$

where

$$\phi_1 = (\mu_s^k - \mu_l^k)\left(1 - \frac{(N+1)^2}{D_s^k}\left|z + O(\varepsilon_k)|z|^2\right|^2\right)/B,$$

$$\phi_2 = \frac{2(N+1)^2}{D_s^k}\delta_k \nabla \mathfrak{h}_k(\delta_k Q_s^k)(Q_l^k - Q_s^k)|z|^2/B,$$

$$\phi_3 = \frac{4(N+1)}{D_s^k B}\,\mathrm{Re}\left((z + O(\varepsilon_k|z|^2))\left(\frac{\bar{p}_s^k - \bar{p}_l^k}{\varepsilon_k}e^{-i\beta_s}\right)\right),$$

$$\phi_4 = \frac{|p_s^k - p_l^k|^2}{\varepsilon_k^2}\left(\frac{2}{D_s^k B} - \frac{2(N+1)^2|z|^2}{D_s^2 B^2} - \frac{2(N+1)^2}{D_s^2 B^2}|z|^2\cos(2\theta - 2\theta_{st} - 2\beta_s)\right),$$

$$B = 1 + \frac{(N+1)^2}{D_s^k}|z + O(\varepsilon_k|z|^2)|^2,$$

and $\mathfrak{R}_k$ is the collection of other insignificant terms. Here we briefly explain the roles of each term. ϕ_1 corresponds to the radial solution in the kernel of the linearized operator of the global equation. In other words, ϕ_1^k/M_k should tend to zero because in Step 1 we have proved $c_0 = 0$. Next ϕ_2^k/M_k is the combination of the other two functions in the kernel. Further ϕ_4 is the second order term which will play a leading role later. Finally, ϕ_3^k comes from the difference of $\mathfrak{h}_k$ at Q_l^k and Q_s^k. The derivation of (4.36) is as follows: First by the expression of Q_s^k in (4.33) we have

$$y^{N+1} = 1 + p_s^k + (N+1)\varepsilon_k z e^{-i\beta_s} + O(\varepsilon_k^2)|z|^2,$$

where $y = Q_s^k + \varepsilon_k z$. Then

$$|y^{N+1} - e_1 - p_s^k|^2 = (N+1)^2\varepsilon_k^2|z + O(\varepsilon_k)z^2|^2 + O(\varepsilon_k^3)|z|^3,$$

$$|y^{N+1} - e_1 - p_l^k|^2 = (N+1)^2\varepsilon_k^2\left|z + \frac{(p_s^k - p_l^k)e^{i\beta_s}}{(N+1)\varepsilon_k} + O(\varepsilon_k)|z|^2\right|^2 + O(\varepsilon_k^3|z|^3).$$

Next by the definition of D_s^k in (4.32),

$$\begin{aligned}
\frac{D_s^k - D_l^k}{D_l^k} &= \delta_k \nabla(\log \mathfrak{h}_k)(0)\cdot(Q_l^k - Q_s^k) + O(\delta_k^2),\\
\frac{e^{\mu_l^k - \mu_s^k}}{D_l^k} &= \frac{1}{D_s^k}\left(1 + \frac{D_s^k - D_l^k}{D_l^k} + \mu_l^k - \mu_s^k + O((\mu_l^k - \mu_s^k)^2) + O(\delta_k^2)\right).\\
&= \frac{1}{D_s^k}(1 + \delta_k \nabla \log \mathfrak{h}_k(0)\cdot(Q_l^k - Q_s^k) + \mu_l^k - \mu_s^k\\
&\qquad + O((\mu_l^k - \mu_s^k)^2) + O(\delta_k^2)). \qquad (4.37)
\end{aligned}$$

Then the expression of A is (for simplicity we omit k in some notations)

$$\begin{aligned}
A = \Bigg(&\frac{e^{\mu_l - \mu_s}}{D_l}(N+1)^2\left(|z|^2 + 2\,\mathrm{Re}\left(z\frac{\overline{p_s - p_l}}{\varepsilon_k(N+1)}e^{-i\beta_s}\right) + \frac{|p_s - p_l|^2}{(N+1)^2\varepsilon_k^2} + O(\varepsilon_k|z|^3)\right)\\
&- \frac{(N+1)^2}{D_s}(|z|^2 + O(\varepsilon_k)|z|^3)\Bigg)/B.
\end{aligned}$$

After using (4.37) we have

$$\begin{aligned}
A = \Bigg(&\frac{1}{D_s}\big(\delta_k \nabla(\log \mathfrak{h}_k)(0)(Q_l - Q_s) + \mu_l - \mu_s + O(\mu_l - \mu_s)^2\big)(N+1)^2|z|^2\\
&+ 2\,\mathrm{Re}\left(z\frac{\bar{p}_s - \bar{p}_l}{\varepsilon_k}e^{-i\beta_s}\right)(N+1)\frac{1}{D_s} + \frac{|p_s - p_l|^2}{\varepsilon_k^2 D_s} + O(\varepsilon_k)|z|^3 + O(\delta_k^2)|z|^2\Bigg)/B.
\end{aligned}$$

so

$$A^2 = \frac{(N+1)^2}{D_s^2}4\left(\mathrm{Re}\left(z\frac{\bar{p}_s - \bar{p}_l}{\varepsilon_k}e^{-i\beta_s}\right)\right)^2/B^2 + \text{other terms}.$$

The numerator of A^2 has the following leading term:

$$\frac{(N+1)^2}{D_s^2}\left(2|z|^2\left(\frac{|p_s - p_l|}{\varepsilon_k}\right)^2(1 + 2\cos(2\theta - 2\theta_{sl}))\right),$$

where $z=|z|e^{i\theta}$ and $p_s-p_l=|p_s-p_l|e^{i\theta_{sl}}$. Using these expressions we can obtain (4.36) by direct computation. Here ϕ_1, ϕ_3 correspond to solutions to the linearized operator. Note that if we set $\varepsilon_{l,k}=e^{-\mu_l^k/2}$, then there is no essential difference between $\varepsilon_{l,k}$ and $\varepsilon_k=e^{-\frac{1}{2}\mu_{1,k}}$ because $\varepsilon_{l,k}=\varepsilon_k+O(\varepsilon_k E)$. If $|\mu_{s,k}-\mu_{l,k}|/M_k\ge C$ there is no way to obtain a limit in the form of $L_{s,l}$ mentioned before. Thus we must have $|\mu_{s,k}-\mu_{l,k}|/M_k\to 0$. After simplification (see ϕ_3 of (4.36)) we have

$$\begin{aligned} c_{1,s,l}&=\lim_{k\to\infty}\frac{|p_s^k-p_l^k|}{2(N+1)M_k\varepsilon_k}\cos(\beta_s+\theta_{sl}),\\ c_{2,s,l}&=\lim_{k\to\infty}\frac{|p_s^k-p_l^k|}{2(N+1)M_k\varepsilon_k}\sin(\beta_s+\theta_{sl}) \end{aligned}\tag{4.38}$$

