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Quantum SL(2) and logarithmic vertex operator algebras at $(p, 1)$-central charge

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Abstract. We provide a ribbon tensor equivalence between the representation category of small quantum SL(2), at parameter $q = e^{\pi i/p}$, and the representation category of the triplet vertex operator algebra at integral parameter $p > 1$. We provide similar quantum group equivalences for representation categories associated to the Virasoro and singlet vertex operator algebras at central charge $c = 1 - 6(p-1)^2/p$. These results resolve a number of fundamental conjectures coming from studies of logarithmic CFTs in type A_1.

Keywords: quantum groups, vertex operator algebras, Kazhdan–Lusztig, tensor categories.

Contents

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1. Introduction

We prove a number of fundamental conjectures which relate quantum group representations for SL(2), and modules for non-rational vertex operator algebras (VOAs) at central charge $c = 1 - 6(1-p)^2/p$. We consider specifically the triplet $\mathcal{W}_p$, singlet $\mathcal{M}_p$, and Virasoro $\mathcal{V}ir_c$ vertex operator algebras.

For the Virasoro VOA, we consider a category of modules which is the logarithmic analog of a rational minimal model. By *logarithmic* we mean, in the simplest sense, that the theories under consideration are non-rational, i.e. non-semisimple. The rational minimal models occur at central charge $c_{p,q} = 1 - 6(p-q)^2/(pq)$, for coprime $p, q > 1$. Their logarithmic analogs, which occur at central charge $c = c_{p,1}$, were first considered in works of Pearce, Rasmussen, and Zuber [72,73], though from a physical perspective.

The triplet and singlet algebras have been studied extensively from both physical and representation-theoretic perspectives (many of these papers are included in our references below), with both algebras appearing first in work of Kausch from the early 90s [56]. Taken together, these three classes of vertex operator algebras, $\mathcal{V}ir_c$, $\mathcal{M}_p$, and $\mathcal{W}_p$, provide the most well-studied non-rational VOAs, or logarithmic chiral CFTs, available to us at the present moment.

In recent works of Creutzig, McRae, and Yang [21,66], and earlier work of Tsuchiya and Wood [77], it was shown that each of the VOAs mentioned above admits a corresponding ribbon tensor category of "affine" representations. We denote these representation categories by

$$\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}, \quad \mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}, \quad \mathrm{rep}(\mathcal{W}_p) \tag{1}$$

respectively. For $\mathrm{rep}(\mathcal{W}_p)$, we simply consider finite length $\mathcal{W}_p$-modules. The constructions of the categories $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ and $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$ are slightly more involved, and are recalled in Sections 10.2 and 11.1 below. Let us say here that the simple objects in $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ and $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$ are those simple modules of integral lowest (conformal) weight $h_{n,s} = \frac{1}{4p}((np-s)^2 - (p-1)^2)$, $n, s \in \mathbb{Z}$. An important point is that the categories (1) are all *affine*, in the sense that they each admit a distinguished tensor generator, or distinguished faithful representation if one prefers (cf. [29, Section II.5, Corollaire 5.2]).

We compare the tensor categories of (1) to categories of representations for quantum SL(2) at parameter $q = \exp(\pi i/p)$. We consider specifically the categories

$$\mathrm{rep}\,\mathrm{SL}(2)_q, \quad \mathrm{rep}(\dot{\mathbf{u}}_q(\mathfrak{sl}_2)), \quad \mathrm{rep}(u_q(\mathfrak{sl}_2))$$

of character graded representations of Lusztig's divided power algebra $U_q^{\mathrm{Lus}}(\mathfrak{sl}_2)$, character graded representations of small quantum SL(2), and usual representations of small quantum SL(2) respectively (see Sections 3.1 and 12.1). We establish the following collection of equivalences, which were conjectured across the works [12,14,16,24,44].

Theorem (9.5/10.1/12.1). *There are equivalences of ribbon tensor categories*

$$\begin{cases} \mathrm{K} : \operatorname{rep} \mathrm{SL}(2)_q \xrightarrow{\sim} \operatorname{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}, \\ \Psi : \operatorname{rep}(\dot{\mathbf{u}}_q(\mathfrak{sl}_2)) \xrightarrow{\sim} \operatorname{rep}(\mathcal{M}_p)_{\mathrm{aff}}, \\ \Theta : \operatorname{rep}(u_q(\mathfrak{sl}_2)) \xrightarrow{\sim} \operatorname{rep}(\mathcal{W}_p). \end{cases}$$

The particular ribbon structures employed on the quantum group sides of Theorems 9.5, 10.1, and 12.1 are those "inverse" to the standard choices of [52,64] (see Section 3.1). We note that in the process of proving the above result we establish modularity of the category $\operatorname{rep}(\mathcal{W}_p)$ of triplet modules (see Theorem 4.7). This point may be of independent interest.

For a clearer historical account, the equivalences K and Θ were originally conjectured to exist in works of Bushlanov, Feigin, Gainutdinov, and Tipunin [12], and Gainutdinov, Semikhatov, Tipunin, and Feigin [44] respectively. The basis for these conjectures was that a number of invariants for quantum groups and their corresponding vertex operator algebras were (essentially) observed to agree. Indeed, it was argued in [36] that the modular group representations for $u_q(\mathfrak{sl}_2)$ and $\mathcal{W}_p$ agree, and also that their fusion rings agree [41]. Furthermore, it was shown in work of Nagatomo and Tsuchiya [68], and subsequently McRae and Yang [66], that there is an *abelian*, non-tensor, equivalence between $\operatorname{rep}(u_q(\mathfrak{sl}_2))$ and $\operatorname{rep}(\mathcal{W}_p)$. At the particular parameter $p = 2$, Creutzig, Lentner, and Rupert verified that this equivalence can in fact be enhanced with the desired tensor structure [20]. The possibility of the equivalence Ψ was alluded to in the works of Creutzig and Milas, and Costantino, Geer, and Patureau-Mirand [14, 24], then was conjectured explicitly in work of Creutzig, Gainutdinov, and Runkel [16].[1]

As one might expect, analogs of Theorems 9.5, 10.1, and 12.1 have been conjectured in arbitrary Dynkin type [37, 59, 76]. The analog of the equivalence Θ at a given almost-simple algebraic group G, for example, proposes an equivalence of modular tensor categories between representations of the small quantum group for G at $q = e^{\pi i/p}$, and modules for the "logarithmic W-algebra" $\mathcal{W}_p(G)$ of [4,76]. Here, for the small quantum group, one should specifically take the cocycle corrected variant of [42,69].

While additional infrastructure is needed to address these conjectures outside of type A_1 – in particular the VOAs $\mathcal{W}_p(G)$ need to be studied further – the results herein give credence to the claim that representations of quantum groups and CFTs should be strongly intertwined, even in the logarithmic context. If we consider type $A_{>1}$ for example, and suppose a sufficiently strong understanding of the algebras $\mathcal{W}_p(\mathrm{SL}(n))$, one could presumably employ the Hecke presentation for $\operatorname{rep} \mathrm{SL}(n)_q$ of [27, Proposition 4.7], and follow the arguments of the present text, using [69, Proposition 7.3] and [15, Proposition 7.4.2], to provide the desired equivalences between $\operatorname{rep}(u_q(\mathfrak{sl}_n))$ and $\operatorname{rep}(\mathcal{W}_p(\mathrm{SL}(n)))$ at arbitrary n.

[1] To be clear, our equivalence Ψ concerns the "integral part" of [16, Conjecture 1.4]. See Remark 12.2 for details.

1.1. Methods

Let us focus on the equivalence $\Theta : \mathrm{rep}(u_q(\mathfrak{sl}_2)) \xrightarrow{\sim} \mathrm{rep}(\mathcal{W}_p)$ of Theorem 9.5, which is the primary target of this work. We first note that our small quantum group $u_q(\mathfrak{sl}_2)$ is the cocycle corrected variant of [16, 43], which we accept at this point as the "correct" version of the small quantum group at an even order root of unity. A point which is essential to this work is the observation [69] that the category of representations $\mathrm{rep}(u_q(\mathfrak{sl}_2))$ can be understood as the de-equivariantization of $\mathrm{rep}\,\mathrm{SL}(2)_q$ along the embedding $\mathrm{Fr} : \mathrm{rep}(\mathrm{PSL}(2)) \to \mathrm{rep}\,\mathrm{SL}(2)_q$ provided by Lusztig's quantum Frobenius functor [64]. Of course, this is a rather technical statement, but the point is the following (Proposition 9.2): One can identify tensor functors $\mathrm{rep}(u_q(\mathfrak{sl}_2)) \to \mathscr{A}$ to a given tensor category $\mathscr{A}$ with a particular class of tensor functors $\mathrm{rep}\,\mathrm{SL}(2)_q \to \mathscr{A}$ out of big quantum SL(2).

With this general framework in mind, we observe furthermore that tensor maps out of $\mathrm{rep}\,\mathrm{SL}(2)_q$ are classified in work of Ostrik [70]. (See Theorems 5.3 and 9.4.) In particular, a tensor map $\mathrm{rep}\,\mathrm{SL}(2)_q \to \mathscr{A}$ to a some category $\mathscr{A}$ is specified by a choice of self-dual object W in the target $\mathscr{A}$, which satisfies certain non-degeneracy properties. So we approach the equivalence Θ by leveraging the works [70] and [69] in tandem.

Of course, in order to produce the equivalence Θ in the suggested manner, we must have a clear understanding of the category of modules $\mathrm{rep}(\mathcal{W}_p)$, and in particular of its self-dual tensor generator X_2^+. One obtains such a concrete understanding of $\mathrm{rep}(\mathcal{W}_p)$ by exploiting relationships between matrix entries of compositions of intertwining operators and differential equations on the sphere. Such relationships go back to the beginnings of CFT, and are also central to the philosophies of, say, Huang [45] and Tsuchiya–Wood [77]. (See Section 8.) This approach is also present in the recent works [21,66]. The equivalence Θ is therefore deduced via a propitious interplay between category-theoretic and analytic techniques.

The equivalence K for the Virasoro is essentially a corollary of our arguments for the triplet, which we employ in conjunction with works of Creutzig, Kanade, McRae, and Yang [19, 66]. The equivalence Ψ is deduced from a non-trivial analysis of the representation category $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$, and an additional analysis of certain rational actions of the torus $\mathbb{C}^\times$ on the categories $\mathrm{rep}(u_q(\mathfrak{sl}_2))$ and $\mathrm{rep}(\mathcal{W}_p)$.

Remark 1.1. In work of Creutzig, Lentner, and Rupert [20], the authors suggest an alternative construction of the equivalence Θ for the triplet, and they realize their construction explicitly when $p = 2$. The methods employed in [20] differ significantly from the ones employed here, and we invite the curious reader to consult the aforementioned text.

Remark 1.2. Our results for the triplet VOA rely on explicit understandings of the indecomposable projectives and fusion rules for $\mathrm{rep}(\mathcal{W}_p)$. Such structural results first appeared in works of Nagatomo, Tsuchiya, and Wood [68, 77], though by most accounts some details are missing from [68]. With this point in mind, we note that independent (re)constructions of the indecomposable projectives and fusion rules for $\mathrm{rep}(\mathcal{W}_p)$ can be found in recent work of McRae and Yang. See in particular [66, Section 7.1].

1.2. Outline

Sections 2–4 cover background material. Section 5 recalls Ostrik's classification of tensor functors out of $\mathrm{rep}\,\mathrm{SL}(2)_q$, and also recalls basic facts about the Temperley–Lieb category. In Section 6 we perform a straightforward calculation of all braidings for the Temperley–Lieb category. Sections 7 and 8 are dedicated to an analysis of the self-dual generator X_2^+ of $\mathrm{rep}(\mathcal{W}_p)$. In Sections 9–12 we establish the equivalences Θ, K, and Ψ for the triplet, Virasoro, and singlet vertex operator algebras, respectively. In the appendices we cover some technical information regarding induction for VOA extensions, and the calculus of (de-)equivariantization for tensor categories equipped with algebraic group actions.

2. (Finite) tensor categories

We cover some basic information about finite tensor categories. Our presentation is based on the texts [10,34], as well as the paper [35]. We work over the base field $k = \mathbb{C}$.

2.1. (Finite) tensor categories and fusion categories

A *tensor category* (over $\mathbb{C}$) is a $\mathbb{C}$-linear, Hom-finite, abelian monoidal category $\mathscr{C}$ which is rigid, has all objects of finite length, and has a simple unit object $\mathbf{1}$. Rigidity means that all objects X in $\mathscr{C}$ have left and right duals X^* and ${}^*X$ [34, Section 2.10]. A *tensor functor* between tensor categories is, by definition, an *exact*, $\mathbb{C}$-linear, monoidal functor. By a natural isomorphism between tensor functors we mean a natural isomorphism which respects the monoidal structures in the expected ways. The following basic observation will be used throughout the text.

Proposition 2.1 ([28, Proposition 1.19]). *Any tensor functor between tensor categories is faithful.*

A tensor category is called *finite* if it has finitely many simple objects, up to isomorphism, and enough projectives. A tensor category is called a *fusion* category if it is finite and semisimple.

Abstractly, any finite tensor category $\mathscr{C}$ admits an abelian equivalence $\mathscr{C} \cong \mathrm{rep}(B)$ to the representation category of a finite-dimensional algebra. (One can take specifically B to be the endomorphism ring of a projective generator.) For some examples, one can consider the category $\mathscr{C} = \mathrm{rep}(\mathcal{W}_p)$ of finite length modules over the triplet vertex operator algebra. One sees from results of [3,13,46,77] that $\mathrm{rep}(\mathcal{W}_p)$ admits a natural finite tensor category structure. Also, for any finite-dimensional quasi-Hopf algebra u [34, Section 5.13], the category $\mathrm{rep}(u)$ of finite-dimensional u-representations has the natural structure of a finite tensor category, with the product $\otimes$ on $\mathrm{rep}(u)$ induced by the coproduct on u.

2.2. *Frobenius–Perron dimension*

For $\mathscr{C}$ a tensor category, the Grothendieck ring $K(\mathscr{C})$ is a $\mathbb{Z}_+$-ring [34, Definition 3.1.1], in the sense that it is a free $\mathbb{Z}$-algebra with specified basis $\{x_i\}_i \subset K(\mathscr{C})$ and non-negative structure coefficients $c_{i,j}^k$, $x_i \cdot x_j = \sum_k c_{i,j}^k x_k$. The basis $\{x_i\}_i$ is provided by the isoclasses of the simples, $\{[X_i] : X_i \text{ simple in } \mathscr{C}\}$, and the unit $[\mathbf{1}]$ in $K(\mathscr{C})$ is provided by the unit object in $\mathscr{C}$. When $\mathscr{C}$ is a finite tensor category, the Grothendieck ring $K(\mathscr{C})$ is of finite rank over $\mathbb{Z}$.

