J. Eur. Math. Soc. 28, 559–618 (2026) DOI 10.4171/JEMS/1566

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Dynamics of skew-products tangent to the identity

Received April 13, 2022; revised January 29, 2024

Abstract. We study the local dynamics of generic skew-products tangent to the identity, i.e. maps of the form $P(z,w) = (p(z), q(z,w))$ with $\mathrm{d}P_0 = \mathrm{Id}$. More precisely, we focus on maps with non-degenerate second differential at the origin; such maps have local normal form $P(z,w) = (z - z^2 + O(z^3), w + w^2 + bz^2 + O(\|(z,w)\|^3))$. We prove the existence of parabolic domains, and prove that inside these parabolic domains the orbits converge non-tangentially if and only if $b \in (1/4, +\infty)$. Furthermore, we prove the existence of a type of parabolic implosion, in which the renormalization limits are different from previously known cases. This has a number of consequences: under a diophantine condition on coefficients of P, we prove the existence of wandering domains with rank 1 limit maps. We also give explicit examples of quadratic skew-products with countably many grand orbits of wandering domains, and we give an explicit example of a skew-product map with a Fatou component exhibiting historic behaviour. Finally, we construct various topological invariants, which allow us to answer a question of Abate.

Keywords: skew-products, germs tangent to identity, parabolic implosion, wandering domains.

1. Introduction

Skew-products are holomorphic self-maps of $\mathbb{C}^2$ of the form

$$P(z,w) = (p(z), q(z,w)).$$

An important feature of these maps is that they preserve the set of vertical lines in $\mathbb{C}^2$. This means that we can view the restriction of P^n to a line $\{z\} \times \mathbb{C}$ as the composition of n entire functions on $\mathbb{C}$, which allows techniques from one-dimensional complex dynamics to be applied. The dynamics of skew-products is therefore in some ways reminiscent of the dynamics of one-variable maps; however, in recent years, several important results have shown that these maps have rich and interesting dynamics [22, 27, 28, 35]. For example,

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Mathematics Subject Classification 2020: 37F40 (primary); 37F46, 37F80, 11K16 (secondary).

in [7], it was shown that there exist polynomial skew-products, i.e. P is a polynomial map, with *wandering Fatou components*, a dynamical phenomenon that is known not to occur for polynomial maps in one complex dimension. The proof of the main result in that paper involves the adaptation of *parabolic implosion* to the skew-product setting (see also [6, 8, 10] for further results on parabolic implosion in several complex variables). Polynomial skew-products were also used in [15, 33] to construct *robust* bifurcations, i.e. open sets contained in the bifurcation locus of the family of endomorphisms of $\mathbb{P}^2$ of given algebraic degree $d \geq 2$.

Given a germ of a holomorphic self-map P of $\mathbb{C}^2$ that fixes the origin, we say that P is *tangent to the identity* if it is of the form $P = \mathrm{Id} + P_k(z, w) + O(\|(z, w)\|^{k+1})$, where $k \geq 2$ and $P_k : \mathbb{C}^2 \to \mathbb{C}^2$ is a non-trivial homogeneous polynomial map of degree k. The study of local dynamics of germs tangent to the identity has received significant attention over the last decades. For general germs of $(\mathbb{C}^2, 0)$ tangent to the identity, a complete description of the dynamics on a full neighborhood of the origin is for now far out of reach. Much effort has been instead devoted to investigating the existence of invariant manifolds or invariant formal curves on which the dynamics converges to the origin (see e.g. [1, 19], and more recently [23, 24]).

In this paper we investigate the local dynamics of skew-products P which are tangent to the identity and have a non-degenerate second order differential at the origin. By this we mean holomorphic maps[1] $P : \mathbb{C}^2 \to \mathbb{C}^2$ of the form

$$P(z, w) = \Big(z + \sum_{i \geq 2} a_i z^i, w + \sum_{i+j \geq 2} b_{i,j} z^i w^j\Big)$$

with $a_2, b_{2,0}, b_{0,2} \neq 0$.

Up to conjugacy by a linear automorphism of $\mathbb{C}^2$, such maps may be reduced to a map of the form

$$P : (z, w) \mapsto \big(z - z^2 + O(z^3), w + w^2 + bz^2 + O(\|(z, w)\|^3)\big),$$

and after a second conjugacy by an automorphism of $\mathbb{C}^2$ of the form

$$(z, w) \mapsto (z, e^{Az} w + Bz^2),$$

we may finally assume that P is of the form $P(z, w) = (p(z), q(z, w))$ with

$$\begin{cases} p(z) := z - z^2 + az^3 + O(z^4), \\ q(z, w) := w + w^2 + bz^2 + b_{0,3} w^3 + b_{3,0} z^3 + O(\|(z, w)\|^4), \end{cases} \tag{1.1}$$

where $a, b, b_{0,3}, b_{3,0} \in \mathbb{C}$.

A study of the local dynamics of skew-products in the case $b = 0$ in (1.1) has been undertaken in [35], where a full description of the dynamics on a neighborhood of a

[1] We believe that with extra care, most of our results could be stated for germs; however, in an effort to keep statements from being excessively technical, we have chosen to work in the setting of globally defined maps.

parabolic fixed point at the origin was achieved. However, most of the difficulty and richness of the dynamics (including the phenomenon of parabolic implosion and the existence of wandering domains) comes precisely from the term bz^2.

In fact, although maps of the form (1.1) are generic among polynomial skew-products which are tangent to the identity (after analytic conjugacy), we will see that they have considerably complicated local dynamics. We see the investigation of those maps (1.1) and the results of this paper as a first step (generic case) towards the systematic analysis of the local dynamics of all polynomial skew-products which are tangent to the identity.

Notation. Throughout this paper, we will be using the notation $q_z(w) := q(z, w)$ (in particular, $q_0 = q(0, \cdot)$).

1.1. Parabolic domains and parabolic implosion

Definition 1.1. Let P be a holomorphic self-map of $\mathbb{C}^2$ with a parabolic fixed point at the origin. A *parabolic domain* of P is a maximal invariant connected domain $\mathcal{U} \subset \mathbb{C}^2$ such that the origin is contained in the boundary of $\mathcal{U}$ and the iterates $P^n_{|\mathcal{U}}$ converge locally uniformly on $\mathcal{U}$ to the origin. Moreover, we say that a parabolic domain is *tangent to a direction* v if each point from the domain is attracted to the origin along trajectories tangent to v.

We begin by discussing the existence of parabolic domains for maps of the form (1.1), which depends only on b:

Theorem 1.2. *Let P be a map of the form* (1.1).

(1) *If $b \in (1/4, \infty)$, the map P has an invariant parabolic domain which is not tangent to any directions.*
(2) *If $b \in \mathbb{C} \setminus (1/4, \infty)$, the map P has an invariant parabolic domain which is tangent to one of its non-degenerate characteristic directions.*

The main novelty here lies in the first statement of this theorem, while the second statement can be deduced from results of Hakim and Vivas. Invariant parabolic domains which are not tangent to any direction are also sometimes called *spiral domains* (see the beginning of Section 3 for a precise definition). Such domains were first constructed by Rivi in her thesis [30, Proposition 4.4.4]. Rong [31, Theorem 1.4] gave sufficient conditions for the existence of spiral domains for some class of maps tangent to the identity. However, his result does not apply to maps of the form (1.1).

From now on we will assume that $b > 1/4$, and we introduce the following notations:

$$c := \frac{\sqrt{4b-1}}{2}, \quad \alpha_0 := e^{\pi/c}, \quad \beta_0 := (b_{0,3} - a)(\alpha_0 - 1). \tag{1.2}$$

Observe that since $b > 1/4$, we have $c > 0$ and $\alpha_0 > 1$.

In what follows we will see that in the case $b > 1/4$ and $\beta_0 \in \mathbb{R}$, there is parabolic implosion, which has many interesting dynamical consequences.

Definition 1.3. Let P be of the form (1.1), and $\alpha, \sigma \in \mathbb{C}$. The *generalized Lavaurs map* of phase σ and parameter α is defined as

$$\mathcal{L}(\alpha, \sigma; z, w) := \psi^o_{q_0}\big(\alpha\phi^\iota_{q_0}(w) + (1-\alpha)\phi^\iota_p(z) + \sigma\big), \tag{1.3}$$

where ϕ^ι_p is the incoming Fatou coordinate of p, $\phi^\iota_{q_0}$ the incoming Fatou coordinate of q_0 and $\psi^o_{q_0}$ the outgoing Fatou parametrization of q_0.

The definitions and basic properties of Fatou coordinates, horn maps and Lavaurs maps are recalled in Section 2. The generalized Lavaurs map is defined for $(z, w) \in \mathcal{B}_p \times \mathcal{B}_{q_0}$, where $\mathcal{B}_p$ and $\mathcal{B}_{q_0}$ are basins of a parabolic fixed point at the origin for p and q_0 respectively. If $\alpha = 1$, then the map $w \mapsto \mathcal{L}(\alpha, \sigma; z, w)$ does not depend on z and coincides with the classical Lavaurs map of phase σ of the one-variable polynomial q_0. Moreover, generalized Lavaurs maps satisfy the following functional relation:

$$\mathcal{L}(\alpha, \sigma; p(z), q_0(w)) = q_0 \circ \mathcal{L}(\alpha, \sigma; z, w) = \mathcal{L}(\alpha, \sigma + 1; z, w) \tag{1.4}$$

for all $(z, w) \in \mathcal{B}_p \times \mathcal{B}_{q_0}$.

Definition 1.4. Given real numbers $\alpha > 1$ and $\beta \in \mathbb{R}$, we say that a strictly increasing sequence $(n_k)_{k\geq 0}$ of positive integers is (α, β)*-admissible* if its *phase sequence* $(\sigma_k)_{k\geq 0}$, defined by $\sigma_k := n_{k+1} - \alpha n_k - \beta \ln n_k$, is bounded. If $\beta = 0$, we will simply call such a sequence α*-admissible*.

Observe that for any $\alpha > 1$ and $\beta \in \mathbb{R}$, there always exists (α, β)-admissible sequences: it suffices to define inductively $n_{k+1} := \lfloor \alpha n_k + \beta \ln n_k \rfloor$ and take $n_0 \in \mathbb{N}$ large enough, where $\lfloor \cdot \rfloor$ denotes the floor function. For this particular type of (α, β)-admissible sequence, we have $\sigma_k \in (-1, 0]$ for all $k \in \mathbb{N}$. However, describing the phase sequence is in general a difficult problem; for instance, even in the particular case of the $\frac{3}{2}$-admissible sequences of the form $n_{k+1} = \lfloor \frac{3}{2} n_k \rfloor$, the phase sequence is not fully understood (see [14]). An interesting question is the existence of (α, β)-admissible sequences with *converging* phase sequence, which will be discussed in detail below.

The following is the main technical result of this paper.

Main Theorem. *Let P be a map of the form* (1.1). *Let α_0, β_0 be as in* (1.2)*, and assume that $b > 1/4$ and $\beta_0 \in \mathbb{R}$. Let $(n_k)_{k\geq 0}$ be an (α_0, β_0)-admissible sequence and let $(\sigma_k)_{k\geq 0}$ denote its phase sequence. Then*

$$P^{n_{k+1}-n_k}(p^{n_k}(z), w) = (0, \mathcal{L}(\alpha_0, \Gamma + \sigma_k; z, w)) + o(1) \quad (\textit{as } k \to \infty)$$

with uniform convergence on compacts in $\mathcal{B}_p \times \mathcal{B}_{q_0}$, and where Γ is a constant depending only on $a, b, b_{0,3}, b_{3,0}$ (*see* (5.1) *for its explicit expression*).

The case where $b > 1/4$ and $\beta_0 \notin \mathbb{R}$ is briefly discussed in Remark 5.14.

The usefulness of this Main Theorem (and of similar results, such as [7, Proposition A] is that by applying it successively, one can estimate more and more precisely certain high iterates of P in terms of iterates of the maps $\mathcal{L}_z : w \mapsto \mathcal{L}(\alpha_0, \Gamma + \sigma_k; z, w)$. Therefore, one can transfer dynamical properties of $\mathcal{L}_z$ to obtain information on the dynamics of P. These maps $\mathcal{L}_z$ are quite complicated (they are non-explicit, transcendental maps, with infinitely many critical points and in general infinitely many critical values). However, by thinking of them as a one-parameter family of maps $(\mathcal{L}_z)_{z\in\mathcal{B}_p}$, we can use ideas from one-dimensional bifurcation theory to obtain information on the dynamics of $\mathcal{L}_z$ for certain values of z. Moreover, under the additional assumption that $\alpha_0 \in \mathbb{N}_{\geq 2}$, we prove in Section 7 that these maps are semi-conjugate to *finite type maps* in the sense of Epstein, which allows us to obtain a more precise understanding of their dynamics, and in turn, of the dynamics of P.

We list below some consequences of the Main Theorem.

1.2. Existence of wandering domains and Pisot numbers

The *Fatou set* is the largest open set in $\mathbb{C}^2$ on which the family of iterates $(P^n)_{n\in\mathbb{N}}$ is normal. A *Fatou component* Ω is a connected component of the Fatou set, and it is called *wandering* if for every $(k,m) \in \mathbb{N} \times \mathbb{N}^*$, we have $P^{k+m}(\Omega) \cap P^k(\Omega) = \emptyset$. A non-wandering Fatou component is a pre-periodic Fatou component. The first examples of polynomial maps with wandering Fatou components were introduced by Buff, Dujardin, Peters, Raissy and the first author [7] in (see also [6]); other examples were constructed by Berger and Biebler [9], by completely different methods, for Hénon maps and polynomial endomorphisms of $\mathbb{P}^2$. In the opposite direction, Ji [20, 21] gave sufficient conditions for the absence of wandering domains near an attracting invariant fiber for a skew-product map.

The examples from [7] are polynomial skew-products of the form

$$(z,w) \mapsto \left(p(z), q(w) + \frac{\pi^2}{4}z\right)$$

with $p(z) = z - z^2 + O(z^3)$ and $q(w) = w + w^2 + O(w^3)$, and are not tangent to the identity at the origin. One can simplify the investigation of these maps by passing to a finite branched cover $y^2 = z$. This brings these maps to a form that is tangent to the identity, but with degenerate second order differential at the origin. In particular, these maps are not of the form (1.1) considered in the present paper, which explains the difference in the dynamical features.

Definition 1.5. Let Ω be a Fatou component of the map P. A *Fatou limit function* on Ω is any limit value of the sequence of maps $(P^n_{|\Omega})_{n\in\mathbb{N}}$.

We define the *rank* of a Fatou component Ω as the maximal rank of $\mathrm{d}h_x$, where $x \in \Omega$ and h ranges over all Fatou limit functions on Ω.

Note that for endomorphisms of $\mathbb{C}^2$, any wandering domain either has rank 0 (all Fatou limits are constant) or rank 1. So far, the only known examples of wandering domains in $\mathbb{C}^2$ have been of rank 0 (that is, the examples constructed in [6, 7, 9]). In other words, Theorem 1.6 below gives the first examples of rank 1 wandering domains in complex dimension 2.

Theorem 1.6. *Let P be a map of the form* (1.1)*, and assume that there exists an* (α_0, β_0)*-admissible sequence with converging phase sequence. Then P has a wandering domain of rank* 1.

We are therefore led to the question: for which values of α and β does such a sequence exist? Before stating an answer, recall the definition of Pisot numbers:

Definition 1.7. A real algebraic integer $\alpha > 1$ is called a *Pisot number* if all of its Galois conjugates are in the open unit disk in $\mathbb{C}$ (in particular, integers ≥ 2 are Pisot numbers).

The next definition is not standard terminology, but it will be convenient for our purposes:

Definition 1.8. We say that $\alpha > 1$ has the *Pisot property* if there exists a real number ζ such that $\|\zeta\alpha^k\| \to 0$, where $\|\cdot\|$ denotes the distance to the nearest integer.

We recall here two classical results from number theory that justify the terminology of "Pisot property":

Theorem ([29]). *Let $\alpha > 1$ be an algebraic number and ζ be a non-zero real number such that $\|\zeta\alpha^k\| \to 0$. Then α is a Pisot number and ζ lies in the field $\mathbb{Q}(\alpha)$.*

Theorem ([29]) *There are only countably many pairs (ζ, α) of real numbers such that $\zeta \neq 0$, $\alpha > 1$, and the sequence $(\{\zeta\alpha^k\})_{k\geq 0}$ has only finitely many limit points. Moreover, if (ζ, α) is such a pair where α is an algebraic number, then α is a Pisot number and ζ lies in the field $\mathbb{Q}(\alpha)$. Here $\{\cdot\}$ denotes the fractional part of the number.*

In particular, an algebraic number has the Pisot property if and only if it is a Pisot number. Moreover, it is a long-standing conjecture known as the *Pisot–Vijayaraghavan problem* that Pisot numbers are the only *real* numbers with the Pisot property.

Definition 1.9. We say that a sequence $(\sigma_k)_{k\geq 0}$ *converges to a cycle of period* ℓ if the subsequence $(\sigma_{k\ell+j})_{k\geq 0}$ converges for every $0 \leq j < \ell$.

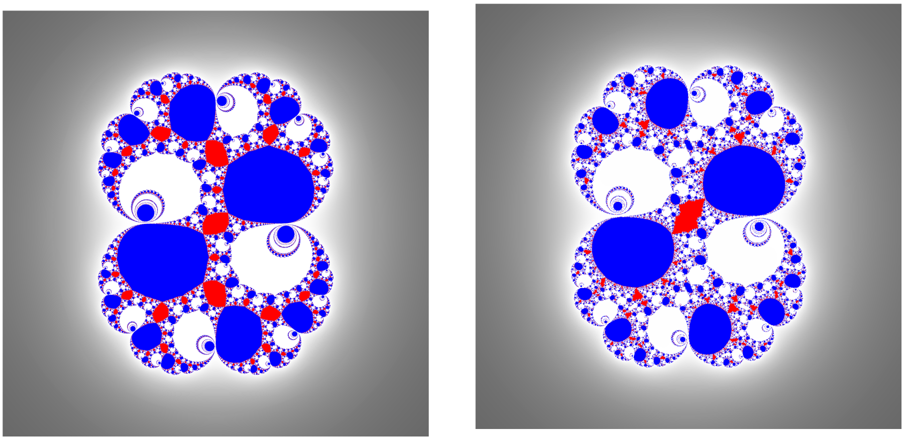
We can now state an almost sharp condition on α and β for the existence of an (α, β)-admissible sequence with converging phase:

Theorem 1.10. *Let $\alpha > 1$ and $\beta \in \mathbb{R}$.*

(1) *There exists an α-admissible sequence with phase sequence converging to a cycle if and only if α has the Pisot property. Moreover, in that case there exists an α-admissible sequence with phase sequence converging to* 0.

(2) (a) *If there exists an (α, β)-admissible sequence with phase sequence converging to a periodic cycle, then α has the Pisot property.*

(b) *Conversely, if α has the Pisot property and $\beta = \frac{\alpha-1}{\ln\alpha}\frac{k_1}{k_2}$, where k_1 and k_2 are coprime integers with $k_2 \geq 1$, then there exists an (α, β)-admissible sequence whose phase sequence converges to a cycle of period k_2.*



(a) $\alpha_0 = 2$ (b) $\alpha_0 = \frac{1+\sqrt{5}}{2}$ (non-integer Pisot number)

Fig. 1. Vertical slices $z =$ constant of quadratic skew-products (1.5) for two different values of α_0. In red, wandering domains; in blue, the two parabolic basins; in shades of grey, the basin of infinity. Observe that figure (a) is q_0-invariant while figure (b) is not.

Note that if the Pisot–Vijayaraghavan conjecture is true, then there exists an α-admissible sequence with converging phase sequence if and only if α is a Pisot number.

It is natural to ask whether the condition of Theorem 1.6 is necessary or not. When there are no (α, β)-admissible sequences whose phase sequence converges to a periodic cycle, the condition means that any wandering Fatou component whose orbit remains in $\mathcal{B}_p \times \mathcal{B}_{q_0}$ would have to remain bounded under a sequence of non-autonomous compositions of generalized Lavaurs maps with non-periodic sequences of phases. Proving rigorously whether such a thing is possible or not is likely to be difficult, but it seems reasonable to expect that for generic values of α it is not the case.

If we now specialize to the case of degree 2, Theorems 1.6 and 1.10 imply that for any Pisot number $\alpha_0 > 1$, the map

$$(z, w) \mapsto \left(z - z^2, w + w^2 + \left(\frac{1}{4} + \frac{\pi^2}{(\ln \alpha_0)^2}\right)z^2\right) \tag{1.5}$$

has a wandering domain of rank 1 (see Figure 1). Those are the first completely explicit[2] examples of polynomial maps with wandering domains, as well as the first examples in degree 2 and the first examples of wandering domains with rank 1.

Recall that two Fatou components Ω_1 and Ω_2 are *in the same grand orbit* (of Fatou components) for P if there exist $n_1, n_2 \in \mathbb{N}$ such that $P^{n_1}(\Omega_1) = P^{n_2}(\Omega_2)$. One may

[2]In [7], there are explicit examples of polynomial maps for which numerical experiments strongly indicate the existence of wandering domains. It is possible that a rigorous argument could be made to prove the existence of wandering domains for these explicit maps as well.

ask whether for polynomial endomorphisms of $\mathbb{P}^2$ there exists a bound on the number of grand orbits of wandering domains that would depend only on the degree. The following theorem gives a negative answer:

Theorem 1.11. *Let P be of the form (1.5) and let $\alpha_0 > 1$ be an integer. Then P has countably many distinct grand orbits of rank 1 wandering domains.*

Note that in contrast to e.g. arguments involving the classical Newhouse phenomenon, we do not use perturbative arguments in the proof of Theorem 1.11, and the maps considered are completely explicit. In fact, more precisely, we construct an injective map from the set of hyperbolic components in a specific family of modified horn maps into the set of grand orbits of wandering Fatou components of P; see Theorem 7.6 and the beginning of Section 7.

1.3. Topological invariants and horn maps

We will now investigate a few consequences of the Main Theorem on the topological classification of skew-products tangent to the identity.

In dimension 1, the topological classification of germs tangent to the identity is simply given by the parabolic multiplicity, that is, the order of vanishing of $f - \mathrm{Id}$ at the origin [12, 32]. However, the *analytic* classification of germs tangent to the identity is considerably more complicated: by a result proved independently by Écalle and Voronin [16, 36] the so-called horn maps (also called Écalle–Voronin invariants) are complete invariants.

To our knowledge, no complete topological classification is available for germs tangent to the identity in $\mathbb{C}^2$. Our results imply that such a classification must also be complicated even in the seemingly simple class of skew-products; in fact, it resembles the analytic classification for one-dimensional parabolic germs.

A first remarkable consequence of the Main Theorem is that the coefficient b is a topological invariant, among maps of the form (1.1):

Theorem 1.12. *Let P_1 and P_2 be two maps of the form (1.1), and assume that there exists a homeomorphism $\mathfrak{h}$ defined near the origin, with $\mathfrak{h}(0,0) = (0,0)$, such that*

$$\mathfrak{h} \circ P_1 = P_2 \circ \mathfrak{h}.$$

Let b_i, α_i, β_i (with $1 \leq i \leq 2$) be as in (1.2), and assume that $b_i > 1/4$ and $\beta_i \in \mathbb{R}$. If both pairs (α_i, β_i) admit an (α_i, β_i)-admissible sequence with a converging phase sequence then $(\alpha_1, \beta_1) = (\alpha_2, \beta_2)$, and so in particular $b_1 = b_2$.

In [2] Abate asked whether maps of the form

$$(3_{u,v,1}): \quad f(z,w) = (z + uz^2 + (1-u)zw, w + vw^2 + (1-v)zw) \quad \text{with } u + v \neq 1 \text{ and } u, v \neq 0$$

are topologically conjugate to each other. Using Theorem 1.12 we can now answer this question negatively. Indeed, observe that for $u = 1$ and $v \neq 0$ this map is conjugate, via a linear automorphism, to the map

$$(z,w) \mapsto \Big(z - z^2, w + w^2 + \frac{1-v^2}{4}z^2\Big), \tag{1.6}$$

which is of the form (1.1). In particular, when $v \in i\mathbb{R}^*$ in (1.6), such maps satisfy $b > 1/4$ and $\beta = 0$. Then Theorem 1.12, together with Theorem 1.10, asserts that all maps of the form $(3_{u,v,1})$ with $u = 1$ and $v = 2\pi i/\ln\rho$, where ρ is a Pisot number, belong to different local topological conjugacy classes.

We now turn to a slightly stronger equivalence relation than local topological conjugacy:

Definition 1.13. We define an equivalence relation $\sim$ on maps of the form (1.1) by declaring that $P_1 \sim P_2 \Leftrightarrow$ there exists a homeomorphism $\mathfrak{h}$ defined near the origin, with $\mathfrak{h}(0,0) = (0,0)$, such that $\mathfrak{h} \circ P_1 = P_2 \circ \mathfrak{h}$ and $\mathfrak{h}$ is of the form $\mathfrak{h}(z,w) = (\mathfrak{f}(z), \mathfrak{g}(z,w))$.

Theorem 1.16 below will provide further information on the equivalence classes of $\sim$. First, let us recall the definition of a horn map of a one-variable holomorphic map, and some of its basic properties; see Section 2 and e.g. [7, Appendix] for more details.

Definition 1.14. Let $f(z) = z + z^2 + O(z^3)$ be an entire map. Let ϕ_f^ι and ϕ_f^o denote its incoming and outgoing Fatou coordinates respectively, and let $\psi_f^o := (\phi_f^o)^{-1}$ (which extends to an entire map).

(1) The *lifted horn map of* f is $\mathcal{E}_f := \phi_f^\iota \circ \psi_f^o$. It is defined on $(\psi_f^o)^{-1}(\mathcal{B}_f)$, and commutes with translation by 1: $\mathcal{E}_f(Z+1) = \mathcal{E}_f(Z) + 1$.

(2) The *horn map of* f is the unique map h such that $h(e^{2i\pi Z}) = e^{2i\pi\mathcal{E}_f(Z)}$. It extends to a holomorphic map fixing both 0 and ∞.

By construction, the lifted horn map is semi-conjugate to the Lavaurs map, and the horn map therefore describes the action of the Lavaurs map on the quotient space $\mathcal{B}_f/\langle f\rangle$.

We now introduce a two-dimensional analogue of horn maps and lifted horn maps:

Definition 1.15. Let P be a map of the form (1.1). Let us define the *lifted horn map of* P of phase σ by

$$\tilde{H}_\sigma(Z,W) := (Z, \alpha_0 \cdot \mathcal{E}_{q_0}(W) + (1-\alpha_0)Z + \sigma) =: (Z, \tilde{H}_{Z,\sigma}(W)). \tag{1.7}$$

The map $\tilde{H}_\sigma$ satisfies the functional relation $\tilde{H}_\sigma(Z+1, W+1) = \tilde{H}_\sigma(Z,W) + (1,1)$, so it descends to a map H_σ defined on $\mathbb{C}^2/\langle(1,1)\rangle$, which we call the *horn map of* P of phase σ.

In fact, we have the following two relations:

$$\tilde{H}_\sigma(Z+1, W) = \tilde{H}_\sigma(Z,W) + (1, 1-\alpha_0),$$
$$\tilde{H}_\sigma(Z, W+1) = \tilde{H}_\sigma(Z,W) + (0, \alpha_0).$$

Therefore, the map $\tilde{H}_\sigma$ descends to a map on $\mathbb{C}^2/\mathbb{Z}^2$ if and only if $\alpha_0 \in \mathbb{N}$. However, even when $\alpha_0 \notin \mathbb{N}$, $\tilde{H}_\sigma$ always descends to the horn map defined above on $\mathbb{C}^2/\langle(1,1)\rangle$.

Theorem 1.16. *Let P_1 and P_2 be of the form (1.1), with $b_i > 1/4$ and $\beta_i \in \mathbb{R}$, and assume that $P_1 \sim P_2$. Let H^i_σ denote their respective horn maps. Then there exist $\sigma_1, \sigma_2 \in \mathbb{C}$ such that $H^1_{\sigma_1}$ and $H^2_{\sigma_2}$ are topologically conjugate on $\mathbb{C}^2/\langle(1,1)\rangle$.*

Finally, using Theorem 1.16, we obtain the following corollary.

Corollary 1.17. *Under the assumptions of Theorem 1.16, the number of grand orbits of critical points of q_i in $\mathcal{B}_{q_i}$ is the same for $i = 1, 2$. In particular, for any $k \in \mathbb{N}$, there exist P_1, P_2 of the form (1.1) such that $P_1(z,w) - P_2(z,w) = O(\|(z,w)\|^k)$, but $P_1 \nsim P_2$.*

If $q : \mathbb{C} \to \mathbb{C}$ is a holomorphic map and $x \in \mathbb{C}$, recall that the *grand orbit* of x is the set of $y \in \mathbb{C}$ such that there exist $n, m \in \mathbb{N}$ with $f^n(x) = f^m(y)$.