We omit k for convenience. It is also important to observe that even if $M_k=o(\varepsilon_k)$ we still have $M_k\sim\max_s|p_s^k-p_l^k|/\varepsilon_k$. Since each $|p_l^k|$ equals E, an upper bound for M_k is

$$M_k\le C\mu_k\varepsilon_k+C\delta_k^2\varepsilon_k^{-1}.\tag{4.39}$$

Equations (4.38) give us a key observation: $|c_{1,s,l}|+|c_{2,s,l}|\sim|p_s^k-p_l^k|/(\varepsilon_k M_k)$. So whenever $|c_{1,s,l}|+|c_{2,s,l}|\ne 0$ we have $|p_s^k-p_l^k|/\varepsilon_k\sim M_k$. In other words, for each l, $M_{l,k}\sim\max_{t\ne l}|p_t^k-p_l^k|/\varepsilon_k$. Hence for any t, if $|p_t^k-p_l^k|/\varepsilon_k\sim M_k$, let $M_{t,k}$ be the maximum of $|v_k-V_{t,k}|$; we have $M_{t,k}\sim M_k$. If $|p_t^k-p_l^k|/\varepsilon_k\sim M_k$ for all k, (4.34) is proved. So we prove that even if some p_t^k is very close to p_l^k, $M_{t,k}$ is still comparable to M_k. The reason is that there exists q such that $\frac{|p_l^k-p_q^k|}{\varepsilon_k}\sim M_k$, so if $\frac{|p_t^k-p_l^k|}{\varepsilon_k}=o(1)M_k$ then

$$|p_t^k-p_q^k|\ge|p_l^k-p_q^k|-|p_t^k-p_l^k|\ge\tfrac{1}{2}|p_l^k-p_q^k|.$$

Thus $|p_t^k-p_q^k|/\varepsilon_k\sim M_k$ and $M_{t,k}\sim M_k$, so (4.34) is established. From now on for convenience we shall just use M_k. Since $M_k\sim\max_{s,t}|p_s^k-p_t^k|/\varepsilon_k$, an upper bound of M_k is

$$M_k\le C\mu_k\varepsilon_k.\tag{4.40}$$

Set $w_{l,k}=v_k-V_{l,k}$. Then $w_{l,k}(Q_l^k)=|\nabla w_{l,k}(Q_l^k)|=0$. Correspondingly, we set

$$\tilde{w}_{l,k}=w_{l,k}/M_k.$$

The equation for $w_{l,k}$ can be written as

$$\begin{aligned} &\Delta w_{l,k}+|y|^{2N}\mathfrak{h}_k(\delta_k Q_l)e^{\xi_l}w_{l,k}\\ &\quad=-\delta_k\nabla\mathfrak{h}_k(\delta_k Q_l)(y-Q_l)|y|^{2N}e^{V_{l,k}}-\delta_k^2\sum_{|\alpha|=2}\frac{\partial^\alpha\mathfrak{h}_k(\delta_k Q_l)}{\alpha!}(y-Q_l)^\alpha|y|^{2N}e^{V_k}\\ &\qquad+O(\delta_k^3)|y-Q_l|^3|y|^{2N}e^{V_k}, \end{aligned}\tag{4.41}$$

where we omitted k in Q_l and ξ_l. Now, ξ_l comes from the mean value theorem and satisfies

$$e^{\xi_l}=e^{V_{l,k}}\big(1+\tfrac{1}{2}w_{l,k}+O(w_{l,k}^2)\big).\tag{4.42}$$

The function $\tilde{w}_{l,k}$ satisfies

$$\lim_{k\to\infty} \tilde{w}_{l,k}(Q_s^k + \varepsilon_k z) = \frac{c_{1,s,l} z_1 + c_{2,s,l} z_2}{1 + \frac{1}{8}|z|^2} \tag{4.43}$$

and around each Q_s^k, (4.35) holds with $M_{s,k}$ replaced by M_k.

Now for $|y| \sim 1$, we use $w_{l,k}(Q_l^k) = 0$ to write $w_{l,k}(y)$ as

$$\begin{aligned} w_{l,k}(y) = \int_{\Omega_k} (G_k(y,\eta) - G_k(Q_l,\eta)) \Big(& \mathfrak{h}_k(\delta_k Q_l)|\eta|^{2N} e^{\xi_l} w_{l,k}(\eta) \\ & + \delta_k \nabla \mathfrak{h}_k(\delta_k Q_l)(\eta - Q_l)|\eta|^{2N} e^{V_{l,k}} \\ & + \delta_k^2 \sum_{|\alpha|=2} \frac{\partial^\alpha \mathfrak{h}_k(\delta_k Q_l)}{\alpha!}(\eta - Q_l)^\alpha |y|^{2N} e^{V_k} \Big) + O(\delta_k^{N+2}). \end{aligned}$$

Note that the last term is $O(\delta_k^{N+2})$ because it comes from the oscillation of $w_{l,k}$ on $\partial\Omega_k$. The harmonic function defined by the boundary value of $w_{l,k}$ has an oscillation of $O(\delta_k^{N+1})$ on $\partial\Omega_k$. The oscillation of this harmonic function in B_R (for any fixed $R > 1$) is $O(\delta_k^{N+2})$. The regular part of the Green's function brings little error in the computation, so we have

$$\begin{aligned} \tilde{w}_{l,k}(y) = -\frac{1}{2\pi} \int_{\Omega_k} \log \frac{|y-\eta|}{|Q_l^k - \eta|} \Big(& \tilde{w}_{l,k}(\eta)\mathfrak{h}_k(\delta_k Q_l^k)|\eta|^{2N} e^{\xi_l} \\ & + \sigma_k \nabla \mathfrak{h}_k(\delta_k Q_l^k)(\eta - Q_l^k)|\eta|^{2N} e^{V_{l,k}} \\ & + \frac{\delta_k^2}{M_k} \sum_{|\alpha|=2} \frac{\partial^\alpha \mathfrak{h}_k(\delta_k Q_l^k)}{\alpha!}(\eta - Q_l^k)^\alpha |\eta|^{2N} e^{V_{l,k}} \Big) d\eta + o(\varepsilon_k) \end{aligned} \tag{4.44}$$

for $|y| \sim 1$.

Around each Q_s^k the e^{ξ_l} can be replaced by $e^{V_{s,k}}$ with controllable error (based on Lemma 4.2 and (4.40)). In order to evaluate the expression of $\tilde{w}_{l,k}$ we need the following identity based on (4.36):

$$\int_{B(Q_s^k,\tau)} (\tilde{w}_{l,k}(\eta)\mathfrak{h}_k(\delta_k Q_l^k)|\eta|^{2N} e^{V_{s,k}} + \sigma_k \nabla \mathfrak{h}_k(\delta_k Q_l^k)(\eta - Q_l)|\eta|^{2N} e^{V_{l,k}})\, d\eta = O(\varepsilon_k^\sigma). \tag{4.45}$$

Note that $e^{V_{l,k}}$ in the first term was replaced by $e^{V_{s,k}}$ but in the second term above this replacement is not necessary. (4.36) is mainly used in the evaluation of the first term. The proof of (4.45) can be found in the appendix.