For any finite rank $\mathbb{Z}_+$-ring A one has a canonically associated dimension function $\mathrm{FPdim} : A \to \mathbb{R}$ called the Frobenius–Perron dimension. For any x in the specified basis for A, the *Frobenius–Perron dimension* $\mathrm{FPdim}(x)$ is defined as the maximal non-negative real eigenvalue of the linear map $x \cdot - : \mathbb{R} \otimes_{\mathbb{Z}} A \to \mathbb{R} \otimes_{\mathbb{Z}} A$. Since multiplication by x is represented by a matrix with non-negative entries, the Frobenius–Perron theorem ensures the existence of such an eigenvalue. Furthermore, $\mathrm{FPdim}(x) \geq 1$ for any such x [34, Proposition 3.3.4].

The function FPdim is a ring homomorphism, and it is in fact the unique character of A which takes positive values on the given basis [34, Proposition 3.3.6]. We apply the above general construction to deduce a dimension function $\mathrm{FPdim} : K(\mathscr{C}) \to \mathbb{R}$ for the Grothendieck ring of any finite tensor category $\mathscr{C}$. The uniqueness properties of the Frobenius–Perron dimension imply the following (standard) result.

Lemma 2.2. *If B is a finite-dimensional quasi-Hopf algebra, then for any V in* $\mathrm{rep}(B)$, $\mathrm{FPdim}(V) = \dim_{\mathbb{C}}(V)$.

Proof. The vector space dimension defines an algebra map $\dim_{\mathbb{C}} : K(\mathrm{rep}(B)) \to \mathbb{R}$ which takes positive values on each class $[V]$ of a non-zero representation V. Since FPdim is the unique character of $K(\mathrm{rep}(B))$ with this property, it follows that $\mathrm{FPdim} = \dim_{\mathbb{C}}$. ∎

We also have a general notion of Frobenius–Perron dimension for $\mathbb{Z}_+$-rings *themselves* [34, Definition 3.3.12], which reduces to the following in our setting.

Definition 2.3 ([34, Definition 6.1.7]). For any finite tensor category $\mathscr{C}$, the *Frobenius–Perron dimension of* $\mathscr{C}$ is defined as

$$\mathrm{FPdim}(\mathscr{C}) := \sum_i \mathrm{FPdim}(P_i)\,\mathrm{FPdim}(X_i),$$

where the sum runs over the isoclasses of simples X_i, and each P_i is the projective cover of X_i.

2.3. *Surjective tensor functors*

A tensor functor $F : \mathscr{C} \to \mathscr{D}$ is called *surjective* if any object in $\mathscr{D}$ is a subquotient of $F(X)$ for some X in $\mathscr{C}$. We have the following two essential results.

Theorem 2.4 ([35, Theorem 2.5]). *If $F : \mathscr{C} \to \mathscr{D}$ is a surjective tensor functor between finite tensor categories, then the image $F(P)$ of any projective object P in $\mathscr{C}$ is projective in $\mathscr{D}$.*

Theorem 2.5 ([35, Proposition 2.20]). *If $F : \mathscr{C} \to \mathscr{D}$ is a surjective tensor functor between finite tensor categories, then* $\mathrm{FPdim}(\mathscr{D}) \leq \mathrm{FPdim}(\mathscr{C})$*, and if equality holds, then any surjective tensor functor $F : \mathscr{C} \to \mathscr{D}$ is an equivalence.*

2.4. Tensor generators

For a collection $\{Y_j\}_j$ of objects in a tensor category $\mathscr{C}$, let $\langle Y_j \rangle_j$ denote the smallest (full) tensor subcategory in $\mathscr{C}$ which contains the Y_j and is closed under taking subquotients. We call this subcategory the tensor subcategory in $\mathscr{C}$ *generated* by the Y_j. We say $\mathscr{C}$ is *tensor generated* by the collection $\{Y_j\}_j$ if $\mathscr{C} = \langle Y_j \rangle_j$.

2.5. Braided tensor categories, ribbon tensor categories, etc.

A *braiding* on a tensor category $\mathscr{C}$ is a chosen collection of natural isomorphisms $c_{X,Y} : X \otimes Y \to Y \otimes X$, for each X and Y in $\mathscr{C}$, which satisfy the equations

$$c_{X,Y\otimes Z} = (\mathrm{id} \otimes c_{X,Z})(c_{X,Y} \otimes \mathrm{id}) \quad \text{and} \quad c_{X\otimes Y,Z} = (c_{X,Z} \otimes \mathrm{id})(\mathrm{id} \otimes c_{Y,Z}) \tag{2}$$

for each triple of objects in $\mathscr{C}$, and $c_{X',Y'}(f \otimes g) = (g \otimes f)c_{X,Y}$ for each pair of morphisms $f : X \to X'$ and $g : Y \to Y'$ in $\mathscr{C}$. We also require that the braidings $c_{\mathbf{1},X}$ and $c_{X,\mathbf{1}}$ composed with the unit isomorphisms are the identity. A tensor category equipped with a particular choice of braiding is called a *braided tensor category*.

Remark 2.6. We have suppressed the associator in the equations (2).

For a braiding c on a tensor category $\mathscr{C}$, we let c^2 denote the square operation $c^2_{X,Y} := c_{Y,X}c_{X,Y}$. The *Müger center* $Z_{\mathrm{Müg}}(\mathscr{C})$ of a braided tensor category $\mathscr{C}$ is the full subcategory consisting of all objects X in $\mathscr{C}$ for which $c^2_{X,-} = \mathrm{id}_{X\otimes -}$. We call a finite braided tensor category *non-degenerate* if the Müger center $Z_{\mathrm{Müg}}(\mathscr{C})$ is just *Vect*, i.e. if any Müger central object is isomorphic to some additive power of the unit.

A *twist* for a braided tensor category $\mathscr{C}$ is a choice of natural automorphism θ of the identity functor, i.e. a collection of natural isomorphisms $\theta_X : X \to X$ for each X such that

$$\theta_{X\otimes Y} = (\theta_X \otimes \theta_Y)c^2_{X,Y}$$

for all X and Y in $\mathscr{C}$. A *ribbon tensor category* is a braided tensor category with a choice of twist θ which is stable under duality, in the sense that $\theta^*_X = \theta_{X^*}$ for all X in $\mathscr{C}$. A *modular tensor category* is a finite, non-degenerate, ribbon tensor category. Note that, unlike some authors, we do not require a modular tensor category to be semisimple. When it is, we call it a *modular fusion category*.

2.6. Tensor categories coming from vertex operator algebras

Let $\mathcal{V}$ be a vertex operator algebra (VOA). VOAs and their modules carry an action of the Virasoro algebra, coming from the conformal vector $\omega \in \mathcal{V}$. By a *generalized $\mathcal{V}$-module* W is meant a module in the obvious algebraic sense, together with a grading $W = \coprod_{r\in\mathbb{C}} W_{(r)}$ into generalized eigenspaces of the Virasoro operator $L(0)$. The presence of the adjective 'generalized' here is unfortunate and historical, and merely refers to the subspaces $W_{(r)}$ being generalized eigenspaces. We are primarily interested in *grading-restricted* W, which means each subspace $W_{(r)}$ is finite-dimensional, and for each $r \in \mathbb{C}$ we have $W_{(r+k)} = 0$ for all sufficiently small $k \in \mathbb{Z}$. For example, $\mathcal{V}$ is a grading-restricted generalized module over itself, where all $V_{(r)}$ are actually eigenspaces for $L(0)$, and where all weights r are integers. A common requirement on generalized modules is C_1-cofiniteness (defined e.g. in [46]).

In the series of papers [49–51], Huang, Lepowsky and Zhang give technical conditions under which a full subcategory of the category of grading-restricted generalized $\mathcal{V}$-modules can be a braided tensor category. In particular, see [51, Theorem 12.15, Corollary 12.16]. In these cases, the VOA $\mathcal{V}$ itself serves as the unit, and all structure maps are deduced via a certain analysis of multivalued functions on the punctured complex plane. We should be clear that, when we speak of a class $\mathcal{C}$ of $\mathcal{V}$-modules admitting *a* braided tensor structure, the braided tensor structure is specified uniquely. In general, the problem is that, for a given class of $\mathcal{V}$-modules, no such structure may exist.

Establishing rigidity is more subtle. The dual of a grading-restricted generalized module W should be the contragredient W^*, which is the natural $\mathcal{V}$-module structure on the restricted dual $\coprod_r W_{(r)}^*$. In a natural sense, $(W^*)^*$ can be identified with W – this is clearly true as a vector space (since the $W_{(r)}$ are finite-dimensional), but as well the formula for the vertex operator of $(W^*)^*$ collapses to that of W. In establishing rigidity, the (co-)evaluation maps are generally clear up to scaling; the challenge is to verify the rescalings are finite.

Such a tensor category of VOA modules comes equipped with a twist provided by the exponential $\theta = e^{2\pi i L(0)}$, which one verifies as in [45, Theorem 4.1]. (A concise recounting of the situation can be found in [19, Section 3].)

We follow the standard VOA practice of distinguishing between modules and representations. The notion of a representation of a VOA $\mathcal{V}$ (i.e. a homomorphism from $\mathcal{V}$ to some sort of VOA canonically associated to the vector space W), and its relation to $\mathcal{V}$-modules, is much more subtle than it is for say associative algebras. This is discussed in more detail in the book [60], which also serves as a standard introduction to VOA theory.

3. Quantum SL(2)

We recall basic information for the category of quantum SL(2)-representations, and give two interpretations of the small quantum group for SL(2) at a root of unity $\zeta \in \mathbb{C}^\times$. Much of our presentation is general, in the sense that the quantum parameter ζ can be an arbitrary

(non-zero) complex number. However, when specificity is needed, we focus on the even order case $\mathrm{ord}(\zeta) = 2p$.

3.1. Big quantum SL(2)

Consider an arbitrary parameter $\zeta \in \mathbb{C}^\times$, and let Λ denote the character lattice for SL(2), so that $\Lambda = \frac{1}{2}\mathbb{Z}\alpha$ with α simple and positive. Let $(-,-)$ denote the normalized Killing form on Λ, so that $(\alpha, \alpha) = 2$. We consider the tensor category

$$\mathrm{rep}\,\mathrm{SL}(2)_\zeta = \left\{ \begin{array}{c} \text{the category of } \Lambda\text{-graded representations} \\ \text{of Lusztig's divided power algebra } U_\zeta^{\mathrm{Lus}}(\mathfrak{sl}_2) \end{array} \right\} = \mathrm{rep}(\dot{U}_\zeta(\mathfrak{sl}_2)).$$

The final algebra $\dot{U}_\zeta(\mathfrak{sl}_2)$ is the modified quantum enveloping algebra of [64, Chapter 31], and for the Λ-grading we require that toral elements in $U_\zeta^{\mathrm{Lus}}(\mathfrak{sl}_2)$ act on the λ-space V_λ via the appropriate eigenfunctions. We refer to objects in $\mathrm{rep}\,\mathrm{SL}(2)_\zeta$ as $\mathrm{SL}(2)_\zeta$*-representations*.

For a finite order parameter, $\mathrm{ord}(\zeta^2) = p$ for example, we require that the toral elements in the quantum group act as

$$K \cdot v = \zeta^{(\alpha,\lambda)} v \quad \text{and} \quad \begin{bmatrix} K_\alpha; 0 \\ p \end{bmatrix} \cdot v = \begin{bmatrix} (\alpha,\lambda) \\ p \end{bmatrix} v, \quad \text{where } \begin{bmatrix} a \\ b \end{bmatrix} \text{ is the } \zeta\text{-binomial.}$$

At such ζ the category $\mathrm{rep}\,\mathrm{SL}(2)_\zeta$ can then be described explicitly as the category of finite-dimensional Λ-graded vector spaces V equipped with linear operators $E, F, E^{(p)}, F^{(p)} : V \to V$ which shift the grading as

$$E \cdot V_\lambda \subset V_{\lambda+\alpha}, \quad E^{(p)} \cdot V_\lambda \subset V_{\lambda+p\alpha}, \quad F \cdot V_\lambda \subset V_{\lambda-\alpha}, \quad F^{(p)} \cdot V_\lambda \subset V_{\lambda-p\alpha}$$

and satisfy the standard quantum group relations of [63,64]. At an infinite order parameter one can deduce a similar (but slightly easier) presentation of $\mathrm{rep}\,\mathrm{SL}(2)_\zeta$.

Let $\Omega : V \otimes W \to V \otimes W$ denote the diagonal endomorphism associated to the normalized Killing form,

$$\Omega(v \otimes w) = \zeta^{(\deg(v),\deg(w))} v \otimes w, \quad (\alpha, \alpha) = 2.$$

When ζ is of finite order, $\mathrm{ord}(\zeta^2) = p$, we have the formal element

$$\begin{aligned} R &= \left(\sum_{n=0}^{p-1} \zeta^{-n(n-1)/2} \frac{(\zeta^{-1} - \zeta)^n}{[n]!} E^n \otimes F^n \right) \Omega^{-1} \\ &= (1 - (\zeta - \zeta^{-1}) E \otimes F + \cdots) \Omega^{-1}, \end{aligned} \tag{3}$$

which acts as a well-defined linear endomorphism on products $V \otimes W$ of quantum group representations. Similarly, at an infinite order parameter we have the element R defined by replacing the finite sum (3) with the evident power series. The element R provides an

R-matrix for the category rep $\mathrm{SL}(2)_\zeta$, so that we have the associated braiding

$$c_{V,W} : V \otimes W \to W \otimes V,$$

$$c_{V,W}(v \otimes w) := R_{21} \cdot (w \otimes v) = \zeta^{-(\deg v, \deg w)} \sum_n \zeta^{-n(n-1)/2} \frac{(\zeta^{-1} - \zeta)^n}{[n]!} F^n w \otimes E^n v,$$

(see [52, 64]).

Remark 3.1. To be clear, the normalized Killing form takes half-integer values on the character lattice Λ, so that the definition of $\Omega = \sum_{\lambda,\mu \in \Lambda} \zeta^{(\lambda,\mu)} 1_\lambda \otimes 1_\mu$ involves a choice of square root for ζ. We simply halve the argument and take the positive square root of the magnitude to define $\zeta^{1/2} := \sqrt{|\zeta|} \exp(i \arg/2)$, where $0 \leq \arg < 2\pi$ is such that $\zeta = |\zeta| e^{i \arg}$. There is, however, another R-matrix for rep $\mathrm{SL}(2)_\zeta$ defined by taking the negative square root of $|\zeta|$ and, accounting for reverse braidings, we observe a total of *four* possible braidings on the category of $\mathrm{SL}(2)_\zeta$-representations.