Note that the maps P_1 and P_2 are by assumption globally defined maps on $\mathbb{C}^2$, assumed to be topologically conjugate only on a neighborhood of the origin. It is natural to ask whether Theorems 1.16 and 1.12 extend to case where P_1 and P_2 would only be germs. As mentioned above, we believe it is the case; however, since the proofs are already technical, we have chosen to restrict ourselves to globally defined maps for simplicity. Observe however that for Corollary 1.17 to make sense it is necessary that the maps P_i are globally defined.

1.4. Fatou components with historic behavior

In [9], Berger and Biebler construct wandering Fatou components Ω for some maps f (which are Hénon maps or endomorphisms of $\mathbb{P}^2$) that have *historic behavior*, meaning that for any $x \in \Omega$, the sequence of empirical measures

$$e_n(x) := \frac{1}{n} \sum_{k=1}^{n} \delta_{f^k(x)}$$

does not converge.

To our knowledge, these are the only known examples so far of Fatou components for endomorphisms of $\mathbb{P}^k$ or for Hénon maps with historic behavior. Note that in the case of the wandering Fatou components constructed in [7] and [6], the sequences $(e_n)_{n\in\mathbb{N}}$ converge to the Dirac mass centered at the parabolic fixed point at the origin. In dimension 1, it follows easily from the Fatou–Sullivan classification that no Fatou component of a rational map on $\mathbb{P}^1$ can have historic behavior; and for moderately dissipative Hénon maps, it follows from the classification of Lyubich and Peters [25] that periodic Fatou components cannot have historic behavior.

Using the Main Theorem of this paper, we give here new, explicit examples of polynomial skew-products (which may be chosen to extend to endomorphisms of $\mathbb{P}^2$) which have a Fatou component with historic behavior:

Corollary 1.18. *Let $P(z,w) = (p(z), q(z,w))$ be a polynomial skew-product satisfying the following properties:*

(1) $p(z) = z - z^2 + O(z^3)$.

(2) *P has two different fixed points tangent to the identity of the form $(0, w_1)$ and $(0, w_2)$, which both satisfy the conditions that $\alpha_i \in \mathbb{N}^*$ and $\beta_i = 0$, with the notations of the Main Theorem and in appropriate local coordinates.*

Then P has a Fatou component Ω with historic behavior. More precisely, for any $(z,w) \in \Omega$, the sequences $(e_n(z,w))_{n\in\mathbb{N}}$ accumulates on

$$\mu_1 := \frac{\alpha_1\alpha_2 - \alpha_2}{\alpha_1\alpha_2 - 1}\delta_{(0,w_1)} + \frac{\alpha_2 - 1}{\alpha_1\alpha_2 - 1}\delta_{(0,w_2)}$$

and on

$$\mu_2 := \frac{\alpha_1 - 1}{\alpha_1\alpha_2 - 1}\delta_{(0,w_1)} + \frac{\alpha_1\alpha_2 - \alpha_1}{\alpha_1\alpha_2 - 1}\delta_{(0,w_2)}.$$

More explicitly, these conditions are given by:

(1) $p(z) = z - z^2 + O(z^3)$.

(2) P has two different fixed points tangent to the identity of the form $(0, w_1)$ and $(0, w_2)$, with $q_0''(w_i) = 2$.

(3) $p'''(0) = q_0'''(w_1) = q_0'''(w_2)$.

(4) If $b_i := \frac{1}{2}\frac{\partial^2 q}{\partial z^2}(0, w_i)$, then $b_i > 1/4$ and $\alpha_i := e^{2\pi/\sqrt{4b_i - 1}} \in \mathbb{N}^*$.

Example 1.19. With

$$p(z) := z - z^2$$

and $q(z,w) := q_0(w) + a(z)$ with

$$q_0(w) := w + w^2 - 5w^4 + 6w^5 - 2w^6$$

and

$$a(z) := \left(\frac{1}{4} + \frac{\pi^2}{(\ln 2)^2}\right) z^2(1-z)^2,$$

the map P satisfies the conditions above, with $w_1 = 0$ and $w_2 = 1$, $\alpha_i = 2$ and $\beta_i = 0$.

Note that we could replace p by $z \mapsto z - z^2 + z^6$ in the previous example to obtain an example which extends to an endomorphism of $\mathbb{P}^2$.

Although we believe that the Fatou component constructed in Corollary 1.18 is wandering, we have not been able to prove so. Note however that if it is not the case, then this would be the first example of an invariant (for some iterate of P) non-recurrent Fatou component whose limit sets depend on the limit map, which would give an affirmative answer to [25, Question 30] for $X = \mathbb{C}^2$ and for $X = \mathbb{P}^2$.

Structure of the paper

We recall in Section 2 the definitions and properties of Fatou coordinates. In Section 3, we recall the classical properties of parabolic curves and prove Theorem 1.2. In Section

4, we introduce approximate Fatou coordinates and prove key estimates on how close the dynamics is to a translation in these approximate Fatou coordinates. The Main Theorem is proved in Section 5. Finally, Sections 6, 7, 8, 9 and 10, are devoted to the proofs of Corollary 1.6, Theorem 1.11, Theorem 1.16, Corollary 1.18 and Theorem 1.10 respectively.

Notations

For $x > 0$, we will use the notation $\ln x$ to denote the natural logarithm of x, and for $z \in \mathbb{C} \setminus \mathbb{R}_-$ we will use $\log z$ for the principal branch of the logarithm at z.

2. Fatou coordinates

We recall in this section the definition of Fatou coordinates of one-variable holomorphic maps, as well as some classical facts about their domains of definition, asymptotic expansion near the parabolic fixed point, and covering properties. The material described in Section 2.1 applies more generally to germs, while the material of Section 2.2 does require a globally defined map. However, for our purposes, it is enough to restrict ourselves to the setting where f is an entire map.

2.1. *Local properties*

Unless otherwise stated, we refer the reader to [7, Appendix] for the proofs of the statements appearing in this subsection.

Consider an entire function $f(z) = z + a_2 z^2 + a_3 z^3 + O(z^4)$ where $a_2 \neq 0$. For $r > 0$ small enough we define the *incoming* and *outgoing petals* by

$$\mathcal{P}_f^\iota = \{|a_2 z + r| < r\} \quad \text{and} \quad \mathcal{P}_f^o = \{|a_2 z - r| < r\}.$$

The incoming petal $\mathcal{P}_f^\iota$ is forward invariant, and all orbits in $\mathcal{P}_f^\iota$ converge to 0. The outgoing petal $\mathcal{P}_f^o$ is backward invariant, with backward orbits converging to 0.

On $\mathcal{P}_f^\iota$ and $\mathcal{P}_f^o$ one can define the *incoming* and *outgoing* univalent *Fatou coordinates* $\phi_f^\iota : \mathcal{P}_f^\iota \to \mathbb{C}$ and $\phi_f^o : \mathcal{P}_f^o \to \mathbb{C}$, solving the functional equations

$$\phi_f^\iota \circ f(z) = \phi_f^\iota(z) + 1 \quad \text{and} \quad \phi_f^o \circ f(z) = \phi_f^o(z) + 1.$$

Moreover, the set $\phi_f^\iota(\mathcal{P}_f^\iota)$ contains a right half-plane and $\phi_f^o(\mathcal{P}_f^o)$ contains a left half-plane.

For the most part, the simple definition of petals given above will be sufficient for our purposes. However, in Lemma 8.3 we will need to work with larger petals, whose union covers a punctured neighborhood of the origin.

Accordingly, for any $R > 0$, let

$$\hat{\mathbf{P}}_R^\iota := \{Z \in \mathbb{C} : R - \operatorname{Re} Z < |\operatorname{Im} Z|\}, \quad \hat{\mathbf{P}}_R^0 := \{Z \in \mathbb{C} : R + \operatorname{Re} Z < |\operatorname{Im} Z|\},$$

and let $\mathbf{P}_R^{\iota/o}$ be the respective images of $\hat{\mathbf{P}}_R^{\iota/o}$ under the map $z \mapsto -1/z$.

By [11, Definition 3], up to taking $R > 0$ large enough, the Fatou coordinates $\phi_f^{\iota/o}$ are defined and univalent on the "fat petals" $\mathbf{P}_R^{\iota/o}$. Observe that $\mathbf{P}_R^\iota \cup \mathbf{P}_R^o$ is a punctured neighborhood of 0.

Neither the incoming nor the outgoing Fatou coordinates may be extended to a meromorphic function in a neighborhood of the origin in general; however, they do satisfy the following asymptotic expansion as $z \to 0$ inside $\mathcal{P}_f^{\iota/o}$ respectively:

$$\phi_f^\iota(z) = -\frac{1}{a_2 z} - \mathfrak{b}\log\left(-\frac{1}{a_2 z}\right) + o(1), \tag{2.1}$$

$$\phi_f^o(z) = -\frac{1}{a_2 z} - \mathfrak{b}\log\left(\frac{1}{a_2 z}\right) + o(1), \tag{2.2}$$

where $\mathfrak{b} := 1 - a_3/a_2^2$.

Fatou coordinates are only unique up to an additive constant; in the rest of the paper, we will work with the unique normalized Fatou coordinates for which the asymptotic expansions above hold, with no constant terms.

From the estimate (2.1), we first deduce that $(\phi_f^\iota)^{-1}(Z) \sim -\frac{1}{a_2 Z}$ as $\operatorname{Re} Z \to \infty$. Then, substituting $(\phi_f^\iota)^{-1}(Z) = -\frac{1}{a_2 Z} + o(\frac{1}{Z})$ in (2.1) again, we obtain

$$(\phi_f^\iota)^{-1}(Z) = -a_2^{-1}\left(Z + \mathfrak{b}\log\left(-\frac{1}{a_2 Z}\right) + o(1)\right)^{-1}. \tag{2.3}$$

Finally, note that for every $z_0 \in \mathcal{B}_f$ we have

$$\begin{aligned} z_k := f^k(z_0) &= (\phi_f^\iota)^{-1}(\phi_f^\iota(z_0) + k) \\ &= -\frac{1}{a_2}\big(\phi_f^\iota(z_0) + k + \mathfrak{b}\ln k + o(1)\big)^{-1} \\ &= -\frac{1}{a_2}\left(\frac{1}{k} - \frac{\mathfrak{b}\ln k}{k^2} - \frac{\phi_f^\iota(z_0)}{k^2}\right) + O\left(\frac{\ln^2 k}{k^3}\right), \end{aligned}$$

hence $\operatorname{Re}(a_2 z_k) = -\frac{1}{k} + O(\frac{\ln k}{k^2})$ and $\operatorname{Im}(a_2 z_k) = O(\frac{1}{k^2})$.

Recall that with our choice of normalization, $p(z) = z - z^2 + O(z^3)$, so that the previous estimates apply to p with $a_2 = -1$; and $q_0(w) = w + w^2 + O(w^3)$, so that they apply to q_0 with $a_2 = 1$.

2.2. *Global properties*

Any orbit which converges to 0 but never lands at 0 must eventually be contained in $\mathcal{P}_f^\iota$. Therefore, we have the following description of the parabolic basin:

$$\mathcal{B}_f = \bigcup_{n \geq 0} f^{-n}(\mathcal{P}_f^\iota).$$

Using the relation $\phi_f^\iota \circ f^n = \phi_f^\iota + n$, the incoming Fatou coordinates can be uniquely extended to the attracting basin $\mathcal{B}_f$. On the other hand, the inverse of ϕ_f^o can be extended to an entire map denoted by ψ_f^o, which satisfies the functional equation

$$f \circ \psi_f^o(Z) = \psi_f^o(Z+1).$$

This entire function is then called an *outgoing Fatou parametrization*.

2.2.1. Covering properties of Fatou coordinates. We first record the covering properties of ϕ_f^ι and ψ_f^o in the next two propositions:

Proposition 2.1 ([11, Proposition 2]). *The set of critical points of the map $\phi_f^\iota : \mathcal{B}_f \to \mathbb{C}$ is exactly*

$$\mathrm{crit}(\phi_f^\iota) = \bigcup_{n\in\mathbb{N}} f^{-n}(\mathrm{crit}(f) \cap \mathcal{B}_f).$$

Moreover, $\phi_f^\iota : \mathcal{B}_f \to \mathbb{C}$ is a branched cover.

Proposition 2.2 ([11, Proposition 3]). *A point $Z \in \mathbb{C}$ is a critical point of ψ_f^o if and only if there exists $n \in \mathbb{N}^*$ such that $\psi_f^o(Z-n) \in \mathrm{crit}(f)$. Moreover, the map $\psi_f^o : \mathbb{C} \setminus (\psi_f^o)^{-1}(P_f) \to \mathbb{C} \setminus P_f$ is a covering, where $P_f := \overline{\bigcup_{n\geq 1} f^n(\mathrm{crit}(f))}$ is the post-critical set of f.*

2.2.2. Lifted horn maps, horn maps and Lavaurs maps.

Definition 2.3. The *Lavaurs map* of phase $\sigma \in \mathbb{C}$ is the map $\mathcal{L}_{f,\sigma} : \mathcal{B}_f \to \mathbb{C}$ defined by $\mathcal{L}_{f,\sigma}(w) := \psi_f^o(\phi_f^\iota(w) + \sigma)$.

In order to better study the dynamics of $\mathcal{L}_{f,\sigma}$, it is convenient to introduce the following map which is semi-conjugate to it:

Definition 2.4. The *lifted horn map* of phase $\sigma \in \mathbb{C}$ is the map defined on $\mathcal{U}_f := (\psi_f^o)^{-1}(\mathcal{B}_f)$ by $\mathcal{E}_{f,\sigma}(W) := \phi_f^\iota \circ \psi_f^o(W) + \sigma$. We will simply denote by $\mathcal{E}_f$ the lifted horn map of phase 0.

The open set $\mathcal{U}_f$ has at least two connected components, one containing an upper half-plane and the other containing a lower half-plane. We record here the following property of the lifted horn maps:

Proposition 2.5 ([11, Proposition 4]). *The set of critical values of $\mathcal{E}_f$ is*

$$\mathrm{CV}(\mathcal{E}_f) = \{\phi_f^\iota(c) + n : c \in \mathrm{crit}(f) \cap \mathcal{B}_f \text{ and } n \in \mathbb{Z}\}.$$

It is not difficult to check that $\mathcal{E}_f(W+1) = \mathcal{E}_f(W) + 1$, so that $\mathcal{E}_f$ (and $\mathcal{E}_{f,\sigma}$, for any $\sigma \in \mathbb{C}$) descends to a well-defined map on the cylinder $\mathbb{C}/\mathbb{Z}$. Then, identifying $\mathbb{C}/\mathbb{Z}$ with $\mathbb{C}^*$, we obtain a unique map $h : U \to \mathbb{C}^*$ such that

$$h(e^{2i\pi W}) = \exp(2i\pi \mathcal{E}_f(W)),$$

where U is the image of $\mathcal{U}_f = (\psi_f^o)^{-1}(\mathcal{B}_f)$ under $W \mapsto e^{2i\pi W}$. The map h is called the *horn map* of f, and the *horn map of phase σ* is $h_\sigma := e^{2i\pi\sigma} h$. It can be proved that it extends holomorphically at 0 and ∞, and the extension fixes both points (see [7, Appendix] and references therein).

3. Parabolic domains and parabolic curves

3.1. Parabolic curves

Let P be a holomorphic germ fixing the origin which is tangent to the identity of order $k \geq 2$, i.e. a map with a homogeneous expansion

$$P = \mathrm{Id} + P_k + P_{k+1} + \cdots ,$$

where $P_k \not\equiv 0$. We say that $v \in \mathbb{C}^2 \setminus \{(0,0)\}$ is a *characteristic direction* for P if there exists a $\lambda \in \mathbb{C}$ such that $P_k(v) = \lambda v$. If $\lambda \neq 0$ then v is said to be *non-degenerate*; otherwise it is *degenerate*. We shall denote by $v \mapsto [v]$ the canonical projection of $\mathbb{C}^2 \setminus \{(0,0)\}$ onto $\mathbb{P}^1$. The *director* of a characteristic direction v is the eigenvalue of the linear operator

$$d(P_k)_{[v]} - \mathrm{Id} : T_{[v]}\mathbb{P}^1 \to T_{[v]}\mathbb{P}^1.$$

A *parabolic curve* for P is an injective holomorphic map $\varphi : \Delta \to \mathbb{C}^2$, satisfying the following properties:

(1) Δ is a simply connected domain in $\mathbb{C}$ with $0 \in \partial\Delta$,

(2) φ is continuous at the origin and $\varphi(0) = (0,0)$,

(3) $\varphi(\Delta)$ is invariant under P and $P^n|_{\varphi(\Delta)} \to (0,0)$ uniformly on compact subsets.

We say that a parabolic curve is *tangent* to $[v] \in \mathbb{P}^1$ if $[\varphi(\xi)] \to [v]$ as $\xi \to 0$ in Δ. This implies that for any given point z in the parabolic curve the orbit $(P^n(z))$ converges to the origin *tangentially* to v, i.e. $[P^n(z)] \to [v]$ in $\mathbb{P}^1$. We now recall the following classical result due to Écalle [16] and Hakim [18, 19]:

Theorem 3.1. *Let $P : \mathbb{C}^2 \to \mathbb{C}^2$ be a holomorphic germ fixing the origin which is tangent to the identity of order $k \geq 2$. Then for any non-degenerate characteristic direction v there exist (at least) $k-1$ parabolic curves for P tangent to $[v]$. Moreover, if the real part of the director of a non-degenerate characteristic direction v is strictly positive, then there exists an invariant parabolic domain in which every point is attracted to the origin along a trajectory tangent to v.*

Additionally, by [19, Section 3], when the director of a non-degenerate parabolic curve is not a natural number, the corresponding parabolic curve is asymptotic to a unique (in general divergent) invariant formal power series.

From now on, let P be a map of the form (1.1) and observe that its characteristic directions are given by the equations

$$\begin{cases} -z^2 = \lambda z, \\ w^2 + bz^2 = \lambda w. \end{cases}$$

It follows that aside from the trivial non-degenerate characteristic direction $(0,1)$, there are two other non-degenerate characteristic directions $(1, c^{\pm})$, where $c^{\pm}$ are the roots of

$$u^2 + u + b = 0. \tag{3.1}$$

Note that $c^{\pm} = -1/2 \pm ic$ where c is the solution of $c^2 = b - 1/4$ with $\mathrm{Re}(c) \geq 0$. Clearly, $\mathrm{Im}\, c = 0$ if and only if $b \geq 1/4$. Moreover, for $b = 1/4$ we have $c^+ = c^- = -1/2$. The directors of the characteristic directions $(1, c^{\pm})$ are $\mp 2ic$; in particular, when $b \in (1/4, \infty)$, neither of them is a natural number.

It follows from Theorem 3.1 that aside from the trivial parabolic curve contained in the invariant line $z = 0$, there are two parabolic curves which are tangent to the non-degenerate characteristic directions $(1, c^{\pm})$ respectively. By Hakim's construction, these parabolic curves may be written as holomorphic graphs $z \mapsto (z, \zeta^{\pm}(z))$ over a small petal $\mathcal{P}_p^{\iota} = \mathbb{D}(r, r)$. Since parabolic curves are invariant under P, the functions $\zeta^{\pm}$ satisfy the functional equations

$$q_z(\zeta^{\pm}(z)) = \zeta^{\pm}(p(z)).$$

From this we can easily compute the first few terms of the formal power series to which they are asymptotic:

$$\zeta^{\pm}(z) := c^{\pm} z + \left(c^{\pm}\Theta + \frac{a + (b-1)b_{0,3}}{2} \right) z^2 + O(z^3), \tag{3.2}$$

where

$$\Theta := b_{0,3} + \frac{a - b_{0,3} + b_{3,0}}{2b}. \tag{3.3}$$

3.2. Parabolic domains and proof of Theorem 1.2

We now turn to the proof of Theorem 1.2, which we break into the following two propositions.

Proposition 3.2. *If $b \in \mathbb{C} \setminus (1/4, \infty)$, then the map P has an invariant parabolic domain, in which each point is attracted to the origin along trajectories tangent to one of its non-degenerate characteristic directions.*

Proof. We have two cases:

Case 1: Let $b \in \mathbb{C} \setminus [1/4, \infty)$. As mentioned above, a straightforward computation proves that the directors of $(0,1)$ and $(1, c^{\pm})$ are -1 and $-1 - 2c^{\pm} = \mp 2ic$ respectively. Note that for our choice of b we have $\mathrm{Im}\, c \neq 0$, hence exactly one of the directions $(1, c^{\pm})$ has a director with a strictly positive real part. By Theorem 3.1, if the real part of the director of a non-degenerate characteristic direction v is strictly positive, then there is an invariant parabolic domain in which each point is attracted to the origin along trajectories tangent to v.

Case 2: Let $b = 1/4$. First, observe that as $b \to 1/4$, the characteristic directions $(1, c^{\pm})$ are getting closer to each other, and in the limit they merge to a single characteristic direction $v = (1, -1/2)$. In the terminology of Abate–Tovena [3], v is an irregular characteristic direction, hence by the result of Vivas [34, Theorem 1.1] there exists an invariant parabolic domain in which each point is attracted to the origin along trajectories tangent to v. ■

Proposition 3.3. *If $b > 1/4$, then each of the two non-vertical parabolic curves is contained in parabolic domains $\mathcal{U}^{\pm}$ which are not tangent to any direction.*

Note that we do not claim that $\mathcal{U}^+ \neq \mathcal{U}^-$, although we believe it is the case.

Proof of Proposition 3.3. Let $\mathbb{D}(x, \rho) := \{z \in \mathbb{C} : |z - x| < \rho\}$ denote the disk centered at x with radius ρ. Let $r > 0$ be so small that $p(\mathbb{D}(r, r)) \subset \mathbb{D}(r, r)$ and p is injective on $\overline{\mathbb{D}(r, r)}$. Note that both P and $F^+(z, w) = (z, w + \zeta^+(z))$ map the set $\mathbb{D}(r, r) \times \mathbb{C}$ into itself. Therefore by conjugating P with F^+ we obtain a well-defined holomorphic map on $\mathbb{D}(r, r) \times \mathbb{C}$ given by

$$\hat{P}(z, w) := \big(p(z), q_z(w + \zeta^+(z)) - \zeta^+(p(z))\big), \tag{3.4}$$

where

$$\begin{aligned} q_z(w + \zeta^+(z)) - \zeta^+(p(z)) &= q_z(w + \zeta^+(z)) - q_z(\zeta^+(z)) \\ &= w + w^2 + 2c^+ zw + O(zw^2, z^2 w, w^3). \end{aligned}$$

Note that in these coordinates, the line $w = 0$ is invariant under $\hat{P}$ and therefore $(1, 0)$ is now a non-degenerate characteristic direction of this map.

By making a blow-up $w = uz$ of the map (3.4), we obtain the map

$$\tilde{P}(z, u) := \big(z - z^2 + O(z^3), u(1 + i2cz) + zu^2 + O(z^2 u)\big) \tag{3.5}$$

that is holomorphic on $\mathbb{D}(r, r) \times \mathbb{C}$. Moreover, if r is sufficiently small then there exists a holomorphic function $h(z)$ defined on $\mathbb{D}(r, r)$ such that

$$\tilde{P}(z, u) = (p(z), \tilde{q}(z, u)) = \big(z - z^2 + O(z^3), ue^{i2cz + z^2 h(z)} + zu^2 + O(z^2 u^2)\big).$$

Let us define $D_r := \{(z, u) : |u| < r,\ z \in \mathbb{D}(r, r)\}$.

Lemma 3.4. *There exists a sequence of real numbers $0 < r_j < r$ such that for any $(z_0, u_0) \in \mathcal{D} := \bigcup_{j \geq 1} \{(z, u) : |u| < r_j,\ z \in \mathbb{D}(r, r\frac{j}{j+1})\}$ we have $\tilde{P}^n(z_0, u_0) \in D_r$ for all $n \geq 0$. Moreover, the sequence $\tilde{P}^n(z_0, u_0)$ is bounded away from the origin.*

Proof of Lemma 3.4. Let $j \in \mathbb{N}^*$ and $z_0 \in K_j := \overline{\mathbb{D}}(r, r\frac{j}{j+1})$. Note that

$$\operatorname{Re} p^n(z_0) = \frac{1}{n} + O\left(\frac{\ln n}{n^2}\right) \quad \text{and} \quad \operatorname{Im} p^n(z_0) = O\left(\frac{\ln n}{n^2}\right),$$

with uniform bounds depending only on the compact K_j and hence on r and j, for all $n \geq 1$ (see Section 2). Using this, we define

$$f_n(u) := \operatorname{proj}_2 \tilde{P}(p^{n-1}(z_0), u) = ue^{\frac{i2c}{n} + \Theta_n(z_0)} + \frac{u^2}{n} + O\left(\frac{u^2 \ln n}{n^2}\right)$$

where $\Theta_n = O(\frac{\ln n}{n^2})$ depends only on z_0 and is uniformly bounded on K_j and the constant in $O(\frac{u^2 \ln n}{n^2})$ is uniform on $K_j \times \overline{\mathbb{D}}(0,r)$.

We need to prove that there exists $0 < r_j < r$ such that for every $u_0 \in \mathbb{D}(0, r_j)$ and every $z_0 \in K_j$ we have $(z_n, u_n) := \tilde{P}^n(z_0, u_0) \in D_r$ for all $n \geq 1$. In particular, we need to prove that $|u_n| < r$ for all $n \geq 1$.

Observe that $u_n = (f_n \circ \cdots \circ f_1)(u_0)$ for all $n \geq 1$ and let $U := \tau(u) = -1/u$. For $n \geq 1$ we define

$$g_n(U) := (\tau \circ f_n \circ \tau^{-1})(U) = Ue^{-\frac{i2c}{n} - \Theta_n} + \frac{1}{n} + O\left(\frac{\ln n}{n^2}, \frac{\ln n}{Un^2}\right).$$

It suffices to prove that there exists $0 < r_j < r$ such that for all (z_0, U_0) where $z_0 \in K_j$ and $|U_0| > 1/r_j$ we have $|g_n \circ \cdots \circ g_1(U_0)| > 1/r$ for all $n \geq 1$.

Observe that since c is real, there exists $\tilde{C}_j > 0$ such that

$$\tilde{C}_j^{-1} < \left|e^{-\sum_{k=1}^{n-1}(\frac{i2c}{k} + \Theta_k)}\right| < \tilde{C}_j$$

on K_j for all $n \geq 1$. By making a non-autonomous change of coordinates

$$\psi_n(U) = e^{-\sum_{k=1}^{n-1}(\frac{i2c}{k} + \Theta_k)} U,$$

we obtain

$$\begin{aligned} G_n(U) &= \psi_{n+1}^{-1} \circ g_n \circ \psi_n(U) \\ &= U + \frac{1}{n} e^{\sum_{k=1}^{n}(\frac{i2c}{k} + \Theta_k)} + O\left(\frac{\ln n}{n^2}, \frac{\ln n}{Un^2}\right) \\ &= U + \frac{1}{n} e^{i2c \ln n + i2c\gamma + \mathfrak{h}(z_0)} + O\left(\frac{\ln n}{n^2}, \frac{\ln n}{Un^2}\right), \end{aligned}$$

where $\mathfrak{h} := \sum_{k=1}^{\infty} \Theta_k$ is a holomorphic function of z_0. Here, we have used the fact that $\sum_{k=1}^{n} \frac{1}{k} = \gamma + \ln n + O(\frac{1}{n})$, where γ is Euler's constant, and that $\sum_{k=1}^{n} \Theta_k(z_1) = \mathfrak{h}(z_0) + O(\frac{\ln n}{n})$, where the bounds are uniform on K_j.

Since $c \neq 0$ is real, it follows from Abel's summation formula that there exists a constant $C > 0$ such that

$$\left|\sum_{k=1}^{n} \frac{1}{k} e^{i2c \ln k}\right| = \left|\sum_{k=1}^{n} k^{-(1-i2c)}\right| < C$$

for all $n \geq 1$. This implies that $G_n \circ \cdots \circ G_1(U) = U + O(1)$ for all $n \geq 1$, where the constant in $O(1)$ depends only on K_j.

Next observe that $g_n \circ \cdots \circ g_1(U) = \psi_{n+1} \circ G_n \circ \cdots \circ G_1(U)$, hence there exists $A_j > 0$ such that for all $|U_0| > 1/r$ and all $z_0 \in K_j$ we have

$$\tilde{C}_j^{-1}|U_0| - A_j < |g_n \circ \cdots \circ g_1(U_0)| < \tilde{C}_j |U_0| + A_j$$

for all $n \geq 1$.