Equation (4.45) also leads to a more accurate estimate of $\tilde{w}_{l,k}$ in regions between bubbling disks. By the Green's representation formula of $\tilde{w}_{l,k}$ it is easy to have, for $|y| \sim 1$,

$$|\tilde{w}_{l,k}(y)| = o(1/\mu_k), \quad y \in B_3 \setminus \bigcup_{s=0}^{N} B(Q_s^k, \tau). \tag{4.46}$$

Indeed, writing the logarithmic term in (4.44) as

$$\log\frac{|y-\eta|}{|Q_l^k-\eta|}=\log\frac{|y-Q_s^k|}{|Q_l^k-Q_s^k|}+\left(\log\frac{|y-\eta|}{|Q_l^k-\eta|}-\log\frac{|y-Q_s^k|}{|Q_l^k-Q_s^k|}\right),$$

we see that the integral in (4.44) related to the second term is $O(\varepsilon_k)$. The integral in (4.44) involving the first term is $O(\varepsilon_k^\sigma)+o(\delta_k^2/M_k)$ for some $\sigma\in(0,1)$ by (4.45) and the definition of M_k. Therefore (4.46) holds. This extra control of $\tilde w_{l,k}$ away from bubbling disks gives a better estimate than (4.35) around Q_l^k. Using the same argument for Lemma 4.2 we have

$$|\tilde w_{l,k}(Q_l^k+\varepsilon_k z)|\le o(\varepsilon_k)\frac{1+|z|}{\log(2+|z|)},\quad |z|<\tau\varepsilon_k^{-1}. \tag{4.47}$$

From the decomposition in (4.36) and with the help of (4.47) we can now estimate the integral of $\tilde w_{l,k}$ more precisely:

$$\begin{aligned}
&\int_{B(Q_s^k,\tau)}\Big(\tilde w_{l,k}(\eta)\mathfrak{h}_k(\delta_k Q_l^k)|\eta|^{2N}e^{\xi_l}+\sigma_k\nabla\mathfrak{h}_k(\delta_k Q_l^k)(\eta-Q_l^k)|\eta|^{2N}e^{V_{l,k}}\\
&\qquad+\frac{\delta_k^2}{M_k}\sum_{|\alpha|=2}\frac{\partial^\alpha\mathfrak{h}_k(\delta_k Q_l^k)}{\alpha!}(\eta-Q_l^k)^\alpha|\eta|^{2N}e^{V_{l,k}}\Big)\,d\eta\\
&=\int_{B(Q_s^k,\tau)}\tilde w_{l,k}(\eta)\mathfrak{h}_k(\delta_k Q_l^k)|\eta|^{2N}e^{V_{s,k}}\,d\eta+8\pi\sigma_k\nabla\log\mathfrak{h}_k(\delta_k Q_s^k)(Q_s^k-Q_l^k)\\
&\quad+2\pi\frac{\delta_k^2}{M_k}\frac{\Delta\mathfrak{h}_k(\delta_k Q_s^k)}{\mathfrak{h}_k(\delta_k Q_s^k)}|Q_s^k-Q_l^k|^2+o(\varepsilon_k)\\
&=D_{s,l}^k+o(\varepsilon_k),
\end{aligned}\tag{4.48}$$

where

$$D_{s,l}^k=\frac{\pi}{(N+1)^2}\frac{|p_s^k-p_l^k|^2}{\varepsilon_k^2M_k^2}M_k+2\pi\frac{\delta_k^2}{M_k}\frac{\Delta\mathfrak{h}_k(\delta_k Q_s^k)}{\mathfrak{h}_k(\delta_k Q_s^k)}|Q_s^k-Q_l^k|^2.$$

Let

$$H_{y,l}(\eta)=\frac{1}{2\pi}\log\frac{|y-\eta|}{|Q_l^k-\eta|}.$$

Then

$$\begin{aligned}
\tilde w_{l,k}(y)&=-\sum_{s\ne l}H_{y,l}(Q_s)D_{s,l}^k\\
&\quad-\sum_{s\ne l}\int_{B(Q_s,\tau)}(\partial_1H_{y,l}(Q_s)\eta_1+\partial_2H_{y,l}(Q_s)\eta_2)\big(\mathfrak{h}_k(\delta_kQ_l)|\eta|^{2N}e^{\xi_l}\tilde w_{l,k}(\eta)\\
&\quad+\sigma_k\nabla\mathfrak{h}_k(\delta_kQ_l)(\eta-Q_l)|\eta|^{2N}e^{V_{l,k}}+O(\delta_k^2M_k^{-1})|\eta-Q_l|^2|\eta|^{2N}e^{V_{l,k}}\big)\,d\eta\\
&\quad+o(\varepsilon_k).
\end{aligned}$$

After evaluation we have

$$
\begin{aligned}
\tilde{w}_{l,k}(y) = & -\frac{1}{2\pi}\sum_{s\neq l}\log\frac{|y-Q_s^k|}{|Q_l^k-Q_s^k|}\, D_{s,l}^k \\
& -\sum_{s\neq l} 8\left(\frac{y_1-Q_s^1}{|y-Q_s|^2}-\frac{Q_l^1-Q_s^1}{|Q_l-Q_s|^2}\right)c_{1,s,l}\varepsilon_k \\
& -8\left(\frac{y_2-Q_s^2}{|y-Q_s|^2}-\frac{Q_l^2-Q_s^2}{|Q_s-Q_l|^2}\right)c_{2,s,l}\varepsilon_k+o(\varepsilon_k),
\end{aligned}
$$

where we have used

$$
\int_{\mathbb{R}^2}\frac{z_1^2}{\left(1+\frac{1}{8}|z|^2\right)^3}\,dz=\int_{\mathbb{R}^2}\frac{z_2^2}{\left(1+\frac{1}{8}|z|^2\right)^3}\,dz=16\pi .
$$

Recall that

$$
\begin{aligned}
c_{1,s,l} &= \frac{|p_s-p_l|}{2(N+1)M_k\varepsilon_k}\cos(\beta_s+\theta_{sl}),\\
c_{2,s,l} &= \frac{|p_s-p_l|}{2(N+1)M_k\varepsilon_k}\sin(\beta_s+\theta_{sl}).
\end{aligned}
$$

For $|y|\sim 1$ but away from the $N+1$ bubbling disks, we have, for $l\neq s$,

$$
v_k(y)=V_{l,k}(y)+M_k\tilde{w}_{l,k}(y)
$$

and

$$
v_k(y)=V_{s,k}(y)+M_k\tilde{w}_{s,k}(y).
$$

Thus for $s\neq l$ we have

$$
\frac{V_{s,k}(y)-V_{l,k}(y)}{M_k}=\tilde{w}_{l,k}(y)-\tilde{w}_{s,k}(y). \tag{4.49}
$$

