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Cohomological χ -independence for Higgs bundles and Gopakumar–Vafa invariants

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Abstract. The aim of this paper is two-fold. Firstly, we prove Toda’s χ -independence conjecture for Gopakumar–Vafa invariants of arbitrary local curves. Secondly, following Davison’s work, we introduce the BPS cohomology for moduli spaces of Higgs bundles of rank r and Euler characteristic χ which are not necessary coprime, and show that it does not depend on χ . This result extends the Hausel–Thaddeus conjecture on the χ -independence of E-polynomials proved by Melit, Groechenig–Wyss–Ziegler and Yu in two ways: We obtain an isomorphism of mixed Hodge modules on the Hitchin base rather than an equality of E-polynomials, and we do not need the coprime assumption. The proof of these results is based on a description of the moduli stack of one-dimensional coherent sheaves on a local curve as a global critical locus which is obtained in the companion paper by the first author and Naruki Masuda.

Keywords: Higgs bundles, Gopakumar–Vafa invariants, moduli spaces, cohomological Donaldson–Thomas theory.

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1. Introduction

We work over the complex number field \mathbb{C} .

1.1. Motivation and results

1.1.1. *Non-abelian Hodge theory.* Let C be a smooth projective curve with genus $g \geq 2$. By the non-abelian Hodge correspondence due to [9,23,30,52], there is a homeomorphism

$$M_{\text{Dol}}(r, m) \simeq M_{\text{B}}(r, m), \tag{1.1}$$

where $M_{\text{Dol}}(r, m)$ is the moduli space of slope semistable Higgs bundles E on C with $\text{rank}(E) = r$, $\chi(E) = m$, and $M_{\text{B}}(r, m)$ is the twisted character variety, i.e., the quotient variety

$$M_{\text{B}}(r, m) := \left\{ A_i, B_i \in \text{GL}_r, i = 1, \dots, g \mid \prod_i [A_i, B_i] = e^{2m\pi\sqrt{-1}/r} \right\} // \text{GL}_r$$

by the conjugate GL_r action.

Assume for a while that r and m are coprime so that the moduli spaces in (1.1) are smooth. Homeomorphism (1.1) induces an isomorphism

$$H^*(M_{\text{Dol}}(r, m)) \simeq H^*(M_{\text{B}}(r, m)) \tag{1.2}$$

between the singular cohomology groups. However, since (1.1) is only a diffeomorphism, isomorphism (1.2) is *not* an isomorphism of mixed Hodge structures. Indeed, the mixed Hodge structure on $H^*(M_{\text{Dol}}(r, m))$ is pure, while that on $H^*(M_{\text{B}}(r, m))$ is not pure. Instead, the cohomology group $H^*(M_{\text{Dol}}(r, m))$ has the so-called *perverse filtration* induced by the Hitchin morphism

$$h: M_{\text{Dol}}(r, m) \rightarrow B.$$

De Cataldo–Hausel–Migliorini [19] conjectured that the perverse filtration on an isomorphism $H^*(M_{\text{Dol}}(r, m))$ matches with the weight filtration on $H^*(M_{\text{B}}(r, m))$ via isomorphism (1.2) ($P = W$ conjecture). This conjecture was recently proved by Hausel–Mellit–Minets–Schiffmann [29] and Maulik–Shen [41] independently.

For character varieties, $M_{\text{B}}(r, m)$ and $M_{\text{B}}(r, m')$ are Galois conjugate to each other, for all $m, m' \in \mathbb{Z}$ with $\gcd(r, m) = \gcd(r, m') = 1$. In particular, we have an isomorphism

$$H^*(M_{\text{B}}(r, m)) \cong H^*(M_{\text{B}}(r, m')) \tag{1.3}$$

of mixed Hodge structures. According to the $P = W$ conjecture, the perverse filtration on $H^*(M_{\text{Dol}}(r, m))$ should be independent of $m \in \mathbb{Z}$, as long as we have $\gcd(r, m) = 1$. We prove this statement using the cohomological Donaldson–Thomas theory.

Theorem 1.1 (Example 5.18). *Let r, m, m' be integers such that $r > 0$ and $\gcd(r, m) = \gcd(r, m') = 1$. Then there exists an isomorphism*

$$H^*(M_{\text{Dol}}(r, m)) \cong H^*(M_{\text{Dol}}(r, m'))$$

preserving the Hodge structure and the perverse filtration.

This kind of statement is called a χ -independence phenomenon, as an invariant of the moduli space of Higgs bundles depends only on the rank r and is independent of the choice of the Euler characteristic χ . Note that the above result for the perverse filtration was obtained by [18] independently, via a completely different method.

Now assume that (r, m) is not coprime. In this case, the moduli spaces $M_{\text{Dol}}(r, m)$ and $M_{\text{B}}(r, m)$ are singular. Hence it is not clear which cohomology theory is the right choice to obtain a $P = W$ type statement. There are two candidates for this:

- (1) intersection cohomology [20, 25, 26, 44];
- (2) BPS cohomology [8].

One advantage of using the intersection cohomology is that it is mathematically defined whereas the BPS cohomology is defined in the physical language. Instead of this, BPS cohomology has its own advantage: whereas the χ -independence phenomena for the intersection cohomology is only expected when we have $\gcd(r, m) = \gcd(r, m')$, the χ -independence for the BPS cohomology is expected to hold without any assumption. Further, the BPS cohomology groups on both sides are expected to carry a Lie algebra structure (see [13]) and the non-abelian Hodge correspondence (1.1) is expected to induce an isomorphism of these Lie algebras [51, Conjecture 1.5]. This suggests that we would have a representation-theoretic approach to the original $P = W$ conjecture.

Following Davison’s idea [12], we propose a definition of the BPS cohomology for the Dolbeault moduli space $H_{\text{BPS}}^*(M_{\text{Dol}}(r, m))$ as a cohomology of a pure Hodge module $\mathcal{BPS}_{r,m}$ on $M_{\text{Dol}}(r, m)$ defined using the cohomological Donaldson–Thomas theory (or refined BPS state counting) for $\text{Tot}_{\mathcal{C}}(\mathcal{O}_{\mathcal{C}} \oplus \omega_{\mathcal{C}})$. We have a split injection $\mathcal{IC}_{M_{\text{Dol}}(r,m)} \hookrightarrow \mathcal{BPS}_{r,m}$ which is an isomorphism when $\gcd(r, m) = 1$, but not necessarily so for general (r, m) . We prove the following χ -independence for the BPS cohomology, which is a non-coprime generalization of Theorem 1.1.

Theorem 1.2 (Corollary 5.15). *Let r, m, m' be integers such that $r > 0$. Then there exists an isomorphism*

$$H_{\text{BPS}}^*(M_{\text{Dol}}(r, m)) \cong H_{\text{BPS}}^*(M_{\text{Dol}}(r, m'))$$

preserving the Hodge structure and the perverse filtration.

Remark 1.3. When we have $\gcd(r, m) = \gcd(r, m')$, the Betti moduli spaces $M_{\text{B}}(r, m)$ and $M_{\text{B}}(r, m')$ are Galois conjugate. Therefore, we expect that there exists an isomorphism

$$H_{\text{BPS}}^*(M_{\text{B}}(r, m)) \cong H_{\text{BPS}}^*(M_{\text{B}}(r, m')) \tag{1.4}$$

preserving the mixed Hodge structure, though we do not discuss the definition of the BPS cohomology for the Betti moduli spaces in this paper. Therefore, Theorem 1.2 gives an evidence of the $P = W$ conjecture for the BPS cohomology. Conversely, $P = W$ conjecture and Theorem 1.2 suggest that isomorphism (1.4) holds without the assumption $\gcd(r, m) = \gcd(r, m')$, which is of independent interest.

Remark 1.4. Recently, Davison–Hennecart–Schlegel Mejia [15] established a theorem relating the BPS cohomology and the intersection cohomology for the moduli space of Higgs bundles and for the character varieties. Their work implies the equivalence of two versions of the $P = W$ conjectures via the BPS cohomology and via the intersection cohomology, and that the χ -independence of the intersection cohomology of the Dolbeault moduli space follows from Theorem 1.2 as long as $\gcd(r, m) = \gcd(r, m')$ holds.

We also establish the cohomological integrality theorem for Higgs bundles, which claims the decomposition of the Borel–Moore homology of the moduli stack of Higgs bundles $H_*^{\text{BM}}(\mathfrak{M}_{\text{Dol}}(r, m))$ into tensor products of the BPS cohomology (see Theorem 5.16 for the precise statement). A similar statement was proved for quivers with potentials in [17, Theorem A] and for preprojective algebras in [12, Theorem D]. As explained in [16, §6.7], a plethystic computation and the cohomological integrality theorem imply that the Euler characteristic of the BPS cohomology is equal to the genus zero BPS invariant introduced by Joyce–Song [34, §6.2]. In particular, the cohomological integrality theorem strengthens the integrality conjecture for the genus zero BPS invariants [34, Conjecture 6.12].

Combining the cohomological integrality theorem and the χ -independence theorem (Theorem 1.2), we obtain the following χ -independence result for the Borel–Moore homology of the moduli stack $\mathfrak{M}_{\text{Dol}}(r, m)$ of Higgs bundles.

Corollary 1.5 (Corollary 5.20). *Let r, m, m' be integers such that $r > 0$ and $\gcd(r, m) = \gcd(r, m')$. Then there exists an isomorphism*

$$H_*^{\text{BM}}(\mathfrak{M}_{\text{Dol}}(r, m)) \cong H_*^{\text{BM}}(\mathfrak{M}_{\text{Dol}}(r, m'))$$

preserving the Hodge structure and the perverse filtration introduced in [14, Proposition 7.24].

1.1.2. *Gopakumar–Vafa invariants.* More generally, we investigate the χ -independence phenomena for curve counting theory on a class of Calabi–Yau (CY) 3-folds called *local curves*. By definition, a local curve is a CY 3-fold of the form $\text{Tot}_C(N)$, where C is a smooth projective curve and N is a rank 2 vector bundle on C such that $\det(N) \cong \omega_C$. To explain our result, we recall some basic background of curve counting theory for CY 3-folds.

There are several ways to count curves in a CY 3-fold X , and one of them is the *Gromov–Witten (GW) theory*: For an integer $g \geq 0$ and a homology class $\beta \in H_2(X, \mathbb{Z})$, denote by $M_{g,\beta}(X)$ the moduli space of stable maps $f: C \rightarrow X$ with C nodal curves of arithmetic genus g and $f_*[C] = \beta$. Then the GW invariant is defined as

$$\text{GW}_{g,\beta} := \int_{[M_{g,\beta}(X)]^{\text{vir}}} 1,$$

where $[M_{g,\beta}(X)]^{\text{vir}}$ denotes the virtual fundamental cycle. Due to the existence of stacky points in the moduli space $M_{g,\beta}(X)$, the GW invariant $\text{GW}_{g,\beta}$ is in general a rational number.

Based on string theory, Gopakumar–Vafa [27] conjectured the existence of *integer valued* invariants $n_{g,\beta} \in \mathbb{Z}$ for $g \geq 0$ and $\beta \in H_2(X, \mathbb{Z})$, satisfying the equation

$$\sum_{g \geq 0, \beta > 0} \text{GW}_{g,\beta} \lambda^{2g-2} t^\beta = \sum_{g \geq 0, \beta > 0, k \geq 1} \frac{n_{g,\beta}}{k} \left(2 \sin \left(\frac{k\lambda}{2} \right) \right)^{2g-2} t^{k\beta}. \quad (1.5)$$

We call the invariants $n_{g,\beta}$ the *Gopakumar–Vafa (GV) invariants* (also known as the *BPS invariants*).

Based on the previous works by Hosono–Saito–Takahashi [31] and Kiem–Li [36], Maulik–Toda [43] and Toda [56] proposed the mathematical definition of the GV invariants. Following the original idea of Gopakumar–Vafa, they consider the moduli space $M_X(\beta, m)$ of slope semistable one-dimensional sheaves E on X satisfying $[E] = \beta \in H_2(X, \mathbb{Z})$ and $\chi(E) = m \in \mathbb{Z}$. The moduli space $M_X(\beta, m)$ admits the *Hilbert–Chow* morphism

$$\pi_M: M_X(\beta, m)^{\text{red}} \rightarrow \text{Chow}_\beta(X),$$

which sends a sheaf to its fundamental cycle. Maulik–Toda [43] and Toda [56] defined the *generalized GV invariants* by the formula

$$\sum_{i \in \mathbb{Z}} \chi(\mathcal{H}^i(\pi_{M*}(\varphi_{M_X(\beta,m)}))) y^i = \sum_{g \geq 0} n_{g,\beta,m} (y^{1/2} + y^{-1/2})^{2g}, \quad (1.6)$$

where $\varphi_{M_X(\beta,m)}$ is a certain perverse sheaf on $M_X(\beta, m)$, see Sections 2.2 and 2.3 for more details.

As the GV invariants are conjecturally equivalent to the GW invariants by formula (1.5), the GV invariants should be independent of the additional choice of the Euler characteristic $m \in \mathbb{Z}$.

Conjecture 1.6 ([56, Conjecture 1.2]). *The generalized GV invariants are independent of the choice of $m \in \mathbb{Z}$, i.e., we have*

$$n_{g,\beta,m} = n_{g,\beta,m'}$$

for all $m, m' \in \mathbb{Z}$.

We call the above conjecture a χ -independence conjecture for GV invariants. In this paper, we prove it for local curves in full generality.

Theorem 1.7 (Theorem 3.1). *Conjecture 1.6 holds for $X = \text{Tot}_C(N)$.*

1.2. Strategy of the proof

1.2.1. Results on local curves. The key ingredient in our arguments is the main result of the companion paper by the first author and Masuda [38] on the construction of a *global d -critical chart* for the moduli space $\mathfrak{M}_X := \mathfrak{M}_X(\beta, m)$ of one-dimensional semistable sheaves on a local curve $X = \text{Tot}_C(N)$, i.e., the description of the moduli space \mathfrak{M}_X as the critical locus inside a certain smooth space: Take an exact sequence

$$0 \rightarrow L_1 \rightarrow N \rightarrow L_2 \rightarrow 0,$$

where L_1, L_2 are line bundles with $\deg(L_2)$ sufficiently large. We denote by $Y := \text{Tot}_C(L_2)$. Then it is shown in [38, Theorem 5.6] that there exists a function $f: M_Y \rightarrow \mathbb{A}^1$ on the good moduli space of one-dimensional semistable sheaves on Y such that we have an isomorphism

$$\mathfrak{M}_X \cong \{d(f \circ p_Y) = 0\} \subset \mathfrak{M}_Y,$$

where $p_Y: \mathfrak{M}_Y \rightarrow M_Y$ is the natural map from the moduli stack of one-dimensional semistable sheaves to its good moduli space.

In this situation, the perverse sheaf appeared in definition (1.6) of the generalized GV invariants coincides with the vanishing cycle sheaf,

$$\varphi_{M_X} \cong \varphi_f(\mathcal{J}\mathcal{C}_{M_Y}),$$

and the proof of Theorem 1.7 is reduced to proving the corresponding statement for the intersection complex $\mathcal{J}\mathcal{C}_{M_Y}$. The latter is proved in the recent paper by Maulik–Shen [42], hence we obtain Theorem 1.7.

1.2.2. Results on Higgs bundles. We define the BPS sheaf $\mathcal{BPS}_{r,m}$ on the moduli space $M_{\text{Dol}}(r, m)$ using the vanishing cycle complex φ_{M_X} for $X = \text{Tot}_C(\mathcal{O}_C \oplus \omega_C)$. Then the argument as above also implies Theorem 1.2. The cohomological integrality theorem for Higgs bundles (Theorem 5.16) is obtained by extending the argument for quivers with potentials [17, Theorem A] using the global critical locus description of \mathfrak{M}_X and applying the first author’s dimensional reduction theorem [37, Theorem 4.14] which relates the vanishing cycle cohomology for \mathfrak{M}_X and the Borel–Moore homology for $\mathfrak{M}_{\text{Dol}}$.

1.3. Relation with existing works

(1) Mellit [46], Groechenig–Wyss–Ziegler [28], and Yu [57] proved that the E-polynomial of $M_{\text{Dol}}(r, m)$ is independent of $m \in \mathbb{Z}$ when $\gcd(r, m) = 1$. These results were proved via the reduction to the positive characteristics.

We extended the result to the non-coprime case and further lifted the equality to an isomorphism of Hodge structures via completely different methods.

(2) Recently, de Cataldo–Maulik–Shen–Zhang [18] used a positive characteristic method to prove that isomorphism (1.3) preserves the perverse filtration induced by the non-abelian Hodge theorem.

At present, we do not know whether our cohomological χ -independence results are compatible with the Galois conjugate.

(3) Toda [56] proved Conjecture 1.6 for primitive classes $\beta \in H_2(X, \mathbb{Z})$ (assuming a technical conjecture on orientation data). For non-primitive classes, Maulik–Shen [42] proved it for local toric del Pezzo surfaces and recently [58] removed the toric assumption from their result. Maulik–Shen [42] also proved the conjecture for local curves of the form $\text{Tot}_C(\mathcal{O}(D) \oplus \omega_C(-D))$ for a divisor D with $\deg(D) > 2g(C) - 2$.

Our Theorem 1.7 proves Conjecture 1.6 for *arbitrary* local curves. In particular, the result for $X = \text{Tot}_C(N)$ with indecomposable N is completely new.

1.4. Structure of the paper

The paper is organized as follows. In Section 2, we recall Joyce’s theory on d-critical structures. Then we recall the definition of the GV invariants, and introduce the notion of local curves and twisted Higgs bundles.

In Section 3, we prove Theorem 1.7. In Section 4, we prove the cohomological integrality theorem for D -twisted Higgs bundles where $\deg(D) > 2g - 2$, which plays an important role in the proofs of Theorems 1.7 and Corollary 1.5. Finally, in Section 5, we discuss applications to Higgs bundles. We prove Theorem 1.2 and the cohomological integrality theorem for Higgs bundles (Theorem 5.16).

In Appendix A, we give a brief overview of the shifted symplectic geometry and prove some technical lemmas that we use in this paper.

In Appendix B, we prove a version of the support lemma of the vanishing cycle complexes which is needed to define the BPS sheaf.

Notation and convention. In this paper, we work over the complex number field \mathbb{C} . We use the following notations:

- We denote by \mathbb{S} the ∞ -category of spaces (see [40, Definition 1.2.16.1]).
- We basically write stacks in Fraktur (e.g., $\mathfrak{X}, \mathfrak{Y}, \dots$), and derived schemes, derived stacks and morphisms between derived stacks in bold (e.g., $\mathbf{X}, \mathfrak{X}, \mathbf{f}, \dots$). We will write $X = t_0(\mathbf{X}), \mathfrak{X} = t_0(\mathfrak{X}), f = t_0(\mathbf{f})$ and so on.
- A derived Artin stack \mathfrak{X} is said to be *quasi-smooth* if the cotangent complex $\mathbb{L}_{\mathfrak{X}}$ has Tor-amplitude $[-1, 1]$.

- All derived/underived Artin stacks are assumed to have quasi-compact and separated diagonals and locally finitely presented over the complex number field. As the fiber product of finite type separated schemes over such a stack is again of finite type and separated, we can consider the category of mixed Hodge modules on such stacks (see Section 4.1 for the detail).
- For a separated complex analytic space X , we let $D_c^b(X)$ denote the bounded derived category of complexes of sheaves in \mathbb{Q} -vector spaces on X with constructible cohomology.
- For a complex analytic stack \mathcal{X} , we let $D_c^b(\mathcal{X})$ denote the bounded derived category of sheaves in \mathbb{Q} -vector spaces on $\mathcal{X}_{\text{lis-an}}$ with constructible cohomology. Here $\mathcal{X}_{\text{lis-an}}$ denotes the lisse-analytic topos of \mathcal{X} (see [54, §3.2.3]).
- If there is no confusion, we use expressions such as f_* and $f_!$ for the derived functors $\mathbf{R}f_*$ and $\mathbf{R}f_!$.

2. Preliminaries

In this section, we collect some basic notions that we use in this paper. Firstly, we recall Joyce’s theory of d-critical locus in Section 2.1. Then we review the construction of vanishing cycle complexes associated with d-critical stacks in Section 2.2. In Section 2.3, we review Maulik–Toda’s construction [43] of Gopakumar–Vafa invariants based on vanishing cycle complexes. In Section 2.4, we collect some basic facts on local curves and recall Maulik–Shen’s cohomological χ -independence theorem [42].

2.1. D-critical structures

In [33], Joyce introduced the notion of d-critical structures which are classical models of derived critical loci. We now briefly recall it.

Let X be a complex analytic space. Joyce [33, Theorem 2.1] introduced a sheaf \mathcal{S}_X on X with the following property: for an open subset $R \subset X$ and an embedding $i: R \hookrightarrow U$ to a complex manifold U , there exists a short exact sequence

$$0 \rightarrow \mathcal{S}_X|_R \rightarrow i^{-1}\mathcal{O}_U/I_{R,U}^2 \xrightarrow{d_{\text{dR}}} i^{-1}\Omega_U/(I_{R,U} \cdot i^{-1}\Omega_U),$$

where $I_{R,U}$ is the ideal sheaf of R in U . One can show that the natural map

$$\mathcal{S}_X|_R \rightarrow i^{-1}\mathcal{O}_U/I_{R,U}^2 \twoheadrightarrow \mathcal{O}_R$$

glues to define a morphism $\mathcal{S}_X \rightarrow \mathcal{O}_X$. We define a subsheaf $\mathcal{S}_X^0 \subset \mathcal{S}_X$ by the kernel of the map

$$\mathcal{S}_X \rightarrow \mathcal{O}_X \twoheadrightarrow \mathcal{O}_{X^{\text{red}}}.$$

If X is the critical locus $\text{Crit}(f)$ of a holomorphic function f on a complex manifold U such that $f|_{X^{\text{red}}} = 0$, then $f + I_{X,U}^2$ defines an element of \mathcal{S}_X^0 .

Definition 2.1. Let X be a complex analytic space. A section $s \in \Gamma(X, \mathcal{S}_X^0)$ is called a *d-critical structure* if for each point $x \in X$, there exist an open neighborhood $R \subset X$, an embedding $i: R \hookrightarrow U$ into a complex manifold, and a holomorphic function f on U with the property $f|_{R^{\text{red}}} = 0$ such that $f + I_{R,U}^2 = s|_R$. The quadruple (R, U, f, i) is called a *d-critical chart* of X . A complex analytic space equipped with a d-critical structure is called a *d-critical analytic space*.

The sheaf \mathcal{S}_X^0 has the following functorial property: for a given morphism of complex analytic spaces $q: X_1 \rightarrow X_2$, there exists natural morphism

$$q^*: q^{-1}\mathcal{S}_{X_2}^0 \rightarrow \mathcal{S}_{X_1}^0.$$

Now assume that q is smooth surjective and take a section $s \in \Gamma(X, \mathcal{S}_X^0)$. Then it is shown in [33, Proposition 2.8] that q^*s is a d-critical structure if and only if s is a d-critical structure.