From this it immediately follows that there exists $0 < r_j < r$ such that for every $|U_0| > 1/r_j$ we have $|g_n \circ \cdots \circ g_1(U_0)| > 1/r$ for all $n \geq 1$. Moreover, for every $|U_0| > 1/r_j$ the sequence $g_n \circ \cdots \circ g_1(U_0)$ is bounded away from infinity.

Thus we have proven that for any $(z_0, u_0) \in K_j \times \mathbb{D}(0, r_j)$, we have $(z_n, u_n) \in D_r$ for all $n \geq 0$, where the sequence $(u_n)_{n\geq 0}$ is bounded away from the origin. This concludes the proof of Lemma 3.4. ∎

Let us resume the proof of Proposition 3.3. Let $\Omega := \{(z, zu + \zeta^+(z)) : (z,u) \in \mathcal{D}\}$; it is a connected open set whose boundary contains the origin, and $P(\Omega) \cap \Omega \neq \emptyset$. Indeed, $\tilde{P}$ maps $\mathbb{D}(r,r) \times \{0\} \subset \mathcal{D}$ into itself, hence $\tilde{P}(\mathcal{D}) \cap \mathcal{D} \neq \emptyset$. From the lemma above, it immediately follows that the iterates $P^n_{|\Omega}$ converge to the origin locally uniformly on Ω, which is therefore contained in some invariant parabolic domain $\mathcal{U}^+$. It remains to prove that this parabolic domain is not tangent to any direction. Let $(z_0, w_0) \in \Omega$ and $(z_n, w_n) = P^n(z_0, w_0)$ and observe that since $z_n \neq 0$, for all $n \in \mathbb{N}$ we have $[z_n : w_n] = [1 : \frac{w_n}{z_n}] = [1 : u_n + c^+ + o(1)]$.

Recall that in Lemma 3.4 we have shown that

$$u_n = \mathrm{proj}_2\, \tilde{P}^n(z_0, u_0) = \tau^{-1} \circ \psi_{n+1}^{-1} \circ G_n \circ \cdots \circ G_1 \circ \tau(u_0),$$

where G_j and ψ_j depend holomorphically on z_0 and $\psi_j(U)$ is linear in U. Moreover,

$$G_n \circ \cdots \circ G_1 \circ \tau(u) = -\frac{1}{u} + O(1)$$

as $u \to 0$ for all $n \geq 1$ where the bound in $O(1)$ depends only on the compact K_j. Hence every limit map of the iterates $(\tilde{P}^n)$ on $\mathcal{D}$ is of the form $(z,u) \mapsto (0, \eta(z,u))$, where η is a non-constant holomorphic function and $\frac{\partial \eta}{\partial u} \not\equiv 0$.

Therefore, there is no vector $v \in \mathbb{C}^2$ such that the sequence $[P^n(z,w)]$ would converge to $[v]$ in $\mathbb{P}^1$ for all $(z,w) \in \Omega$.

The proof of the existence of an invariant parabolic domain $\mathcal{U}^-$ that contains the parabolic curve associated to the characteristic direction $(1, c^-)$ follows verbatim with an appropriate change of sign. ∎

4. The error functions

Here, we introduce and study properties for one of the main objects to appear in our arguments: the functions $\tilde{A}(z,w)$, $A(z,w)$ and $A_0(z)$, which measure how far the dynamics is from a translation in certain local coordinates.

Let P be a skew-product of the form (1.1), and recall that $v = (1, c^{\pm})$ are two non-degenerate characteristic directions of P, where $c^{\pm} := -\frac{1}{2} \pm ic$.

Definition 4.1. Let

$$\psi_z(w) := \frac{1}{2ic} \log\left(\frac{\zeta^+(z) - w}{w - \zeta^-(z)}\right)$$

where log is the principal branch of logarithm and let[3]

$$\psi_z^{\iota/o}(w) := \psi_z(w) \pm \frac{\pi}{2c}.$$

Note that with this choice of branch, ψ_z is defined on $\mathbb{C} \setminus L_z$, where L_z is the real line through $\zeta^+(z)$ and $\zeta^-(z)$ minus the segment $[\zeta^-(z), \zeta^+(z)]$. In particular, ψ_z^ι and ψ_z^o are both defined in a disk centered at $w = \frac{1}{2}(\zeta^+(z) + \zeta^-(z))$ whose radius is of order z.

Remark 4.2. Observe that if $\operatorname{Im} \frac{\zeta^+(z)-w}{w-\zeta^-(z)} < 0$ (or equivalently, if w is in the half-plane to the left of L_z), then

$$\psi_z^\iota(w) = \frac{1}{2ic} \log\left(\frac{w-\zeta^+(z)}{w-\zeta^-(z)}\right),$$

and if $\operatorname{Im} \frac{\zeta^+(z)-w}{w-\zeta^-(z)} > 0$ (or equivalently, if w is in the half-plane to the right of L_z), then

$$\psi_z^o(w) = \frac{1}{2ic} \log\left(\frac{w-\zeta^+(z)}{w-\zeta^-(z)}\right).$$

Definition 4.3. Let

$$A(z,w) := \psi_{p(z)}^{\iota/o} \circ q_z(w) - \psi_z^{\iota/o}(w) - z, \quad A_0(w) := -\frac{1}{q_0(w)} + \frac{1}{w} - 1.$$

Note that the formula for $A(z,w)$ does not depend on whether the ingoing or outgoing coordinate ψ_z is used, and is therefore well-defined. The map A is for now defined on the open set $\{(z,w) \in \mathcal{P}_p^\iota \times \mathbb{C} : w \notin L_z \text{ and } q_z(w) \notin L_{p(z)}\}$; however, we will see below that it extends analytically to a bi-disk $\mathbb{D}(r,r) \times \mathbb{D}(0,r)$.

Proposition 4.4. (1) *A_0 is analytic near 0, and $A_0(w) = (b_{0,3} - 1)w + O(w^2)$.*

(2) *There exists $r > 0$ such that for all $z \in \mathbb{D}(r,r)$, $A(z,\cdot)$ extends analytically to the disk $\mathbb{D}(0,r)$.*

Proof. Item (1) is an easy computation. For (2), observe that if $r > 0$ is small enough, $z \in \mathbb{D}(r,r)$ and $w \notin L_z$, $q_z(w) \notin L_{p(z)}$, we have

$$\begin{aligned} A(z,w) &= \frac{1}{2ic} \log\left(\frac{q_z(w) - \zeta^+(p(z))}{q_z(w) - \zeta^-(p(z))}\right) - \frac{1}{2ic} \log\left(\frac{w-\zeta^+(z)}{w-\zeta^-(z)}\right) - z \\ &= \frac{1}{2ic} \log\left(\frac{q_z(w) - \zeta^+(p(z))}{w-\zeta^+(z)} : \frac{q_z(w) - \zeta^-(p(z))}{w-\zeta^-(z)}\right) - z \\ &= \frac{1}{2ic} \log\left(\frac{q_z(w) - q_z(\zeta^+(z))}{w-\zeta^+(z)} : \frac{q_z(w) - q_z(\zeta^-(z))}{w-\zeta^-(z)}\right) - z. \end{aligned}$$

[3]The map ψ_z^o should not be confused with ψ_f^o, the latter being the outgoing Fatou parametrization. In our notation the outgoing Fatou parametrization will always have a function in its subscript, whereas the map defined above will always have a point in its subscript.

Taking $r > 0$ even smaller if necessary, we may assume further that for all $z \in \mathbb{D}(r,r)$, we have $q_z(w) \neq q_z(\zeta^{\pm}(z))$ if $w \in \mathbb{D}(0,r) \setminus \{\zeta^{\pm}(z)\}$, and that q_z has no critical point on $\mathbb{D}(0,r)$. For this choice of r, it is then clear that $w \mapsto A(z,w)$ has removable singularities at $w = \zeta^{\pm}(z)$. ∎

Proposition 4.5. *Let Θ be as in (3.3). We have*

$$A(z,w) = zA_0(w) + (\Theta + 1/2 - b_{0,3})z^2 + O(z^3, z^2 w),$$

where the O-constants are uniform for $(z,w) \in \mathbb{C}^2$ near $(0,0)$ (with $z \in \mathcal{P}_p^{\iota}$).

Proof. Let K be a compact in $\mathbb{C}^*$, and let $w \in K$. By a straightforward computation we obtain

$$\begin{aligned} \frac{1}{2ic}\log\left(\frac{w-\zeta^+(z)}{w-\zeta^-(z)}\right) &= \frac{1}{2ic}\left(\frac{\zeta^-(z)-\zeta^+(z)}{w} - \frac{(\zeta^+(z))^2-(\zeta^-(z))^2}{2w^2}\right) + O(z^3) \\ &= -\frac{z}{w} - \frac{\Theta z^2}{w} + \frac{z^2}{2w^2} + O(z^3). \end{aligned} \tag{4.1}$$

Using this we can now show that

$$\begin{aligned} \psi^{\iota/o}_{p(z)} \circ q_z(w) &= -\frac{p(z)}{q_z(w)} - \frac{\Theta(p(z))^2}{q_z(w)} + \frac{(p(z))^2}{2(q_z(w))^2} + O(z^3) \\ &= -\frac{z-z^2}{q_0(w)} - \frac{\Theta z^2}{q_0(w)} + \frac{z^2}{2(q_0(w))^2} + O(z^3). \end{aligned}$$

This implies that

$$\begin{aligned} A(z,w) &= zA_0(w) + \Theta z^2\left(\frac{1}{w} - \frac{1}{q_0(w)}\right) \\ &\quad + \frac{z^2}{2}\left(\frac{1}{(q_0(w))^2} - \frac{1}{w^2} + \frac{2}{q_0(w)}\right) + O(z^3) \\ &= zA_0(w) + \Theta z^2 + \frac{z^2}{2}(1-2b_{0,3}) + O(z^3, z^2 w) \\ &= zA_0(w) + (\Theta + 1/2 - b_{0,3})z^2 + O(z^3, z^2 w). \end{aligned}$$

Here, we have used the fact that $A(z,\cdot)$ is analytic, hence all terms in w with negative powers are canceled.

Note that the constant in $O(z^3, z^2 w)$ a priori depends on $K \subset \mathbb{C}^*$. Let $\phi_z(w) := \frac{A(z,w)-zA_0(w)}{z^2}$; by Proposition 4.4 it is holomorphic on $\mathbb{D}(0,r)$. We have proved that for compact $K \subset \mathbb{C}^*$, for all $w \in K$, and for all $z \in \mathcal{P}_p^{\iota}$, we have $|\phi_z(w)| \leq C_K$. By taking $K = \{|w| = r/2\}$ we therefore obtain the same estimate $|\phi_z(w)| \leq C_K$ for all $|w| \leq r/2$ because of the maximum modulus principle. This gives the desired uniformity. ∎

Definition 4.6. As in [7], let $\nu \in (1/2, 2/3)$ and let

$$\mathcal{R}_z := \{W \in \mathbb{C} : |z|^{1-\nu}/10 < \operatorname{Re} W < \pi/c - |z|^{1-\nu}/10 \quad \text{and } -1/2 < \operatorname{Im} W < 1/2\}.$$

Definition 4.7. Let $\chi_z(W) = W + (b_{0,3} - 1)R(z, W)$, where

$$R(z, W) := cze^W F_c(W) \tag{4.2}$$

and F_c is the primitive on $\mathcal{R}_0$ of $W \mapsto e^{-W} \cot(cW)$ vanishing at $\frac{\pi}{2c}$.

A straightforward computation shows that $R(z, W)$ is a solution of the linear PDE

$$-z\frac{\partial R}{\partial z} + \frac{\partial R}{\partial W} = cz\cot(cW). \tag{4.3}$$

Lemma 4.8. *We have*

$$(\psi_z^{\iota/o})^{-1}(W) = -cz\cot(cW) - z/2 + O(z^2\cot(cW), z^2).$$

Proof. We have

$$(\psi_z^{\iota/o})^{-1}(W) = \frac{\zeta^+(z) - \zeta^-(z)e^{2icW}}{1 - e^{2icW}} \tag{4.4}$$

and using the fact that $\zeta^{\pm}(z) := (-1/2 \pm ic)z + (\frac{a+(b-1)b_{0,3}-\Theta}{2} \pm ic\Theta)z^2 + O(z^3)$, we get the conclusion. ∎

Lemma 4.9. *Assume that $\psi_z^\iota(w) \in \mathcal{R}_z$, and let $W := \psi_z^\iota(w)$, $W_1 := \psi_{p(z)}^\iota \circ q_z(w)$ and $z_1 := p(z)$. Then*

$$|R(z_1, W_1) - R(z, W) - cz^2\cot(cW)| = O(|z|^{2+\delta}) \tag{4.5}$$

with $\delta := 2\nu - 1 > 0$.

Proof. In the computations that follow, we will frequently use the bound $\cot(cW) = O(\sin(cW)^{-1})$, valid for $W \in \mathcal{R}_0$. Let $x := (z, W)$ and $h := (z_1, W_1) - (z, W)$. Then by Taylor–Lagrange's formula, we have

$$R(x + h) - R(x) - \mathrm{d}R_x(h) = \int_0^1 \frac{(1-t)^2}{2}\mathrm{d}^2R_{x+th}(h, h)\,dt \tag{4.6}$$

and

$$\mathrm{d}^2R_y(h, h) = R_{zz}(y)h_1^2 + 2R_{zW}(y)h_1h_2 + R_{WW}(y)h_2^2$$

where $R_{zz} := \frac{\partial^2 R}{\partial z^2}$, etc. Moreover, $R_{zz} = 0$, and

$$R_{zW}(z, W) = ce^W F_c(W) + c\cot(cW),$$

$$R_{WW}(z, W) = cz\left(e^W F_c(W) + \cot(cW) - \frac{c}{\sin^2(cW)}\right).$$

Since $e^W = O(1)$ and $F_c(W) = O(\log W, \log(W - \pi/c)) = O(\sin(cW)^{-1})$ in $\mathcal{R}_0$, we have

$$R_{zW}(z, W) = O(\sin(cW)^{-1}).$$

Similarly, $R_{WW}(z, W) = O(z\sin(cW)^{-2})$.

Using $h_1 = O(z^2)$ and $h_2 = O(z)$, we deduce

$$d^2 R_y(h, h) = O(z^3 \sin(cW)^{-1}) + O(z^3 \sin(cW)^{-2}) \tag{4.7}$$

Since $W \in \mathcal{R}_z$ by assumption, we have $z^3 \sin(cW)^{-2} = O(|z|^{1+2\nu}) = O(|z|^{2+\delta})$ with $\delta := 2\nu - 1 > 0$. Therefore

$$|R(z_1, W_1) - R(z, W) - dR_x(h)| = O(|z|^{2+\delta}).$$

It now remains to compare $dR_x(h)$ and $cz^2 \cot(cW)$. First, note that

$$h = (-z^2 + O(z^3), z + O(zw, z^2)) = \big(-z^2 + O(z^3), z + O(z^2 \sin(cW)^{-1})\big).$$

Indeed, by Lemma 4.8 and the assumption that $W \in \mathcal{R}_z$, we have $w = O(z \sin(cW)^{-1})$. Therefore

$$\begin{aligned} dR_x(h) &= R_z(x)h_1 + R_W(x)h_2 \\ &= -z^2 R_z(x) + zR_W(x) + O(z^3 R_z, z^2 \sin(cW)^{-1} R_W) \\ &= cz^2 \cot(cW) + O(z^3 \sin(cW)^{-1}, z^3 \sin(cW)^{-2}), \end{aligned}$$

hence we have

$$|dR_x(h) - cz^2 \cot(cW)| = O(|z|^{2+\delta}).$$ ■

Definition 4.10. We define $\tilde{A}(z, w) := \chi_{p(z)} \circ \psi^\iota_{p(z)} \circ q_z(w) - \chi_z \circ \psi^\iota_z(w) - z$.

Proposition 4.11 (Almost translation property). *Let $\delta := 2\nu - 1 > 0$. Then*

$$|\tilde{A}(z, w) - \Lambda z^2| = O(|z|^{2+\delta})$$

for all (z, w) such that $\psi^\iota_z(w) \in \mathcal{R}_z$, where $\Lambda := \Theta + 1 - 3b_{0,3}/2$.

Proof. Let $z_1 := p(z)$, $W := \psi^\iota_z(w)$ and $W_1 := \psi^\iota_{z_1} \circ q_z(w)$. We have

$$\begin{aligned} \tilde{A}(z, w) &= \chi_{z_1} \circ \psi^\iota_{z_1} \circ q_z(w) - \chi_z \circ \psi^\iota_z(w) - z \\ &= \psi^\iota_{z_1} \circ q_z(w) - \psi^\iota_z(w) - z + (b_{0,3} - 1)(R(z_1, W_1) - R(z, W)). \end{aligned}$$

By Lemma 4.9,

$$|\tilde{A}(z, w) - A(z, w) - cz^2(b_{0,3} - 1) \cot(cW)| = O(|z|^{2+\delta}).$$

On the other hand, by Proposition 4.5 we have

$$\begin{aligned} A(z, w) &= zA_0(w) + (\Theta + 1/2 - b_{0,3})z^2 + O(z^2 w, z^3) \\ &= (b_{0,3} - 1)zw + (\Theta + 1/2 - b_{0,3})z^2 + O(zw^2, z^2 w, z^3), \end{aligned}$$

so Lemma 4.8 yields

$$A(z, w) = (1 - b_{0,3})cz^2 \cot(cW) + z^2(\Theta + 1 - 3b_{0,3}/2) + O(zw^2, z^2 w, z^3, z^3 \cot(cW)).$$

Putting all of these estimates together, we get

$$|\tilde{A}(z,w) - \Lambda z^2| = O(|zw^2|, |z^2 w|, |z|^3, |z|^{2+\delta}). \tag{4.8}$$

Finally, since by assumption $\psi_z(w) \in \mathcal{R}_z$, we have $|w| = O(|z|^\nu)$. Indeed, by Lemma 4.8 and the definition of $\mathcal{R}_z$, $w = O(z \sin(cW)^{-1}) = O(|z|/|z|^{1-\nu}) = O(|z|^\nu)$.

Finally, using $\nu \in (1/2, 2/3)$ we have

- $|zw^2| = O(|z|^{1+2\nu}) = O(|z|^{2+\delta})$, since $\delta = 2\nu - 1$ by definition;
- $|z^2 w| = O(|z|^{2+\nu}) = O(|z|^{1+2\nu})$ since $2 + \nu > 1 + 2\nu$;
- $|z^3| = O(|z|^{1+2\nu})$ (again, since $1 + 2\nu < 3$).

Therefore (4.8) gives the required estimate

$$|\tilde{A}(z,w) - \Lambda z^2| = O(|z|^{2+\delta}).$$

■

Lemma 4.12. *As $W \to 0$ in $\mathcal{R}_0$, we have*

$$F_c(W) = \frac{1}{c}\log(cW) - \frac{1}{c}\int_0^{\pi/(2c)} e^{-u} \log\sin(cu)\, du + o(1). \tag{4.9}$$

Similarly, as $W \to \pi/c$ in $\mathcal{R}_0$, we have

$$F_c(W) = e^{-\pi/c}\frac{1}{c}\log(\pi - cW) + \frac{1}{c}\int_{\pi/(2c)}^{\pi/c} e^{-u} \log\sin(cu)\, du + o(1). \tag{4.10}$$

Proof. Recall that

$$F_c(W) = \int_{\pi/(2c)}^{W} e^{-u}\cot(cu)\, du.$$

An integration by parts gives

$$F_c(W) = \frac{1}{c}e^{-W}\log\sin(cW) + \frac{1}{c}\int_{\pi/(2c)}^{W} e^{-u}\log\sin(cu)\, du,$$

which implies both (4.9) and (4.10). ■

5. Proof of the main theorem

We begin this section by explaining how the map ψ_z, defined in the previous section, transforms the complex plane.

Let D_z be the disk of radius $\frac{1}{2}|\zeta^+(z) - \zeta^-(z)| = c|z| + O(z^2)$ centered at $\frac{1}{2}(\zeta^+(z) + \zeta^-(z))$. Let $\mathcal{S}(z, R)$ be the union of the two disks of radius R that both contain the points $\zeta^+(z), \zeta^-(z)$ on their boundary. The radius R will be sufficiently small, to be fixed later. The definition of $\mathcal{S}(z, R)$ of course only makes sense when the distance between $\zeta^+(z)$ and $\zeta^-(z)$ is less than $2R$, which once R is fixed will be satisfied for $|z|$ sufficiently small. Our choice of R will depend on the map q_0, but not on z.

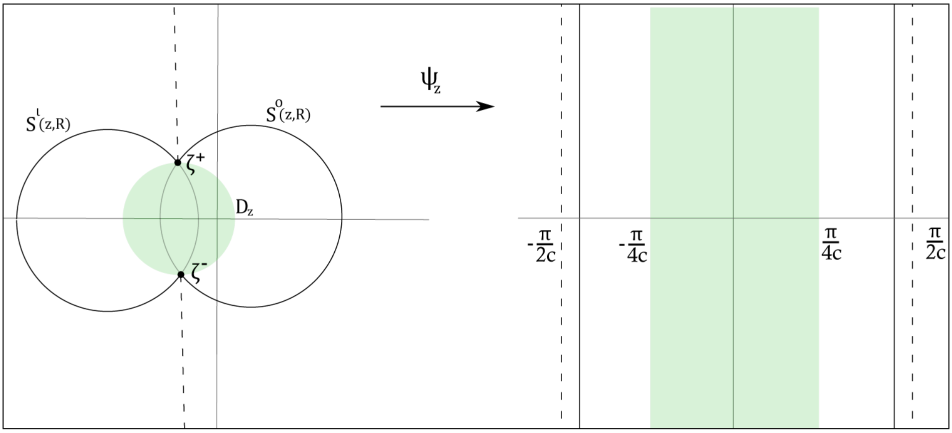


Fig. 2

The line L_z through $\zeta^+(z)$ and $\zeta^-(z)$ cuts the complex plane into the left half-plane H_z^ι and the right half-plane H_z^o. We define $\mathcal{S}^{\iota/o}(z,R) := \mathcal{S}(z,R) \cap H_z^{\iota/o}$. The map ψ_z maps the disk D_z to the shaded strip $[-\frac{\pi}{4c}, \frac{\pi}{4c}] \times i\mathbb{R}$. The image of $\mathcal{S}(z,R)$ is bounded by two vertical lines, intersecting the real line in points of the form $\mp\frac{\pi}{2c} + O(z)$; see Figure 2. Next we define $\mathcal{P}_R^{\iota/o} := \mathbb{D}(\mp R, R)$ and observe that $\mathcal{S}^{\iota/o}(p^n(z), R) \xrightarrow{n\to\infty} \mathcal{P}_R^{\iota/o}$ for all $z \in \mathcal{B}_p$.

Fixing constants: We choose constants $\nu \in (1/2, 2/3)$ and $R > 0$ such that

(i) the inverse q_0^{-1} is well defined on $\mathbb{D}(0, 2R)$;

(ii) $q_0(\mathcal{P}_R^\iota) \subset \mathcal{P}_R^\iota$ and $q_0^{-1}(\mathcal{P}_R^o) \subset \mathcal{P}_R^o$.

Fixing a compact: For the rest of this section we fix a compact subset $K' \times K \subset \mathcal{B}_p \times \mathcal{B}_{q_0}$.

Fixing an integer: We fix $n_0 \in \mathbb{N}$ large enough that for every $n > n_0$ we have $p^n(K') \subset \mathcal{P}_p^\iota$ and $q_0^{k_n}(K) \subset \mathcal{P}_R^\iota$, where $k_n := \lfloor n^\nu \rfloor$.

Notations: Given a point $(z_0, w_0) \in K' \times K$, we will write $\epsilon_j := p^{n+j}(z_0)$ and $w_j := q_{\epsilon_{j-1}} \circ q_{\epsilon_{j-2}} \circ \cdots \circ q_{\epsilon_0}(w_0)$.

Remark 5.1. Unless otherwise stated, all the constants appearing in estimates depend only on the compact $K' \times K$, but not on the point (z_0, w_0) or the integer n.

In computations throughout this section, we will frequently use the following Euler–Maclaurin formula for the estimate of a finite sum:

$$\sum_{j=n}^{m-1} f(j) = \int_n^m f(x)\,dx + \tfrac{1}{2}(f(n) - f(m)) + \tfrac{1}{12}(f'(m) + f'(n)) + \rho(f, m, n),$$

where f is a smooth function and $|\rho(f, m, n)| \le \frac{1}{120}\int_n^m |f'''(x)|\,dx$.

5.1. Entering the eggbeater

We start with the following lemma which tells us that the first k_n elements of the non-autonomous orbit w_j will be suitably close to the autonomous orbit $\{q_0^j(w_0)\}$, assuming that we have chosen sufficiently large n.

Lemma 5.2. *There exists $n_0 > 0$ such that for every $0 \leq \ell \leq k_n$ (where $n > n_0$) we have $\phi_{q_0}^{\iota}(w_\ell) = \phi_{q_0}^{\iota}(w_0) + \ell + o(1)$ (as $n \to \infty$) and $w_{k_n} = -\frac{1}{k_n} + O(\frac{\ln n}{k_n^2}) \in \mathcal{P}_R^{\iota}$.*

Proof. We use induction on ℓ. For $\ell = 0$ the claim clearly holds. Now we assume that $\phi_{q_0}^{\iota}(w_j) = \phi_{q_0}^{\iota}(w_0) + j + o(1)$ for all $0 \leq j \leq \ell < k_n$ and we proceed to prove that the same holds for $j = \ell + 1 \leq k_n$. Note that $\phi_{q_0}^{\iota}(w_j) = \phi_{q_0}^{\iota}(w_0) + j + o(1)$ implies $w_j = q_0^j(w_0) + o(1) = O(1/j)$

Observe that for all $0 \leq j \leq \ell + 1$ we have

$$\begin{aligned}
\phi_{q_0}^{\iota}(w_{j+1}) &= \phi_{q_0}^{\iota}\big(q_0(w_j) + b\epsilon_j^2 + O(\epsilon_j^3, \epsilon_j w_j^3, \epsilon_j^2 w_j^2)\big) \\
&= \phi_{q_0}^{\iota}(w_j) + 1 + O\big((\phi_{q_0}^{\iota})'(w_j)\epsilon_j^2, (\phi_{q_0}^{\iota})'(w_j)\epsilon_j w_j^3, (\phi_{q_0}^{\iota})'(w_j)\epsilon_j^2 w_j^2\big) \\
&= \phi_{q_0}^{\iota}(w_j) + 1 + O\left(\frac{\epsilon_j^2}{w_j^2}, \epsilon_j w_j, \epsilon_j^2\right) \\
&= \phi_{q_0}^{\iota}(w_j) + 1 + O\left(\frac{j^2}{n^2}, \frac{1}{nj}, \frac{1}{n^2}\right) \\
&= \phi_{q_0}^{\iota}(w_j) + 1 + O\left(\frac{k_n^2}{n^2}, \frac{1}{nj}\right).
\end{aligned}$$

In the last two equalities we have used the fact that $\epsilon_j = O(\frac{1}{n+j}) \leq O(\frac{1}{n})$ and $w_j = O(\frac{1}{j}) \geq O(\frac{1}{k_n})$. It follows that

$$\begin{aligned}
\phi_{q_0}^{\iota}(w_{\ell+1}) &= \phi_{q_0}^{\iota}(w_0) + \ell + 1 + \sum_{j=0}^{\ell} O\left(\frac{k_n^2}{n^2}, \frac{1}{nj}\right) \\
&= \phi_{q_0}^{\iota}(w_0) + \ell + 1 + O\left(\frac{k_n^2 \ell}{n^2}, \frac{\log \ell}{n}\right) \\
&= \phi_{q_0}^{\iota}(w_0) + \ell + 1 + O\left(\frac{k_n^3}{n^2}\right) \\
&= \phi_{q_0}^{\iota}(w_0) + \ell + 1 + o(1),
\end{aligned}$$

where the last equality holds since $k_n = \lfloor n^\nu \rfloor$ for $\nu \in (1/2, 2/3)$. Note these computations are only valid if $w_j \in \mathcal{B}_{q_0}$ for all $0 \leq j \leq \ell + 1$, but clearly this is the case for all sufficiently large n. The fact that the choice of good n's does not depend on ℓ is derived from the above $o(1)$ estimate. Therefore by induction we obtain $\phi_{q_0}^{\iota}(w_{k_n}) = \phi_{q_0}^{\iota}(w_0) + k_n + o(1)$. Furthermore, since $\phi_{q_0}^{\iota}(w) = -1/w + (1 - b_{0,3}) \log(-w) + o(1)$, we also obtain $w_{k_n} = -\frac{1}{k_n} + O(\frac{\ln n}{k_n^2})$ and hence $w_{k_n} \in \mathcal{P}_R^{\iota}$ for all sufficiently large n. ∎

Lemma 5.3. *For all sufficiently large n we have*

$$\psi^{\iota}_{\epsilon_{k_n}}(w_{k_n}) = -\frac{\epsilon_{k_n}}{w_{k_n}} + \frac{\epsilon^2_{k_n}}{2w^2_{k_n}} + o(\epsilon_{k_n}).$$

Proof. First observe that $\psi_{\epsilon_{k_n}}(w_{k_n})$ is well defined for all sufficiently large n. Indeed, for all large n we have $\epsilon_{k_n} \in \mathcal{P}^{\iota}_p$ and $w_{k_n} \in \mathcal{S}^{\iota}(\epsilon_{k_n}, R)$. From the computation in the proof of Proposition 4.5 we can conclude that

$$\begin{aligned}\psi^{\iota}_{\epsilon_{k_n}}(w_{k_n}) &= -\frac{\epsilon_{k_n}}{w_{k_n}} + \frac{\epsilon^2_{k_n}}{2w^2_{k_n}} + O\left(\frac{\epsilon^3_{k_n}}{w^3_{k_n}}\right)\\ &= -\frac{\epsilon_{k_n}}{w_{k_n}} + \frac{\epsilon^2_{k_n}}{2w^2_{k_n}} + O\left(\epsilon_{k_n}\cdot\frac{k^3_n}{n^2}\right)\\ &= -\frac{\epsilon_{k_n}}{w_{k_n}} + \frac{\epsilon^2_{k_n}}{2w^2_{k_n}} + o(\epsilon_{k_n}),\end{aligned}$$

where the last two equalities follow from the same argument as in the proof of the previous lemma. ∎

Remark 5.4. By Lemma 5.2 we have $w_{k_n} \in \mathcal{P}^{\iota}_R$ and since $\mathcal{S}^{\iota}(\epsilon_{k_n}, R) \to \mathcal{P}^{\iota}_R$ as $n \to \infty$ we can conclude that $w_{k_n} \in \mathcal{S}^{\iota}(\epsilon_{k_n}, R)$ for all $(z_0, w_0) \in K' \times K$ and all sufficiently large n. Moreover, by combining Lemmas 5.2 and 5.3 we also get

$$w_{k_n} = -|\epsilon_{k_n}|^{\nu} + o(\epsilon^{\nu}_{k_n})$$

and hence $\psi^{\iota}_{\epsilon_{k_n}}(w_{k_n}) \in \mathcal{R}_{\epsilon_{k_n}}$ for all $(z_0, w_0) \in K' \times K$ and all sufficiently large n (recall that $\mathcal{R}_z$ was introduced in Definition 4.6).

