



Michael K. Brown · Daniel Erman

## Tate resolutions on toric varieties

Received 28 March 2022; revised 5 October 2023

**Abstract.** We develop an analogue of Eisenbud–Fløystad–Schreyer’s Tate resolutions for toric varieties. Our construction, which is given by a noncommutative analogue of a Fourier–Mukai transform, works quite generally and provides a new perspective on the relationship between Tate resolutions and Beilinson’s resolution of the diagonal. We also develop a Beilinson-type resolution of the diagonal for toric varieties.

**Keywords:** Bernstein–Gel’fand–Gel’fand correspondence, Fourier–Mukai transform, resolution of the diagonal, Tate resolution, toric variety.

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### 1. Introduction

Eisenbud–Fløystad–Schreyer’s Tate resolutions are a powerful tool in the study of sheaves on projective space [39], closely connected with both the Bernstein–Gel’fand–Gel’fand correspondence [16] and Beilinson monads [12] and with many applications to commutative algebra, algebraic geometry, and computation [1, 5, 13, 28, 32, 34, 40–44, 46, 47, 66]. The main goal of this paper is to develop an analogue of Tate resolutions over projective toric varieties. It is reasonable to wonder whether such a theory should even exist, as other related homological features of  $\mathbb{P}^n$  – such as the existence of full, strongly exceptional collections of line bundles – do not extend to all projective toric varieties [55]. Indeed, key aspects of Eisenbud–Fløystad–Schreyer’s approach to Tate resolutions over  $\mathbb{P}^n$  simply do not generalize to other toric varieties.

We therefore give a totally new construction of Tate resolutions, based on a noncommutative analogue of a Fourier–Mukai transform. Our toric Tate resolutions exhibit both the known features of Tate resolutions on  $\mathbb{P}^n$  and subtle new behaviour, such as exactness properties that are parametrized by the combinatorics of the toric variety. The Tate

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*Mathematics Subject Classification 2020:* 14M25 (primary); 13D02, 14F06, 14F08 (secondary).

resolutions we obtain over  $\mathbb{P}^n$  recover the original notion of [39], though our construction is novel even in this case, yielding new proofs of the main results in [39] and clarifying the relationship between Tate resolutions and Beilinson's resolution of the diagonal. We also construct a toric analogue of Beilinson's resolution of the diagonal, generalizing a result of [27] and yielding an analogue of Beilinson monads in the toric setting.

Throughout the introduction,  $X$  denotes a smooth projective toric variety over a field  $k$ , and  $S = k[x_0, \dots, x_n]$  denotes the Cox ring of  $X$ , graded by the divisor class group  $\text{Cl}(X)$ .<sup>1</sup> Tate resolutions are a geometric manifestation of the Koszul duality between a polynomial ring and an exterior algebra, which is often called the *Bernstein–Gel'fand–Gel'fand (BGG) correspondence*. We briefly recall a multigraded version of the BGG correspondence. Let  $E = \Lambda_k(e_0, \dots, e_n)$ , equipped with the  $\text{Cl}(X) \oplus \mathbb{Z}$ -grading where  $\deg(e_i) = (-\deg(x_i); -1)$ . The Koszul duality between  $S$  and  $E$  takes the form of an adjunction

$$\mathbf{L}: \text{DM}(E) \rightleftarrows \text{Com}(S) : \mathbf{R},$$

where  $\text{Com}(S)$  is the category of complexes of  $\text{Cl}(X)$ -graded  $S$ -modules, and  $\text{DM}(E)$  is the category of *differential  $E$ -modules*, i.e.,  $E$ -modules equipped with a square 0 endomorphism of degree  $(0; -1)$  [11, 52]. See Section 2 for details.

In [39], the *Tate resolution* functor

$$\mathbf{T}: \text{Coh}(\mathbb{P}^n) \rightarrow \text{Com}(E)$$

is introduced as a geometric refinement of the (standard graded) BGG functor  $\mathbf{R}$ . Tate resolutions provide an explicit link between the BGG correspondence and Beilinson monads, and they lead to an efficient algorithm for computing sheaf cohomology [34], among many other applications. Our main goal is to extend the theory of Tate resolutions to more general toric varieties.

Our construction of toric Tate resolutions differs significantly from that of [39]. First, as in the multigraded BGG correspondence, toric Tate resolutions are differential  $E$ -modules, rather than complexes. Second, the previous constructions of Tate resolutions simply do not extend to the general toric setting; see Remark 3.9 for just one problem that arises. Instead, we build toric Tate resolutions via a Fourier–Mukai transform associated to the diagram

$$\begin{array}{ccc} & X \times \text{“Spec}(E)'' & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ X & & \text{“Spec}(E)''. \end{array}$$

<sup>1</sup>The smoothness assumption on  $X$  can be removed throughout at the cost of working with an associated toric *stack*, rather than a toric variety, and our main results are proven in this level of generality. This is a fairly secondary point, so we stick to the smooth case in the rest of the introduction, unless otherwise noted.

Of course, this diagram does not fully make sense:  $E$  is a noncommutative ring, and so “ $\text{Spec}(E)$ ” is not well defined. But, one may define the toric Tate resolution functor  $\mathbf{T}$  via the corresponding categorical diagram

$$\begin{array}{ccccc}
 & \text{Coh}(X_E) & \xrightarrow{- \otimes \mathcal{K}} & \text{DM}(X_E) & \\
 \pi_1^* \nearrow & & & & \searrow R\pi_2_* \\
 \text{Coh}(X) & \xrightarrow{\mathbf{T}} & & & K_{\text{DM}}(E).
 \end{array} \tag{1.1}$$

Here,  $\text{Coh}(X_E)$  (resp.  $\text{DM}(X_E)$ ) is the category of coherent (resp. differential)  $\mathcal{O}_X$ - $E$  bimodules,  $K_{\text{DM}}(E)$  is the homotopy category of differential  $E$ -modules, and the object  $\mathcal{K}$  is a Koszul complex; see Section 3.2 for a full explanation of the categories and functors in (1.1).

Our main result shows that toric Tate resolutions have properties analogous to those over  $\mathbb{P}^n$  (see Theorem 3.3 for a statement including the nonsmooth case).

**Theorem 1.1.** *Let  $X$  be a smooth, projective toric variety with Cox ring  $S$ , and let  $E$  be the  $\text{Cl}(X) \oplus \mathbb{Z}$ -graded Koszul dual exterior algebra of  $S$ , as in the multigraded BGG correspondence. Let  $\mathcal{F}$  be a coherent sheaf on  $X$ . The Tate resolution  $\mathbf{T}(\mathcal{F})$ , defined as in (1.1), has the following properties:*

- (1)  *$\mathbf{T}(\mathcal{F})$  is a minimal, exact, free differential  $E$ -module.*
- (2) *The Tate resolution encodes the sheaf cohomology groups of  $\mathcal{F}$ : more precisely, for any  $a \in \text{Cl}(X)$  and  $j \in \mathbb{Z}$ , we have  $H^j(X, \mathcal{F}(a)) = \underline{\text{Hom}}_E(k, \mathbf{T}(\mathcal{F}))(a; -j)$ .*
- (3) *Let  $B$  denote the irrelevant ideal of  $X$ , and let  $M$  be a  $\text{Cl}(X)$ -graded  $S$ -module such that  $\tilde{M} = \mathcal{F}$ . Assume  $H_B^0(M) = 0$ . The injective map  $M \hookrightarrow \bigoplus_{a \in \text{Cl}(X)} H^0(X, \mathcal{F}(a))$  induces an embedding  $\mathbf{R}(M) \hookrightarrow \mathbf{T}(\mathcal{F})$  of differential  $E$ -modules.*

Theorem 1.1 shows that our toric Tate resolutions have the same key characteristics as those of [39]. There is a categorical shift to differential modules, but when  $X = \mathbb{P}^n$ , the “folding” and “unfolding” functors from Section 2.1 recover the original results. For toric varieties other than  $\mathbb{P}^n$ , Tate resolutions also have rich, new structures; the reader may want to look to Section 3.4, where several examples of toric Tate resolutions are worked out in detail.

Moreover, our Fourier–Mukai construction provides an independent – and conceptually quite different – proof of the main results on Tate resolutions from [39]. For instance, Eisenbud–Fløystad–Schreyer’s proof of Theorem 1.1 (2) over  $\mathbb{P}^n$  relies upon their theory of the linear part of a free complex, as developed in [39, §3]; by contrast, Theorem 1.1 (2) is an elementary consequence of our construction. Our work also reveals a deeper connection between Tate resolutions and Beilinson’s resolution of the diagonal: when  $X = \mathbb{P}^n$ , the Fourier–Mukai kernel  $\mathcal{K}$  “lifts” Beilinson’s resolution of the diagonal from  $\mathbb{P}^n \times \mathbb{P}^n$  to “ $\mathbb{P}^n \times E$ ”. See Section 3.3 for details, and see Section 5 for how this generalizes to other toric varieties.

The toric variety  $X$  is determined by its Cox ring  $S$  and its irrelevant ideal. Since the Tate resolution is a fundamentally geometric object, it is natural to ask: how is the

irrelevant ideal of  $X$  reflected in the properties of toric Tate resolutions? Our next result provides the answer, demonstrating a phenomenon that was not present on  $\mathbb{P}^n$ . To state the result, we need the following definition: if  $I \subseteq \{0, 1, \dots, n\}$ , we set  $P_I := \langle x_i : i \in I \rangle \subseteq S$ , and we say that  $I$  is *irrelevant* if  $P_I$  contains the irrelevant ideal of  $X$ .

**Theorem 1.2** (Exactness properties). *Let  $I \subseteq \{0, 1, \dots, n\}$ , and let  $E_I = E/\langle e_i : i \notin I \rangle$ . If  $I$  is an irrelevant subset, then, for any coherent sheaf  $\mathcal{F}$  on  $X$ ,  $\mathbf{T}(\mathcal{F}) \otimes_E E_I$  is exact.*

Thus, not only are Tate resolutions exact, but this exactness is robust and nuanced. For instance, if  $X$  is the blowup of  $\mathbb{P}^1 \times \mathbb{P}^1$  at a torus fixed point, then the irrelevant ideal is

$$\langle x_0, x_2 \rangle \cap \langle x_1, x_3 \rangle \cap \langle x_2, x_4 \rangle \cap \langle x_0, x_3 \rangle \cap \langle x_1, x_4 \rangle.$$

The theorem says that any Tate resolution  $\mathbf{T}(\mathcal{F})$  for a sheaf on  $X$  will remain exact even if we restrict the differential to just the variables  $\{e_0, e_2\}$ , or to just  $\{e_1, e_3\}$ , and so on. This result highlights the fact that Tate resolutions depend not just on the Cox ring of  $X$ , but also on the combinatorics of the irrelevant ideal; the examples in Section 3.4 illustrate this point in some detail. These exactness properties are invisible over  $\mathbb{P}^n$ , as there are no nontrivial irrelevant subsets of  $\{0, 1, \dots, n\}$  in that case. However, a precursor to this result can be found in the “exact strands” result from [38, §3] for products of projective spaces.

A main result of [39] is that a coherent sheaf on  $\mathbb{P}^n$  can be completely recovered from its Tate resolution [39, Theorem 6.1]. We generalize this to toric varieties in Corollary 5.2. En route to this result, we develop a toric analogue of Beilinson’s resolution of the diagonal in Theorem 4.1. Our resolution of the diagonal is based on a fairly simple idea. Let  $S' = k[x_0, \dots, x_n, y_0, \dots, y_n]$  be the Cox ring of  $X \times X$ . The image of the diagonal embedding  $X \hookrightarrow X \times X$  is essentially the locus where “ $x_i = y_i$ ” for all  $i$ , though those equations are not homogeneous and hence do not yield well-defined equations on  $X \times X$ . However, we can force these equations to be homogeneous by imposing them not on  $S'$  itself, but on a certain  $S'$ -module. This recovers a result of [27] and has implications for the study of derived categories of toric varieties and virtual resolutions in multigraded commutative algebra; see also Conjecture 7.2 and the surrounding discussion.

### Overview

In Section 2, we discuss the multigraded BGG correspondence in detail. While many of the results in this section are known, some key results, like Proposition 2.11, are new. In Section 3, we introduce our construction of the toric Tate resolution, and we prove Theorems 1.1 and 1.2. In Section 4, we construct our toric resolution of the diagonal, and we explain in Section 5 how one can recover a sheaf from its toric Tate resolution. As an application of our results, we give in Section 6 an interpretation of the bounded derived category of a weighted projective stack in terms of its Koszul dual exterior algebra. Finally, we discuss some questions and conjectures in Section 7. We provide in Appendix A some background on the notion of a “positively” multigraded ring, and Appendix B is a collection of necessary technical results on differential modules.

## 2. The multigraded Bernstein–Gel’fand–Gel’fand correspondence

We recall the multigraded BGG correspondence discussed in the introduction, and we prove a number of related results. Throughout this section,  $S$  denotes a polynomial ring  $k[x_0, \dots, x_n]$  that is positively graded by some abelian group  $A$  (Definition A.1). In particular,  $S$  may be the Cox ring of a projective toric variety (Example A.2). As in the introduction, we let  $E = \Lambda_k(e_0, \dots, e_n)$ , equipped with the  $A \oplus \mathbb{Z}$ -grading given by  $\deg(e_i) = (-\deg(x_i); -1)$ . We call the  $\mathbb{Z}$ -grading on  $E$  the *auxiliary grading*.

**Conventions 2.1.** Throughout the paper, all  $E$ -modules are right modules. However, since  $E$  is graded commutative (with respect to the auxiliary grading), any right  $E$ -module  $M$  can be considered as a left  $E$ -module. In detail, the left  $E$ -action on a right  $E$ -module  $M$  is given by  $em = (-1)^{\text{aux}(e)\text{aux}(m)}me$ , where  $\text{aux}(-)$  denotes the degree in the auxiliary grading. Given a graded ring  $R$ , we let  $\underline{\text{Hom}}_R(-, -)$  denote the internal Hom in the category of graded  $R$ -modules.

### 2.1. Differential $E$ -modules

**Definition 2.2.** A *differential  $E$ -module* is an  $A \oplus \mathbb{Z}$ -graded right  $E$ -module  $D$  equipped with a degree  $(0; -1)$  endomorphism  $\partial$  such that  $\partial^2 = 0$ . We restrict attention to differentials with this particular degree because they are the only ones that will arise in our work. See [20] for a more general treatment of differential modules.<sup>2</sup> We have  $\partial(de) = \partial(d)e$  for all  $d \in D$  and  $e \in E$ ; bearing in mind Conventions 2.1, we also have  $\partial(ed) = (-1)^{\text{aux}(e)}e\partial(d)$ .

A *morphism*  $D \rightarrow D'$  of differential  $E$ -modules is a degree 0 map  $f: D \rightarrow D'$  satisfying  $f\partial = \partial'f$ . Let  $\text{DM}(E)$  denote the category of differential  $E$ -modules. The *homology* of an object  $D \in \text{DM}(E)$  is the subquotient  $\ker(\partial: D \rightarrow D(0; -1)) / \text{im}(\partial: D(0; 1) \rightarrow D)$ , denoted by  $H(D)$ . A morphism in  $\text{DM}(E)$  is a *quasi-isomorphism* if it induces an isomorphism on homology. A *homotopy* of morphisms  $f, f': D \rightarrow D'$  in  $\text{DM}(E)$  is a morphism  $h: D \rightarrow D'(0; 1)$  of  $E$ -modules such that  $f - f' = h\partial + \partial'h$ . The *mapping cone* of a morphism  $f: D \rightarrow D'$  in  $\text{DM}(E)$  is the module  $D' \oplus D(0, -1)$  equipped with the differential  $\begin{pmatrix} \partial' & f \\ 0 & -\partial \end{pmatrix}$ .

**Remark 2.3.** The ring  $E$  may be considered as an  $A$ -graded dg  $k$ -algebra with trivial differential and homological grading induced by the auxiliary  $\mathbb{Z}$ -grading. The category  $\text{DM}(E)$  is equivalent (in fact, isomorphic) to the category of dg-modules over this dg-algebra. This is the perspective taken in the proof of the multigraded BGG correspondence in [52]. We use differential modules because they are more amenable to computing in Macaulay2.

Yet another way of thinking of the category  $\text{DM}(E)$  is as follows. Let  $\text{Com}_{\text{per}}(E)$  denote the category of complexes of  $E$ -modules of the form  $\dots \rightarrow N(0; 1) \xrightarrow{\partial} N \xrightarrow{\partial} \dots$

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<sup>2</sup>Differential modules also appear in Rouquier’s analogue of the BGG correspondence for non-graded polynomial and exterior algebras in [65, Section 4].