For $|y|\sim 1$ away from bubbling disks, we have

$$
\begin{aligned}
& V_{s,k}(y)-V_{l,k}(y)=4\log|y^{N+1}-e_1-p_l|-4\log|y^{N+1}-e_1-p_s|+o(\varepsilon_k M_k),\\
& |y^{N+1}-e_1-p_l|^2\\
& \quad =|y^{N+1}-e_1-p_s|^2+2\,\mathrm{Re}\big((y^{N+1}-e_1-p_s)(\bar{p}_s-\bar{p}_l)\big)+|p_l-p_s|^2.
\end{aligned}
$$

Thus

$$
\frac{V_{s,k}(y)-V_{l,k}(y)}{M_k}=4\,\mathrm{Re}\left(\frac{y^{N+1}-1}{|y^{N+1}-1|^2}\,\frac{\bar{p}_l-\bar{p}_s}{M_k\varepsilon_k}\right)\varepsilon_k+o(\varepsilon_k).
$$

We are going to derive a contradiction based on (4.49). For this purpose we choose $s = 0$ in (4.49), which means $\tilde{w}_{s,k} = o(\varepsilon_k)$; then we have

$$\begin{aligned}
&\mathrm{Re}\left(\frac{y^{N+1}-1}{|y^{N+1}-1|^2}\frac{\bar{p}_l}{\varepsilon_k M_k}\right)\varepsilon_k + o(\varepsilon_k)\\
&\quad = -\frac{1}{2\pi}\sum_{s\neq l}\log\frac{|y-Q_s^k|}{|Q_l^k-Q_s^k|}D_{s,l}^k\\
&\qquad + \sum_{s\neq l} 8\left(\left(\frac{y_1-Q_l^1}{|y-Q_l|^2}-\frac{Q_l^1-Q_s^1}{|Q_l-Q_s|^2}\right)\left(\frac{|p_s-p_l|}{2(N+1)M_k\varepsilon_k}\right)\cos(\beta_s+\theta_{sl})\right.\\
&\qquad \left. + \left(\frac{y_2-Q_l^2}{|y-Q_l|^2}-\frac{Q_l^2-Q_s^2}{|Q_l-Q_s|^2}\right)\frac{|p_s-p_l|}{2(N+1)\varepsilon_k M_k}\sin(\beta_s+\theta_{sl})\right)\varepsilon_k, \quad \forall l. \qquad (4.50)
\end{aligned}$$

If $M_k \geq C\varepsilon_k$, the first term on the right hand side will dominate all other terms when $|y| \gg 1$, violating the equality in (4.50). But when $M_k = o(\varepsilon_k)$, the equality cannot hold either if we choose $|y| \gg 1$ because the last two terms of (4.50) will majorize the left hand side. Lemma 4.3 is established. ■

Proposition 4.1 is an immediate consequence of Lemma 4.3. ■

Now we finish the proof of Theorem 4.1.

Let $\hat{w}_k = w_k/\tilde{\delta}_k$. (Recall that $\tilde{\delta}_k = \delta_k|\nabla\mathfrak{h}_k(0)| + \delta_k^2\mu_k$.) If $|\nabla\mathfrak{h}_k(0)|/(\delta_k\mu_k) \to \infty$, we see that in this case $\tilde{\delta}_k \sim \delta_k\mu_k|\nabla\mathfrak{h}_k(0)|$. The equation for $\hat{w}_k$ is

$$\Delta\hat{w}_k + |y|^{2N}e^{\xi_k}\hat{w}_k = a_k\cdot(e_1-y)|y|^{2N}e^{V_k} + b_k e^{V_k}|y-e_1|^2|y|^{2N} \qquad (4.51)$$

in Ω_k, where $a_k = \delta_k\nabla\mathfrak{h}_k(0)/\tilde{\delta}_k$, $b_k = o(1/\mu_k)$. By Proposition 4.1, $|\hat{w}_k(y)| \leq C$. Before we carry out the remaining part of the proof we observe that $\hat{w}_k$ converges to a harmonic function in $\mathbb{R}^2$ minus finitely many singular points. Since $\hat{w}_k$ is bounded, all these singularities are removable. Thus $\hat{w}_k$ converges to a constant. Based on the information around e_1, we shall prove that this constant is 0. However, looking at the right hand side of the equation,

$$(e_1-y)|y|^{2N}e^{V_k} \rightharpoonup \sum_{l=1}^{N} 8\pi(e_1-e^{i\beta_l})\delta_{e^{i\beta_l}},$$

we will get a contradiction by comparing the Pohozaev identities for v_k and V_k, respectively.

Now we use the notation W_k again and use Proposition 4.1 to rewrite the equation for W_k. Let

$$W_k(z) = \hat{w}_k(e_1+\varepsilon_k z), \quad |z| < \delta_0\varepsilon_k^{-1},$$

for $\delta_0 > 0$ small. Then from Proposition 4.1 we have

$$\mathfrak{h}_k(\delta_k y) = \mathfrak{h}_k(\delta_k e_1) + \delta_k \nabla \mathfrak{h}_k(\delta_k e_1)(y - e_1) + O(\delta_k^2)|y - e_1|^2, \tag{4.52}$$
$$|y|^{2N} = |e_1 + \varepsilon_k z|^{2N} = 1 + O(\varepsilon_k)|z|, \tag{4.53}$$
$$V_k(e_1 + \varepsilon_k z) + 2\log \varepsilon_k = U_k(z) + O(\varepsilon_k)|z| + O(\varepsilon_k^2)(\log(1 + |z|))^2, \tag{4.54}$$
$$\xi_k(e_1 + \varepsilon_k z) + 2\log \varepsilon_k = U_k(z) + O(\varepsilon_k)(1 + |z|). \tag{4.55}$$

Using (4.52)–(4.55) in (4.51) we write the equation for W_k as

$$\Delta W_k + \mathfrak{h}_k(\delta_k e_1) e^{U_k(z)} W_k = -\varepsilon_k a_k \cdot z e^{U_k(z)} + E_w, \quad 0 < |z| < \delta_0 \varepsilon_k^{-1}, \tag{4.56}$$

where

$$E_w(z) = O(\varepsilon_k)(1 + |z|)^{-3}, \quad |z| < \delta_0 \varepsilon_k^{-1}. \tag{4.57}$$

Since $\hat{w}_k$ obviously converges to a global harmonic function with removable singularity, we have $\hat{w}_k \to \bar{c}$ for some $\bar{c} \in \mathbb{R}$. We claim the following.

Lemma 4.4. $\bar{c} = 0$.