Now let \mathfrak{X} be a complex analytic stack. Then it is shown in [33, Corollary 2.52] that there exists a sheaf $\mathcal{S}_{\mathfrak{X}}^0$ on the lisse-analytic site of \mathfrak{X} with the following property:

- For a smooth morphism $t: T \rightarrow \mathfrak{X}$, there exists a natural isomorphism $\eta_t: \mathcal{S}_{\mathfrak{X}}^0|_T \cong \mathcal{S}_T^0$.
- For a morphism

$$q: (t_1: T_1 \rightarrow \mathfrak{X}) \rightarrow (t_2: T_2 \rightarrow \mathfrak{X})$$

between complex analytic spaces smooth over \mathfrak{X} , the natural map $q^{-1}(\mathcal{S}_{\mathfrak{X}}^0|_{T_2}) \rightarrow \mathcal{S}_{\mathfrak{X}}^0|_{T_1}$ is identified with q^* .

For a smooth morphism $t: T \rightarrow \mathfrak{X}$ from a scheme and a section $s \in \Gamma(\mathfrak{X}, \mathcal{S}_{\mathfrak{X}}^0)$, we write $t^*s := \eta_t(s|_T) \in \Gamma(T, \mathcal{S}_T^0)$.

Definition 2.2. For a complex analytic stack \mathfrak{X} , a section $s \in \Gamma(\mathfrak{X}, \mathcal{S}_{\mathfrak{X}}^0)$ is called a *d-critical structure* if for any smooth surjective morphism $t: T \rightarrow \mathfrak{X}$, the element t^*s is a d-critical structure on T . A *d-critical stack* is a complex analytic stack \mathfrak{X} equipped with a d-critical structure.

For a complex analytic stack \mathfrak{X} equipped with a d-critical structure s , Joyce [33, §§2.4 and 2.8] defines a line bundle $K_{\mathfrak{X},s}^{\text{vir}}$ on $\mathfrak{X}^{\text{red}}$ called the *virtual canonical bundle* of (\mathfrak{X}, s) . If there is no confusion, we simply write $K_{\mathfrak{X}}^{\text{vir}} = K_{\mathfrak{X},s}^{\text{vir}}$. We now recall some of its basic properties. Firstly, assume \mathfrak{X} is a complex analytic space and write $\mathfrak{X} = X$. Take a d-critical chart $\mathcal{R} = (R, U, f, i)$ of (X, s) . Then there exists a natural isomorphism

$$\iota_{\mathcal{R}}: K_{X,s}^{\text{vir}}|_{R^{\text{red}}} \cong K_U^{\otimes 2}|_{R^{\text{red}}}.$$

Let $q: X_1 \rightarrow X_2$ be a smooth morphism and s_2 be a d-critical structure on X_2 . Write $s_1 = q^*s_2$. Then it is shown in [33, Proposition 2.30] that there exists a natural isomorphism

$$\Upsilon_q: q^{\text{red},*} K_{X_2,s_2}^{\text{vir}} \otimes \det(\Omega_{X_1/X_2})|_{X_1^{\text{red}}}^{\otimes 2} \cong K_{X_1,s_1}^{\text{vir}}$$

with the following property: if we are given d-critical charts $\mathcal{R}_1 = (R_1, U_1, f_1, i_1)$ of (X_1, s_1) and $\mathcal{R}_2 = (R_2, U_2, f_2, i_2)$ of (X_2, s_2) such that $q(R_1) \subset R_2$, and a smooth

morphism $\tilde{q}: U_1 \rightarrow U_2$ such that $f_1 = f_2 \circ \tilde{q}$ and $i_2 \circ q|_{R_1} = \tilde{q} \circ i_1$, the following diagram of line bundles on R_1^{red} commutes:

$$\begin{array}{ccc}
 q^{\text{red},*} K_{X_2, s_2}^{\text{vir}}|_{R_1^{\text{red}}} \otimes \det(\Omega_{X_1/X_2})|_{R_1^{\text{red}}}^{\otimes 2} & \xrightarrow{\Upsilon_q|_{R_1^{\text{red}}}} & K_{X_1, s_1}^{\text{vir}}|_{R_1^{\text{red}}} \\
 \downarrow (q|_{R_1})^{\text{red},*} \iota_{R_2} \otimes \text{id} & & \downarrow \iota_{R_1} \\
 \tilde{q}^* K_{U_2}^{\otimes 2}|_{R_1^{\text{red}}} \otimes \det(\Omega_{X_1/X_2})|_{R_1^{\text{red}}}^{\otimes 2} & \longrightarrow & K_{U_1}^{\otimes 2}|_{R_1^{\text{red}}}.
 \end{array}$$

Here the bottom horizontal arrow is defined by the natural isomorphism $\Omega_{X_1/X_2}|_{R_1} \cong \Omega_{U_1/U_2}|_{R_1}$.

Now we treat the stacky case. Let \mathfrak{X} be a complex analytic space and $t: T \rightarrow \mathfrak{X}$ be a smooth morphism from an analytic space. Then there exists a natural isomorphism

$$\Upsilon_t: t^{\text{red},*} K_{\mathfrak{X}, s}^{\text{vir}} \otimes \det(\Omega_{T/\mathfrak{X}})|_{T^{\text{red}}}^{\otimes 2} \cong K_{T, t^*s}^{\text{vir}},$$

see [33, Theorem 2.56]. For a morphism $q: (t_1: T_1 \rightarrow \mathfrak{X}) \rightarrow (t_2: T_2 \rightarrow \mathfrak{X})$ between complex analytic spaces smooth over \mathfrak{X} , the following diagram commutes:

$$\begin{array}{ccc}
 t_1^{\text{red},*} K_{\mathfrak{X}, s}^{\text{vir}} \otimes \det(\Omega_{T_1/\mathfrak{X}})|_{T_1^{\text{red}}}^{\otimes 2} & & \\
 \downarrow & \searrow \Upsilon_{t_1} & \\
 q^{\text{red},*} (t_2^{\text{red},*} K_{\mathfrak{X}, s}^{\text{vir}} \otimes \det(\Omega_{T_2/\mathfrak{X}})|_{T_2^{\text{red}}}^{\otimes 2}) \otimes \det(\Omega_{T_1/T_2})|_{T_1^{\text{red}}}^{\otimes 2} & & (2.1) \\
 \downarrow q^{\text{red},*} \Upsilon_{t_2} \otimes \text{id} & & \\
 q^{\text{red},*} K_{T_2, t_2^*s}^{\text{vir}} \otimes \det(\Omega_{T_1/T_2})|_{T_1^{\text{red}}}^{\otimes 2} & \xrightarrow{\Upsilon_q} & K_{T_1, t_1^*s}^{\text{vir}}.
 \end{array}$$

For a d-critical stack (\mathfrak{X}, s) , an *orientation* is a choice of a line bundle L on $\mathfrak{X}^{\text{red}}$ and an isomorphism $o: L^{\otimes 2} \cong K_{\mathfrak{X}, s}^{\text{vir}}$. For a smooth morphism $t: T \rightarrow \mathfrak{X}$, we define an orientation

$$t^*o: (t^{\text{red},*} L \otimes \det(\Omega_{T/\mathfrak{X}})|_{T^{\text{red}}})^{\otimes 2} \cong K_{T, t^*s}^{\text{vir}}$$

using Υ_t . If we are given a smooth morphism $q: (t_1: T_1 \rightarrow \mathfrak{X}) \rightarrow (t_2: T_2 \rightarrow \mathfrak{X})$ between analytic spaces smooth over \mathfrak{X} , there exists a natural isomorphism

$$t_1^*o \cong q^*t_2^*o.$$

2.2. Vanishing cycle complexes on d-critical stacks

In this subsection, we recall some basic properties of the vanishing cycle functors and the vanishing cycle complexes associated with oriented d-critical stacks.

Let U be a complex manifold and f be a holomorphic function on U . Write $U_0 = f^{-1}(0)$. Then the *vanishing cycle functor*

$$\varphi_f: D_c^b(U) \rightarrow D_c^b(U_0)$$

is defined by the following formula:

$$\varphi_f := (U_0 \hookrightarrow U_{\leq 0})^*(U_{\leq 0} \hookrightarrow U)^!$$

where $U_{\leq 0} \subset U$ denotes the closed subset $\text{Re}(f)^{-1}(\mathbb{R}_{\leq 0})$. It is shown in [35, Corollary 10.3.13] that the functor φ_f preserves the perversity. If there is no confusion, we write $\varphi_f := \varphi_f(\mathbb{Q}_U[\dim U])$.

Let $q: V \rightarrow U$ be a holomorphic map between complex manifolds. Write $V_0 := (f \circ q)^{-1}(0)$ and we let $q_0: V_0 \rightarrow U_0$ be the restriction of q . By the definition of the vanishing cycle functor, we have the following base change morphisms:

$$\varphi_f \circ q_* \rightarrow q_{0*} \circ \varphi_{f \circ q}, \quad q_0^* \varphi_f \rightarrow \varphi_{f \circ q} \circ q^*.$$

The first morphism is an isomorphism if q is proper, and the latter morphism is an isomorphism if q is smooth. These are direct consequences of the proper/smooth base change theorem.

Now let \mathfrak{U} be a smooth complex analytic stack and f be a holomorphic function on \mathfrak{U} . Write $\mathfrak{U}_0 := f^{-1}(0)$. For a perverse sheaf $\mathcal{P} \in \text{Perv}(\mathfrak{X})$, we define the perverse sheaf

$$\varphi_f(\mathcal{P}) \in \text{Perv}(\mathfrak{U}_0)$$

as follows: Take a smooth surjective morphism $q: U \rightarrow \mathfrak{U}$. We let $\text{pr}_i: U \times_{\mathfrak{U}} U \rightarrow U$ denote the i -th projection and $\text{pr}_{i,0}: (f \circ q \circ \text{pr}_i)^{-1}(0) \rightarrow (f \circ q)^{-1}(0)$ denote the restriction of pr_i . Then we have a natural isomorphism

$$\text{pr}_{1,0}^* \varphi_{f \circ q}(q^* \mathcal{P}) \cong \varphi_{f \circ q \circ \text{pr}_1}(\text{pr}_1^* q^* \mathcal{P}) \cong \varphi_{f \circ q \circ \text{pr}_2}(\text{pr}_2^* q^* \mathcal{P}) \cong \text{pr}_{2,0}^* \varphi_{f \circ q}(q^* \mathcal{P}).$$

This isomorphism satisfies the cocycle condition, hence the shifted perverse sheaf $\varphi_{f \circ q}(q^* \mathcal{P})$ descends to a perverse sheaf $\varphi_f(\mathcal{P}) \in \text{Perv}(\mathfrak{U}_0)$. One can show that the construction does not depend on the choice of the smooth morphism q .

Now we recall the vanishing cycle complex associated with an oriented d -critical stack constructed in [3, Theorem 4.8].

First we treat the non-stacky case. Let (X, s, o) be an oriented d -critical analytic space. Then it is shown in [4, Theorem 6.9] that there is a natural perverse sheaf

$$\varphi_{X,s,o} \in \text{Perv}(X)$$

called the *vanishing cycle complex* associated with (X, s, o) . We sometimes omit s and o and write $\varphi_X = \varphi_{X,s,o}$ if there is no confusion. For a d -critical chart $\mathcal{R} = (R, U, f, i)$ of (X, s) , we have a natural isomorphism

$$\omega_{\mathcal{R}}: \varphi_{X,s,o}|_R \cong i^* \varphi_f \otimes_{\mathbb{Z}/2\mathbb{Z}} Q_{\mathcal{R}}^o,$$

where $Q_{\mathcal{R}}^o$ is a $\mathbb{Z}/2\mathbb{Z}$ -local system on R parametrizing local square roots of the isomorphism

$$L^{\otimes 2}|_{R^{\text{red}}} \xrightarrow{\cong} K_{X,s}^{\text{vir}}|_{R^{\text{red}}} \cong i^* K_U^{\otimes 2}|_{R^{\text{red}}}.$$

Example 2.3. Let U be a complex manifold and $f: U \rightarrow \mathbb{A}^1$ be a holomorphic function such that $f|_{\text{Crit}(f)^{\text{red}}} = 0$. Write $X = \text{Crit}(f)$ and equip it with the canonical d-critical structure s and the canonical orientation $o: K_U|_{X^{\text{red}}}^{\otimes 2} \cong K_{X,s}^{\text{vir}}$. Then $\mathcal{R} := (X, U, f, X \hookrightarrow U)$ defines a d-critical chart. In this case, the local system $\mathcal{Q}_{\mathcal{R}}^o$ is trivial. Therefore, we have a natural isomorphism

$$\varphi_{X,s,o} \cong \varphi_f|_X.$$

Let $q: X_1 \rightarrow X_2$ be a smooth morphism and equip X_2 with a d-critical structure s_2 and an orientation o_2 . Write $s_1 = q^*s_2$ and $o_1 = q^*o_2$. Then there exists a natural isomorphism of perverse sheaves

$$\Theta_q: \varphi_{X_1,s_1,o_1} \cong q^*\varphi_{X_2,s_2,o_2}[\dim q]$$

with the following property: If we are given d-critical charts $\mathcal{R}_1 = (R_1, U_1, f_1, i_1)$ of (X_1, s_1) and $\mathcal{R}_2 = (R_2, U_2, f_2, i_2)$ of (X_2, s_2) such that $q(R_1) \subset R_2$, and a smooth morphism $\tilde{q}: U_1 \rightarrow U_2$ such that $f_1 = f_2 \circ \tilde{q}$ and $i_2 \circ q|_R = \tilde{q} \circ i_1$, the following diagram in $\text{Perv}(R_1)$ commutes:

$$\begin{CD} \varphi_{X_1,s_1,o_1}|_{R_1} @>\omega_{\mathcal{R}_1}>> i_1^*\varphi_{f_1} \otimes_{\mathbb{Z}/2\mathbb{Z}} \mathcal{Q}_{\mathcal{R}_1}^{o_1} \\ @V\Theta_q|_{R_1}VV @VVV \\ (q|_{R_1})^*(\varphi_{X_2,s_2,o_2})[\dim q] @>(q|_{R_1})^*\omega_{\mathcal{R}_2}[\dim q]>> (q|_{R_1})^*(i_2^*\varphi_{f_2} \otimes_{\mathbb{Z}/2\mathbb{Z}} \mathcal{Q}_{\mathcal{R}_2}^{o_2})[\dim q], \end{CD} \tag{2.2}$$

where the right vertical arrow is defined using the natural isomorphisms

$$\varphi_{f_1} \cong (\tilde{q}|_{(f_2 \circ \tilde{q})^{-1}(0)})^*\varphi_{f_2}[\dim q] \quad \text{and} \quad \mathcal{Q}_{\mathcal{R}_1}^{o_1} \cong (q|_{R_1})^*\mathcal{Q}_{\mathcal{R}_2}^{o_2}.$$

Now we move to the stacky case. Let (\mathfrak{X}, s, o) be a d-critical stack. Then it is shown in [3, Theorem 4.8] that there exists a natural perverse sheaf

$$\varphi_{\mathfrak{X},s,o} \in \text{Perv}(\mathfrak{X})$$

with the following property: If we are given a smooth morphism $t: T \rightarrow \mathfrak{X}$ from a complex analytic space, then there exists a natural isomorphism

$$\Theta_t: \varphi_{T,t^*o,t^*o} \cong t^*\varphi_{\mathfrak{X},s,o}[\dim t].$$

Furthermore, if we are given a smooth morphism $q: (t_1: T_1 \rightarrow \mathfrak{X}) \rightarrow (t_2: T_2 \rightarrow \mathfrak{X})$ between schemes smooth over \mathfrak{X} , the following diagram commutes:

$$\begin{CD} \varphi_{T_1,t_1^*s,t_1^*o} @>\sim>> \varphi_{T_1,q^*t_2^*s,q^*t_2^*o} @>\Theta_q>> q^*\varphi_{T_2,t_2^*s,t_2^*o}[\dim q] \\ @V\Theta_{t_1}VV @VVV @VVq^*\Theta_{t_2}V \\ t_1^*\varphi_{\mathfrak{X},s,o}[\dim t_1] @>\sim>> q^*t_2^*\varphi_{\mathfrak{X},s,o}[\dim t_1]. \end{CD} \tag{2.3}$$

Let \mathcal{U} be a smooth Artin stack and $f: \mathcal{U} \rightarrow \mathbb{A}^1$ be a regular function on it. Then it is shown in Example A.7 that $\mathbf{Crit}(f)$ carries a natural (-1) -shifted symplectic structure, hence there exists a natural d-critical structure s on its classical truncation $\mathfrak{X} := \mathbf{Crit}(f)$. We will see in Lemma A.10 that the d-critical analytic stack (\mathfrak{X}, s) admits a canonical orientation $o: K_{\mathcal{U}}^{\otimes 2}|_{\mathfrak{X}^{\text{red}}} \cong K_{\mathfrak{X},s}^{\text{vir}}$.

Proposition 2.4. *There exists a natural isomorphism of perverse sheaves,*

$$\theta: \varphi_{\mathfrak{X},s,o} \cong \varphi_f(\mathbb{Q}_{\mathcal{U}}[\dim \mathcal{U}]).$$

We postpone the proof to Section A.2.

Remark 2.5. The argument in [3, Theorem 4.8] shows that the perverse sheaf $\varphi_{\mathfrak{X},s,o}$ naturally extends to a mixed Hodge module $\varphi_{\mathfrak{X},s,o}^{\text{mhm}}$ and to a monodromic mixed Hodge module $\varphi_{\mathfrak{X},s,o}^{\text{mmhm}}$. Proposition 2.4 extends to an isomorphism of monodromic mixed Hodge modules with the same proof. We refer the reader to Section 4.2 for a brief discussion on monodromic mixed Hodge modules.

2.3. Maulik–Toda’s construction of Gopakumar–Vafa invariants

In this subsection, we recall the definition of generalized Gopakumar–Vafa (GV) invariants following [43, 56]. Let X be a smooth quasi-projective Calabi–Yau threefold and H be an ample divisor on X .

Definition 2.6. Let E be a pure one-dimensional coherent sheaf with compact support on X .

(1) We define the H -slope to be

$$\mu_H(E) := \frac{\chi(E)}{H \cdot [E]},$$

where $[E] \in H_2(X, \mathbb{Z})$ denotes the second homology class of E .

(2) The sheaf E is μ_H -semistable (resp. stable) if for any saturated subsheaf $0 \neq F \subsetneq E$, the inequality

$$\mu_H(F) \leq \mu_H(E) \quad (\text{resp. } \mu_H(F) < \mu_H(E))$$

holds.

For a given element $v = (\beta, m) \in H_2(X, \mathbb{Z}) \times \mathbb{Z}$, we denote by $\mathfrak{M}_H(v)$ the moduli stack of μ_H -semistable one-dimensional sheaves E satisfying

$$[E] = \beta, \quad \chi(E) = m.$$

The stack $\mathfrak{M}_H(v)$ admits the good moduli space $p: \mathfrak{M}_H(v) \rightarrow M_H(v)$, and we have the Hilbert–Chow morphism

$$\pi_M: M_H(v)^{\text{red}} \rightarrow \text{Chow}_{\beta}(X) \tag{2.4}$$

sending a sheaf E to its fundamental one cycle. Here, $\text{Chow}_\beta(X)$ denotes the Chow variety of compactly supported effective one cycles with homology class β (see [39] for the definition; note that it is denoted as $\text{Chow}'(X)$ in [39]). We denote by $\pi_{\mathfrak{M}}$ the composition

$$\pi_{\mathfrak{M}}: \mathfrak{M}_H(v)^{\text{red}} \rightarrow M_H(v)^{\text{red}} \rightarrow \text{Chow}_\beta(X).$$

Recall from Example A.2 that the stack $\mathfrak{M}_H(v)$ is the classical truncation of a (-1) -shifted derived Artin stack. In particular, the stack $\mathfrak{M}_H(v)$ carries a natural d-critical structure and (A.2) implies that there exists a natural isomorphism

$$K_{\mathfrak{M}_H(v)}^{\text{vir}} \cong \det(\mathbf{R}p_{\mathfrak{M}*} \mathbf{R}\mathcal{H}om(\mathcal{E}, \mathcal{E}))|_{\mathfrak{M}_H(v)^{\text{red}}},$$

where

$$p_{\mathfrak{M}}: \mathfrak{M}_H(v) \times X \rightarrow \mathfrak{M}_H(v)$$

denotes the projection and \mathcal{E} denotes the universal sheaf on $\mathfrak{M}_H(v) \times X$. In order to define the well-defined notion of Gopakumar–Vafa invariants, Maulik–Toda [43] and Toda [56] proposed the following conjecture on the virtual canonical bundle of the stack $\mathfrak{M}_H(v)$.

Conjecture 2.7 ([56, Conjecture 2.10]). *The stack $\mathfrak{M}_H(v)$ is Calabi–Yau at any point $\gamma \in \text{Chow}_\beta(X)$, i.e., there exists an analytic open neighborhood $\gamma \in U \subset \text{Chow}_\beta(X)$ such that the virtual canonical bundle $K_{\mathfrak{M}}^{\text{vir}}$ is trivial on $\pi_{\mathfrak{M}}^{-1}(U)$.*

Suppose that Conjecture 2.7 holds. Then we can take an orientation of $\pi_{\mathfrak{M}}^{-1}(U)$ with

$$(K_{\mathfrak{M}}^{\text{vir}}|_{\pi_{\mathfrak{M}}^{-1}(U)})^{1/2} \cong \mathcal{O}_{\pi_{\mathfrak{M}}^{-1}(U)},$$

which we call a *Calabi–Yau (CY) orientation*. As we have seen in Section 2.2, we have the associated perverse sheaf

$$\varphi_{\mathfrak{M}_H(v)|_U} \in \text{Perv}(\mathfrak{M}_H(v)|_U).$$

We then define the perverse sheaf on the good moduli space as

$$\varphi_{M_H(v)|_U} := \mathcal{H}^1(p_* \varphi_{\mathfrak{M}_H(v)|_U}) \in \text{Perv}(M_H(v)|_U), \tag{2.5}$$

where we denote by $\mathfrak{M}_H(v)|_U$, $M_H(v)|_U$ the pull-back of $\mathfrak{M}_H(v)$, $M_H(v)$ along the open embedding $U \subset \text{Chow}_\beta(X)$, respectively. Note that we denote by $\mathcal{H}^i(-)$ the i -th perverse cohomology.

Definition 2.8. Suppose Conjecture 2.7 holds. For an element $\gamma \in \text{Chow}_\beta(X)$, we define a Laurent polynomial $\Phi_H(\gamma, m)$ as follows:

$$\Phi_H(\gamma, m) := \sum_{i \in \mathbb{Z}} \chi(\mathcal{H}^i(\pi_{M_*} \varphi_{M_H(v)|_U})) y^i \in \mathbb{Z}[y^{\pm 1}], \tag{2.6}$$

where the perverse sheaf $\varphi_{M_H(v)|_U}$ is defined as in (2.5).

Remark 2.9. (1) By [56, Lemma 2.14], the Laurent polynomial (2.6) is independent of the choice of a CY orientation on $\mathfrak{M}_H(v)|_U$.

(2) The definition of the perverse sheaf in (2.5) is motivated by the notion of BPS sheaves for quivers with super-potentials introduced by Davison–Meinhardt [17]. See [56, Section 2.8] for the detailed discussion.