$N(0; -1) \rightarrow \dots$ , where the differentials are all identical. A morphism is a chain map that is the same in each homological degree. There is an equivalence (in fact, isomorphism) of categories  $\text{Ex: } \text{DM}(E) \xrightarrow{\sim} \text{Com}_{\text{per}}(E)$  that sends a differential module  $(D, \partial)$  to the complex

$$\text{Ex}(D) = (\dots \rightarrow D(0; 1) \xrightarrow{\partial} D \xrightarrow{\partial} D(0; -1) \rightarrow \dots).$$

We call  $\text{Ex}(D)$  the *expansion* of  $D$ , following the terminology in [7].

**Definition 2.4.** The *derived category of differential  $E$ -modules*  $\text{D}_{\text{DM}}(E)$  is obtained by inverting quasi-isomorphisms in  $\text{DM}(E)$ . The category  $\text{D}_{\text{DM}}(E)$  is triangulated, with shift functor given by  $D[1] = D(0; 1)$ ; to prove this, one may use the identification between differential  $E$ -modules and dg-modules over a certain dg-algebra explained in Remark 2.3. The *bounded derived category of differential  $E$ -modules*  $\text{D}_{\text{DM}}^b(E)$  is the subcategory of  $\text{D}_{\text{DM}}(E)$  given by objects with finitely generated homology. Denote by  $\text{D}_{\text{DM}}^{\text{fg}}(E)$  the subcategory of  $\text{D}_{\text{DM}}^b(E)$  given by objects whose underlying module is finitely generated.

**Proposition 2.5.** *The inclusion  $\text{D}_{\text{DM}}^{\text{fg}}(E) \hookrightarrow \text{D}_{\text{DM}}^b(E)$  is an equivalence.*

*Proof.* Let  $D \in \text{D}_{\text{DM}}^b(E)$ . Since  $S$  is positively  $A$ -graded,  $E$  is as well; choose a positive  $A$ -grading on  $E$ , as in Appendix A. For the rest of this proof, we will consider  $E$  as a non-negatively  $\mathbb{Z}$ -graded  $k$ -algebra and  $D$  as a  $\mathbb{Z}$ -graded  $E$ -module; notice that the differential on  $D$  is degree 0 with respect to this grading. By [20, Theorem 1.3 (b)],  $D$  admits a free resolution  $F$  such that  $\dim_k F_j < \infty$  for all  $j$ , and  $F_j = 0$  for  $j \ll 0$ . Choose  $N \gg 0$  such that the homology of  $F$  lives in degrees  $< N$ . Let  $F'$  be the  $E$ -submodule of  $F$  generated by the elements of degree  $< N$ . The submodule  $F'$  is finitely generated, and it is in fact a differential submodule of  $F$ , since the differential on  $F$  is degree 0. Finally, observe that the inclusion  $F' \hookrightarrow F$  is a quasi-isomorphism. ■

**Definition 2.6** (Folding and unfolding differential modules). Suppose  $S$  is equipped with the standard grading, i.e.,  $\deg(x_i) = 1$  for all  $i$ . Denote by  $\text{Com}(E)$  the category of complexes of  $\mathbb{Z}$ -graded  $E$ -modules. The *folding functor*  $\text{Fold: } \text{Com}(E) \rightarrow \text{DM}(E)$  is given by  $(\dots \rightarrow C_j \xrightarrow{\partial_C} C_{j-1} \rightarrow \dots) \mapsto (\bigoplus_{j \in \mathbb{Z}} C_j(0; -j), \partial_C)$ . Here, we think of each  $\mathbb{Z}$ -graded module  $C_j$  as  $\mathbb{Z} \times \mathbb{Z}$ -graded by setting

$$(C_j)_{(a,b)} = \begin{cases} (C_j)_a, & a = b, \\ 0, & a \neq b. \end{cases}$$

Going the other direction, if  $D \in \text{DM}(E)$ , set  $D_j = \{d \in D : \deg(d) = (a; i)\}$ , where  $i - a = j\}$ . Notice that the  $D_j$  are  $E$ -modules. Since  $\partial_D$  is a map from  $D$  to  $D(0; -1)$ ,  $\partial_D$  induces a map from  $D_j$  to  $D_{j-1}$  for all  $j$ . Note that any  $\mathbb{Z} \times \mathbb{Z}$ -graded  $E$ -module  $M$  can be considered as a  $\mathbb{Z}$ -graded  $E$ -module with components  $M_a = \bigoplus_{i \in \mathbb{Z}} M_{(a;i)}$ . The *unfolding functor*  $\text{Unfold: } \text{DM}(E) \rightarrow \text{Com}(E)$  is given by

$$D \mapsto (\dots \xrightarrow{\partial_D} D_j \xrightarrow{\partial_D} D_{j-1} \xrightarrow{\partial_D} \dots).$$

One easily verifies that the functors  $\text{Fold}$  and  $\text{Unfold}$  are inverse equivalences.

## 2.2. The multigraded Bernstein–Gel’fand–Gel’fand functors

Let  $\text{Com}(S)$  denote the category of complexes of  $A$ -graded  $S$ -modules. We begin by defining the BGG functors  $\mathbf{L}: \text{DM}(E) \rightleftarrows \text{Com}(S) : \mathbf{R}$ . Given  $D \in \text{DM}(E)$ , the complex  $\mathbf{L}(D)$  has terms and differential given by

$$\mathbf{L}(D)_j = \bigoplus_{a \in A} S(-a) \otimes_k D_{(a;j)} \quad \text{and} \quad s \otimes d \mapsto \left( \sum_{i=0}^n x_i s \otimes e_i d \right) - s \otimes \partial_D(d).$$

Here, while  $D$  is a right  $E$ -module, the  $e_i$  act on  $d$  on the left via the formula in Conventions 2.1. Let  $\omega_E$  denote the  $E$ -module  $\underline{\text{Hom}}_k(E, k) \cong E(-\sum_{i=0}^n \deg(x_i); -n - 1)$ . Given  $C \in \text{Com}(S)$ , the object  $\mathbf{R}(C) \in \text{DM}(E)$  has underlying  $E$ -module and differential

$$\bigoplus_{j \in \mathbb{Z}} \bigoplus_{a \in A} (C_j)_a \otimes_k \omega_E(-a; -j) \quad \text{and} \quad c \otimes f \mapsto (-1)^j \left( \sum_{i=0}^n x_i c \otimes e_i f \right) + \partial_C(c) \otimes f,$$

where we have assumed that  $c$  lies in  $C_j$ . The following theorem follows from work of Hawwa–Hoffman–Wang [52].

**Theorem 2.7.** *The functors  $\mathbf{L}$  and  $\mathbf{R}$  satisfy the following:*

- (a) *The functor  $\mathbf{L}$  is left adjoint to  $\mathbf{R}$ , and both  $\mathbf{L}$  and  $\mathbf{R}$  are exact.*
- (b) *Let  $C \in \text{Com}(S)$  and  $D \in \text{DM}(E)$ . The unit and counit of adjunction give quasi-isomorphisms  $D \xrightarrow{\sim} \mathbf{R}\mathbf{L}(D)$  and  $\mathbf{L}\mathbf{R}(C) \xrightarrow{\sim} C$ .*
- (c) *The functors  $\mathbf{L}$  and  $\mathbf{R}$  induce an equivalence  $\text{D}_{\text{DM}}(E) \simeq \text{D}(S)$ .*

**Example 2.8.** Given pairwise relatively prime positive integers  $w_0, \dots, w_n$ , the *weighted projective stack*  $\mathbb{P}(w_0, \dots, w_n)$  is the stack quotient of  $\mathbb{A}^{n+1} \setminus \{0\}$  by the  $\mathbb{G}_m$ -action  $t \cdot (a_0, \dots, a_n) = (t^{w_0}a_0, \dots, t^{w_n}a_n)$  (cf. [2, Definition 2.1.1]). Say  $S = k[x_0, x_1, x_2]$  is equipped with the  $\mathbb{Z}$ -grading given by  $\deg(x_0) = 1 = \deg(x_1)$  and  $\deg(x_2) = 2$ . Geometrically,  $S$  is the Cox ring of the weighted projective stack  $\mathbb{P}(1, 1, 2)$ . The differential module  $\mathbf{R}(S)$  is infinitely generated, and the degrees of its generators correspond to the degrees  $i \in \mathbb{Z}$  such that  $S_i \neq 0$ . It has the form

$$\omega_E \xrightarrow{\quad} \omega_E(-1; 0)^{\oplus 2} \xrightarrow{\quad} \omega_E(-2; 0)^{\oplus 4} \xrightarrow{\quad} \cdots.$$

(horizontal arrows) (curved arrows)

As bases, we choose the monomials  $\{1\}$ ,  $\{x_0, x_1\}$  and  $\{x_0^2, x_0x_1, x_1^2, x_2\}$ . The horizontal arrows are multiplication by  $e_0$  and  $e_1$ , and the curved arrows are multiplication by  $e_2$ ; for instance,  $\omega_E$  maps to  $\omega_E(-1; 0)^{\oplus 2}$  via  $(e_0, e_1)^t$  and to  $\omega_E(-2; 0)^{\oplus 4}$  via  $(0, 0, 0, e_2)^t$ .

**Remarks 2.9.** We highlight the following observations.

- (1) Given  $C \in \text{Com}(S)$ , it can be helpful to interpret  $\mathbf{R}(C)$  as the totalization of a bicomplex. Form a bicomplex with  $q$ -th row given by

$$\cdots \leftarrow \mathbf{R}(C_q)(0; -1) \xleftarrow{(-1)^q \sum_{i=0}^n x_i \otimes e_i} \mathbf{R}(C_q) \xleftarrow{(-1)^q \sum_{i=0}^n x_i \otimes e_i} \mathbf{R}(C_q)(0; 1) \leftarrow \cdots$$

and vertical differentials induced by  $\partial_C$ . Notice that the  $q$ -th row is the expansion of  $\mathbf{R}(C_q)$ , up to a sign on the differential (see Remark 2.3). Totalizing this bicomplex gives an object in  $\text{Com}_{\text{per}}(E)$ ; applying the inverse of the expansion functor gives the differential module  $\mathbf{R}(C)$ . Notice that the homological grading on  $C$  plays a crucial role here; for this reason, the functor  $\mathbf{L}$  is not given by the totalization of a bicomplex.

(2) When  $\deg(x_i) = 1$  for all  $i$ , the multigraded BGG functors are essentially the same as the original ones from [39]. More precisely, letting  $\mathbf{L}_{\text{classical}}$  and  $\mathbf{R}_{\text{classical}}$  denote the original BGG functors over  $\mathbb{P}^n$ , and using the notation of Definition 2.6, we have  $\mathbf{R} = \text{Fold} \circ \mathbf{R}_{\text{classical}}$  and  $\mathbf{L} = \mathbf{L}_{\text{classical}} \circ \text{Unfold}$ .

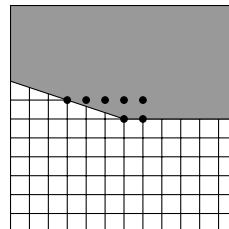
(3) Given a graded  $S$ -module  $M$ , the object  $\mathbf{R}(M)$  in  $\mathrm{DM}(E)$  is an injective coflag, which is like an injective resolution in the setting of differential modules; see Definition B.1.

(4) One could define similar functors between the categories  $\text{Com}(E)$  and  $\text{DM}(S)$  or between  $\text{DM}(E)$  and  $\text{DM}(S)$ . However, our primary interest is in complexes of  $S$ -modules, and so it is natural (for us) to preserve the homological grading on the  $S$ -module side.

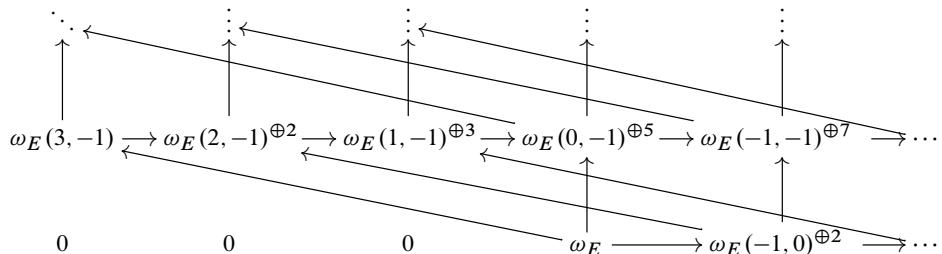
**Example 2.10.** Let  $X = \mathbb{F}_3$  be the Hirzebruch surface of type 3. Let  $S = k[x_0, x_1, x_2, x_3]$  be its Cox ring, where the  $\text{Cl}(X) = \mathbb{Z}^{\oplus 2}$ -grading is given by

$$\deg(x_0) = (1, 0), \quad \deg(x_1) = (-3, 1), \quad \deg(x_2) = (1, 0), \quad \text{and} \quad \deg(x_3) = (0, 1).$$

The degrees of the generators of  $\mathbf{R}(S)$  correspond to the degrees of the effective cone of  $X$ , as illustrated by the shaded region below:



Focusing on the free summands of  $\mathbf{R}(S)$  corresponding to the degrees marked with black dots, we get the diagram below. The twists in the auxiliary grading are all 0, so we have omitted them. The arrows indicate the effect of the differential:



For instance, if we choose the monomials in  $S$  as generators for the free module  $\mathbf{R}(S)$ , then the differential sends  $\omega_E$  to:  $\omega_E(3, -1)$  via  $e_1$ , to a summand of  $\omega_E(0, -1)^{\oplus 5}$  via  $e_3$ ; and to  $\omega_E(-1, 0)^{\oplus 2}$  via the column vector  $(e_0, e_2)^t$ .

The following calculation of the homology of the functors  $\mathbf{L}$  and  $\mathbf{R}$  extends [39, Proposition 2.3]. See Appendix B.2 for the definitions of tensor product, internal Hom, and Ext for differential modules used in the statement of this result and its proof.

**Proposition 2.11.** *Let  $C \in \text{Com}(S)$  and  $D \in \text{DM}(E)$ . Assume  $D$  is finitely generated as an  $E$ -module. We have*

- (a)  $H(\mathbf{R}(C))_{(a;j)} = \text{Tor}_j^S(C, k)_a$ , and
- (b)  $H_j(\mathbf{L}(D))_a = \text{Ext}_E^{\text{DM}}(k, D)_{(a;j)}$ .

*Proof.* We can view the Koszul complex  $K$  on the variables of  $S$  as the complex of  $A$ -graded  $S$ -modules with  $j$ -th term  $\bigoplus_{d \in A} S(-d) \otimes_k (\omega_E)_{(d;j)}$  and differential given by multiplication on the left by  $\sum_{i=0}^n x_i \otimes e_i$ . We have

$$\begin{aligned} \mathbf{R}(C)_{(a;j)} &= \bigoplus_{i \in \mathbb{Z}} \bigoplus_{d \in A} (C_i)_d \otimes_k (\omega_E)_{(-d+a;j-i)} \cong \left( \bigoplus_{i \in \mathbb{Z}} C_i \otimes_S K_{j-i} \right)_a \\ &= ((C \otimes_S K)_j)_a. \end{aligned}$$

Thus,  $\mathbf{R}(C) \cong C \otimes_S K$  as  $A \oplus \mathbb{Z}$ -graded  $k$ -vector spaces. Moreover, this isomorphism preserves the differentials; this proves (a). As for (b), setting  $w = \sum_{i=0}^n \deg(x_i)$ , we have

$$\begin{aligned} (\mathbf{L}(D)_j)_a &= \bigoplus_{d \in A} S_{-d} \otimes_k D_{(d+a;j)} \\ &\xrightarrow{\sigma} \bigoplus_{d \in A} S_{-d} \otimes_k (\omega_E(d + w; n + 1) \otimes_E D)_{(a;j)} = (\mathbf{R}(S(w)[n + 1]) \otimes_E^{\text{DM}} D)_{(a;j)}, \end{aligned}$$

where the  $k$ -linear isomorphism  $\sigma$  sends  $s \otimes d$  to  $(-1)^{|s|+j} s \otimes (x_0 \cdots x_n) \otimes d$ . Here,  $|s|$  denotes the degree of  $s$  in the standard grading on  $S$ , and  $x_0 \cdots x_n$  is the element of  $\omega_E$  dual to  $e_0 \cdots e_n \in E$ . This gives an isomorphism  $\mathbf{L}(D) \cong \mathbf{R}(S(w)[n + 1]) \otimes_E^{\text{DM}} D$  of  $A \oplus \mathbb{Z}$ -graded  $k$ -vector spaces that preserves the differentials.