Definition 5.5 (Approximate Fatou coordinate). Let $\Phi_z := \chi_z \circ \psi^{\iota}_z$.

Lemma 5.6 (Comparison with incoming Fatou coordinates). *We have*

$$\frac{1}{\epsilon_{k_n}}\Phi_{\epsilon_{k_n}}(w_{k_n}) = \phi^{\iota}_{q_0}(w_{k_n}) + \frac{k^2_n}{2n} + (1 - b_{0,3})\ln n + E^{\iota} + o(1),$$

where

$$E^{\iota} := (b_{0,3} - 1)\left(\ln c - \int_0^{\pi/(2c)} e^{-u} \ln \sin(cu)\, du\right).$$

Proof. Recall that by Lemmas 5.2 and 5.3 we have

$$w_{k_n} = -\frac{1}{k_n} + O\left(\frac{\ln n}{k^2_n}\right)$$

and

$$\psi^{\iota}_{\epsilon_{k_n}}(w_{k_n}) = -\frac{\epsilon_{k_n}}{w_{k_n}} + \frac{\epsilon^2_{k_n}}{2w^2_{k_n}} + o(\epsilon_{k_n}).$$

Next, we have

$$\begin{aligned}\frac{1}{\epsilon_{k_n}}\Phi_{\epsilon_{k_n}}(w_{k_n}) &= \frac{1}{\epsilon_{k_n}}\chi_{\epsilon_{k_n}}\circ\psi^\iota_{\epsilon_{k_n}}(w_{k_n})\\ &= \frac{W}{\epsilon_{k_n}} - c(1-b_{0,3})e^W F_c(W) \quad \text{with } W := \psi^\iota_{\epsilon_{k_n}}(w_{k_n})\\ &= -\frac{1}{w_{k_n}} + \frac{k_n^2}{2n} - c(1-b_{0,3})e^W F_c(W) + o(1).\end{aligned}$$

Note that $e^W = 1 + O(W) = 1 + O(\epsilon_{k_n}/w_{k_n})$ and therefore by Lemma 4.12,

$$\begin{aligned}ce^W F_c(W) &= \log\left(-\frac{\epsilon_{k_n}}{w_{k_n}}\right) + \ln c - \int_0^{\pi/(2c)} e^{-u}\ln\sin(cu)du + o(1)\\ &= -\log(-w_{k_n}) - \ln n + \ln c - \int_0^{\pi/(2c)} e^{-u}\ln\sin(cu)du + o(1).\end{aligned}$$

Putting all together we get

$$\begin{aligned}\frac{1}{\epsilon_{k_n}}\Phi_{\epsilon_{k_n}}(w_{k_n}) &= -\frac{1}{w_{k_n}} + (1-b_{0,3})\log(-w_{k_n}) + \frac{k_n^2}{2n} + (1-b_{0,3})\ln n + E^\iota + o(1)\\ &= \phi^\iota_{q_0}(w_{k_n}) + \frac{k_n^2}{2n} + (1-b_{0,3})\ln n + E^\iota + o(1).\end{aligned}$$ ∎

5.2. *Passing through the eggbeater*

Definition 5.7. Let α_0, β_0 be as in (1.2) and define $M_n := \lfloor(\alpha_0 - 1)n + \beta_0 \ln n\rfloor$, where $\lfloor\cdot\rfloor$ is the floor function. Let $\ell_n := \lfloor e^{\pi/c} k_n\rfloor$ and $\rho_n := \{(\alpha_0 - 1)n + \beta_0 \ln n\}$, where $\{\cdot\}$ denotes the fractional part. Finally, we define $W_j := \Phi_{\epsilon_j}(w_j)$.

Lemma 5.8. *For $k_n \le i \le M_n - \ell_n$, we have $W_i \in \mathcal{R}_{\epsilon_i}$ and*

$$W_i = W_{k_n} + \sum_{j=k_n}^{i-1} \epsilon_j + \tilde{A}(\epsilon_j, w_j).$$

Proof. We prove this by induction on i.

- Initialization: It comes from the fact that $W_{k_n} = k_n/n + O(k_n^2/n^2)$ (see Lemmas 5.2, 5.3 and 5.6).
- Inductive step: Let $k_n \le i \le M_n - \ell_n$ and assume that $W_j \in \mathcal{R}_{\epsilon_j}$ for all $k_n \le j < i$. We need to prove that also $W_i \in \mathcal{R}_{\epsilon_i}$. First recall that by Proposition 4.11 we have

$$\begin{aligned}\left|W_i - W_{k_n} - \sum_{j=k_n}^{i-1}\epsilon_j\right| &= \left|\sum_{j=k_n}^{i-1}\tilde{A}(\epsilon_j, w_j)\right|\\ &\le \sum_{j=k_n}^{i-1}\frac{C}{(n+j)^2} = \frac{C(i-k_n)}{(n+k_n)(n+i)} + O\left(\frac{1}{n}\right),\end{aligned}$$

where we have used the Euler–Maclaurin formula to compute the sum $\sum_{j=k_n}^{i-1} \frac{1}{(n+j)^2}$ and the fact that $\epsilon_j = \frac{1}{n+j} + O(\frac{\ln(n+j)}{(n+j)^2})$, and that $\tilde{A}(\epsilon_j, w_j) = O(\epsilon_j^2)$ whenever $\Phi_{\epsilon_j}(w_j) \in \mathcal{R}_{\epsilon_j}$. Moreover, since $i \leq M_n - \ell_n$ we have

$$\frac{C(i-k_n)}{(n+k_n)(n+i)} = O\left(\frac{1}{n}\right)$$

and therefore

$$\begin{aligned} W_i &= W_{k_n} + \sum_{j=k_n}^{i-1} \epsilon_j + O\left(\frac{1}{n}\right) \\ &= \frac{k_n}{n} + \sum_{j=k_n}^{i-1} \frac{1}{n+j} + O\left(\frac{k_n^2}{n^2}\right) \\ &= \ln\left(\frac{n+i}{n+k_n}\right) + \frac{k_n}{n} + O\left(\frac{k_n^2}{n^2}\right) \\ &= \ln\left(1+\frac{i}{n}\right) + O\left(\frac{k_n^2}{n^2}\right). \end{aligned}$$

Now observe that

$$\ln\left(1+\frac{i}{n}\right) \leq \frac{\pi}{c} - e^{-\pi/c}\frac{\ell_n}{n} = \frac{\pi}{c} - \frac{k_n}{n} + O\left(\frac{1}{n}\right),$$

and therefore

$$\begin{aligned} |\epsilon_i|^{1-\nu} + o(\epsilon_i^{1-\nu}) = \operatorname{Re} W_{k_n} < \operatorname{Re} W_i &\leq \frac{\pi}{c} - \frac{k_n}{n} + O\left(\frac{k_n^2}{n^2}\right) \\ &= \frac{\pi}{c} - |\epsilon_i|^{1-\nu} + o(\epsilon_i^{1-\nu}) \end{aligned}$$

and $\operatorname{Im} W_i = O(k_n^2/n^2)$. Finally, note that all the bounds in the $O(\cdot)$ terms above can be chosen to be independent of i. Therefore, there exists $N > 0$ (independent of i), such that for every $n > N$ we have $W_i \in \mathcal{R}_{\epsilon_i}$ as long as $W_j \in \mathcal{R}_{\epsilon_j}$ for all $k_n \leq j < i$. ∎

In the above proof we have seen that $W_{M_n-\ell_n} = \pi/c - |\epsilon_{M_n-\ell_n}|^{1-\nu} + o(\epsilon_{M_n-\ell_n}^{1-\nu})$, but for our purposes we will need the following sharper estimate.

Lemma 5.9. *We have*

$$W_{M_n-\ell_n} = \frac{\pi}{c} + \frac{G_n}{n} + o\left(\frac{1}{n}\right),$$

where

$$G_n := -e^{-\pi/c}\ell_n + \frac{k_n^2}{2n} + (1-b_{0,3})e^{-\pi/c}\ln n - e^{-\pi/c}\rho_n + \phi_{q_0}^{\iota}(w_0) + \tilde{C}$$

and

$$\tilde{C} := (1-a)e^{-\pi/c}\frac{\pi}{c} + (1-e^{-\pi/c})\big(\Theta + \tfrac{3}{2}(1-b_{0,3}) + (a-1) - \phi_p^{\iota}(z_0)\big) + E^{\iota}.$$

Proof. First recall that by Proposition 4.11 and Lemma 5.8 we have

$$W_{M_n-\ell} = W_{k_n} + \sum_{j=k_n}^{M_n-\ell_n-1} \epsilon_j + \tilde{A}(\epsilon_j, w_j)$$
$$= W_{k_n} + \sum_{j=k_n}^{M_n-\ell_n-1} \left(\epsilon_j + \Lambda\epsilon_j^2 + O(\epsilon_j^{2+\delta})\right).$$

Also recall that by Lemmas 5.2 and 5.6 we have

$$W_{k_n} = \frac{1}{n}\left(\phi^\iota_{q_0}(w_0) + k_n + \frac{k_n^2}{2n} + (1-b_{0,3})\ln n + E^\iota\right) + o\left(\frac{1}{n}\right)$$

and

$$\epsilon_j = \frac{1}{n+j} - \frac{(1-a)\ln(n+j) + \phi^\iota_p(z_0)}{(n+j)^2} + O\left(\frac{\ln^2 n}{n^3}\right).$$

First observe that since $M_n = O(n)$ and $\epsilon_j = O(\frac{1}{j+n}) \le O(\frac{1}{n})$ we have

$$\sum_{j=k_n}^{M_n-\ell_n-1} \tilde{A}(\epsilon_j, w_j) = \sum_{j=k_n}^{M_n-\ell_n-1} \Lambda\epsilon_j^2 + O\left(\frac{1}{n^{1+\delta}}\right) = \sum_{j=k_n}^{M_n-\ell_n-1} \Lambda\epsilon_j^2 + o\left(\frac{1}{n}\right).$$

Next we define the functions

$$\delta_1(j) := \frac{1}{n+j}, \quad \delta_2(j) := -\frac{(1-a)\ln(n+j) + \phi^\iota_p(z_0)}{(n+j)^2},$$
$$\delta_3(j) := \epsilon_j - \delta_1(j) - \delta_2(j) = O\left(\frac{\ln^2 n}{n^3}\right).$$

Then

$$\sum_{j=k_n}^{M_n-\ell_n-1} \delta_3(j) = O\left(\frac{\ln^2 n}{n^2}\right) = o\left(\frac{1}{n}\right),$$

and therefore

$$\sum_{j=k_n}^{M_n-\ell_n-1} \epsilon_j = \sum_{j=k_n}^{M_n-\ell_n-1} (\delta_1(j) + \delta_2(j)) + O\left(\frac{\ln^2 n}{n^2}\right).$$

Furthermore, by the Euler–Maclaurin formula applied to $\delta_1 + \delta_2$, we get

$$\sum_{j=k_n}^{M_n-\ell_n-1} (\delta_1(j) + \delta_2(j))$$
$$= \int_{k_n}^{M_n-\ell_n} (\delta_1(j) + \delta_2(j))\, dj + \frac{1}{2}(\delta_1(k_n) - \delta_1(M_n - \ell_n))$$
$$+ \rho(\delta_1 + \delta_2, k_n, M_n - \ell_n - 1)$$
$$= \int_{k_n}^{M_n-\ell_n} \delta_1(j)\, dj + \int_{k_n}^{M_n-\ell_n} \delta_2(j)\, dj + \frac{1}{2n}(1 - e^{-\pi/c}) + o\left(\frac{1}{n}\right).$$

Next we compute the two integrals in the above expression:

$$\begin{aligned}\int_{k_n}^{M_n-\ell_n}\delta_1(j)\,dj &= \ln\left(\frac{n+M_n-\ell_n}{n+k_n}\right)\\ &= \frac{\pi}{c}+(b_{0,3}-a)(1-e^{-\pi/c})\frac{\ln n}{n}-e^{-\pi/c}\frac{1}{n}\rho_n-\frac{k_n}{n}-e^{-\pi/c}\frac{\ell_n}{n}\\ &\quad+\frac{1}{2n^2}(k_n^2-e^{-2\pi/c}\ell_n^2)+O\left(\frac{1}{n^{3(1-\nu)}}\right)\\ &= \frac{\pi}{c}+(b_{0,3}-a)(1-e^{-\pi/c})\frac{\ln n}{n}-e^{-\pi/c}\frac{1}{n}\rho_n-\frac{k_n}{n}-e^{-\pi/c}\frac{\ell_n}{n}+o\left(\frac{1}{n}\right),\end{aligned}$$

where we have used the fact that $1/2<\nu<2/3$ and $k_n^2-e^{-2\pi/c}\ell_n^2=O(n^\nu)$. For the other integral we have

$$\begin{aligned}\int_{k_n}^{M_n-\ell_n}\delta_2(j)\,dj &= ((1-a)+\phi_p^\iota(z_0))\left(\frac{1}{n+M_n-\ell_n}-\frac{1}{n+k_n}\right)\\ &\quad+(1-a)\left(\frac{\ln(n+M_n-\ell_n)}{n+M_n-\ell_n}-\frac{\ln(n+k_n)}{n+k_n}\right)\\ &= \frac{1}{n}(1-a)(e^{-\pi/c}-1)+\frac{1}{n}(e^{-\pi/c}-1)\phi_p^\iota(z_0)+\frac{1}{n}(1-a)e^{-\pi/c}\frac{\pi}{c}\\ &\quad+(1-a)(e^{-\pi/c}-1)\frac{\ln n}{n}+o\left(\frac{1}{n}\right)\\ &= \frac{1}{n}\left((1-a+\phi_p^\iota(z_0))(e^{-\pi/c}-1)+(1-a)e^{-\pi/c}\frac{\pi}{c}\right)\\ &\quad+(1-a)(e^{-\pi/c}-1)\frac{\ln n}{n}+o\left(\frac{1}{n}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{j=k_n}^{M_n-\ell_n-1}\epsilon_j &= \frac{\pi}{c}+(b_{0,3}-a)(1-e^{-\pi/c})\frac{\ln n}{n}-e^{-\pi/c}\frac{1}{n}\rho_n-\frac{k_n}{n}-e^{-\pi/c}\frac{\ell_n}{n}\\ &\quad+(1-a)(e^{-\pi/c}-1)\frac{\ln n}{n}+o\left(\frac{1}{n}\right).\end{aligned}$$

Next, observe that $\epsilon_j^2=\delta_1^2(j)+O(\frac{\ln n}{n^3})$, so that

$$\sum_{j=k_n}^{M_n-\ell_n-1}\epsilon_j^2=\sum_{j=k_n}^{M_n-\ell_n-1}\delta_1(j)^2+O\left(\frac{\ln n}{n^2}\right).$$

By the Euler–Maclaurin formula, we have

$$\begin{aligned}\sum_{j=k_n}^{M_n-\ell_n-1}\delta_1(j)^2 &= \int_{k_n}^{M_n-\ell_n}\frac{1}{(n+j)^2}\,dj+O\left(\frac{1}{n^2}\right)\\ &= \frac{1}{n+k_n}-\frac{1}{n+M_n-\ell_n}+o\left(\frac{1}{n}\right)=\frac{1}{n}(1-e^{-\pi/c})+o\left(\frac{1}{n}\right).\end{aligned}$$

Therefore

$$\sum_{j=k_n}^{M_n-\ell_n-1} \epsilon_j^2 = \frac{1}{n}(1-e^{-\pi/c}) + o\left(\frac{1}{n}\right).$$

Putting all together we obtain

$$\begin{aligned} W_{M_n-\ell_n} &= W_{k_n} + \sum_{j=k_n}^{M_n-\ell_n-1} (\epsilon_j + \Lambda\epsilon_j^2) + o\left(\frac{1}{n}\right) \\ &= \frac{\pi}{c} - e^{-\pi/c}\frac{\ell_n}{n} + (1-b_{0,3})e^{-\pi/c}\frac{\ln n}{n} + \frac{k_n^2}{2n^2} \\ &\quad + \frac{1}{n}(\phi_{q_0}^{\iota}(w_0) + \tilde{C} - e^{-\pi/c}\rho_n) + o\left(\frac{1}{n}\right). \end{aligned}$$

■

Remark 5.10. Since $G_n = e^{-\pi/c}\ell_n + o(\ell_n)$, we have $\operatorname{Re} G_n < 0$ for all large n.

5.3. *Exiting the eggbeater*

Lemma 5.11 (Comparison with outgoing Fatou coordinates). *We have $w_{M_n-\ell_n} \in \mathcal{P}_R^o$, and*

$$\frac{1}{\epsilon_{M_n-\ell_n}}\left(\Phi_{\epsilon_{M_n-\ell_n}}(w_{M_n-\ell_n}) - \frac{\pi}{c}\right)$$
$$= \phi_{q_0}^o(w_{M_n-\ell_n}) + e^{-\pi/c}\frac{\ell_n^2}{2n} + (1-b_{0,3})\ln n + E^o + o(1),$$

where

$$E^o := (1-b_{0,3})\left(\frac{\pi}{c} - \ln c - e^{\pi/c}\int_{\pi/(2c)}^{\pi/c} e^{-u}\ln\sin(cu)\,du\right).$$

Proof. By Lemmas 5.8 and 5.9 we know that $W_{M_n-\ell_n} \in \mathcal{R}_{\epsilon_{M_n-\ell_n}}$ and $W_{M_n-\ell_n} = \pi/c - e^{-\pi/c}\ell_n/n + o(\ell_n/n)$. Since $w_{M_n-\ell_n} = -c\epsilon_{M_n-\ell}\cot(cW_{M_n-\ell_n}) + O(1/n)$, we have $w_{M_n-\ell_n} \sim 1/\ell_n$ and hence $w_{M_n-\ell_n} \in \mathcal{P}_R^o$ for all sufficiently large n. By the same computation as in the incoming case, we have

$$\begin{aligned} \psi^o_{\epsilon_{M_n-\ell_n}}(w_{M_n-\ell_n}) &= -\frac{\epsilon_{M_n-\ell_n}}{w_{M_n-\ell_n}} + \frac{\epsilon^2_{M_n-\ell_n}}{2w^2_{M_n-\ell_n}} + o(\epsilon_{M_n-\ell_n}) \\ &= \epsilon_{M_n-\ell_n}\left(-\frac{1}{w_{M_n-\ell_n}} + e^{-\pi/c}\frac{\ell_n^2}{2n} + o(1)\right). \end{aligned}$$

Recall that $\phi_{q_0}^o(w) = -1/w + (1-b_{0,3})\log w + o(1)$.

Next, we have

$$\begin{aligned} \Phi_{\epsilon_{M_n-\ell_n}}(w_{M_n-\ell_n}) &= \chi_{\epsilon_{M_n-\ell_n}} \circ \psi^{\iota}_{\epsilon_{M_n-\ell_n}}(w_{M_n-\ell_n}) \\ &= \chi_{\epsilon_{M_n-\ell_n}}(\psi^o_{\epsilon_{M_n-\ell_n}}(w_{M_n-\ell_n}) + \pi/c) \\ &= \pi/c + W - c\epsilon_{M_n-\ell_n}(1-b_{0,3})e^{W+\pi/c}F_c(W+\pi/c), \end{aligned}$$

where $W := \psi^o_{\epsilon_{M_n-\ell_n}}(w_{M_n-\ell_n})$, and by Lemma 4.12,

$$ce^{W+\pi/c}F_c(W+\pi/c) = \log\left(\frac{\epsilon_{M_n-\ell_n}}{w_{M_n-\ell_n}}\right) + \ln c + e^{\pi/c}\int_{\pi/(2c)}^{\pi/c} e^{-u}\ln\sin(cu)du + o(1)$$
$$= -\log w_{M_n-\ell_n} - \ln(e^{\pi/c}n) + \ln c + e^{\pi/c}\int_{\pi/(2c)}^{\pi/c} e^{-u}\ln\sin(cu)du + o(1).$$

Putting all together we obtain

$$\frac{1}{\epsilon_{M_n-\ell_n}}(\Phi_{\epsilon_{M_n-\ell_n}}(w_{M_n-\ell_n}) - \pi/c)$$
$$= -\frac{1}{w_{M_n-\ell_n}} + (1-b_{0,3})\log w_{M_n-\ell_n} + e^{-\pi/c}\frac{\ell_n^2}{2n} + (1-b_{0,3})\ln n + E^o + o(1)$$
$$= \phi^o_{q_0}(w_{M_n-\ell_n}) + e^{-\pi/c}\frac{\ell_n^2}{2n} + (1-b_{0,3})\ln n + E^o + o(1). \qquad \blacksquare$$

Lemma 5.12. *For $M_n - \ell_n \le j \le M_n$, we have $w_j \in \mathcal{P}^o_R$ and*

$$\phi^o_{q_0}(w_j) = \phi^o_{q_0}(w_{M_n-\ell_n}) + j - (M_n - \ell_n) + o(1).$$

Proof. Recall that $w_{M_n-\ell_n} = O(1/\ell_n)$.

For $M_n - \ell_n \le j \le M_n - 1$ we have

$$\phi^o_{q_0}(w_{j+1}) = \phi^o_{q_0}(q_0(w_j) + b\epsilon_j^2 + O(\epsilon_j^3)) = \phi^o_{q_0}(w_j) + 1 + O((\phi^o_{q_0})'(w_j)\epsilon_j^2)$$
$$= \phi^o_{q_0}(w_j) + 1 + O(\epsilon_j^2/w_j^2).$$

Now, similarly to Lemma 5.2, induction on j proves that $w_j = O(\frac{1}{M_n-j}) \ge O(\frac{1}{\ell_n})$. This implies that

$$\phi^o_{q_0}(w_{j+1}) = \phi^o_{q_0}(w_j) + 1 + O(\ell_n^2/n^2)$$

and therefore, again by induction, $\phi^o_{q_0}(w_{M_n}) = \phi^o_{q_0}(w_{M_n-\ell_n}) + \ell_n + o(1)$, where we have used the fact that $O(\ell_n^3/n^2) = o(1)$ since $\ell_n \sim n^\nu$ for $\nu \in (1/2, 2/3)$. $\blacksquare$

5.4. Conclusion

Our Main Theorem is a corollary of the following more general theorem which we prove first.

Theorem 5.13. *We have*

$$P^{M_n}(p^n(z), w) = \big(p^{M_n+n}(z), \mathcal{L}(e^{\pi/c}, \Gamma - \rho_n; z, w) + o(1)\big),$$

where

$$\Gamma := (e^{\pi/c}-1)\left(\frac{a - b_{0,3} + b_{3,0}}{2b} + a + \frac{1}{2}(1-b_{0,3}) + (b_{0,3}-1)\ln c\right) + (b_{0,3}-a)\frac{\pi}{c}$$
$$+ e^{\pi/c}(1-b_{0,3})\int_0^{\pi/c} e^{-u}\ln\sin(cu)\,du. \qquad (5.1)$$

Proof. We have

$$\begin{aligned}\phi^o_{q_0}(w_{M_n}) &= \phi^o_{q_0}(w_{M_n-\ell_n}) + \ell_n + o(1)\\ &= \frac{1}{\epsilon_{M_n-\ell_n}}(\Phi_{\epsilon_{M_n-\ell_n}}(w_{M_n-\ell_n}) - \pi/c) - e^{-\pi/c}\frac{\ell_n^2}{2n} - (1-b_{0,3})\ln n - E^o + \ell_n + o(1)\\ &= e^{\pi/c}\phi^\iota_{q_0}(w_0) - \rho_n + e^{\pi/c}\tilde{C} - E^o + o(1),\end{aligned}$$

where the first equality follows from Lemma 5.12, the second from Lemma 5.11, and the last one from Lemmas 5.9 and 5.8. Note that in this computation we have used the fact that $\frac{1}{2n}(e^{\pi/c}k_n^2 - e^{-\pi/c}\ell_n^2) = o(1)$.

Finally, recall that

$$\begin{aligned}\Theta &= b_{0,3} + \frac{a - b_{0,3} + b_{3,0}}{2b},\\ E^\iota &= (b_{0,3}-1)\left(\ln c - \int_0^{\pi/(2c)} e^{-u}\ln\sin(cu)du\right),\\ \tilde{C} &= (1-a)e^{-\pi/c}\frac{\pi}{c} + (1-e^{-\pi/c})\big(\Theta + \tfrac{3}{2}(1-b_{0,3}) + (a-1) - \phi^\iota_p(z_0)\big) + E^\iota,\\ E^o &= (1-b_{0,3})\left(\frac{\pi}{c} - \ln c - e^{\pi/c}\int_{\pi/(2c)}^{\pi/c} e^{-u}\ln\sin(cu)du\right).\end{aligned}$$

A quick computation now gives

$$e^{\pi/c}\tilde{C} - E^o = -(e^{\pi/c}-1)\phi^\iota_p(z_0) + \Gamma,$$

hence

$$\phi^o_{q_0}(w_{M_n}) = e^{\pi/c}\phi^\iota_{q_0}(w_0) - (e^{\pi/c}-1)\phi^\iota_p(z_0) - \rho_n + \Gamma + o(1). \qquad \blacksquare$$

Remark 5.14. Theorem 5.13 has been proved under the assumption that $\beta_0 \in \mathbb{R}$. Following essentially the same proof with $\beta_0 \in \mathbb{C}$ (only replacing the definition of M_n and ρ_n in Definition 5.7 by $M_n := \lfloor(\alpha_0 - 1)n + \mathrm{Re}(\beta_0)\ln n\rfloor$ and $\rho_n := \{(\alpha_0-1)n + \mathrm{Re}(\beta_0)\ln n\}$), one could prove that

$$\begin{aligned}w_{M_n} = (\phi^o_{q_0})^{-1}\big(&e^{\pi/c}\phi^\iota_{q_0}(w_0) - (e^{\pi/c}-1)\phi^\iota_p(z_0) - \rho_n + \Gamma + i\,\mathrm{Im}(b_{0,3}-a)\ln n\big)\\ &+ o(1).\end{aligned}$$

It then seems likely that (z_{n+M_n}, w_{M_n}) belongs to one of the parabolic domains $\mathcal{U}^\pm$ from Theorem 1.2, which in turn would imply that (z_n, w) belongs to the parabolic basin of $(0,0)$ for all n large enough. This also seems to be supported by numerical experiments.