The following χ -independence conjecture is the main subject of this paper.

Conjecture 2.10 ([56, Conjecture 2.15]). *The Laurent polynomial (2.6) is independent of $m \in \mathbb{Z}$.*

At this moment, the above conjecture is known to hold in the following cases:

- $X = \text{Tot}_S(\omega_S)$, where S is a smooth projective surface, and γ is primitive [56].
- $X = \text{Tot}_S(\omega_S)$, where S is a del Pezzo surface, and γ is arbitrary [42, 58].
- $X = \text{Tot}_C(\mathcal{O}(D) \oplus \omega_C(-D))$, where C is a smooth projective curve, D is a divisor with $\deg(D) > 2g(C) - 2$, and γ is arbitrary [42].

Remark 2.11. Suppose that Conjecture 2.10 holds. Then we may write the Laurent polynomial (2.6) as $\Phi(\gamma) := \Phi_H(\gamma, 1) = \Phi_H(\gamma, m)$ for $m \in \mathbb{Z}$. Note that we can drop the subscript H in the notation since for $m = 1$, the moduli space is independent of the choice of an ample divisor H .

Furthermore, for $m = 1$, we know that the perverse sheaf $\varphi_{M_H(v)}$ is Verdier self-dual. Hence there exist integers $n_{g,\gamma} \in \mathbb{Z}$ for $g \geq 0$ such that the equation

$$\Phi(\gamma) = \sum_{g \geq 0} n_{g,\gamma} (y^{1/2} + y^{-1/2})^{2g}$$

holds. Following Maulik–Toda [43], we call the integers $n_{g,\gamma}$ as the *GV invariants* of X .

2.4. Local curves and twisted Higgs bundles

In this section, we introduce a class of Calabi–Yau threefolds which we call local curves. Then we review the results on the twisted Higgs bundles due to Maulik–Shen [42].

2.4.1. Spectral correspondence for local curves. Let C be a smooth projective curve and N be a rank two vector bundle on C with $\det N \cong \omega_C$. Then the total space $X := \text{Tot}_C(N)$ of the bundle N gives an example of quasi-projective Calabi–Yau threefolds, which we call a *local curve*. Denote by $p: X \rightarrow C$ the projection.

In this section, we recall the spectral-type correspondence for coherent sheaves on local curves. See, e.g., [53] for the details.

Lemma 2.12. *Giving a compactly supported pure one-dimensional coherent sheaf on X is equivalent to giving a pair (E, φ) of a locally free sheaf E on C and a morphism $\varphi \in \text{Hom}(E, E \otimes N)$ satisfying $\varphi \wedge \varphi = 0$.*

We call a pair (E, φ) in the above lemma as an N -Higgs bundle. We can define the slope semistability for N -Higgs bundles as in Definition 2.6.

Definition 2.13. Let (E, φ) be an N -Higgs bundle.

(1) We define the *slope* of (E, φ) as

$$\mu(E) := \frac{\chi(E)}{\text{rk}(E)}.$$

(2) The N -Higgs bundle (E, φ) is μ -semistable (resp. stable) if for any saturated subsheaf $0 \neq F \subsetneq E$ with $\varphi(F) \subset F \otimes N$, the inequality

$$\mu(F) \leq \mu(E) \quad (\text{resp. } \mu(F) < \mu(E))$$

holds.

Lemma 2.14. Take an ample divisor H on C . Let \mathcal{E} be a pure one-dimensional coherent sheaf on X and (E, φ) be the corresponding N -Higgs bundle. Then the sheaf \mathcal{E} is μ_{p^*H} -(semi)stable if and only if the N -Higgs bundle (E, φ) is μ -(semi)stable.

Let $\mathfrak{M}_X^{\text{ss}}(r, m) := \mathfrak{M}_{p^*H}(r[C], m)$ be the moduli stack of μ_{p^*H} -semistable sheaves \mathcal{E} on X satisfying $[\mathcal{E}] = r[C]$ and $\chi(\mathcal{E}) = m$. Let $M_X^{\text{ss}}(r, m)$ be the good moduli space of $\mathfrak{M}_X^{\text{ss}}(r, m)$. By the above lemma, \mathbb{C} -valued points of $\mathfrak{M}_X^{\text{ss}}(r, m)$ correspond to μ -semistable N -Higgs bundles.

The moduli space $M_X^{\text{ss}}(r, m)$ admits a Hitchin type morphism: Define a *Hitchin base* as

$$B_X := \bigoplus_{i=1}^r H^0(C, \text{Sym}^i(N)),$$

and a *Hitchin morphism* as follows:

$$h_X: M_X^{\text{ss}}(r, m) \rightarrow B_X, \quad (E, \varphi) \mapsto (\text{tr}(\varphi^i))_{i=1}^r, \tag{2.7}$$

where

$$\varphi^i: E \rightarrow E \otimes \text{Sym}^i(N)$$

is obtained by the i -th iteration of φ .

Remark 2.15. We can construct a bijection between the sets of closed points of $\text{im}(h_X)$ and $\text{im}(\pi_M)$ by sending a point in $\text{im}(h_X)$ to its spectral curve, where

$$\pi_M: M_X^{\text{ss}}(r, m)^{\text{red}} \rightarrow \text{Chow}_{r[C]}(X)$$

denotes the Hilbert–Chow morphism defined in (2.4). Moreover, by the properness of the morphisms h_X and π_M , the spaces $\text{im}(h_X)$ and $\text{im}(\pi_M)$ are homeomorphic.

As a result, the GV invariants do not change if we replace the Hilbert–Chow morphism with the Hitchin morphism. Hence we use the Hitchin morphism for the GV theory of local curves in this paper.

2.4.2. *Twisted Higgs bundles.* Let L be a line bundle on a smooth projective curve C . Denote by $Y := \text{Tot}_C(L)$ the total space of L .

An L -Higgs bundle is a pair (E, θ) consisting of a locally free sheaf E on C and a homomorphism $\theta \in \text{Hom}(E, E \otimes L)$. For the canonical divisor $L = K_C$, the notion of K_C -Higgs bundles agrees with the usual notion of Higgs bundles.

As in Definition 2.13, we can define the notion of μ -semistability for L -Higgs bundles. We denote by $\mathfrak{M}_Y^{\text{ss}}(r, m)$ the moduli stack of μ -semistable L -Higgs bundles (E, θ) with $\text{rk}(E) = r$, $\chi(E) = m$, and by $M_Y^{\text{ss}}(r, m)$ its good moduli space. Similarly to (2.7), we have a Hitchin morphism

$$h_Y: M_Y^{\text{ss}}(r, m) \rightarrow B_Y := \bigoplus_{i=1}^r H^0(C, L^{\otimes i}) \tag{2.8}$$

sending an L -Higgs bundle (E, θ) to $(\text{tr}(\theta^i))_{i=1}^r$.

We denote by $\tilde{h}_Y: \mathfrak{M}_Y^{\text{ss}}(r, m) \rightarrow B_Y$ the composition

$$\tilde{h}_Y: \mathfrak{M}_Y^{\text{ss}}(r, m) \rightarrow M_Y^{\text{ss}}(r, m) \rightarrow B_Y. \tag{2.9}$$

Given an element $a \in B_Y$, we denote by $C_a \subset Y$ its spectral curve. Define an open dense subset $U \subset B_Y$ as

$$U := \{a \in B_Y : C_a \text{ is smooth}\},$$

and let $g: \mathcal{C} \rightarrow U$ be the universal spectral curve. The following result plays a key role in this paper.

Theorem 2.16 ([42, Theorem 0.4]). *Suppose that $\text{deg}(L) > 2g(C) - 2$. Then we have an isomorphism*

$$h_{Y*} \mathcal{I}\mathcal{C}_{M_Y^{\text{ss}}(r, m)} \cong \bigoplus_{i=0}^{2d} \mathcal{I}\mathcal{C}(\wedge^i \mathbf{R}^1 g_* \mathbb{Q}_{\mathcal{C}})[-i + d],$$

where d denotes the genus of the fibers of $g: \mathcal{C} \rightarrow U$.

In particular, we have isomorphisms

$$h_{Y*} \mathcal{I}\mathcal{C}_{M_Y^{\text{ss}}(r, m)} \cong h_{Y*} \mathcal{I}\mathcal{C}_{M_Y^{\text{ss}}(r, m')}$$

for all $m, m' \in \mathbb{Z}$.

3. Cohomological χ -independence for local curves

Let C be a smooth projective curve of genus g and N be a rank two vector bundle on C with $\det N \cong \omega_C$. We put $X := \text{Tot}_C(N)$. The goal of this section is to prove the following theorem.

Theorem 3.1. *Let $X = \text{Tot}_C(N)$ be a local curve. For every positive integer $r \in \mathbb{Z}_{>0}$ and a class $\gamma \in B_X$, Conjecture 2.10 holds.*

3.1. Global d -critical charts for moduli spaces on local curves

We first recall the main result of the companion paper [38].

Theorem 3.2 ([38, Theorem 5.6 and Proposition 5.7]). *Let C be a smooth projective curve and take a short exact sequence*

$$0 \rightarrow L_1 \rightarrow N \rightarrow L_2 \rightarrow 0 \tag{3.1}$$

of locally free sheaves on C , where L_1 and L_2 are of rank one. Suppose that there exists an isomorphism $\det(N) \cong \omega_C$, and the inequality $\deg(L_2) > 2g(C) - 2$ holds. Write $X = \text{Tot}_C(N)$ and $Y = \text{Tot}_C(L_2)$. Let \mathfrak{M}_X and \mathfrak{M}_Y be the derived moduli stack of compactly supported coherent sheaves on X and Y , respectively.

(i) *There exists a function f on \mathfrak{M}_Y such that the projection from X to Y induces an equivalence of (-1) -shifted symplectic derived Artin stacks*

$$\mathfrak{M}_X \simeq \text{Crit}(f). \tag{3.2}$$

(ii) *Let (E, φ) be an L_2 -Higgs bundle. Then we have an equality*

$$f([(L_2, \varphi)]) = \frac{1}{2} \cdot \alpha(\text{tr}(\varphi^2)),$$

where $\alpha \in H^0(C, L_2^{\otimes 2})^\vee \cong \text{Ext}^1(L_2, L_1)$ is the class corresponding to the short exact sequence (3.1).

We now want to describe the moduli stack of *semistable* N -Higgs bundle as a global critical locus. We begin with the following easy lemma.

Lemma 3.3. *Let C be a smooth projective curve and N be a rank two vector bundle on C . Then we can take the short exact sequence (3.1) such that $\deg(L_2) > 2g(C) - 2$ holds. More generally, we can take L_2 so that its degree is arbitrarily large.*

Proof. Let $\mathcal{O}_C(1)$ be an ample line bundle on C . Then there exists an integer $l_0 > 0$ such that for every integer $l \geq l_0$, the bundle $N^\vee(l)$ is globally generated. Then a general element $s \in \text{Hom}(N, \mathcal{O}_C(l)) \cong H^0(C, N^\vee(l))$ is surjective. Putting $L_2 := \mathcal{O}_C(l)$ and $L_1 := \text{Ker}(s)$, we get the desired exact sequence as in (3.1). ■

Lemma 3.4. *Take integers $r, m \in \mathbb{Z}$ with $r > 0$. Then there exists an integer $k(r) > 2g(C) - 2$ depending only on r , such that, for any short exact sequence (3.1) with $\deg(L_2) \geq k(r)$, the following statement holds: For every μ -semistable N -Higgs bundle $(E, \varphi) \in \mathfrak{M}_X^{\text{ss}}(r, m)$, the L_2 -Higgs bundle $(E, s \circ \varphi)$ is μ -semistable.*

Proof. Let $(E, \varphi) \in \mathfrak{M}_X^{\text{ss}}(r, m)$ be a μ -semistable N -Higgs bundle. Suppose that the L_2 -Higgs bundle $(E, s \circ \varphi)$ is not μ -semistable. We claim that there exists a saturated subsheaf $F \subset E$ such that

$$\mu(F) > \mu(E) \quad \text{and} \quad \text{Hom}(F, E/F \otimes L_1) \neq 0.$$

Indeed, let $F \subset E$ be the maximal destabilizing subsheaf of $(E, s \circ \varphi)$. This means that we have $\mu(F) > \mu(E)$ and $(s \circ \varphi)(F) \subset F \otimes L_2$. The latter condition is equivalent that the composition

$$F \hookrightarrow E \xrightarrow{s \circ \varphi} E \otimes L_2 \rightarrow E/F \otimes L_2$$

is zero. On the other hand, by the μ -semistability of (E, φ) , we have $\varphi(F) \not\subset F \otimes N$, i.e., the composition

$$F \hookrightarrow E \xrightarrow{\varphi} E \otimes N \rightarrow E/F \otimes N$$

is non-zero. As a result, we obtain the following diagram:

$$\begin{array}{ccccc}
 & & F & & \\
 & \swarrow \text{dotted} & \downarrow \neq 0 & \searrow 0 & \\
 E/F \otimes L_1 & \longrightarrow & E/F \otimes N & \longrightarrow & E/F \otimes L_2,
 \end{array}$$

hence we have $\text{Hom}(F, E/F \otimes L_1) \neq 0$.

By Lemma 3.5 below, we can replace an exact sequence (3.1) so that

$$\text{Hom}(F, E/F \otimes L_1) = 0$$

for all μ -semistable N -Higgs bundles $(E, \varphi) \in \mathfrak{M}_\chi^{\text{ss}}(r, m)$ and all saturated subsheaves $F \subset E$ with $\mu(F) > \mu(E)$. Hence the above argument shows that $(E, s \circ \varphi)$ remains μ -semistable for such a choice of the exact sequence (3.1). ■

Lemma 3.5. *Take integers $r, m \in \mathbb{Z}$ with $r > 0$. Then the following sets are bounded:*

$$\begin{aligned}
 \mathcal{S} &:= \{F, E/F : (E, \varphi) \in \mathfrak{M}_\chi^{\text{ss}}(r, m), F \subset E \text{ is saturated with } \mu(F) > \mu(E)\}, \\
 \text{HN}(\mathcal{S}) &:= \{A : A \text{ is a Harder–Narasimhan factor of } G \in \mathcal{S}\}.
 \end{aligned}$$

Moreover, there exists an integer $k'(r) \in \mathbb{Z}$, depending only on r , such that for all line bundles L_1 with $\text{deg}(L_1) \leq k'(r)$ and for all $F, E/F \in \mathcal{S}$, we have the vanishing $\text{Hom}(F, E/F \otimes L_1) = 0$.

Proof. The boundedness of the sets $\mathcal{S}, \text{HN}(\mathcal{S})$ follows from the boundedness of $\mathfrak{M}_\chi^{\text{ss}}(r, m)$ and Grothendieck’s boundedness lemma (cf. [32, Lemma 1.7.9]).

In particular, there exist integers a, b such that the inequalities $\mu_{\min}(F) \geq a$ and $\mu_{\max}(F/E) \leq b$ hold for all $F, E/F \in \mathcal{S}$. By setting $k'(r, m) := a - b - 1$, we obtain the inequality

$$\mu_{\min}(F) > \mu_{\max}(E/F \otimes L_1)$$

for all line bundles L_1 with $\text{deg}(L_1) \leq k'(r, m)$ and for all $F, E/F \in \mathcal{S}$.

Finally, observe that we have an isomorphism $\mathfrak{M}_\chi^{\text{ss}}(r, m) \cong \mathfrak{M}_\chi^{\text{ss}}(r, m + r)$ by tensoring with a degree one line bundle. Hence by putting $k'(r) := \min\{k'(r, m) : m = 0, \dots, r - 1\}$, the second assertion holds. ■

Proposition 3.6. *Let $r, m \in \mathbb{Z}$ be integers with $r > 0$. Take an exact sequence (3.1) as in Lemma 3.4. Let $\alpha \in H^0(C, L_2^{\otimes 2})^\vee \cong \text{Ext}^1(L_2, L_1)$ be the corresponding class. Denote $Y := \text{Tot}_C(L_2)$. Define the function $g: B_Y \rightarrow \mathbb{A}^1$ as*

$$g: B_Y = \bigoplus_{i=1}^r H^0(C, L_2^{\otimes i}) \rightarrow \mathbb{A}^1, \quad (a_i)_{i=1}^r \mapsto \frac{1}{2} \cdot \alpha(a_2). \tag{3.3}$$

Then we have an isomorphism

$$\mathfrak{M}_X^{\text{ss}}(r, m) \cong \{d(g \circ \tilde{h}_Y) = 0\} \subset \mathfrak{M}_Y^{\text{ss}}(r, m)$$

which preserves the d -critical structure.

Proof. By Lemma 3.4, isomorphism (3.2) restricts to the semistable loci. Then the claim follows from Theorem 3.2 (ii), the fact that the derived moduli stack $\mathfrak{M}_Y^{\text{ss}}(r, m)$ is a smooth (classical) stack, and that the classical truncation of the derived critical locus of a function on a smooth stack coincides with the classical critical locus. ■

For the vanishing cycle sheaves on the good moduli spaces, we have the following result.

Proposition 3.7. *Let $r, m, m' \in \mathbb{Z}$ be integers with $r > 0$. Let $g: B_Y \rightarrow \mathbb{A}^1$ be a function as in Proposition 3.6. Then we have an isomorphism*

$$h_{Y*}(\varphi_{g \circ h_Y}(\mathcal{J}^{\mathcal{C}}_{M_Y^{\text{ss}}(r, m)})) \cong h_{Y*}(\varphi_{g \circ h_Y}(\mathcal{J}^{\mathcal{C}}_{M_Y^{\text{ss}}(r, m')})), \tag{3.4}$$

where

$$h_Y: M_Y^{\text{ss}}(r, m) \rightarrow B_Y, \quad h_Y: M_Y^{\text{ss}}(r, m') \rightarrow B_Y$$

denote the Hitchin morphisms (2.8) on the good moduli spaces.

Proof. Since the Hitchin morphism $h_Y: M_Y^{\text{ss}}(r, m) \rightarrow B_Y$ is proper, the result follows from Theorem 2.16 together with the commutativity of the vanishing cycle functors and proper push forwards. ■

In the following subsections, we will show that the complexes in (3.4) compute the generalized GV invariants for the local curve $X = \text{Tot}_C(N)$.

3.2. Calabi–Yau property for local curves

In this subsection, we fix integers $r, m \in \mathbb{Z}$ with $r > 0$, and an exact sequence (3.1). We assume that the line bundle L_2 satisfies the following conditions:

- $\text{deg}(L_2) \geq k(m)$ (see Lemma 3.4),
- L_2 is globally generated.

Recall that we denote $X := \text{Tot}_C(N)$ and $Y := \text{Tot}_C(L_2)$. By Proposition 3.6, the moduli stack $\mathfrak{M}_X^{\text{ss}}(r, m)$ is written as the global critical locus inside $\mathfrak{M}_Y^{\text{ss}}(r, m)$.

Proposition 3.8. *The canonical bundle $K_{\mathfrak{M}_Y^{\text{ss}}(r,m)}$ of the stack $\mathfrak{M}_Y^{\text{ss}}(r,m)$ is trivial, and hence so is the virtual canonical bundle $K_{\mathfrak{M}_X^{\text{ss}}(r,m)}^{\text{vir}}$ of the stack $\mathfrak{M}_X^{\text{ss}}(r,m)$. In particular, the stack $\mathfrak{M}_X^{\text{ss}}(r,m)$ is Calabi–Yau at any point $\gamma \in B_X$, i.e., Conjecture 2.7 holds for $\mathfrak{M}_X^{\text{ss}}(r,m)$.*

Proof. A similar argument can be found in [56, Theorem 7.1]. Take a morphism $T \rightarrow \mathfrak{M}_Y^{\text{ss}}(r,m)$ from a scheme T . Let $\mathcal{E} \in \text{Coh}(Y \times T)$ be the corresponding family of μ -semistable one-dimensional sheaves on Y . We consider the following diagram:

$$\begin{array}{ccc} Y_T := Y \times T & \xrightarrow{\pi_T} & T \\ p_T \downarrow & \nearrow q_T & \\ C_T := C \times T & & \end{array}$$

We need to construct an isomorphism

$$\det \mathbf{R}\mathcal{H}om_{\pi_T}(\mathcal{E}, \mathcal{E}) \cong \mathcal{O}_T,$$

which is functorial in T . We have the following exact sequence:

$$0 \rightarrow p_T^*(L_2^{-1} \boxtimes p_{T*}\mathcal{E}) \rightarrow p_T^*p_{T*}\mathcal{E} \rightarrow \mathcal{E} \rightarrow 0. \tag{3.5}$$

Applying the functor $\mathbf{R}\mathcal{H}om_{\pi_T}(-, \mathcal{E})$ to the exact sequence (3.5), we obtain the exact triangle

$$\mathbf{R}\mathcal{H}om_{\pi_T}(\mathcal{E}, \mathcal{E}) \rightarrow \mathbf{R}\mathcal{H}om_{q_T}(\mathcal{F}, \mathcal{F}) \rightarrow \mathbf{R}\mathcal{H}om_{q_T}(\mathcal{F} \boxtimes L_2^{-1}, \mathcal{F}),$$

where we put $\mathcal{F} := p_{T*}\mathcal{E}$. By taking the determinants, we get

$$\det \mathbf{R}\mathcal{H}om_{\pi_T}(\mathcal{E}, \mathcal{E}) \cong \det \mathbf{R}\mathcal{H}om_{q_T}(\mathcal{F}, \mathcal{F}) \otimes (\det \mathbf{R}\mathcal{H}om_{q_T}(\mathcal{F} \boxtimes L_2^{-1}, \mathcal{F}))^{-1}. \tag{3.6}$$

On the other hand, we have an exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow L_2 \rightarrow \mathcal{O}_Z \rightarrow 0,$$

where $Z \in |L_2|$ is a finite set of points. Applying the functor $\mathbf{R}\mathcal{H}om_{q_T}(\mathcal{F}, \mathcal{F} \boxtimes (-))$ and taking the determinants, we get

$$\begin{aligned} \det \mathbf{R}\mathcal{H}om_{q_T}(\mathcal{F} \boxtimes L_2^{-1}, \mathcal{F}) &\cong \det \mathbf{R}\mathcal{H}om_{q_T}(\mathcal{F}, \mathcal{F} \boxtimes L_2) \\ &\cong \det \mathbf{R}\mathcal{H}om_{q_T}(\mathcal{F}, \mathcal{F}) \otimes \det \mathbf{R}\mathcal{H}om_{q_T}(\mathcal{F}, \mathcal{F}_Z), \end{aligned} \tag{3.7}$$

where we put $\mathcal{F}_Z := \mathcal{F}|_{Z \times T}$. Combining equations (3.6) and (3.7), we obtain the desired isomorphism

$$\begin{aligned} \det \mathbf{R}\mathcal{H}om_{\pi_T}(\mathcal{E}, \mathcal{E}) &\cong \det \mathbf{R}\mathcal{H}om_{q_T}(\mathcal{F}, \mathcal{F}_Z)^{-1} \cong \det \mathbf{R}\mathcal{H}om_{r_T}(\mathcal{F}_Z, \mathcal{F}_Z)^{-1} \\ &\cong \bigotimes_{i=1}^k \det \mathbf{R}\mathcal{H}om(\mathcal{F}_{p_i}, \mathcal{F}_{p_i}) \cong \mathcal{O}_T, \end{aligned}$$

where we denote by $r_T: Z \times T \rightarrow T$ the projection. For the third isomorphism, we put $Z = \{p_1, \dots, p_k\}$ and $\mathcal{F}_{p_i} := \mathcal{F}|_{\{p_i\} \times T}$.