Given  $N \in \text{DM}(E)$ , let  $N^\vee = \underline{\text{Hom}}_E^{\text{DM}}(N, E)$ . Since  $D$  is finitely generated, we have an isomorphism  $(\mathbf{R}(S(w)[n + 1]) \otimes_E^{\text{DM}} D)_{(a;j)} \cong \underline{\text{Hom}}_E^{\text{DM}}(\mathbf{R}(S(-w)[n + 1])^\vee, D)_{(a;j)}$  in  $\text{DM}(E)$ . Note that  $\mathbf{R}(S(-w)[n + 1])^\vee$  is a free resolution (in the sense of Definition B.1) of the residue field  $k$ , considered as an object in  $\text{DM}(E)$  with trivial differential. This proves (b).  $\blacksquare$

**Remark 2.12.** In the above proof, the finitely generated assumption is used when pulling a direct sum out of the first component of Hom. This subtlety does not arise in [39], due to the presence of the homological grading.

**Corollary 2.13.** *Let  $C \in \text{Com}(S)$ . If  $C$  is bounded, and each term of  $C$  is finitely generated, then  $\dim_k H(\mathbf{R}(C)) < \infty$ .*

**Corollary 2.14.** *The equivalence  $D_{DM}(E) \simeq D(S)$  induced by  $\mathbf{L}$  and  $\mathbf{R}$  in Theorem 2.7 (c) restricts to an equivalence  $D_{DM}^b(E) \simeq D^b(S)$ .*

*Proof.* Immediate from Proposition 2.5 and Corollary 2.13. ■

We next observe that a multigraded generalization of Eisenbud–Fløystad–Schreyer’s reciprocity theorem [39, Theorem 3.7 (a)] follows immediately from Theorem 2.7.

**Theorem 2.15.** *Let  $S$  be a positively graded polynomial ring over  $k$ , and let  $E$  denote its Koszul dual exterior algebra. Let  $M$  be a finitely generated  $S$ -module and  $N$  a finitely generated  $E$ -module. The complex  $\mathbf{L}(N)$  is a free resolution of  $M$  if and only if the differential module  $\mathbf{R}(M)$  is an injective resolution of  $N$  (see Definition B.1 for the notion of an injective resolution of a differential module).*

*Proof.* If there is a quasi-isomorphism  $\mathbf{L}(N) \xrightarrow{\sim} M$ , then, as  $\mathbf{R}$  is exact (Theorem 2.7 (a)), we have a quasi-isomorphism  $\mathbf{RL}(N) \xrightarrow{\sim} \mathbf{R}(M)$ . Composing with the quasi-isomorphism  $N \xrightarrow{\sim} \mathbf{RL}(N)$  from Theorem 2.7 (b) gives a quasi-isomorphism  $N \xrightarrow{\sim} \mathbf{R}(M)$ . Since  $\mathbf{R}(M)$  is an injective coflag (Remarks 2.9 (3)),  $N \xrightarrow{\sim} \mathbf{R}(M)$  is an injective resolution. The converse is similar. ■

**Remark 2.16.** A main emphasis of [39] is the relationship between BGG and linear complexes over polynomial and exterior algebras [39, Section 3]. While some of this theory extends naturally to the multigraded setting, there are also significant distinctions; we explore this in detail in our follow-up papers [21, 22]. For instance, one of the fundamental facts used by Eisenbud–Fløystad–Schreyer in their construction of Tate resolutions for  $\mathbb{P}^n$  is that, for any finitely generated module  $M$  and any  $d \gg 0$ , the minimal free resolution of the truncated module  $M_{\geq d}$  is linear, in the sense that it is of the form  $\mathbf{L}(N)$  for some  $E$ -module  $N$  [39, §4]. A similar fact plays a key role in Eisenbud–Erman–Schreyer’s construction for products of projective spaces [38, proof of Corollary 1.14]. However, this fundamentally fails to hold for more general toric varieties, motivating the need for a new approach to constructing Tate resolutions. For instance, let  $S$  be the Cox ring of a Hirzebruch surface of type 3, with notation as in Example 2.10. Consider  $M = S/(x_3 - x_0^3 x_1, x_2)$ . For any  $d = (d_1, d_2)$  with  $d_1, d_2 \geq 0$ , the minimal free resolution of  $M_{\geq d}$  is a twist of the Koszul complex on  $x_3 - x_0^3 x_1$  and  $x_2$ , which is not of the form  $\mathbf{L}(N)$  for any  $E$ -module  $N$ .

### 3. Tate resolutions on toric stacks

In this section, we extend the notion of a Tate resolution from  $\mathbb{P}^n$  to more general toric varieties, and we prove (a generalization of) Theorem 1.1. We recall some background on toric stacks in Section 3.1, prove our main results in Section 3.2, examine a connection to Beilinson’s resolution of the diagonal in Section 3.3, and discuss a number of examples in Section 3.4.

Let us start by briefly recalling Eisenbud–Fløystad–Schreyer’s approach to defining the Tate resolution in [39] and how it differs from ours. Their definition of the Tate resolution of a sheaf  $\mathcal{F}$  over  $\mathbb{P}^n$  is as follows: write  $\mathcal{F} = \tilde{M}$  for some  $S$ -module  $M$ . By [39, Corollary 2.4], when  $d$  is at least the Castelnuovo–Mumford regularity  $\text{reg}(M)$ , there is a quasi-isomorphism  $H_d(\mathbf{R}_{\text{classical}}(M_{\geq d})) \xrightarrow{\sim} \mathbf{R}_{\text{classical}}(M_{\geq d})$ .<sup>3</sup> Letting  $F$  be the minimal free resolution of this  $d$ -th homology module, they define the Tate resolution  $\mathbf{T}(\mathcal{F})$  to be the mapping cone of the composition  $F \xrightarrow{\sim} \mathbf{R}_{\text{classical}}(M_{\geq d})$ .

Attempting to adapt this recipe to the toric setting raises significant challenges. First, applying  $\mathbf{R}$  to “high” truncations of modules need not yield a differential module that is quasi-isomorphic to its homology; see Remark 3.9. While one could use the theory of [20] to resolve the entire differential module  $\mathbf{R}(M)$  (in the sense of Definition B.1), rather than the homology of  $\mathbf{R}(M_{\geq d})$  for some  $d$ , this raises further complications for adapting the arguments of [39], which often make use of the fact that the resolution of  $M_{\geq d}$  has the form  $\mathbf{L}(P)$  for some module  $P$ . There is also a second, more fundamental complication: over a projective toric variety  $X$  with  $\text{Cl}(X) \not\simeq \mathbb{Z}$ , taking a minimal free resolution of  $\mathbf{R}(M)$  is a purely algebraic construction that cannot differentiate between toric varieties with the same Cox ring. Thus, any construction of  $\mathbf{T}(\tilde{M})$  from  $\mathbf{R}(M)$  must incorporate features of the irrelevant ideal; this is not an issue on  $\mathbb{P}^n$ . And finally, even if one produced such a construction, verifying the properties in Theorems 1.1 and 1.2 would require nontrivial arguments, as it did in [38, 39].

We therefore use an approach to Tate resolutions that is fundamentally different from that of [39]. The central novelty is the introduction of a noncommutative analogue of a Fourier–Mukai transform that we use to construct Tate resolutions in great generality and that easily implies Theorems 1.1 and 1.2.

### 3.1. Setup

Let  $N$  be a finitely generated free abelian group and  $\Sigma$  a rational fan inside  $N \otimes_{\mathbb{Z}} \mathbb{Q}$ . We will write  $X = X_{\Sigma}$  for the toric variety determined by  $\Sigma$ . Let  $\Sigma(1) = \{\rho_0, \dots, \rho_n\}$  denote the rays of  $\Sigma$ , and let  $S = k[x_i : \rho_i \in \Sigma(1)]$  denote the *Cox ring* of  $\Sigma$ . The Cox ring  $S$  is equipped with a natural  $\text{Cl}(X)$ -grading. The *irrelevant ideal* of  $\Sigma$  is the ideal

$$B = \left\langle \prod_{\rho_i \not\subseteq \sigma} x_i : \sigma \in \Sigma \right\rangle$$

of  $S$ . The variety  $Z := \text{Spec}(S) \setminus V(B)$  admits an action by a torus  $G$ . Taking the stack quotient by this action gives the *toric stack*

$$\mathcal{X} = \mathcal{X}_{\Sigma} := [Z/G].$$

---

<sup>3</sup>Recall that  $\mathbf{R}_{\text{classical}}$  denotes the functor whose output is a complex of  $E$ -modules, as opposed to a differential  $E$ -module. It thus makes sense to discuss the  $d$ -th homology module of  $\mathbf{R}_{\text{classical}}(M_{\geq d})$ .

If  $X$  is smooth, then  $\mathcal{X} = X$  and we need not worry about the stack. When  $X$  is singular, we will generally apply adjectives to  $\mathcal{X}$  that could be applied to  $X$  or  $\Sigma$ . For instance, we say  $\mathcal{X}$  is *simplicial* (resp. *projective*) if  $X$  is simplicial (resp. projective).

When  $X$  is singular, it turns out that the stack  $\mathcal{X}$  is more closely connected to the BGG correspondence, Tate resolutions, and other topics considered in this paper. This aligns with a common philosophy: sheaves are often better behaved on a stack quotient (like  $\mathcal{X}$ ) than on a GIT quotient (like  $X$ ). There are many examples of this in the toric setting, e.g., [6, 10, 18, 26, 27, 48, 67].

While we do not require our toric stacks to be simplicial in this paper, we note that such stacks have particularly nice properties (see [17]): if  $k = \mathbb{C}$  and  $\mathcal{X}$  is simplicial, then  $\mathcal{X}$  is Deligne–Mumford, and the canonical map  $\pi: \mathcal{X} \rightarrow X$  exhibits  $X$  as a *good moduli scheme*, in the sense of [3, Section 1.2]. This implies, for instance, that  $\pi_*$  is exact on quasi-coherent sheaves, allowing cohomology computations to pass from  $\mathcal{X}$  to  $X$ .

**Notation 3.1.** If  $\mathcal{X}$  is a toric stack, then  $X$  will denote the corresponding variety.

**Remark 3.2.** It would be interesting to develop the theory of toric Tate resolutions in greater generality, by considering, for example, the more general toric Deligne–Mumford stacks studied by Borisov–Chen–Smith and Fantechi–Mann–Nironi [17, 45], the extended stacky fans introduced by Jiang [56], or the toric Artin stacks of Geraschenko–Satriano [48]. We do not pursue these routes in detail.

### 3.2. Main results

The goal of this section is to prove the following result, which combines and generalizes Theorems 1.1 and 1.2. In part (4) of the theorem, the notion of an irrelevant subset  $I$  of  $\{0, \dots, n\}$  and the notation  $E_I$  are as introduced in and above Theorem 1.2.

**Theorem 3.3.** *Let  $\mathcal{X}$  be a projective toric stack, and let  $\mathcal{F}$  be a coherent sheaf on  $\mathcal{X}$ . There exists  $\mathbf{T}(\mathcal{F}) \in \mathrm{DM}(E)$  with the following properties:*

- (1)  *$\mathbf{T}(\mathcal{F})$  is a minimal, exact, free differential  $E$ -module.*
- (2) *The Tate resolution encodes the sheaf cohomology groups of  $\mathcal{F}$ : more precisely, for any  $a \in \mathrm{Cl}(X)$  and  $j \in \mathbb{Z}$ , we have*

$$H^j(X, \mathcal{F}(a)) = \underline{\mathrm{Hom}}_E(k, \mathbf{T}(\mathcal{F}))(a; -j).$$

- (3) *Let  $B$  denote the irrelevant ideal of  $\mathcal{X}$ , and let  $M$  be a  $\mathrm{Cl}(X)$ -graded  $S$ -module such that  $\tilde{M} = \mathcal{F}$ . Assume  $H_B^0(M) = 0$ . The injective map  $M \hookrightarrow \bigoplus_{a \in \mathrm{Cl}(X)} H^0(X, \mathcal{F}(a))$  induces an embedding  $\mathbf{R}(M) \hookrightarrow \mathbf{T}(\mathcal{F})$  of differential  $E$ -modules.*
- (4) *Let  $I \subseteq \{0, \dots, n\}$ . If  $I$  is irrelevant, then  $\mathbf{T}(\mathcal{F}) \otimes_E E_I$  is exact.*

We call  $\mathbf{T}(\mathcal{F})$  the *Tate resolution* of  $\mathcal{F}$ , though as we will see, saying “the” Tate resolution is an abuse of terminology, because our construction is only well defined up to homotopy equivalence. We construct toric Tate resolutions via a “noncommutative

Fourier–Mukai transform” involving  $\mathcal{X}$  and the exterior algebra  $E$ . That is, we will define a functor  $\Phi_{\mathcal{K}}: \text{Coh}(\mathcal{X}) \rightarrow \text{DM}(E)$  via the following diagram:

$$\begin{array}{ccccc}
 & & \text{Coh}(\mathcal{X}_E) & \xrightarrow{- \otimes^{\text{DM}, \mathcal{K}}} & \text{DM}(\mathcal{X}_E) \\
 & \nearrow \pi_1^* & & & \searrow R\pi_2* \\
 \text{Coh}(\mathcal{X}) & \xrightarrow{\Phi_{\mathcal{K}}} & & & \text{DM}(E).
 \end{array} \tag{3.1}$$

Let us start by defining the categories in diagram (3.1). Loosely speaking, the category  $\text{Coh}(\mathcal{X}_E)$  has objects given by  $\mathcal{O}_{\mathcal{X}}\text{-}E$  bimodules that are  $\text{Cl}(X) \oplus \mathbb{Z}$ -graded as  $E$ -modules and coherent as  $\mathcal{O}_{\mathcal{X}}$ -modules, and the category  $\text{DM}(\mathcal{X}_E)$  has objects given by  $\mathcal{O}_{\mathcal{X}}\text{-}E$  bimodules that are equipped with a  $\text{Cl}(X) \oplus \mathbb{Z}$ -grading as  $E$ -modules and a square 0 endomorphism of degree  $(0; -1)$  with respect to the grading on  $E$ . More precisely, these categories are defined as follows. We note first that, since  $\mathcal{X}$  is the quotient stack  $[Z/G]$ , where  $Z$  and  $G$  are as in Section 3.1,  $\mathcal{O}_{\mathcal{X}}$ -modules may be identified with  $G$ -equivariant  $\mathcal{O}_Z$ -modules.

**Definition 3.4.** A  $G$ -equivariant  $\mathcal{O}_Z$ -module  $\mathcal{E}$  is an  $\mathcal{O}_{\mathcal{X}}\text{-}E$  bimodule if, for each open set  $U \subseteq Z$ ,  $\Gamma(U, \mathcal{E})$  has the structure of a  $\text{Cl}(X) \oplus \text{Cl}(X) \oplus \mathbb{Z}$ -graded  $S\text{-}E$  bimodule such that

- (1) the first  $\text{Cl}(X)$ -grading on  $\Gamma(U, \mathcal{E})$  is associated to its  $S$ -module structure, and the complementary  $\text{Cl}(X) \oplus \mathbb{Z}$ -grading makes it a graded  $E$ -module;
- (2) the restriction maps for  $\mathcal{E}$  are morphisms of bimodules; and
- (3) the  $E$ -actions on each  $\Gamma(U, \mathcal{E})$  are  $G$ -equivariant, which in this case just means that the actions of the exterior variables  $e_i$  on each  $\Gamma(U, \mathcal{E})$  are homogeneous of degree 0 with respect to the  $\text{Cl}(X)$ -grading associated to the  $S$ -module structure.

We say an  $\mathcal{O}_{\mathcal{X}}\text{-}E$  bimodule is *coherent* if it is so as an  $\mathcal{O}_Z$ -module;  $\text{Coh}(\mathcal{X}_E)$  is the category of coherent  $\mathcal{O}_{\mathcal{X}}\text{-}E$  bimodules. A *differential  $\mathcal{O}_{\mathcal{X}}\text{-}E$  bimodule* is an  $\mathcal{O}_{\mathcal{X}}\text{-}E$ -bimodule equipped with a square zero  $\mathcal{O}_Z$ -linear endomorphism that is  $G$ -equivariant, (i.e., homogeneous of degree 0 with respect to the  $\text{Cl}(X)$ -grading associated to the  $S$ -module structure) and that makes each  $\Gamma(U, \mathcal{E})$  a degree  $(0; -1)$  differential  $E$ -module, i.e., an object in  $\text{DM}(E)$ . Denote by  $\text{DM}(\mathcal{X}_E)$  the category of differential  $\mathcal{O}_{\mathcal{X}}\text{-}E$  bimodules.