We refer to the above braiding on rep $\mathrm{SL}(2)_\zeta$, defined by taking the positive square root of the magnitude $|\zeta|$ and corresponding R-matrix (3), as the *standard braiding* on rep $\mathrm{SL}(2)_\zeta$.

Suppose now, for the sake of specificity, that ζ is of even order $\mathrm{ord}(\zeta) = 2p$. The category rep $\mathrm{SL}(2)_\zeta$ is pivotal [34, Definition 4.7.7], with slightly unorthodox pivotal element provided by the grouplike K^{p-1}. We consider the Drinfeld morphism

$$u_W : W \to W, \quad u_W(w) = \zeta^{(\deg w, \deg w)} \sum_{n=0}^{p-1} \zeta^{n(n+1)/2} \zeta^{n(\deg w, \alpha)} \frac{(\zeta^2 - 1)^n}{[n]!} F^n E^n w,$$

defined by R to find the corresponding twist

$$\theta_W^{-1} = u_W K^{-p+1} : W \to W,$$

$$\theta_W^{-1}(w) = (-1)^{(\deg w, \alpha)} \zeta^{(\deg w, \deg w)} \sum_{n=0}^{p-1} \zeta^{n(n+1)/2} \zeta^{(n+1)(\deg w, \alpha)} \frac{(\zeta^2 - 1)^n}{[n]!} F^n E^n w.$$

This twist provides the category rep $\mathrm{SL}(2)_\zeta$ of quantum group representations with a ribbon structure.

3.2. *Simple objects in* rep $\mathrm{SL}(2)_\zeta$ *and the standard representation*

For any dominant weight $\lambda \in \Lambda$, i.e. any weight of the form $\lambda = n\alpha/2$ with n positive, we have a uniquely associated simple module $L(\lambda)$ in rep $\mathrm{SL}(2)_\zeta$ of highest weight λ. Furthermore, these representations exhaust all of the simple objects in rep $\mathrm{SL}(2)_\zeta$ [62, Proposition 6.4]. Each $L(\lambda)$ appears as

$$L(\lambda) = \mathbb{C} v_\lambda \oplus \mathbb{C} v_{\lambda - m_1 \alpha} \oplus \cdots \oplus \mathbb{C} v_{\lambda - m_{n(\lambda)} \alpha},$$

where the m_i are positive integers which depend on λ. At $\lambda = \alpha/2$ we obtain a 2-dimensional simple representation

$$\mathbb{V} := L\left(\tfrac{1}{2}\alpha\right) = \mathbb{C}v_1 \oplus \mathbb{C}v_{-1} \quad \text{with} \quad v_{\pm 1} = v_{\pm\alpha/2}.$$

We call $\mathbb{V}$ the *standard representation* for $\mathrm{SL}(2)_\zeta$.

When ζ is of infinite order, each simple $L(n\alpha/2)$ is of dimension $n+1$ and the category $\mathrm{rep}\,\mathrm{SL}(2)_\zeta$ is semisimple. Indeed, at such ζ we have the obvious abelian equivalence between classical representations $\mathrm{rep}\,\mathrm{SL}(2)$ and $\mathrm{rep}\,\mathrm{SL}(2)_\zeta$ which induces an isomorphism on Grothendieck rings.

Consider now ζ of finite order, and take $p = \mathrm{ord}(\zeta^2)$. We adopt special notations for the first p simples:

$$V_1 = \mathbf{1} = L(0),\ V_2 = \mathbb{V} = L\left(\tfrac{1}{2}\alpha\right),\ V_3 = L(\alpha),\ \ldots,\ V_p = L\left(\frac{p-1}{2}\alpha\right). \tag{4}$$

Each simple V_s is of dimension s, and has non-vanishing weight spaces in degrees $(s-1-2j)\alpha/2$, for all $j = 0, \ldots, s-1$.

By considering the behaviors of weight spaces under the tensor product, it is easy to see that each simple $L(\lambda)$ is a quotient of some power $\mathbb{V}^{\otimes n}$ of the standard representation. The following is well-known, and is a consequence of the fact that the simple representation V_p is also projective [12, Lemma 3.2.1].

Lemma 3.2. *The standard representation* $\mathbb{V}$ *generates* $\mathrm{rep}\,\mathrm{SL}(2)_\zeta$ *as a tensor category.*

As for the simples of highest weight $> \frac{p-1}{2}\alpha$, each simple $L(r\frac{p}{2}\alpha)$ is $(r+1)$-dimensional, is supported in weight spaces $(r-2j)\frac{p}{2}\alpha$ for $0 \le j \le r$, and is annihilated by E and F. We have the following basic result.

Lemma 3.3 ([6, Theorem 1.10]). *Each simple* $L(\lambda)$ *admits a unique decomposition* $L(\lambda) = L(\mu) \otimes V_s$, *where* $\mu \in p\mathbb{Z}\frac{\alpha}{2}$ *and* $\lambda = \mu + \frac{s-1}{2}\alpha$.

Proof. The general version of this theorem is covered in [6]. In our case, one can examine the product $L(\mu) \otimes V_s$ directly to see that it admits no submodules, and so is simple. Since $L(\mu) \otimes V_s$ is of highest weight $\lambda = \mu + \frac{s-1}{2}\alpha$, we conclude $L(\mu) \otimes V_s = L(\lambda)$. ∎

3.3. Lusztig's quantum Frobenius

For the remainder of the section we suppose ζ *is of even order* $\mathrm{ord}(\zeta) = 2p$, again for the sake of specificity.

Let us consider the sublattice $\Lambda^M := p\mathbb{Z}\alpha$ in Λ. The collection of objects in $\mathrm{rep}\,\mathrm{SL}(2)_\zeta$ whose Λ-grading is supported on the sublattice Λ^M forms a tensor subcategory in $\mathrm{rep}\,\mathrm{SL}(2)_\zeta$. This subcategory is semisimple and is equivalent to the category of representations of classical PSL(2). Indeed, Lusztig's quantum Frobenius provides a braided tensor embedding

$$\mathrm{Fr} : \mathrm{rep}(\mathrm{PSL}(2)) \to \mathrm{rep}\,\mathrm{SL}(2)_\zeta$$

(see [64, Chapter 35]). Furthermore, the image of the functor Fr is precisely the Müger center of $\operatorname{rep}\mathrm{SL}(2)_\zeta$ [69, Theorem 5.3], so that we have an identification

$$Z_{\mathrm{Müg}}(\operatorname{rep}\mathrm{SL}(2)_\zeta) = \operatorname{rep}(\mathrm{PSL}(2)).$$

3.4. Small quantum SL(2) *as a quasi-Hopf algebra* [16]

Recall our standing assumption $\operatorname{ord}(\zeta) = 2p$. As an algebra, the small quantum group $u_\zeta(\mathfrak{sl}_2) := u_\zeta(\mathrm{SL}(2))$ has a presentation

$$u_q(\mathfrak{sl}_2)$$
$$= \mathbb{C}\langle E, F, K\rangle / \left(K^{2p} - 1, E^p, F^p, [E,F] - \frac{K - K^{-1}}{\zeta - \zeta^{-1}},\ KE - \zeta^2 E,\ KF - \zeta^{-2} FK \right).$$

We let $G = \langle K \rangle$ denote the subgroup of units in $u_\zeta(\mathfrak{sl}_2)$ generated by K.

The quasi-Hopf structure on $u_\zeta(\mathfrak{sl}_2)$ is a "toral perturbation" of the usual Hopf structure induced by the algebra inclusion $u_\zeta(\mathfrak{sl}_2) \to U_\zeta^{\mathrm{Lus}}(\mathfrak{sl}_2)$ into Lusztig's divided power algebra [63, 64]. While we will not explicitly describe the quasi-Hopf structure here, let us say that the (co)associator $\phi \in u_\zeta(\mathfrak{sl}_2)^{\otimes 3}$ lies in the group of grouplikes $(\mathbb{C}G)^{\otimes 3}$, and the coproduct is of the form

$$\Delta(E) = E \otimes L + M(K \otimes E), \quad \Delta(F) = F \otimes L^{-1} + M'(K^{-1} \otimes F), \quad \Delta(K) = K \otimes K$$

with $L \in \mathbb{C}G$ and $M, M' \in \mathbb{C}G \otimes \mathbb{C}G$. The counit $u_\zeta(\mathfrak{sl}_2) \to \mathbb{C}$ is the usual one. (More details can be found in [16, 42, 69].)

The category of $u_\zeta(\mathfrak{sl}_2)$-representations admits a ribbon structure for which the restriction functor

$$\mathrm{res}^\omega = (\mathrm{res}, T^\omega) : \operatorname{rep}\mathrm{SL}(2)_\zeta \to \operatorname{rep}(u_\zeta(\mathfrak{sl}_2))$$

becomes a map of ribbon tensor categories, after we introduce a non-trivial tensor compatibility $T^\omega_{V,W} : \mathrm{res}(V) \otimes \mathrm{res}(W) \to \mathrm{res}(V \otimes W)$ [69, Proposition 6.3].

The particular form of the quasi-Hopf structure on $u_\zeta(\mathfrak{sl}_2)$ will not be important for us. Indeed, when addressing the tensor category $\operatorname{rep}(u_\zeta(\mathfrak{sl}_2))$ we prefer the presentation of Section 3.5 below. There are, however, some advantages to considering the quasi-Hopf interpretation of representations for the small quantum group. Namely, simple facts about the representation theory of $u_\zeta(\mathfrak{sl}_2)$ are clear from this concrete perspective.

We observe for $u_\zeta(\mathfrak{sl}_2)$ the anomalous 1-dimensional representation

$$\chi = \mathbb{C}v_p, \quad E \cdot v_p = F \cdot v_p = 0, \quad K \cdot v_p = -v_p.$$

Since χ is 1-dimensional, one sees abstractly that it must be invertible in the tensor category $\operatorname{rep}(u_\zeta(\mathfrak{sl}_2))$, and since χ is in fact the *unique* non-trivial 1-dimensional representation for $u_\zeta(\mathfrak{sl}_2)$, we see that it is self-dual. So $\chi \otimes \chi \cong \mathbf{1}$.

There are furthermore a total of $2p$ simple $u_\zeta(\mathfrak{sl}_2)$-representations which correspond to the $2p$ characters of the group of grouplikes G. The first p simples are provided by

the images of the simples V_s in $\mathrm{rep}\,\mathrm{SL}(2)_\xi$ under restriction. The final p simples are given as the products $\chi \otimes V_s$, so that we have a complete list of simple representations, $\{V_s,\ \chi \otimes V_s : 1 \leq s \leq p\}$, up to isomorphism.

The fact that the restriction functor $\mathrm{rep}\,\mathrm{SL}(2)_\xi \to \mathrm{rep}(u_\xi(\mathfrak{sl}_2))$ is a tensor functor, or rather admits the structure of a tensor functor, tells us that the fusion rules for $\mathrm{rep}(u_\xi(\mathfrak{sl}_2))$ are the expected ones. To be more precise, the fusion rules are those one would calculate from the naïve Hopf structure on $u_\xi(\mathfrak{sl}_2)$ induced by that of $U_\xi^{\mathrm{Lus}}(\mathfrak{sl}_2)$.

3.5. Small quantum SL(2) *as Frobenius de-equivariantizaton* [9, 69]

We have the quantum Frobenius $\mathrm{Fr} : \mathrm{rep}(\mathrm{PSL}(2)) \to \mathrm{rep}\,\mathrm{SL}(2)_\xi$ of Section 3.3, and can consider the central algebra object $\mathscr{O} := \mathrm{Fr}\,\mathscr{O}(\mathrm{PSL}(2))$ in the category $\mathrm{Rep}\,\mathrm{SL}(2)_\xi$ of infinite-dimensional quantum group representations. (Specifically, $\mathrm{Rep}\,\mathrm{SL}(2)_\xi$ is the category of arbitrary $U_\xi^{\mathrm{Lus}}(\mathfrak{sl}_2)$-modules M which are the unions $\bigcup_\mu M_\mu$ of finite-dimensional subobjects M_μ in $\mathrm{rep}\,\mathrm{SL}(2)_\xi$.) We consider the de-equivariantization

$$(\mathrm{rep}\,\mathrm{SL}(2)_\xi)_{\mathrm{PSL}(2)} := \left\{\begin{matrix}\text{the monoidal category of finitely presented} \\ \mathscr{O}\text{-modules in } \mathrm{Rep}\,\mathrm{SL}(2)_\xi\end{matrix}\right\}$$

(see [9, 26]). The category $(\mathrm{rep}\,\mathrm{SL}(2)_\xi)_{\mathrm{PSL}(2)}$ is monoidal, with product $\otimes_{\mathscr{O}}$, and we have the monoidal functor

$$dE : \mathrm{rep}\,\mathrm{SL}(2)_\xi \to (\mathrm{rep}\,\mathrm{SL}(2)_\xi)_{\mathrm{PSL}(2)}, \quad dE(V) := \mathscr{O} \otimes V$$

(see [57, Theorem 1.6]). There is a unique ribbon structure on $(\mathrm{rep}\,\mathrm{SL}(2)_\xi)_{\mathrm{PSL}(2)}$ such that the de-equivariantization map dE is a map of ribbon monoidal categories, and we consider $(\mathrm{rep}\,\mathrm{SL}(2)_\xi)_{\mathrm{PSL}(2)}$ as a ribbon category with its induced ribbon structure.

Theorem 3.4 ([69]). *The category* $(\mathrm{rep}\,\mathrm{SL}(2)_\xi)_{\mathrm{PSL}(2)}$ *is finite, rigid, and non-degenerate, and hence modular. Furthermore, there is a ribbon tensor equivalence*

$$\mathbb{C} \otimes_{\mathscr{O}}^{\omega} - : (\mathrm{rep}\,\mathrm{SL}(2)_\xi)_{\mathrm{PSL}(2)} \xrightarrow{\sim} \mathrm{rep}(u_\xi(\mathfrak{sl}_2))$$

which fits into a diagram

$$\begin{array}{ccc} \mathrm{rep}\,\mathrm{SL}(2)_\xi & \xrightarrow{\quad \mathrm{res}^{\omega} \quad} & \mathrm{rep}(u_\xi(\mathfrak{sl}_2)) \\ & {\scriptstyle dE} \searrow \quad \nearrow {\scriptstyle \mathbb{C}\otimes_{\mathscr{O}}^{\omega} -} & \\ & (\mathrm{rep}\,\mathrm{SL}(2)_\xi)_{\mathrm{PSL}(2)} & \end{array} .$$

The above result is comprised specifically of [69, Corollary 5.6, Proposition 6.3, Corollary 7.2]. We argue in the present paper that one should observe an equivalence

$$F : (\mathrm{rep}\,\mathrm{SL}(2)_\xi)_{\mathrm{PSL}(2)} \xrightarrow{\sim} \mathrm{rep}(\mathcal{W}_p)$$

directly *from the de-equivariantization* of $\mathrm{rep}\,\mathrm{SL}(2)_\xi$, rather than from the representation category of the quasi-Hopf algebra $u_\xi(\mathfrak{sl}_2)$. This is very natural from the VOA perspective, as we will see starting next section. Indeed, we will interpret the de-equivariantization dE as the induction of $\mathcal{V}ir_c$-modules to $\mathcal{W}_p$-modules.