The triviality of the virtual canonical bundle $K_{\mathfrak{M}_X^{\text{ss}}(r,m)}^{\text{vir}}$ follows from Lemma A.10. ■

3.3. Proof of Theorem 3.1

In this subsection, we finish the proof of Theorem 3.1.

Consider the following commutative diagram:

$$\begin{array}{ccc}
 M_X^{ss}(r, m) & \xrightarrow{\iota} & M_Y^{ss}(r, m) \\
 h_X \downarrow & & \downarrow h_Y \\
 B_X & \xrightarrow{b} & B_Y,
 \end{array} \tag{3.8}$$

where the morphism $b: B_X \rightarrow B_Y$ is induced by the surjection $\text{Sym}^k(N) \rightarrow L_2^{\otimes k}$ for $k = 1, \dots, r$.

Lemma 3.9. *The morphism*

$$b|_{\text{im}(h_X)}: \text{im}(h_X) \rightarrow B_Y \tag{3.9}$$

is finite.

Proof. It is enough to show that the morphism in (3.9) is proper and affine. The composition $h_Y \circ \iota = b \circ h_X: M_X^{ss}(r, m) \rightarrow B_Y$ is proper as so are h_Y and ι . Furthermore, the morphism $h_X: M_X^{ss}(r, m) \rightarrow \text{im}(h_X)$ is proper and surjective. Hence morphism (3.9) is proper.

On the other hand, by the properness of the Hitchin morphism h_X , the inclusion $\text{im}(h_X) \hookrightarrow B_X$ is closed. As the morphism $b: B_X \rightarrow B_Y$ is just the projection of affine spaces, it is also affine. We conclude that the composition

$$\text{im}(h_X) \hookrightarrow B_X \xrightarrow{b} B_Y$$

is affine, as required. ■

Recall from (2.8) and (2.9) that we denote by

$$h_Y: M_Y^{ss}(r, m) \rightarrow B_Y, \quad \tilde{h}_Y: \mathfrak{M}_Y^{ss}(r, m) \rightarrow B_Y$$

the Hitchin morphisms. Recall also that we have the function $g: B_Y \rightarrow \mathbb{A}^1$ defined in Proposition 3.6. We equip $\mathfrak{M}_X^{ss}(r, m)$ with the orientation defined by the global critical chart description in Proposition 3.6 and Lemma A.10. We define the vanishing cycle complex $\varphi_{\mathfrak{M}_X^{ss}(r, m)} \in \text{Perv}(\mathfrak{M}_X^{ss}(r, m))$ using this orientation. We set

$$\varphi_{M_X^{ss}(r, m)} := \mathcal{H}^1(p_{X*} \varphi_{\mathfrak{M}_X^{ss}(r, m)}) \in \text{Perv}(M_X^{ss}(r, m)). \tag{3.10}$$

We need the following proposition.

Proposition 3.10. *We have isomorphisms*

$$\varphi_{M_X^{ss}(r, m)} \cong \mathcal{H}^1(p_{X*}(\varphi_{g \circ \tilde{h}_Y}(\mathcal{I}\mathcal{C}_{\mathfrak{M}_Y^{ss}(r, m)}))) \cong \varphi_{g \circ h_Y}(\mathcal{I}\mathcal{C}_{M_Y^{ss}(r, m)}).$$

We postpone the proof of this proposition until the next section.

Proof of Theorem 3.1. Let us take integers $r, m, m' \in \mathbb{Z}$ with $r > 0$. Let $k(r) \in \mathbb{Z}$ be integers as in Lemma 3.4. We take an exact sequence (3.1) such that L_2 is globally generated and $\deg(L_2) \geq k(r)$ holds. Then by Proposition 3.6, the moduli stacks $\mathfrak{M}_X^{\text{ss}}(r, m)$, $\mathfrak{M}_X^{\text{ss}}(r, m')$ are written as the global critical loci inside the stacks $\mathfrak{M}_Y^{\text{ss}}(r, m)$, $\mathfrak{M}_Y^{\text{ss}}(r, m')$, respectively.

Recall from Proposition 3.8 that the canonical bundles $K_{\mathfrak{M}_Y^{\text{ss}}(r, m)}$ and $K_{\mathfrak{M}_Y^{\text{ss}}(r, m')}$ are trivial, hence the natural orientation data in Lemma A.10 is a CY orientation data.

Let $\varphi_{M_X^{\text{ss}}(r, m)}, \varphi_{M_X^{\text{ss}}(r, m')}$ be the associated perverse sheaves on $M_X^{\text{ss}}(r, m)$, $M_X^{\text{ss}}(r, m')$, defined in (3.10). By using the embeddings $M_X^{\text{ss}}(r, m) \hookrightarrow M_Y^{\text{ss}}(r, m)$ and $M_X^{\text{ss}}(r, m') \hookrightarrow M_Y^{\text{ss}}(r, m')$ in (3.8), we can regard them as perverse sheaves on $M_Y^{\text{ss}}(r, m)$ and $M_Y^{\text{ss}}(r, m')$, respectively. We have an isomorphism

$$h_{Y*}(\varphi_{M_X^{\text{ss}}(r, m)}) \cong h_{Y*}(\varphi_{M_X^{\text{ss}}(r, m')}) \tag{3.11}$$

by Proposition 3.7. By using the commutative diagram (3.8), we can rewrite the left-hand side of (3.11) as

$$h_{Y*}(\varphi_{M_X^{\text{ss}}(r, m)}) \cong b_* h_{X*}(\varphi_{M_X^{\text{ss}}(r, m)}).$$

By Lemma 3.9, map 3.9 is finite. Since the pushforward along a finite morphism preserves the perverse t-structures, we obtain

$$\mathcal{H}^i(h_{Y*}(\varphi_{M_X^{\text{ss}}(r, m)})) \cong b_* \mathcal{H}^i(h_{X*}(\varphi_{M_X^{\text{ss}}(r, m)})) \tag{3.12}$$

for all $i \in \mathbb{Z}$, and we have the same isomorphisms if we replace the integer m with m' .

Combining isomorphisms (3.11) and (3.12), and taking the Euler characteristics, we conclude that

$$\chi(\mathcal{H}^i(h_{X*}(\varphi_{M_X^{\text{ss}}(r, m)}))) = \chi(\mathcal{H}^i(h_{X*}(\varphi_{M_X^{\text{ss}}(r, m')})))$$

as desired. ■

4. Cohomological integrality theorem for twisted Higgs bundles

In this section, we prove the cohomological integrality theorem in the sense of [17, §1.3] for twisted Higgs bundles. Since semistable twisted Higgs bundles form a category of homological dimension one, we can prove the cohomological integrality theorem using the techniques of [17, 45], which treat the case of quivers.

4.1. Mixed Hodge modules on stacks

Here we give a quick introduction to a mixed Hodge module, which is a sheaf theoretic version of a mixed Hodge structure introduced by Saito [50]. An advantage of working with mixed Hodge modules (rather than perverse sheaves) is the fact that the category of pure Hodge modules is semisimple. In particular, two pure Hodge modules are isomorphic if and only if they are equal in the Grothendieck group of the category of pure Hodge modules.

This was used by Davison–Meinhardt [17] in their proof of the cohomological integrality theorem for quivers with potentials, and will be used in the proof of Theorem 4.6.

Let X be a separated scheme locally of finite type over a complex number whose connected components are quasi-compact. For such X , we can define the category of mixed Hodge modules $\text{MHM}(X)$ and its bounded derived category $D^b(\text{MHM}(X))$ which admits a six-functor formalism (see [49] for an overview). There exists an exact functor

$$\text{rat}: D^b(\text{MHM}(X)) \rightarrow D^b(\text{Perv}(X))$$

which restricts to a faithful functor $\text{MHM}(X) \rightarrow \text{Perv}(X)$. The functor rat is compatible with all six functors. A mixed Hodge module M is equipped with an increasing filtration called the *weight filtration* which we denote by $W \cdot M$. A mixed Hodge module $M \in \text{MHM}(X)$ is called *pure of weight i* if $W_{i-1}M = 0$ and $W_i M = M$ holds, and an object $M^\bullet \in D^b(\text{MHM}(X))$ is called *pure* if the i -th cohomology mixed Hodge module $\mathcal{H}^i(M)$ is pure of weight i .

The category of mixed Hodge modules over a point is equivalent to the category of graded polarizable mixed Hodge structures. Let $a_X: X \rightarrow \text{Spec } \mathbb{C}$ be the constant map to a point. Then we define objects $\mathbb{Q}_X, \mathbb{D}\mathbb{Q}_X \in D^b(\text{MHM}(X))$ as

$$\mathbb{Q}_X := a_X^* \mathbb{Q}, \quad \mathbb{D}\mathbb{Q}_X := a_X^! \mathbb{Q},$$

where \mathbb{Q} denotes the mixed Hodge structure of weight zero and dimension one.

The category of mixed Hodge modules forms a stack in the smooth topology (see [1, Theorem 2.3]). This motivates the following definition of the category of mixed Hodge modules on an Artin stack \mathfrak{X} .

Definition 4.1. Let \mathfrak{X} be a complex Artin stack. We let $\text{Sch}_{/\mathfrak{X}}^{\text{sm,sep}}$ denote the category of separated schemes smooth and of finite type over \mathfrak{X} . A *mixed Hodge module* on \mathfrak{X} is a pair consisting of an assignment

$$\text{Sch}_{/\mathfrak{X}}^{\text{sm,sep}} \ni (t: T \rightarrow \mathfrak{X}) \mapsto M_t \in \text{MHM}(T)$$

and a choice of an isomorphism

$$\theta_q: q^* M_{t_2}[\dim q] \cong M_{t_1}$$

for each smooth morphism $q: (t_1: T_1 \rightarrow \mathfrak{X}) \rightarrow (t_2: T_2 \rightarrow \mathfrak{X})$ in $\text{Sch}_{/\mathfrak{X}}^{\text{sm,sep}}$ satisfying the associativity relation. Mixed Hodge modules on \mathfrak{X} form a category $\text{MHM}(\mathfrak{X})$ in a natural way. We have a natural forgetful functor

$$\text{rat}: \text{MHM}(\mathfrak{X}) \rightarrow \text{Perv}(\mathfrak{X}).$$

Take a smooth surjective morphism from a separated finite type scheme $t: T \rightarrow \mathfrak{X}$, and let $\text{pr}_i: T \times_{\mathfrak{X}} T \rightarrow T$ denote the i -th projection. Then we can identify $\text{MHM}(\mathfrak{X})$ with the category of pairs (M, σ) , where M is a mixed Hodge module on T and σ is an isomorphism

$$\sigma: \text{pr}_1^* M \cong \text{pr}_2^* M$$

satisfying the cocycle condition.

At present, we do not have a full six-functor formalism for mixed Hodge modules on Artin stacks. However, we have some part of it which is sufficient for applications in this paper. Firstly, if we are given a smooth morphism $q: \mathfrak{X}_1 \rightarrow \mathfrak{X}_2$ between Artin stacks, we can define a functor

$$q^*[\dim q]: \text{MHM}(\mathfrak{X}_2) \rightarrow \text{MHM}(\mathfrak{X}_1)$$

in the standard way.

Now assume that we are given a finite type morphism $p: \mathfrak{X} \rightarrow X$ from an Artin stack to a separated finite type scheme. We want to define the functor

$$\mathcal{H}^n(p_*(-)): \text{MHM}(\mathfrak{X}) \rightarrow \text{MHM}(X)$$

compatible with the functor rat . Here \mathcal{H}^n denotes the n -th cohomology with respect to the perverse t-structure on $D(\text{MHM}(X))$. We assume that the morphism p satisfies the following assumption:

- (*) For each object $\mathcal{F} \in D_c^b(\mathfrak{X})$ in the bounded derived category of sheaves on \mathfrak{X} with constructible cohomology and integer N , there exists a smooth morphism from a scheme $q_N: T_N \rightarrow \mathfrak{X}$ such that the natural map

$$\mathcal{H}^n(\mathcal{F}) \rightarrow \mathcal{H}^n(q_{N*}q_N^*\mathcal{F})$$

is isomorphism for each $n \leq N$. Here \mathcal{H}^n denotes the perverse t-structure on $D_c^b(\mathfrak{X})$.

This assumption is automatically satisfied when \mathfrak{X} is of the form $[Y/G]$ for some scheme Y and a linear algebraic group G (see [14, §2.3.2]). Let $p: \mathfrak{X} \rightarrow X$ be a morphism to a scheme. For a mixed Hodge module $M \in \text{MHM}(\mathfrak{X})$ and $n \in \mathbb{Z}$, we define a mixed Hodge module

$$\mathcal{H}^n(p_*M) := \mathcal{H}^n((p \circ q_N)_*q_N^*M),$$

where N is a sufficiently large integer. We can show that $\mathcal{H}^n(p_*M)$ is independent of the choice of N and q_N . If we take p as the constant map $a_{\mathfrak{X}}: \mathfrak{X} \rightarrow \text{Spec } \mathbb{C}$, we can construct a mixed Hodge structure

$$H^n(\mathfrak{X}, M) := \mathcal{H}^n(a_{\mathfrak{X}*}M).$$

Similarly, we can extend the perverse sheaves $\mathcal{H}^n(p_*\mathbb{Q}_{\mathfrak{X}})$ and $\mathcal{H}^n(p_*\mathbb{D}_{\mathfrak{X}})$ to mixed Hodge modules, and the vector spaces $H^n(\mathfrak{X})$ and $H_n^{\text{BM}}(\mathfrak{X})$ to mixed Hodge structures.

For a complex of mixed Hodge modules $M \in D(\text{MHM}(X))$, we define

$$\mathcal{H}(M) := \bigoplus_{i \in \mathbb{N}} \mathcal{H}^i(M)[-i].$$

Lemma 4.2. *Let \mathfrak{X} be a stack satisfying condition (*), $p: \mathfrak{X} \rightarrow X$ be a morphism to a separated finite type complex scheme, and $h: X \rightarrow B$ be a proper morphism between separated finite type complex schemes. Take $M \in D^b(\text{MHM}(X))$ and assume that $\mathcal{H}(p_*M)$ is pure. Then we have an isomorphism*

$$\mathcal{H}(h_*\mathcal{H}(p_*M)) \cong \mathcal{H}((h \circ p)_*M).$$

Proof. Take an integer n and an integer N such that $N > n + \dim h^{-1}(x)$ holds for each $x \in B$. Then [21, Corollary 5.2.14] implies that we have an isomorphism

$$\mathcal{H}^n(h_*\mathcal{H}(p_*M)) \cong \mathcal{H}^n(h_*\tau^{\leq N}\mathcal{H}(p_*M)).$$

Take a smooth morphism $q: T \rightarrow X$ such that we have isomorphisms

$$\begin{aligned} \tau^{\leq N}\mathcal{H}(p_*M) &\cong \tau^{\leq N}\mathcal{H}((p \circ q)_*q^*M), \\ \mathcal{H}^n((h \circ p)_*M) &\cong \mathcal{H}^n((h \circ p \circ q)_*q^*M). \end{aligned}$$

Then it is enough to prove the following isomorphism:

$$\mathcal{H}^n(h_*\tau^{\leq N}\mathcal{H}((p \circ q)_*q^*M)) \cong \mathcal{H}^n((h \circ p \circ q)_*q^*M).$$

Saito’s decomposition theorem implies an isomorphism

$$\mathcal{H}^n(h_*\tau^{\leq N}\mathcal{H}((p \circ q)_*q^*M)) \cong \mathcal{H}^n(h_*\tau^{\leq N}(p \circ q)_*q^*M).$$

Then using [21, Corollary 5.2.14] again, we obtain the desired isomorphism. ■

4.2. Monodromic mixed Hodge modules

Here we recall some basic properties of monodromic mixed Hodge modules. We do not give the precise definition here and refer the reader to [17, §2] and [3, §2.9] for the detailed discussion. Let X be a separated scheme locally of finite type over a complex number whose connected components are quasi-compact. Then we can define an abelian category $\text{MMHM}(X)$ of *monodromic mixed Hodge modules* on X . Roughly speaking, a monodromic mixed Hodge module consists of its underlying mixed Hodge module M and a monodromy operator acting on it. We have a natural functor

$$\text{MMHM}(X) \rightarrow \text{MHM}(X)$$

forgetting the monodromy operator and a fully faithful functor

$$\text{MHM}(X) \hookrightarrow \text{MMHM}(X)$$

which associates a mixed Hodge module M to a monodromic mixed Hodge module whose underlying mixed Hodge module is M and the monodromy operator is trivial. Similar to the usual mixed Hodge modules, monodromic mixed Hodge modules are also equipped with weight filtrations.

The bounded derived category $D^b(\text{MMHM}(X))$ admits a six-functor formalism, similarly to the usual mixed Hodge modules. The inclusion functor

$$D^b(\text{MHM}(X)) \rightarrow D^b(\text{MMHM}(X))$$

is compatible with these six operations. The forgetful functor

$$D^b(\text{MMHM}(X)) \rightarrow D^b(\text{MHM}(X))$$

is compatible with four operations f_* , $f_!$, f^* , $f^!$ for a morphism f between separated finite type complex schemes. However, the tensor product of monodromic mixed Hodge modules is not compatible with the tensor product of the underlying mixed Hodge modules.

For a regular function $f: X \rightarrow \mathbb{A}^1$, we can define the *monodromic vanishing cycle functor* for (possibly unbounded) mixed Hodge module complexes

$$\phi_f^{\text{mmhm}}: D(\text{MHM}(X)) \rightarrow D(\text{MMHM}(X)), \tag{4.1}$$

which enhances the usual vanishing cycle functor by incorporating the monodromy operator acting on it.

The essential difference between monodromic and the usual mixed Hodge modules are the following:

- *Thom–Sebastiani isomorphism* holds for the monodromic vanishing cycle functors (4.1). See [17, Proposition 2.13] for the precise statement and other basic properties.
- There exists an object $\mathbb{L}^{1/2} \in D^b(\text{MMHM}(\text{pt}))$ with an isomorphism

$$(\mathbb{L}^{1/2})^{\otimes 2} \cong \mathbb{L} \in D^b(\text{MMHM}(\text{pt})),$$

where we put $\mathbb{L} := H_c^*(\mathbb{A}^1, \mathbb{Q}) \in D^b(\text{MHM}(\text{pt})) \subset D^b(\text{MMHM}(\text{pt}))$, which is concentrated in cohomological degree two, and is pure of weight two.

When X is an irreducible variety, we define the object $\mathcal{I}\mathcal{C}_X \in \text{MMHM}(X)$ as follows:

$$\mathcal{I}\mathcal{C}_X := \mathbb{L}^{-\dim X/2} \otimes \widetilde{\mathcal{I}\mathcal{C}}_X \in \text{MMHM}(X),$$

where $\widetilde{\mathcal{I}\mathcal{C}}_X$ denotes the intermediate extension of $\mathbb{Q}_{X_{\text{reg}}}$ on the regular locus $X_{\text{reg}} \subset X$. We will also use the following object:

$$H^*(\text{BC}^*)_{\text{vir}} := \mathbb{L}^{1/2} \otimes H^*(\text{BC}^*) \in D(\text{MMHM}(\text{pt})). \tag{4.2}$$

As in the previous subsection, we can define the notion of monodromic mixed Hodge modules for an Artin stack. In particular, we can define the object $\mathcal{I}\mathcal{C}_{\mathcal{X}} \in \text{MMHM}(\mathcal{X})$ for a smooth Artin stack \mathcal{X} . Moreover, we can define the functor

$$\mathcal{H}^n(p_*(-)): \text{MMHM}(\mathcal{X}) \rightarrow \text{MMHM}(X)$$

for a morphism $p: \mathcal{X} \rightarrow X$ from an Artin stack \mathcal{X} to a scheme X satisfying condition (*) in Section 4.1.

Let X be a separated scheme locally of finite type over \mathbb{C} whose connected components are quasi-compact. A (possibly unbounded) complex $M \in D(\text{MMHM}(X))$ is *locally finite* if for each connected component $Z \subset X$, the following conditions hold:

- For each $n \in \mathbb{Z}$, the set $\{i \in \mathbb{Z} : \text{gr}_n^W \mathcal{H}^i(M)|_Z \neq 0\}$ is finite.
- There exists an integer n such that $W_n \mathcal{H}^i(M)|_Z = 0$ for all i .

We let $D^{\geq, \text{lf}}(\text{MMHM}(X)) \subset D^{\geq}(\text{MMHM}(X))$ denote the full subcategory consisting of locally finite monodromic mixed Hodge complexes. We can see that the Grothendieck group $K_0(D^{\geq, \text{lf}}(\text{MMHM}(X)))$ is isomorphic to the completion of $K_0(\text{MMHM}(X))$ with respect to ideals $\{I_i\}_{i \in \mathbb{Z}}$, where I_i is generated by objects whose weight is greater than i .

Let (X, m) be a monoid scheme, where X is a separated and locally of finite type over complex number whose connected components are quasi-compact, and $m: X \times X \rightarrow X$ is a finite morphism. For objects $M, N \in D^{\geq, \text{lf}}(\text{MMHM}(X))$, we define

$$M \boxtimes_m N := m_*(M \boxtimes N) \in D^{\geq, \text{lf}}(\text{MMHM}(X)).$$

The functor \boxtimes_m defines a symmetric monoidal structure on $D^{\geq, \text{lf}}(\text{MMHM}(X))$. Therefore, for each $n \in \mathbb{Z}_{>0}$, we can define the symmetric product functor

$$\text{Sym}_{\boxtimes_m}^n : D^{\geq, \text{lf}}(\text{MMHM}(X)) \rightarrow D^{\geq, \text{lf}}(\text{MMHM}(X)).$$

4.3. Approximation by proper morphisms

Let L be a line bundle on a smooth projective curve C with $\text{deg}(L) > 2g(C) - 2$, and put $Y := \text{Tot}_C(L)$. For given integers $r, m \in \mathbb{Z}$ with $r > 0$, we denote by $p: \mathfrak{M}_Y^{\text{ss}}(r, m) \rightarrow M_Y^{\text{ss}}(r, m)$ the morphism from the moduli stack to its good moduli space. We fix a regular function $F: M_Y^{\text{ss}}(r, m) \rightarrow \mathbb{A}^1$ and denote by $\tilde{F} := F \circ p: \mathfrak{M}_Y^{\text{ss}}(r, m) \rightarrow \mathbb{A}^1$ the composition.