As for the functors in diagram (3.1),  $\pi_1^*$  sends a sheaf  $\mathcal{F} \in \text{Coh}(\mathcal{X})$  to  $\mathcal{F} \otimes_k E \in \text{Coh}(\mathcal{X}_E)$ . The object  $\mathcal{K} \in \text{DM}(\mathcal{X}_E)$  has underlying  $G$ -equivariant  $\mathcal{O}_Z$ -module given by  $\bigoplus_{a \in \text{Cl}(X)} \mathcal{O}(a) \otimes_k \omega_E(-a; 0)$  and differential given by multiplication on the left by  $\sum_{i=0}^n x_i \otimes e_i$ . The functor  $R\pi_2*$  is defined as follows. Let  $\mathcal{D}$  be an object in  $\text{DM}(\mathcal{X}_E)$  with differential  $\partial$ . Here is the basic idea: first, we take the derived global sections of  $\mathcal{D}$ , thought of as a 1-periodic complex of  $G$ -equivariant  $\mathcal{O}_Z$ -modules. We then observe that the resulting object is a 1-periodic complex of  $E$ -modules and therefore determines an object in  $\text{DM}(E)$ . Let us now make this precise. Let  $\mathcal{C}_{\mathcal{D}}^{\bullet}$  be the Čech resolution of the  $G$ -equivariant  $\mathcal{O}_Z$ -module  $\mathcal{D}$  with respect to the  $G$ -invariant affine open cover of  $Z$  described in [17, Proposition 4.3]. Recall that  $\mathcal{C}_{\mathcal{D}}^{\bullet}$  is a complex of  $\text{Cl}(X) \oplus \text{Cl}(X) \oplus \mathbb{Z}$ -

graded  $S$ - $E$ -bimodules. Let  $\mathcal{C}_{\mathcal{D}}^{G,\bullet}$  denote the  $G$ -invariant subcomplex of  $\mathcal{C}_{\mathcal{D}}^\bullet$ , i.e., the strand of  $\mathcal{C}_{\mathcal{D}}^\bullet$  that lies in degrees of the form  $(0, a; i) \in \text{Cl}(X) \oplus \text{Cl}(X) \oplus \mathbb{Z}$ . We observe that  $\mathcal{C}_{\mathcal{D}}^{G,\bullet}$  computes the derived global sections of the  $G$ -equivariant  $\mathcal{O}_Z$ -module  $\mathcal{D}$ . Consider the bicomplex

$$\begin{array}{ccccccc} \cdots & \longleftarrow & \mathcal{C}_{\mathcal{D}}^{G,0}(0; -1) & \xleftarrow{\delta} & \mathcal{C}_{\mathcal{D}}^{G,0} & \xleftarrow{\delta} & \mathcal{C}_{\mathcal{D}}^{G,0}(0; 1) & \longleftarrow \cdots \\ & & \downarrow d_{\text{Cech}} & & \downarrow d_{\text{Cech}} & & \downarrow d_{\text{Cech}} & \\ \cdots & \longleftarrow & \mathcal{C}_{\mathcal{D}}^{G,1}(0; -1) & \xleftarrow{-\delta} & \mathcal{C}_{\mathcal{D}}^{G,1} & \xleftarrow{-\delta} & \mathcal{C}_{\mathcal{D}}^{G,1}(0; 1) & \longleftarrow \cdots \\ & & \downarrow d_{\text{Cech}} & & \downarrow d_{\text{Cech}} & & \downarrow d_{\text{Cech}} & \\ & & \vdots & & \vdots & & \vdots & \end{array}$$

Since the  $E$ -action on  $\mathcal{C}_{\mathcal{D}}^\bullet$  is  $G$ -equivariant, this is a bicomplex of  $E$ -modules. We define  $R\pi_{2*}(\mathcal{D})$  to be the object in  $\text{DM}(E)$  obtained by totalizing this bicomplex and then applying the equivalence  $\text{Com}_{\text{per}}(E) \xrightarrow{\sim} \text{DM}(E)$  from Remark 2.3.

In summary, given  $\mathcal{F} \in \text{Coh}(\mathcal{X})$ ,  $\Phi_{\mathcal{K}}(\mathcal{F})$  is given by totalizing the bicomplex

$$\begin{array}{ccccccc} \cdots & \longleftarrow & \bigoplus_{a \in \text{Cl}(X)} \mathcal{C}_{\mathcal{F}(a)}^{G,0} \otimes_k \omega_E(-a; -1) & \xleftarrow{\sum_{i=0}^n x_i \otimes e_i} & \bigoplus_{a \in \text{Cl}(X)} \mathcal{C}_{\mathcal{F}(a)}^{G,0} \otimes_k \omega_E(-a; 0) & \longleftarrow \cdots & \\ & & \downarrow d_{\text{Cech}} \otimes 1 & & \downarrow d_{\text{Cech}} \otimes 1 & & \\ \cdots & \longleftarrow & \bigoplus_{a \in \text{Cl}(X)} \mathcal{C}_{\mathcal{F}(a)}^{G,1} \otimes_k \omega_E(-a; -1) & \xleftarrow{-\sum_{i=0}^n x_i \otimes e_i} & \bigoplus_{a \in \text{Cl}(X)} \mathcal{C}_{\mathcal{F}(a)}^{G,1} \otimes_k \omega_E(-a; 0) & \longleftarrow \cdots & (3.2) \\ & & \downarrow d_{\text{Cech}} \otimes 1 & & \downarrow d_{\text{Cech}} \otimes 1 & & \\ & & \vdots & & \vdots & & \end{array}$$

and then applying the equivalence  $\text{Com}_{\text{per}}(E) \xrightarrow{\sim} \text{DM}(E)$  from Remark 2.3.

*Proof of Theorem 3.3.* Let us write the terms of bicomplex (3.2) as  $Y_{i,j}$ . The columns of this bicomplex split  $E$ -linearly; choose an  $E$ -linear decomposition

$$Y_{i,j} = B_{i,j} \oplus H_{i,j} \oplus L_{i,j}$$

for each  $i, j$  such that  $B_{i,j} \oplus H_{i,j} = Z_{i,j}^{\text{vert}}$ , where  $Z_{i,j}^{\text{vert}}$  denotes the vertical cycles in  $Y_{i,j}$ . Observe that there is an isomorphism

$$H_{i,j} \cong \bigoplus_{a \in \text{Cl}(X)} H^{-j}(\mathcal{X}, \mathcal{F}(a)) \otimes_k \omega_E(-a; i).$$

Let  $\sigma_H: Y_{\bullet,\bullet} \rightarrow H_{\bullet,\bullet}$  and  $\sigma_B: Y_{\bullet,\bullet} \rightarrow B_{\bullet,\bullet}$  denote the projections,  $g: L_{\bullet,\bullet} \xrightarrow{\cong} B_{\bullet,\bullet-1}$  the isomorphism induced by the vertical differential, and  $\pi = g^{-1}\sigma_B$ . By [39, Lemma 3.5], the object  $\Phi_{\mathcal{K}}(\mathcal{F}) \in \text{DM}(E)$  is homotopy equivalent to the differential  $E$ -module  $\mathbf{T}(\mathcal{F})$

with underlying module  $\bigoplus_{i=0}^{\dim X} \bigoplus_{a \in \text{Cl}(X)} H^i(\mathcal{X}, \mathcal{F}(a)) \otimes_k \omega_E(-a; i)$  and differential given by

$$\partial_{\mathbf{T}(\mathcal{F})} = \sum_{j \geq 0} \sigma_H(d_{\text{hor}} \pi)^j d_{\text{hor}},$$

where  $d_{\text{hor}}$  is the horizontal differential in bicomplex (3.2). We make the following observations about the differential  $\partial_{\mathbf{T}(\mathcal{F})}$ :

- (i) The  $j = 0$  term of  $\partial_{\mathbf{T}(\mathcal{F})}$  applied to an element in  $H^i(\mathcal{X}, \mathcal{F}(a)) \otimes_k \omega_E(-a; i)$  is given by multiplication on the left by  $(-1)^i \sum_{i=0}^n x_i \otimes e_i$ .
- (ii) The  $j > 0$  terms of  $\partial_{\mathbf{T}(\mathcal{F})}$  decrease the sheaf cohomology degree. That is, given  $z \in H^i(\mathcal{X}, \mathcal{F}(a)) \otimes_k \omega_E(-a; i)$ , the  $j > 0$  terms of  $\partial_{\mathbf{T}(\mathcal{F})}$  map  $z$  to

$$\bigoplus_{\ell < i} H^\ell(\mathcal{X}, \mathcal{F}(a)) \otimes_k \omega_E(-a; \ell - 1).$$

- (iii) Combining (i) and (ii), we conclude that, given  $z \in H^0(\mathcal{X}, \mathcal{F}(a)) \otimes_k \omega_E(-a; 0)$ , we have

$$\partial_{\mathbf{T}(\mathcal{F})}(z) = \left( \sum_{i=0}^n x_i \otimes e_i \right) \cdot z.$$

Since the rows of (3.2) are exact, and its columns are bounded, the differential module  $\Phi_{\mathcal{K}}(\mathcal{F})$  is exact, and so  $\mathbf{T}(\mathcal{F})$  is as well. The underlying  $E$ -module of  $\mathbf{T}(\mathcal{F})$  is free, and the minimality of  $d_{\text{hor}}$  as a morphism of  $E$ -modules implies the minimality of  $\partial_{\mathbf{T}(\mathcal{F})}$ . This proves (1). Part (2) is immediate from our description of the underlying module of  $\mathbf{T}(\mathcal{F})$ . It is also clear that the embedding  $M \hookrightarrow \bigoplus_{a \in \text{Cl}(X)} H^0(\mathcal{X}, \mathcal{F}(a))$  induces an embedding  $\mathbf{R}(M) \hookrightarrow \mathbf{T}(\mathcal{F})$  on underlying  $E$ -modules, and (iii) above implies that this is a morphism of differential  $E$ -modules; this proves (3). As for (4), let  $I \subseteq \{0, \dots, n\}$  be irrelevant, and let  $\mathcal{K}_I := \mathcal{K} \otimes_E E_I$ . The homology of  $\mathcal{K}_I$  is just a direct sum of twists of the homology of the Koszul complex on the variables in  $\{x_i : i \in I\}$ , so  $\mathcal{K}_I$  is exact. It follows that, if we tensor bicomplex (3.2) with  $E_I$ , the result has exact rows. Since the columns of this bicomplex are bounded, its totalization is therefore exact, i.e.,  $\Phi_{\mathcal{K}}(\mathcal{E}) \otimes_E E_I$  is exact. Since  $\Phi_{\mathcal{K}}(\mathcal{F}) \otimes_E E_I$  is homotopy equivalent to  $\mathbf{T}(\mathcal{F}) \otimes_E E_I$ , we conclude that  $\mathbf{T}(\mathcal{F}) \otimes_E E_I$  is exact. ■

**Remark 3.5.** Let  $\mathcal{F} \in \text{Coh}(\mathcal{X})$ . We may identify the underlying module of  $\mathbf{T}(\mathcal{F})$  with  $\bigoplus_{i=0}^{\dim X} T_i$ , where  $T_i = \bigoplus_{a \in \text{Cl}(X)} H^i(\mathcal{X}, \mathcal{F}(a)) \otimes_k \omega_E(-a, i)$ . By degree considerations, each  $T_{\leq i} = \bigoplus_{j \leq i} T_j$  is a differential submodule of  $\mathbf{T}(\mathcal{F})$ , yielding a filtration

$$T_0 \subseteq T_{\leq 1} \subseteq \dots \subseteq T_{\leq \dim(X)} = \mathbf{T}(\mathcal{F}).$$

**Remark 3.6.** Theorem 3.3 has a subtle implication. A theme in research on syzygies is that the geometry of  $\mathbb{P}^n$  can influence the algebraic properties of graded free resolutions. In the toric situation, where varying the irrelevant ideal can yield different toric varieties/stacks, one can ask which of these geometric objects affects the algebraic properties of modules over the Cox ring? Of course, the answer is: all of them. For example,

if  $X$  and  $X'$  have the same Cox ring  $S$ , and if  $M$  is an  $S$ -module, then Theorem 3.3 (3) shows that the differential  $E$ -module  $\mathbf{R}(M)$  can be completed to a Tate resolution in distinct ways, each of which imposes constraints on the homology of  $\mathbf{R}(M)$ . Via Proposition 2.11, this shows that the Betti numbers of  $M$  are influenced by both  $X$  and  $X'$  and, more generally, by the toric varieties/stacks arising in the Mori chamber decomposition of the effective cone in  $X$ .

As one can see from the proof of Theorem 3.3, our construction of the Tate resolution is, *a priori*, only well defined up to homotopy equivalence: the differential we get depends on the choices of splittings of the columns of bicomplex (3.2) that we use in our application of [39, Lemma 3.5]. However, Theorems 3.7 and 3.10 below imply that our construction of the Tate resolution is well defined up to isomorphism in some special cases. In particular, we will see that our Tate resolutions agree with those of [38, 39].

We say a toric stack is a *generalized weighted projective stack* if its associated toric variety is a fake weighted projective space, i.e., a quotient of a weighted projective space by the action of a finite abelian group. We remind the reader that we only consider weighted projective spaces whose weights are pairwise relatively prime. The divisor class group of a fake weighted projective space is isomorphic to  $\mathbb{Z} \oplus A$  for some finite abelian group  $A$ .

**Theorem 3.7.** *Suppose  $\mathcal{X}$  is a generalized weighted projective stack, and let  $\mathcal{F}$  be a coherent sheaf on  $\mathcal{X}$ . Choose a  $B$ -saturated  $S$ -module  $M$  such that  $\tilde{M} = \mathcal{F}$ . Any differential module  $T$  with properties (1)–(4) in Theorem 3.3 is isomorphic to  $\text{cone}(F \xrightarrow{\sim} \mathbf{R}(M))$ , where  $F$  is the minimal free resolution of  $\mathbf{R}(M)$ , in the sense of Definition B.1. In particular, properties (1)–(4) in Theorem 3.3 determine the Tate resolution of  $\mathcal{F}$  up to isomorphism.*

*Proof.* Consider the embedding  $\mathbf{R}(M) \hookrightarrow T$  in Theorem 3.3 (3). Bearing part (2) of Theorem 3.3 in mind, we conclude that, as a map of  $E$ -modules, this embedding may be interpreted as the inclusion  $\mathbf{R}(M) \hookrightarrow \mathbf{R}(M) \oplus N$ , where  $N = \bigoplus_{i=1}^{\dim X} \bigoplus_{a \in \text{Cl}(X)} H^i(\mathcal{X}, \mathcal{F}(a)) \otimes_k \omega_E(-a; i)$ . By degree considerations, we see that, with respect to this decomposition, the differential on  $T$  has the form  $\begin{pmatrix} \partial_{\mathbf{R}} & \alpha \\ 0 & \beta \end{pmatrix}$ , and so  $T$  is isomorphic to the mapping cone of a morphism  $\alpha: (N, -\beta) \rightarrow \mathbf{R}(M)$  of differential modules. Notice that  $(N, -\beta)$  is a minimal free differential  $E$ -module. To conclude that  $(N, -\beta)$  is the minimal free resolution of  $\mathbf{R}(M)$ , in the sense of Definition B.1, it suffices to show that  $(N, -\beta)$  is a free flag. Let  $\pi$  be a surjection  $\text{Cl}(X) \rightarrow \mathbb{Z}$ ; note that  $\ker(\pi)$  is a finite abelian group. For  $j \in \mathbb{Z}$ , define

$$N_j = \bigoplus_{i=1}^{\dim X} \bigoplus_{\pi(a) \geq -j} H^i(\mathcal{X}, \mathcal{F}(a)) \otimes_k \omega_E(-a; i).$$

It follows from an analogue of the Serre vanishing theorem that  $N_j = 0$  for  $j \ll 0$ . By the minimality of  $T$ , the differential maps each  $N_j$  to  $\bigoplus_{\ell < j} N_{\ell}$ . Thus,  $N = \bigoplus N_j$  is a free flag differential module. The last statement follows from uniqueness of minimal free resolutions of differential modules (see Theorem B.2). ■

**Corollary 3.8.** *When  $\mathcal{X} = \mathbb{P}^n$ , we have  $\mathbf{T} = \text{Fold} \circ \mathbf{T}_{\text{EFS}}$ , where  $\mathbf{T}_{\text{EFS}}$  denotes Eisenbud–Fløystad–Schreyer’s Tate resolution functor, and the functor Fold is as in Definition 2.6.*

*Proof.* The proof follows from Theorem 3.7 and the relation  $\mathbf{R} = \text{Fold} \circ \mathbf{R}_{\text{classical}}$  from Remarks 2.9 (2). ■

**Remark 3.9.** Suppose  $\mathcal{X} = \mathbb{P}^n$ . As discussed in the beginning of this section, if  $M$  is a finitely generated  $S$ -module, then  $\mathbf{R}(M_{\geq d})$  is quasi-isomorphic to its homology when  $d \geq \text{reg}(M)$ . Thus, the Tate resolution of  $\tilde{M}$  can be defined by resolving  $H(\mathbf{R}(M_{\geq d}))$ , rather than all of  $\mathbf{R}(M_{\geq d})$ ; this is the approach taken in [39] and also in [38] over multiprojective spaces. Unfortunately, this does not work more generally; for instance, one can check that this fails in the example discussed in Remark 2.16, over a Hirzebruch surface of type 3.