Proof of the Main Theorem from Theorem 5.13. It only remains to rephrase Theorem 5.13 in terms of admissible sequences. Let $(n_k)_{k\geq 0}$ be an (α_0, β_0)-admissible sequence. By definition of M_n and ρ_n, we have

$$M_{n_k} = \lfloor(\alpha_0 - 1)n_k + \beta_0 \ln n_k\rfloor,$$

and $\rho_{n_k} = \{(\alpha_0 - 1)n_k + \beta_0 \ln n_k\}$. Therefore, by definition of an (α_0, β_0)-admissible sequence, there exists a bounded sequence $(m_k)_{k\geq 0}$ of integers such that

$$n_{k+1} - n_k = M_{n_k} + m_k,$$

and the phase sequence of $(n_k)_{k\geq 0}$ is given by

$$\begin{aligned}\sigma_k = n_{k+1} - \alpha_0 n_k - \beta_0 \ln n_k &= n_{k+1} - (M_{n_k} + n_k + \rho_{n_k}) \\ &= m_k - \rho_{n_k}.\end{aligned}$$

By Theorem 5.13, we have

$$P^{M_{n_k}}(p^{n_k}(z), w) = \big(p^{n_k + M_{n_k}}(z), \mathcal{L}(\alpha_0, \Gamma - \rho_{n_k}; z, w) + o(1)\big)$$

and therefore, by the functional equation satisfied by $\mathcal{L}$,

$$\begin{aligned}P^{n_{k+1}-n_k}(p^{n_k}(z), w) &= P^{M_{n_k}+m_k}(p^{n_k}(z), w) \\ &= \big(p^{n_k+M_{n_k}+m_k}(z), \mathcal{L}(\alpha_0, \Gamma + m_k - \rho_{n_k}; z, w) + o(1)\big) \\ &= \big(p^{n_{k+1}}(z), \mathcal{L}(\alpha_0, \Gamma + \sigma_k; z, w) + o(1)\big).\end{aligned}$$

Finally, since $z \in \mathcal{B}_p$ we have $p^{n_{k+1}}(z) = o(1)$ and hence we obtain the desired result

$$P^{n_{k+1}-n_k}(p^{n_k}(z), w)) = (0, \mathcal{L}(\alpha_0, \Gamma + \sigma_k; z, w)) + o(1). \qquad \blacksquare$$

Remark 5.15. The proof of the Main Theorem does not require that the phase sequence (σ_k) is bounded, but as we will see later this property is crucial for its application.

6. Wandering domains of rank 1

The aim of this section is to prove Theorem 1.6.

Proof of Theorem 1.6. By our assumption, if $(\sigma_k)_{k\in\mathbb{N}}$ denotes the phase sequence associated to the (α_0, β_0)-admissible sequence $(n_k)_{k\in\mathbb{N}}$, then

$$\sigma_k := n_{k+1} - \alpha_0 n_k - \beta_0 \ln n_k \xrightarrow{k\to\infty} \theta,$$

and hence by the Main Theorem we have $P^{n_{k+1}-n_k}(p^{n_k}(z), w) \xrightarrow{k\to\infty} (0, \mathcal{L}_z(w))$, where $\mathcal{L}_z(w) := \mathcal{L}(\alpha_0, \Gamma + \theta; z, w)$.

Let $\mathcal{E}(W) := \phi^{\iota}_{q_0} \circ \psi^{o}_{q_0}(W)$ be the *lifted horn map* of q_0 (see Section 2). Let us define $\sigma := \Gamma + \theta$, where Γ is the constant from the Main Theorem, and

$$\tilde{H}_{Z,\sigma}(W) := \alpha_0 \mathcal{E}(W) + (1 - \alpha_0)Z + \sigma$$

as in Definition 1.15.

Lemma 6.1. *There exists a point* $(z_0, w_0) \in \mathcal{B}_p \times \mathcal{B}_{q_0}$ *such that* w_0 *is a superattracting fixed point of the map* $\mathcal{L}_{z_0}(w)$.

Proof. First observe that $\mathcal{L}_z$ is semi-conjugate to $\tilde{H}_{Z,\sigma}$, where $Z := \phi_p^\iota(z)$. Indeed, we have $\mathcal{L}_z \circ \psi_{q_0}^o = \psi_{q_0}^o \circ \tilde{H}_{Z,\sigma}$, where $\psi_{q_0}^o$ is the outgoing Fatou parametrization, hence it suffices to prove that the map $\tilde{H}_{Z,\sigma}$ has a superattracting fixed point for appropriate choice of $Z \in \mathbb{C}$.

Let W_0 be a critical point of $\mathcal{E}$ and observe that since $\mathcal{E}$ commutes with translation by 1, for every $N \in \mathbb{N}$ the point $W_0 + N$ is also a critical point of $\mathcal{E}$.

Next, observe that

$$\frac{\alpha_0 \mathcal{E}(W_0 + N) - (W_0 + N) + \sigma}{\alpha_0 - 1} = \frac{\alpha_0 \mathcal{E}(W_0) - W_0 + \sigma}{\alpha_0 - 1} + N,$$

hence for sufficiently large $N_0 \in \mathbb{N}$ there exists $z_0 \in \mathcal{B}_p$ such that

$$Z_0 := \phi_p^\iota(z_0) = \frac{\alpha_0 \mathcal{E}(W_0 + N_0) - (W_0 + N_0) + \sigma}{\alpha_0 - 1}.$$

It is then straightforward to check that $W_0 + N_0$ is a superattracting fixed point of $\tilde{H}_{Z_0,\sigma}(W)$. ∎

Let $(z_0, w_0) \in \mathcal{B}_p \times \mathcal{B}_{q_0}$ be such that w_0 is a superattracting fixed point of $\mathcal{L}_{z_0}(w)$. Let $\mathcal{A} := \{(z, w) \in \mathcal{B}_p \times \mathcal{B}_{q_0} : \mathcal{L}_z(w) = w\}$. The analytic set $\mathcal{A}$ has pure dimension 1, and since w_0 is a superattracting fixed point of $\mathcal{L}_{z_0}(w)$, the Implicit Function Theorem implies that the point (z_0, w_0) is contained in a regular part of $\mathcal{A}$. Therefore, there exists a small disk Δ_{z_0} centered at z_0 and a holomorphic function $\eta : \Delta_{z_0} \to \mathcal{B}_{q_0}$ that satisfies $\eta(z_0) = w_0$ and $h(\Delta_{z_0}) \subset \mathcal{A}$ where $h(z) := (z, \eta(z))$. Moreover, by restricting that disk if necessary, we can assume that $|\mathcal{L}_z'(\eta(z))| < 1/2$ on Δ_{z_0}.

Lemma 6.2. *The map $\eta : \Delta_{z_0} \to \mathbb{C}$ is non-constant.*

Proof. Recall that we constructed $Z_0, W_0 \in \mathbb{C}$ such that $\tilde{H}_{Z_0,\sigma}(W_0) = W_0$, and $Z_0 = \phi_p^\iota(z_0)$, $\eta(z_0) = \psi_{q_0}^o(W_0)$. Again by the Implicit Function Theorem, there exists a holomorphic map $\tilde{\eta} : \Delta_{Z_0} \to \mathbb{C}$ such that $\tilde{\eta}(Z)$ is a fixed point of $\tilde{H}_{Z,\sigma}$ for all $Z \in \Delta_{Z_0}$, where Δ_{Z_0} is a small disk centered at Z_0. Moreover, $\eta = \psi_{q_0}^o \circ \tilde{\eta} \circ \phi_p^\iota$. From the expression of $\tilde{H}_{Z,\sigma}$, it is not difficult to find that $\tilde{\eta}'(Z_0) = 1 - \alpha_0 \neq 0$, therefore $\tilde{\eta}$ and also η are non-constant. ∎

By the Main Theorem, for each $z \in \Delta_{z_0}$ there exist a disk $D_z \subset \mathcal{B}_{q_0}$ centered at $\eta(z)$ and $k_0 > 0$ such that

$$\mathrm{proj}_2(P^{n_{k+1}-n_k}(p^{n_k}(z) \times D_z)) \Subset D_z \tag{6.1}$$

for all $k \geq k_0$, where $\mathrm{proj}_2 : \mathbb{C}^2 \to \mathbb{C}$ denotes the projection on the second coordinate. Moreover, we can find a continuously varying family of disks $\{z\} \times D_z \subset \mathcal{B}_p \times \mathcal{B}_{q_0}$ and a uniform constant k_0 with respect to the parameter $z \in \Delta_{z_0}$ for which (6.1) holds. Let us define an open set

$$V := \bigcup_{z \in \Delta_{z_0}} \{p^{n_{k_0}}(z)\} \times D_z, \tag{6.2}$$

and let U be a connected component of the open set $P^{-n_{k_0}}(V)$ containing a point (z_0, w')

for which $P^{n_{k_0}}(z_0, w') = (p^{n_{k_0}}(z_0), w_0)$. Observe that for all $(z, w) \in U$ we have

$$\begin{aligned} P^{n_k}(z,w) &= P^{n_k - n_{k-1}} \circ P^{n_{k-1}-n_{k-2}} \circ \cdots \circ P^{n_{k_0+1}-n_{k_0}} \circ P^{k_0}(z,w) \\ &= \big(z_{n_k}, \mathcal{L}_{z_{n_{k_0}}}^{k-k_0}(w_{n_{k_0}}) + o(1)\big), \end{aligned}$$

where $z_j = p^j(z)$ and $w_j = \mathrm{proj}_2 P^j(z, w)$ and where the last equality follows from the Main Theorem. Hence it follows from our construction of the set U that the sequence $(P^{n_k})_{k\geq 0}$ converges uniformly on compacts in U to a holomorphic map $\varphi(z, w) := (0, \eta(z))$ where η is as above. Moreover, it follows from the proof of the Main Theorem (see Lemmas 5.2, 5.8, 5.12) and (6.1) that for every compact $K \subset U$ the sequence $P^j(P^{n_k}(K))$ is bounded for all $0 \leq j \leq n_{k+1} - n_k = M_k$ and all $k \geq 0$. Hence by Cauchy estimates, $(P^n|_U)_{n\geq 0}$ is normal and therefore U is contained in some Fatou component $\Omega \subset \mathbb{C}^2$.

Lemma 6.3. *The map η extends holomorphically to a map $\eta : \mathrm{proj}_1(\Omega) \to \mathcal{B}_{q_0}$, and there exists a subsequence $(P^{m_k})_{k\geq 0}$ that converges locally uniformly on Ω to the map $\Phi : \Omega \to \{0\} \times \mathcal{B}_{q_0}$ defined by $\Phi(z, w) = (0, \eta(z))$.*

Proof. Since (P^{n_k}) is normal on Ω, it has a convergent subsequence, say (P^{m_k}). Moreover, $\Omega \subset \mathcal{B}_p \times \mathbb{C}$ and therefore any limit function of a convergent subsequence of (P^{n_k}) must be of the form $\Phi(z, w) = (0, \kappa(z, w))$, and $\kappa(z, w) = \eta(z)$ for all $(z, w) \in U$. By the identity principle, we therefore have $\frac{\partial \kappa}{\partial w} = 0$ on Ω, and so κ gives a holomorphic continuation of η on $\mathrm{proj}_1(\Omega)$, which we still denote by η. Finally, let us argue that $\eta : \mathrm{proj}_1(\Omega) \to \mathcal{B}_{q_0}$.

First, observe that if $(z, w) \in \Omega$, then any ω-limit point of the orbit $(P^n(z, w))_{n\geq 0}$ has bounded orbit under P. This implies that η takes values in the non-escaping locus $\mathbb{C} \setminus I(q_0)$ (which is the filled-in Julia set $K(q_0)$ if q_0 is a polynomial) where $I(q_0) = \{w \in \mathbb{C} : q_0^n(w) \to \infty, \text{ as } n \to \infty\}$ denotes the escaping set of q_0. Moreover, by Lemma 6.2, η is non-constant and therefore open; and by definition, $\eta(\Delta_{z_0}) \subset \mathcal{B}_{q_0}$. Note that $\mathcal{B}_{q_0}$ is a regular open set, i.e. $\mathrm{int}(\overline{\mathcal{B}_{q_0}}) = \mathcal{B}_{q_0}$. Indeed, by Montel's Theorem, for a non-linear entire function the union of the forward images of an open set having non-empty intersection with the Julia set can omit at most one value of the complex plane. Finally, since $\partial \mathcal{B}_{q_0} = J(q_0) = \partial I(q_0)$, the map η must therefore take values in $\mathcal{B}_{q_0}$. ∎

Since $\mathcal{E}(W) = W - \pi i(1 - b_{0,3}) + o(1)$ as $|\mathrm{Im}\, W| \to \infty$ (see [7, Appendix]), we have

$$\tilde{H}_{Z,\sigma}(W) = \alpha_0 W + (1 - \alpha_0)Z + C + o(1) \quad \text{as } |\mathrm{Im}\, W| \to \infty \tag{6.3}$$

for some constant $C \in \mathbb{C}$. Let $\tilde{H}_\sigma(Z, W) := (Z, \tilde{H}_{Z,\sigma}(W))$ be the lifted horn map of P, with the notations of the introduction, and recall that it commutes with the map $T(Z, W) = (Z + 1, W + 1)$. This map is well defined on $\mathbb{C} \times \mathcal{U}_{q_0}$. The set of fixed points of $\tilde{H}_\sigma$ can be explicitly written as

$$\mathrm{Fix}_{\tilde{H}_\sigma} := \left\{ \left(\frac{\alpha_0 \mathcal{E}(W) - W}{\alpha_0 - 1} + \frac{\sigma}{\alpha_0 - 1}, W \right) : W \in \mathcal{U}_{q_0} \right\}. \tag{6.4}$$

Let us define $\chi(Z,W) = (Z - W, e^{2\pi i W})$ and $\tilde{U} := \chi(\mathbb{C} \times \mathcal{U}_{q_0}) \subset \mathbb{C} \times \mathbb{C}^*$. Observe that there is a small punctured disk Δ^* such that $\mathbb{C} \times \Delta^* \subset \tilde{U}$ and there exists a holomorphic map $\Psi : \tilde{U} \to \mathbb{C} \times \mathbb{C}^*$ such that $\Psi \circ \chi = \chi \circ \tilde{H}_\sigma$. This map Ψ is holomorphically conjugate to the horn map H_σ of P (see Definition 1.15). It can be expressed more explicitly as

$$\Psi(X,Y) = \left(\alpha_0 X - \alpha_0\left(\mathcal{E}\left(\frac{\log Y}{2\pi i}\right) - \frac{\log Y}{2\pi i}\right) - \sigma, Y^{1-\alpha_0}(h(Y))^{\alpha_0} e^{2\pi i((1-\alpha_0)X+\sigma)}\right),$$

where h is the horn map of q_0. Moreover, $\mathcal{E}(\frac{\log Y}{2\pi i}) - \frac{\log Y}{2\pi i}$ is a single-valued function since $\mathcal{E}(W+1) = \mathcal{E}(W) + 1$. It extends holomorphically over $\mathbb{C} \times \{0\}$ with $\Psi(X,0) = (\alpha_0 X + \alpha_0 \pi i(1 - b_{0,3}) - \sigma, 0)$. We still denote this extended map by Ψ.

Lemma 6.4. *Ω is a wandering domain.*

Proof. Let $\Phi(z,w) = (0, \eta(z))$ be the limit function as in the lemma above and define $\Lambda := \mathrm{proj}_1(\Omega) \subset \mathcal{B}_p$. Observe that $\eta(\Lambda) = \mathrm{proj}_2(\Phi(\Omega))$ and $\Sigma := \{(z, \eta(z)) : z \in \Lambda\}$ is connected.

Let Fix_Ψ be the analytic variety of fixed points of Ψ and observe that Fix_Ψ is closed in the domain of definition of Ψ. Moreover,

$$\chi\left(\frac{\alpha_0 \mathcal{E}(W) - W}{\alpha_0 - 1} + \frac{\sigma}{\alpha_0 - 1}, W\right) = \left(\frac{\alpha_0(\mathcal{E}(W) - W)}{\alpha_0 - 1} + \frac{\sigma}{\alpha_0 - 1}, e^{2\pi i W}\right), \qquad (6.5)$$

and hence

$$\chi\left(\frac{\alpha_0 \mathcal{E}(W) - W}{\alpha_0 - 1} + \frac{\sigma}{\alpha_0 - 1}, W\right) \xrightarrow{\mathrm{Im}\, W \to \infty} \left(\frac{-\alpha_0 \pi i(1-b) + \sigma}{\alpha_0 - 1}, 0\right). \qquad (6.6)$$

Since Fix_Ψ is closed, it follows that $(\frac{-\alpha_0 \pi i(1-b)+\sigma}{\alpha_0 - 1}, 0) \in \mathrm{Fix}_\Psi$.

Let $B_z(w) := \alpha_0 \phi^\iota_{q_0}(w) + (1-\alpha_0)\phi^\iota_p(z) + \sigma$. Observe that $\mathcal{L}_z = \psi^o_{q_0} \circ B_z$, and if $Z := \phi^\iota_p(z)$, then $\tilde{H}_{Z,\sigma} = B_z \circ \psi^o_{q_0}$. In other words, B_z also semi-conjugates $\mathcal{L}_z$ and $\tilde{H}_{Z,\sigma}$. We let $\Xi(z,w) := (\phi^\iota_p(z), B_z(w))$, and let

$$\Sigma' := \Xi(\Sigma) \subset \mathrm{Fix}_{\tilde{H}_\sigma}$$

be the "lift" of Σ. Since Ξ is continuous and Σ is connected, so is Σ'.

Let us assume that Ω is not wandering. Up to replacing Ω with $P^\ell(\Omega)$ we may assume that it is periodic, i.e. $P^m(\Omega) = \Omega$. Observe that this implies that Σ' is forward invariant under the translation T^m. Let $\gamma : I \to \Sigma'$ be a smooth curve such that $\gamma(0) = (Z_0, W_0)$ and $\gamma(1) = (Z_0 + m, W_0 + m)$ (this is possible since Σ' is connected), and such that $\chi(\gamma(I))$ is a Jordan curve.

Now observe that by (6.4), $\mathrm{Fix}_{\tilde{H}_\sigma}$ is a holomorphic graph above $\mathcal{U}_{q_0}$ and therefore is conformally equivalent to the upper half-plane; and by (6.5) and (6.6), its image under χ is conformally equivalent to a punctured disk. After the addition of the fixed point $(\frac{-\alpha_0 \pi i(1-b)+\sigma}{\alpha_0 - 1}, 0)$, we therefore see that Fix_Ψ is conformally equivalent to a disk. The curve $\chi(\gamma)$ is a Jordan curve around $(\frac{-\alpha_0\pi i(1-b)+\sigma}{\alpha_0-1}, 0)$ in that disk. Next by the chain rule we have

$$\mathrm{d}\Psi(\psi(Z,W)) \circ \mathrm{d}\chi(Z,W) = \mathrm{d}\chi(\tilde{H}_\sigma(Z,W)) \circ \mathrm{d}\tilde{H}_\sigma(Z,W), \tag{6.7}$$

hence at fixed points of $\tilde{H}_\sigma$ the linear endomorphisms $\mathrm{d}\Psi(Z,W)$ and $\mathrm{d}\tilde{H}_\sigma(Z,W)$ have the same eigenvalues. Since $\tilde{H}_\sigma(Z,W) = (Z, \tilde{H}_{Z,\sigma})$ it follows by construction of Σ' that the two eigenvalues of $\mathrm{d}\tilde{H}_\sigma$ at fixed points from Σ' are 1 and $\lambda \in \mathbb{D}(0,1)$.

Now let us consider the holomorphic map $\det \mathrm{d}\Psi : \mathrm{Fix}_\Psi \to \mathbb{C}$. Since $\gamma \subset \Sigma'$, it clearly follows from (6.7) that $|\det \mathrm{d}\Psi| < 1$ on $\chi(\gamma)$. Moreover, by (6.6) and (6.7) we have

$$\begin{aligned}
\det \mathrm{d}\Psi\left(\frac{-\alpha_0\pi i(1-b)+\sigma}{\alpha_0-1}, 0\right) &= \lim_{\substack{\operatorname{Im} W\to\infty \\ W\in\mathcal{U}_{q_0}}} \det \mathrm{d}\tilde{H}_\sigma\left(\frac{\alpha_0\mathcal{E}(W)-W}{\alpha_0-1} + \frac{\sigma}{\alpha_0-1}, W\right) \\
&= \lim_{\substack{\operatorname{Im} W\to\infty \\ W\in\mathcal{U}_{q_0}}} \frac{\partial \tilde{H}_{Z,\sigma}}{\partial W}\left(\frac{\alpha_0\mathcal{E}(W)-W}{\alpha_0-1} + \frac{\sigma}{\alpha_0-1}, W\right) \\
&= \lim_{\substack{\operatorname{Im} W\to\infty \\ W\in\mathcal{U}_{q_0}}} \alpha_0 + o(1) = \alpha_0 > 1,
\end{aligned}$$

where the last line is computed using (6.3). But this contradicts the maximum principle, hence Ω must be a wandering domain. ■

This completes the proof of Theorem 1.6. ■

7. Wandering domains for higher periods

7.1. Simply connected hyperbolic components

In this section we assume that $\alpha_0 \in \mathbb{N}^*$ and $q_0(w) = w + w^2$. We let $\hat{h}$ denote the classical horn map of q_0 (see Section 2), and recall that

$$e^{2i\pi(1-\alpha_0)Z+2i\pi\sigma}\hat{h}(e^{2i\pi W})^{\alpha_0} = e^{2i\pi \tilde{H}_{Z,\sigma}(W)}. \tag{7.1}$$

We let $h := \hat{h}^{\alpha_0}$, and consider the family $(h_\lambda)_{\lambda\in\mathbb{C}^*}$ defined by $h_\lambda := \lambda h$. By the choice of q_0, the maps h_λ have exactly three singular values:

(1) 0 and ∞, which are asymptotic values that are also superattracting fixed points;

(2) one free critical value $v_\lambda := \lambda v$, where $v := e^{2i\pi\phi^\iota_{q_0}(-1/2)}$.

Indeed, these last two properties follow from the classical fact that the map $\hat{h}$ has exactly three singular values: 0 and ∞ which are fixed asymptotic values, and v which is a critical value. We refer the reader to [7, Appendix] for a proof of these facts. In particular, if h_λ has an attracting cycle different from 0 and ∞, then it must capture v_λ.

Note that h is a finite type map in the sense of Epstein and it is clear from its definition that it is semi-conjugate to $\mathcal{L}_z$.

Definition 7.1. A *hyperbolic component* of period m in the family $(h_\lambda)_{\lambda\in\mathbb{C}^*}$ is a connected component of the set of $\lambda \in \mathbb{C}^*$ such that h_λ has an attracting cycle of period m different from 0 and ∞.

In order to prove that the Fatou components that we construct are indeed wandering, we will use the following result, which also has intrinsic interest:

Theorem 7.2. *Hyperbolic components in the family $(h_\lambda)_{\lambda\in\mathbb{C}^*}$ are simply connected.*

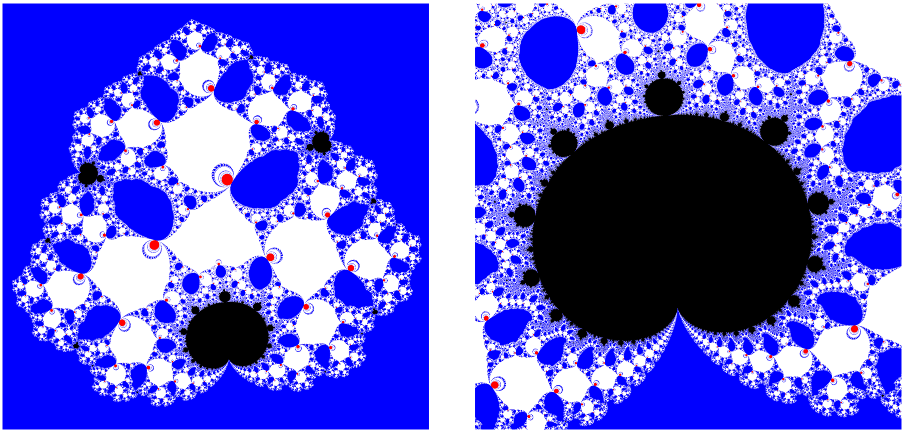


Fig. 3. Parameter space of $(h_\lambda)_{\lambda\in\mathbb{C}^*}$. Hyperbolic components are in black. Red and blue correspond to parameters λ for which v_λ is captured by 0 or ∞ respectively, and white to λ such that v_λ eventually exits the domain of h_λ. Observe that for all $|\lambda|$ large enough, v_λ is captured by ∞ (blue). Right: a zoom on a copy of the Mandelbrot set (bottom center of the left figure).

Before proving Theorem 7.2, we introduce some further notations:

Definition 7.3. We let $P_m := \{(\lambda, z) \in \mathbb{C}^* \times \mathbb{C}^* : z = h_\lambda^m(z)\}$, and $\tilde{\rho} : P_m \to \mathbb{C}$ be the map defined by $\tilde{\rho}(\lambda, z) = (h_\lambda^m)'(z)$.

Let U be a hyperbolic component of period m and $\mathbb{D} \subset \mathbb{C}$ the unit disk. Then $U = \mathrm{proj}_1(\Pi)$, where Π is a connected component of $\tilde{\rho}^{-1}(\mathbb{D})$. Since for every $\lambda \in \mathbb{C}^*$, h_λ has only one free singular value, it may have at most one attracting cycle different from 0 and ∞; therefore if (λ, z_1) and (λ, z_2) are in the same fiber of the map $\mathrm{proj}_1 : \Pi \to U$, then z_1 and z_2 must be periodic points of the same attracting cycle. This means that the function $\tilde{\rho} : \Pi \to \mathbb{D}$ descends to a well-defined holomorphic function $\rho : U \to \mathbb{D}$ satisfying $\tilde{\rho} = \rho \circ \mathrm{proj}_1$.

Lemma 7.4. *Let $U_0 := U \setminus \rho^{-1}(\{0\})$. The map $\rho : U_0 \to \mathbb{D}^*$ is locally invertible.*

Proof. We will prove this using a classical surgery argument, originally due to Douady–Hubbard [13] for the quadratic family. Let $\lambda_0 \in U_0$, and let V be a simply connected open subset of $\mathbb{D}^*$ containing $\rho(\lambda_0)$. Using a standard surgery procedure, we construct for any

$t \in V$ a quasiconformal homeomorphism g_t such that $g_t \circ h_{\lambda_0} \circ g_t^{-1}$ is holomorphic, and $g_t(z_0)$ is a periodic point of period m and multiplier t. We refer to [6, Proposition 6.7] for the details (see also e.g. [17, Theorem 6.4]).

We let $\phi : V \to \mathrm{Teich}(h_{\lambda_0})$ be the holomorphic map induced by $t \mapsto \mu_t$, where μ_t is the Beltrami form associated to g_t and $\mathrm{Teich}(h_{\lambda_0})$ is the dynamical Teichmüller space of h_{λ_0}. For the definition of the dynamical Teichmüller space, see [4, 26]. Let $\hat{V} \subset U_0$ be a simply connected domain containing λ_0. Since for all $\lambda \in \hat{V}$ the free critical value v_λ remains captured by the attracting cycle, the family $(h_\lambda)_{\lambda \in \hat{V}}$ is J-stable by [5, Theorem E]. In fact, since there are no non-persistent singular relations for the family $(h_\lambda)_{\lambda \in \hat{V}}$, by [26, Theorem 7.4] (stated for rational maps, but whose proof carries over verbatim in this setting) the map h_{λ_0} is in fact *structurally stable* on $\mathbb{P}^1$: there is a second holomorphic family $\hat{g}_\lambda$ of quasiconformal homeomorphisms $\hat{g}_\lambda : \mathbb{P}^1 \to \mathbb{P}^1$ such that $h_\lambda := \hat{g}_\lambda \circ h_{\lambda_0} \circ \hat{g}_\lambda^{-1}$ for all $\lambda \in \hat{V}$, and $\hat{g}_{\lambda_0} = \mathrm{Id}$.