Let us recall the construction of moduli spaces of *framed objects* following [17, 42, 45]. We follow the notations in [42, §3.3]. By construction, we have $\mathfrak{M}_Y^{\text{ss}}(r, m) = [\text{Quot}^{\text{ss}} / \text{GL}_n]$, where

$$\text{Quot}^{\text{ss}} \subset \text{Quot}$$

is the GIT semistable locus inside a certain quot scheme Quot with respect to a certain GL_n -linearization. For a given integer $f > n$, we put

$$\mathbb{A} := \text{Hom}(\mathbb{C}, \mathbb{C}^n)^f, \quad G_n := \mathbb{C}^* \times \text{GL}_n.$$

We have a G_n -action on Quot which passes through the GL_n -action. We define a G_n -action on \mathbb{A} as follows:

$$(t, g) \cdot (a_i) := (t^{-1} a_i g), \quad (t, g) \in G_n, (a_i)_{i=1}^f \in \mathbb{A}.$$

By choosing certain G_n -linearizations on \mathbb{A} and $\text{Quot} \times \mathbb{A}$, we obtain the diagram

$$\begin{array}{ccccc}
 U_f & \hookrightarrow & M_f & \hookrightarrow & \mathfrak{X}_f \\
 & \searrow & \downarrow \pi_f & \swarrow & \\
 & & M_Y^{\text{ss}}(r, m) & &
 \end{array} \tag{4.3}$$

where we put

$$\begin{aligned}
 U_f &:= (\text{Quot}^{\text{ss}} \times \mathbb{A}^{\text{ss}}) / \text{PG}_n, \\
 M_f &:= (\text{Quot} \times \mathbb{A})^{\text{ss}} / \text{PG}_n, \\
 \mathfrak{X}_f &:= [\text{Quot}^{\text{ss}} \times \mathbb{A} / \text{PG}_n].
 \end{aligned}$$

Diagram (4.3) satisfies the following properties (cf. [42, Proposition 3.6]):

- the horizontal morphisms are open immersions,
- U_f and M_f are smooth schemes and the morphism π_f is projective,
- $\lim_{f \rightarrow \infty} \text{codim}_{\mathcal{X}_f}(\mathcal{W}_f) = \infty$, where $\mathcal{W}_f \subset \mathcal{X}_f$ denotes the complement of U_f .

We can compute the cohomology objects $\mathcal{H}^n(p_*\varphi_{\tilde{F}}^{\text{mmhm}} \mathcal{I}\mathcal{C}_{\mathfrak{M}_Y^{\text{ss}}(r,m)})$ by the following proposition, using the pushforward along the proper morphism π_f .

Proposition 4.3 (cf. [17, Lemma 4.1 and Proposition 4.3]). *The following statements hold:*

- (1) *For each $n \in \mathbb{Z}$, there exists $f \gg 0$ such that*

$$\mathcal{H}^n(p_*\varphi_{\tilde{F}}^{\text{mmhm}} \mathcal{I}\mathcal{C}_{\mathfrak{M}_Y^{\text{ss}}(r,m)}) \cong \mathcal{H}^n(\pi_{f*}\varphi_{F \circ \pi_f}^{\text{mmhm}} \mathcal{I}\mathcal{C}_{M_f}).$$

- (2) *We have an isomorphism*

$$\mathcal{H}(p_*\varphi_{\tilde{F}}^{\text{mmhm}} \mathcal{I}\mathcal{C}_{\mathfrak{M}_Y^{\text{ss}}(r,m)}) \cong \varphi_F^{\text{mmhm}} \mathcal{H}(p_* \mathcal{I}\mathcal{C}_{\mathfrak{M}_Y^{\text{ss}}(r,m)}).$$

Proof. We just give an outline of the proof. See [17, Lemma 4.1, Proposition 4.3] for the details. Using the fact that $\lim_{f \rightarrow \infty} \text{codim}_{\mathcal{X}_f}(\mathcal{W}_f) = \infty$, we can check that the morphism $\pi_f: M_f \rightarrow M_Y^{\text{ss}}(r, m)$ approximates the map $p: \mathfrak{M}_Y^{\text{ss}}(r, m) \rightarrow M_Y^{\text{ss}}(r, m)$ in the sense of (*) in Section 4.1. Hence the first assertion holds.

The second assertion now follows from the natural isomorphism

$$\varphi_F^{\text{mmhm}} \circ \pi_{f*} \cong \pi_{f*} \circ \varphi_{\pi_f \circ F}^{\text{mmhm}}$$

between functors, which holds since the morphism $\pi_f: M_f \rightarrow M_Y^{\text{ss}}(r, m)$ is proper. ■

The following statement will be used in Section 5.3.

Proposition 4.4. *Let $\mathfrak{Z} \subset \mathfrak{M}_Y^{\text{ss}}(r, m)$ be the critical locus of \tilde{F} and $Z \subset M_Y^{\text{ss}}(r, m)$ be its good moduli space. Given a morphism $q: Z \rightarrow W$ between schemes, we have an isomorphism*

$$\mathcal{H}(q_* \mathcal{H}((p|_{\mathfrak{Z}})_* \varphi_{\tilde{F}}^{\text{mmhm}} \mathcal{I}\mathcal{C}_{\mathfrak{M}_Y^{\text{ss}}(r,m)})) \cong \mathcal{H}((q \circ p|_{\mathfrak{Z}})_* \varphi_{\tilde{F}}^{\text{mmhm}} \mathcal{I}\mathcal{C}_{\mathfrak{M}_Y^{\text{ss}}(r,m)}).$$

Proof. Fix an integer n . We let Z_f denote the critical locus of the function $F \circ \pi_f$. Take a sufficiently large integer f such that the following isomorphism holds:

$$\mathcal{H}^n((q \circ p|_{\mathfrak{Z}})_* \varphi_{\tilde{F}}^{\text{mmhm}} \mathcal{I}\mathcal{C}_{\mathfrak{M}_Y^{\text{ss}}(r,m)}) \cong \mathcal{H}^n((q \circ \pi_f|_{Z_f})_* \varphi_{F \circ \pi_f}^{\text{mmhm}} \mathcal{I}\mathcal{C}_{M_f}).$$

We have the following isomorphisms:

$$\begin{aligned} \mathcal{H}^n((q \circ \pi_f|_{Z_f})_* \varphi_{F \circ \pi_f}^{\text{mmhm}} \mathcal{I}\mathcal{C}_{M_f}) &\cong \mathcal{H}^n(q_* \varphi_{F \circ \pi_f}^{\text{mmhm}}(\pi_{f*} \mathcal{I}\mathcal{C}_{M_f})) \\ &\cong \mathcal{H}^n(q_* \varphi_{F \circ \pi_f}^{\text{mmhm}}(\mathcal{H}(\pi_{f*} \mathcal{I}\mathcal{C}_{M_f}))) \\ &\cong \mathcal{H}^n(q_* \mathcal{H}(\varphi_{F \circ \pi_f}^{\text{mmhm}}(\pi_{f*} \mathcal{I}\mathcal{C}_{M_f}))), \end{aligned}$$

where the second isomorphism follows from Saito’s decomposition theorem. If f is sufficiently large, we also have an isomorphism

$$\mathcal{H}^n(q_*\mathcal{H}(\varphi_{F \circ \pi_f}^{\text{mmhm}}(\pi_{f*} \mathcal{I}\mathcal{C}_{M_f}))) \cong \mathcal{H}^n(q_*\mathcal{H}(p_*\varphi_{\tilde{F}}^{\text{mmhm}} \mathcal{I}\mathcal{C}_{\mathfrak{M}_Y^{\text{ss}}(r,m)})),$$

so we obtain the claim. ■

4.4. Cohomological integrality theorem for L -Higgs bundles

Here we prove the cohomological integrality theorem for $Y := \text{Tot}_C(L)$, where L is a line bundle on a smooth projective curve C with $\text{deg}(L) > 2g(C) - 2$. Since the category of μ -semistable one-dimensional sheaves on Y is homological of dimension one, this can be proved in the same manner as [17, Theorem A] by applying the main result of [45]. However, we give a sketch of the proof for the reader’s convenience.

For a given rational number $\mu \in \mathbb{Q}$, we set

$$M_Y^{\text{ss}}(\mu) := \coprod_{m/r=\mu} M_Y^{\text{ss}}(r, m).$$

For each positive integer $n \in \mathbb{Z}_{>0}$, we have the following morphism:

$$\oplus: (M_Y^{\text{ss}}(\mu))^{\times n} \rightarrow M_Y^{\text{ss}}(\mu), \quad (E_i)_{i=1}^n \mapsto \bigoplus_i E_i,$$

which is finite (cf. [16, Examples 2.14 and 2.16]). We define functors

$$\text{Sym}_{\boxplus}^n, \text{Sym}_{\boxtimes}^n: D^{\geq, lf}(\text{MMHM}(M_Y^{\text{ss}}(\mu))) \rightarrow D^{\geq, lf}(\text{MMHM}(M_Y^{\text{ss}}(\mu)))$$

as follows:

$$\text{Sym}_{\boxplus}^n(\mathcal{F}) := (\oplus_* \mathcal{F}^{\boxplus n})^{\mathcal{S}_n}, \quad \text{Sym}_{\boxtimes}^n(\mathcal{F}) := \bigoplus_{n \geq 1} \text{Sym}_{\boxplus}^n(\mathcal{F}).$$

Proposition 4.5. *The following statements hold:*

- (1) *The functor Sym_{\boxplus}^n is exact.*
- (2) *The functor Sym_{\boxtimes}^n sends pure objects to pure objects.*
- (3) *Let $F_\mu: M_Y^{\text{ss}}(\mu) \rightarrow \mathbb{A}^1$ be a regular function satisfying $F_\mu(A \oplus B) = F_\mu(A) + F_\mu(B)$ for all $A, B \in M_Y^{\text{ss}}(\mu)$. Then the functors $\varphi_{F_\mu}^{\text{mmhm}}$ and Sym_{\boxplus}^n commute.*

Proof. The same proofs as in [17, Propositions 3.5, 3.8 and 3.11] work by using the finiteness of the morphism $\oplus: (M_Y^{\text{ss}}(\mu))^{\times n} \rightarrow M_Y^{\text{ss}}(\mu)$ and Thom–Sebastiani isomorphism for the vanishing cycle functors $\varphi_{(-)}^{\text{mmhm}}$. ■

We use the following notations:

$$\mathcal{I}\mathcal{C}_{\mathfrak{M}_Y^{\text{ss}}(\mu)} := \bigoplus_{m/r=\mu} \mathcal{I}\mathcal{C}_{\mathfrak{M}_Y^{\text{ss}}(r,m)}, \quad \mathcal{I}\mathcal{C}_{M_Y^{\text{ss}}(\mu)} := \bigoplus_{m/r=\mu} \mathcal{I}\mathcal{C}_{M_Y^{\text{ss}}(r,m)}.$$

Recall that we denote by $p: \mathfrak{M}_Y^{\text{ss}}(r, m) \rightarrow M_Y^{\text{ss}}(r, m)$ the canonical morphism to the good moduli space. Recall also the definition of the object $H^*(\text{BC}^*)_{\text{vir}}$ from (4.2).

Theorem 4.6. *We have the following isomorphisms in $D^{\geq,lf}(\text{MMHM}(M_Y^{\text{ss}}(\mu)))$:*

$$\mathcal{H}(p_* \mathcal{I}\mathcal{C}_{\mathfrak{M}_Y^{\text{ss}}(\mu)}) \cong \text{Sym}_{\boxtimes_{\oplus}}(\mathbf{H}^*(\mathbf{BC}^*)_{\text{vir}} \otimes \mathcal{I}\mathcal{C}_{M_Y^{\text{ss}}(\mu)}), \tag{4.4}$$

$$\mathcal{H}(p_* \varphi_{F_{\mu} \circ p}^{\text{mmhm}} \mathcal{I}\mathcal{C}_{\mathfrak{M}_Y^{\text{ss}}(\mu)}) \cong \text{Sym}_{\boxtimes_{\oplus}}(\mathbf{H}^*(\mathbf{BC}^*)_{\text{vir}} \otimes \varphi_{F_{\mu}}^{\text{mmhm}} \mathcal{I}\mathcal{C}_{M_Y^{\text{ss}}(\mu)}) \tag{4.5}$$

for a regular function $F_{\mu}: M_Y^{\text{ss}}(\mu) \rightarrow \mathbb{A}^1$ satisfying $F_{\mu}(A \oplus B) = F_{\mu}(A) + F_{\mu}(B)$ for all $A, B \in M_Y^{\text{ss}}(\mu)$.

Proof. We first construct isomorphism (4.4). By the exactness of the functor $\text{Sym}_{\boxtimes_{\oplus}}$ (see Proposition 4.5 (1)), the right-hand side is isomorphic to its total cohomology. Hence it is enough to prove the isomorphism for each cohomology. By Proposition 4.3 (1), for each $n \in \mathbb{Z}$ and $(r, m) \in \mathbb{Z}_{>0} \times \mathbb{Z}$, there exists $f \gg 0$ such that we have an isomorphism

$$\mathcal{H}^n(p_* \mathcal{I}\mathcal{C}_{\mathfrak{M}_Y^{\text{ss}}(r,m)}) \cong \mathcal{H}^n(\pi_{f*} \mathcal{I}\mathcal{C}_{M_f}). \tag{4.6}$$

Since the morphism π_f is proper and the object $\mathcal{I}\mathcal{C}_{M_f}$ is pure, it follows that the object in (4.6) is a pure mixed Hodge module.

On the other hand, since $\mathcal{I}\mathcal{C}_{M_Y^{\text{ss}}(r,m)}$ is pure, Proposition 4.5 (2) implies that the n -th cohomology of the right-hand side of (4.4) is also pure.

Now n -th cohomologies of both sides of (4.4) are direct sums of simple pure mixed Hodge modules. Hence it is enough to prove the equality in the Grothendieck group $K_0(D^{\geq,lf}(M_Y^{\text{ss}}(\mu)))$, which holds by the main theorem of [45].

The second isomorphism (4.5) follows by applying the vanishing cycle functor $\varphi_{F_{\mu}}^{\text{mmhm}}$ to both sides of isomorphism (4.4) and then using Propositions 4.3 (2) and 4.5 (3). ■

We end this section by proving Proposition 3.10 from the previous section.

Proof of Proposition 3.10. Fix integers $r > 0$ and $m \in \mathbb{Z}$, and put $\mu := m/r$. Let $g: B_Y \rightarrow \mathbb{A}^1$ be the function defined in (3.3). Recall from Proposition 3.6 that we have

$$\mathfrak{M}_X^{\text{ss}}(r, m) \cong \{d(\tilde{h}_Y \circ g) = 0\} \subset \mathfrak{M}_Y^{\text{ss}}(r, m)$$

for a line bundle L_2 with $\text{deg}(L_2) \gg 0$. Hence the first isomorphism in Proposition 3.10 follows from Proposition 2.4.

For the second isomorphism, let us put

$$F_{\mu} := g \circ h_Y: M_Y^{\text{ss}}(\mu) \rightarrow \mathbb{A}^1.$$

By taking the first cohomology of isomorphism (4.5), we obtain

$$\mathcal{H}^1(p_* \varphi_{F_{\mu} \circ p} \mathcal{I}\mathcal{C}_{\mathfrak{M}_Y^{\text{ss}}(\mu)}) \cong \varphi_{F_{\mu}} \mathcal{I}\mathcal{C}_{M_Y^{\text{ss}}(\mu)}.$$

Restricting it to the component $M_Y^{\text{ss}}(r, m) \subset M_Y^{\text{ss}}(\mu)$, we get the second isomorphism in Proposition 3.10. ■

Remark 4.7. It is clear from the proof that Proposition 3.10 naturally extends to an isomorphism of monodromic mixed Hodge modules.

5. The case of Higgs bundles

In this section, we prove the cohomological integrality theorem and the cohomological χ -independence theorem for Higgs bundle moduli spaces on curves using the dimensional reduction theorem due to the first author [37].

5.1. Dimensional reduction theorem

Let \mathfrak{Y} be a quasi-smooth derived Artin stack and $\mathbf{T}^*[-1]\mathfrak{Y}$ be its (-1) -shifted cotangent stack. We write $\mathfrak{Y} := t_0(\mathfrak{Y})$ and $\tilde{\mathfrak{Y}} := t_0(\mathbf{T}^*[-1]\mathfrak{Y})$, and let $\pi: \tilde{\mathfrak{Y}} \rightarrow \mathfrak{Y}$ be the natural projection. As we have seen in Example A.3, $\mathbf{T}^*[-1]\mathfrak{Y}$ carries a natural (-1) -shifted symplectic structure. Further, as was proved in [55, Lemma 3.3.3], there exists a natural orientation

$$o: \det \mathbb{L}_{\mathbf{T}^*[-1]\mathfrak{Y}|\mathfrak{Y}^{\text{red}}} \cong (\pi^{\text{red}})^* \det(\mathbb{L}_{\mathfrak{Y}})^{\otimes 2}. \tag{5.1}$$

We let $\varphi_{\mathbf{T}^*[-1]\mathfrak{Y}}$ denote the perverse sheaf on $\tilde{\mathfrak{Y}}$ recalled in Section 2.2 with respect to this (-1) -shifted symplectic structure and orientation. The following theorem is called the *dimensional reduction theorem*.

Theorem 5.1 ([37, Theorem 4.14]). *There exists a natural isomorphism in $D_c^b(\mathfrak{Y})$*

$$\pi_* \varphi_{\mathbf{T}^*[-1]\mathfrak{Y}} \cong \mathbb{D}\mathbb{Q}_{\mathfrak{Y}}[-\text{vdim } \mathfrak{Y}]. \tag{5.2}$$

Here $\text{vdim } \mathfrak{Y} := \text{rank } \mathbb{L}_{\mathfrak{Y}}$ denotes the virtual dimension of \mathfrak{Y} .

We now discuss the generalization of this theorem to the level of complexes in mixed Hodge modules. Firstly, we discuss the case when \mathfrak{Y} is a derived scheme. To specify that \mathfrak{Y} is schematic, we write $Y = \mathfrak{Y}$, $Y = \mathfrak{Y}$ and $\tilde{Y} = \tilde{\mathfrak{Y}}$. The following lemma is useful.

Lemma 5.2. *Let X be an algebraic variety and take complexes of mixed Hodge module $M, N \in D^b(\text{MHM}(X))$ such that there exists an isomorphism $\eta: \text{rat}(M) \cong \text{rat}(N)$ in $D^b(X)$. Assume that for each $i < 0$, the group $\text{Ext}^i(\text{rat}(M), \text{rat}(N))$ vanishes and we have an isomorphism of mixed Hodge structures $H^0(X, \mathcal{H}\text{om}(M, N)) \cong \mathbb{Q}$. Then η naturally extends to an isomorphism $M \cong N$ in $D^b(\text{MHM}(X))$.*

Proof. Consider the natural map of mixed Hodge complexes

$$\tau_{\leq 0} \text{RHom}(M, N) \rightarrow \text{RHom}(M, N).$$

The assumption implies an isomorphism $\mathbb{Q} \cong \tau_{\leq 0} \text{RHom}(M, N)$, hence we obtain a map in $D^b(\text{MMHM}(X))$

$$\mathbb{Q}_X \rightarrow \mathcal{H}\text{om}(M, N)$$

by adjunction. Then the following composition of morphisms in $D^b(\text{MHM}(X))$

$$M = M \otimes \mathbb{Q}_X \rightarrow M \otimes \mathcal{H}\text{om}(M, N) \rightarrow N$$

upgrades the isomorphism η up to scalar. ■

Proposition 5.3. *Assume that the virtual dimension $\mathrm{vdim} \mathbf{Y}$ is even. Then the dimensional reduction isomorphism (5.2) for a quasi-smooth derived scheme \mathbf{Y} naturally upgrades to an isomorphism in $D^b(\mathrm{MHM}(\mathbf{Y}))$:*

$$\pi_* \varphi_{\mathbf{T}^*[-1]\mathbf{Y}}^{\mathrm{mhm}} \cong \mathbb{L}^{\mathrm{vdim} \mathbf{Y}/2} \otimes \mathbb{D}\mathbb{Q}_{\mathbf{Y}}. \tag{5.3}$$

Proof. Using Lemma 5.2, we only need to prove that the mixed Hodge structure on

$$\mathrm{Hom}(\pi_* \varphi_{\mathbf{T}^*[-1]\mathbf{Y}}, \mathbb{D}\mathbb{Q}_{\mathbf{Y}}[-\mathrm{vdim} \mathbf{Y}])$$

is pure of weight zero. As this statement can be checked locally, using [5, Theorem 4.1], we may assume that there exists a smooth scheme U which admits a global étale coordinate, a trivial vector bundle E on U , and a section $s \in \Gamma(U, E)$ such that \mathbf{Y} is isomorphic to the derived zero locus $\mathbf{Z}(s)$. In this case, the proof of [37, Theorem 3.1] shows that the dimensional reduction isomorphism (5.2) can be identified with Davison’s local dimensional reduction theorem [11, Theorem A.1]. As the proof of this theorem works verbatim for complexes of mixed Hodge modules, we conclude that the claim holds. ■

Remark 5.4. We expect that isomorphism (5.3) further upgrades to an isomorphism $D^b(\mathrm{MMHM}(\mathbf{Y}))$

$$\pi_* \varphi_{\mathbf{T}^*[-1]\mathbf{Y}}^{\mathrm{mmhm}} \cong \mathbb{L}^{\mathrm{vdim} \mathbf{Y}/2} \otimes \mathbb{D}\mathbb{Q}_{\mathbf{Y}}.$$

However, we could not prove this since we do not know whether the tensor-hom adjunction holds for monodromic mixed Hodge modules. Instead, we can easily see that we have an isomorphism in $D^b(\mathrm{MMHM}(\mathbf{Y}))$

$$\mathcal{H}(\pi_* \varphi_{\mathbf{T}^*[-1]\mathbf{Y}}^{\mathrm{mmhm}}) \cong \mathcal{H}(\mathbb{L}^{\mathrm{vdim} \mathbf{Y}/2} \otimes \mathbb{D}\mathbb{Q}_{\mathbf{Y}})$$

since the monodromy operator acts trivially on both sides (see [12, Remark 3.9]). It is enough for our purposes.

Now we discuss the stacky case of this proposition. Let \mathfrak{Y} be a quasi-smooth derived Artin stack such that its classical truncation $\mathfrak{Y} = t_0(\mathfrak{Y})$ is of the form $[Y/G]$ for some scheme Y and a linear algebraic group G . In this case, we can upgrade the dimensional reduction theorem to an isomorphism of mixed Hodge structures.