It is not always the case that  $\text{cone}(F \xrightarrow{\sim} \mathbf{R}(M))$ , where  $F$  is the minimal free resolution of  $\mathbf{R}(M)$ , yields the Tate resolution. For instance, if  $X = \mathbb{P}^1 \times \mathbb{P}^1$ , then that mapping cone is the corner complex as defined in [38, §3]. However, we do have the following.

**Theorem 3.10.** *When  $\mathcal{X}$  is a product of projective spaces, any differential module  $T$  with properties (1)–(4) in Theorem 3.3 satisfies  $T = \text{Fold} \circ \mathbf{T}_{\text{EES}}$ , where the functor Fold is as introduced in Definition 2.6, and  $\mathbf{T}_{\text{EES}}$  denotes Eisenbud–Erman–Schreyer’s Tate resolution functor. In particular, properties (1)–(4) in Theorem 3.3 determine the Tate resolution up to isomorphism in this case.*

*Proof.* The differential module  $T$  can be realized as the folding of a complex of  $E$ -modules, as in Definition 2.6. We may then assume that  $T$  is a complex. The exactness properties – that is, Theorem 3.3 (4) – imply that  $T$  has exact strands. By [38, Theorem 3.3], this implies that every corner complex of  $T$  is also exact. Let  $T_{d-\text{tail}} \rightarrow T_{d-\text{head}}$  be the degree  $d$  corner complex of  $T$ , as defined in [38, Section 3]. Theorem 3.3 (2) and (3) imply that, for  $d \gg 0$ ,  $T_{d-\text{head}}$  is precisely  $\mathbf{R}(M_{\geq d})$ . Exactness of the corner complex and minimality of  $T$  imply that  $T_{d-\text{tail}}$  is the minimal free resolution of  $\mathbf{R}(M_{\geq d})$  and is thus uniquely determined. It follows that  $T_{d-\text{tail}}$  is isomorphic to the analogous degree  $d$  tail of  $\mathbf{T}(\mathcal{F})$  for any  $d \gg 0$ ; we conclude that  $T$  and  $\mathbf{T}(\mathcal{F})$  are isomorphic complexes. ■

While the toric stacks considered in Theorems 3.7 and 3.10 are relatively simple, we believe that similar results should hold much more broadly; see Conjecture 7.1 below. But carrying out such a generalization will require new homological methods. The proofs of Theorems 3.7 and 3.10 rely on the fact that we can construct  $\mathbf{T}(\mathcal{F})$  from  $\mathbf{R}(M)$  by taking minimal free resolutions of complexes or differential modules in these cases. However, this approach makes no explicit reference to the irrelevant ideal of  $X$ . By considering situations where one has two toric varieties  $X$  and  $X'$  with the same Cox ring but with different irrelevant ideals, one can see that characterizing the Tate resolution up to isomorphism in general will require an approach that makes greater use of the exactness properties of Theorem 3.3 (3). To underscore the algebraic challenge: a free resolution yields an object that is exact, but how does one produce an object with all of the exactness properties from Theorem 3.3 (3)?

### 3.3. Tate resolutions over $\mathbb{P}^n$ and Beilinson’s resolution of the diagonal

A major emphasis of [39] is the relationship between Tate resolutions and Beilinson monads on  $\mathbb{P}^n$ . Our “noncommutative Fourier–Mukai” construction of Tate resolutions over  $\mathbb{P}^n$  takes this relationship a step further. The goal of this subsection is to explain this, and in doing so provide some intuition for our construction of toric Tate resolutions.

Let  $\mathcal{B}$  denote Beilinson’s resolution of  $\mathcal{O}_\Delta$  in  $\text{Coh}(\mathbb{P}^n \times \mathbb{P}^n)$ , which has the form

$$0 \leftarrow \mathcal{O} \boxtimes \mathcal{O} \leftarrow \mathcal{O}(-1) \boxtimes \Omega^1(1) \leftarrow \cdots \leftarrow \mathcal{O}(-n) \boxtimes \Omega^n(n) \leftarrow 0;$$

it is a Koszul complex on the section  $\sum_{i=0}^n x_i \boxtimes \frac{\partial}{\partial y_i}$  of  $\mathcal{O}_{\mathbb{P}^n}(1) \boxtimes \mathcal{T}_{\mathbb{P}^n}(-1)$ . Let  $\Phi_{\mathcal{B}}$  denote the Fourier–Mukai transform given by  $\mathcal{F} \mapsto R\pi_{2*}(\pi_1^*(\mathcal{F}) \otimes \mathcal{B})$ . Since  $\mathcal{B}$  is a resolution of the diagonal,  $\Phi_{\mathcal{B}}(\mathcal{F})$  is isomorphic to  $\mathcal{F}$  in  $D^b(\mathbb{P}^n)$ . From [12], there is a natural “Beilinson spectral sequence” whose  $E^1$  terms have the form  $H^i(\mathbb{P}^n, \mathcal{F}(-j)) \otimes \Omega_{\mathbb{P}^n}^j(j)$ . But one can go further and produce an actual complex of sheaves whose terms are sums of these sheaves and whose 0-th homology is  $\mathcal{F}$ . Such a complex is called a *Beilinson monad* for  $\mathcal{F}$ ; see [4, 39, 57] for details.

To explicitly construct the Beilinson monad, Eisenbud–Fløystad–Schreyer define a functor  $\Omega$  which sends a free complex of  $E$ -modules<sup>4</sup> to a complex of  $\mathcal{O}_{\mathbb{P}^n}$ -modules [39, §6]. The functor is defined by the formula  $\Omega(\omega_E(j)) = \Omega_{\mathbb{P}^n}^j(j)$ , with the key point being that there is a canonical map

$$\text{Hom}_E(\omega_E(i), \omega_E(j)) \rightarrow \text{Hom}_{\mathbb{P}^n}(\Omega_{\mathbb{P}^n}^i(i), \Omega_{\mathbb{P}^n}^j(j)).$$

They prove that applying  $\Omega$  to the Tate resolution of  $\mathcal{F}$  yields a Beilinson monad for  $\mathcal{F}$ .

Our construction demonstrates that this connection can be “lifted” one level further. Indeed, the object  $\mathcal{K} \in \text{DM}(\mathbb{P}_E^n)$  is equivalent, via the Fold functor (Definition 2.6), to

$$\cdots \leftarrow \mathcal{O}_{\mathbb{P}^n} \otimes_k \omega_E \leftarrow \mathcal{O}_{\mathbb{P}^n}(1) \otimes_k \omega_E(-1) \leftarrow \cdots \leftarrow \mathcal{O}_{\mathbb{P}^n}(-n) \otimes_k \omega_E(-n) \leftarrow \cdots. \quad (3.3)$$

Since  $\Omega(\omega_E(i)) = 0$  if  $i > n$  or  $i < 0$ , replacing each  $\omega_E(i)$  by  $\Omega(\omega_E(i))$  transforms the doubly infinite complex (3.3) into a bounded one whose terms are identical to those of Beilinson’s resolution. The intuition is that applying “ $\text{id} \times \Omega$ ” to (3.3) recovers Beilinson’s resolution.<sup>5</sup> The situation may be summarized by the following diagram:

$$\begin{array}{ccccc} K & \xrightarrow{\quad} & \mathcal{K} & \xrightarrow{\text{“id} \times \Omega”} & \text{Beilinson’s resolution of } \mathcal{O}_\Delta \\ \downarrow \rho & & \downarrow R\pi_{2*} & & \downarrow R\pi_{2*} \\ \mathbf{R}(S) & \xrightarrow{\text{inclusion}} & \mathbf{T}(\mathcal{O}_{\mathbb{P}^n}) & \xrightarrow{\Omega} & \text{Beilinson monad of } \mathcal{O}_{\mathbb{P}^n}. \end{array} \quad (3.4)$$

Here,  $K$  denotes the Koszul complex on the variables in  $S$ , considered as an  $S$ – $E$ -bimodule; the map  $\rho$  denotes restriction of scalars along the inclusion  $E \hookrightarrow S \otimes_k E$ ; and the

<sup>4</sup>In [39], the exterior algebra has the  $\mathbb{Z}$ -grading where  $\deg(e_i) = -1$  for all  $i$ .

<sup>5</sup>As an aside, we note that one may also think of bicomplex (3.2) above as a “lift from  $\mathbb{P}^n$  to  $E$ ” of the bicomplex that gives the Beilinson spectral sequence.

map  $K \rightarrow \mathcal{K}$  is given by sheafification and inclusion. The bottom row follows from [39, Theorem 6.1], and our construction of the Tate resolution provides the rest of the diagram. So, while the authors of [39] use a Tate resolution to recover the Beilinson monad, our construction directly connects to Beilinson's resolution itself. Theorem 5.1 and Proposition 5.4 below make the ideas conveyed by this diagram precise and extend them to any projective toric stack.

### 3.4. Examples of toric Tate resolutions

The exactness properties in Theorem 3.3 (4) underscore a key point: if  $\mathcal{F}$  is a sheaf on  $\mathcal{X}$ , then  $\mathbf{T}(\mathcal{F})$  depends not just on the Cox ring of  $\mathcal{X}$  but on its irrelevant ideal. The following examples are intended to illustrate this point.

**Example 3.11.** Let  $\mathcal{X} = \mathbb{P}(1, 1, 2)$ . In Example 2.8, we computed  $\mathbf{R}(S)$ ; let us now compute  $\mathbf{T}(\mathcal{O}_{\mathcal{X}})$ . By Theorem 3.7, we need only compute the minimal free resolution  $F$  of  $\mathbf{R}(S)$  and take the mapping cone of the augmentation  $F \rightarrow \mathbf{R}(S)$ . The result looks like

$$\cdots \rightarrow \omega_E(5; 2)^{\oplus 2} \rightarrow \omega_E(4; 2) \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \omega_E \rightarrow \omega_E(-1; 0)^{\oplus 2} \rightarrow \cdots$$

$e_0 e_1 e_2$

In detail, consider the  $S$ -module  $M = \bigoplus_{a \in \mathbb{Z}} H^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(a))$  as a complex concentrated in homological degree  $-2$ . The minimal free resolution  $F$  of  $\mathbf{R}(S)$  is given by  $\mathbf{R}(M)$ . The augmentation is given by a single map from  $F$  to  $\mathbf{R}(S)$  that sends  $\omega_E(4; 2)$  to  $\omega_E(0; 0)$  via multiplication by  $e_0 e_1 e_2$ . Note, for instance, that the  $\omega_E(5; 2)^{\oplus 2}$  term encodes the fact that  $H^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(-5)) = k^2$ .

**Example 3.12.** Continuing with  $\mathcal{X} = \mathbb{P}(1, 1, 2)$ , let  $C \subseteq \mathcal{X}$  be the genus 1 curve  $C = V(x_0^4 + x_1^4 + x_2^2)$ . The Tate resolution  $\mathbf{T}(\mathcal{O}_C)$  has the form

$$\cdots \rightarrow \omega_E(2; 1)^{\oplus 4} \rightarrow \omega_E(1; 1)^{\oplus 2} \rightarrow \omega_E(0; 1) \oplus \omega_E(0; 0) \rightarrow \omega(-1; 0)^{\oplus 2} \rightarrow \omega(-2; 0)^{\oplus 4} \rightarrow \cdots$$

The terms from  $\mathbf{R}(\frac{S}{x_0^4+x_1^4+x_2^2})$  are those with no twist in the auxiliary degree, i.e., those of the form  $\omega_E(i; 0)$  for some  $i$ . Note also that the middle term  $\omega_E(0; 1) \oplus \omega_E(0; 0)$  corresponds to the fact that, because  $C$  is a genus one curve, we have  $H^1(\mathcal{X}, \mathcal{O}_C) = k = H^0(\mathcal{X}, \mathcal{O}_C)$ .

**Example 3.13.** For another example with  $\mathcal{X} = \mathbb{P}(1, 1, 2)$ , let  $P$  denote the stacky point  $V(x_0, x_1)$ . Of course,  $\mathcal{O}_P$  has no higher cohomology groups, and we have

$$H^0(\mathcal{X}, \mathcal{O}_P(d)) = k$$

when  $d$  is even and 0 when  $d$  is odd. The Tate resolution  $\mathbf{T}(\mathcal{O}_P)$  thus looks as follows:

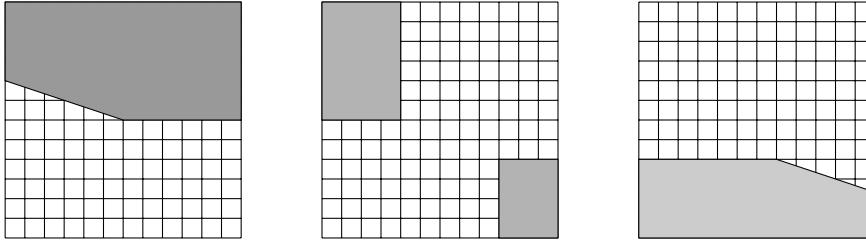
$$\cdots \rightarrow \omega_E(2; 0) \rightarrow 0 \rightarrow \omega_E \rightarrow 0 \rightarrow \omega_E(-2; 0) \rightarrow 0 \rightarrow \omega_E(-4; 0) \rightarrow \cdots$$

$e_2$

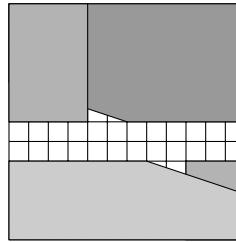
**Example 3.14.** Let  $\mathcal{X} = X$  be the Hirzebruch surface of type 3, with Cox ring  $S$  as described in Example 2.10. The irrelevant ideal is

$$B = (x_0, x_2) \cap (x_1, x_3).$$

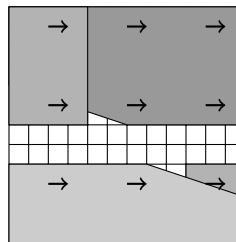
The Tate resolution  $\mathbf{T}(\mathcal{O}_X)$  has summands corresponding to  $H^0$ ,  $H^1$  and  $H^2$  groups. We illustrate these in the pictures below, with the left picture depicting the degrees where the sheaf has  $H^0$ ; the middle picture the  $H^1$ 's; and the right picture the  $H^2$ 's:



Putting these together, we get a picture that looks like



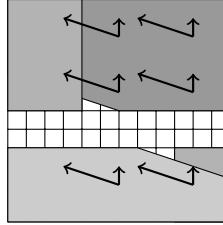
To illustrate the exactness properties, let us consider the irrelevant subset  $I = \{0, 2\}$ . Theorem 1.2 implies that the Tate resolution remains exact after modding out by  $e_1, e_3$ . Since the variables  $x_0$  and  $x_2$  have degree  $(1, 0)$ , the differential on this restricted Tate resolution will be “purely horizontal”; that is to say, the differential will send each generator only to elements to the right and in the same row.



The differential module  $\mathbf{T}(\mathcal{O}_X) \otimes_E E_{\{0,2\}}$  therefore decomposes as a direct sum of exact differential modules corresponding to the rows of the above picture.

Now consider the irrelevant subset  $I = \{1, 3\}$ . Since the variables  $x_1$  and  $x_3$  have degrees  $(-3, 1)$  and  $(0, 1)$ , the differential on this restricted Tate resolution will be “vertical”

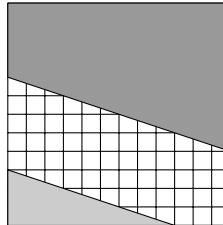
and also “vertical and to the left”. That is, the differential on  $\mathbf{T}(\mathcal{O}_X) \otimes_E E_{\{1,3\}}$  looks like this:



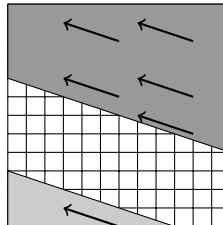
**Example 3.15.** Let  $S$  be as in the previous example, but alter the irrelevant ideal so that it is now

$$B' := (x_1) \cap (x_0, x_2, x_3).$$

While this does not fit into the setup of Theorem 3.3 – the irrelevant ideal of a toric variety cannot have a principal minimal prime – our main construction easily extends to this case. Let  $\mathcal{X}'$  be the corresponding stack quotient; the corresponding toric variety  $X'$  is the weighted projective space  $\mathbb{P}(1, 1, 3)$  obtained by contracting the exceptional divisor  $V(x_1)$  on the Hirzebruch surface  $X$ . Since  $S$  has remained unchanged from the previous example,  $\mathbf{R}(S)$  is the same as before. However, the Tate resolution  $\mathbf{T}(\mathcal{O}_{\mathcal{X}'})$  looks quite different, with no  $H^1$  degrees, and  $H^0$  and  $H^2$  degrees illustrated in dark and light gray as below:



This Tate resolution and the one in Example 3.14 satisfy different exactness properties. For instance, consider  $I = \{1\}$ , which is irrelevant for  $\mathcal{X}'$  but not for the Hirzebruch surface. Since  $\deg(x_1) = (-3, 1)$ , the differential on  $\mathbf{T}(\mathcal{O}) \otimes_E E_{\{1\}}$  looks like this:



The diagonals with slope  $-\frac{1}{3}$  yield exact “strands” of the Tate resolution  $\mathbf{T}(\mathcal{O}_{\mathcal{X}'})$ . Far enough out along any such diagonal in the  $H^0$  region, this Tate resolution will agree with  $\mathbf{R}(S)$ ; in particular, any diagonal of  $\mathbf{R}(S)$  is “eventually exact”. By contrast, the “rows” of  $\mathbf{T}(\mathcal{O}_{\mathcal{X}'})$  are not exact, which is to be expected, since  $\{0, 2\}$  is not an irrelevant subset for  $\mathcal{X}'$ .