4. Modules over the triplet algebra

We provide basic information regarding the tensor category of modules for the triplet vertex operator algebra $\mathcal{W}_p$. In this section, by module we mean finite length generalized module. Most of our presentation is deducible from works of Adamović–Milas [3], Nagatomo–Tsuchiya [68], and Tsuchiya–Wood [77], although some of our conclusions are not explicitly in the literature.

As explained in the introduction, we compare modules over the triplet algebra to representations of quantum SL(2) at a particular complex parameter q.

$$\textit{Throughout this work we fix } q = e^{\pi i/p}.$$

The parameter q appears in a number of formulas related to modules for the triplet algebra.

$$\textit{We similarly fix the central charge } c = 1 - 6(p-1)^2/p.$$

4.1. Tensor categories for C_2-cofinite VOAs

To keep this subsection relatively short, we will use some standard technical terminology without defining it – see e.g. the series of papers [49–51] for details in a very general context.

By a *strongly-finite VOA* we mean a simple C_2-cofinite VOA of positive energy with $\mathcal{V}$ isomorphic to $\mathcal{V}^*$ as $\mathcal{V}$-modules. By $\mathrm{rep}(\mathcal{V})$ we mean the category whose objects are finite length generalized $\mathcal{V}$-modules, and whose morphisms are homomorphisms. For simplicity we will just call these *modules*. In this context, 'finite length' is equivalent to 'grading-restricted' or 'quasi-finite-dimensional' (see [46, Proposition 4.3]).

Recall from Section 2.6 that VOA modules carry actions of the Virasoro algebra. When the VOA is strongly finite, the eigenvalues of $L(0)$ are rational. Any module W has a minimum such eigenvalue, called the *conformal weight* $h(W)$.

The category $\mathrm{rep}(\mathcal{V})$ for strongly finite $\mathcal{V}$ has finitely many simples [30, Proposition 3.6] and enough projectives [46, Theorem 3.23]. In fact, $\mathrm{rep}(\mathcal{V})$ is a braided monoidal category [46, Theorem 4.11] (we use monoidal here rather than tensor because the latter usually requires rigidity), with the tensor unit being $\mathcal{V}$ itself. It is a finite tensor category provided the simple modules (hence all modules) are in addition rigid (see [21, Theorem 4.4.1]).

4.2. The triplet algebra and some modules

We consider the triplet vertex operator algebra $\mathcal{W}_p$ at an arbitrary integer parameter $p > 1$. Let $V_{\sqrt{p}Q}$ be the lattice VOA associated to the $\sqrt{p}$-scaling of the root lattice $Q \subset \mathfrak{h}^*$ for SL(2), where Q is given its normalized Killing form $(\alpha, \alpha) = 2$, and we employ the non-standard conformal vector

$$\omega = \frac{1}{4}\alpha_{(-1)}^2|0\rangle + \frac{p-1}{2\sqrt{p}}\alpha_{(-2)}|0\rangle \in V_{\sqrt{p}Q}.$$

Then $\mathcal{W}_p$ is a vertex operator subalgebra of $V_{\sqrt{p}Q}$, and both algebras are of central charge $c = 1 - 6(p-1)^2/p$. We therefore have a sequence of inclusions

$$\mathcal{V}ir_c \to \mathcal{W}_p \to V_{\sqrt{p}Q},$$

where $\mathcal{V}ir_c$ is the simple Virasoro VOA at the prescribed central charge. (Details can be found in [3].)

rep$(\mathcal{W}_p)$ is a braided finite tensor category: that $\mathcal{W}_p$ is strongly finite is proved in [3, 13], while rigidity is proved in [77] (with some details fleshed out in [21]). The categorical duals ${}^*X$ and X^* of X are both identified with its contragredient. In Theorem 4.7 we prove that rep$(\mathcal{W}_p)$ is a modular tensor category, as was expected. We denote the tensor product on rep$(\mathcal{W}_p)$ via the generic notation $\otimes$.

The VOA $\mathcal{W}_p$ has precisely $2p$ simple modules $X_s^{\pm}$, $1 \le s \le p$, and the triplet algebra itself appears as the simple $\mathcal{W}_p = X_1^+$. As modules for the Virasoro algebra we have

$$X_s^+ = \bigoplus_{m\ge 1} (2m-1)\mathcal{L}_{2m-1,s}, \quad X_s^- = \bigoplus_{m\ge 1} 2m\,\mathcal{L}_{2m,s},$$

where $\mathcal{L}_{n,s}$ is the simple Virasoro module of lowest conformal weight

$$h_{n,s} = \frac{1}{4p}((np-s)^2 - (p-1)^2) \tag{5}$$

(see [3]) and central charge c. Therefore X_s^+ and X_s^- have respective conformal weights

$$h_s^+ := h_{1,s} = \frac{1}{4p}(s^2-1) - \frac{1}{2}(s-1) \quad \text{and} \quad h_s^- := h_{2,s} = \frac{3}{4}p + \frac{1}{4p}(s^2-1) - s + \frac{1}{2}.$$

In the case $s = 2$ the above expression reduces to $h_2^+ = \frac{3}{4p} - \frac{1}{2}$.

The triplet algebra $\mathcal{W}_p = X_1^+$ is generated as a VOA by the conformal vector ω together with three vectors of conformal weight $2p-1$. The conformal vector generates the subalgebra $\mathcal{V}ir_c$. Each of those three vectors generates over $\mathcal{V}ir_c$ a copy of $\mathcal{L}_{3,1}$. We establish in this paper the ribbon tensor equivalence of rep $\mathrm{SL}(2)_q$ with a subcategory of rep$(\mathcal{V}ir_c)$; under this equivalence, $\mathcal{L}_{3,1}$ is identified with the image by Lusztig's quantum Frobenius of the adjoint representation of PSL(2), and $\mathcal{W}_p$ (as a $\mathcal{V}ir_c$-module) with the regular representation of PSL(2). From the point of view of VOAs, this happens because $\mathrm{Aut}(\mathcal{W}_p) \cong \mathrm{PSL}(2,\mathbb{C})$ [2, Theorem 2.3], and the orbifold (fixed-point subalgebra) of this $\mathrm{PSL}(2,\mathbb{C})$-action on $\mathcal{W}_p$ is $\mathcal{V}ir_c$.

Remark 4.1. With the two notations PSL(2) and $\mathrm{PSL}(2,\mathbb{C})$ we are simply distinguishing between PSL(2) considered as an algebraic group, or group scheme, and its corresponding discrete group, or Lie group, of $\mathbb{C}$-points.

4.3. The fusion ring for rep$(\mathcal{W}_p)$ *and Frobenius–Perron dimensions*

Let us recall some basic information from [77].

Proposition 4.2 ([3, Theorems 4.4, 5.9], [68, Theorem 5.1]). *The object X_p^+ is projective (and simple) in* $\mathrm{rep}(\mathcal{W}_p)$.

An independent verification of the projectivity of X_p^+ can also be found in [66, Proposition 7.7].

Theorem 4.3 ([77]). *In the fusion ring $K(\mathrm{rep}(\mathcal{W}_p))$, the class $[X_1^+]$ is the unit, the class $[X_1^-]$ acts as $[X_1^-]\cdot[X_s^\pm] = [X_s^\mp]$, and the class $[X_2^+]$ acts as*

$$\begin{cases} [X_2^+]\cdot[X_s^\pm] = [X_{s-1}^\pm] + [X_{s+1}^\pm] & \text{when } 1 < s < p, \\ [X_2^+]\cdot[X_p^\pm] = 2[X_1^\mp] + 2[X_{p-1}^\pm]. \end{cases}$$

The above two results imply the following.

Corollary 4.4. *The tensor category* $\mathrm{rep}(\mathcal{W}_p)$ *is generated by X_2^+.*

Proof. By Theorem 4.3 each simple object in $\mathrm{rep}(\mathcal{W}_p)$ appears as a subquotient of some power $(X_2^+)^{\otimes n}$, so that each simple object lies in the subcategory $\langle X_2^+\rangle$ of $\mathrm{rep}(\mathcal{W}_p)$ generated by X_2^+. By Proposition 4.2, the projective cover $\mathscr{P}$ of some simple object X also appears in $\langle X_2^+\rangle$. Via the composite

$$X^*\otimes\mathscr{P} \to X^*\otimes X \xrightarrow{\mathrm{ev}} \mathbf{1},$$

exactness of the tensor product, and projectivity of the product $X^*\otimes\mathscr{P}$ in $\mathrm{rep}(\mathcal{W}_p)$ [35, Proposition 2.1], we find that the unit object $\mathbf{1}$ admits a surjection from a projective object which lies in $\langle X_2^+\rangle$. It follows that the projective cover of $\mathbf{1}$ lies in $\langle X_2^+\rangle$, and subsequently that all projectives in $\mathrm{rep}(\mathcal{W}_p)$ lie in $\langle X_2^+\rangle$. Hence the subcategory $\langle X_2^+\rangle$ is all of $\mathrm{rep}(\mathcal{W}_p)$. ∎

Via associativity, the formulas of Theorem 4.3 determine the fusion rules for $K(\mathrm{rep}(\mathcal{W}_p))$ completely. Furthermore, one can explicitly calculate the fusion ring of $u_q(\mathfrak{sl}_2)$-representations to find that there is a unique isomorphism of $\mathbb{Z}^+$-rings

$$T : K(\mathrm{rep}(u_q(\mathfrak{sl}_2))) \xrightarrow{\sim} K(\mathrm{rep}(\mathcal{W}_p)) \tag{6}$$

which sends the class $[\mathbb{V}]$ of the generator to $[X_2^+]$. We consider the Frobenius–Perron dimension of the category $\mathrm{rep}(\mathcal{W}_p)$ (see Section 2.2).

Lemma 4.5. $\mathrm{FPdim}(\mathrm{rep}(\mathcal{W}_p)) = \mathrm{FPdim}(\mathrm{rep}(u_q(\mathfrak{sl}_2))) = 2p^3$.

Proof. The isomorphism (6) sends each simple $[V_s]$ over $u_q(\mathfrak{sl}_2)$ to $[X_s^+]$, and each simple $[\chi\otimes V_s]$ to $[X_s^-]$. Since the Frobenius–Perron dimension of an object is determined by its action on the Grothendieck ring, we find now

$$\mathrm{FPdim}(X_s^\pm) = \mathrm{FPdim}(\chi\otimes V_s) = \mathrm{FPdim}(V_s) = s.$$

Furthermore, by a direct comparison of composition factors for the projective covers $P_s \to V_s$ and $\mathscr{P}_s^{\pm} \to X_s^{\pm}$ ([68, Proposition 4.5], [66, Theorem 7.9], [58, Proposition 2.3.5]), one sees also that for the projective covers of the simples we have

$$T[P_s] = [\mathscr{P}_s^+], \quad T[\chi \otimes P_s] = [\mathscr{P}_s^-],$$

so that the Frobenius–Perron dimensions of the projective covers agree as well. It follows that

$$\begin{aligned}\operatorname{FPdim}(\operatorname{rep}(\mathcal{W}_p)) &= \operatorname{FPdim}\Big(\bigoplus_{s,\pm} \mathscr{P}_s^{\pm} \otimes X_s^{\pm}\Big) \\ &= 2 \cdot \operatorname{FPdim}\Big(\bigoplus_{s} P_s \otimes V_s\Big) = \operatorname{FPdim}(\operatorname{rep} u_q(\mathfrak{sl}_2)) = 2p^3. \qquad \blacksquare\end{aligned}$$

4.4. The ribbon structure and modularity of $\operatorname{rep}(\mathcal{W}_p)$

We recall that the category of $\mathcal{W}_p$-modules is ribbon, with twist provided by the exponential of the zero mode of the conformal vector

$$\theta = e^{2\pi i L(0)}.$$

Applying this to the simples described in Section 4.2 gives the following.

Lemma 4.6. *The twist θ acts on the simple modules $X_s^{\pm}$ as the scalars*

$$\theta_{X_s^+} = (-1)^{(s-1)} q^{(s^2-1)/2} \quad \text{and} \quad \theta_{X_s^-} = -e^{3\pi i p/2} q^{(s^2-1)/2}.$$

In particular, $\theta_{X_2^+} = -q^{3/2}$.

Proof. One simply checks the value of $e^{2\pi i L(0)}$ on the vector $v_s^{\pm}$ of lowest conformal weight in $X_s^{\pm}$,

$$e^{2\pi i L(0)} v_s^{\pm} = e^{2\pi i h_s^{\pm}} v_s^{\pm}.$$

We plug in the weights $h_s^{\pm}$ to obtain

$$e^{2\pi i h_s^+} = \exp\left(\frac{\pi i (s^2-1)}{2p}\right) \exp(-\pi i (s-1)) = (-1)^{(s-1)} q^{(s^2-1)/2}$$

and

$$e^{2\pi i h_s^-} = \exp\left(\pi i \frac{3}{2} p + \pi i \frac{1}{2p}(s^2-1)\right) - 2\pi i s + \pi i) = -e^{3\pi i p/2} q^{(s^2-1)/2},$$

as claimed. $\blacksquare$

One can use the above formula for the twist to establish modularity of the category $\operatorname{rep}(\mathcal{W}_p)$ of triplet modules.

Theorem 4.7. *The category* $\mathrm{rep}(\mathcal{W}_p)$ *is a non-degenerate, and hence modular, tensor category.*

Proof. By naturality of the braiding, if a given object X is Müger central, then all of its simple composition factors are also Müger central. Additionally, the unit object in a finite tensor category admits no extensions [35, Theorem 2.17]. So $Z_{\mathrm{Müg}}(\mathrm{rep}(\mathcal{W}_p)) = \mathit{Vect}$ if and only if the only Müger central simple is the unit object. So we can simply check the braidings of X_2^+ against the simples, using the formula for the twist given in Lemma 4.6, to observe the claimed triviality of the Müger center. Let us compute.