Proof. If $\bar{c} \neq 0$, we use $W_k(z) = \bar{c} + o(1)$ on $B(0, \delta_0 \varepsilon_k^{-1}) \setminus B(0, \frac{1}{2}\delta_0 \varepsilon_k^{-1})$ and consider the projection of W_k on 1:

$$g_0(r) = \frac{1}{2\pi} \int_0^{2\pi} W_k(re^{i\theta})\, d\theta.$$

If we use F_0 to denote the projection to 1 of the right hand side, using the rough estimate of E_w in (4.57) we have

$$g_0''(r) + \frac{1}{r} g_0'(r) + \mathfrak{h}_k(\delta_k e_1) e^{U_k(r)} g_0(r) = F_0, \quad 0 < r < \delta_0 \varepsilon_k^{-1},$$

where

$$F_0(r) = O(\varepsilon_k)(1 + |z|)^{-3}.$$

In addition,

$$\lim_{k \to \infty} g_0(\delta_0 \varepsilon_k^{-1}) = \bar{c} + o(1).$$

For simplicity we omit k in some notations. By the same argument as in Lemma 4.1, we have

$$g_0(r) = O(\varepsilon_k) \log(2 + r), \quad 0 < r < \delta_0 \varepsilon_k^{-1}.$$

Thus $\bar{c} = 0$. Lemma 4.4 is established. ■

Based on Lemma 4.4 and the standard Harnack inequality for elliptic equations, we have

$$\tilde{w}_k(x) = o(1), \quad \nabla \tilde{w}_k(x) = o(1), \quad x \in B_3 \setminus \bigcup_{l=1}^{N} (B(e^{i\beta_l}, \delta_0) \setminus B(e^{i\beta_l}, \delta_0/8)). \tag{4.58}$$

This is equivalent to $w_k = o(\tilde{\delta}_k)$ and $\nabla w_k = o(\tilde{\delta}_k)$ in the same region.

In the next step we consider the difference between two Pohozaev identities. For $s = 1, \dots, N$ we consider the Pohozaev identity around Q_s^k. Let $\Omega_{s,k} = B(Q_s^k, r)$ for small $r > 0$. For v_k we have

$$\int_{\Omega_{s,k}} \partial_\xi(|y|^{2N} \mathfrak{h}_k(\delta_k y))e^{v_k} - \int_{\partial\Omega_{s,k}} e^{v_k}|y|^{2N}\mathfrak{h}_k(\delta_k y)(\xi\cdot\nu) = \int_{\partial\Omega_{s,k}} \big(\partial_\nu v_k \partial_\xi v_k - \tfrac{1}{2}|\nabla v_k|^2(\xi\cdot\nu)\big)\,dS, \tag{4.59}$$

where ξ is an arbitrary unit vector. Correspondingly, the Pohozaev identity for V_k is

$$\int_{\Omega_{s,k}} \partial_\xi(|y|^{2N} \mathfrak{h}_k(\delta_k e_1))e^{V_k} - \int_{\partial\Omega_{s,k}} e^{V_k}|y|^{2N}\mathfrak{h}_k(\delta_k e_1)(\xi\cdot\nu) = \int_{\partial\Omega_{s,k}} \big(\partial_\nu V_k \partial_\xi V_k - \tfrac{1}{2}|\nabla V_k|^2(\xi\cdot\nu)\big)\,dS. \tag{4.60}$$

Using $w_k = v_k - V_k$ and $|w_k(y)| \le C\tilde{\delta}_k$ we have

$$\begin{aligned}
&\int_{\partial\Omega_{s,k}} \big(\partial_\nu v_k \partial_\xi v_k - \tfrac{1}{2}|\nabla v_k|^2(\xi\cdot\nu)\big)\,dS \\
&\quad = \int_{\partial\Omega_{s,k}} \big(\partial_\nu V_k \partial_\xi V_k - \tfrac{1}{2}|\nabla V_k|^2(\xi\cdot\nu)\big)\,dS \\
&\qquad + \int_{\partial\Omega_{s,k}} \big(\partial_\nu V_k \partial_\xi w_k + \partial_\nu w_k \partial_\xi V_k - (\nabla V_k\cdot\nabla w_k)(\xi\cdot\nu)\big)\,dS + o(\tilde{\delta}_k).
\end{aligned}$$

If we just use the crude estimate $\nabla w_k = o(\tilde{\delta}_k)$, we have

$$\int_{\partial\Omega_{s,k}} \big(\partial_\nu v_k \partial_\xi v_k - \tfrac{1}{2}|\nabla v_k|^2(\xi\cdot\nu)\big)\,dS - \int_{\partial\Omega_{s,k}} \big(\partial_\nu V_k \partial_\xi V_k - \tfrac{1}{2}|\nabla V_k|^2(\xi\cdot\nu)\big)\,dS = o(\tilde{\delta}_k).$$

The difference of the second terms is minor: If we use the expansion of $v_k = V_k + w_k$ and that of $\mathfrak{h}_k(\delta_k y)$ around e_1, it is easy to obtain

$$\int_{\partial\Omega_{s,k}} e^{v_k}|y|^{2N}\mathfrak{h}_k(\delta_k y)(\xi\cdot\nu) - \int_{\partial\Omega_{s,k}} e^{V_k}|y|^{2N}\mathfrak{h}_k(\delta_k e_1)(\xi\cdot\nu) = o(\tilde{\delta}_k).$$

To evaluate the first term, we use

$$\begin{aligned}
&\partial_\xi(|y|^{2N}\mathfrak{h}_k(\delta_k y))e^{v_k} \\
&= \partial_\xi\big(|y|^{2N}\mathfrak{h}_k(\delta_k e_1) + |y|^{2N}\delta_k\nabla\mathfrak{h}_k(\delta_k e_1)(y-e_1) + O(\delta_k^2)\big)e^{V_k}(1 + w_k + O(\delta_k^2\mu_k)) \\
&= \partial_\xi(|y|^{2N})\mathfrak{h}_k(\delta_k e_1)e^{V_k} + \delta_k\partial_\xi\big(|y|^{2N}\nabla\mathfrak{h}_k(\delta_k e_1)(y-e_1)\big)e^{V_k} \\
&\quad + \partial_\xi(|y|^{2N}\mathfrak{h}_k(\delta_k e_1))e^{V_k}w_k + O(\delta_k^2\mu_k)e^{V_k}.
\end{aligned} \tag{4.61}$$

For the third term on the right hand side of (4.61) we use the equation for w_k:

$$\Delta w_k + \mathfrak{h}_k(\delta_k e_1)e^{V_k}|y|^{2N} w_k = -\delta_k \nabla \mathfrak{h}_k(\delta_k e_1)\cdot (y-e_1)|y|^{2N}e^{V_k} + O(\delta_k^2)e^{V_k}|y|^{2N}.$$