**Remark 3.16.** Examples 3.14 and 3.15 point toward a new feature in the multigraded case: when  $S$  is the Cox ring of a toric variety  $X$ , the properties of multigraded  $S$ -modules are related not just to the geometry of  $X$ , but also to the other toric varieties that can arise in the Mori chamber decomposition of the effective cone of  $X$ . In particular, we used a Tate resolution on the Hirzebruch surface to understand the asymptotic exactness properties of  $\mathbf{R}(S)$  along rows and columns, and we used a Tate resolution for  $\mathbb{P}(1, 1, 3)$  to understand the exactness properties of  $\mathbf{R}(S)$  along the diagonals of slope  $-\frac{1}{3}$ . This is an example of the sort of behaviour to which we alluded in Remark 3.6.

#### 4. A linear resolution of the diagonal for toric stacks

In this section, we construct a Beilinson-type resolution of the diagonal for projective toric stacks. As discussed in Section 3.3, Beilinson's resolution of the diagonal over  $\mathbb{P}^n$  arises as a Koszul complex, and our toric version does as well.

Let  $\mathcal{X}$  be a projective toric stack with Cox ring  $S = k[x_0, \dots, x_n]$  and associated toric variety  $X$ . Denote the Cox ring of  $\mathcal{X} \times \mathcal{X}$  by  $S' = k[x_0, \dots, x_n, y_0, \dots, y_n]$ . Let  $\Delta: \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$  be the diagonal morphism, and set  $\mathcal{O}_\Delta := \Delta_*(\mathcal{O}_X)$ . The naive equations " $x_i = y_i$ " are not homogeneous and hence do not yield well-defined equations in  $S'$ . However, we can force these naive equations to be homogeneous using a simple trick. Let  $V \subseteq \text{Cl}(X)$  be the semigroup generated by  $\deg(x_0), \dots, \deg(x_n)$ , and consider the semigroup ring  $R = S'[V]$  as a quotient of the polynomial ring  $S'[u_0, \dots, u_n]$ , where each  $u_i$  maps to  $\deg(x_i) \in V$ . Equip  $R$  with the  $\text{Cl}(X) \oplus \text{Cl}(X)$ -grading such that  $\deg(u_i) = (\deg(x_i), -\deg(x_i))$ . Each relation  $x_i - y_i u_i$  is now homogeneous in  $R$ , of degree  $(\deg(x_i), 0)$ .

**Theorem 4.1.** *Let  $F$  be the Koszul complex on  $x_0 - y_0 u_0, \dots, x_n - y_n u_n \in R$ . We have*

- (1)  *$F$  is acyclic; that is,  $H_i(F) = 0$  for  $i > 0$ .*
- (2) *Viewing  $H_0(F)$  as an  $S'$ -module, we have  $\widetilde{H_0(F)} = \mathcal{O}_\Delta$ .*

*In particular, viewing  $\tilde{F}$  as a complex of quasi-coherent  $\mathcal{O}_{\mathcal{X} \times \mathcal{X}}$ -modules, we have that  $\tilde{F}$  is a resolution of  $\mathcal{O}_\Delta$ .*

We can write the terms of the complex in Theorem 4.1 as

$$F_i = \bigoplus_{a \in V} \bigoplus_{b \in \text{Cl}(X)} (\omega_E)_{(b; i)} \otimes_k S'(-a - b, a). \quad (4.1)$$

Here, as above,  $\omega_E$  is the  $k$ -dual of the  $\text{Cl}(X) \oplus \mathbb{Z}$ -graded exterior algebra  $E = \Lambda_k(e_0, \dots, e_n)$ . From this view, the differential on  $F$  is multiplication on the left by  $\sum_{i=0}^n (e_i \otimes x_i - e_i \otimes y_i)$ .

Another way of thinking of the complex  $F$  is as follows. Let  $K$  be the Koszul complex on  $x_0, \dots, x_n \in S$ . For any  $d \in \text{Cl}(X)$ , write  $K_d$  for the subcomplex of  $K$  given by summands of the form  $S(-e)$  with  $d - e \in V$ . We have

$$F = \bigoplus_{d \in V} S(-d) \boxtimes K_d(d);$$

with differential  $\sum_{i=0}^n (x_i \otimes e_i - y_i \otimes e_i)$ , where each  $e_i$  shifts terms in the Koszul complex factor, each  $y_i \in S'$  maps a summand to itself, and each  $x_i \in S'$  maps between summands.

This latter point of view clarifies the relationship between our resolution and Beilinson's resolution of the diagonal over  $\mathbb{P}^n$  (see Section 3.3). When  $X = \mathbb{P}^n$ , each complex  $\widetilde{K}_i(i)$  of sheaves is a free resolution of  $\Omega^i(i)$ . In particular,  $\widetilde{K}_i(i)$  is exact for  $i > n$ , and so the summands  $S(-i) \boxtimes K_i(i)$  of  $F$  with  $i > n$  are superfluous; this is an idea to which we will return in Section 4.2.

**Remark 4.2.** We can give yet another way of constructing the complex  $F$  from Theorem 4.1, by applying the BGG functor  $\mathbf{L}$  to a module over the  $\mathrm{Cl}(X) \oplus \mathrm{Cl}(X) \oplus \mathbb{Z}$ -graded exterior algebra  $E' = \Lambda_k(e_0, \dots, e_n, f_0, \dots, f_n)$ . Let  $N$  denote the free  $E'$ -module  $\bigoplus_{d \in V} \omega_{E'}(-d, d; 0)$ , where  $\omega_{E'} = \underline{\mathrm{Hom}}_{E'}(E', k)$ . It will be helpful to reinterpret  $N$  as  $\omega_{E'}[V]$ , just like the ring  $R$ , except that we declare here that  $\deg(u_i) = (\deg(x_i), -\deg(x_i); 0) \in \mathrm{Cl}(X) \oplus \mathrm{Cl}(X) \oplus \mathbb{Z}$ . Let  $N_\Delta$  be the kernel of the map  $N \rightarrow \bigoplus_{i=0}^n N(0, -\deg(x_i); -1)$  whose  $i$ -th component is  $f_i + e_i u_i$ . In other words,  $N_\Delta$  is the  $E'[V]$ -submodule of  $N$  generated by  $\gamma_0 \cdots \gamma_n$ , where  $\gamma_i = x_i - y_i u_i$ . One can check that  $\mathbf{L}(N_\Delta) = F$ .

**Example 4.3.** Say  $X = \mathbb{P}(1, 2)$ . Our complex  $F$  has the following form:

$$\begin{array}{ccccc}
 & \vdots & & \vdots & \vdots \\
 & \swarrow & & \swarrow & \\
 S'(-3, 3) & \leftarrow & S'(-5, 3) \oplus S'(-4, 3) & \leftarrow & S'(-6, 3) \\
 & \uparrow & & \uparrow & \\
 & S'(-2, 2) & \leftarrow & S'(-4, 2) \oplus S'(-3, 2) & \leftarrow S'(-5, 2) \\
 & \uparrow & & \uparrow & \\
 & S'(-1, 1) & \leftarrow & S'(-3, 1) \oplus S'(-2, 1) & \leftarrow S'(-4, 1) \\
 & \uparrow & & \uparrow & \\
 S' & \leftarrow & S'(-2, 0) \oplus S'(-1, 0) & \leftarrow & S'(-3, 0).
 \end{array}$$

The  $x_i$ 's map each summand horizontally;  $y_0$  maps horizontally and up one position, while  $y_1$  maps horizontally and up two positions. The complexes  $S(-d) \boxtimes K_d(d)$  arise among the northwest diagonals; e.g., the bold summands (and corresponding arrows) form that complex for  $d = 3$ . When  $d \geq 3$ , truncating has no effect, and the strand is simply  $S(-d) \boxtimes K(d)$ .

Unlike Beilinson's resolution of the diagonal over  $\mathbb{P}^n$  [12], or the resolution of the diagonal for a weighted projective stack in [27], the resolution in Theorem 4.1 has infinite rank terms. However, we show in Section 4.2 that, in these cases, our resolution can be pruned down to a finite rank complex that is nearly equivalent to those resolutions.

#### 4.1. Proof of Theorem 4.1

Denote by  $S_\Delta$  the  $S'$ -module

$$\bigoplus_{(d,d') \in \mathrm{Cl}(X) \oplus \mathrm{Cl}(X)} H^0(\mathcal{X} \times \mathcal{X}, \mathcal{O}_\Delta(d, d')).$$

Notice that  $(S_\Delta)_{(d,d')} = H^0(\mathcal{X} \times \mathcal{X}, \mathcal{O}_\Delta(d, d')) = H^0(\mathcal{X}, \Delta^* \mathcal{O}_{\mathcal{X} \times \mathcal{X}}(d, d')) = S_{d+d'}$ . Equip  $S$  with the  $\mathrm{Cl}(X) \oplus \mathrm{Cl}(X)$ -grading such that  $f \in S$  has degree  $(0, \deg_S(f))$ . The module  $S_\Delta$  is a  $\mathrm{Cl}(X) \oplus \mathrm{Cl}(X)$ -graded  $S$ -module via the map  $S \hookrightarrow S'$  sending  $y_i$  to  $y_i$ . In fact, for a fixed  $d \in \mathrm{Cl}(X)$ ,  $(S_\Delta)_{d,*}$  is a free  $\mathrm{Cl}(X) \oplus \mathrm{Cl}(X)$ -graded  $S$ -module with generator in degree  $(d, -d)$ ; it follows that, as an  $S$ -module, we have

$$S_\Delta = \bigoplus_{d \in \mathrm{Cl}(X)} S \cdot \varepsilon_d,$$

where  $\varepsilon_d$  is a generator with  $\deg(\varepsilon_d) = (d, -d)$ . Denote by  $S_\Delta^+$  the submodule of  $S_\Delta$  generated by the  $\varepsilon_d$  such that  $d \in V$ . In other words,

$$S_\Delta^+ = \bigoplus_{(d,d') \in \mathrm{Cl}(X), d \in V} H^0(\mathcal{X} \times \mathcal{X}, \mathcal{O}_\Delta(d, d')).$$

Since  $S_\Delta$  and  $S_\Delta^+$  agree in high degrees (for instance, they agree in all degrees  $(d, d')$ , where  $d, d' \in V$ ), they determine the same sheaf on  $\mathcal{X} \times \mathcal{X}$ , which is to say that  $S_\Delta^+ = \mathcal{O}_\Delta$ .

We can identify multiplication by  $y_i$  and  $x_i$  on  $S_\Delta$  via the relations

$$x_i \varepsilon_d = y_i \varepsilon_{d+\deg_S(y_i)}.$$

We claim that this gives a presentation of  $S_\Delta^+$ .

**Proposition 4.4.** *The  $S'$ -module  $S_\Delta^+$  has generators  $\varepsilon_d$  for all  $d \geq 0$  and relations  $x_i \varepsilon_d = y_i \varepsilon_{d+\deg(x_i)}$  for all  $d \geq 0$  and all  $i = 0, \dots, n$ . In particular, the  $S'$ -module  $S_\Delta^+$  has a free presentation  $F_1 \xrightarrow{\varphi} F_0 \rightarrow S_\Delta^+ \rightarrow 0$ , where  $F_0 = \bigoplus_{d \in V} S'(-d, d)$ , and  $F_1 = \bigoplus_{i=0}^n F_0(-\deg_S(x_i), 0)$ .*

*Proof.* We have a surjection  $\mathrm{coker}(\varphi) \twoheadrightarrow S_\Delta^+$ , and the source and target are identical as free  $\mathrm{Cl}(X) \oplus \mathrm{Cl}(X)$ -graded  $S$ -modules. Choosing a positive  $\mathrm{Cl}(X)$ -grading  $\theta: \mathrm{Cl}(X) \rightarrow \mathbb{Z}$  for  $S$  gives a positive  $\mathrm{Cl}(X) \oplus \mathrm{Cl}(X)$ -grading  $\theta': \mathrm{Cl}(X) \oplus \mathrm{Cl}(X) \rightarrow \mathbb{Z}$  of  $S$  given by  $\theta'(a, b) = a + b$ . Since  $S_\Delta^+$  is a positively  $\mathrm{Cl}(X) \oplus \mathrm{Cl}(X)$ -graded  $S$ -module with respect to  $\theta'$ , it follows from Nakayama's lemma that the surjection  $\mathrm{coker}(\varphi) \twoheadrightarrow S_\Delta^+$  is an isomorphism. ■

*Proof of Theorem 4.1.* Observe that  $F \otimes S'/(y_0, \dots, y_n)$  is a direct sum of Koszul complexes, one for each  $d \geq 0$ . In particular,  $F \otimes S'/(y_0, \dots, y_n)$  has homology concentrated in degree 0. It follows from Lemma A.4 and induction that  $F$  is exact in positive degrees, i.e., the sequence  $x_0 - y_0 u_0, \dots, x_n - y_n u_n$  of elements in  $R$  is regular. Now observe that the differential  $F_1 \rightarrow F_0$  is exactly the presentation map in Proposition 4.4. ■

#### 4.2. Finite rank resolutions of the diagonal

We now show that, when  $\mathcal{X}$  is a generalized weighted projective stack (as defined in Section 3.2),  $F$  has a finite rank subcomplex of length  $n$  that is also a resolution of  $\mathcal{O}_\Delta$ . Recall that the divisor class group of the fake weighted projective space  $X$  associated to  $\mathcal{X}$  is  $\mathbb{Z} \oplus A$  for some finite abelian group  $A$ . As in Section 3.2, let  $\pi$  be a surjective map  $\text{Cl}(X) \rightarrow \mathbb{Z}$ , and write  $w = \sum_{i=0}^n \pi(\deg(x_i))$ . Using the expression of  $F$  in (4.1), let  $F'$  be the subcomplex of  $F$  with terms  $F'_i = \bigoplus_{\pi(a) \geq 0} \bigoplus_{\pi(b) < w - \pi(a)} (\omega_E)_{(b,i)} \otimes_k S'(-a - b, a)$ . Notice that  $F'$  is finite rank.

**Example 4.5.** When  $\mathcal{X} = \mathbb{P}(1, 2)$ ,  $F'$  is the subcomplex spanned by the northwest diagonal strands  $S(-d) \boxtimes K_d(d)$  from Example 4.3 for  $d < 3$ . The key idea is that, when  $d \geq 3$ , the strand  $S(-d) \boxtimes K_d(d)$  makes an irrelevant contribution to the homology of  $F$ , and so the cokernel of  $F' \rightarrow F$  is entirely supported on the irrelevant ideal.

The following theorem yields a finite rank resolution of the diagonal for a generalized weighted projective stack, giving a slight generalization of a result of Canonaco–Karp for weighted projective stacks [27].

**Theorem 4.6.** *The complex  $\tilde{F}'$  of sheaves on  $\mathcal{X} \times \mathcal{X}$  is a resolution of the diagonal.*

*Proof.* Observe that the terms of the complex  $F$  are positively graded. Let  $F'' = F/F'$ . The quotient  $F''/(x_0, \dots, x_n)F''$  is a direct sum of twists of Koszul complexes on the  $x_i$ 's; it thus follows from Lemma A.4 and induction that  $H_i(F'') = 0$  for  $i > 0$ . Thus,  $H_i(F') = 0$  for  $i > 0$  as well. As for  $i = 0$ , let  $G^j$  be the subcomplex of  $F''$  with  $t$ -th term,

$$G_t^j = \bigoplus_{\pi(a) \geq 0} \bigoplus_{\pi(b) = w - \pi(a)}^{w - \pi(a) + j} (\omega_E)_{(b,t)} \otimes_k S'(-a - b, a).$$

This is the sum of the first  $j + 1$  full Koszul strands in the  $x_i$ 's. A direct calculation yields  $H_0(\tilde{G}^j) = 0$  for each  $j$ . Since  $F''$  is the colimit of the  $G^j$ 's, we get

$$H_0(\tilde{F}') = H_0(\tilde{F}) = \mathcal{O}_\Delta. \quad \blacksquare$$

**Remark 4.7.** We believe that finite rank resolutions of the diagonal constructed as in Theorem 4.6 exist more generally: see Conjecture 7.2 below. For instance, this has been verified in the case where  $\mathcal{X}$  is a smooth projective toric variety of Picard rank 2 [23].