The fusion

$$X_2^+ \otimes X_s^\pm \cong X_{s-1}^\pm \oplus X_{s+1}^\pm,$$

at $1 < s < p$, implies that the twist $\theta_{X_2^+ \otimes X_s^+}$ has eigenvalues

$$\theta_{X_{s-1}^+} = (-1)^s q^{(s^2-2s)/2} \quad \text{and} \quad \theta_{X_{s+1}^+} = (-1)^s q^{(s^2+2s)/2}.$$

Similarly $\theta_{X_2^+ \otimes X_s^-}$ has eigenvalues

$$\theta_{X_{s-1}^-} = -e^{3\pi i p/2} q^{(s^2-2s)/2} \quad \text{and} \quad \theta_{X_{s+1}^+} = -e^{3\pi i p/2} q^{(s^2+2s)/2}.$$

Since $q^s = q^{-s}$ if and only if $s = p$, we see that each twist $\theta_{X_2^+ \otimes X_s^\pm} \in \mathrm{End}_{\mathcal{W}_p}(X_2^+ \otimes X_s^\pm)$ has two distinct eigenvalues when $1 < s < p$. The competing endomorphism $\theta_{X_2^+} \otimes \theta_{X_s^\pm}$ acts as a scalar, so that the square braiding

$$\theta_{X_2^+ \otimes X_s^\pm} \circ (\theta_{X_2^+}^{-1} \otimes \theta_{X_s^\pm}^{-1}) = c^2_{X_2^+, X_s^\pm}$$

is *not* the identity at $1 < s < p$. It follows that no such simple $X_s^\pm$ is Müger central.

At $s = 1$ and $\pm = -$, we have $X_2^+ \otimes X_1^- \cong X_2^-$ and $\theta_{X_2^-} = -e^{3\pi i p/2} q^{3/2}$ while $\theta_{X_2^+} \otimes \theta_{X_1^-} = e^{3\pi i p/2} q^{3/2}$. Hence

$$-1 = \theta_{X_2^+ \otimes X_1^-} \circ (\theta_{X_2^+}^{-1} \otimes \theta_{X_1^-}^{-1}) = c^2_{X_2^+, X_1^-},$$

and we conclude that X_1^- is not Müger central.

Finally, for $X_p^\pm$ we note that the twist $\theta_{X_2^+ \otimes X_p^\pm}$ has an eigenvalue $-e^{3\pi i p/2}$ or 1, while $\theta_{X_2^+} \otimes \theta_{X_p^\pm}$ is scaling by

$$(-1)^p q^{(p^2-1)/2} q^{3/2} = e^{3\pi i p/2} q \quad \text{or} \quad e^{3\pi i p/2} q^{(p^2-1)/2} q^{3/2} = q,$$

respectively. From this information we conclude, just as above, that the square braiding $c^2_{X_2^+, X_p^\pm}$ has an eigenvalue $\pm q^{-1}$, so that the objects $X_p^\pm$ are not Müger central. This exhausts all non-trivial simples in $\mathrm{rep}(\mathcal{W}_p)$, and we conclude that the Müger center of $\mathrm{rep}(\mathcal{W}_p)$ is precisely *Vect*. ∎

5. Ostrik's theorem

We recall a theorem of Ostrik, which classifies tensor functors $\operatorname{rep} \mathrm{SL}(2)_\zeta \to \mathscr{A}$ from quantum SL(2) into an arbitrary tensor category $\mathscr{A}$.

Our presentation is relatively robust, and we discuss various mechanisms employed in the proof of Ostrik's theorem in detail. In particular, we recall the construction of the Temperley–Lieb category $\mathrm{TL}(d)$ and its universal properties. The Temperley–Lieb category will play a prominent role in many of the arguments in the sections that follow.

5.1. Ostrik's theorem

By a *self-dual object* W in a tensor category $\mathscr{A}$ we mean a triple $(W, \mathrm{coev}, \mathrm{ev})$ of an object W, and maps $\mathrm{coev} : \mathbf{1} \to W \otimes W$ and $\mathrm{ev} : W \otimes W \to \mathbf{1}$ which identify W as its own left and right dual (in the sense of [34, Section 2.10]).

Definition 5.1. For a self-dual object W in a tensor category $\mathscr{A}$, the *intrinsic dimension* of W is the composite endomorphism

$$d(W) := (\mathbf{1} \xrightarrow{\mathrm{coev}} W \otimes W \xrightarrow{\mathrm{ev}} \mathbf{1}).$$

Since $\mathrm{End}_{\mathscr{A}}(\mathbf{1}) = \mathbb{C}\,\mathrm{id}_{\mathbf{1}}$, we can identify the endomorphism $d(W)$ with a scalar. Also, when W is simple, this scalar $d(W)$ is independent of choice of structure maps for W, and so is an invariant of W as an object in $\mathscr{A}$. When W is *not* simple however, this dimension does depend on the choices of coevaluation and evaluation. So calling such a dimension intrinsic is a slight abuse of language in this case.

Remark 5.2. More generally, when W is simple and not self-dual, all that can be defined is an intrinsic squared dimension. When $\mathscr{A}$ has a pivotal structure, the square of the intrinsic dimension of W equals the product of the left and right categorical dimensions of W.

For a self-dual object W of intrinsic dimension $d(W)$, there is a unique-up-to-inversion scalar $\zeta \in \mathbb{C}^\times$ such that $d(W) = -(\zeta + \zeta^{-1})$. Hence the order $\mathrm{ord}(\zeta^2)$ is an invariant of W as a self-dual object, or simply as an object in $\mathscr{A}$ when W is simple. Let us call this number the *quantum order of* $\mathscr{A}$ *at* W, or just the *quantum order of* W.

As we explain below, when W is of (finite) quantum order p, the coevaluation and evaluation maps provide a collection of self-dual objects $W_1 = \mathbf{1}, W_2 = W, W_3, \ldots, W_p$ in $\mathscr{A}$ and splittings

$$W \otimes W_r \cong W_{r-1} \oplus W_{r+1} \quad \text{for all } r < p.$$

Each object W_r is of intrinsic dimension $\pm(\zeta^{r-1} + \zeta^{r-3} + \cdots + \zeta^{-r+1})$ for appropriate choice of signs. In particular, $d(W_p) = 0$. We say W is *non-reduced* (resp. *reduced*) if the object W_p is non-vanishing (resp. vanishes) in $\mathscr{A}$.

Theorem 5.3 (Ostrik's Theorem [70, Section 2.6]). *Suppose $\mathscr{A}$ is a tensor category, and W is a self-dual object in $\mathscr{A}$ of intrinsic dimension $-(\zeta+\zeta^{-1})$. Suppose also that* $\mathrm{ord}(\zeta^2) < \infty$, *and that W is non-reduced. Then there is a uniquely associated* (*exact*) *tensor functor*

$$F_W : \mathrm{rep}\,\mathrm{SL}(2)_\zeta \to \mathscr{A} \quad \textit{with} \quad F_W(\mathbb{V}) = W.$$

Furthermore, any tensor functor $F : \mathrm{rep}\,\mathrm{SL}(2)_\zeta \to \mathscr{A}$ is isomorphic to F_W for some non-reduced, self-dual object W.

As we recall below, the standard representation $\mathbb{V}$ in $\mathrm{rep}\,\mathrm{SL}(2)_\zeta$ is self-dual, and the equality $F_W(\mathbb{V}) = W$ in the statement of Theorem 5.3 should be interpreted as an identification of self-dual objects. So, F_W is required to send the (co)evaluation maps for $\mathbb{V}$ to the (co)evaluation maps for W. Theorem 5.3 appears precisely in [70, Remark 2.10]. We sketch the proof of Ostrik's theorem below.

For consistency, we call a self-dual object non-reduced whenever $\mathrm{ord}(\zeta^2) = \infty$ as well. In the infinite order (or generic) case, the appropriate analog of the above theorem holds. Namely, a self-dual object W in $\mathscr{A}$ which is of infinite quantum order uniquely specifies a map $F_W : \mathrm{rep}\,\mathrm{SL}(2)_\zeta \to \mathscr{A}$. This generic version of Theorem 5.3 is well-known, and is easily deduced from our presentation.

Remark 5.4. Consider the situation presented in Theorem 5.3. In the case in which W is self-dual and *reduced*, with $d(W) = -(\zeta+\zeta^{-1})$, we obtain a uniquely associated tensor functor $\bar{F}_W : \mathscr{C}(\mathrm{SL}(2), \zeta) \to \mathscr{A}$ from the semisimplified representation category of $\mathrm{SL}(2)$ at ζ. The category $\mathscr{C}(\mathrm{SL}(2), \zeta)$ is explicitly the quotient of the category of tilting modules in $\mathrm{rep}\,\mathrm{SL}(2)_\zeta$ by the ideal of negligible morphisms [10, Section 3.3]. An example of a tensor category $\mathscr{A}$ realizing this case is the modular fusion category $\mathrm{rep}(L_{\widehat{\mathfrak{sl}_2}}(k, 0))$ with $\zeta = e^{\pi i/(k+2)}$, where $L_{\widehat{\mathfrak{sl}_2}}(k, 0)$ is the rational C_2-cofinite VOA associated to affine $\mathfrak{sl}_2$ at level $k \in \mathbb{Z}_{>0}$ [60, Chapter 6]. So, a more complete recounting of Ostrik's theorem would include the reduced case as well. This point, however, is inessential for our study.

5.2. The Temperley–Lieb category

Consider $\mathrm{TL}(d)$, the Temperley–Lieb category at parameter $d \in \mathbb{C}$ ([55], [39, Appendix]). This category has objects $[n]$, for each non-negative integer n, and morphisms $\mathrm{Hom}_{\mathrm{TL}}([m], [n])$ given as the $\mathbb{C}$-linear span of non-crossing pairings (non-crossing planar string diagrams) between m points and n points, up to isotopy. So, a general morphism from [5] to [7] is a sum of diagrams which look like

etc.

Composition is given by stacking diagrams vertically, where we reduce any diagram with loops to a diagram without loops by replacing each loop with the scalar d, $\bigcirc = d \cdot \mathrm{id} \in \mathrm{Hom}_{\mathrm{TL}}([0], [0])$.

The category $\mathrm{TL}(d)$ is a $\mathbb{C}$-linear, rigid monoidal category with products given by placing diagrams alongside each other horizontally, so that $[n] \otimes [m] = [n + m]$. The generating object $[1]$ is self-dual with coevaluation and evaluation given by the cup and cap morphisms

$$\mathrm{coev}_{[1]} = \cup \in \mathrm{Hom}_{\mathrm{TL}}([0], [2]), \quad \mathrm{ev}_{[1]} = \cap \in \mathrm{Hom}_{\mathrm{TL}}([2], [0]).$$

Hence all products $[m] = [1]^{\otimes m}$ are also self-dual, and the object $[1]$ is of intrinsic dimension

$$d([1]) = \mathrm{ev}_{[1]} \circ \mathrm{coev}_{[1]} = \bigcirc = d.$$

The category $\mathrm{TL}(d)$ has a universal property among monoidal categories equipped with a chosen self-dual object. We recall this universal property momentarily, after introducing a preferred additive completion of $\mathrm{TL}(d)$.

Let $\mathcal{TL}(d)$ denote the Karoubi, followed by additive, completion of the Temperley–Lieb category

$$\mathcal{TL}(d) = Kar(\mathrm{TL}(d))^{\oplus}.$$

So, we construct $\mathcal{TL}(d)$ by splitting idempotents in $\mathrm{TL}(d)$, then adding in finite sums of objects.

Theorem 5.5 ([40, Theorem 4.1.1], [82, Lemma 6.1]). *For $\mathscr{A}$ a $\mathbb{C}$-linear, pre-additive monoidal category, and W a self-dual object in $\mathscr{A}$ of intrinsic dimension d, there is a unique linear monoidal functor $F_W : \mathrm{TL}(d) \to \mathscr{A}$ with $F_W([1]) = W$, $F_W(\mathrm{coev}_{[1]}) = \mathrm{coev}_W$, and $F_W(\mathrm{ev}_{[1]}) = \mathrm{ev}_W$.*

When $\mathscr{A}$ is a tensor category, the aforementioned universal map extends uniquely to an additive monoidal functor $F_W : \mathcal{TL}(d) \to \mathscr{A}$.

By "unique" we mean unique up to natural isomorphism of monoidal functors.

Proof. As an alternative to the cited texts, one can note that $\mathrm{TL}(d)$ admits a monoidal presentation by generators and relations [78, Section 2.2], with generating object $[1]$ and generating morphisms given by evaluation and coevaluation. One then applies [78, Lemma 4.3.1] to obtain the claimed universal property. ∎

5.3. *The self-dual generator for* $\mathrm{SL}(2)_\zeta$

The object $\mathbb{V}$ in $\mathrm{SL}(2)_\zeta$ is self-dual, with (co)evaluation maps

$$\begin{aligned} \mathrm{coev} : \mathbf{1} \to \mathbb{V} \otimes \mathbb{V}, &\quad \mathrm{coev}(1) = \zeta^{-1/2} v_1 \otimes v_{-1} - \zeta^{1/2} v_{-1} \otimes v_1, \\ \mathrm{ev} : \mathbb{V} \otimes \mathbb{V} \to \mathbf{1}, &\quad \mathrm{ev}(v_1 \otimes v_1) = \mathrm{ev}(v_{-1} \otimes v_{-1}) = 0, \\ &\quad \mathrm{ev}(v_1 \otimes v_{-1}) = -\zeta^{-1/2}, \\ &\quad \mathrm{ev}(v_{-1} \otimes v_1) = \zeta^{1/2}. \end{aligned}$$

One deduces these morphisms via the (co)evaluation maps for the usual dual $\mathbb{V}^*$ and the explicit $\mathrm{SL}(2)_\zeta$-isomorphism

$$\phi : \mathbb{V} \xrightarrow{\cong} \mathbb{V}^*, \quad \phi(v_1) = -\zeta^{-1/2} v^{-1}, \ \phi(v_{-1}) = \zeta^{1/2} v^1,$$

where v^i denotes the dual function to v_i in the above expression. We then calculate the dimension

$$d(\mathbb{V}) = \mathrm{ev} \circ \mathrm{coev} = -(\zeta + \zeta^{-1}).$$

The following result is fundamental.