From integration by parts we have

$$\begin{aligned}
&\int_{\Omega_{s,k}} \partial_\xi(|y|^{2N})\mathfrak{h}_k(\delta_k e_1)e^{V_k} w_k = 2N\int_{\Omega_{s,k}} |y|^{2N-2} y_\xi \mathfrak{h}_k(\delta_k e_1)e^{V_k} w_k \\
&\quad = 2N\int_{\Omega_{s,k}} \frac{y_\xi}{|y|^2}\big(-\Delta w_k - \delta_k \nabla \mathfrak{h}_k(\delta_k e_1)(y-e_1)|y|^{2N}e^{V_k} + O(\delta_k^2)e^{V_k}|y|^{2N}\big) \\
&\quad = -2N\delta_k \int_{\Omega_{s,k}} \frac{y_\xi}{|y|^2}\nabla \mathfrak{h}_k(\delta_k e_1)(y-e_1)|y|^{2N}e^{V_k} \\
&\qquad + 2N\int_{\partial\Omega_{s,k}} \left(\partial_\nu\left(\frac{y_\xi}{|y|^2}\right) w_k - \partial_\nu w_k \frac{y_\xi}{|y|^2}\right) + o(\tilde{\delta}_k) \\
&\quad = \nabla \mathfrak{h}_k(\delta_k e_1)\big(-16N\delta_k \pi (e^{i\beta_s}\cdot \xi)(e^{i\beta_s} - e_1) + O(\mu_k \varepsilon_k^2)\big) + o(\tilde{\delta}_k),
\end{aligned} \tag{4.62}$$

where we have used $\nabla w_k, w_k = o(\tilde{\delta}_k)$ on $\partial\Omega_{s,k}$. For the second term on the right hand side of (4.61), we have

$$\begin{aligned}
&\int_{\Omega_{s,k}} \delta_k \partial_\xi(|y|^{2N}\nabla \mathfrak{h}_k(\delta_k e_1)(y-e_1))e^{V_k} \\
&\quad = 2N\delta_k \int_{\Omega_{s,k}} y_\xi |y|^{2N-2}\nabla \mathfrak{h}_k(\delta_k e_1)(y-e_1)e^{V_k} + \delta_k \int_{\Omega_{s,k}} |y|^{2N}\partial_\xi \mathfrak{h}_k(\delta_k e_1)e^{V_k} \\
&\quad = \nabla \mathfrak{h}_k(\delta_k e_1)\big(16N\pi\delta_k (e^{i\beta_s}\cdot\xi)(e^{i\beta_s} - e_1) + O(\mu_k\varepsilon_k^2)\big) \\
&\qquad + \delta_k \partial_\xi \mathfrak{h}_k(\delta_k e_1)(8\pi + O(\mu_k \varepsilon_k^2)) + o(\tilde{\delta}_k).
\end{aligned} \tag{4.63}$$

Using (4.62) and (4.63) in the difference between (4.59) and (4.60), we have

$$\delta_k \partial_\xi \mathfrak{h}_k(\delta_k e_1)(1 + O(\mu_k\varepsilon_k^2)) = o(\tilde{\delta}_k).$$

Thus $\nabla \mathfrak{h}_k(\delta_k e_1) = O(\delta_k \mu_k)$. Theorem 4.1 is established. ∎

5. Proof of Theorem 1.1

First we handle the case $N \geq 2$. In [26] we have already proved that

$$\Delta(\log \mathfrak{h}_k)(0) = O(\delta_k^{-2}\mu_k e^{-\mu_k}) + O(\delta_k).$$

Therefore if $\delta_k/(\mu_k^{1/2}\varepsilon_k) \to \infty$ there is nothing to prove. So we only consider the case when $\delta_k \leq C\mu_k^{1/2}\varepsilon_k$. In this case $\varepsilon_k^{-1}\delta_k^2 \leq C\varepsilon_k^{\varepsilon}$ for some $\varepsilon \in (0,1)$. The whole argument of Proposition 4.1 can be employed to prove

$$|w_k(y)| \leq C\delta_k^2 \mu_k^{7/4}. \tag{5.1}$$

In order to employ the same strategy of the proof, one needs to have three things: First, $\varepsilon_k^{-1}\delta_k^2 = O(\varepsilon_k^{\varepsilon})$. This is clear from the definition of δ_k. Second, in the proof of Lemma 4.3 we need

$$O(\delta_k^{N+2}/M_k) = o(\varepsilon_k),$$

where $M_k > \delta_k^2\mu_k^{7/4}$. Since $\delta_k \le C\mu_k^{1/2}\varepsilon_k$ and $N \ge 1$, the required inequality holds. Thirdly, we need to have

$$\delta_k^3/M_k = o(\varepsilon_k).$$

This is used in (7.1). From the requirement on δ_k and the definition of M_k this clearly also holds. The proof of Proposition 4.1 follows. Thus for $N \ge 2$ we also have (5.1).

The precise upper bound of w_k in (5.1) leads to the vanishing rate of the Laplacian estimate for $N \ge 2$ and some cases of $N = 1$: If we use

$$W_k(z) = w_k(e_l + \varepsilon_k z)/(\delta_k^2\mu_k^{7/4}), \quad |z| < \tau\varepsilon_k^{-1},$$

where $e_l \ne e_1$. We shall show that the projection of W_k over 1 is not bounded when $|z| \sim \varepsilon_k^{-1}$, which gives the desired contradiction.

We write the equation for w_k as

$$\Delta w_k + |y|^{2N}e^{\xi_k}w_k = \big(\mathfrak{h}_k(\delta_k e_1) - \mathfrak{h}_k(\delta_k y)\big)|y|^{2N}e^{V_k}.$$

Then for $l \ne 1$,

$$\begin{aligned}\Delta W_k(z) + e^{U_k}W_k(z) &= a_0 e^{U_k} + a_1 z e^{U_k} + \frac{1}{2\mu_k^{7/4}}\Delta\mathfrak{h}_k(0)|z|^2 e^{U_k}\\ &\quad + \frac{1}{\mu_k^{7/4}}R_2(\theta)|z|^2 e^{U_k} + O(\varepsilon_k^{\varepsilon}(1+|z|)^{-3}),\end{aligned}$$

where

$$\begin{aligned}a_0 &= \big(\mathfrak{h}_k(\delta_k e_1) - \mathfrak{h}_k(\delta_k e_l)\big)/(\delta_k^2\mu_k^{7/4}),\\ a_1 &= -\nabla\mathfrak{h}_k(\delta_k e_l)/(\delta_k\mu_k^{7/4}),\end{aligned}$$

and R_2 is the sum of spherical harmonic functions of degree 2. Note that the assumption $l \ne 1$ means there is no appearance of ε_k or ε_k^2 in the equation for W_k.