#### 5. Recovering a sheaf from its toric Tate resolution

As discussed above, Eisenbud–Fløystad–Schreyer explain in [39] how to recover a sheaf on projective space from its Tate resolution [39, Theorem 6.1]. Our next goal is to generalize this result to toric varieties. In fact, we will prove the following stronger result.

**Theorem 5.1.** *Let  $\mathcal{X}$  be a projective toric stack with Cox ring  $S$ , and let  $E$  denote the Koszul dual exterior algebra of  $S$ . Let  $\mathcal{R}$  denote the resolution of the diagonal for  $\mathcal{X}$  from Theorem 4.1 and  $\mathcal{K}$  the kernel of the Fourier–Mukai transform in the definition of the toric Tate resolution functor (Section 3.2). There exists a functor  $\mathbf{U}: \mathrm{DM}(E) \rightarrow \mathrm{Com}(\mathcal{X})$  making*

$$\begin{array}{ccc}
 & \mathrm{DM}(\mathcal{X}_E) & \xrightarrow{R\pi_{2*}} \mathrm{DM}(E) \\
 \pi_1^*(-) \otimes \mathcal{K} \nearrow & & \downarrow \mathbf{U} \\
 \mathrm{Coh}(\mathcal{X}) & & \\
 \searrow \pi_1^*(-) \otimes \mathcal{R} & & \\
 & \mathrm{Com}(\mathcal{X} \times \mathcal{X}) & \xrightarrow{R\pi_{2*}} \mathrm{Com}(\mathcal{X})
 \end{array}$$

commute up to isomorphism in  $\mathrm{D}^b(\mathcal{X})$ .

Recall that the top row of the diagram in Theorem 5.1 gives the Tate resolution, up to homotopy equivalence. The following generalization of [39, Theorem 6.1] is therefore immediate from Theorems 4.1 and 5.1.

**Corollary 5.2.** *Let  $\mathcal{F} \in \mathrm{Coh}(\mathcal{X})$ . There is an isomorphism  $\mathbf{UT}(\mathcal{F}) \cong \mathcal{F}$  in  $\mathrm{D}^b(\mathcal{X})$ .*

*Proof of Theorem 5.1.* We begin by defining the functor  $\mathbf{U}$ . For  $D \in \mathrm{DM}(E)$ , we let  $D'$  denote the submodule  $\bigoplus_{-a \in V, i \in \mathbb{Z}} D_{(a;i)}$  of  $D$ , where  $V$  is as defined in Section 4. We define  $\mathbf{U}: \mathrm{DM}(E) \rightarrow \mathrm{Com}(\mathcal{X})$  by sending  $D$  to  $\mathbf{L}(D') \in \mathrm{Com}(S)$  and then sheafifying to obtain a complex of  $\mathcal{O}_{\mathcal{X}}$ -modules.

Recall that  $\mathbf{T}(\mathcal{F})$  is homotopy equivalent to the differential module  $Y$  that one gets by totalizing bicomplex (3.2) and then applying the equivalence  $\mathrm{Com}_{\mathrm{per}}(E) \xrightarrow{\sim} \mathrm{DM}(E)$  from Remark 2.3. It follows that there is a homotopy equivalence  $\mathbf{UT}(\mathcal{F}) \xrightarrow{\sim} \mathbf{U}(Y)$ . The underlying  $E$ -module of  $Y$  is  $\bigoplus_{\ell=0}^{\dim X} \bigoplus_{a \in \mathrm{Cl}(X)} \mathcal{C}_{\mathcal{F}(a)}^{G,\ell} \otimes_k \omega_E(-a; \ell)$ , and so  $\mathbf{U}(Y)_j$  is

$$\bigoplus_{b \in V} \bigoplus_{\ell=0}^{\dim X} \bigoplus_{a \in \mathrm{Cl}(X)} (\omega_E)_{(-a-b, \ell+j)} \otimes_k \mathcal{C}_{\mathcal{F}(a)}^{G,\ell} \otimes_k \mathcal{O}(b).$$

Reindexing, we may write this sum as

$$\bigoplus_{a \in V} \bigoplus_{\ell=0}^{\dim X} \bigoplus_{b \in \mathrm{Cl}(X)} (\omega_E)_{(b, \ell+j)} \otimes_k \mathcal{C}_{\mathcal{F}(-a-b)}^{G,\ell} \otimes_k \mathcal{O}(a).$$

The differential on  $\mathbf{U}(Y)$  sends a section  $f \otimes c \otimes s$  of  $(\omega_E)_{(b, \ell+j)} \otimes_k \mathcal{C}_{\mathcal{F}(-a-b)}^{G,\ell} \otimes_k \mathcal{O}(a)$  to

$$\sum_{i=0}^n (e_i f \otimes c \otimes x_i s + (-1)^{\ell+1} e_i f \otimes y_i c \otimes s) - f \otimes \partial_C(c) \otimes s,$$

where  $\partial_C$  denotes the Čech differential. We can view  $\mathbf{U}(Y)$  as the totalization of the bicomplex with  $(p, q)$  entry  $\bigoplus_{a \in V} \bigoplus_{b \in \text{Cl}(X)} (\omega_E)_{(b; p)} \otimes_k \mathcal{C}_{\mathcal{F}(-a-b)}^{G, -q} \otimes_k \mathcal{O}(a)$ , horizontal differential given by sending  $f \otimes c \otimes s$  to  $\sum_{i=0}^n (e_i f \otimes c \otimes x_i s + (-1)^{q+1} e_i f \otimes y_i c \otimes s)$ , and vertical differential induced by  $-\partial_C$ . For instance, notice that the  $q = 0$  row is given by applying  $\mathcal{C}^{G, 0}$  to the first tensor factor of every term in  $\pi_1^*(\mathcal{F}) \otimes \mathcal{R}$ , where  $\mathcal{R}$  denotes the resolution of the diagonal from Theorem 4.1. By the projection formula and flat base change, the complex  $\mathbf{U}(Y)$  is a model for the Fourier–Mukai transform  $\Phi_{\mathcal{R}}(\mathcal{F}) := R\pi_{2*}(\pi_1^*(\mathcal{F}) \otimes \mathcal{R})$  in  $D^b(\mathcal{X})$ . Finally, observe that  $\Phi_{\mathcal{R}}(\mathcal{F})$  is isomorphic to  $\mathcal{F}$  in  $D^b(\mathcal{X})$ . ■

**Question 5.3.** In [38, §7] and [66], an analogue of the functor  $\mathbf{U}$  for products of projective spaces is used to give generalized Horrocks-type criteria for when a vector bundle splits as a sum of line bundles. Do similar criteria hold for more general toric varieties? In a different direction: can  $\mathbf{U}$  be used to study vector bundles on  $\mathcal{X}$  in the manner of [41]?

A Horrocks splitting criterion in this vein for smooth projective toric varieties of Picard rank 2 is obtained in [23].

Applying the functor  $\mathbf{U}$  from the proof of Theorem 5.1 to the  $E$ -module  $\omega_E(d; i)$  gives the truncated and shifted Koszul complex  $\tilde{K}_d(d)[i]$ . When  $\mathcal{X} = \mathbb{P}^n$ ,  $\mathbf{U}(\omega_E(d; 0))$  is therefore quasi-isomorphic to  $\Omega_{\mathbb{P}^n}^d(d)$ , and thus this functor is closely related<sup>6</sup> to the functor  $\Omega$  defined in [39] and discussed in Section 3.3. We now use this observation to make precise the rough intuition from Section 3.3 that, over  $\mathbb{P}^n$ , applying “ $\text{id} \times \Omega$ ” to (3.3) gives Beilinson’s resolution of the diagonal. In fact, the following result gives an analogous statement over any projective toric stack; its proof is a straightforward calculation.

**Proposition 5.4.** *Let  $\mathbf{U}$  be the functor defined in the proof of Theorem 5.1. The chain complex with  $i$ -th term  $\bigoplus_{a \in \text{Cl}(X)} \mathcal{O}(a) \otimes_k \mathbf{U}(\omega_E(-a; 0))_i$  and differential given by  $\sum_{i=0}^n x_i \otimes \mathbf{U}(e_i) - \text{id} \otimes \partial_{\mathbf{U}}$  is precisely the resolution of the diagonal from Theorem 4.1.*

The idea is that the complex in Proposition 5.4 is obtained by applying “ $\text{id} \times \mathbf{U}$ ” to the noncommutative Fourier–Mukai kernel  $\mathcal{K}$  from Section 3.2.

## 6. The bounded derived category of a weighted projective stack

We recall that Tate resolutions over  $\mathbb{P}^n$  can be used to interpret  $D^b(\mathbb{P}^n)$  in terms of the exterior algebra, giving a geometric refinement of the classical BGG equivalence  $D^b(S) \simeq D^b(E)$ . In detail, let  $K^{\text{ex}}(E)$  denote the homotopy category of (unbounded)

<sup>6</sup>The functor  $\mathbf{U}$  can behave differently over general projective toric stacks than over  $\mathbb{P}^n$ . For instance, taking  $\mathcal{X}$  to be the weighted projective stack  $\mathbb{P}(1, 1, 2)$ , we have  $\mathbf{U}(\omega_E(1; 0)) = (\mathcal{O}(-1))^{\oplus 2} \xrightarrow{(x_0 x_1)} \mathcal{O}$ , which has homology that is nonzero in both positions and is not given by a vector bundle in degree 0.

exact complexes of finitely generated free  $\mathbb{Z}$ -graded  $E$ -modules. It follows from results of [25, 39] that the Tate resolution functor implements an equivalence  $D^b(\mathbb{P}^n) \simeq K_{\text{DM}}^{\text{ex}}(E)$ . The goal of this section is to use our results on toric Tate resolutions to generalize this equivalence to weighted projective stacks.

Let us first fix some notation. Let  $\mathcal{X}$  be a weighted projective stack with Cox ring  $S$  and Koszul dual exterior algebra  $E$ . We will say an object  $D \in \text{DM}(E)$  is *locally finite* if  $\sum_{i \in \mathbb{Z}} \dim_k D_{(a;i)} < \infty$  for all  $a \in \text{Cl}(X)$ . We let  $K_{\text{DM}}^{\text{ex}}(E)$  denote the homotopy category of exact, free, locally finite differential  $E$ -modules. We will prove the following result.

**Theorem 6.1.** *For a weighted projective stack  $\mathcal{X}$ , there is an equivalence  $D^b(\mathcal{X}) \xrightarrow{\sim} K_{\text{DM}}^{\text{ex}}(E)$  that sends a coherent sheaf concentrated in homological degree 0 to its Tate resolution.*

**Remark 6.2.** The results in this section all generalize in an evident way to generalized weighted projective stacks; we omit the details.

Suppose  $\mathcal{X}$  is a weighted projective stack. Let  $\text{Perf}_{\text{DM}}(E) \subseteq D_{\text{DM}}^b(E)$  be the smallest triangulated subcategory of  $D_{\text{DM}}^b(E)$  containing all summands of finitely generated free flag differential  $E$ -modules, and define the *singularity category of differential  $E$ -modules* to be the Verdier quotient  $D_{\text{DM}}^{\text{sing}}(E) := D_{\text{DM}}^b(E) / \text{Perf}_{\text{DM}}(E)$ . We will prove Theorem 6.1 by constructing a chain of equivalences

$$D^b(\mathcal{X}) \xrightarrow{\sim} D_{\text{DM}}^{\text{sing}}(E) \xrightarrow{\sim} K_{\text{DM}}^{\text{ex}}(E). \quad (6.1)$$

The second equivalence is an analogue of a theorem of Buchweitz [25, Theorem 4.4.1].

We start with the first link in the chain. We say a graded  $S$ -module is *torsion* if it is annihilated by a power of the maximal ideal  $\langle x_0, \dots, x_n \rangle$ . Denote by  $D_{\text{tors}}^b(S)$  the subcategory of  $D^b(S)$  given by complexes with torsion homology.

**Proposition 6.3.** *The equivalence  $D_{\text{DM}}^b(E) \simeq D^b(S)$  of Corollary 2.14 induces equivalences*

$$D_{\text{tors}}^b(S) \simeq \text{Perf}_{\text{DM}}(E) \quad \text{and} \quad D^b(\mathcal{X}) \xrightarrow{\sim} D_{\text{DM}}^{\text{sing}}(E).$$

*Proof.* We recall that a triangulated subcategory  $\mathcal{T}'$  of a triangulated category  $\mathcal{T}$  is called *thick* if  $\mathcal{T}'$  is closed under taking summands. Since a finitely generated free flag is precisely the same thing as a finite iterated extension of rank 1 free modules, one concludes that the subcategory  $\text{Perf}_{\text{DM}}(E) \subseteq D_{\text{DM}}^b(E)$  is the thick subcategory generated by  $E(a; 0)$  for all  $a \in \mathbb{Z}$ . The first equivalence follows immediately, since  $D_{\text{tors}}^b(S)$  is the thick subcategory of  $D^b(S)$  generated by  $k(a)$  for all  $a \in \mathbb{Z}$ , and  $\mathbf{R}(k(a)) = E(a; 0)$ .

As for the second equivalence, it suffices to observe that  $D^b(\mathcal{X})$  is equivalent to the Verdier quotient  $D^b(S) / D_{\text{tors}}^b(S)$ . To see this, apply [63, Proposition 2.17] to see that  $\text{Coh}(\mathcal{X})$  is equivalent to the abelian quotient of the category of finitely generated graded  $S$ -modules by the Serre subcategory given by torsion modules. Then recall that, for any abelian category  $\mathcal{A}$  and Serre subcategory  $\mathcal{S}$  of  $\mathcal{A}$ , there is an equivalence  $D^b(\mathcal{A}) / D_{\mathcal{S}}^b(\mathcal{A}) \simeq D^b(\mathcal{A} / \mathcal{S})$ , where  $D_{\mathcal{S}}^b(\mathcal{A}) = \{X \in D^b(\mathcal{A}) : H_n(X) \in \mathcal{S} \text{ for all } n \in \mathbb{Z}\}$ . ■

Any object in  $\mathrm{DM}(E)$  with finitely generated homology admits a locally finite free resolution by Theorem B.2. Let  $D$  be a finitely generated object in  $\mathrm{DM}(E)$ ,  $F$  a locally finite free resolution of  $D$ , and  $G$  a locally finite free resolution of  $D^\vee := \underline{\mathrm{Hom}}_E(D, E)$ . Since dualizing over  $E$  is exact, and every finitely generated module over  $E$  is maximal Cohen–Macaulay, we have an induced quasi-isomorphism  $F \xrightarrow{\sim} D \cong (D^\vee)^\vee \xrightarrow{\sim} G^\vee$ . Denote the mapping cone of this quasi-isomorphism by  $\mathbf{C}(D)$ . The object  $\mathbf{C}(D)$  is contained in  $K_{\mathrm{DM}}^{\mathrm{ex}}(E)$ , and it is well defined up to the choices of  $F$  and  $G$ . If  $f: D \rightarrow D'$  is a quasi-isomorphism in  $\mathrm{DM}(E)$ , there is an induced homotopy equivalence  $\mathbf{C}(D) \xrightarrow{\sim} \mathbf{C}(D')$ , and so there is an induced functor  $\mathbf{C}: \mathrm{D}_{\mathrm{DM}}^b(E) \rightarrow K_{\mathrm{DM}}^{\mathrm{ex}}(E)$  (recall from Proposition 2.5 that every object in  $\mathrm{D}_{\mathrm{DM}}^b(E)$  is isomorphic to a finitely generated differential module). Moreover, if  $D \in \mathrm{Perf}_{\mathrm{DM}}(E)$ , then  $D$  and  $D^\vee$  are free resolutions of themselves, so  $\mathbf{C}(D) = \mathrm{cone}(D \xrightarrow{\sim} (D^\vee)^\vee)$ , which is contractible. It follows that we have an induced functor  $\mathbf{C}: \mathrm{D}_{\mathrm{DM}}^{\mathrm{sing}}(E) \rightarrow K_{\mathrm{DM}}^{\mathrm{ex}}(E)$ . The definition of the map  $\mathbf{C}$  closely resembles Buchweitz’s construction of *complete resolutions* in [25]; hence the notation “ $\mathbf{C}$ ” for this functor.