Proposition 5.5. *Assume that $\mathrm{vdim} \mathfrak{Y}$ is even. Then the dimensional reduction isomorphism $H^*(\tilde{\mathfrak{Y}}, \varphi_{\mathbf{T}^*[-1]\mathfrak{Y}}) \cong H_{-*+\mathrm{vdim} \mathfrak{Y}}^{\mathrm{BM}}(\mathfrak{Y})$ upgrades naturally to an isomorphism of mixed Hodge structures*

$$H^*(\tilde{\mathfrak{Y}}, \varphi_{\mathbf{T}^*[-1]\mathfrak{Y}}^{\mathrm{mhm}}) \cong \mathbb{L}^{\mathrm{vdim} \mathfrak{Y}/2} \otimes H_{-*}^{\mathrm{BM}}(\mathfrak{Y}).$$

Proof. For a fixed i , take a smooth morphism $q: T \rightarrow \mathfrak{Y}$ of relative dimension d such that

$$q^!: H_{-i+\mathrm{vdim} \mathfrak{Y}}^{\mathrm{BM}}(\mathfrak{Y}) \rightarrow H_{-i+\mathrm{vdim} \mathfrak{Y}+2d}^{\mathrm{BM}}(T)$$

and the map

$$\tilde{q}^!: H^i(\tilde{\mathfrak{Y}}, \varphi_{\mathbf{T}^*[-1]\mathfrak{Y}}) \rightarrow H^i(\tilde{T}, \tilde{q}^* \varphi_{\mathbf{T}^*[-1]\mathfrak{Y}})$$

are isomorphisms. Here $\tilde{q}: \tilde{T} \rightarrow \tilde{\mathfrak{Y}}$ is the base change of q . Therefore, we need to show that the following composition of isomorphisms of vector spaces

$$H^i(\tilde{T}, \tilde{q}^* \varphi_{T^*[-1]\mathfrak{Y}}) \cong H^i(\tilde{\mathfrak{Y}}, \varphi_{T^*[-1]\mathfrak{Y}}) \cong H_{-i+\text{vdim } \mathfrak{Y}}^{\text{BM}}(\mathfrak{Y}) \cong H_{-i+\text{vdim } \mathfrak{Y}+2d}^{\text{BM}}(T)$$

upgrades to an isomorphism of mixed Hodge structures. To prove this, we will show that the following morphism in $D^b(T)$

$$\begin{aligned} \eta_q: \pi_{T*} \tilde{q}^* \varphi_{T^*[-1]\mathfrak{Y}} &\cong q^* \pi_{\mathfrak{Y}*} \varphi_{T^*[-1]\mathfrak{Y}} \cong q^* \mathbb{D}\mathbb{Q}_{\mathfrak{Y}}[-\text{vdim } \mathfrak{Y}] \\ &\cong \mathbb{D}\mathbb{Q}_T[-\text{vdim } \mathfrak{Y} - 2d] \end{aligned}$$

upgrades to an isomorphism in $D^b(\text{MHM}(T))$. Here $\pi_T: \tilde{T} \rightarrow T$ and $\pi_{\mathfrak{Y}}: \tilde{\mathfrak{Y}} \rightarrow \mathfrak{Y}$ are natural projections, and the second isomorphism is the dimensional reduction isomorphism. Using Lemma 5.2, we need to show that the mixed Hodge structure of

$$\text{Hom}(\pi_{T*} \tilde{q}^* \varphi_{T^*[-1]\mathfrak{Y}}, \mathbb{D}\mathbb{Q}_T[-\text{vdim } \mathfrak{Y} - 2d]) \tag{5.4}$$

is of zero weight. To prove this, take a smooth surjective morphism from a derived scheme $h: U \rightarrow \mathfrak{Y}$ such that $\text{vdim } U$ is even. Write $h = t_0(h)$, $t_0(U) = U$, and $\tilde{X} := t_0(T^*[-1]U)$. Let $\widetilde{T \times_{\mathfrak{Y}} U}$ be the fiber product $(T \times_{\mathfrak{Y}} U) \times_U \tilde{U}$. We let

$$\pi_{T \times_{\mathfrak{Y}} U}: \widetilde{T \times_{\mathfrak{Y}} U} \rightarrow T \times_{\mathfrak{Y}} U \quad \text{and} \quad \pi_U: \tilde{U} \rightarrow U$$

denote the natural projections, and let $q_U: T \times_{\mathfrak{Y}} U \rightarrow U$ and $\tilde{q}_U: \widetilde{T \times_{\mathfrak{Y}} U} \rightarrow \tilde{U}$ be the base changes of q . Then we can construct a natural isomorphism

$$\eta_{q_U}: \pi_{T \times_{\mathfrak{Y}} U*} \tilde{q}_U^* \varphi_{T^*[-1]U} \cong \mathbb{D}\mathbb{Q}_{T \times_{\mathfrak{Y}} U}[-\text{vdim } U - 2d]$$

in the same manner as η_q . As we have seen in Proposition 5.3, the map η_{q_U} upgrades to an isomorphism in $D^b(\text{MHM}(T \times_{\mathfrak{Y}} U))$,

$$\pi_{T \times_{\mathfrak{Y}} U*} \tilde{q}_U^* \varphi_{T^*[-1]U}^{\text{mhm}} \cong \mathbb{L}^{d+\text{vdim } U/2} \otimes \mathbb{D}\mathbb{Q}_{T \times_{\mathfrak{Y}} U}.$$

Let $h_{\tilde{\mathfrak{Y}}}: \tilde{U} \rightarrow \tilde{\mathfrak{Y}}$, $h_T: T \times_{\mathfrak{Y}} U \rightarrow T$ and $h_{T \times_{\mathfrak{Y}} U}: \widetilde{T \times_{\mathfrak{Y}} U} \rightarrow \tilde{T}$ be the base changes of h . Then we have a natural isomorphism

$$h_T^* \pi_{T*} \tilde{q}^* \varphi_{T^*[-1]\mathfrak{Y}}^{\text{mhm}} \cong \pi_{T \times_{\mathfrak{Y}} U*} \tilde{q}_U^* h_{\tilde{\mathfrak{Y}}}^* \varphi_{T^*[-1]\mathfrak{Y}}^{\text{mhm}} \cong \mathbb{L}^{\dim h/2} \otimes \pi_{T \times_{\mathfrak{Y}} U*} \tilde{q}_U^* \varphi_{T^*[-1]U}^{\text{mhm}},$$

where the latter isomorphism follows from [37, Proposition 4.10]. We also have a natural isomorphism

$$\mathbb{L}^{d+\text{vdim } \mathfrak{Y}/2} \otimes h_T^* \mathbb{D}\mathbb{Q}_T \cong \mathbb{L}^{d+h/2+\text{vdim } U/2} \otimes \mathbb{D}\mathbb{Q}_{T \times_{\mathfrak{Y}} U}.$$

Under these identifications, the proof of [37, Theorem 4.14] implies that $h_T^* \eta_q$ is equal to $\eta_{q_U}[-\dim h]$ up to a certain choice of the sign. This and the fact that η_{q_U} upgrades to an isomorphism in $D^b(\text{MHM}(T \times_{\mathfrak{Y}} U))$ imply that the weight of the mixed Hodge structure (5.4) is zero. ■

The following statement can be proved in the same manner as the previous proposition.

Proposition 5.6. *We keep the notation from the previous proposition. Let $p: \mathfrak{Y} \rightarrow Z$ be a morphism to a separated finite type complex scheme. Then we have an isomorphism of mixed Hodge modules*

$$\mathcal{H}((p \circ \pi_{\mathfrak{Y}})_* \varphi_{\mathbf{T}^*[-1]\mathfrak{Y}}^{\text{mhm}}) \cong \mathbb{L}^{\text{vdim } \mathfrak{Y}/2} \otimes \mathcal{H}(p_* \mathbb{D}\mathbb{Q}_{\mathfrak{Y}}), \tag{5.5}$$

where $\pi_{\mathfrak{Y}}: \tilde{\mathfrak{Y}} \rightarrow \mathfrak{Y}$ is the natural projection.

Remark 5.7. The argument as in Remark 5.4 implies that isomorphism (5.5) upgrades to an isomorphism in $D^b(\text{MMHM}(Z))$

$$\mathcal{H}((p \circ \pi_{\mathfrak{Y}})_* \varphi_{\mathbf{T}^*[-1]\mathfrak{Y}}^{\text{mmhm}}) \cong \mathbb{L}^{\text{vdim } \mathfrak{Y}/2} \otimes \mathcal{H}(p_* \mathbb{D}\mathbb{Q}_{\mathfrak{Y}}).$$

5.2. BPS cohomology for Higgs bundles

In [10], Davison defined BPS sheaves and BPS cohomology for preprojective algebras. In this section, we introduce Higgs counterpart of these notions.

Let C be a smooth projective curve of genus g . We write $S = \text{Tot}_C(\omega_C)$ and $X = \text{Tot}_C(\mathcal{O}_C \oplus \omega_C)$. Recall that $\mathfrak{M}_X^{\text{ss}}(r, m)$ (resp. $\mathfrak{M}_S^{\text{ss}}(r, m)$) denotes the moduli stack of one-dimensional semistable sheaves of rank r and Euler characteristic m on X (resp. S), and $M_X^{\text{ss}}(r, m)$ (resp. $M_S^{\text{ss}}(r, m)$) denotes the good moduli space of $\mathfrak{M}_X^{\text{ss}}(r, m)$ (resp. $\mathfrak{M}_S^{\text{ss}}(r, m)$). We have the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{M}_X^{\text{ss}}(r, m) & \xrightarrow{\pi} & \mathfrak{M}_S^{\text{ss}}(r, m) \\ \downarrow p_X & & \downarrow p_S \\ M_X^{\text{ss}}(r, m) & \xrightarrow{\bar{\pi}} & M_S^{\text{ss}}(r, m). \end{array}$$

It is shown in [37, Theorem 5.1] that there exists a natural equivalence of (-1) -shifted symplectic derived Artin stacks $\mathfrak{M}_X^{\text{ss}}(r, m) \cong \mathbf{T}^*[-1]\mathfrak{M}_S^{\text{ss}}(r, m)$, where $\mathfrak{M}_X^{\text{ss}}(r, m)$ (resp. $\mathfrak{M}_S^{\text{ss}}(r, m)$) denotes the derived enhancement of $\mathfrak{M}_X^{\text{ss}}(r, m)$ (resp. $\mathfrak{M}_S^{\text{ss}}(r, m)$). Therefore, (5.1) implies that there exists a canonical orientation

$$o: (\pi^{\text{red},*} \det(\mathbb{L}\mathfrak{m}_S^{\text{ss}}(r, m)|_{\mathfrak{m}_X^{\text{ss}}(r, m)^{\text{red}}})^{\otimes 2} \cong K_{\mathfrak{m}_X^{\text{ss}}(r, m)}^{\text{vir}}.$$

On the other hand, we have seen in Proposition 3.6 that there exist a line bundle L with $\text{deg } L > 2g - 2$ and a function f on the moduli stack $\mathfrak{M}_{\text{Tot}_C(L)}^{\text{ss}}(r, m)$ of semistable sheaves on $\text{Tot}_C(L)$ such that there exists an equivalence of (-1) -shifted symplectic derived Artin stacks $\mathfrak{M}_X^{\text{ss}}(r, m) \cong \text{Crit}(f)$. Therefore, there exists an orientation

$$o': K_{\mathfrak{M}_{\text{Tot}(L)}(r, m)}|_{\mathfrak{m}_X^{\text{ss}}(r, m)^{\text{red}}}^{\otimes 2} \cong K_{\mathfrak{m}_X^{\text{ss}}(r, m)}^{\text{vir}}.$$

Proposition 5.8. *There exists an isomorphism of orientations $o \cong o'$.*

Proof. We have seen in Proposition 3.8 that there exists a trivialization

$$K_{\mathfrak{M}_{\text{Tot } C}^{\text{ss}}(r,m)} \cong \mathcal{O}_{\mathfrak{M}_{\text{Tot } C}^{\text{ss}}(r,m)}.$$

On the other hand, we also have a trivialization

$$\det(\mathbb{L}\mathfrak{m}_S^{\text{ss}}(r,m)|_{\mathfrak{m}_S^{\text{ss}}(r,m)^{\text{red}}}) \cong \mathcal{O}_{\mathfrak{m}_S^{\text{ss}}(r,m)^{\text{red}}}$$

since there exists an open immersion

$$\mathfrak{M}_S^{\text{ss}}(r,m) \hookrightarrow \mathbf{T}^*\mathfrak{M}_C,$$

where \mathfrak{M}_C denotes the moduli stack of coherent sheaves on C . Therefore, we need to show that the composition

$$\begin{aligned} \mathcal{O}_{\mathfrak{M}_X^{\text{ss}}(r,m)^{\text{red}}}^{\otimes 2} &\cong (\pi^{\text{red},*} \det(\mathbb{L}\mathfrak{m}_S^{\text{ss}}(r,m)|_{\mathfrak{m}_X^{\text{ss}}(r,m)^{\text{red}}}))^{\otimes 2} \cong K_{\mathfrak{M}_X^{\text{ss}}(r,m)^{\text{red}}}^{\text{vir}} \\ &\cong K_{\mathfrak{M}_{\text{Tot } C}^{\text{ss}}(r,m)}|_{\mathfrak{m}_X^{\text{ss}}(r,m)^{\text{red}}}^{\otimes 2} \cong \mathcal{O}_{\mathfrak{M}_X^{\text{ss}}(r,m)^{\text{red}}}^{\otimes 2} \end{aligned}$$

has a square root. More strongly, we will show that any invertible function

$$f \in \Gamma(\mathfrak{M}_X^{\text{ss}}(r,m)^{\text{red}}, \mathcal{O}_{\mathfrak{M}_X^{\text{ss}}(r,m)^{\text{red}}}^{\times}) \cong \Gamma(M_X^{\text{ss}}(r,m)^{\text{red}}, \mathcal{O}_{M_X^{\text{ss}}(r,m)^{\text{red}}}^{\times})$$

is constant, and hence admits a square root.

We say that a reduced finite type complex scheme X satisfies the property (P) if every invertible regular function on X is locally constant. What we need to prove is that the scheme $M_X^{\text{ss}}(r,m)^{\text{red}}$ satisfies the property (P). Property (P) satisfies the following:

- If we are given a surjective morphism between reduced finite type complex schemes $X \rightarrow Y$ and X satisfies the property (P), then Y satisfies the property (P).
- For reduced finite type complex schemes X and Y satisfying the property (P), $X \times Y$ also satisfies the property (P).

Write $k = \text{gcd}(r,m)$ and $(r,m) = (kr_0, km_0)$. Take a partition $\lambda_{\bullet} = (\lambda_1, \lambda_2, \dots, \lambda_t)$ of k , i.e., λ_i is a positive integer with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_t$ such that $\sum_i \lambda_i = k$ holds. Let $M_X^{\text{ss}}(r,m)_{\lambda_{\bullet}}^{\text{red}}$ be the subscheme consisting of points corresponding to polystable sheaves which can be written as

$$\bigoplus_i E_i,$$

where E_i is a stable sheaf on X such that $\text{rank}(\pi_{X*} E_i) = \lambda_i r_0$, where $\pi_X: X \rightarrow C$ is the projection. We let $\overline{M_X^{\text{ss}}(r,m)_{\lambda_{\bullet}}^{\text{red}}}$ denote the closure of $M_X^{\text{ss}}(r,m)_{\lambda_{\bullet}}^{\text{red}}$. Since we have an equality

$$M_X^{\text{ss}}(r,m)^{\text{red}} = \bigcup_{\lambda_{\bullet}} \overline{M_X^{\text{ss}}(r,m)_{\lambda_{\bullet}}^{\text{red}}},$$

we need to show that the scheme $\overline{M_X^{\text{ss}}(r,m)_{\lambda_{\bullet}}^{\text{red}}}$ satisfies the property (P).

Consider the map

$$\prod_{i=1}^t \overline{M_X^{ss}(\lambda_i r_0, \lambda_i m_0)_{(\lambda_i)}^{\text{red}}} \rightarrow M_X^{ss}(r, m)^{\text{red}}$$

taking the direct sum. The image of this map is nothing but $\overline{M_X^{ss}(r, m)_{\lambda}^{\text{red}}}$. Hence it is enough to show that $\overline{M_X^{ss}(\lambda_i r_0, \lambda_i m_0)_{(\lambda_i)}^{\text{red}}}$ satisfies the property (P). As we have an isomorphism

$$\overline{M_X^{ss}(\lambda_i r_0, \lambda_i m_0)_{(\lambda_i)}^{\text{red}}} \cong M_S^{ss}(\lambda_i r_0, \lambda_i m_0) \times \mathbb{A}^1,$$

we need to prove that $M_S^{ss}(\lambda_i r_0, \lambda_i m_0)$ satisfies property (P).

Let $g: M_S^{ss}(\lambda_i r_0, \lambda_i m_0) \rightarrow \mathbb{A}^1$ be an invertible function. We need to prove that g is constant. Let $h_S: M_S^{ss}(\lambda_i r_0, \lambda_i m_0) \rightarrow B_S$ be the Hitchin fibration. Since the general fiber of h_S is connected, we have an isomorphism $h_{S,*} \mathcal{O}_{M_S^{ss}(\lambda_i r_0, \lambda_i m_0)} \cong \mathcal{O}_{B_S}$. Therefore, there exists an invertible function g' on B_S such that $g = g' \circ h_S$. Since B_S is an affine space, g' is a constant function. ■

Remark 5.9. The proof shows that any orientation $o'': L^{\otimes 2} \cong K_{\mathfrak{M}_X^{vir}(r, m)}^{\text{vir}}$ such that L is trivial is isomorphic to o .

From now, we always equip $\mathfrak{M}_X^{ss}(r, m)$ with the orientation o . Define a monodromic mixed Hodge module $\varphi_{M_X^{ss}(r, m)}^{\text{mmhm}}$ on $M_X^{ss}(r, m)$ by

$$\varphi_{M_X^{ss}(r, m)}^{\text{mmhm}} := \mathcal{H}^0(\mathbb{L}^{-1/2} \otimes p_X * \varphi_{\mathfrak{M}_X^{ss}(r, m)}^{\text{mmhm}}).$$

For a given rational number μ , we define

$$M_X^{ss}(\mu) := \coprod_{m/r=\mu} M_X^{ss}(r, m),$$

$$\varphi_{\mathfrak{M}_X^{ss}(\mu)}^{\text{mmhm}} := \bigoplus_{m/r=\mu} \varphi_{\mathfrak{M}_X^{ss}(r, m)}^{\text{mmhm}}, \quad \varphi_{M_X^{ss}(\mu)}^{\text{mmhm}} := \bigoplus_{m/r=\mu} \varphi_{M_X^{ss}(r, m)}^{\text{mmhm}}.$$

Recall that we have constructed a symmetric monoidal structure \boxtimes_{\oplus} on the category $D^{\geq, lf}(\text{MMHM}(M_X^{ss}(\mu)))$ in Section 4.2. The following proposition is the cohomological integrality theorem (in the sense of [17, Theorem A]) for the Calabi–Yau threefold X .

Proposition 5.10. *We have an isomorphism*

$$\mathcal{H}(p_X * \varphi_{\mathfrak{M}_X^{ss}(\mu)}^{\text{mmhm}}) \cong \text{Sym}_{\boxtimes_{\oplus}}(\mathbb{H}^*(\text{BC}^*)_{\text{vir}} \otimes \varphi_{M_X^{ss}(\mu)}^{\text{mmhm}})$$

in $D^{\geq, lf}(\text{MMHM}(M_X^{ss}(\mu)))$.

Proof. Using Proposition 5.8, we may use the orientation o' instead of o . Then the claim follows from Proposition 3.10 and Theorem 4.6. ■

Now we state the Higgs version of the support lemma [10, Lemma 4.1].

Proposition 5.11. *Let $\ell: M_S^{\text{ss}}(r, m) \times \mathbb{A}^1 \rightarrow M_X^{\text{ss}}(r, m)$ be the map given by*

$$M_S^{\text{ss}}(r, m) \times \mathbb{A}^1 \ni ([E], t) \mapsto [i_{t*}E] \in M_X^{\text{ss}}(r, m),$$

where i_t is the composition $S = S \times \{t\} \hookrightarrow X$. Then the support of the perverse sheaf $\varphi_{M_X^{\text{ss}}(r, m)}$ is contained in the image of ℓ .

The proof will be given in Appendix B.

Proposition 5.12. *The monodromic mixed Hodge module $\varphi_{M_X^{\text{ss}}(r, m)}^{\text{mmhm}}$ is \mathbb{A}^1 -equivariant with respect to the natural \mathbb{A}^1 -action on $M_X^{\text{ss}}(r, m)$. Further, there exists a monodromic mixed Hodge module $\mathcal{BPS}_{r, m} \in \text{MMHM}(M_S^{\text{ss}}(r, m))$ such that*

$$\varphi_{M_X^{\text{ss}}(r, m)}^{\text{mmhm}} \cong \mathbb{L}^{-1/2} \otimes l_* \text{pr}_1^* \mathcal{BPS}_{r, m},$$

where $\text{pr}_1: M_S^{\text{ss}}(r, m) \times \mathbb{A}^1 \rightarrow M_S^{\text{ss}}(r, m)$ is the projection.

Proof. In general, a monodromic mixed Hodge module M on $T \times \mathbb{A}^1$ for an algebraic variety T is \mathbb{A}^1 -equivariant if and only if the counit map $\text{pr}_1^* \text{pr}_{1*} M \rightarrow M$ is isomorphic, where $\text{pr}_1: T \times \mathbb{A}^1 \rightarrow T$ is the first order projection. Therefore, the \mathbb{A}^1 -equivariance of M is equivalent to the \mathbb{A}^1 -equivariance of $\text{rat}(M)$. This is further equivalent to the condition $\sigma_1^* \text{rat}(M) \cong \text{rat}(M)$, where $\sigma_1: T \times \mathbb{A}^1 \cong T \times \mathbb{A}^1$ is the map translating in the \mathbb{A}^1 -direction by $1 \in \mathbb{A}^1$.

Now we return to the proposition. Let $\sigma_1: \mathfrak{M}_X^{\text{ss}}(r, m) \cong \mathfrak{M}_X^{\text{ss}}(r, m)$ be the map induced by the translation map on $X = S \times \mathbb{A}^1$ in the \mathbb{A}^1 -direction by $1 \in \mathbb{A}^1$. We need to show that there exists an isomorphism of perverse sheaves

$$\sigma_1^* \varphi_{\mathfrak{M}_X^{\text{ss}}(r, m)} \cong \varphi_{\mathfrak{M}_X^{\text{ss}}(r, m)}.$$

To do this, it is enough to show that there exists an isomorphism of orientations $\sigma_1^* o \cong o$, where o is the natural orientation on $\mathfrak{M}_X^{\text{ss}}(r, m)$. But this is a consequence of Remark 5.9. The latter statement follows from Proposition 5.11. ■

The object $\mathcal{BPS}_{r, m} \in \text{MMHM}(M_S^{\text{ss}}(r, m))$ is called the *BPS sheaf*. We will see that it is a pure Hodge module in the next subsection. We write

$$\text{BPS}_{r, m} := H^*(M_S^{\text{ss}}(r, m), \mathcal{BPS}_{r, m}),$$

and it is called the *BPS cohomology*.

5.3. Cohomological integrality and χ -independence for Higgs bundles

In this section, we prove the χ -independence theorem and cohomological integrality theorem for Higgs bundles using the dimensional reduction theorem.

We first need the following lemma.