Let $g_k(r)$ be the projection of W_k on 1. By the same ODE analysis as before, we see that g_k satisfies

$$g_k'' + \frac{1}{r}g_k'(r) + e^{U_k}g_k = E_k,$$

where

$$E_k(r) = O(\varepsilon_k^{\varepsilon})(1+r)^{-3} + \frac{1}{2\mu_k^{7/4}}\Delta(\log\mathfrak{h}_k)(0)r^2 e^{U_k}.$$

Using the same argument as in Lemma 4.1, we have

$$g_k(r) \sim \Delta(\log\mathfrak{h}_k)(0)(\log r)^2\mu_k^{-7/4}, \quad r > 10.$$

Clearly if $\Delta(\log \mathfrak{h}_k(0) \neq 0$ we obtain a violation of the bound of w_k for $r \sim \varepsilon_k^{-1}$. Theorem 1.1 for $N \geq 2$ is proved under the assumption

$$\varepsilon_k^{-1}|Q_s^k - e^{i\beta_s}| \leq \varepsilon_k^{\varepsilon}, \quad s = 1, \ldots, N. \tag{5.2}$$

We need this assumption because the ξ_k function that comes from the equation of w_k has to tend to U after scaling. From [26, (3.13)], $|Q_s^k - e^{i\beta_s}| = O(\delta_k^2) + O(\mu_k e^{-\mu_k})$. If $\delta_k^2 \varepsilon_k^{-1} \geq C$, the argument in Theorem 4.1 cannot be used because either ξ_k does not tend to U or $c_0 = 0$ cannot be proved. For $N \geq 2$, this is not a problem because we only consider $\delta_k \leq C\mu_k^{1/2}\varepsilon_k$.

Next we prove Theorem 1.1 for $N = 1$ and $\delta_k \leq \mu_k \varepsilon_k$. The reader can see immediately that the proof for $N \geq 2$ still works.

So we now handle the only remaining case.

Proof of Theorem 1.1 for $N = 1$ and $\delta_k \geq \mu_k \varepsilon_k$. In this case we write the equation for w_k as

$$\Delta w_k + |y|^2 \mathfrak{h}_k(\delta_k y) e^{v_k} - |y|^2 \mathfrak{h}_k(\delta_k e_1) e^{V_k} = 0.$$

From $0 = \nabla w_k(e_1)$ we have

$$0 = \int_{\Omega_k} \nabla_1 G_k(e_1, \eta) |\eta|^2 \big(\mathfrak{h}_k(\delta_k \eta) e^{v_k} - \mathfrak{h}_k(\delta_k e_1) e^{V_k}\big)\, d\eta + O(\delta_k^3). \tag{5.3}$$

Note that v_k is close to another global solution $\bar{V}_k$ which matches with a local maximum of v_k at Q_2^k. Evaluating the right hand side of (5.3) we have

$$\nabla_1 G_k(e_1, Q_2^k) - \nabla_1 G_k(e_1, e^{i\pi}) = O(\varepsilon_k^2 \mu_k) + O(\delta_k^3).$$

This expression gives

$$Q_2^k - e^{i\pi} = O(\delta_k^3) + O(\mu_k \varepsilon_k^2).$$

This estimate will lead to a better estimate of w_k outside the two bubbling disks. From the Green's representation for w_k we now obtain

$$\begin{aligned} w_k(y) = &\int_{\Omega_k} (G_k(y,\eta) - G_k(e_1,\eta))|\eta|^2 \big(\mathfrak{h}_k(\delta_k \eta) e^{v_k(y)} - \mathfrak{h}_k(\delta_k e_1) e^{V_k}\big)\, d\eta \\ &+ O(\delta_k^2), \end{aligned}$$

where $O(\delta_k^2)$ comes from the oscillation of w_k on $\partial\Omega_k$. Then we have

$$\begin{aligned} w_k(y) &= -\frac{1}{2\pi} \int_{\Omega_k} \log \frac{|y-\eta|}{|e_1-\eta|} |\eta|^2 \big(\mathfrak{h}_k(\delta_k \eta) - \mathfrak{h}_k(\delta_k e_1) e^{V_k}\big) + O(\delta_k^2) \\ &= -4 \log \frac{|y - Q_2^k|}{|e_1 - Q_2^k|} + 4 \log \frac{|y - e^{i\pi}|}{2} + O(\delta_k^2 \mu_k). \end{aligned}$$

Since $|Q_2^k - e^{i\pi}| = O(\delta_k^2)$ we see that $w_k(y) = O(\delta_k^2)$ on $|y - e^{i\pi}| = \tau$.

The standard pointwise estimate for singular equations (see [16, 30]) gives

$$v_k(Q_2^k+\varepsilon_k z)+2\log\varepsilon_k = \log\frac{e^{\mu_k}}{\left(1+\frac{e^{\mu_k}}{8\mathfrak{h}_k(\delta_k Q_k)}|z|^2\right)^2}+\phi_1^k+C\delta_k^2\Delta(\log\mathfrak{h}_k)(0)(\log(1+|z|))^2,\quad |z|\sim\varepsilon_k^{-1}.$$

and

$$V_k(e^{i\pi}+\varepsilon_k z)+2\log\varepsilon_k = \log\frac{e^{\mu_k}}{\left(1+\frac{e^{\mu_k}}{8\mathfrak{h}_k(\delta_k e_1)}|z|^2\right)^2}+\phi_2^k+O(\varepsilon_k^2(\log\varepsilon_k)^2),\quad |z|\sim\varepsilon_k^{-1}.$$

Thus

$$w_k(Q_2^k+\varepsilon_k z)=O(\varepsilon_k^2(\log\varepsilon_k)^2)+\phi_1^k-\phi_2^k+C\Delta(\log\mathfrak{h}_k)(0)\delta_k^2(\log(1+|z|))^2$$

for $|z|\sim\varepsilon_k^{-1}$. Taking the average around the origin, the spherical averages of the two harmonic functions are zero and $O(\delta_k^2)$ respectively, since they take zero at the origin and at a point at most $O(\delta_k^2)$ away from the origin. So the spherical average of w_k is comparable to

$$\Delta(\log\mathfrak{h}_k)(0)\delta_k^2(\log\varepsilon_k)^2$$

for $|z|\sim\varepsilon_k^{-1}$. Thus we know $\Delta(\log\mathfrak{h}_k)(0)=o(1)$ because $w_k=O(\delta_k^2\mu_k)$ in this region. Theorem 1.1 is thus established in all cases.