Theorem 5.6 ([33, Lemma A.7], [70]). *For $d = -(\zeta + \zeta^{-1})$, the map*

$$F_{\mathbb{V}} : \mathcal{TL}(d) \to \mathrm{rep}\,\mathrm{SL}(2)_\zeta,$$

determined by the self-dual object $\mathbb{V}$, is fully faithful, and restricts to an equivalence

$$F_{\mathbb{V}} : \mathcal{TL}(d) \xrightarrow{\sim} \mathcal{T}_\zeta$$

onto the subcategory of tilting objects in $\mathrm{rep}\,\mathrm{SL}(2)_\zeta$.

While the tilting category $\mathcal{T}_\zeta$ in $\mathrm{rep}\,\mathrm{SL}(2)_\zeta$ admits a rich representation-theoretic structure (see e.g. [5, 7, 8, 75]), for us it is simply the full, additive, Karoubian, monoidal subcategory in $\mathrm{rep}\,\mathrm{SL}(2)_\zeta$ generated by $\mathbb{V}$. This category is all of $\mathrm{rep}\,\mathrm{SL}(2)_\zeta$ precisely when ζ has infinite order, or is ± 1.

Sketch proof of Theorem 5.6. Fully faithfulness of the initial functor

$$F_{\mathbb{V}} : \mathrm{TL}(d) \to \mathrm{rep}\,\mathrm{SL}(2)_\zeta$$

follows from [33, Lemma A.7] via base change, and implies that the corresponding map from the completion $\mathcal{TL}(d)$ is also fully faithful. Since the indecomposable objects in $\mathcal{T}_\zeta$ are summands of the powers $F_{\mathbb{V}}([n]) = \mathbb{V}^{\otimes n}$ of the standard representation [75, Proposition 4], the image of $\mathcal{TL}(d)$ is precisely the subcategory $\mathcal{T}_\zeta$ of tilting objects. ∎

From the above identification $\mathcal{TL}(d) \cong \mathcal{T}_\zeta$ we deduce a universal property for the subcategory of tilting modules in $\mathrm{rep}\,\mathrm{SL}(2)_\zeta$.

Corollary 5.7 ([70, Theorem 2.4]). *For any tensor category $\mathscr{A}$, and self-dual object W in $\mathscr{A}$ of dimension $d(W) = -(\zeta + \zeta^{-1})$, there is a unique additive monoidal functor $F_W : \mathcal{T}_\zeta \to \mathscr{A}$ with $F_W(\mathbb{V}) = W$, $F_W(\mathrm{coev}_{\mathbb{V}}) = \mathrm{coev}_W$, and $F_W(\mathrm{ev}_{\mathbb{V}}) = \mathrm{ev}_W$.*

As stated above, $\mathcal{T}_\zeta = \mathrm{rep}\,\mathrm{SL}(2)_\zeta$ when $\mathrm{ord}(\zeta) = \infty$. So Corollary 5.7 already implies the generic version of Ostrik's theorem.

When ζ is of finite order, $\mathrm{ord}(\zeta^2) = p$, recall that we have the p simple $\mathrm{SL}(2)_\zeta$-representations V_s of respective highest weights $(s-1)\alpha/2$ for $s \leq p$, as in (4). These simples V_s all lie in the subcategory of tilting modules in $\mathrm{rep}\,\mathrm{SL}(2)_\zeta$. It follows that for any self-dual object W in $\mathscr{A}$ which is of dimension $d(W) = -(\zeta + \zeta^{-1})$, the functor F_W defines distinguished objects W_s in $\mathscr{A}$ by taking

$$W_1 = \mathbf{1}, \quad W_2 = W, \quad \text{and} \quad W_s = F_W(V_s) \quad \text{for all } s \leq p. \tag{7}$$

These W_s are alternatively defined via certain idempotent endomorphisms in $\mathrm{TL}(d)$ [53, 81].

5.4. The proof of Ostrik's theorem

We paraphrase the proof from [70]. Consider W a self-dual object in $\mathscr{A}$ of dimension $d(W) = -(\zeta + \zeta^{-1})$, and $\mathrm{ord}(\zeta^2) = p < \infty$. Suppose also that W is non-reduced. In the arguments below, by a tensor triangulated category we simply mean a triangulated category with a compatible rigid monoidal structure.

Fix $F = F_W : \mathcal{T}_\zeta \to \mathscr{A}$. We consider the bounded homotopy category $K^b(\mathcal{T}_\zeta)$ of tilting complexes and have maps

$$A : K^b(\mathcal{T}_\zeta) \to D^b(\mathrm{SL}(2)_\zeta) \quad \text{and} \quad RF : K^b(\mathcal{T}_\zeta) \to D^b(\mathscr{A}) \tag{8}$$

induced by the inclusion $\mathcal{T}_\zeta \to \mathrm{rep}\,\mathrm{SL}(2)_\zeta$ and the additive map $F : \mathcal{T}_\zeta \to \mathscr{A}$. Here $D^b(\mathscr{C})$ denotes the bounded derived category [80, Chapter 10] of the given abelian category $\mathscr{C}$, and RF denotes the composite

$$K^b(\mathcal{T}_\zeta) \xrightarrow{K^b F} K^b(\mathscr{A}) \xrightarrow{\text{localize}} D^b(\mathscr{A}).$$

All of the functors of (8) are maps of tensor triangulated categories.

By a general result of Beilinson, Bezrukavnikov, and Mirković ([11], [8, Proposition 2.4], [70, Proposition 2.7]), the functor $A : K^b(\mathcal{T}_\zeta) \to D^b(\mathrm{SL}(2)_\zeta)$ is an equivalence of tensor triangulated categories. We consider the candidate extension of F to all of $\mathrm{rep}\,\mathrm{SL}(2)_\zeta$ defined via the composite

$$\mathscr{F} := \big(\mathrm{rep}\,\mathrm{SL}(2)_\zeta \to D^b(\mathrm{SL}(2)_\zeta) \xrightarrow{A^{-1}} K^b(\mathcal{T}_\zeta) \xrightarrow{RF} D^b(\mathscr{A}) \xrightarrow{H^0} \mathscr{A}\big). \tag{9}$$

The functor $\mathscr{F}$ is a perfectly well-defined map of additive categories, and satisfies $\mathscr{F}|_{\mathcal{T}_\zeta} = F$. We must argue now that $\mathscr{F}$ is exact, and preserves tensor products, provided $F(V_p) \neq 0$.

We claim that, for any V in $\mathrm{rep}\,\mathrm{SL}(2)_\zeta$,

$$\text{the cohomology } H^*(RF \circ A^{-1}(V)) \text{ vanishes in all non-zero degrees.} \tag{10}$$

If we can verify this claim, then the tensor map $\mathrm{rep}\,\mathrm{SL}(2)_\zeta \to D^b(\mathscr{A})$ appearing in (9) has image in the abelian subcategory $D^b(\mathscr{A})^\heartsuit$ of objects with cohomology concentrated in degree 0, and it follows that $\mathscr{F}$ is an exact, $\mathbb{C}$-linear, monoidal functor as desired. Indeed, in this case we have an exact tensor functor $\mathrm{rep}\,\mathrm{SL}(2)_\zeta \to D^b(\mathscr{A})^\heartsuit$, and composing with the tensor equivalence $H^0 : D^b(\mathscr{A})^\heartsuit \xrightarrow{\cong} \mathscr{A}$ we see that $\mathscr{F}$ is exact and monoidal.

As all objects in $\mathrm{rep}\,\mathrm{SL}(2)_\zeta$ are obtainable from the simples via extension, it suffices to prove the desired vanishing (10) for all of the simples $L(\lambda)$ in $\mathrm{rep}\,\mathrm{SL}(2)_\zeta$. As explained in [70, Section 2.6], it furthermore suffices to prove the desired vanishing (only) at the 2-dimensional simple $L(\frac{p}{2}\alpha)$ in $\mathrm{rep}\,\mathrm{SL}(2)_\zeta$. For this final claim one writes out $L(\frac{p}{2}\alpha)$ as a 3-term complex

$$A^{-1}\big(L\big(\tfrac{p}{2}\alpha\big)\big) = 0 \to L \to \mathbb{V} \otimes L' \to L \to 0$$

of tilting modules, and observes that after tensoring with the projective V_p, the com-

plex $A^{-1}(L(\frac{p}{2}\alpha)) \otimes V_p$ is isomorphic to a tilting module L'' concentrated in degree 0 in $K^b(\mathcal{T}_\zeta)$ [70, Sublemma 1, Section 2.5]. We therefore calculate

$$RF\left(A^{-1}\left(L\left(\frac{p}{2}\alpha\right)\right)\right) \otimes F(V_p) \cong RF\left(A^{-1}\left(L\left(\frac{p}{2}\alpha\right)\right) \otimes V_p\right) \cong F(L''),$$

and find that the leftmost object has cohomology concentrated in degree 0. Via exactness and faithfulness of the product on $\mathscr{A}$ [10, Proposition 2.1.8], and the fact that $F(V_p)$ is non-zero in $\mathscr{A}$ by assumption, we conclude that $RF(A^{-1}(L(\frac{p}{2}\alpha)))$ has cohomology concentrated in degree 0, as desired. So we establish the desired vanishing (10).

We conclude finally that when $W_p = F(V_p) \neq 0$ in $\mathscr{A}$, the map $\mathscr{F} : \operatorname{rep} \mathrm{SL}(2)_\zeta \to \mathscr{A}$ of (9) provides the desired extension of $F : \mathcal{T}_\zeta \to \mathscr{A}$ to all of $\operatorname{rep} \mathrm{SL}(2)_\zeta$. Uniqueness follows from the fact that, by the above information, any two such extensions $\mathscr{F}$ and $\mathscr{F}'$ of F must have isomorphic derived functors, and hence must themselves be isomorphic.

6. Braidings for TL(d) and quantum group representations

We directly calculate all possible braidings on $\mathrm{TL}(d)$, and on $\operatorname{rep} \mathrm{SL}(2)_\zeta$. The results of this section are used to determine when a given tensor functor $F_W : \operatorname{rep} \mathrm{SL}(2)_\zeta \to \mathscr{A}$ into a braided tensor category $\mathscr{A}$ is in fact a *braided* tensor functor.

6.1. *Calculating braidings for* TL(d)

Consider $d \in \mathbb{C}$ and write $d = -(\zeta + \zeta^{-1})$ for some $\zeta \in \mathbb{C}^\times$. Recall that there is a unique solution to the equation $d = -(X + X^{-1})$ up to inversion, so that the choice here is between ζ and ζ^{-1}. We define the square root $\zeta^{1/2}$ unambiguously by halving the argument of ζ and taking the positive square root of the magnitude $|\zeta|$. Throughout the section we abuse notation and write coev (resp. ev) for the coevaluation morphism $\mathrm{coev}_{[1]}$ (resp. evaluation morphism $\mathrm{ev}_{[1]}$) in $\mathrm{TL}(d)$.

Consider the Temperley–Lieb category $\mathrm{TL}(d)$ and write, for any non-negative integer m, $\mathrm{TL}_m(d) = \mathrm{Hom}_{\mathrm{TL}}([m], [m])$. We have the map $f : [2] \to [2]$ given by

$$f = \cup \circ \cap = \mathrm{coev} \circ \mathrm{ev}$$

and the two elements $\{\mathrm{id}_{[2]}, f\}$ form a basis for the endomorphism ring $\mathrm{TL}_2(d)$. When $d \neq 0$ we furthermore have the idempotent $d^{-1} f$ which realizes [0] as a summand of [2] in the Karoubi completion of $\mathrm{TL}(d)$.

In $\mathrm{TL}(d)$ one can calculate the (standard) equations

$$f \circ \mathrm{coev} = d \cdot \mathrm{coev}, \quad (f \otimes 1)(1 \otimes \mathrm{coev}) = (\mathrm{coev} \otimes 1),$$
$$(1 \otimes f)(\mathrm{coev} \otimes 1) = (1 \otimes \mathrm{coev}).$$

For an element $c \in \mathrm{TL}_2(d) = \mathrm{End}_{\mathrm{TL}}([1] \otimes [1])$ let us define

$$c_{[2],[1]} = (c \otimes 1)(1 \otimes c) \quad \text{and} \quad c_{[0],[1]} = \mathrm{id}.$$

The following two results are straightforward, and certainly well-known.

Lemma 6.1. *There are precisely four solutions $c \in \mathrm{TL}_2(d)$ to the equation*

$$c_{[2],[1]}(\mathrm{coev} \otimes 1) = (1 \otimes \mathrm{coev})c_{[0],[1]} \tag{11}$$

in $\mathrm{Hom}_{\mathrm{TL}}([1], [3])$*, namely*

$$c = \pm(\zeta^{1/2} f + \zeta^{-1/2}) \quad \textit{and} \quad c = \pm(\zeta^{-1/2} f + \zeta^{1/2}).$$

Proof. Write $c = af + b$. We have directly

$$\begin{aligned} c_{[2],[1]}(\mathrm{coev} \otimes 1) &= (c_{[1],[1]} \otimes 1)(1 \otimes c_{[1],[1]})(\mathrm{coev} \otimes 1) \\ &= a(c_{[1],[1]} \otimes 1)(1 \otimes f)(\mathrm{coev} \otimes 1) + b(c_{[1],[1]} \otimes 1)(\mathrm{coev} \otimes 1) \\ &= a^2(f \otimes 1)(1 \otimes f)(\mathrm{coev} \otimes 1) + ab(1 \otimes f + f \otimes 1)(\mathrm{coev} \otimes 1) + b^2(\mathrm{coev} \otimes 1) \\ &= (a^2 + b^2)(\mathrm{coev} \otimes 1) + ab(1 \otimes f + f \otimes 1)(\mathrm{coev} \otimes 1) \\ &= (a^2 + b^2 + dab)(\mathrm{coev} \otimes 1) + ab(1 \otimes \mathrm{coev}). \end{aligned}$$

Since the two representing diagrams for $(\mathrm{coev} \otimes 1)$ and $(1 \otimes \mathrm{coev})$ are non-isotopic, and hence these maps are linearly independent in $\mathrm{Hom}_{\mathrm{TL}(d)}([1], [3])$, the identity (11) implies the equations

$$ab = 1 \;\Rightarrow\; b = a^{-1} \quad \text{and subsequently} \quad d = -(a^2 + a^{-2}).$$

The final equation gives $a^2 = \zeta^{\pm 1}$. Hence $a = \pm\zeta^{\pm 1/2}$. ∎

Note that for any braiding c on $\mathrm{TL}(d)$ the endomorphism $c_{[1],[1]} \in \mathrm{TL}_2(d)$ solves (11). Note also that any braiding on $\mathrm{TL}(d)$ is specified uniquely by its value on $[1] \otimes [1]$. So the above lemma says that there are precisely four possible braidings on $\mathrm{TL}(d)$. In the above formulas, inverting ζ corresponds to replacing the braiding by its opposite. Changing the sign of the braiding corresponds to choosing a different square root of ζ. The existence of the Kauffman bracket implies that all four possibilities are realized as braidings on $\mathrm{TL}(d)$.