We let $\hat{\phi} : \hat{V} \to \mathrm{Teich}(h_{\lambda_0})$ denote the map induced by $\lambda \mapsto \hat{\mu}_\lambda$, where $\hat{\mu}_\lambda$ is the Beltrami form associated to $\hat{g}_\lambda$. Let $\xi := \frac{d}{d\lambda}|_{\lambda=\lambda_0} \hat{g}_\lambda$, and observe that since $\hat{g}_\lambda(v_{\lambda_0}) = v_\lambda = \lambda v$, we have $\xi(v_{\lambda_0}) \neq 0$. By [4, Proposition 5], the derivative $\hat{\phi}'(\lambda_0)$ is therefore non-zero. Therefore, up to restricting V, we may assume that $\phi(V) \subset \hat{\phi}(\hat{V})$ and there exists a well-defined inverse branch $\hat{\phi}^{-1} : \phi(V) \to \hat{V}$. Let $c : V \to \hat{V}$ be the map defined by $c := \hat{\phi}^{-1} \circ \phi$. Then c is a holomorphic local inverse of ρ, which maps $\rho(\lambda_0)$ to λ_0; the lemma is proved. ■

Lemma 7.5. *The map $\rho : U_0 \to \mathbb{D}^*$ is a covering map of finite degree.*

Proof. We start by claiming that Π is relatively compact in P_m. Indeed, $U = \mathrm{proj}_1(\Pi)$ is relatively compact in $\mathbb{C}^*$ because if $|\lambda|$ is small (respectively large) enough, v_λ is captured by the superattracting fixed point 0 (respectively ∞). Moreover, by [5, Theorem A], the map $\mathrm{proj}_1 : P_m \to \mathbb{C}^*$ is proper, because the only two asymptotic values in the family $(h_\lambda)_{\lambda \in \mathbb{C}^*}$ are persistently fixed. Therefore Π is relatively compact in P_m. Since $\tilde{\rho}$ is analytic (hence continuous) on P_m, and since the set $\Pi_0 := \Pi \setminus \tilde{\rho}^{-1}(\{0\})$ is a connected component of $\tilde{\rho}^{-1}(\mathbb{D}^*)$, this proves that $\tilde{\rho} : \Pi_0 \to \mathbb{D}^*$ is proper. Consequently, so is $\rho : U_0 \to \mathbb{D}^*$.

By Lemma 7.4, the map $\rho : U_0 \to \mathbb{D}^*$ is also locally invertible; therefore it is a finite degree covering map. ■

Proof of Theorem 7.2. By the lemma above, $\rho : U_0 \to \mathbb{D}^*$ is a finite degree covering map. This implies that there exists $\lambda_0 \in U$ such that $U_0 = U \setminus \{\lambda_0\}$, and U_0 is isomorphic to a punctured disk and U to a disk. ■

7.2. Proof of Theorem 1.11

We give here is a slightly more precise statement of Theorem 1.11:

Theorem 7.6. *To each hyperbolic component U of the family $(h_\lambda)_{\lambda \in \mathbb{C}^*}$, we can associate a wandering Fatou component Ω_U of P. Moreover, if $U_1 \neq U_2$, then Ω_{U_1} and Ω_{U_2} are in different grand orbits of P.*

Since α_0 is an integer, we may choose an α_0-admissible sequence (n_k) to be simply $n_k = \alpha_0^k$, which has zero phase sequence, and where $n_0 \in \mathbb{N}^*$. By the Main Theorem, $P^{n_{k+1}-n_k}(p^{n_k}(z), w) \to (0, \mathcal{L}(\alpha_0, \sigma; z, w))$ uniformly on compacts in $\mathcal{B}_p \times \mathcal{B}_{q_0}$. Here σ is simply the constant Γ from the Main Theorem, since the phase sequence of $(n_k)_{k\in\mathbb{N}}$ is zero. Let $(\lambda_0, x_0) \in \mathbb{C}^* \times \mathbb{C}^*$ be such that x_0 is a superattracting periodic point of exact period ℓ for h_{λ_0}. Let $(z_0, w_0) \in \mathcal{B}_p \times \mathcal{B}_{q_0}$ be such that $e^{2i\pi(1-\alpha_0)\phi_p^\iota(z_0)+2i\pi\sigma} = \lambda_0$ and $e^{2i\pi\phi_{q_0}^o(w_0)} = x_0$: then w_0 is a superattracting periodic point of $\mathcal{L}(\alpha_0, \sigma; z_0, \cdot)$. The existence of z_0 and w_0 follows from the fact that $\phi_p^\iota(\mathcal{P}_p^\iota)$ contains a right half-plane and $\phi_{q_0}^o(\mathcal{P}_{q_0}^o)$ contains a left half-plane; see Section 2.

Since w_0 is a superattracting periodic point of $\mathcal{L}_{z_0} := \mathcal{L}(\alpha_0, \sigma; z_0, \cdot)$ of period ℓ, there exists $\epsilon > 0$ such that $|\mathcal{L}_z^\ell(w) - w_0| \le |w - w_0|/4$ for all $(z, w) \in \mathbb{D}(z_0, 2\epsilon) \times \mathbb{D}(w_0, \epsilon)$. By the Main Theorem, there exists $k_0 \in \mathbb{N}$ such that for all $k \ge k_0$ and all $(z, w) \in \mathbb{D}(z_0, \epsilon) \times \mathbb{D}(w_0, \epsilon)$, we have

$$P^{n_{k+\ell}-n_k}(p^{n_k}(z), w) \in p^{n_{k+\ell}}(\mathbb{D}(z_0, \epsilon)) \times \mathbb{D}(w_0, \epsilon/2).$$

It will be convenient to choose $n_0 \ge \alpha_0^{k_0}$ and replace the α_0-admissible sequence $(\alpha_0^k)_k$ by $(n_0\alpha_0^k)_k$, which we still denote by $(n_k)_k$; then the above inclusion holds for all $k \ge 0$. Let $\mathcal{U} := p^{n_0}(\mathbb{D}(z_0, \epsilon)) \times \mathbb{D}(w_0, \epsilon)$; an immediate induction proves that

$$P^{n_{k\ell}-n_0}(\mathcal{U}) \subset p^{n_{k\ell}-n_0}(\mathbb{D}(z_0, \epsilon)) \times \mathbb{D}(w_0, \epsilon)$$

for all $k \ge 0$; in particular, $\mathcal{U}$ is in the Fatou set of P. For any $j \ge 0$, let $z_j := p^j(z_0)$.

Let $\Omega = \Omega(z_0, w_0, n_0)$ denote the Fatou component containing $(z_{n_0}, w_0) \in U_0$. By the identity principle and the Main Theorem, $P_{|\Omega}^{n_{k\ell}-n_0}(z, w) \to (0, \eta(z))$, where $\eta(z)$ is an attracting periodic point of period ℓ of $\mathcal{L}_z = \mathcal{L}(\alpha_0, \sigma; z, \cdot)$, with $\eta(z_{n_0}) = w_0$.

Let us now sum up the construction above: given (z_0, w_0) such that w_0 is a superattracting periodic point of period ℓ for $\mathcal{L}_{z_0}$, and given any $n_0 \in \mathbb{N}$ large enough, we have constructed a Fatou component $\Omega(z_0, w_0, n_0)$ of P which contains $(z_{n_0}, w_0) = (p^{n_0}(z_0), w_0)$, and such that $P_{|\Omega(z_0,w_0,n_0)}^{n_{k\ell}-n_0} \to (0, \eta(z))$, where $n_k = n_0\alpha_0^k$.

Lemma 7.7. *The Fatou component $\Omega := \Omega(z_0, w_0, n_0)$ is wandering.*

Proof. The proof is similar to the one in Section 6, and we use some of the same notations. We assume for a contradiction that Ω is not wandering: then $P^{p+m}(\Omega) = P^p(\Omega)$ for some $m \in \mathbb{N}$ and $p \in \mathbb{N}^*$. Up to replacing Ω by $P^p(\Omega)$, we may assume $p = 0$.

There exists some continuous curve joining (z_{n_0}, w_0) and $P^\ell(z_{n_0}, w_0)$ inside Ω. Using the convergence of $P^{n_{k\ell}-n_0}$ to $(0, \eta)$, we obtain a curve joining $\eta(z_{n_0})$ and $\eta(z_{n_0+p})$ inside $\Sigma := \eta(\Lambda)$, where $\Lambda := \mathrm{proj}_1(\Omega)$. We let Σ' be as in Section 6; it is an open subset of $\mathrm{Per}_m(\tilde{H}_\sigma) := \{(Z, W) \in \mathbb{C}^2 : \tilde{H}_\sigma^m(Z, W) = (Z, W)\}$. Then there is a curve in Σ' joining (Z_0, W_0) and $(Z_0 + p, W_0 + p)$, where $Z_0 := \phi_p^\iota(z_{n_0})$ and $W_0 := \alpha_0\phi_{q_0}^\iota(w_0) + (1 - \alpha_0)Z_0 + \sigma$.

Finally, we consider the image of this curve under the map

$$e : (Z, W) \mapsto (e^{2i\pi(1-\alpha_0)Z+2i\pi\sigma}, e^{2i\pi W}). \tag{7.2}$$

It now becomes a closed loop in $\Pi := e(\Sigma')$, which we denote by $\gamma := (\gamma_1, \gamma_2)$. By construction, the loop γ_2 is non-contractible in $\mathbb{C}^*$; however, it is contained in the hyperbolic component $U = \mathrm{proj}_1(\Pi)$, so this contradicts Theorem 7.2. ∎

Lemma 7.8. *If (z_i, w_i) $(i = 1, 2)$ are such that $\lambda_i = e^{2i\pi(1-\alpha_0)\phi_p^\iota(z_i)+2i\pi\sigma}$ are centers of two different hyperbolic components U_i, and if $n_0 \in \mathbb{N}$ is large enough, then the wandering Fatou components $\Omega_i := \Omega(z_i, w_i, n_0)$ are not in the same grand orbit.*

Proof. The idea of the proof is similar. Recall that w_i is a superattracting periodic point of $\mathcal{L}_{z_i} := \mathcal{L}(\alpha_0, \sigma; z_i, \cdot)$. Let us denote by ℓ_i the period of w_i under $\mathcal{L}_{z_i}$.

Assume towards a contradiction that Ω_1 and Ω_2 are in the same grand orbit of Fatou components for P; then there exist $m_i \in \mathbb{N}$ such that $P^{m_1}(\Omega_1) = P^{m_2}(\Omega_2) =: \Omega$. Moreover, by construction, $P_{|\Omega_i}^{n_k\ell_i-n_0}$ converges to the map $(z, w) \mapsto (0, \eta_i(z))$, where for all $z \in \Lambda_i := \mathrm{proj}_1(\Omega_i)$, $\eta_i(z_i)$ is a periodic point of period ℓ_i of the maps $\mathcal{L}_{z_i} := \mathcal{L}(\alpha_0, \sigma; z_i, \cdot)$. In particular, if we let $n'_k := n_{k\ell}$ with $\ell := \mathrm{lcm}(\ell_1, \ell_2)$, then we have $P_{|\Omega_i}^{n'_k-n_0} \to (0, \eta_i)$.

By normality, it is easy to see that the multipliers of those fixed points cannot be repelling: $\rho_i(z) := (\mathcal{L}_z^{\ell_i})'(\eta_i(z)) \in \overline{\mathbb{D}}$ for all $z \in \Lambda_i$. Moreover, $\rho_i(z_i) = 0$. If ρ_i is constant, then in particular ρ_i takes its values in $\mathbb{D}$; and if it is non-constant, then its image is an open set contained in $\overline{\mathbb{D}}$, therefore it is also contained in $\mathbb{D}$.

Next, we claim that there exists $\xi : \Lambda := \mathrm{proj}_1(\Omega) \to \mathbb{C}$ such that $\eta_i = q_0^{N_i} \circ \xi \circ p^{m_i}$ for some $N_i \in \mathbb{N}$. Indeed, since $P_{|\Omega_i}^{n'_k-n_0} \to (0, \eta_i)$ on Ω_i, there exist functions $\xi_i : \Lambda \to \mathbb{C}$ such that up to replacing (n'_k) by a subsequence, $P^{n'_k-n_0-m_i} \to (0, \xi_i)$ on Ω, and

$$\eta_i = \xi_i \circ p^{m_i}.$$

Assume without loss of generality that $N_0 := m_1 - m_2 \geq 0$. Then

$$(0, \xi_2) = \lim_k P^{n_k-n_0-m_2} = \lim_k P^{n_k-n_0-m_1+N_0} = P^{N_0} \circ (0, \xi_1)$$

so that $q_0^{N_0} \circ \xi_1 = \xi_2$. So we can take $\xi := \xi_1$, $N_1 := N_0$ and $N_2 := 0$.

Recall now that

$$\Xi(z, w) := \big(\phi_p^\iota(z), \alpha_0\phi_{q_0}^\iota(w) + (1-\alpha_0)\phi_p^\iota(z) + \sigma\big),$$
$$\Sigma_i := \{(z, \eta(z)) : z \in \Lambda_i\}, \quad \Sigma'_i := \Xi(\Sigma_i).$$

Let $\gamma = (\gamma_1, \gamma_2) : [0, 1] \to \Omega$ be a continuous curve joining $P^{m_1}(p^{n_0}(z_1), w_1)$ and $P^{m_2}(p^{n_0}(z_2), w_2)$ in Ω. Let $(Z_i, W_i) := \Xi(p^{n_0}(z_i), w_i)$. Then for all $k, p \in \mathbb{N}$ and $(z, w) \in \mathcal{B}_p \times \mathcal{B}_{q_0}$, we have $\Xi(p^k(z), q_0^p(w)) - \Xi(z, w) \in \mathbb{Z}^2$. In particular, $\tilde{\gamma}(t) := \Xi(\gamma_1(t), \xi(\gamma_1(t)))$ gives a continuous curve with the following properties:

(1) $\tilde{\gamma}(0) - \Xi(z_1, \eta_1(z_1)) \in \mathbb{Z}^2$;

(2) $\tilde{\gamma}(1) - \Xi(z_2, \eta_2(z_2)) \in \mathbb{Z}^2$;

(3) for all $t \in [0, 1]$, $\tilde{\gamma}(t) \in \mathrm{Per}_\ell(\tilde{H}_\sigma)$, where $\ell := \mathrm{lcm}(\ell_1, \ell_2)$, $\tilde{H}_\sigma(Z, W)$ is the lifted horn map defined in (1.7), and $(\frac{\partial}{\partial W}\tilde{H}_{Z,\sigma}^\ell)(\tilde{\gamma}(t)) \in \mathbb{D}$.

(Property (3) comes from the previous observation that $\rho_i(z) \in \mathbb{D}$ for all $z \in \Lambda_i$.)

Finally, we consider a curve $(\hat{\gamma}_1, \hat{\gamma}_2) := e \circ \tilde{\gamma}$, were e is given by (7.2). Then $\hat{\gamma}_1$ is a continuous curve joining λ_1 and λ_2 inside a hyperbolic component of period (dividing) ℓ for the family $(h_\lambda)_{\lambda \in \mathbb{C}^*}$, which is a contradiction. Thus Lemma 7.8 and Theorem 7.6 are proved. ∎

Finally, to deduce Theorem 1.11 from Theorem 7.6, we just need to know that there are countably many hyperbolic components in the family $(h_\lambda)_{\lambda \in \mathbb{C}^*}$. Since we have proved that the multiplier map is a conformal uniformization of any hyperbolic component on the unit disk, it is enough to prove that there are countably many $\lambda \in \mathbb{C}^*$ such that h_λ has a superattracting periodic point (different from 0 or ∞). But this follows from e.g. [5, Proposition 5.1].

8. Proofs of Theorems 1.12 and 1.16

8.1. Proof of Theorem 1.12

Lemma 8.1. *Let P be a map of the form (1.1). Let U be a neighborhood of $(0,0)$ in $\mathbb{C}^2$, and $\sigma \in \mathbb{C}$. Then there exists $(z_0, w_0) \in (\mathcal{B}_p \times \mathcal{B}_{q_0}) \cap U$ such that w_0 is an attracting fixed point of $\mathcal{L}_{z_0} := \mathcal{L}(\alpha_0, \sigma; z_0, \cdot)$.*

Proof. By Lemma 6.1, there exists $(z', w') \in \mathcal{B}_p \times \mathcal{B}_{q_0}$ such that w' is a superattracting fixed point of $\mathcal{L}_{z'}$. By the Implicit Function Theorem, for all z in the neighborhood of z', there exists $\eta(z)$ such that $w = \eta(z)$ is an attracting fixed point of $\mathcal{L}_z$. Moreover, we have proved in Lemma 6.2 that η is a non-constant holomorphic map, hence it is open. In particular, we may choose $\hat{z}$ close to z' such that the forward orbit of $\hat{w} := \eta(\hat{z})$ under q_0 does not meet $\mathrm{crit}(q_0)$. Let $N \in \mathbb{N}$ be large enough that $(z_0, w_0) := (p^N(\hat{z}), q_0^N(\hat{w})) \in U$; we will prove that w_0 is an attracting fixed point of $\mathcal{L}_{z_0}$.

Indeed, by (1.4), we have

$$\mathcal{L}(\alpha_0, \sigma; z_0, w_0) = q_0^N \circ \mathcal{L}(\alpha_0, \sigma; \hat{z}, \hat{w}) = q_0^N(\hat{w}) = w_0,$$

so w_0 is indeed a fixed point of $\mathcal{L}_{z_0}$.

Secondly,

$$\begin{aligned}
\frac{\partial}{\partial w}\bigg|_{(z,w)=(\hat{z},\hat{w})} \mathcal{L}(\alpha_0, \sigma; p^N(z), q_0^N(w)) &= \frac{\partial}{\partial w}\bigg|_{(z,w)=(\hat{z},\hat{w})} q_0^N \circ \mathcal{L}(\alpha_0, \sigma; z, w), \\
\frac{\partial \mathcal{L}}{\partial w}(\alpha_0, \sigma; z_0, w_0) \times (q_0^N)'(\hat{w}) &= (q_0^N)' \circ \mathcal{L}(\alpha_0, \sigma; \hat{z}, \hat{w}) \times \frac{\partial \mathcal{L}}{\partial w}(\alpha_0, \sigma; \hat{z}, \hat{w}) \\
&= (q_0^N)'(\hat{w}) \frac{\partial \mathcal{L}}{\partial w}(\alpha_0, \sigma; \hat{z}, \hat{w}).
\end{aligned}$$

Therefore, since $(q_0^N)'(\hat{w}) \neq 0$, we have

$$(\mathcal{L}_{z_0})'(w_0) = \frac{\partial \mathcal{L}}{\partial w}(\alpha_0, \sigma; z_0, w_0) = \frac{\partial \mathcal{L}}{\partial w}(\alpha_0, \sigma; \hat{z}, \hat{w}) \in \mathbb{D}$$

and w_0 is indeed an attracting fixed point of $\mathcal{L}_{z_0}$. ∎

Lemma 8.2. *Let P be a map of the form (1.1), which admits an (α_0, β_0)-admissible sequence $(n_k)_{k\geq 0}$ with a converging phase sequence, whose limit we denote by θ. Let U be a neighborhood of $(0,0)$ in $\mathbb{C}^2$, and let (z_0, w_0) be as in the lemma above. Then for $k_0 \in \mathbb{N}$ large enough, $P^j(p^{n_{k_0}}(z_0), w_0) \in U$ for all $j \in \mathbb{N}$.*

Proof. Assume without loss of generality that $U = \mathbb{D}(0,r) \times \mathbb{D}(0,r)$ for some $r > 0$. Let (z_0, w_0) be given by Lemma 8.1, so that $\max(|z_0|, |w_0|) < r$. For k_0 large enough, we have $|p^{j+n_{k_0}}(z_0)| < r$ for all $j \geq 0$, since $\lim_{k\to\infty} p^{n_k}(z_0) = 0$. In particular, we may choose k_0 large enough that $(p^{n_{k_0}}(z_0), w_0) \in U$, and $|p^{j+n_{k_0}}(z_0)| < r$ for all $j \geq 0$. It then only remains to prove that for a choice of k_0 large enough, $|\mathrm{proj}_2 \circ P^j(p^{n_{k_0}}(z_0), w_0)| < r$ for all $j \geq 0$.

Let $\sigma := \theta + \Gamma$. Since w_0 is attracting, there exists $0 < \epsilon < r - |w_0|$ such that $\mathcal{L}_{z_0}(\mathbb{D}(w_0,\epsilon)) \Subset \mathbb{D}(w_0,\epsilon)$. By the Main Theorem, we may choose k_0 large enough that for all $k \geq k_0$,

$$\mathrm{proj}_2 \circ P^{n_{k+1}-n_k}(\{p^{n_k}(z_0)\}, \mathbb{D}(w_0,\epsilon)) \Subset \mathbb{D}(w_0,\epsilon).$$

Then, by induction on k, we have $\mathrm{proj}_2 \circ P^{n_{k+1}-n_{k_0}}(p^{n_{k_0}}(z_0), w_0) \in \mathbb{D}(w_0,\epsilon) \subset \mathbb{D}(0,r)$.

Finally, we claim that for all $k \geq k_0$ and all $0 \leq j \leq n_{k+1} - n_k$,

$$\mathrm{proj}_2 \circ P^j(p^{n_{k_0}}(z_0), w_0) \in \mathbb{D}(0,r).$$

For $0 \leq j \leq t_k := \lfloor (n_k)^\nu \rfloor$, this follows from Lemma 5.2.

For $t_k \leq j \leq n_{k+1} - n_k - \lfloor \alpha_0 t_k \rfloor$, it follows from Lemma 5.8. Indeed, by Lemma 5.8, we have $W_j \in \mathcal{R}_{\epsilon_i}$ for all $t_k \leq j \leq n_{k+1} - n_k - \lfloor \alpha_0 t_k \rfloor$. Then, by Lemma 4.8,

$$w_j = \psi_{z_j}^{-1}(W_j) = -cz_j \cot(cW_j) + O(z_j) = O(|z_j|^\nu) = o(1).$$

Finally, for $n_{k+1} - n_k - \lfloor \alpha_0 t_k \rfloor \leq j \leq n_{k+1} - n_k$, the claim follows from Lemma 5.11. ∎

Lemma 8.3. *Let $p(z) = z - z^2 + O(z^3)$ be an entire map. There exists $r > 0$ such that if $p^n(z) \in \mathbb{D}(0,r)$ for all $n \in \mathbb{N}$, then $z \in \mathcal{B}_p \cup \{0\}$. In particular, $\lim_{n\to\infty} p^n(z) = 0$.*

Proof. We consider the "fat petals" $\mathbf{P}_p^{\iota/o}$, introduced in Section 2. Take $r > 0$ small enough that $\mathbb{D}(0,r) \setminus \{0\}$ is contained in $\mathbf{P}_p^\iota \cup \mathbf{P}_p^o$. Since $\mathbf{P}_p^\iota \subset \mathcal{B}_p$, we may assume that for all $n \in \mathbb{N}$, $p^n(z) \in \mathbf{P}_p^o$ (for otherwise we are done).

We then have $\phi_p^o(p^n(z)) = \phi_p^o(z) + n \to \infty$, therefore $p^n(z) \to 0$ and $z \in \mathcal{B}_p$. ∎

Proof of Theorem 1.12. Let $\mathfrak{h} : U \to V$ be a homeomorphism such that $\mathfrak{h} \circ P_1 = P_2 \circ \mathfrak{h}$ and $\mathfrak{h}(0,0) = (0,0)$, where U, V are open neighborhoods of $(0,0)$. Let us begin by proving that $\alpha_1 = \alpha_2$. If not, assume without loss of generality that $\alpha_1 < \alpha_2$. Let $(n_k^1)_{k\in\mathbb{N}}$ be an (α_1, β_1)-admissible sequence with converging phase sequence of limit θ, and let (z_0, w_0) be given by Lemma 8.1 applied to P_1 and with $\sigma := \theta + \Gamma_1$.

By Lemma 8.2, there exists $k_0 \in \mathbb{N}$ such that for all $j \geq 0$,

$$P_1^j\big(p_1^{n^1_{k_0}}(z_0), w_0\big) \in U.$$

Therefore, for all $j \geq 0$,

$$\mathfrak{h} \circ P_1^j\big(p_1^{n^1_{k_0}}(z_0), w_0\big) = P_2^j \circ \mathfrak{h}\big(p_1^{n^1_{k_0}}(z_0), w_0\big). \tag{8.1}$$

Let $r > 0$ be small enough that $\mathcal{L}_{z_0}(\mathbb{D}(w_0, r)) \Subset \mathbb{D}(z_0, r)$.

Up to taking k_0 large enough, the Main Theorem implies that for all $k \geq k_0$,

$$\mathrm{proj}_2 \circ P_1^{n^1_{k+1} - n^1_k}(\{p_1^{n^1_k}(z_0)\} \times \mathbb{D}(w_0, r)) \Subset \mathbb{D}(w_0, r).$$

An easy induction then gives

$$\lim_{k \to \infty} P_1^{n^1_k - n^1_{k_0}}\big(p_1^{n^1_{k_0}}(z_0), w_0\big) = (0, w_0).$$

Let $(z_2, w_2) := \mathfrak{h}\big(p_1^{n^1_{k_0}}(z_0), w_0\big)$. Then, by (8.1),

$$\lim_{k \to \infty} P_2^{n^1_k - n^1_{k_0}}(z_2, w_2) = \mathfrak{h}(0, w_0). \tag{8.2}$$

We now claim that $\mathfrak{h}(0, w_0) = (0, w_3)$ for some $w_3 \in \mathcal{B}_{q_2}$. Indeed, by Lemma 8.3 and (8.2), we have (up to taking U and V small enough)

$$\mathrm{proj}_1 \circ \mathfrak{h}(0, w_0) = \lim_{j \to +\infty} p_2^j(z_2) = 0.$$

Therefore, $\mathfrak{h}(0, w_0) = (0, w_3)$ for some $w_3 \in \mathbb{C}$, and since for all $j \in \mathbb{N}$ we have

$$\mathfrak{h} \circ P_1^j(0, w_0) = P_2^j \circ \mathfrak{h}(0, w_0) = (0, q_2^j(w_3)),$$

it follows that $\lim_{j \to \infty} q_2^j(w_3) = 0$, hence either $w_3 = 0$ or $w_3 \in \mathcal{B}_{q_2}$. But since $w_0 \neq 0$ and $\mathfrak{h}$ is injective, we cannot have $\mathfrak{h}(0, w_2) = (0, 0)$, therefore $w_3 \in \mathcal{B}_{q_2}$.

Let $\epsilon > 0$ be small enough that $\overline{\mathbb{D}}(w_3, \epsilon) \Subset \mathcal{B}_{q_2}$, and let $k_1 \in \mathbb{N}$ be large enough that $\mathrm{proj}_2 \circ P_2^{n^1_k - n^1_{k_0}}(z_2, w_2) \in \mathbb{D}(w_3, \epsilon)$ for all $k \geq k_1$.

Let $k \geq k_1$ and let $n := n^1_k - n^1_{k_0}$. Since we have assumed for a contradiction that $\alpha_2 > \alpha_1$, we have

$$k_n := \lfloor n^\nu \rfloor < n^1_{k+1} - n^1_k < M_n - \ell_n := \lfloor (\alpha_2 - 1)n + \beta_2 \ln n \rfloor - \lfloor e^{\pi/c_2} k_n \rfloor.$$

By Lemma 5.8 (applied with $P := P_2$, $z := z_2$ and $w := \mathrm{proj}_2 \circ P^{n^1_k - n^1_{k_0}}(z_2, w_2) \in \mathbb{D}(w_3, \epsilon) \Subset \mathcal{B}_{q_2}$),

$$\begin{aligned}\Phi_{\epsilon_k}\big(\mathrm{proj}_2 \circ P_2^{n^1_{k+1} - n^1_{k_0}}(z_2, w_2)\big) &= \Phi_{\epsilon_k}\big(\mathrm{proj}_2 \circ P_2^{n^1_{k+1} - n^1_k}\big(P^{n^1_k - n^1_{k_0}}(z_2, w_2)\big)\big) \\ &= \Phi_{\epsilon_k}\big(\mathrm{proj}_2 \circ P_2^{n^1_{k+1} - n^1_k}(p_2^n(z_2), w)\big) \in \mathcal{R}_{\epsilon_k},\end{aligned}$$

where $\epsilon_k := p_2^{n+n^1_{k+1}-n^1_k}(z_2)$. Therefore, by Lemma 4.8,

$$\mathrm{proj}_2 \circ P_2^{n^1_{k+1}-n^1_{k_0}}(z_2, w_2) = O(|p_2^n(z_2)|^\nu) = o(1).$$

But this contradicts (8.2).

Therefore, $\alpha_2 \le \alpha_1$ and by symmetry, $\alpha_2 = \alpha_1$.