Lemma 5.13. *The map $b|_{\text{im}(h_X)}: \text{im}(h_X) \rightarrow B_Y$ considered in Lemma 3.9 is injective for $X = \text{Tot}_C(\mathcal{O}_C \oplus \omega_C)$.*

Proof. Let $\gamma, \gamma' \in \text{im}(h_X)$ be cycles on X such that the pushforward cycles $\sigma_*\gamma$ and $\sigma_*\gamma'$ define the same cycle on $Y = \text{Tot}_C(L)$, where σ is the projection from X to Y . We want to show $\gamma = \gamma'$. Write

$$\gamma = \sum_i \gamma_{t_i}, \quad \gamma' = \sum_i \gamma'_{t'_i},$$

where γ_{t_i} is supported on $S \times \{t_i\} \subset X$ and similarly for $\gamma'_{t'_i}$. Now we take a point $p \in \text{Supp}(\text{coker}(\omega_C \hookrightarrow L))$. Then the restriction of σ at the fiber of p is given by

$$\mathbb{A}^1 \oplus \omega_C|_p \rightarrow \mathbb{A}^1 \cong L_p,$$

where the first map is the projection to the first factor and the latter map is induced from the composition $\mathcal{O}_C \hookrightarrow \mathcal{O}_C \oplus \omega_C \rightarrow L$. Therefore, the cycle $\sigma_*\gamma_{t_i}|_{L_p}$ is concentrated in $\{t_i\} \subset \mathbb{A}^1 \cong L_p$. Hence we may assume that γ and γ' are contained in $S \times \{t\}$ for some $t \in \mathbb{A}^1$. Then the claim follows since the map $S \times \{t\} \subset X \rightarrow Y$ defines an injection on the set of cycles. ■

The following corollary is an immediate consequence of isomorphism (3.11) and the above lemma.

Corollary 5.14. *Let us take integers r, m, m' such that $r > 0$. Then there exists an isomorphism in $D^b(\text{MMHM}(B_X))$,*

$$h_{X*}\varphi_{M_X^{\text{ss}}(r,m)}^{\text{mmhm}} \cong h_{X*}\varphi_{M_X^{\text{ss}}(r,m')}^{\text{mmhm}}.$$

Corollary 5.15. *Let r, m, m' be as in the previous corollary. Then there exists an isomorphism in $D^b(\text{MMHM}(B_S))$,*

$$h_{S*} \mathcal{BPS}_{r,m} \cong h_{S*} \mathcal{BPS}_{r,m'}.$$

We now prove the cohomological integrality theorem for Higgs bundles. Recall that we have the following diagram:

$$\begin{array}{ccc} \mathfrak{M}_X^{\text{ss}}(\mu) & \xrightarrow{\pi} & \mathfrak{M}_S^{\text{ss}}(\mu) \\ \downarrow p_X & & \downarrow p_S \\ M_X^{\text{ss}}(\mu) & \xrightarrow{\bar{\pi}} & M_S^{\text{ss}}(\mu). \end{array}$$

For a rational number μ , we write

$$\mathcal{BPS}_\mu := \bigoplus_{m/r=\mu} \mathcal{BPS}_{r,m}.$$

Theorem 5.16. *The monodromic mixed Hodge module \mathcal{BPS}_μ is contained in the category $\text{MHM}(M_S^{\text{ss}}(\mu))$, i.e., it has a trivial monodromy operator. Further, we have an isomorphism*

$$\bigoplus_{m/r=\mu} \mathcal{H}(p_{S*} \mathbb{D}\mathbb{Q}\mathfrak{M}_S^{\text{ss}}(r,m)) \otimes \mathbb{L}^{r^2(g-1)} \cong \text{Sym}_{\boxplus}(\mathbb{H}^*(\text{BC}^*) \otimes \mathcal{BPS}_\mu) \quad (5.6)$$

in $D^{\geq,lf}(\text{MHM}(M_S^{\text{ss}}(\mu)))$.

Proof. Proposition 4.4 and Proposition 5.10 imply isomorphisms

$$\begin{aligned} \mathcal{H}((p_S \circ \pi)_* \varphi_{\mathfrak{M}_X^{\text{ss}}(\mu)}^{\text{mmhm}}) &\cong \mathcal{H}((\bar{\pi} \circ p_X)_* \varphi_{\mathfrak{M}_X^{\text{ss}}(\mu)}^{\text{mmhm}}) \\ &\cong \mathcal{H}(\bar{\pi}_* \mathcal{H}(p_X_* \varphi_{\mathfrak{M}_X^{\text{ss}}(\mu)}^{\text{mmhm}})) \\ &\cong \mathcal{H}(\bar{\pi}_* \text{Sym}_{\boxtimes_{\oplus}}(\mathbb{H}^*(\mathbb{B}\mathbb{C}^*)_{\text{vir}} \otimes \varphi_{M_X^{\text{ss}}(\mu)}^{\text{mmhm}})) \\ &\cong \text{Sym}_{\boxtimes_{\oplus}}(\mathbb{H}^*(\mathbb{B}\mathbb{C}^*) \otimes \mathcal{BPS}_{\mu}). \end{aligned}$$

As we have seen in Remark 5.7, the left-hand side is monodromy-free, hence so is the BPS sheaf. Isomorphism (5.6) follows from the above isomorphism and an isomorphism

$$\mathcal{H}((p_S \circ \pi)_* \varphi_{\mathfrak{M}_X^{\text{ss}}(\mu)}^{\text{mmhm}}) \cong \bigoplus_{m/r=\mu} \mathcal{H}(p_{S*} \mathbb{D}\mathbb{Q}\mathfrak{M}_S^{\text{ss}}(r,m)) \otimes \mathbb{L}^{r^2(g-1)}$$

which is a consequence of Proposition 5.6 and the equality

$$\text{vdim } \mathfrak{M}_S^{\text{ss}}(r,m) = 2r^2(g-1). \quad \blacksquare$$

Corollary 5.17. *The mixed Hodge module \mathcal{BPS}_{μ} is pure.*

Proof. The above theorem implies that there exists an embedding

$$\mathcal{BPS}_{r,m} \hookrightarrow \mathcal{H}(p_{S*} \mathbb{D}\mathbb{Q}\mathfrak{M}_S^{\text{ss}}(r,m)) \otimes \mathbb{L}^{r^2(g-1)}.$$

The purity of the right-hand side is proved in [14, Proposition 7.20], so we obtain the claim. ■

Example 5.18. Assume that (r,m) is coprime, in which case $\mathfrak{M}_S^{\text{ss}}(r,m)$ is smooth and p_S is a \mathbb{C}^* -gerbe. In this case, we have an isomorphism

$$\mathcal{H}^0(p_{S*} \mathbb{D}\mathbb{Q}\mathfrak{M}_S^{\text{ss}}(r,m)) \otimes \mathbb{L}^{r^2(g-1)} \cong \mathcal{IC}_{M_S^{\text{ss}}(r,m)}.$$

Therefore, we have an isomorphism

$$\mathcal{IC}_{M_S^{\text{ss}}(r,m)} \cong \mathcal{BPS}_{r,m}.$$

In particular, for coprime pairs (r,m) and (r,m') , Corollary 5.15 implies an isomorphism

$$h_{S*} \mathcal{IC}_{M_S^{\text{ss}}(r,m)} \cong h_{S*} \mathcal{IC}_{M_S^{\text{ss}}(r,m')}.$$

Now let (r,m) be a non-coprime pair. It follows from [53, Theorem 11.1] and [14, Theorem 5.11] that $M_S^{\text{ss}}(r,m)$ is normal. The connectedness of $M_S^{\text{ss}}(r,m)$ is proved in [22, Claim 3.5 (iii)]. Therefore, the moduli space $M_S^{\text{ss}}(r,m)$ is irreducible. Then using [14, Theorem 6.6], we can construct an inclusion

$$\mathcal{IC}_{M_S^{\text{ss}}(r,m)} \hookrightarrow \mathcal{BPS}_{r,m} \tag{5.7}$$

but it is not necessary an isomorphism (see Section 5.4).

Write B_S^* for the disjoint union of all the Hitchin bases (i.e., B_S^* is the moduli space of all one-dimensional cycles on S). We let

$$\oplus_B: B_S^* \times B_S^* \rightarrow B_S^* \quad \text{and} \quad +: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$$

denote the canonical monoid structures. The following statement is a direct consequence of Theorem 5.16 and Lemma 4.2.

Corollary 5.19. *We have isomorphisms*

$$\bigoplus_{m/r=\mu} \mathcal{H}((h_S \circ p_S)_* \mathbb{D}\mathbb{Q}\mathfrak{M}_S^{\text{ss}}(r,m)) \otimes \mathbb{L}^{r^2(g-1)} \cong \text{Sym}_{\boxtimes_{\oplus_B}}(\mathbb{H}^*(\text{BC}^*) \otimes h_{S*} \mathcal{BPS}_\mu),$$

$$\bigoplus_{m/r=\mu} \mathbb{H}^{\text{BM}}(\mathfrak{M}_S^{\text{ss}}(r,m)) \otimes \mathbb{L}^{r^2(g-1)} \cong \text{Sym}_{\boxtimes_+}(\mathbb{H}^*(\text{BC}^*) \otimes \text{BPS}_\mu)$$

in $D^{\geq,lf}(\text{MHM}(B_S^*))$ and $D^{\geq,lf}(\text{MHM}(\mathbb{N}))$, respectively.

Combining the above corollary and the χ -independence theorem for BPS cohomology (Corollary 5.14), we obtain the following χ -independence theorem for the Borel–Moore homology.

Corollary 5.20. *Let r, m, m' be integers such that $r > 0$ and $\text{gcd}(r, m) = \text{gcd}(r, m')$. Then there exist isomorphisms*

$$\mathcal{H}((h_S \circ p_S)_* \mathbb{D}\mathbb{Q}\mathfrak{M}_S^{\text{ss}}(r,m)) \cong \mathcal{H}((h_S \circ p_S)_* \mathbb{D}\mathbb{Q}\mathfrak{M}_S^{\text{ss}}(r,m')),$$

$$\mathbb{H}^{\text{BM}}(\mathfrak{M}_S^{\text{ss}}(r,m)) \cong \mathbb{H}^{\text{BM}}(\mathfrak{M}_S^{\text{ss}}(r,m'))$$

in $D^{\geq,lf}(\text{MHM}(B_S))$ and $D^{\geq,lf}(\text{MHM}(\text{pt}))$, respectively.

Remark 5.21. Based on $P = W$ conjecture, it is conjectured in [26] that there exists an isomorphism of intersection cohomology groups

$$\text{IH}(M_S^{\text{ss}}(r,m)) \cong \text{IH}(M_S^{\text{ss}}(r,m'))$$

preserving the perverse filtration for r, m, m' such that $\text{gcd}(r, m) = \text{gcd}(r, m')$. At present, we do not know how to prove this conjecture. However, once Davison’s conjecture [13, Conjecture 7.7] on the structure of the BPS sheaf is established, it would be possible to deduce the χ -independence for intersection cohomology from the χ -independence for BPS cohomology (Corollary 5.14).

5.4. An example: $g = 2, r = 2$

Here we give an example where the intersection cohomology and the BPS cohomology are different. Let C be a smooth projective curve of genus 2, and put $S := \text{Tot}_C(\omega_C)$. We consider the moduli space $M_S(2, 0)$. By taking the cohomology of inclusion (5.7), we have an inclusion

$$\text{IH}(M_S(2, 0)) \hookrightarrow \text{BPS}_{2,0}. \tag{5.8}$$

We will check that the above inclusion is not an isomorphism. Note that by Corollary 5.15 and Example 5.18, we have an isomorphism

$$\text{BPS}_{2,0} \cong \text{BPS}_{2,1} \cong \text{IH}(M_S(2, 1)). \tag{5.9}$$

We denote

$$\begin{aligned} \Phi_{\text{IC}}(t) &:= \sum_{i \in \mathbb{Z}} h^i(\text{IH}(M_S(2, 0)))t^i, \\ \Phi_{\text{BPS}}(t) &:= \sum_{i \in \mathbb{Z}} h^i(\text{BPS}_{2,0})t^i. \end{aligned}$$

By [24, Theorem 1.2] and [48, Exercise 4.1], we have

$$\begin{aligned} \Phi_{\text{IC}}(t) &= t^{-10}(t + 1)^4(2t^6 + 2t^4 + t^2 + 1), \\ \Phi_{\text{BPS}}(t) &= t^{-10}(t + 1)^4(2t^6 + 4t^5 + 2t^4 + 4t^3 + t^2 + 1). \end{aligned} \tag{5.10}$$

For the formula of Φ_{BPS} , we used isomorphisms (5.9). Note that the term t^{-10} appears by our shift convention so that the intersection and the BPS complexes are perverse sheaves, together with the fact $\dim M_S(2, 0) = 10$. From formulas (5.10), it is obvious that inclusion (5.8) is not an isomorphism.

On the other hand, assume that [13, Conjecture 7.7] is true. Then we have an isomorphism

$$\mathcal{BPS}_{2,0} \cong \mathcal{IC}_{M_S(2,0)} \oplus \wedge_{\oplus}^2(\mathcal{IC}_{M_S(1,0)}),$$

where $\wedge^2(-)$ is the graded wedge product in the category of graded vector spaces. Since the moduli space $M_S(1, 0)$ is isomorphic to the cotangent bundle of the Jacobian $\text{Jac}(C)$ of C , we have

$$\text{IH}(M_S(1, 0)) \cong \text{H}^*(\text{Jac}(C))[4] = \mathbb{Q}[4] \oplus \mathbb{Q}^4[3] \oplus \mathbb{Q}^6[2] \oplus \mathbb{Q}^4[1] \oplus \mathbb{Q},$$

and hence we conclude that

$$\sum_{i \in \mathbb{Z}} h^i(\wedge^2(\text{IH}(M_S(1, 0))))t^i = t^{-7}(t + 1)^4(t^2 + 1) = \Phi_{\text{BPS}}(t) - \Phi_{\text{IC}}(t).$$

This computation gives an evidence of [13, Conjecture 7.7]. At the same time, we can see that the χ -independence for the intersection cohomology does not necessarily hold when $\gcd(r, m) \neq \gcd(r, m')$.

Appendix A. Shifted symplectic structure and vanishing cycles

In this appendix, we briefly recall the theory of shifted symplectic geometry and prove some technical lemmas including Proposition 2.4.

A.1. *Shifted symplectic structures*

We recall the notion of shifted symplectic structures introduced in [47]. Let \mathfrak{X} be a derived Artin stack. We define the space of n -shifted p -forms $\mathcal{A}^p(\mathfrak{X}, n) \in \mathbb{S}$ by

$$\mathcal{A}^p(\mathfrak{X}, n) := \text{Map}(\mathcal{O}_{\mathfrak{X}}, \wedge^p \mathbb{L}_{\mathfrak{X}}[n]).$$

We can also define the space of n -shifted closed p -forms $\mathcal{A}^{p,\text{cl}}(\mathfrak{X}, n) \in \mathbb{S}$ (see [47, Definition 1.12]). It satisfies the étale descent and for a connective commutative differential graded algebra A , we have an equivalence

$$\mathcal{A}^{p,\text{cl}}(\text{Spec } A, n) \simeq \left| \prod_{i \geq 0} \wedge^{p+i} \mathbb{L}_A[-i+n], d + d_{\text{dR}} \right|,$$

where d is the internal differential, d_{dR} is the de Rham differential, and $| - |$ is the geometric realization functor. The space of n -shifted p -forms and the space of n -shifted closed p -forms are functorial with respect to morphisms between derived Artin stacks, i.e., if we are given a morphism $f: \mathfrak{X} \rightarrow \mathfrak{Y}$, there exist natural maps

$$f^*: \mathcal{A}^p(\mathfrak{Y}, n) \rightarrow \mathcal{A}^p(\mathfrak{X}, n), \quad f^*: \mathcal{A}^{p,\text{cl}}(\mathfrak{Y}, n) \rightarrow \mathcal{A}^{p,\text{cl}}(\mathfrak{X}, n).$$

We have a natural forgetful map

$$\pi: \mathcal{A}^{p,\text{cl}}(-, n) \rightarrow \mathcal{A}^p(-, n)$$

and the de Rham differential map

$$d_{\text{dR}}^{\text{cl}}: \mathcal{A}^p(-, n) \rightarrow \mathcal{A}^{p+1,\text{cl}}(-, n).$$

Definition A.1. An n -shifted closed 2-form $\omega_{\mathfrak{X}}$ is called an n -shifted symplectic form if its underlying n -shifted 2-form is non-degenerate, i.e., the natural map

$$\pi(\omega_{\mathfrak{X}}): \mathbb{L}_{\mathfrak{X}}^{\vee} \rightarrow \mathbb{L}_{\mathfrak{X}}[n]$$

is an equivalence.

In this paper, we are only interested in (-1) -shifted symplectic structures.

Example A.2. Let X be a Calabi–Yau threefold, i.e., a three-dimensional smooth variety with trivial canonical bundle. Then the derived moduli stack \mathfrak{M}_X of compactly supported coherent sheaves on X carries a canonical (-1) -shifted symplectic structure. See [47, Theorem 0.1] and [6, Main theorem].

Example A.3. Let \mathfrak{Y} be a derived Artin stack and

$$\mathbf{T}^*[n]\mathfrak{Y} := \text{Spec}_{\mathfrak{Y}}(\text{Sym}(\mathbb{L}_{\mathfrak{Y}}^{\vee}[-n]))$$

be its n -shifted cotangent stack. Let $\lambda \in \mathcal{A}^1(\mathbf{T}^*[n]\mathfrak{Y}, n)$ be the tautological 1-form. Then it is shown in [47, Proposition 1.21] and [7, Theorem 2.2] that the n -shifted closed 2-form $d_{\text{dR}}^{\text{cl}}\lambda$ is shifted symplectic.

Example A.4. Let $U = \text{Spec } A$ be a smooth affine scheme which admits an étale coordinate (x_1, \dots, x_n) and $f: U \rightarrow \mathbb{A}^1$ be a regular function. Let B be the commutative differential graded algebra (cdga) defined by the Koszul complex

$$B := (\cdots \rightarrow \Omega_A^\vee \xrightarrow{d_{\text{dR}} f} A).$$

Then $\mathbf{Spec } B$ is equivalent to the derived critical locus $\mathbf{Crit}(f)$. We let $y_i \in B^{-1}$ be the element of degree -1 corresponding to $\partial/\partial x_i$. Then the (-1) -shifted closed 2-form

$$\omega := (d_{\text{dR}} x_1 \wedge d_{\text{dR}} y_1 + \cdots + d_{\text{dR}} x_n \wedge d_{\text{dR}} y_n, 0, 0, \dots) \in \mathcal{A}^{2,\text{cl}}(\mathbf{Spec } B, -1)$$

defines a (-1) -shifted symplectic structure on $\mathbf{Crit}(f)$.

It is shown in [5, Theorem 5.18] that any (-1) -shifted symplectic derived scheme is Zariski locally of this form.

Now we discuss the canonical (-1) -shifted symplectic structure on the derived critical locus of a function on a general derived Artin stack. To do this, we need to recall the notion of Lagrangian structures.

Definition A.5. Let $(\mathfrak{X}, \omega_{\mathfrak{X}})$ be an n -shifted symplectic derived Artin stack and $\tau: L \rightarrow \mathfrak{X}$ be a morphism of derived Artin stacks. An *isotropic structure* is a path from 0 to $\tau^* \omega_{\mathfrak{X}}$ in $\mathcal{A}^{2,\text{cl}}(L, n)$. An isotropic structure η is called a *Lagrangian structure* if it induces an equivalence $\mathbb{L}_L^\vee \simeq \mathbb{L}_\tau[n-1]$. See [47, §2.2] for details.

Example A.6. (1) Let \mathfrak{Y} be a derived Artin stack and $\lambda \in \mathcal{A}^1(\mathbf{T}^*[n]\mathfrak{Y}, n)$ be the tautological 1-form. Then $\lambda|_{\mathfrak{Y}}$ is naturally equivalent to zero, hence so is $d_{\text{dR}}^{\text{cl}} \lambda|_{\mathfrak{Y}}$. Therefore, the zero section map $\mathfrak{Y} \rightarrow \mathbf{T}^*[n]\mathfrak{Y}$ carries a natural isotropic structure. It is shown in [7, Theorem 2.2] that this isotropic structure is a Lagrangian structure.

(2) Let \mathfrak{Y} and λ be as above, and take a function $f \in \Gamma(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}[n])$ of degree n . Let $\overline{d_{\text{dR}} f}: \mathfrak{Y} \rightarrow \mathbf{T}^*[n]\mathfrak{Y}$ be the map corresponding to the section $d_{\text{dR}} f \in \Gamma(\mathfrak{Y}, \mathbb{L}_{\mathfrak{Y}}[n])$. Then the natural homotopy

$$(\overline{d_{\text{dR}} f})^* d_{\text{dR}}^{\text{cl}} \lambda \sim d_{\text{dR}}^{\text{cl}} \circ d_{\text{dR}} f \sim 0$$

defines an isotropic structure on $(\overline{d_{\text{dR}} f})$. It is shown in [7, Theorem 2.15] that this isotropic structure is a Lagrangian structure.

Let \mathfrak{X} be an n -shifted symplectic derived Artin stack and $\tau_1: L_1 \rightarrow \mathfrak{X}$ and $\tau_2: L_2 \rightarrow \mathfrak{X}$ be Lagrangians. These Lagrangian structures define a loop in $\mathcal{A}^{2,\text{cl}}(L_1 \times_{\mathfrak{X}} L_2, n)$ hence a point in $\mathcal{A}^{2,\text{cl}}(L_1 \times_{\mathfrak{X}} L_2, n-1)$. It is shown in [47, Theorem 2.9] that this $(n-1)$ -shifted closed 2-form is shifted symplectic.

Example A.7. Let \mathfrak{Y} be a derived Artin stack and $f \in \Gamma(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}[n])$ be a function of degree n . The derived critical locus $\mathbf{Crit}(f)$ is defined to be the intersection

$$\mathbf{Crit}(f) := \mathfrak{Y} \times_{0, \mathbf{T}^*[n]\mathfrak{Y}, \overline{d_{\text{dR}} f}} \mathfrak{Y}.$$

Example A.6 and the above discussion implies that $\mathbf{Crit}(f)$ carries a canonical $(n-1)$ -shifted symplectic structure.

The (-1) -shifted symplectic structure constructed in Example A.4 is a special case of the above example.

Lemma A.8. *Let $f: U \rightarrow \mathbb{A}^1$ be a regular function on a smooth affine scheme. Assume that U admits a global étale coordinate. Then the (-1) -shifted symplectic structure on $\mathbf{Crit}(f)$ constructed in Example A.4 is equivalent to the (-1) -shifted symplectic structure constructed in Example A.7.*

Proof. We write $U = \text{Spec } A$ and take a global étale coordinate $x_1, x_2, \dots, x_n \in A$. We let B be the cdga appeared in Example A.4 whose underlying graded algebra is $A[y_1, \dots, y_n]$. As we have seen in Example A.4, $\mathbf{Spec } B$ gives a model for $\mathbf{Crit}(f)$. Consider the element

$$\alpha = \sum_i y_i \cdot d_{\text{dR}} x_i \in \Omega_B^{-1}.$$

Then we have an identity $d\alpha = d_{\text{dR}} f$ which corresponds to the natural homotopy

$$d_{\text{dR}} f|_{\mathbf{Crit}(f)} \sim 0$$

in $\mathcal{A}^1(\mathbf{Crit}(f), 0)$. Therefore, the element

$$d_{\text{dR}}^{\text{cl}} \alpha = \left(\sum_i d_{\text{dR}} y_i \cdot d_{\text{dR}} x_i, 0, \dots \right)$$

corresponds to the (-1) -shifted symplectic structure constructed in Example A.7. ■

Now we discuss the relation of the (-1) -shifted symplectic structure and the d-critical structure. Let (\mathfrak{X}, ω) be a (-1) -shifted symplectic derived Artin stack. Then it is shown in [3, Theorem 3.18 (a)] that the classical truncation $\mathfrak{X} = t_0(\mathfrak{X})$ carries a natural d-critical structure s . We now recall some of its basic properties.