Proposition 6.2. *The category $\mathrm{TL}(d)$ has precisely four braidings, and precisely two braidings modulo inversion. These two braidings (up to inversion) are specified uniquely by their values on $[1] \otimes [1]$, and differ only by a sign:*

$$c_{[1],[1]} = \pm(\zeta^{1/2} f + \zeta^{-1/2}).$$

Proof. The Kauffman bracket (see e.g. [79, Section XII.2.2]) realizes all four of the possible braidings suggested in Lemma 6.1 as braidings on $\mathrm{TL}(d)$. (These braidings are also realized as braidings on $\mathrm{rep}\, \mathrm{SL}(2)_\zeta$, which one recalls involves a choice of square root for ζ.) To pair inverses we check

$$(\zeta^{-1/2} f + \zeta^{1/2})(\zeta^{1/2} f + \zeta^{-1/2}) = (d + \zeta + \zeta^{-1}) f + 1 = 1.$$ ∎

When $d \neq 0$, we have the two idempotents $e_1 = d^{-1} f$ and $e_3 = 1 - d^{-1} f$ which provide an alternative basis for the endomorphisms $\mathrm{TL}_2(d)$. One translates directly to find $\zeta^{1/2} f + \zeta^{-1/2} = -\zeta^{3/2} e_1 + \zeta^{-1/2} e_3$. So, in terms of this basis of idempotents, Proposition 6.2 yields the following.

Proposition 6.3. *When $d \neq 0$, the category* $\mathrm{TL}(d)$ *has precisely two braidings, modulo inversion, which are specified by the values*

$$c_{[1],[1]} = \pm(-\zeta^{3/2} e_1 + \zeta^{-1/2} e_3).$$

6.2. Calculations with the generator $\mathbb{V}$ *in* $\mathrm{rep}\,\mathrm{SL}(2)_\zeta$

Via the equivalence $K^b(\mathcal{TL}(d)) \cong D^b(\mathrm{SL}(2)_\zeta)$ we observe a bijection between braidings on $\mathrm{rep}\,\mathrm{SL}(2)_\zeta$ and braidings on the Temperley–Lieb category (see Section 5.4). In particular, restricting along the fully faithful functor $F_{\mathbb{V}} : \mathrm{TL}(d) \to \mathrm{rep}\,\mathrm{SL}(2)_\zeta$ induces a bijection between the respective collections of braidings. So we may compare the standard braiding of Section 3.1, for the category of quantum group representations, with the possibilities of Proposition 6.2. We perform some calculations with the generator $\mathbb{V}$.

Consider the braiding c on $\mathrm{rep}\,\mathrm{SL}(2)_\zeta$ determined by the R-matrix (3). Since $\mathbb{V}$ is annihilated by the second powers of E and F, we have

$$c_{\mathbb{V},\mathbb{V}}(v \otimes v') = \zeta^{-(\deg(v),\deg(v'))} v' \otimes v - \zeta^{-(\deg(v),\deg(v'))}(\zeta - \zeta^{-1}) F v' \otimes E v$$

so that

$$c_{\mathbb{V},\mathbb{V}} = \begin{cases} v_1 \otimes v_1 \mapsto \zeta^{-1/2} v_1 \otimes v_1, \\ v_{-1} \otimes v_{-1} \mapsto \zeta^{-1/2} v_{-1} \otimes v_{-1}, \\ v_1 \otimes v_{-1} \mapsto \zeta^{1/2} v_{-1} \otimes v_1, \\ v_{-1} \otimes v_1 \mapsto \zeta^{1/2} v_1 \otimes v_{-1} - \zeta^{1/2}(\zeta - \zeta^{-1}) v_{-1} \otimes v_1. \end{cases}$$

Also, the element $f_{\mathbb{V}} = \mathrm{coev} \circ \mathrm{ev} \in \mathrm{End}_{\mathrm{rep}\,\mathrm{SL}(2)_\zeta}(\mathbb{V} \otimes \mathbb{V})$ is such that

$$\begin{aligned} f_{\mathbb{V}}(v_1 \otimes v_1) &= f_{\mathbb{V}}(v_{-1} \otimes v_{-1}) = 0, \\ f_{\mathbb{V}}(v_1 \otimes v_{-1}) &= -\zeta^{-1} v_1 \otimes v_{-1} + v_{-1} \otimes v_1, \\ f_{\mathbb{V}}(v_{-1} \otimes v_1) &= v_1 \otimes v_{-1} - \zeta v_{-1} \otimes v_1. \end{aligned}$$

One observes an expression for the standard braiding directly from the above calculations.

Lemma 6.4. $c_{\mathbb{V},\mathbb{V}} = \zeta^{1/2} f_{\mathbb{V}} + \zeta^{-1/2}$.

7. An analysis of the generator X_2^+

The object X_2^+ is self-dual [77, Theorem 37], and so the material of Section 5.1 implies the existence of a tensor functor from some quantum group category to $\mathrm{rep}(\mathcal{W}_p)$. As explained in Theorem 5.3 and Remark 5.4, the precise domain of this functor depends on the behavior of X_2^+ in $\mathrm{rep}(\mathcal{W}_p)$. In this section we prove the following.

Proposition 7.1. *The object X_2^+ is of intrinsic dimension $d(X_2^+) = -(q + q^{-1})$, and is non-reduced.*

Proposition 7.1 implies, via Ostrik's theorem, the existence of a unique tensor functor

$$F_{X_2^+} : \operatorname{rep} \mathrm{SL}(2)_q \to \operatorname{rep}(\mathcal{W}_p)$$

with $F_{X_2^+}(\mathbb{V}) = X_2^+$. This functor is examined in more detail in Section 9.

We note that the dimension $d(X_2^+)$ can be calculated directly by following the analysis of Tsuchiya and Wood [77, Section 4.2]. We will, however, determine the dimension via abstract arguments which simultaneously address the non-reduced property of X_2^+, and also provide information on the braiding for $\operatorname{rep}(\mathcal{W}_p)$. At the conclusion of the section we determine the braiding $c_{X_2^+,X_2^+}$, up to a sign. We complete our analysis of the braiding in Section 8.

7.1. *Proof of Proposition 7.1*

Recall that at $p = 2$ the product $X_2^+ \otimes X_2^+$ is the minimal projective cover of the unit $\mathbf{1} = X_1^+$ [77, Theorem 37], and the evaluation map $X_2^+ \otimes X_2^+ \to \mathbf{1}$ identifies $\mathbf{1}$ with the cosocle of this projective module. In particular, evaluation is the unique non-zero map in $\operatorname{Hom}_{\mathcal{W}_p}(X_2^+ \otimes X_2^+, \mathbf{1})$, up to scaling. This statement also holds at $p > 2$, since we have a decomposition $X_2^+ \otimes X_2^+ \cong \mathbf{1} \oplus X_3^+$. Therefore at arbitrary p there is a unique linear map

$$_{\mathbf{1}}|(-) : \operatorname{End}_{\mathcal{W}_p}(X_2^+ \otimes X_2^+) \to \operatorname{End}_{\mathcal{W}_p}(\mathbf{1})$$

defined by composing functions with any non-zero morphism $X_2^+ \otimes X_2^+ \to \mathbf{1}$. In terms of evaluation for example, this map is uniquely defined by the property $(_{\mathbf{1}}|f) \circ \mathrm{ev} = \mathrm{ev} \circ f$.

Lemma 7.2. *At $p > 2$ the natural braiding on $\operatorname{rep}(\mathcal{W}_p)$ is such that*

$$c^2_{X_2^+,X_2^+} = q^{-3}e_1 + qe_3,$$

and at all p the corestriction of the braiding along evaluation is $_{\mathbf{1}}|c^2_{X_2^+,X_2^+} = q^{-3}$.

To be clear, the e_i above are the (unique) idempotents in $\operatorname{End}_{\mathcal{W}_p}(X_2^+ \otimes X_2^+)$ corresponding to the simple factors X_1^+ and X_3^+ in the product $X_2^+ \otimes X_2^+$, $e_i := (X_2^+ \otimes X_2^+ \to X_i^+ \to X_2^+ \otimes X_2^+)$.

Proof. When $p > 2$ we have, by naturality of the twist,

$$\theta_{X_2^+ \otimes X_2^+} = \theta_{\mathbf{1}}e_1 + \theta_{X_3^+}e_3 = e_1 + q^4 e_3$$

and $\theta^2_{X_2^+} = q^3$. Hence

$$c^2_{X_2^+,X_2^+} = \theta_{X_2^+ \otimes X_2^+}(\theta^{-1}_{X_2^+} \otimes \theta^{-1}_{X_2^+}) = q^{-3}e_1 + qe_3.$$

In general, naturality of the twist gives

$$_{\mathbf{1}}|\theta_{X_2^+ \otimes X_2^+}(\theta^{-1}_{X_2^+} \otimes \theta^{-1}_{X_2^+}) = q^{-3}(_{\mathbf{1}}|\theta_{X_2^+ \otimes X_2^+}) = q^{-3}\theta_{\mathbf{1}} = q^{-3}.$$ ∎

Lemma 7.3. *The category* $\mathrm{rep}(\mathcal{W}_p)$ *is of finite quantum order at* X_2^+ (*see Section* 5.1). *Furthermore, this order divides* p.

Proof. Take $\zeta \in \mathbb{C}^\times$ such that $d(X_2^+) = -(\zeta + \zeta^{-1})$. Then by the fusion rule for $\mathrm{rep}(\mathcal{W}_p)$, and induction, we observe

$$d(X_r^+) = \pm(\zeta^{r-1} + \zeta^{r-3} + \cdots + \zeta^{-r+1})$$

for all $1 \leq r \leq p$. But now, the simple X_p^+ is projective, and so must be of dimension 0. To make this point more clear, simply note that any composite $\mathbf{1} \to X_p^+ \otimes X_p^+ \to \mathbf{1}$ must be 0, since otherwise $\mathbf{1}$ would appear as a summand of the projective object $X_p^+ \otimes X_p^+$, and would therefore be projective – but we know this is not the case.

The vanishing $d(X_p^+) = 0$ implies

$$0 = \zeta^{p-1} d(X_p^+) = \sum_{m=0}^{p-1} (\zeta^2)^m,$$

so

$$0 = (\zeta^2 - 1)\Big(\sum_{m=0}^{p-1} (\zeta^2)^m\Big) = \zeta^{2p} - 1.$$

Hence $\mathrm{ord}(\zeta^2)$, which is the quantum order of X_2^+ in $\mathrm{rep}(\mathcal{W}_p)$, is finite and divides p. ∎

Lemma 7.4. *The object* X_2^+ *is non-reduced in* $\mathrm{rep}(\mathcal{W}_p)$ (*see Section* 5.1).

Proof. Take ζ such that $d = d(X_2^+) = -(\zeta + \zeta^{-1})$. We already saw at Lemma 7.3 that $\mathrm{ord}(\zeta^2) \leq p$. Take $p' = \mathrm{ord}(\zeta^2)$. In the arguments to follow, the fact that $p' \leq p$ is of significance.

We have the universal map $F_{X_2^+} : \mathcal{T}_\zeta \cong \mathcal{TL}(d) \to \mathrm{rep}(\mathcal{W}_p)$, $F(\mathbb{V}) = X_2^+$, and one sees by a recursive argument that $F_{X_2^+}(V_s) \cong X_s^+$ for all $s \leq p'$. Namely, $F_{X_2^+}(\mathbb{V}) = F_{X_2^+}(V_2) = X_2^+$, and if we assume $F_{X_2^+}(V_r) \cong X_r^+$ for all $r < s$ then

$$\begin{aligned} X_{s-2}^+ \oplus F_{X_2^+}(V_s) &\cong F_{X_2^+}(\mathbb{V} \otimes V_{s-1}) \\ &\cong F_{X_2^+}(\mathbb{V}) \otimes F_{X_2^+}(V_{s-1}) \cong X_2^+ \otimes X_{s-1}^+ \cong X_{s-2}^+ \oplus X_s^+. \end{aligned}$$

Uniqueness of Jordan–Hölder series therefore forces $F_{X_2^+}(V_s) \cong X_s^+$, as proposed. These identifications imply, in particular, that $F_{X_2^+}(V_{p'})$ is isomorphic to the (non-zero) simple $X_{p'}^+$. So we see that X_2^+ is non-reduced. ∎

We can now prove the claimed result.

Proof of Proposition 7.1. We have already seen that X_2^+ is non-reduced, in Lemma 7.4. So we only have to determine the dimension. When $p = 2$ the object $X_2^+ \otimes X_2^+$ is projective, and thus the intrinsic dimension of X_2^+ is $0 = -(i - i^{-1})$. Suppose now $p > 2$.

Fix ζ such that $d = d(X_2^+) = -(\zeta + \zeta^{-1})$. We want to show that $\zeta = q^{\pm 1}$. We have the universal map $F_{X_2^+} : \mathrm{TL}(d) \to \mathrm{rep}(\mathcal{W}_p)$, $F([1]) = X_2^+$. Since X_2^+ is non-reduced,

the map $F_{X_2^+}$ extends to a tensor functor from $\operatorname{rep} \mathrm{SL}(2)_\zeta$ and is therefore faithful [28, Proposition 1.19]. Faithfulness, and the fact that

$$\dim_{\mathbb{C}} \operatorname{End}_{\mathcal{W}_p}(X_2^+ \otimes X_2^+) = \dim_{\mathbb{C}} \operatorname{End}_{\mathcal{W}_p}(X_1^+ \oplus X_3^+) = 2,$$

implies that the map on morphisms $F_{X_2^+} : \mathrm{TL}_2(d) \to \operatorname{End}_{\mathcal{W}_p}(X_2^+ \otimes X_2^+)$ is an isomorphism.

Now, by Lemma 6.1 and Proposition 6.2, the preimage $c \in \mathrm{TL}_2(d)$ of the element $c_{X_2^+,X_2^+} \in \operatorname{End}_{\mathcal{W}_p}(X_2^+ \otimes X_2^+)$ specifies a unique braiding on $\mathrm{TL}(d)$ under which the functor $F_{X_2^+}$ is braided monoidal. By the calculation of Lemma 7.2, this braiding satisfies

$$c^2_{[1],[1]} = q^{-3}e_1 + qe_3.$$

By the constraints placed on braidings on $\mathrm{TL}(d)$ given in Proposition 6.3, we see that $\zeta = q^{\pm 1}$ and $d = d(X_2^+) = -(q + q^{-1})$. ∎

7.2. The braiding on $\operatorname{rep}(\mathcal{W}_p)$*, up to a sign*

In keeping with our general convention, we let $f_{X_2^+} : X_2^+ \otimes X_2^+ \to X_2^+ \otimes X_2^+$ denote the composite of the evaluation and coevaluation maps

$$X_2^+ \otimes X_2^+ \to \mathbf{1} \to X_2^+ \otimes X_2^+.$$

Since X_2^+ is simple, the endomorphism $f_{X_2^+}$ is independent of the specific choice of evaluation and coevaluation.