It remains to prove that $\beta_1 = \beta_2$. Similarly, we assume, for the sake of contradiction, that $\beta_2 > \beta_1$. With the same notations as above, we now have, for k (and hence n) large enough,

$$n^1_{k+1} - n^1_k = (\alpha_1 - 1)n + \beta_1 \ln n + O(1)$$

and

$$M_n = (\alpha_1 - 1)n + \beta_2 \ln n + O(1) = n^1_{k+1} - n^1_k + (\beta_2 - \beta_1) \ln n + O(1).$$

Therefore,

$$M_n - \ell_n < n^1_{k+1} - n^1_k < M_n,$$

and by Lemma 5.12 and Theorem 5.13,

$$\phi^o_{q_2} \circ \mathrm{proj}_2 \circ P_2^{n^1_{k+1}-n^1_k}(z_2, w_2) = (\beta_1 - \beta_2) \ln n + O(1).$$

In particular,

$$\mathrm{proj}_2 \circ P_2^{n^1_{k+1}-n^1_k}(z_2, w_2) \sim \frac{1}{(\beta_2 - \beta_1) \ln n},$$

which again contradicts (8.2). Therefore, $\beta_1 = \beta_2$, and we are done. ■

8.2. Proof of Theorem 1.16

In order to prove Theorem 1.16 we first need to introduce two intermediate results which are in the same spirit as Lemma 8.2 and Theorem 1.12. We will see it is possible to drop the assumption on the convergence of the phase sequences if we know that the skew-products P_1 and P_2 are topologically conjugated in a neighborhood of the origin by a homeomorphism $\mathfrak{h} : U \to V$ of the form $\mathfrak{h}(z, w) = (\mathfrak{f}(z), \mathfrak{g}(z, w))$, where U, V are open neighborhoods of $(0, 0)$ in $\mathbb{C}^2$. That is, assume that $\mathfrak{h} \circ P_1 = P_2 \circ \mathfrak{h}$ on U. We may assume that U, V are bounded in $\mathbb{C}^2$ and that $U = \mathbb{D}(0, r) \times \mathbb{D}(0, r)$ for some $r > 0$.

As before, we will also denote by $\mathcal{L}_i$, α_i, β_i for $i \in \{1, 2\}$ the quantities appearing in the Main Theorem, and by $(n^i_k)_{k\in\mathbb{N}}$ two (α_i, β_i)-admissible sequences defined by $n^i_{k+1} := \lfloor \alpha_i n^i_k + \beta_i \ln n^i_k \rfloor$, where $n^i_0 = n_0$ is chosen large enough that both sequences are strictly increasing, and let σ^i_k denote their phase sequences.

In what follows we write $q_i(w) := \mathrm{proj}_2 \circ P_i(0, w)$ for $i \in \{1, 2\}$.

Lemma 8.4. *Let $z \in \mathcal{B}_{p_1}$, $w \in \mathcal{B}_{q_1} \cap \mathbb{D}(0, r)$, and for any $n \in \mathbb{N}$, let $z_n := p_1^n(z)$. Then there exists $m \in \mathbb{N}$ such that $P_1^j(z_{n^1_k}, w) \in U$ for all k large enough and all $0 \le j \le n^1_{k+1} - n^1_k - m$.*

Proof. First, since $\lim_{n\to\infty} z_n = 0$, z_{j+n_k} belongs to an arbitrary neighborhood of 0 for $j \ge 0$ and k large enough. Therefore, if we let w_j denote the second component of

$P_1^j(z_{n_k^1}, w)$, it is enough to prove that for k and m large enough, w_j remains in $\mathbb{D}(0,r)$ for all $0 \le j \le n_{k+1}^1 - n_k^1 - m$. For $0 \le j \le t_k := \lfloor (n_k^1)^\nu \rfloor$, this follows from Lemma 5.2.

For $t_k \le j \le n_{k+1}^1 - n_k^1 - \lfloor \alpha_1 t_k \rfloor$, it follows from Lemma 5.8. Indeed, by Lemma 5.8, we have $W_j \in \mathcal{R}_{\epsilon_i}$ for all $t_k \le j \le n_{k+1}^1 - n_k^1 - \lfloor \alpha_1 t_k \rfloor$. Then, by Lemma 4.8,

$$w_j = \psi_{z_j}^{-1}(W_j) = -cz_j \cot(cW_j) + O(z_j) = O(|z_j|^\nu) = o(1).$$

Finally, the existence of $m > 0$ (independent of n_k^1) such that for all $n_{k+1}^1 - n_k^1 - \lfloor \alpha_1 t_k \rfloor \le j \le n_{k+1}^1 - n_k^1 - m$ we have $w_j \in \mathbb{D}(0,r)$ follows from Lemma 5.11. ■

Proposition 8.5. *Let P_1 and P_2 be of the form (1.1) with $b_i > 1/4$ and $\beta_i \in \mathbb{R}$, and assume that $P_1 \sim P_2$. Then $(\alpha_1, \beta_1) = (\alpha_2, \beta_2)$ and so in particular $b_1 = b_2$.*

Proof. Let $z \in \mathcal{B}_p \cap \mathbb{D}(0,r)$ and $w \in \mathcal{B}_{q_1} \cap \mathbb{D}(0,r)$; in particular, $(z,w) \in U$. By Lemma 8.4,

$$\mathfrak{h} \circ P_1^j(p_1^{n_k^1}(z), w) = P_2^j \circ \mathfrak{h}(p_1^{n_k^1}(z), w) \tag{8.3}$$

for all $0 \le j \le n_{k+1}^1 - n_k^1 - m$. In particular, both sides of the equation belong to V.

Let $M_k := \lfloor (\alpha_2 - 1)n_k^1 + \beta_2 \ln n_k^1 \rfloor$ and $\rho_k := \{(\alpha_2 - 1)n_k^1 + \beta_2 \ln n_k^1\}$. Choose $R > 0$ large enough that $V \subset \mathbb{D}(0,R)^2$, and choose $(z,w) \in U$ so that

$$|\mathcal{L}_2(\alpha_2, \Gamma_2 - \rho_k; \mathfrak{f}(z), \mathfrak{g}_0(w))| > R$$

for arbitrarily large values of k. We will show that this is always possible. Indeed, let $\rho \in [0,1)$ be an accumulation point of the sequence ρ_k. Since 0 is in the Julia set of the entire map q_2, there exists $\tilde{w} \in V_0 := V \cap (\{0\} \times \mathbb{C})$ such that $q_2^{m_j}(\tilde{w}) \to \infty$ for some increasing sequence $(m_j)_{j \in \mathbb{N}}$. It is easy to see that there exists $(z', w') \in V$ such that $\mathcal{L}_2(\alpha_2, \Gamma_2 - \rho; z', w') = \tilde{w}$. Then, using the functional equation

$$\mathcal{L}_2(\alpha_2, \Gamma_2 - \rho; p_2(z), q_2(w)) = q_2 \circ \mathcal{L}_2(\alpha_2, \Gamma_2 - \rho; z, w),$$

we have $|\mathcal{L}_2(\alpha_2, \Gamma_2 - \rho; p_2^{m_j}(z'), q_2^{m_j}(w'))| > 2R$ for some fixed j large enough. Moreover, by taking j even larger if necessary, we may assume that $z = \mathfrak{f}^{-1} \circ p_2^{m_j}(z')$ and $w := \mathfrak{g}_0^{-1} \circ q_2^{m_j}(w')$ are well-defined and in U. Since ρ is an accumulation point of the sequence ρ_k, there are arbitrarily large values of k for which

$$|\mathcal{L}_2(\alpha_2, \Gamma_2 - \rho_k; \mathfrak{f}(z), \mathfrak{g}_0(w))| > R$$

as required.

Next, it follows from Theorem 5.13 that

$$\begin{aligned} P_2^{M_k} \circ \mathfrak{h}(p_1^{n_k^1}(z), w) &= P_2^{M_k}\big(p_2^{n_k^1}(\mathfrak{f}(z)), \mathfrak{g}_0(w) + o(1)\big) \\ &= \big(0, \mathcal{L}_2(\alpha_2, \Gamma_2 - \rho_k; \mathfrak{f}(z), \mathfrak{g}_0(w)\big) + o(1). \end{aligned}$$

Therefore, by (8.3) and our choice of R, z and w, we must have $M_k > n_{k+1}^1 - n_k^1 - m$ for arbitrarily large values of k. Therefore $\alpha_2 \ge \alpha_1$; but then by symmetry, $\alpha_2 = \alpha_1$. Then, using again the fact that $M_k > n_{k+1}^1 - n_k^1 - m$, we find $\beta_2 \ge \beta_1$, and therefore we finally have, again by symmetry, $\beta_1 = \beta_2$. ■

Proof of Theorem 1.16. By Proposition 8.5, we have $n_k^1 = n_k^2 =: n_k$, $\sigma_k^1 = \sigma_k^2 =: \sigma_k$ and $\alpha_1 = \alpha_2 =: \alpha$, so we can apply (8.3) with $j = n_{k+1} - n_k - m$, k large enough, $z \in \mathcal{B}_p$, and $w \in U_0 \cap \mathcal{B}_{q_0}$:

$$\mathfrak{h} \circ P_1^{n_{k+1}-n_k-m}(p_1^{n_k}(z), w) = P_2^{n_{k+1}-n_k-m} \circ \mathfrak{h}(p_1^{n_k}(z), w)$$
$$= P_2^{n_{k+1}-n_k-m}\big(p_2^{n_k} \circ \mathfrak{f}(z), \mathfrak{g}(p_2^{n_k} \circ \mathfrak{f}(z), w)\big).$$

By continuity of $\mathfrak{g}$, we have $\mathfrak{g}(p_2^{n_k} \circ \mathfrak{f}(z), w) = \mathfrak{g}_0(w) + o(1)$. By the Main Theorem,

$$P_1^{n_{k+1}-n_k}(p_1^{n_k}(z), w) = \big(o(1), \mathcal{L}_1(\alpha, \Gamma_1 + \sigma_k; z, w) + o(1)\big)$$

and

$$P_2^{n_{k+1}-n_k}\big(p_2^{n_k} \circ \mathfrak{f}(z), \mathfrak{g}(p_2^{n_k} \circ \mathfrak{f}(z), w)\big) = P_2^{n_{k+1}-n_k}\big(p_2^{n_k} \circ \mathfrak{f}(z), \mathfrak{g}_0(w) + o(1)\big)$$
$$= \big(o(1), \mathcal{L}_2(\alpha, \Gamma_2 + \sigma_k; \mathfrak{f}(z), \mathfrak{g}_0(w) + o(1)\big).$$

Finally, let $(z_j, w_j) := P_1^j(p_1^{n_k}(z), w)$. By Lemma 5.12, $\phi_{q_1}^o(w_{n_{k+1}-n_k-m}) = \phi_{q_1}^o(w_{n_{k+1}-n_k}) - m + o(1)$, so that

$$w_{n_{k+1}-n_k-m} = \mathcal{L}_1(\alpha, \Gamma_1 + \sigma_k - m; z, w) + o(1).$$

Similarly,

$$P_2^{n_{k+1}-n_k-m}(p_2^{n_k} \circ \mathfrak{f}(z), \mathfrak{g}(p_2^{n_k} \circ \mathfrak{f}(z), w))$$
$$= \big(o(1), \mathcal{L}_2(\alpha, \Gamma_2 + \sigma_k - m; \mathfrak{f}(z), \mathfrak{g}_0(w) + o(1)\big).$$

Putting all this together, we obtain

$$\mathfrak{g}_0(\mathcal{L}_1(\alpha, \Gamma_1 + \sigma_k - m; z, w)) = \mathcal{L}_2(\alpha, \Gamma_2 + \sigma_k - m; \mathfrak{f}(z), \mathfrak{g}_0(w)) + o(1). \tag{8.4}$$

Therefore, for any accumulation point σ of the sequence $(\sigma_k)_{k \geq 0}$, we have

$$\mathfrak{g}_0(\mathcal{L}_1(\alpha, \Gamma_1 + \sigma - m; z, w)) = \mathcal{L}_2(\alpha, \Gamma_2 + \sigma - m; \mathfrak{f}(z), \mathfrak{g}_0(w)). \tag{8.5}$$

Let us write for simplicity $\mathcal{L}_i(z, w) := \mathcal{L}_i(\alpha, \Gamma_i + \sigma - m; z, w)$. Observe that since $\mathfrak{f}$ and $\mathfrak{g}_0$ conjugate p_1 to p_2 and q_1 to q_2 respectively, there exist homeomorphisms $\tilde{\mathfrak{f}} : \mathbb{C} \to \mathbb{C}$ and $\tilde{\mathfrak{g}}_0 : \mathbb{C} \to \mathbb{C}$ commuting with translation by 1 such that

$$\tilde{\mathfrak{g}}_0 \circ \phi_1^o = \phi_2^o \circ \mathfrak{g}_0, \tag{8.6}$$
$$\tilde{\mathfrak{f}} \circ \phi_{p_1}^\iota = \phi_{p_2}^\iota \circ \mathfrak{f}, \tag{8.7}$$

where ϕ_i^o denotes the outgoing Fatou coordinate of q_i. Indeed, the map $\tilde{\mathfrak{f}}$ is first defined on a left half-plane, since $\phi_{p_1}^\iota$ is univalent on $\mathcal{P}_{p_1}^\iota$ and its image contains a right half-plane (see Section 2). Then, using the functional relation $\tilde{\mathfrak{f}}(Z+1) = \tilde{\mathfrak{f}}(Z) + 1$, we extend $\tilde{\mathfrak{f}}$ to all of $\mathbb{C}$. The case of $\tilde{\mathfrak{g}}_0$ is analoguous, using the fact that $\phi_1^o(\mathcal{P}_{q_1}^o)$ contains a left half-plane.

For z, w as above, let $Z := \phi_{p_1}^\iota(z)$ and $W := \phi_1^\iota(w)$. Let us compute

$$\begin{aligned}
\tilde{\mathfrak{g}}_0 \circ \tilde{H}^1_{Z,\sigma_1}(W) &= \phi_2^o \circ \mathfrak{g}_0 \circ (\phi_1^o)^{-1} \circ \tilde{H}^1_{Z,\sigma_1}(W) \\
&= \phi_2^o \circ \mathfrak{g}_0 \circ \mathcal{L}_1(z, (\phi_1^o)^{-1}(W)) \\
\text{(by (8.5))} \quad &= \phi_2^o \circ \mathcal{L}_2(\mathfrak{f}(z), \mathfrak{g}_0 \circ (\phi_1^o)^{-1}(W)) \\
&= \alpha\phi_2^\iota \circ \mathfrak{g}_0 \circ (\phi_1^o)^{-1}(W) + (1-\alpha)\phi_{p_2}^\iota(\mathfrak{f}(z)) + \sigma_2 \\
&= \alpha\phi_2^\iota \circ (\phi_2^o)^{-1} \circ \tilde{\mathfrak{g}}_0(W) + (1-\alpha)\tilde{\mathfrak{f}}(Z) + \sigma_2 \\
&= \tilde{H}^2_{\tilde{\mathfrak{f}}(Z),\sigma_2}(\tilde{\mathfrak{g}}_0(W)),
\end{aligned}$$

where $\sigma_i = \sigma + \Gamma_i - m$.

Therefore, if we let $G(Z, W) = (\tilde{\mathfrak{f}}(Z), \tilde{\mathfrak{g}}_0(W))$ we have proved that

$$G \circ \tilde{H}^1_{\sigma_1}(Z, W) = \tilde{H}^2_{\sigma_2} \circ G(Z, W).$$

This relation holds for all $z \in \mathcal{B}_{p_1}$ and all $w \in \mathcal{B}_{q_1} \cap U_0$; therefore it holds for all $Z \in \mathbb{C}$ and all $W \in \mathbb{C}$ with $\operatorname{Re} Z$ and $\operatorname{Re} W$ large enough.

But since the lifted horn maps $\tilde{H}^i_{\sigma_i}$ commute with translation by the vector $(1, 1)$, this conjugacy descends to a conjugacy of the horn maps on $\mathbb{C}^2/\langle(1, 1)\rangle$. ■

Proof of Corollary 1.17. Let $\mathcal{E}_i := \phi_{q_i}^\iota \circ \psi_{q_i}^o$ be the *lifted horn map* of q_i. Then $(Z, W) \in \mathbb{C}^2$ is a critical point of $\tilde{H}^i_{\sigma_i}$ if and only if W is a critical point of $\mathcal{E}_i$. Therefore, the set of critical values of $\tilde{H}^i_{\sigma_i}$ is

$$\mathrm{CV}(\tilde{H}^i_{\sigma_i}) = \{(Z, \alpha W + (1-\alpha)Z + \sigma_i) : (Z, W) \in \mathbb{C} \times \mathrm{CV}(\mathcal{E}_i)\},$$

which is a union of affine lines in $\mathbb{C}^2$. Now recall that

$$\mathrm{CV}(\mathcal{E}_i) = \{\phi_{q_i}^\iota(c) + n : c \in \mathrm{crit}(q_i) \cap \mathcal{B}_{q_i} \text{ and } n \in \mathbb{Z}\}.$$

Therefore, if we let

$$I_i := \{\phi_{q_i}^\iota(c) + n : c \in \mathrm{crit}(q_i) \cap \mathcal{B}_{q_i} \text{ and } n \in \mathbb{Z} \text{ such that } 0 \leq \operatorname{Re} \phi_{q_i}^\iota(c) + n < 1\},$$

and

$$L^i_{W,n} := \{(Z, \alpha(W + n) + (1-\alpha)Z + \sigma_i) : Z \in \mathbb{C}\}.$$

then the set

$$\mathrm{CV}(\tilde{H}^i_{\sigma_i}) = \bigcup_{(W,n)\in I_i \times \mathbb{Z}} L^i_{W,n}$$

is a countable union of affine lines. Next, for any $(W, n) \in I_i \times \mathbb{Z}$ we have $L^i_{W,n+1} = L^i_{W,n} + (1, 1)$. Let $\pi : \mathbb{C}^2 \to \mathbb{C}^2/\langle(1, 1)\rangle$ denote the quotient map. Since $\pi \circ \tilde{H}^i_{\sigma_i} = H_{\sigma_i} \circ \pi$ and π is a covering, the set of critical values of $H^i_{\sigma_i}$ is $\mathrm{CV}(H^i_{\sigma_i}) = \pi(\mathrm{CV}(\tilde{H}^i_{\sigma_i}))$. It is straightforward to check that $\pi(L^i_{W,n})$ is an irreducible curve in $\mathbb{C}^2/\langle(1, 1)\rangle$. Moreover, given $(W_1, n_1), (W_2, n_2) \in I_i \times \mathbb{Z}$, we have $\pi(L^i_{W_1,n_1}) = \pi(L^i_{W_2,n_2})$ if and only if $W_1 - W_2 \in \mathbb{Z}$, which by the definition of I_i is equivalent to $W_1 = W_2$. Furthermore, if $\pi(L^i_{W_1,n_1}) \neq \pi(L^i_{W_2,n_2})$ then $\pi(L^i_{W_1,n_1}) \cap \pi(L^i_{W_2,n_2}) = \emptyset$. Therefore, $\mathrm{CV}(H^i_{\sigma_i})$ has exactly card I_i connected components.

By Theorem 1.16, the maps $H^1_{\sigma_1}$ and $H^2_{\sigma_2}$ are topologically conjugate. The topological conjugacy must map $\mathrm{CV}(H^1_{\sigma_1})$ to $\mathrm{CV}(H^2_{\sigma_2})$; therefore they must have the same number of connected components.

Finally, we conclude the proof of the first assertion of Corollary 1.17 by observing that card I_i is exactly the number of grand orbits of critical points in $\mathcal{B}_{q_i}$.

For the second assertion, it suffices to observe that this number cannot depend on any k-jet of q_i at $w = 0$. ∎

We end this section with the following natural question.

Question 8.6. *Let P_1 and P_2 be of the form (1.1), with $b_i > 1/4$ and $\beta_i \in \mathbb{R}$. If P_1 and P_2 are topologically conjugate in some neighborhood of the origin, does that imply $P_1 \sim P_2$?*

Note that the positive answer to the above question, together with Proposition 8.5, would imply that the constants α_0, β_0 and b are in fact topological invariants.

9. Proof of Corollary 1.18

Let $\mathcal{L}^{(i)}$ denote the extended Lavaurs maps associated to both parabolic fixed points $(0, w_i)$, and let $\mathcal{L}^{(i)}_z(w) := \mathcal{L}^{(i)}(\alpha_i, \Gamma_i; z, w)$. Let $\mathcal{M}_z := \mathcal{L}^{(2)}_z \circ \mathcal{L}^{(1)}_z$. We denote by $\mathcal{B}_i$ the parabolic basins of w_i for q_0, so that $(z, w) \mapsto \mathcal{M}_z(w)$ is defined on $\mathcal{B}_p \times \mathcal{B}_1$. We start by recalling the notion of islands, named after Ahlfors's famous Five Islands Theorem.

Definition 9.1. Let $f : U \to \mathbb{P}^1$ be a holomorphic map, where $U \subset \mathbb{P}^1$ is a domain. Let $D \subset \mathbb{P}^1$ be a Jordan domain. We say that $\tilde{D} \subset U$ is an *island* for f over D if $f : \tilde{D} \to D$ is a conformal isomorphism.

Lemma 9.2. *Let $f(z) = z + z^2 + O(z^3)$ be a polynomial map with a parabolic fixed point, and let $\phi^\iota_f : \mathcal{B}_f \to \mathbb{C}$ and $\psi^o_f : \mathbb{C} \to \mathbb{C}$ denote its incoming Fatou coordinate and outgoing Fatou parametrization respectively.*

(1) *For every Jordan domain $D \subset \mathbb{C}$ such that $(\phi^\iota_f)^{-1}(D)$ does not intersect critical orbits of f, and for every open set Ω intersecting $\partial\mathcal{B}_f$, ϕ^ι_f has an island $\tilde{D} \Subset \Omega$ over D.*

(2) *For every Jordan domain $D \subset \mathbb{C}$ that does not intersect the postcritical set of f, ψ^o_f has an island $\tilde{D}$ over D.*

Proof. Let $D \subset \mathbb{C}$ be a Jordan domain such that $(\phi^\iota_f)^{-1}(D)$ does not intersect critical orbits of f, and let Ω be an open set intersecting $\partial\mathcal{B}_f$.

Let $D_k := D + k$. By Proposition 2.1, $\phi^\iota_f : \mathcal{B}_f \to \mathbb{C}$ is a branched cover whose critical points are the precritical orbits of f in $\mathcal{B}_f$; therefore, by the assumptions on $D = D_0$,

D_k is simply connected and does not contain any critical value of ϕ_f^ι, so ϕ_f^ι has an island U_0 above D_k.

By assumption, U_0 does not meet any critical orbits of f, and it is simply connected, so we may define univalent inverse branches of f^{-k} for all k, and for k large enough, at least one such branch g_k will map U_0 compactly into Ω (by normality and the equidistribution of preimages). Let $U_k := g_k(U_0)$. We then have

$$D_0 + k = \phi_f^\iota(U_0) = \phi_f^\iota \circ f^k(U_k) = \phi_f^\iota(U_k) + k$$

so that $\phi_f^\iota(U_k) = D_0$. The domain U_k is the desired island above D_0.

The second item follows immediately from Proposition 2.2, which implies that

$$\psi_f^o : \mathbb{C} \setminus (\psi_f^o)^{-1}(P_f) \to \mathbb{C} \setminus P_f$$

is a covering map, where P_f denotes the postcritical set of f. ■

Lemma 9.3. *There exists $z_0 \in \mathcal{B}_p$ such that $\mathcal{M}_{z_0}$ has a superattracting fixed point w_0.*

Proof. The difficulty is that we cannot apply Montel's theorem, as the domain of $\mathcal{M}_z^n$ shrinks as $n \to \infty$. Instead, we will follow closely the proof of the Shooting Lemma from [5]. Let ϕ_i^ι (with $i = 1, 2$) denote the incoming Fatou coordinates of w_i for q_0, and let ψ_i^o denote the outgoing Fatou parametrizations associated to w_i for q_0. Let $Z := \phi_p^\iota(z)$, $A_{i,Z}(W) := \alpha_i W + (1-\alpha_i)Z + \Gamma_i$, so that

$$\mathcal{M}_z = \psi_2^o \circ A_{2,Z} \circ \phi_2^\iota \circ \psi_1^o \circ A_{1,Z} \circ \phi_1^\iota. \tag{9.1}$$

Let $c \in \mathcal{B}_1$ be a critical point for ϕ_1^ι. Let $x \in (\psi_2^o)^{-1}(\{c\})$. Let $\gamma(Z) := A_{1,Z} \circ \phi_1^\iota(c)$, and let $g_Z := A_{2,Z} \circ \phi_2^\iota \circ \psi_1^o$. If we can find $Z \in \mathbb{C}$ such that $g_Z \circ \gamma(Z) = x$, then this will mean that $\mathcal{M}_z(c) = c$, where $\phi_p^\iota(z) = Z$, which will prove the lemma.

Let $U_0 := (\psi_1^o)^{-1}(\mathcal{B}_2)$. We claim that there exists $Z_0 \in \mathbb{C}$ such that $\gamma(Z_0) \in \partial U_0$ and x is not a critical value of g_{Z_0}.

Since $\psi_1^o : \mathbb{C} \to \mathbb{C}$ is an entire function, $U_0 \subset \mathbb{C}$ is an open set whose boundary contains a continuum. From the expression of γ, if we fix any $W_0 \in \mathbb{C}$, then for $Z_0 := \frac{1}{1-\alpha_1}(W_0 - \alpha_1\phi_1^\iota(c) - \Gamma_1)$ we have $\gamma(Z_0) = W_0$; therefore, the set of $Z \in \mathbb{C}$ such that $\gamma(Z) \in \partial U_0$ is the image of ∂U_0 under an affine map and also contains a continuum.

On the other hand, the critical values of g_Z are of the form $A_{2,Z}(v)$, where v is a critical value of $\phi_2^\iota \circ \psi_1^o$. Therefore, the set of $Z \in \mathbb{C}$ such that x is a critical value of g_Z is a countable set, so we can indeed find $Z_0 \in \mathbb{C}$ such that $\gamma(Z_0) \in \partial U_0$ and x is not a critical value of g_{Z_0}.

Next, observe that $g_Z = g_{Z_0} + (1-\alpha_2)(Z - Z_0)$. Therefore, if we define $h(Z) := x + (\alpha_2 - 1)(Z - Z_0)$, the equation $g_Z \circ \gamma(Z) = x$ becomes equivalent to

$$g_{Z_0} \circ \gamma(Z) = h(Z). \tag{9.2}$$

Let D be a disk centered at x such that D contains no critical values of g_{Z_0}. This is possible because of our choice of Z_0 and because the set of critical values of g_{Z_0} is discrete, in fact finite in $\mathbb{C}/\mathbb{Z}$. Let $\epsilon > 0$ be small enough that $h(\mathbb{D}(Z_0, \epsilon)) \Subset D$. Let $\Omega := \gamma(\mathbb{D}(Z_0, \epsilon))$; it is an open neighborhood of $W_0 \in \partial U_0$. By Lemma 9.2, there exists $D_1 \Subset \Omega \cap U_0$ such that $g_{Z_0} : D_1 \to D$ is a conformal isomorphism. In particular, $g_{Z_0} \circ \gamma : V \to D$ is a conformal isomorphism, where $V := \gamma^{-1}(D_1)$ is a conformal disk that is compactly contained in $\mathbb{D}(Z_0, \epsilon)$. By the definition of ϵ and V, we therefore have $h(V) \Subset g_{Z_0} \circ \gamma(V) = D$, and D, V are conformal disks with smooth boundaries. It then follows from the Argument Principle that there exists $Z \in V$ satisfying (9.2), and the lemma is proved. ∎

Proof of Corollary 1.18. We consider an inductive sequence of integers defined by $n_{k+1} = \alpha_1 n_k$ if k is even and $n_{k+1} = \alpha_2 n_k$ if k is odd.

By the Main Theorem applied twice, we have

$$P^{n_{k+2}-n_k}(z_{n_k}, w) = (z_{n_{k+2}}, \mathcal{M}_z(w)) + o(1)$$

with local uniform convergence for (z, w) sufficiently close to the point (z_0, w_0) given by Lemma 9.3.

Since w_0 is a superattracting fixed point for $\mathcal{M}_{z_0}$, there exists $r > 0$ such that $\mathcal{M}_{z_0}(\mathbb{D}(w_0, r)) \Subset \mathbb{D}(w_0, r/2)$, and by continuity there exists $\eta > 0$ such that for all $z \in \mathbb{D}(z_0, \eta)$ we have $\mathcal{M}_z(\mathbb{D}(w_0, r)) \Subset \mathbb{D}(w_0, r)$.