6. Singular mean field equation

In this section we prove Theorem 1.2. First it is well known that if p is a blowup point that has a non-quantized singular source ($\alpha_p=0$ or $\alpha_p\notin\mathbb{N}$), the profile of the bubbling solutions around p is a simple blowup (see [30, 31]). So we only need to focus on the case $\alpha_p\in\mathbb{N}$. Let $G(\cdot,\cdot)$ be the Green's function corresponding to $-\Delta_g$:

$$-\Delta_y G(p,y)=\delta_p-1,\qquad \int G(p,y)\,dV_g=0.$$

By setting

$$G_1(y)=4\pi\sum_{t=1}^M\alpha_t G(p_t,y),$$

we have

$$-\Delta G_1=4\pi\sum_t\alpha_t(\delta_{p_t}-1).$$

Then the function $v_k=u_k+G_1$ satisfies

$$\Delta_g v_k+\rho^k\left(\frac{he^{v_k}e^{-G_1}}{\int he^{u_k}}-1\right)=0.$$

Let p_1 be a quantized singular source, which means $\alpha_{p_1} \in 4\pi\mathbb{N}$. In the neighborhood of p_1 we have

$$\Delta_g v_k + |y - p_1|^{2\alpha_{p_1}} H_k e^{v_k} = \rho_k,$$

where

$$H_k = \frac{-\rho_k h e^{4\pi\alpha_1\gamma(p_1,y) - 4\pi\sum_{t\neq 1}\alpha_t G(p_t,y)}}{\int_M h e^{u_k}},$$

where γ is the regular part of the Green's function. In local coordinates around p_1, the equation can be written as

$$\Delta v_k + |x|^{2\alpha_{p_1}} H e^{\phi} e^{v_k} = \rho e^{\phi},$$

where $\phi(0) = |\nabla\phi(0)| = 0$ and $\Delta\phi(0) = -2K(p_1)$. Finally, we use f to remove the right hand side:

$$\Delta f = \rho^k e^{\phi}, \quad f(0) = 0, \quad f = \text{constant on } \partial B_\tau,$$

for $\tau > 0$ small. When we consider $v^k - f$ as the blowup solutions, we have

$$\Delta(v_k - f) + |y|^{2\alpha_{p_1}} H_k e^f e^{v_k - f} = 0.$$

It is a standard result that H_k is uniformly bounded above and below. From the definition of H_k we have

$$\Delta H_k(0) = \Delta h(p_1) - \sum_{t=1}^{M} 4\pi\alpha_t.$$

Using Theorem 1.1 we would have

$$\Delta \log H_k(0) + \Delta\phi + \Delta f = o(1)$$

if non-simple blowup happens at p_1, which is

$$\Delta \log h(p_1) - 2K(p_1) - 4\pi\sum_{t=1}^{M}\alpha_t + \rho^k = o(1). \tag{6.1}$$

Since $\rho^k \to \rho \in \Lambda$, we see from (1.8) that (6.1) cannot hold. Theorem 1.2 is established. ∎

7. Appendix

In this section we prove (4.45). Here we recall that v_k is close to $V_{s,k}$ near Q_s^k (see 4.35)). That is why (4.36) is used here. The terms of ϕ_1 and ϕ_3 lead to $o(\varepsilon_k)$, the integration involving ϕ_2 cancels with the second term of (4.45). The computation of ϕ_2 is based on this equation:

$$\int_{\mathbb{R}^2} \frac{\frac{\mathfrak{h}_k(\delta_k Q_l^k)}{4}\sigma_k \nabla\mathfrak{h}_k(\delta_k Q_l^k)(Q_l^k - Q_s^k)|z|^2}{\left(1 + \frac{\mathfrak{h}_k(\delta_k Q_l^k)}{8}|z|^2\right)^3} dz = 8\pi\sigma_k\nabla(\log\mathfrak{h}_k)(\delta_k Q_l^k)(Q_l^k - Q_s^k),$$

and by (4.2),

$$\nabla \log \mathfrak{h}_k(\delta_k Q_l^k) - \nabla \log \mathfrak{h}_k(\delta_k Q_s^k) = O(\delta_k) = o(\varepsilon_k). \tag{7.1}$$

The integration involving ϕ_4 provides the leading term. More detailed information is the following: First, for a global solution

$$V_{\mu,p} = \log \frac{e^\mu}{\left(1 + \frac{e^\mu}{\lambda}|z^{N+1} - p|^2\right)^2}$$

of

$$\Delta V_{\mu,p} + \frac{8(N+1)^2}{\lambda}|z|^{2N} e^{V_{\mu,p}} = 0 \quad \text{in } \mathbb{R}^2,$$

by differentiation with respect to μ we have

$$\Delta(\partial_\mu V_{\mu,p}) + \frac{8(N+1)^2}{\lambda}|z|^{2N} e^{V_{\mu,p}} \partial_\mu V_{\mu,p} = 0 \quad \text{in } \mathbb{R}^2.$$

By the expression of $V_{\mu,p}$ we see that

$$\partial_r(\partial_\mu V_{\mu,p})(x) = O(|x|^{-2N-3}).$$

Thus

$$\int_{\mathbb{R}^2} \partial_\mu V_{\mu,p}|z|^{2N} e^{V_{\mu,p}} = \int_{\mathbb{R}^2} \frac{\left(1 - \frac{e^\mu}{\lambda}|z^{N+1} - P|^2\right)|z|^{2N}}{\left(1 + \frac{e^\mu}{\lambda}|z^{N+1} - P|^2\right)^3} dz = 0. \tag{7.2}$$

We also have

$$\int_{\mathbb{R}^2} \partial_P V_{\mu,p}|y|^{2N} e^{V_{\mu,p}} = \int_{\mathbb{R}^2} \partial_{\bar{P}} V_{\mu,p}|y|^{2N} e^{V_{\mu,p}} = 0,$$

which gives

$$\int_{\mathbb{R}^2} \frac{\frac{e^\mu}{\lambda}(\bar{z}^{N+1} - \bar{P})|z|^{2N}}{\left(1 + \frac{e^\mu}{\lambda}|z^{N+1} - P|^2\right)^3} = \int_{\mathbb{R}^2} \frac{\frac{e^\mu}{\lambda}(z^{N+1} - P)|z|^{2N}}{\left(1 + \frac{e^\mu}{\lambda}|z^{N+1} - P|^2\right)^3} = 0. \tag{7.3}$$

Now we need more precise expressions of ϕ_1, ϕ_3 and B:

$$\begin{aligned}
\phi_1 &= (\mu_s^k - \mu_l^k)\left(1 - \frac{(N+1)^2}{D_s^k}\left|z + \frac{N}{2}\varepsilon_k z^2 e^{-i\beta_s}\right|^2\right)/B, \\
\phi_3 &= \frac{4(N+1)}{D_s^k B} \operatorname{Re}\left(\left(z + \frac{N}{2}\varepsilon_k e^{-i\beta_s} z^2\right)\left(\frac{\bar{p}_s^k - \bar{p}_l^k}{\varepsilon_k} e^{-i\beta_s}\right)\right), \\
B &= 1 + \frac{(N+1)^2}{D_s^k}\left|z + \frac{N}{2} z^2 e^{-i\beta_s}\varepsilon_k\right|^2,
\end{aligned}$$

We now use scaling and cancellation to obtain

$$\int_{B(0,\tau\varepsilon_k^{-1})} \frac{\phi_1}{M_k} B^{-2} = o(\varepsilon_k), \quad \int_{B(0,\tau\varepsilon_k^{-1})} \frac{\phi_3}{M_k} B^{-2} = o(\varepsilon_k).$$

Thus (4.45) holds.

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