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# Corrigendum to "The Klein–Gordon equation, the Hilbert transform, and dynamics of Gauss-type maps"

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**Abstract.** We fix an erroneous step in the earlier work [J. Eur. Math. Soc. **22**, 1703–1757 (2020)] on solutions to the $(1 + 1)$-dimensional Klein–Gordon equation having Fourier transforms supported on one branch of the hyperbola associated with the Klein–Gordon equation.

*Keywords:* Fourier uniqueness, transfer operator, completeness, Klein–Gordon equation.

## 1. Background

In [3], uniqueness problems for solutions to the Klein–Gordon equation in one temporal and one spatial dimensions were studied. Follow-up work in [4, 5] further sharpened the results and methods, which were based on a certain reduction to ergodic theory. A further direction opened up recently with the interpolation methods of [1], which connect with the Fourier interpolation methods employed in, e.g., [6, 7] (cf. [2]).

## 2. The issue

It was pointed out to me a couple of years ago by Rajesh Srivastava that the proof of Theorem 1.6.1 in [4] contains an error. Here we fix that error.

### *2.1. Fixing the erroneous equivalence*

We begin with the following assertion.

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**Lemma 2.1.** *Suppose that $g \in L^1(\mathbb{R}_+)$, that is, $g \in L^1(\mathbb{R})$ and $g(t) = 0$ for a.e. $t \in \mathbb{R}_-$. Then we have the equivalence*

$$\int_{\mathbb{R}_+} \mathrm{e}^{\mathrm{i}2\pi m t} g(t)\,\mathrm{d}t = 0 \ \ \forall m \in \mathbb{Z} \ \Leftrightarrow \ \sum_{j=0}^{+\infty} g(t+j) = 0 \ \ \textit{a.e. on } [0,1].$$

*Proof.* The calculation

$$\int_{\mathbb{R}_+} \mathrm{e}^{\mathrm{i}2\pi m t} g(t)\,\mathrm{d}t = \sum_{j=0}^{+\infty} \int_{[j,j+1]} \mathrm{e}^{\mathrm{i}2\pi m t} g(t)\,\mathrm{d}t = \sum_{j=0}^{+\infty} \int_{[0,1]} \mathrm{e}^{\mathrm{i}2\pi m t} g(t+j)\,\mathrm{d}t$$
$$= \int_{[0,1]} \mathrm{e}^{\mathrm{i}2\pi m t} \sum_{j=0}^{+\infty} g(t+j)\,\mathrm{d}t$$

combined with the uniqueness theorem for Fourier series gives the claimed equivalence. ∎

We remark that the assertion of Lemma 2.1 replaces equation (6.1.5) in [4] which is erroneous, as it claims that the sum in the lemma should vanish on all of $\mathbb{R}_+$, which need not be true.

We proceed to indicate how the proof of Theorem 1.6.1 in [4] runs with the indicated change. It is given that $f \in L^1(\mathbb{R}_+)$, and that the relations

$$\int_{\mathbb{R}_+} \mathrm{e}^{\mathrm{i}2\pi m t} f(t)\,\mathrm{d}t = \int_{\mathbb{R}_+} \mathrm{e}^{\mathrm{i}2\pi \gamma n/t} f(t)\,\mathrm{d}t = 0, \quad m,n \in \mathbb{Z}, \tag{2.1}$$

hold for a given real parameter $\gamma$ with $0 < \gamma \le 1$. We may apply Lemma 2.1 to the vanishing of the left-hand side expression in (2.1) with $g = f$ to conclude that

$$\sum_{j=0}^{+\infty} f(t+j) = 0 \quad \text{a.e. on } [0,1],$$

which is the same as the relation

$$f(t) = -\sum_{j=1}^{+\infty} f(t+j) \quad \text{a.e. on } [0,1]. \tag{2.2}$$

As for the vanishing of the second integral in (2.1), we apply a change of variables in the integral, which gives that

$$\int_{\mathbb{R}_+} \mathrm{e}^{\mathrm{i}2\pi \gamma n/t} f(t)\,\mathrm{d}t = \int_{\mathbb{R}_+} \mathrm{e}^{\mathrm{i}2\pi n t} f\Big(\frac{\gamma}{t}\Big)\frac{\mathrm{d}t}{t^2} = 0, \quad m,n \in \mathbb{Z}. \tag{2.3}$$