Going the other direction, let  $D \in K_{\mathrm{DM}}^{\mathrm{ex}}(E)$ , and write  $D = \bigoplus_{(a;i) \in \mathbb{Z} \times \mathbb{Z}} E(a; i)^{\oplus r_{a,i}}$ . For all  $a \in \mathbb{Z}$ , write  $D_a = \bigoplus_{i \in \mathbb{Z}} E(a; i)^{\oplus r_{a,i}}$ . Let  $D' = \bigoplus_{a \leq 0} D_a$  and  $D'' = \bigoplus_{a > 0} D_a$ . Decomposing  $D$  as  $D' \oplus D''$ , we can write the differential  $\partial$  on  $D$  as a matrix of the form  $\begin{pmatrix} \partial' & \alpha \\ 0 & \beta \end{pmatrix}$ . Notice that  $\alpha$  exhibits  $(D''(0; 1), -\beta)$  as a free resolution of  $(D', \partial')$ , and  $D = \mathrm{cone}(\alpha)$ . Observe also that, since  $\dim_k \bigoplus_{i \in \mathbb{Z}} D_{(a;i)} < \infty$  for all  $a \in \mathbb{Z}$ , the differential module  $D'$  has finitely generated homology. We thus have a functor  $\tau_{\leq 0}: K_{\mathrm{DM}}^{\mathrm{ex}}(E) \rightarrow \mathrm{D}_{\mathrm{DM}}^{\mathrm{sing}}(E)$  given by  $D \mapsto D'$ .

**Proposition 6.4.** *The functors  $\mathbf{C}: \mathrm{D}_{\mathrm{DM}}^{\mathrm{sing}}(E) \rightleftarrows K_{\mathrm{DM}}^{\mathrm{ex}}(E) : \tau_{\leq 0}$  are inverse equivalences.*

*Proof.* Let  $D \in \mathrm{D}_{\mathrm{DM}}^{\mathrm{sing}}(E)$  be a finitely generated differential module. Let  $F$  be a locally finite free resolution of  $D$  and  $G$  a locally finite free resolution of  $D^\vee$ , so that  $\mathbf{C}(D) = F(0; -1) \oplus G^\vee$ . Write

$$\mathbf{C}(D) = \bigoplus_{(a;i) \in \mathbb{Z} \times \mathbb{Z}} E(a; i)^{\oplus r_{a,i}} \quad \text{and} \quad \mathbf{C}(D)_a = \bigoplus_{i \in \mathbb{Z}} E(a; i)^{\oplus r_{a,i}},$$

and choose  $a \ll 0$  such that  $\mathbf{C}(D)_b \subseteq F$  for all  $b \leq a$ . Let  $N = \bigoplus_{b \leq a} \mathbf{C}(D)_b$ . The natural maps

$$N \rightarrow (\tau_{\leq 0} \circ \mathbf{C})(D) \quad \text{and} \quad N \rightarrow F$$

are both isomorphisms in  $\mathrm{D}_{\mathrm{DM}}^{\mathrm{sing}}(E)$ ; it follows that  $(\tau_{\leq 0} \circ \mathbf{C})(D) \cong D$  in  $\mathrm{D}_{\mathrm{DM}}^{\mathrm{sing}}(E)$ .

On the other hand, let  $D \in K_{\mathrm{DM}}^{\mathrm{ex}}(E)$ , and let  $D'$  and  $D''$  be as in the above construction of the functor  $\tau_{\leq 0}$ . The differential module  $D'$  has finitely generated homology: let  $Y$  be the  $E$ -submodule of  $D'$  generated by elements of  $\mathrm{Cl}(X)$ -degree  $a$  such that  $H(D')_{(a;i)} \neq 0$  for some  $i \in \mathbb{Z}$ . The object  $Y$  is finitely generated, and it is a differential submodule of  $D'$ , because the differential is degree 0 with respect to the  $\mathrm{Cl}(X)$ -degree. Moreover, the inclusion  $Y \hookrightarrow D'$  is a quasi-isomorphism; in fact, the dual  $(D')^\vee \rightarrow Y^\vee$  of this inclusion is a free resolution of  $Y^\vee$ . Choose locally finite free resolutions  $F$  of  $Y$

and  $G$  of  $Y^\vee$ , so that  $(\mathbf{C} \circ \tau_{\leq 0})(D)$  is isomorphic to  $\text{cone}(F \xrightarrow{\sim} G^\vee)$ . Since  $G^\vee$  is homotopy equivalent to  $D'$  and  $F$  is homotopy equivalent to  $(D''(0; 1), -\beta)$ ,  $D$  is homotopy equivalent to  $(\mathbf{C} \circ \tau_{\leq 0})(D)$ .  $\blacksquare$

*Proof of Theorem 6.1.* Our equivalence is given by the composition

$$\mathrm{D}^b(\mathcal{X}) \xrightarrow{\mathbf{R}} \mathrm{D}_{\mathrm{DM}}^{\mathrm{sing}}(E) \xrightarrow{\mathbf{C}} K_{\mathrm{DM}}^{\mathrm{ex}}(E)$$

of the equivalences in Propositions 6.3 and 6.4. By Theorem 3.7, applying this equivalence to a sheaf  $\mathcal{F}$ , concentrated in degree 0, gives the Tate resolution of  $\mathcal{F}$ .  $\blacksquare$

## 7. Future directions

A running theme in this work, and elsewhere [14, 15, 24, 29, 37, 38, 50, 51, 53, 54, 61, 62, 68], is that the multigradings on the Cox rings of toric varieties make many homological constructions more subtle than their well-known counterparts over  $\mathbb{P}^n$ . We propose a few questions in this vein.

**Conjecture 7.1.** *The properties in Theorem 3.3 characterize  $\mathbf{T}(\mathcal{F})$  up to isomorphism of differential modules.*

In general, we have only shown that  $\mathbf{T}(\mathcal{F})$  is well defined up to homotopy, but Theorems 3.7 and 3.10 imply that Conjecture 7.1 holds over generalized weighted projective stacks and products of projective spaces. See the paragraph beneath the proof of Theorem 3.10 for a discussion of the difficulty of extending these results to general projective toric stacks.

**Conjecture 7.2.** *The resolution  $F$  in Theorem 4.1 always admits a finite rank subcomplex, whose sheafification is a resolution of the diagonal. In fact, we can always find such a subcomplex whose length is at most  $\dim(X)$ .*

Conjecture 7.2 would resolve a question of Berkesch–Erman–Smith on the minimal length of a virtual resolution of a module (see [14, Question 6.5]), and it would imply a large swath of new cases of a conjecture of Orlov concerning the Rouquier dimension of the bounded derived category of a quasi-projective variety [64, Conjecture 10]. Conjecture 7.2 has been proven by the first author and Sayrafi in the case of smooth projective toric varieties of Picard rank 2 [23].

One application of the Tate resolutions from [39] was to the development of an efficient algorithm for computing sheaf cohomology on  $\mathbb{P}^n$ ; see also [34].

**Question 7.3.** Can one use toric Tate resolutions to develop an exterior algebra algorithm for computing sheaf cohomology on any projective toric stack?

In followup work, we will show that this question has a positive answer for weighted projective stacks, using the theory of minimal free resolutions of differential modules

from [20]. Just as with the problem of characterizing the toric Tate resolution up to isomorphism (Conjecture 7.1), we expect that extending such an algorithm to a more general toric variety will rely heavily upon the exactness properties in Theorem 3.3 (4).

**Question 7.4.** Can one generalize Theorem 6.1 above by relating the bounded derived category of a projective toric stack  $\mathcal{X}$  to an appropriate homotopy category of exact differential  $E$ -modules?

A positive answer to Question 7.4 could provide new connections to the study of derived categories of toric varieties and stacks, e.g., [8–10, 19, 30, 33, 35, 36, 58–60, 63]. For instance, an exceptional object/collection on the  $E$ -module side would immediately yield the same for  $D^b(\mathcal{X})$ .

A fundamental challenge underlying Question 7.4 is determining the correct analogue of  $K_{\text{DM}}^{\text{ex}}(E)$  in the case of an arbitrary projective toric stack  $\mathcal{X}$ . To explain the difficulty: notice that passing from  $D^b(S)$  to  $D^b(\mathcal{X})$  requires one to take a quotient not just by complexes supported in the maximal ideal  $\langle x_0, \dots, x_n \rangle$ , but complexes supported in the irrelevant ideal. The counterpart of  $D^b(\mathcal{X})$  will therefore typically be strictly smaller than  $K_{\text{DM}}^{\text{ex}}(E)$ . One can see this from another point of view: by Theorem 3.3 (3), the Tate resolution satisfies more subtle exactness properties than a typical object in  $K_{\text{DM}}^{\text{ex}}(E)$ , and so we should not expect every object in  $K_{\text{DM}}^{\text{ex}}(E)$  to correspond to an object in  $D^b(\mathcal{X})$  via the Tate resolution functor.

## Appendix A. Positive multigradings

Let  $A$  be an abelian group and  $R$  an  $A$ -graded ring with  $R_0$  a field.

**Definition A.1.** A group homomorphism  $\theta: A \rightarrow \mathbb{Z}$  is called a *positive  $A$ -grading* on  $R$  if, for all  $x \in R \setminus \{0\}$ ,

- (1)  $\theta(\deg(x)) \geq 0$ , and
- (2)  $\theta(\deg(x)) = 0$  if and only if  $x$  is a unit.

The ring  $R$  is called *positively  $A$ -graded* if a positive  $A$ -grading exists. The above properties ensure that  $R$  is, via  $\theta$ , a nonnegatively  $\mathbb{Z}$ -graded ring such that the ideal in  $R$  generated by elements of positive degree is maximal.

**Example A.2.** Let  $X$  be a projective toric variety with  $\text{Cl}(X)$ -graded Cox ring  $S$ . We claim that  $S$  is positively  $\text{Cl}(X)$ -graded. Indeed, letting  $H$  be an ample divisor on  $X$ , the map  $\theta: \text{Cl}(X) \rightarrow \mathbb{Z}$  given by intersecting with  $H^{\dim X - 1}$  is a positive  $\text{Cl}(X)$ -grading. The projectivity assumption cannot be removed; see, e.g., [31, Example 5.2.3].

**Definition A.3.** Let  $M$  be an  $A$ -graded  $R$ -module. A homomorphism  $\theta: A \rightarrow \mathbb{Z}$  determines a  $\mathbb{Z}$ -grading on  $M$  by setting  $M_i = \bigoplus_{\theta(a)=i} M_a$ . We say  $M$  is *positively  $A$ -graded* if there is a positive  $A$ -grading on  $R$  such that this associated  $\mathbb{Z}$ -grading of  $M$  is bounded below.

We record the following homological variant of Nakayama's lemma.

**Lemma A.4.** *Let  $R$  be positively  $A$ -graded, and let  $C$  be a complex of free  $R$ -modules such that  $H_i(C)$  is positively  $A$ -graded. Let  $x \in R$  be homogeneous of nonzero degree. If  $H_i(C/xC) = 0$ , then  $H_i(C) = 0$ .*

## Appendix B. Further background on differential $E$ -modules

As in Section 2, let  $S = k[x_0, \dots, x_n]$  be positively graded by an abelian group  $A$ , and let  $E = \Lambda_k(e_0, \dots, e_n)$  be equipped with the  $A \oplus \mathbb{Z}$ -grading given by  $\deg(e_i) = (-\deg(x_i); -1)$ .

### B.1. Resolutions of differential $E$ -modules

We recall from [20] the notion of a free resolution of a differential module.

**Definition B.1** ([20, Section 1]). A differential  $E$ -module  $F$  is a *free flag* if  $F$  is a free module that may be equipped with a decomposition  $\bigoplus_{i \geq 0} F_i$  such that  $\partial_F(F_i) \subseteq \bigoplus_{j < i} F_j$ . Given  $D \in \text{DM}(E)$ , a *free flag resolution* of  $D$  is a quasi-isomorphism  $F \xrightarrow{\sim} D$ , where  $F$  is a free flag. A *free resolution* of  $D$  is a quasi-isomorphism  $F \xrightarrow{\sim} D$  that factors as  $F \rightarrow \tilde{F} \rightarrow D$ , where  $\tilde{F} \rightarrow D$  is a free flag resolution, and  $F \rightarrow \tilde{F}$  is a split injection. We say a free resolution  $F \xrightarrow{\sim} D$  is *minimal* if  $\partial_F(F) \subseteq \mathfrak{m}F$ , where  $\mathfrak{m} = \langle e_0, \dots, e_n \rangle \subseteq E$ .

Reversing arrows, one can define injective resolutions. A differential  $E$ -module  $I$  is an *injective coflag* if  $I$  is an injective module that may be equipped with a decomposition  $\bigoplus_{i \leq 0} I_i$  such that  $\partial_I(I_i) \subseteq \bigoplus_{j < i} I_j$  (note that graded injective, projective, and free modules coincide over  $E$ ). One defines an *injective coflag resolution* in the evident way, and an *injective resolution* of a differential module  $D$  is a quasi-isomorphism  $D \xrightarrow{\sim} I$  that factors as  $D \rightarrow \tilde{I} \rightarrow I$ , where  $D \rightarrow \tilde{I}$  is an injective coflag resolution, and  $\tilde{I} \rightarrow I$  is a split surjection.

The following fact plays a key role in the proof of Theorem 3.7 above.

**Theorem B.2.** *Any differential  $E$ -module  $D$  whose homology is finitely generated admits a minimal free resolution  $F \xrightarrow{\sim} D$ , and this minimal free resolution is unique up to isomorphism of differential modules. Moreover, we have  $\sum_{i \in \mathbb{Z}} \dim_k F_{(a,i)} < \infty$  for all  $a \in \text{Cl}(X)$ , and  $F$  is positively  $A$ -graded, in the sense of Definition A.3.*

*Proof.* By [20, Theorem 1.2], minimal free resolutions exist and are unique for any differential module with finitely generated homology and degree 0 differential over a (possibly noncommutative)  $\mathbb{Z}$ -graded local ring  $R$  such that  $R_0$  is a field. Our result does not immediately follow from this theorem, because  $E$  is  $A \oplus \mathbb{Z}$ -graded, not  $\mathbb{Z}$ -graded, and the differential on  $D$  has degree  $(0, -1)$ , not 0. However, a slight modification of the arguments in [20] gives the result we want. In detail, the positive  $A$ -grading on  $S$  induces a positive  $A$ -grading on  $E$  in the evident way. We can use this induced  $\mathbb{Z}$ -grading to

construct a minimal free resolution of  $D$  exactly as in [20, Remark 5.7]; this proves existence. Uniqueness follows from an argument identical to the proof of the uniqueness part of [20, Theorem 4.2 (b)].  $\blacksquare$

### B.2. Tensor product and internal Hom for differential $E$ -modules

We can use the auxiliary  $\mathbb{Z}$ -grading on  $E$  to define a tensor product, internal Hom, Tor, and Ext for differential  $E$ -modules; these coincide with the usual notions for dg-modules via the first equivalence discussed in Remark 2.3. Letting  $D, D' \in \mathrm{DM}(E)$ , we define the tensor product  $D \otimes_E^{\mathrm{DM}} D'$  to be the differential module with underlying module  $D \otimes_E D'$  and differential

$$d \otimes d' \mapsto \partial_D(d) \otimes d' + (-1)^{\mathrm{aux}(d)} d \otimes \partial_{D'}(d'),$$

where, as stated in Conventions 2.1,  $\mathrm{aux}(-)$  denotes the auxiliary degree. Recall also from Conventions 2.1 that any right  $E$ -module may be considered as a left  $E$ -module in a canonical way, so the tensor product  $D \otimes_E D'$  makes sense. The internal Hom object  $\underline{\mathrm{Hom}}_E^{\mathrm{DM}}(D, D')$  is defined to be the differential module with underlying  $E$ -module  $\mathrm{Hom}_E(D, D')$  and differential  $f \mapsto \partial_{D'} \circ f - (-1)^{\mathrm{aux}(f)} f \circ \partial_D$ . Let  $F$  be a free resolution of  $D$  (Definition B.1). We define

$$\mathrm{Tor}_{\mathrm{DM}}^E(D, D') = H(F \otimes_E^{\mathrm{DM}} D') \quad \text{and} \quad \mathrm{Ext}_E^{\mathrm{DM}}(D, D') = H(\underline{\mathrm{Hom}}_E^{\mathrm{DM}}(F, D')).$$

One can also define Tor (resp Ext) using a free (resp. injective) resolution of  $D'$ .

*Acknowledgements.* Many ideas in this paper had origins in joint conversations with David Eisenbud and Frank-Olaf Schreyer, and we are tremendously grateful for their generous insights. This project evolved over several years, and along the way we benefited from conversations with many researchers including: Christine Berkesch, Juliette Bruce, Lauren Cranton Heller, Milena Hering, Lutz Hille, Srikanth Iyengar, Brian Lehmann, Diane Maclagan, Steven V Sam, Mahrud Sayrafi, Gregory G. Smith, Hal Schenck, Mike Stillman, and Mark Walker. The computer algebra system Macaulay2 [49] provided valuable assistance. Finally, we thank the anonymous referees for their careful reading and their many suggestions that have improved this paper.

*Funding.* The first author was supported by NSF-RTG grant 150263. The second author was supported by NSF grants DMS-1601619 and DMS-1902123.

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