Let V be a connected component of $P^{-n_0}(p^{n_0}(\mathbb{D}(z_0, \eta)) \times \mathbb{D}(w_0, r))$. For n_0 large enough and (n_k) satisfying the induction relation above, we have, for any $k \in \mathbb{N}$ and $(z, w) \in V$,

$$P^{n_{2k}}(z, w) \in \mathcal{B}_p \times \mathbb{D}(w_0, r). \tag{9.3}$$

In particular, $V \subset K(P)$, where

$$K(P) := \{(z, w) \in \mathbb{C}^2 : (P^n(z, w))_{n \in \mathbb{N}} \text{ is bounded}\}.$$

Therefore, V is contained in the Fatou set of P. Let Ω be the Fatou component of P containing V.

Finally, let us prove that Ω satisfies the historicity property. We claim that

$$\lim_{k \to \infty} \frac{1}{n_{2k+1} - n_{2k}} \sum_{j=n_{2k}+1}^{n_{2k+1}} \delta_{P^j(z,w)} = \delta_{(0,w_1)} \tag{9.4}$$

and

$$\lim_{k \to \infty} \frac{1}{n_{2k+2} - n_{2k+1}} \sum_{j=n_{2k+1}+1}^{n_{2k+2}} \delta_{P^j(z,w)} = \delta_{(0,w_2)}. \tag{9.5}$$

Informally speaking, this follows from the fact that it takes $n_{2k+1} - n_{2k}$ iterations to "pass through the eggbeater" associated to $(0, w_1)$, and $n_{2k+2} - n_{2k+1}$ to pass through the one associated to $(0, w_2)$.

Let us give a more precise justification. Let $k \in \mathbb{N}$. By the above argument, we know that $P^{n_{2k}}(z, w) \in \mathcal{B}_p \times \mathbb{D}(w_0, r)$. Therefore, with $n := n_{2k}$ and $(\hat{z}_j, \hat{w}_j) := P^j(z, w)$, Lemma 5.8 states that for all k large enough, $\Phi_{\hat{z}_j}(\hat{w}_j) \in \mathcal{R}_{\hat{z}_j}$ for all $t_n \leq j \leq M_n - \ell_n$, where $t_n := \lfloor n^\nu \rfloor = \lfloor n_{2k}^\nu \rfloor$, $\ell_n = \lfloor e^{\pi/c} t_n \rfloor$, and $M_n = \lfloor (\alpha_1 - 1)n \rfloor = (\alpha_1 - 1)n_{2k}$ (since we have assumed $\beta_1 = 0$ and $\alpha_1 \in \mathbb{N}^*$). In particular, by Lemma 4.8,

$$\hat{w}_j - w_1 = O(|\hat{z}_j|^\nu) = O\left(\frac{1}{n_{2k}^\nu}\right) = o_{k\to\infty}(1)$$

for $t_n \leq j \leq M_n - \ell_n$. Moreover, $M_n + n_{2k} = n_{2k+1}$, so that $M_n - t_n - \ell_n = n_{2k+1} - n_{2k} + O(n_{2k}^\nu) \sim_{k\to\infty} n_{2k+1} - n_{2k}$. This proves (9.4).

The proof of (9.5) is similar, replacing n_{2k} by n_{2k+1}, n_{2k+1} by n_{2k+2}, and w_1 by w_2.

Let $(z, w) \in V$, and let us consider $e_n = e_n(z, w) := \frac{1}{n}\sum_{j=0}^{n-1} \delta_{P^j(z,w)}$. By (9.4), we have

$$\begin{aligned} e_{n_{2k+1}} &= e_{n_{2k}} \frac{n_{2k}}{n_{2k+1}} + (1 - \frac{n_{2k}}{n_{2k+1}})\delta_{(0,w_1)} + o(1) \\ &= \frac{1}{\alpha_1} e_{n_{2k}} + \left(1 - \frac{1}{\alpha_1}\right)\delta_{(0,w_1)} + o(1) \end{aligned}$$

and similarly, using (9.5),

$$e_{n_{2k}} = \frac{1}{\alpha_2} e_{n_{2k-1}} + \left(1 - \frac{1}{\alpha_2}\right)\delta_{(0,w_2)} + o(1).$$

Putting the last two equations together, we find

$$\begin{aligned} e_{n_{2k}} &= \frac{\alpha_1\alpha_2 - \alpha_2}{\alpha_1\alpha_2 - 1}\delta_{(0,w_1)} + \frac{\alpha_2 - 1}{\alpha_1\alpha_2 - 1}\delta_{(0,w_2)} + o(1), \\ e_{n_{2k+1}} &= \frac{\alpha_1 - 1}{\alpha_1\alpha_2 - 1}\delta_{(0,w_1)} + \frac{\alpha_1\alpha_2 - \alpha_1}{\alpha_1\alpha_2 - 1}\delta_{(0,w_2)} + o(1). \end{aligned} \quad \blacksquare$$

10. Admissible sequences and Pisot numbers

Finally, we will give the proof of Theorem 1.10. Let us recall that for given $\alpha > 1$ and $\beta \in \mathbb{R}$, we say that a strictly increasing sequence $(n_k)_{k\geq 0}$ of positive integers is (α, β)-admissible if its phase sequence $\sigma_k = n_{k+1} - \alpha n_k - \beta \ln n_k$ is bounded.

Lemma 10.1. *For every (α, β)-admissible sequence $(n_k)_{k\geq 0}$ there exist a real number $\zeta > 0$ and a bounded sequence $(d_k)_{k\geq 0}$ of real numbers such that*

$$n_k = \zeta\alpha^k - k\frac{\beta \ln \alpha}{\alpha - 1} + d_k, \quad \forall k \geq 0.$$

Moreover, if we let $\rho_k := n_{k+1} - \alpha n_k - k\beta \ln \alpha$, then

$$\begin{aligned} \rho_k &= \sigma_k + \beta \ln \zeta + o(1), \\ \zeta &= n_0 + \frac{\beta \ln \alpha}{(\alpha - 1)^2} + \frac{1}{\alpha}\sum_{j=0}^{\infty} \frac{\rho_j}{\alpha^j}, \\ d_k &= -\frac{\beta \ln \alpha}{(\alpha - 1)^2} - \frac{1}{\alpha}\sum_{j=0}^{\infty} \frac{\rho_{j+k}}{\alpha^j}. \end{aligned}$$

Proof. First we study the asymptotic behavior of (α, β)-admissible sequences.

Claim 1. *For every (α, β)-admissible sequence $(n_k)_{k\geq 0}$ there exist constants $\zeta, C \geq 0$ such that $|n_k - \zeta\alpha^k| \leq Ck$ for all $k \geq 1$.*

Proof of Claim 1. Let us first define a sequence $v_k := n_{N+k}/\alpha^k$, where we have chosen N sufficiently large so that

$$n_N - \sum_{j=0}^{\infty}\left(\frac{|\beta|}{\alpha^{j+1}}\ln\left(\frac{n_N}{2}\right) + \frac{j|\beta|\ln\alpha}{\alpha^{j+1}} + \frac{|\sigma_{N+j}|}{\alpha^{j+1}}\right) > \frac{n_N}{2},$$

$$n_N + \sum_{j=0}^{\infty}\left(\frac{|\beta|}{\alpha^{j+1}}\ln(2n_N) + \frac{j|\beta|\ln\alpha}{\alpha^{j+1}} + \frac{|\sigma_{N+j}|}{\alpha^{j+1}}\right) < 2n_N.$$

Such an N exists because $(n_k)_{k\geq 0}$ is strictly increasing and the phase sequence $(\sigma_k)_{k\geq 0}$ is bounded.

Observe that

$$v_k = v_{k-1} + \frac{\beta}{\alpha^k}\ln v_{k-1} + \frac{\beta(k-1)\ln\alpha}{\alpha^k} + \frac{\sigma_{N+k-1}}{\alpha^k}$$

and let us prove that

$$n_N/2 < v_k < 2n_N, \quad \forall k \geq 0.$$

Clearly this holds for $k = 0$ since $v_0 = n_N$. Assume that these bounds hold for all $0 \leq j < k$. Since

$$v_k = v_0 + \sum_{j=0}^{k-1}(v_{j+1} - v_j) = n_N + \sum_{j=0}^{k-1}\left(\frac{\beta}{\alpha^{j+1}}\ln v_j + \frac{j\beta\ln\alpha}{\alpha^{j+1}} + \frac{\sigma_{N+j}}{\alpha^{j+1}}\right),$$

the bounds for v_k are clearly guaranteed by our choice of N. Moreover, this also implies that the sequence $(v_k)_{k\geq 0}$ converges to some positive real number

$$\tilde{\zeta} = v_0 + \sum_{j=0}^{\infty}(v_{j+1} - v_j) = n_N + \sum_{j=0}^{\infty}\left(\frac{\beta}{\alpha^{j+1}}\ln v_j + \frac{j\beta\ln\alpha}{\alpha^{j+1}} + \frac{\sigma_{N+j}}{\alpha^{j+1}}\right),$$

where the sum converges absolutely.

Now we define a new sequence $u_k := n_k/\alpha^k$ and observe that $u_{N+k}\alpha^N = v_k$. From the above computation it follows that the sequence $(u_k)_{k\geq 0}$ converges to $\zeta := \tilde{\zeta}/\alpha^N > 0$, and since

$$u_k = u_{k-1} + \frac{\beta}{\alpha^k}\ln u_{k-1} + \frac{\beta(k-1)\ln\alpha}{\alpha^k} + \frac{\sigma_{k-1}}{\alpha^k},$$

we have

$$\zeta = u_0 + \sum_{j=0}^{\infty}(u_{j+1} - u_j) = n_0 + \sum_{j=0}^{\infty}\left(\frac{\beta}{\alpha^{j+1}}\ln u_j + \frac{j\beta\ln\alpha}{\alpha^{j+1}} + \frac{\sigma_j}{\alpha^{j+1}}\right).$$

Finally, observe that

$$n_k = \alpha^k u_k = \zeta\alpha^k - \frac{1}{\alpha}\sum_{j=0}^{\infty}\left(\frac{\beta}{\alpha^j}\ln u_{j+k} + \frac{(j+k)\beta\ln\alpha}{\alpha^j} + \frac{\sigma_{j+k}}{\alpha^j}\right)$$

and there exists $C > 0$ such that

$$\left|\frac{1}{\alpha}\sum_{j=0}^{\infty}\left(\frac{\beta}{\alpha^j}\ln u_{j+k} + \frac{(j+k)\beta\ln\alpha}{\alpha^j} + \frac{\sigma_{j+k}}{\alpha^j}\right)\right| < Ck, \quad \forall k \geq 1. \qquad \blacksquare$$

Claim 2. *We have* $\rho_k = \sigma_k + \beta\ln\zeta + o(1)$.

Proof of Claim 2. Observe that by the previous claim, we have

$$\begin{aligned}\rho_k &= n_{k+1} - \alpha n_k - k\beta\ln\alpha = \sigma_k + \beta\ln n_k - k\beta\ln\alpha \\ &= \sigma_k + \beta\ln u_k = \sigma_k + \beta\ln\zeta + o(1).\end{aligned} \qquad \blacksquare$$

Claim 3. *We have*

$$\zeta = n_0 + \frac{\beta\ln\alpha}{(\alpha-1)^2} + \frac{1}{\alpha}\sum_{j=0}^{\infty}\frac{\rho_j}{\alpha^j}, \quad d_k = -\frac{\beta\ln\alpha}{(\alpha-1)^2} - \frac{1}{\alpha}\sum_{j=0}^{\infty}\frac{\rho_{j+k}}{\alpha^j}.$$

Proof of Claim 3. Recall that $\rho_k = \sigma_k + \beta\ln u_k$. Clearly $(\rho_k)_{k\geq 0}$ is bounded because $(\sigma_k)_{k\geq 0}$ is bounded and $(u_k)_{k\geq 0}$ converges to ζ. From the proof of Claim 1 it now follows that

$$\begin{aligned}\zeta &= n_0 + \sum_{j=0}^{\infty}\frac{\beta}{\alpha^{j+1}}\ln u_j + \frac{j\beta\ln\alpha}{\alpha^{j+1}} + \frac{\sigma_j}{\alpha^{j+1}} \\ &= n_0 + \frac{1}{\alpha}\sum_{j=0}^{\infty}\frac{j\beta\ln\alpha}{\alpha^j} + \frac{\rho_j}{\alpha^j} = n_0 + \frac{\beta\ln\alpha}{(\alpha-1)^2} + \frac{1}{\alpha}\sum_{j=0}^{\infty}\frac{\rho_j}{\alpha^j}\end{aligned}$$

and

$$\begin{aligned}d_k &= k\frac{\beta\ln\alpha}{\alpha-1} + n_k - \zeta\alpha^k = k\frac{\beta\ln\alpha}{\alpha-1} - \frac{1}{\alpha}\sum_{j=0}^{\infty}\frac{\beta}{\alpha^j}\ln u_{j+k} + \frac{(j+k)\beta\ln\alpha}{\alpha^j} + \frac{\sigma_{j+k}}{\alpha^j} \\ &= k\frac{\beta\ln\alpha}{\alpha-1} - \frac{1}{\alpha}\sum_{j=0}^{\infty}\frac{(j+k)\beta\ln\alpha}{\alpha^j} + \frac{\rho_{j+k}}{\alpha^j} = -\frac{\beta\ln\alpha}{(\alpha-1)^2} - \frac{1}{\alpha}\sum_{j=0}^{\infty}\frac{\rho_{j+k}}{\alpha^j}.\end{aligned}$$

Clearly the sequence $(d_k)_{k\geq 0}$ is bounded since $(\rho_k)_{k\geq 0}$ is bounded. ■

This completes the proof of Lemma 10.1. ■

Remark 10.2. By Lemma 10.1 we have $\rho_k = \sigma_k + \beta\ln u_k$, where $(u_k)_{k\geq 0}$ converges to $\zeta > 0$. Hence the phase sequence $(\sigma_k)_{k\geq 0}$ of an (α, β)-admissible sequence converges to a cycle if and only if the sequence $(\rho_k)_{k\geq 0}$ converges to a cycle of the same period.

Moreover, since

$$d_k = -\frac{\beta \ln \alpha}{(\alpha-1)^2} - \frac{1}{\alpha}\sum_{j=0}^{\infty}\frac{\rho_{j+k}}{\alpha^j} \quad \text{and} \quad \sigma_k = d_{k+1} - \alpha d_k - \beta\frac{\ln\alpha}{\alpha-1} - \beta \ln u_k,$$

the phase sequence $(\sigma_k)_{k\geq 0}$ converges to a cycle if and only if $(d_k)_{k\geq 0}$ converges to a cycle of the same period.

Corollary 10.3. *Let $(n_k)_{k\geq 0}$ be an α-admissible sequence whose phase sequence converges to zero. Then α has the Pisot property.*

Proof. Since $(n_k)_{k\geq 0}$ is an α-admissible sequence, $\beta = 0$ and $\rho_k = \sigma_k$ (using the notation from Lemma 10.1). Moreover, since $(\sigma_k)_{k\geq 0}$ converges to zero, the same holds for $(d_k)_{k\geq 0}$, and hence $\|\zeta\alpha^k\| \to 0$. ∎

Lemma 10.4. *Let $(n_k)_{k\geq 0}$ be an (α,β)-admissible sequence and $(\sigma_k)_{k\geq 0}$ its phase sequence. Then $(\sigma_k)_{k\geq 0}$ converges to a cycle of period ℓ if and only if $m_k := n_{k+\ell} - n_k$ is an α-admissible sequence whose phase sequence converges to $\ell\beta\ln\alpha$.*

Proof. Observe that

$$\begin{aligned} m_{k+1} - \alpha m_k &= n_{k+1+\ell} - n_{k+1} - \alpha(n_{k+\ell} - n_k) \\ &= (n_{k+1+\ell} - \alpha n_{k+\ell}) - (n_{k+1} - \alpha n_k) \\ &= \sigma_{\ell+k} - \sigma_k + \beta \ln\frac{n_{k+\ell}}{n_k} = \sigma_{\ell+k} - \sigma_k + \ell\beta\ln\alpha + o(1). \end{aligned}$$ ∎

Corollary 10.5. *If $(n_k)_{k\geq 0}$ is α-admissible with converging phase sequence, then $m_k := n_{k+1} - n_k$ is α-admissible and has phase converging to zero.*

Proof. This follows from the previous lemma with $\beta = 0$. ∎

Lemma 10.6. *Let α have the Pisot property. Then there exists an α-admissible sequence whose phase sequence converges to* 0.

Proof. Since α has the Pisot property, there is $\zeta > 0$ such that $\|\zeta\alpha^k\| \to 0$. Now we can define a sequence of integers

$$n_k := \begin{cases} \zeta\alpha^k - \|\zeta\alpha^k\| & \text{if } 0 \leq \{\zeta\alpha^k\} < 1/2, \\ \zeta\alpha^k + \|\zeta\alpha^k\| & \text{otherwise,} \end{cases}$$

for which clearly $n_{k+1} - \alpha n_k \to 0$. ∎

We shall denote by $\sigma(n_\bullet)$ the phase sequence associated to the sequence (n_k), and by $\sigma(n_\bullet)_k$ its k-th element.

Lemma 10.7. *Let (n_k), (m_k) be two α-admissible sequences, and let $j, j_1, j_2 \in \mathbb{Z}$. Then*

(1) *(n_{k+j}) is again an α-admissible sequence, and $\sigma(n_{\bullet+j})_k = \sigma(n_\bullet)_{k+j}$;*

(2) *if* $(j_1 n_k + j_2 m_k)$ *is strictly increasing, then it is an* α*-admissible sequence, and* $\sigma(j_1 n_\bullet + j_2 m_\bullet) = j_1 \sigma(n_\bullet) + j_2 \sigma(m_\bullet)$*;*

(3) *if* (m_k) *is* α*-admissible and* $\epsilon_k \in \ell^\infty$*, then* $n_k := m_k + \epsilon_k$ *is* α*-admissible, and* $\sigma(n_\bullet)_k = \sigma(m_\bullet)_k + \epsilon_{k+1} - \alpha\epsilon_k$.

Proof. This is a direct computation. ■

Observe that Corollaries 10.3 and 10.5 and Lemmas 10.6 and 10.7 imply the following result which settles claim (1) of Theorem 1.10.

Corollary 10.8. *Let* $\alpha > 1$ *and* $m \in \mathbb{N}^*$ *be arbitrary. The following are equivalent:*

(1) α *has the Pisot property;*

(2) *there exists an* α*-admissible sequence whose phase sequence converges;*

(3) *there exists an* α*-admissible sequence whose phase sequence converges to a cycle of exact period* m*.*

Let us mention that for a very special type of α-admissible sequences similar conclusions were already made by Dubickas [14].

Remark 10.9. Let $(n_k)_{k\geq 0}$ be an (α, β)-admissible sequence and denote $\theta = \frac{\beta \ln \alpha}{\alpha - 1}$ and $m_k = n_k + \lfloor k\theta \rfloor$. By Lemma 10.1 we have

$$\begin{aligned} n_{k+1} - \alpha n_k - \beta \ln n_k &= n_{k+1} - \alpha n_k - k\beta \ln \alpha - \beta \ln \zeta + o(1) \\ &= m_{k+1} - \alpha m_k + \{(k+1)\theta\} - \alpha\{k\theta\} - \theta - \beta \ln \zeta + o(1). \end{aligned}$$

It follows that the phase sequence of $(n_k)_{k\geq 0}$ converges to a cycle if and only if the sequence $(m_k)_{k\geq 0}$ is α-admissible and the sequence $m_{k+1} - \alpha m_k + \{(k+1)\theta\} - \alpha\{k\theta\}$ converges to a cycle of the same period as $\sigma(n_k)$.

Finally, claim (2) of Theorem 1.10 follows from Lemma 10.4, Corollary 10.8 and the following observation. Let $(m_k)_{k\geq 0}$ be an α-admissible sequence whose phase sequence converges to zero (note that such always exists since α has the Pisot property) and let $\theta := \frac{\beta \ln \alpha}{\alpha - 1}$. If $\theta = \frac{k_1}{k_2} \in \mathbb{Q}$ then by the above remark the sequence $n_k := m_k - \lfloor k\theta \rfloor$ is an (α, β)-admissible sequence whose phase sequence converges to a cycle of period k_2. This completes the proof of Theorem 1.10.

We conclude this section with the following question.

Question 10.10. *Let* $\alpha > 1$ *have the Pisot property. We have seen that* $\theta \in \mathbb{Q}$ *is a sufficient condition for the existence of an* α*-admissible sequence* $(m_k)_{k\geq 0}$ *such that the sequence* $m_{k+1} - \alpha m_k + \{(k+1)\theta\} - \alpha\{k\theta\}$ *converges to a cycle. Is this condition also necessary?*

Acknowledgments. We thank Marco Mancini for his invaluable help in writing the code used to produce Figure 3, and Arnaud Chéritat for helpful discussions. We are grateful to the anonymous referees for a careful reading of the manuscript, which led to a great improvement in the exposition.

Funding. The second author is partially supported by the research program P1-0291 and the grant N1-0237 from ARRS, Republic of Slovenia.

References

[1] Abate, M.: The residual index and the dynamics of holomorphic maps tangent to the identity. Duke Math. J. **107**, 173–207 (2001) Zbl 1015.37035 MR 1815255

[2] Abate, M.: Holomorphic classification of 2-dimensional quadratic maps tangent to the identity. Surikaisekikenkyusho Kokyuroku **1447**, 1–14 (2005)

[3] Abate, M., Tovena, F.: Poincaré–Bendixson theorems for meromorphic connections and holomorphic homogeneous vector fields. J. Differential Equations **251**, 2612–2684 (2011) Zbl 1241.32012 MR 2825343

[4] Astorg, M.: The Teichmüller space of a rational map immerses into moduli space. Adv. Math. **313**, 991–1023 (2017) Zbl 1508.37059 MR 3649243

[5] Astorg, M., Benini, A. M., Fagella, N.: Bifurcation loci of families of finite type meromorphic maps. arXiv:2107.02663v3 (2023)

[6] Astorg, M., Boc Thaler, L., Peters, H.: Wandering domains arising from Lavaurs maps with Siegel disks. Anal. PDE **16**, 35–88 (2023) Zbl 1518.32004 MR 4578525

[7] Astorg, M., Buff, X., Dujardin, R., Peters, H., Raissy, J.: A two-dimensional polynomial mapping with a wandering Fatou component. Ann. of Math. (2) **184**, 263–313 (2016) Zbl 1368.37055 MR 3505180

[8] Bedford, E., Smillie, J., Ueda, T.: Semi-parabolic bifurcations in complex dimension two. Comm. Math. Phys. **350**, 1–29 (2017) Zbl 1365.37044 MR 3606468

[9] Berger, P., Biebler, S.: Emergence of wandering stable components. J. Amer. Math. Soc. **36**, 397–482 (2023) Zbl 1522.37056 MR 4536902

[10] Bianchi, F.: Parabolic implosion for endomorphisms of $\mathbb{C}^2$. J. Eur. Math. Soc. **21**, 3709–3737 (2019) Zbl 1435.32020 MR 4022713

[11] Buff, X., Epstein, A. L.: A parabolic Pommerenke–Levin–Yoccoz inequality. Fund. Math. **172**, 249–289 (2002) Zbl 1115.37323 MR 1898687

[12] Camacho, C.: On the local structure of conformal mappings and holomorphic vector fields in $\mathbf{C}^2$. In: Journées Singulières de Dijon (Dijon, 1978), Astérisque **59**, 3, 83–94 (1978) Zbl 0415.30015 MR 0542732

[13] Douady, A., Hubbard, J. H.: Étude dynamique des polynômes complexes. Partie I. Publications Mathématiques d'Orsay 84, Université de Paris-Sud, Département de Mathématiques, Orsay (1984) Zbl 0552.30018 MR 0762431

[14] Dubickas, A.: On integer sequences generated by linear maps. Glasgow Math. J. **51**, 243–252 (2009) Zbl 1196.11034 MR 2500747

[15] Dujardin, R.: Non-density of stability for holomorphic mappings on $\mathbb{P}^k$. J. École Polytech. Math. **4**, 813–843 (2017) Zbl 1406.37041 MR 3694096

[16] Écalle, J.: Les fonctions résurgentes. Tome III. Publications Mathématiques d'Orsay 85, Université de Paris-Sud, Département de Mathématiques, Orsay (1985) Zbl 0602.30029 MR 0852210

[17] Fagella, N., Keen, L.: Stable components in the parameter plane of transcendental functions of finite type. J. Geom. Anal. **31**, 4816–4855 (2021) Zbl 1466.37039 MR 4244887

[18] Hakim, M.: Attracting domains for semi-attractive transformations of $\mathbf{C}^p$. Publ. Mat. **38**, 479–499 (1994) Zbl 0832.58031 MR 1316642

[19] Hakim, M.: Analytic transformations of $(\mathbf{C}^p, 0)$ tangent to the identity. Duke Math. J. **92**, 403–428 (1998) Zbl 0952.32012 MR 1612730

[20] Ji, Z.: Non-wandering Fatou components for strongly attracting polynomial skew products. J. Geom. Anal. **30**, 124–152 (2020) Zbl 1461.37050 MR 4058508

[21] Ji, Z.: Non-uniform hyperbolicity in polynomial skew products. Int. Math. Res. Notices **2023**, 8755–8799 Zbl 07690541 MR 4589086

[22] Jonsson, M.: Dynamics of polynomial skew products on $\mathbf{C}^2$. Math. Ann. **314**, 403–447 (1999) Zbl 0940.37018 MR 1704543

[23] López-Hernanz, L., Raissy, J., Ribón, J., Sanz-Sánchez, F.: Stable manifolds of two-dimensional biholomorphisms asymptotic to formal curves. Int. Math. Res. Notices **2021**, 12847–12887 Zbl 1487.32159 MR 4307676

[24] López-Hernanz, L., Ribón, J., Sanz-Sánchez, F., Vivas, L.: Stable manifolds of biholomorphisms in $\mathbb{C}^n$ asymptotic to formal curves. Proc. London Math. Soc. (3) **125**, 277–317 (2022) Zbl 1523.32035 MR 4466236

[25] Lyubich, M., Peters, H.: Classification of invariant Fatou components for dissipative Hénon maps. Geom. Funct. Anal. **24**, 887–915 (2014) Zbl 1301.32013 MR 3213832

[26] McMullen, C. T., Sullivan, D. P.: Quasiconformal homeomorphisms and dynamics. III. The Teichmüller space of a holomorphic dynamical system. Adv. Math. **135**, 351–395 (1998) Zbl 0926.30028 MR 1620850

[27] Peters, H., Smit, I. M.: Fatou components of attracting skew-products. J. Geom. Anal. **28**, 84–110 (2018) Zbl 1397.32003 MR 3745850

[28] Peters, H., Vivas, L. R.: Polynomial skew-products with wandering Fatou-disks. Math. Z. **283**, 349–366 (2016) Zbl 1339.37039 MR 3489070

[29] Pisot, C.: Répartition (mod 1) des puissances successives des nombres réels. Comment. Math. Helv. **19**, 153–160 (1946) Zbl 0063.06259 MR 0017744

[30] Rivi, M.: Local behaviour of discrete dynamical systems. Ph.D. thesis, Università di Firenze (1998)

[31] Rong, F.: The non-dicritical order and attracting domains of holomorphic maps tangent to the identity. Internat. J. Math. **25**, article no. 1450003, 10 pp. (2014) Zbl 1288.32024 MR 3189760

[32] Shcherbakov, A. A.: Topological classification of germs of conformal mappings with an identical linear part. Vestnik Moskov. Univ. Ser. I Mat. Mekh. **1982**, no. 3, 52–57, 111 (1982) (in Russian) Zbl 0508.30015 MR 0671059

[33] Taflin, J.: Blenders near polynomial product maps of $\mathbb{C}^2$. J. Eur. Math. Soc. **23**, 3555–3589 (2021) Zbl 1484.32025 MR 4310812

[34] Vivas, L.: Degenerate characteristic directions for maps tangent to the identity. Indiana Univ. Math. J. **61**, 2019–2040 (2012) Zbl 1280.37041 MR 3129100

[35] Vivas, L.: Local dynamics of parabolic skew-products. Pro Math. **31**, 53–71 (2020) MR 4561233

[36] Voronin, S. M.: Analytic classification of germs of conformal mappings $(\mathbf{C}, 0) \to (\mathbf{C}, 0)$ with identity linear part. Funktsional. Anal. i Prilozhen. **15**, no. 1, 1–17, 96 (1981) (in Russian) Zbl 0463.30010 MR 0609790