Next, we apply Lemma 2.1 with $g \in L^1(\mathbb{R}_+)$ given by $g(t) = t^{-2} f(\gamma/t)$, and find an equivalent formulation for the vanishing of the second expression in (2.3):

$$\sum_{j=0}^{+\infty} \frac{1}{(t+j)^2} f\Big(\frac{\gamma}{t+j}\Big) = 0 \quad \text{a.e. on } [0,1]. \tag{2.4}$$

We single out the first term in the sum, and rewrite (2.4) further,

$$\frac{1}{t^2} f\Big(\frac{\gamma}{t}\Big) = -\sum_{j=1}^{+\infty} \frac{1}{(t+j)^2} f\Big(\frac{\gamma}{t+j}\Big), \tag{2.5}$$

a.e. on [0, 1]. After the change of variables $t \mapsto \gamma/t$, (2.5) becomes

$$f(t) = -\sum_{j=1}^{+\infty} \frac{\gamma^2}{(\gamma+jt)^2} f\Big(\frac{\gamma t}{\gamma+jt}\Big), \tag{2.6}$$

now a.e. on $[\gamma, +\infty[$. By combining the equality in (2.2) on [0, 1] with the equality in (2.6) on $[\gamma, +\infty[$, using that if $j \geq 1$ and $t \in [0, 1]$, we get $t + j \geq 1 \geq \gamma$, and that

$$f(t) = \sum_{j,l=1}^{+\infty} \frac{\gamma^2}{[\gamma + l(j+t)]^2} f\Big(\frac{\gamma(t+j)}{\gamma + l(t+j)}\Big) \quad \text{a.e. on } [0,1]. \tag{2.7}$$

Finally, since $I_1^+ = ]0, 1[$, condition (2.7) amounts to having

$$f = \mathbf{S}_\gamma^2 f \quad \text{a.e. on } I_1^+,$$

where $\mathbf{S}_\gamma$ is the subtransfer operator used in [4]. The rest of the proof of Theorem 1.6.1 remains unchanged.

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## References

[1] Bakan, A., Hedenmalm, H., Montes-Rodríguez, A.: Hyperbolic Fourier series and the Klein–Gordon equation. arXiv:2401.06871v2 (2024)

[2] Bakan, A., Hedenmalm, H., Montes-Rodríguez, A., Radchenko, D., Viazovska, M.: Fourier uniqueness in even dimensions. Proc. Natl. Acad. Sci. USA **118**, article no. 2023227118 (2021) Zbl 1556.42039 MR 4294062

[3] Hedenmalm, H., Montes-Rodríguez, A.: Heisenberg uniqueness pairs and the Klein–Gordon equation. Ann. of Math. (2) **173**, 1507–1527 (2011) Zbl 1227.42002 MR 2800719

[4] Hedenmalm, H., Montes-Rodríguez, A.: The Klein–Gordon equation, the Hilbert transform, and dynamics of Gauss-type maps. J. Eur. Math. Soc. (JEMS) **22**, 1703–1757 (2020) Zbl 1437.42010 MR 4092897

[5] Hedenmalm, H., Montes-Rodríguez, A.: The Klein–Gordon equation, the Hilbert transform and Gauss-type maps: $H^\infty$ approximation. J. Anal. Math. **144**, 119–190 (2021) Zbl 1481.42010 MR 4361892

[6] Radchenko, D., Viazovska, M.: Fourier interpolation on the real line. Publ. Math. Inst. Hautes Études Sci. **129**, 51–81 (2019) Zbl 1455.11075 MR 3949027

[7] Viazovska, M. S.: The sphere packing problem in dimension 8. Ann. of Math. (2) **185**, 991–1015 (2017) Zbl 1373.52025 MR 3664816