Firstly, assume that \mathfrak{X} is a derived scheme and write $X = \mathfrak{X}$ and $X = \mathfrak{X}$. Take an open embedding $\iota: \mathbf{Crit}(f) \hookrightarrow X$, where f is a regular function on a smooth scheme U such that $f|_{\mathbf{Crit}(f)^{\text{red}}} = 0$ and U has a global étale coordinate. We equip $\mathbf{Crit}(f)$ with the (-1) -shifted symplectic structure constructed in Example A.4 and assume that ι preserves the (-1) -shifted symplectic structures. We let R denote the image of $t_0(\iota)$ and $i: R \hookrightarrow U$ denote the natural inclusion. Then (R, U, f, i) defines a d-critical chart of (X, s) , see [5, Theorem 6.6].

Now we remove the assumption that \mathfrak{X} is a derived scheme. Take a smooth morphism $q: T \rightarrow \mathfrak{X}$. Assume that there exist a morphism between derived schemes $\tau: T \rightarrow \hat{T}$, a (-1) -shifted symplectic structure $\omega_{\hat{T}}$ on \hat{T} , and an equivalence $q^* \omega \sim \tau^* \omega_{\hat{T}}$. Then there exists an equality

$$t_0(q)^* s = t_0(\tau)^* s_{\hat{T}} \tag{A.1}$$

of d-critical structures, where $s_{\hat{T}}$ is the d-critical structure on $\hat{T} = t_0(\hat{T})$ induced from the (-1) -shifted symplectic structure $\omega_{\hat{T}}$. See [37, Theorem 4.6] for the proof.

It is shown in [3, Theorem 3.18 (b)] that there exists a natural isomorphism

$$K_{\mathfrak{X}, s}^{\text{vir}} \cong \det(\mathbb{L}_{\mathfrak{X}}|_{\mathfrak{X}^{\text{red}}}). \tag{A.2}$$

A.2. Proof of Proposition 2.4

Here we give the proof of Proposition 2.4.

Lemma A.9. *Let $q: U \rightarrow \mathfrak{U}$ be a smooth morphism from a smooth scheme. We let s_X denote the canonical d -critical structure on $X := \text{Crit}(f \circ q)$. Then the following equality holds: $(q|_X)^*s = s_X$.*

Proof. We let $\omega_{\text{Crit}(f)}$ (resp. $\omega_{\text{Crit}(f \circ q)}$) denote the natural (-1) -shifted symplectic structure on $\mathbf{Crit}(f)$ (resp. $\mathbf{Crit}(f \circ q)$). Consider the following diagram of derived Artin stacks:

$$\begin{array}{ccc} \mathbf{Crit}(f) \times_{\mathfrak{U}} U & \xrightarrow{\iota} & \mathbf{Crit}(f \circ q) \\ \downarrow q_0 & & \\ \mathbf{Crit}(f) & & \end{array}$$

where the map q_0 is induced by q and ι is the natural map which is identity on the truncation. We claim that there is an equivalence of (-1) -shifted symplectic structures

$$q_0^* \omega_{\text{Crit}(f)} \sim \iota^* \omega_{\text{Crit}(f \circ q)}. \tag{A.3}$$

We have a natural homotopy

$$\alpha: d_{\text{dR}}(f)|_{\mathbf{Crit}(f)} \sim 0$$

in $\mathcal{A}^1(\mathbf{Crit}(f), 0)$. By definition, the symplectic form $\omega_{\text{Crit}(f)}$ corresponds to the loop

$$0 \sim d_{\text{dR}}^{\text{cl}} \circ d_{\text{dR}}(f)|_{\mathbf{Crit}(f)} \sim 0$$

in $\mathcal{A}^{2,\text{cl}}(\mathbf{Crit}(f), 0)$, where the first homotopy is defined by the equivalence $d_{\text{dR}}^{\text{cl}} \circ d_{\text{dR}} \sim 0$ and the latter homotopy is defined by $d_{\text{dR}}^{\text{cl}} \alpha$. Therefore, the closed (-1) -shifted 2-form $q_0^* \omega_{\text{Crit}(f)}$ corresponds to the loop

$$0 \sim d_{\text{dR}}^{\text{cl}} \circ d_{\text{dR}}(f \circ q)|_{\mathbf{Crit}(f) \times_{\mathfrak{U}} U} \sim 0$$

in $\mathcal{A}^{2,\text{cl}}(\mathbf{Crit}(f) \times_{\mathfrak{U}} U, 0)$. A similar argument shows that $\iota^* \omega_{\text{Crit}(f \circ q)}$ has the same description, hence we obtain equivalence (A.3).

Combining this equivalence and equality (A.1), we obtain the desired equality. ■

Lemma A.10. *There exists a natural orientation of (\mathfrak{X}, s)*

$$o: K_{\mathfrak{U}}^{\otimes 2}|_{\mathfrak{X}^{\text{red}}} \cong K_{\mathfrak{X},s}^{\text{vir}}.$$

Proof. Take a smooth surjective morphism $q: U \rightarrow \mathfrak{U}$ and write $X = \text{Crit}(f \circ q)$. Consider the following composition:

$$\begin{aligned} o_q: q^* K_{\mathfrak{U}}^{\otimes 2}|_{X^{\text{red}}} &\cong (K_U^{\otimes 2} \otimes \det(\Omega_{U/\mathfrak{U}})^{\otimes -2})|_{X^{\text{red}}} \cong K_{X,s_X} \otimes \det(\Omega_{X/\mathfrak{X}})^{\otimes -2}|_{X^{\text{red}}} \\ &\cong (q|_{X^{\text{red}}})^* K_{\mathfrak{X},s}, \end{aligned}$$

where we set $s_X = (q|_X)^*s$. We claim that the isomorphism o_q descends to an orientation for (\mathfrak{X}, s) . To do this, take an étale surjective morphism $\eta: V \rightarrow U \times_{\mathbb{U}} U$ from a scheme V . We let $\text{pr}_i: U \times_{\mathbb{U}} U$ be the i -th projection for $i = 1, 2$. Write $Y = \text{Crit}(f \circ q \circ \text{pr}_1 \circ \eta)$ and define

$$o_{q \circ \text{pr}_i \circ \eta}: ((q \circ \text{pr}_i \circ \eta)^* K_{\mathbb{U}}^{\otimes 2})|_{Y^{\text{red}}} \rightarrow (q \circ \text{pr}_i \circ \eta|_{Y^{\text{red}}})^* K_{\mathfrak{X}, s}$$

in the same manner as o_q . It is enough to prove the commutativity of the following diagram for each $i = 1, 2$:

$$\begin{CD} ((q \circ \text{pr}_i \circ \eta)^* K_{\mathbb{U}}^{\otimes 2})|_{Y^{\text{red}}} @>o_{q \circ \text{pr}_i \circ \eta}>> (q \circ \text{pr}_i \circ \eta|_{Y^{\text{red}}})^* K_{\mathfrak{X}, s} \\ @VVV @VVV \\ (\text{pr}_i \circ \eta)^{\text{red},*} (q^* K_{\mathbb{U}}^{\otimes 2}|_{X^{\text{red}}}) @>(\text{pr}_i \circ \eta)^{\text{red},*} o_q>> (\text{pr}_i \circ \eta)^{\text{red},*} (q|_{X^{\text{red}}})^* K_{\mathfrak{X}, s}. \end{CD}$$

This follows from the commutativity of diagram (2.1). ■

Proof of Proposition 2.4. We keep the notation as in the proof of the previous lemma. Let o_X and o_Y be natural orientations on (X, q^*s) and $(Y, \eta^* \text{pr}_1^* q^*s)$ coming from the descriptions as global critical loci. The construction of the orientation o in the previous lemma implies that we have the following natural commutative diagram of orientations:

$$\begin{CD} o_Y @>>> \eta^* \text{pr}_1^* o_X \\ @V \searrow VV @VV \swarrow V \\ \eta^* \text{pr}_1^* q^* o \end{CD} \tag{A.4}$$

for each $i = 1, 2$.

Now define an isomorphism

$$\theta_q: q^* \varphi_{\mathfrak{X}, s, o} \cong q^* \varphi_f(\mathbb{Q}_{\mathbb{U}}[\dim \mathbb{U}])$$

by the following composition

$$\begin{aligned} q^* \varphi_{\mathfrak{X}, s, o} &\cong \varphi_{X, q^*s, q^*o}[-\dim q] \cong \varphi_{X, s_X, o_X}[-\dim q] \\ &\cong \varphi_{f \circ q}(\mathbb{Q}_U[\dim U - \dim q]) \cong q^* \varphi_f(\mathbb{Q}_{\mathbb{U}}[\dim \mathbb{U}]) \end{aligned}$$

for $i = 1, 2$. Here the third isomorphism follows from Lemma A.8 and Example 2.3. Similarly, we can define an isomorphism

$$\theta_{q \circ \text{pr}_i \circ \eta}: (q \circ \text{pr}_i \circ \eta)^* \varphi_{\mathfrak{X}, s, o} \cong (q \circ \text{pr}_i \circ \eta)^* \varphi_f(\mathbb{Q}_X[\dim \mathfrak{X}]).$$

The commutativity of diagrams (A.4), (2.3), and (2.2) implies an equality

$$(\text{pr}_i \circ \eta)^* \theta_q = \theta_{q \circ \text{pr}_i \circ \eta},$$

hence θ_q descends to the desired isomorphism. ■

Appendix B. Proof of the support lemma

We will give the proof of the support lemma (Proposition 5.11) here.

Fix positive integers r_1, r_2 and integers m_1, m_2 such that $\mu = m_1/r_1 = m_2/r_2$. Write $r = r_1 + r_2$ and $m = m_1 + m_2$. Define $\mathfrak{W} \subset \mathfrak{M}_X(r_1, m_1) \times \mathfrak{M}_X(r_2, m_2)$ to be the substack consisting of pairs $([E], [F])$ such that $p(\text{Supp } E) \cap p(\text{Supp } F) = \emptyset$, where $p: X = S \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$ is the projection. Define a map $w: \mathfrak{W} \rightarrow \mathfrak{M}_X(r, m)$ by taking direct sum. The map w is an étale map.

Lemma B.1. *Suppose that ω_1 (resp. ω_2, ω) is the (-1) -shifted symplectic structure on $\mathfrak{M}_X(r_1, m_1)$ (resp. $\mathfrak{M}_X(r_2, m_2), \mathfrak{M}_X(r, m)$). Then there exists an equivalence*

$$(\omega_1 \boxplus \omega_2)|_{\mathfrak{W}} \simeq w^* \omega.$$

Proof. It follows from [6, Corollary 6.5] that there exists a Lagrangian structure on the morphism

$$(\iota, w): \mathfrak{W} \rightarrow \mathfrak{M}_X(r_1, m_1) \times \mathfrak{M}_X(r_2, m_2) \times \mathfrak{M}_X(r, m),$$

where ι is the natural inclusion, and we equip $\mathfrak{M}_X(r_1, m_1) \times \mathfrak{M}_X(r_2, m_2) \times \mathfrak{M}_X(r, m)$ with the (-1) -shifted symplectic structure $\omega_1 \boxplus \omega_2 \boxplus (-\omega)$. The Lagrangian structure induces the desired equivalence. ■

For an open subset $U \subset \mathbb{C}$ in the analytic topology, we define $\mathfrak{M}_X^{ss}(r, m)^U \subset \mathfrak{M}_X^{ss}(r, m)$ to be the complex analytic open substack consisting of points corresponding to sheaves whose supports are contained in $S \times U \subset X$. The following statement is a straightforward consequence of the above lemma.

Corollary B.2. *Let $U_1, U_2 \subset \mathbb{C}$ be disjoint open subsets in the analytic topology. Consider the following open immersion*

$$w_{U_1, U_2}: \mathfrak{M}_X^{ss}(r_1, m_1)^{U_1} \times \mathfrak{M}_X^{ss}(r_2, m_2)^{U_2} \rightarrow \mathfrak{M}_X^{ss}(r, m)^{U_1 \amalg U_2}$$

induced from w . We let s_i denote the d -critical structure on $\mathfrak{M}_X^{ss}(r_i, m_i)^{U_i}$ and s denote the d -critical structure on the right-hand side. Then we have an equality $w_{U_1, U_2}^(s) = s_1 \boxplus s_2$.*

We now want to prove that the map w_{U_1, U_2} preserves the canonical orientation following the idea of the proof of Proposition 5.8. Let W be the image of $t_0(\mathfrak{W})$ along the map

$$\mathfrak{M}_X^{ss}(r_1, m_1) \times \mathfrak{M}_X^{ss}(r_2, m_2) \rightarrow M_X^{ss}(r_1, m_1) \times M_X^{ss}(r_2, m_2).$$

Write $(r_1, m_1) = (k_1 r_0, k_1 m_0)$ and $(r_2, m_2) = (k_2 r_0, k_2 m_0)$, where (r_0, m_0) is coprime. Define an open subspace

$$\mathbb{A}_W \subset \text{Sym}^{k_1}(\mathbb{A}^1) \times \text{Sym}^{k_2}(\mathbb{A}^1)$$

consisting of configurations (P, Q) such that $P \cap Q = \emptyset$. There exists a natural map

$$W^{\text{red}} \hookrightarrow M_X^{ss}(r_1, m_1)^{\text{red}} \times M_X^{ss}(r_2, m_2)^{\text{red}} \rightarrow \text{Sym}^{r_1}(\mathbb{A}^1) \times \text{Sym}^{r_2}(\mathbb{A}^1),$$

where the latter map is induced by the projection $X = S \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$. Note that the above map factors through the inclusion

$$\mathbb{A}_W \hookrightarrow \text{Sym}^{k_1}(\mathbb{A}^1) \times \text{Sym}^{k_2}(\mathbb{A}^1) \hookrightarrow \text{Sym}^{r_1}(\mathbb{A}^1) \times \text{Sym}^{r_2}(\mathbb{A}^1),$$

where the latter map is the diagonal embedding. Therefore, we obtain a surjective map

$$\eta_W: W^{\text{red}} \rightarrow \mathbb{A}_W.$$

Lemma B.3. *Let f be an invertible regular function on W^{red} . Then there exists a regular function g on \mathbb{A}_W such that $f = g \circ \eta_W$.*

Proof. We first claim that the function f is constant along the reduced parts of fibers of η_W . Take a configuration $(P, Q) \in \mathbb{A}_W$. Write $P \cup Q = \{p_1, \dots, p_t\}$ and we let l_i be the multiplicity of $P \cup Q$ at p_i . Then the fiber $\eta_W^{-1}((P, Q))$ is isomorphic to the scheme

$$\prod_{i=1}^t M_S^{\text{ss}}(l_i r_0, l_i m_0).$$

So it follows from the proof of Proposition 5.8 that f is constant along $\eta_W^{-1}((P, Q))^{\text{red}}$. Therefore, it is enough to prove that the map η_W admits a section.

Take an arbitrary stable sheaf $E \in M_S^{\text{ss}}(r_0, m_0)$. Consider the map $s_W: \mathbb{A}_W \rightarrow W^{\text{red}}$ defined by

$$(P = \{p_1, \dots, p_t\}, Q = \{q_1, \dots, q_s\}) \mapsto \left(\bigoplus_j (i_{p_j,*} E)^{\oplus \text{mult}(p_j)}, \bigoplus_j (i_{q_j,*} E)^{\oplus \text{mult}(q_j)} \right),$$

where i_{p_j} denotes the embedding $S \times \{p_j\} \hookrightarrow X$ and $\text{mult}(p_j)$ denotes the multiplicity of p_j in P , and similarly for i_{q_j} and $\text{mult}(q_j)$. We can see that s_W is a section of η_W . Thus we obtain the claim. ■

Assume that $s_{\mathfrak{W}}$ is the natural d -critical structure on $\mathfrak{W} = t_0(\mathfrak{W})$. Let $o_{\mathfrak{W}}: M_1^{\otimes 2} \cong K_{\mathfrak{W}, s_{\mathfrak{W}}}^{\text{vir}}$ be the orientation on \mathfrak{W} induced from the canonical orientation on $\mathfrak{M}_X^{\text{ss}}(r_1, m_1) \times \mathfrak{M}_X^{\text{ss}}(r_2, m_2)$ and $o'_{\mathfrak{W}}: M_2^{\otimes 2} \cong K_{\mathfrak{W}, s_{\mathfrak{W}}}^{\text{vir}}$ be the orientation on \mathfrak{W} induced from the canonical orientation on $\mathfrak{M}_X^{\text{ss}}(r, m)$. We have seen in the proof of Proposition 5.8 that there exist trivializations

$$M_1 \cong \mathcal{O}_{\mathfrak{W}^{\text{red}}} \quad \text{and} \quad M_2 \cong \mathcal{O}_{\mathfrak{W}^{\text{red}}}.$$

Therefore, the composition $(o'_{\mathfrak{W}})^{-1} \circ o_{\mathfrak{W}}$ defines an element

$$\alpha \in \Gamma(\mathfrak{W}^{\text{red}}, \mathcal{O}_{\mathfrak{W}^{\text{red}}}) \cong \Gamma(W^{\text{red}}, \mathcal{O}_{W^{\text{red}}}).$$

Corollary B.4. *We use the notations as in Corollary B.2. Assume that each connected component of U_1 and U_2 is homeomorphic to the disk. Then there exists an isomorphism of orientations*

$$o_{\mathfrak{W}}|_{\mathfrak{M}_X^{\text{ss}}(r_1, m_1)^{U_1} \times \mathfrak{M}_X^{\text{ss}}(r_2, m_2)^{U_2}} \cong o'_{\mathfrak{W}}|_{\mathfrak{M}_X^{\text{ss}}(r_1, m_1)^{U_1} \times \mathfrak{M}_X^{\text{ss}}(r_2, m_2)^{U_2}}.$$

Proof. Let $M_X^{ss}(r_1, m_1)^{U_1} \subset M_X^{ss}(r_1, m_1)$ (resp. $M_X^{ss}(r_2, m_2)^{U_2} \subset M_X^{ss}(r_2, m_2)$) be the image of $\mathfrak{M}_X^{ss}(r_1, m_1)^{U_1}$ (resp. $\mathfrak{M}_X^{ss}(r_2, m_2)^{U_2}$) along the map p_X . We need to show that $\alpha|_{M_X^{ss}(r_1, m_1)^{U_1} \times M_X^{ss}(r_2, m_2)^{U_2}}$ admits a square root. Lemma B.3 implies that there exists a regular function g on \mathbb{A}_W such that $\alpha = g \circ \eta_W$. Let $\mathbb{A}_{U_1, U_2} \subset \mathbb{A}_W$ be an open subset consisting of configurations (P, Q) such that $P \subset U_1$ and $Q \subset U_2$. Since the image of $M_X^{ss}(r_1, m_1)^{U_1} \times M_X^{ss}(r_2, m_2)^{U_2}$ under the map η_W is contained in \mathbb{A}_{U_1, U_2} , we need to show that $g|_{\mathbb{A}_{U_1, U_2}}$ admits a square root. But this follows from the connectedness of \mathbb{A}_{U_1, U_2} . ■

Proof of Proposition 5.11. The proof is almost identical to the proof of the support lemma for preprojective algebras [10]. Take disjoint open subsets $U_1, U_2 \subset \mathbb{A}^1$ whose connected components are homeomorphic to the disk. It follows from Proposition 5.10 that there exists an isomorphism

$$\mathcal{H}((p_X * \varphi_{\mathfrak{M}_X^{ss}(\mu)})|_{M_X^{ss}(\mu)^{U_i}}) \cong \text{Sym}_{\boxtimes \oplus}(\mathbf{H}^*(\mathbf{BC}^*)_{\text{vir}} \otimes \varphi_{M_X^{ss}(\mu)}|_{M_X^{ss}(\mu)^{U_i}})$$

for each $i = 1, 2$. Then the Thom–Sebastiani theorem [2, Remark 5.23], Corollaries B.2 and B.4 imply the following isomorphism:

$$\begin{aligned} &\mathcal{H}((p_X * \varphi_{\mathfrak{M}_X^{ss}(\mu)})|_{M_X^{ss}(\mu)^{U_1 \amalg U_2}}) \\ &\cong \mathcal{H}((p_X * \varphi_{\mathfrak{M}_X^{ss}(\mu)})|_{M_X^{ss}(\mu)^{U_1}}) \boxtimes_{\oplus} \mathcal{H}((p_X * \varphi_{\mathfrak{M}_X^{ss}(\mu)})|_{M_X^{ss}(\mu)^{U_2}}) \\ &\cong \text{Sym}_{\boxtimes \oplus}(\mathbf{H}^*(\mathbf{BC}^*)_{\text{vir}} \otimes \varphi_{M_X^{ss}(\mu)}|_{M_X^{ss}(\mu)^{U_1}}) \\ &\quad \boxtimes_{\oplus} \text{Sym}_{\boxtimes \oplus}(\mathbf{H}^*(\mathbf{BC}^*)_{\text{vir}} \otimes \varphi_{M_X^{ss}(\mu)}|_{M_X^{ss}(\mu)^{U_2}}) \\ &\cong \text{Sym}_{\boxtimes \oplus}(\mathbf{H}^*(\mathbf{BC}^*)_{\text{vir}} \otimes (\varphi_{M_X^{ss}(\mu)}|_{M_X^{ss}(\mu)^{U_1}} \oplus \varphi_{M_X^{ss}(\mu)}|_{M_X^{ss}(\mu)^{U_2}})). \end{aligned}$$

On the other hand, Proposition 5.10 implies an isomorphism

$$\mathcal{H}((p_X * \varphi_{\mathfrak{M}_X^{ss}(\mu)})|_{M_X^{ss}(\mu)^{U_1 \amalg U_2}}) \cong \text{Sym}_{\boxtimes \oplus}(\mathbf{H}^*(\mathbf{BC}^*)_{\text{vir}} \otimes \varphi_{M_X^{ss}(\mu)}|_{M_X^{ss}(\mu)^{U_1 \amalg U_2}}).$$

Therefore, we obtain an isomorphism

$$\varphi_{M_X^{ss}(\mu)}|_{M_X^{ss}(\mu)^{U_1}} \oplus \varphi_{M_X^{ss}(\mu)}|_{M_X^{ss}(\mu)^{U_2}} \cong \varphi_{M_X^{ss}(\mu)}|_{M_X^{ss}(\mu)^{U_1 \amalg U_2}}. \tag{B.1}$$

Now take a point $[E] \in \text{Supp}(\varphi_{M_X^{ss}(\mu)})$, where E is a polystable sheaf on X . Assume that the support of E is contained in $S \times (U_1 \amalg U_2)$ for some disjoint open subsets $U_1, U_2 \subset \mathbb{C}$ (or equivalently, $[E] \in M_X^{ss}(\mu)^{U_1 \amalg U_2}$). Then isomorphism (B.1) implies that the support of E is contained in either of U_1 or U_2 . Therefore, there exists some $t \in \mathbb{C}$ such that $\text{Supp}(E) \subset S \times \{t\}$, which implies the proposition. ■

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