Lemma 7.5. *The braiding on* $\operatorname{rep}(\mathcal{W}_p)$ *is specified, up to a sign, as*

$$c_{X_2^+,X_2^+} = \pm(q^{-1/2} f_{X_2^+} + q^{1/2}). \tag{12}$$

Proof. At $p > 2$ the endomorphism (12) can be rewritten as $\pm(-q^{-3/2}e_1 + q^{1/2}e_3)$ and squares to the appropriate formula according to Lemma 7.2. At $p = 2$ the square of this endomorphism is $2f + q = 2f + q^{-3}$, since $q^4 = 1$ and $f^2 = 0$, where we write $f = f_{X_2^+}$. (At $p = 2$ the square of f is zero since $\mathbf{1}$ is not projective, and so does not appear as a summand of the projective object $X_2^+ \otimes X_2^+$.)

As explained in the proof of Proposition 7.1, there is a unique braiding on $\mathrm{TL}(d)$ under which the (faithful) monoidal functor $F_{X_2^+} : \mathrm{TL}(d) \to \operatorname{rep}(\mathcal{W}_p)$ is braided monoidal. By the constraints of Propositions 6.2, and the fact that ${}_{\mathbf{1}}|c^2_{X_2^+,X_2^+} = q^{-3}$ by Lemma 7.2, we see that the only possible values for $c_{X_2^+,X_2^+}$ are those of the form (12). ∎

At general p, these two possibilities for the braiding can be distinguished by their compositions along any non-zero projection $X_2^+ \otimes X_2^+ \to \mathbf{1}$.

Corollary 7.6. *We have* ${}_{\mathbf{1}}|c_{X_2^+,X_2^+} = \pm(-q^{-3/2})$*, with sign* $\pm$ *corresponding precisely to the sign at* (12).

Proof. One calculates directly

$$\mathrm{ev} \circ (q^{-1/2} f_{X_2^+} + q^{1/2}) = (q^{-1/2} d + q^{1/2})\mathrm{ev} = -q^{-3/2} ev. \qquad \blacksquare$$

8. The braiding for rep($\mathcal{W}_p$)

We calculate the braiding on rep($\mathcal{W}_p$) explicitly via its vertex tensor structure.

8.1. Categorical data for strongly finite VOAs

Let $\mathcal{V}$ be a strongly finite VOA, and $W \in \mathrm{rep}(\mathcal{V})$ be a module (recall the discussion in Section 4.1). Then as in Section 2.6 we obtain the decompositions $\mathcal{V} = \coprod_{n=0}^{\infty} \mathcal{V}_{(n)}$ and $W = \coprod_r W_{(r)}$ by $L(0)$. For each $v \in \mathcal{V}_{(n)}$ we have the vertex operator $Y_W(v,z) = \sum_{m\in\mathbb{Z}} v_m z^{-m-1}$, where the linear map v_m sends $W_{(r)}$ to $W_{(r+n-m-1)}$.

To understand the tensor product of $\mathcal{V}$-modules, we need a generalization of vertex operators called (logarithmic) intertwining operators $\mathcal{Y}$. If W^1, W^2, W^3 are simple, then an *intertwining operator of type* $\binom{W^3}{W^1 W^2}$ has the shape

$$\mathcal{Y}(w^1, z) = z^{h(W^3)-h(W^1)-h(W^2)} \sum_{m\in\mathbb{Z}} w^1_m z^{-m-1} \quad \text{for any } w^1 \in W^1,$$

where w^1_m sends $W^2_{(s)}$ to $W^3_{(r-h(W^1)+s-h(W^2)-m-1+h(W^3))}$ when $w^1 \in W^1_{(r)}$. More generally, when W^i are merely indecomposable, each term in the sum will include polynomials in $\log(z)$. More precisely, choosing $w^1 \in W^1_{(r)}$ and $w^2 \in W^2_{(s)}$, let k_i be the smallest values such that $(L(0)-r)^{k_1} w^1 = 0 = (L(0)-s)^{k_2} w^2$; then the mth term in the sum is $z^{h(W^3)-h(W^1)-h(W^2)-m-1}$ times a polynomial in $\log(z)$ of degree $k_1+k_2+k_3-3$ with coefficients in $W^3_{(t)}$, where

$$t = h(W^3) + r - h(W^1) + s - h(W^2) - m - 1$$

and k_3 is the smallest power such that $(L(0)-t)^{k_3} w^3 = 0$ for all $w^3 \in W^3_{(t)}$.

We denote by $\mathcal{V}\binom{W^3}{W^1 W^2}$ the space of intertwining operators of that type. Its dimension equals that of $\mathrm{Hom}_{\mathcal{V}}(W^1 \otimes W^2, W^3)$ [50, Proposition 4.17]. Thus when W^3 is simple and $W^1 \otimes W^2$ is semisimple, $\dim \mathcal{V}\binom{W^3}{W^1 W^2}$ will equal the multiplicity of W^3 in $W^1 \otimes W^2$.

The tensor unit is $\mathcal{V}$. There is a unique non-degenerate invariant bilinear form on $\mathcal{V}$ up to scaling; let (u,v) denote the one normalized so that $(\mathbf{1},\mathbf{1}) = 1$, where $\mathcal{V}_0 = \mathbb{C}\mathbf{1}$. We will need an explicit intertwining operator $\mathcal{Y}$ of type $\binom{\mathcal{V}}{W\ W^*}$ (for W simple, this space is always 1-dimensional). It is defined by

$$(v, \mathcal{Y}(w,z)w') = \langle e^{\pi i (h(W)-\mathrm{wt}\, w)} z^{-2\mathrm{wt}\, w} e^{L(-1)/z} Y_W(v, e^{-\pi i} z^{-1}) e^{zL(1)} w, w' \rangle \quad (13)$$

for any $v \in \mathcal{V}$, $w \in W$ and $w' \in W^*$. The Virasoro operator $L(1)$ lowers weights by 1 (so automatically kills lowest weight vectors), while $L(-1)$ raises them by 1.

For VOAs, the braiding isomorphism $c_{W^1,W^2} : W^1 \otimes W^2 \to W^2 \otimes W^1$ is skew-symmetry. More precisely, suppose $\mathcal{V}$ is strongly finite and W^1, W^2 are simple. When $W^1 \otimes W^2$ is semisimple (i.e. isomorphic to a sum of simples), we can identify the $\mathcal{V}$-module $W^1 \otimes W^2$ with $\bigoplus_{W^3} \mathcal{V}\binom{W^3}{W^1 W^2} \otimes_{\mathbb{C}} W^3$, where the (finite) sum is over inequivalent simple $\mathcal{V}$-modules W^3. In this case we can interpret c_{W^1,W^2} as a linear map between spaces of intertwining operators $\mathcal{V}\binom{W^3}{W^1 W^2} \to \mathcal{V}\binom{W^3}{W^2 W^1}$:

$$c_{W^1,W^2}(\mathcal{Y})(w^2, z)w^1 = e^{zL(-1)}\mathcal{Y}(w^1, e^{\pi i} z)w^2 \tag{14}$$

for any $\mathcal{Y} \in \mathcal{V}\binom{W^3}{W^1 W^2}$, where we are using the skew-symmetry operator Ω_r defined in [48, equation (7.1)] (for us, $r = 0$). The notation $e^{\pi i}$ here indicates the appropriate choice of branch of logarithm, needed to evaluate fractional powers. We also use (14) when $W^1 \otimes W^2$ is not semisimple, though in this case the relation of the intertwining operator to the tensor product can be slightly more subtle.

8.2. *The sign of the braiding*

In Proposition 7.1 we determined the dimension $d(X_2^+)$ of X_2^+ by "abstract nonsense". In Lemma 7.5 we similarly determined the braiding on $\mathrm{rep}(\mathcal{W}_p)$ up to a sign, again by "nonsense". Here we specify the braiding on the category of triplet modules precisely, by dealing directly with the vertex tensor structure on $\mathrm{rep}(\mathcal{W}_p)$.

Proposition 8.1. *The braiding on* $\mathrm{rep}(\mathcal{W}_p)$ *satisfies*

$$c_{X_2^+,X_2^+} = q^{-1/2} f_{X_2^+} + q^{1/2}.$$

Proof. By Corollary 7.6, in order to specify uniquely the braiding of $\mathrm{rep}(\mathcal{W}_p)$ it suffices to apply (14) to the intertwining operator $\mathcal{Y} \in \mathcal{W}_p\binom{\mathcal{W}_p}{X_2^+ X_2^+}$ defined in (13). For all p, this $\mathcal{Y}$ corresponds to the (unique up to scaling) homomorphism $X_2^+ \otimes X_2^+ \to \mathcal{W}_p$.

Take $w^1 = w^2 = w$, a non-zero vector in the lowest weight space of X_2^+, which is 1-dimensional. Note that $\mathcal{Y}(w, z)w \in z^{1-3/(2p)}(x\mathbf{1} + z\,\mathcal{W}_p[[z]])$ for some $x \in \mathbb{C}$. It is clear from (13) that $x \neq 0$. We find

$$\begin{aligned} c_{X_2^+,X_2^+}(\mathcal{Y})(w, z)w \in e^{zL(-1)} e^{\pi i(1-3/(2p))} z^{1-3/(2p)}(x\mathbf{1} + z\,\mathcal{W}_p[[z]]) \\ = -e^{-3\pi i/(2p)} z^{1-3/(2p)}(x\mathbf{1} + z\,\mathcal{W}_p[[z]]). \end{aligned}$$

This must be a multiple of $\mathcal{Y}$ (since $\dim \mathcal{W}_p\binom{\mathcal{V}}{X_2^+ X_2^+} = 1$), and because $x \neq 0$, that multiple is thus $-q^{-3/2}$. ∎

For $p = 2$ the product $X_2^+ \otimes X_2^+$ is not semisimple, but rather is the minimal projective cover P_1^+ of the unit. This plays no role in the proof, however. When $p = 2$, for what it is worth, $\mathcal{Y}$ is the part of the full logarithmic intertwining operator of type $\binom{P_1^+}{X_2^+ X_2^+}$ coming with the highest powers of $\log(z)$.

For $p > 2$ we can determine the rest of the braiding in a similar way. Take any non-zero $\mathcal{Y} \in \mathcal{W}_p\binom{X_3^+}{X_2^+ X_2^+}$. Then $\mathcal{Y}(w,z)w \in z^{1/(2p)}(w_3 + zX_3^+[[z]])$, so

$$c_{X_2^+,X_2^+}(\mathcal{Y})(w,z)w \in e^{zL(-1)}e^{\pi i/(2p)}z^{1/(2p)}(w_3 + zX_3^+[[z]]) \\ = e^{\pi i/(2p)}z^{1/(2p)}(w_3 + zX_3^+[[z]])$$

for some $w_3 \in X_3^+$. This must be a multiple of $\mathcal{Y}$, and provided $w_3 \neq 0$, that multiple is thus $q^{1/2}$, as desired. It is elementary to show that $w_3 \neq 0$ using the hypergeometric function calculations in [21,77]. But we will skip the details as we already know $q^{1/2}$ is correct from Lemma 7.5 together with the calculation in the preceding proof.

We note that, by naturality of the braiding and the braid relation, the braiding on the category $\mathrm{rep}(\mathcal{W}_p)$ is determined uniquely by its value $c_{X_2^+,X_2^+}$ on the generator, as implied by Lemma 7.5.

9. A modular tensor equivalence $\mathrm{rep}(u_q(\mathfrak{sl}_2)) \cong \mathrm{rep}(\mathcal{W}_p)$

At Theorem 9.5 below, we prove that there is an equivalence of (non-semisimple) modular tensor categories $\mathrm{rep}(u_q(\mathfrak{sl}_2))^{\mathrm{rev}} \cong \mathrm{rep}(\mathcal{W}_p)$, at quantum parameter $q = e^{\pi i/p}$. Our proof relies on a number of technical points. First, we identify tensor functors out of the de-equivariantization $(\mathrm{rep}\,\mathrm{SL}(2)_q)_{\mathrm{PSL}(2)}$ with a certain class of tensor functors out of $\mathrm{rep}\,\mathrm{SL}(2)_q$. We then prove a braided version of Ostrik's theorem, which is given in Theorem 9.4 below. These results, in conjunction with the analysis of $\mathrm{rep}(\mathcal{W}_p)$ provided in Sections 7 and 8, will imply the claimed equivalence.

9.1. *De-equivariantization as categorical base change*

Suppose $\zeta \in \mathbb{C}^\times$ is of even order $2p$, for the sake of specificity. We say a functor $\mathrm{M} : \mathrm{rep}\,\mathrm{SL}(2)_\zeta \to \mathscr{A}$ *annihilates* the central subcategory $\mathrm{rep}(\mathrm{PSL}(2)) \subset \mathrm{rep}\,\mathrm{SL}(2)_\zeta$ if the image of $\mathrm{rep}(\mathrm{PSL}(2))$ in $\mathscr{A}$ is precisely the subcategory $\mathit{Vect} \subset \mathscr{A}$ generated by the unit, and for any W in $\mathrm{rep}(\mathrm{PSL}(2))$ and V in $\mathrm{rep}\,\mathrm{SL}(2)_\zeta$ the diagram

$$\begin{array}{ccc} \mathrm{M}(W \otimes V) & \xrightarrow{\mathrm{M}(c_{V,W})} & \mathrm{M}(V \otimes W) \\ \cong \downarrow & & \downarrow \cong \\ \mathrm{M}(W) \otimes \mathrm{M}(V) & \xrightarrow{\tau_{\mathrm{M}(W),\mathrm{M}(V)}} & \mathrm{M}(V) \otimes \mathrm{M}(W) \end{array} \tag{15}$$

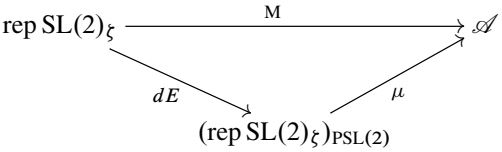
commutes. Here the vertical maps are the structure maps for M, and $\tau_{\mathrm{M}(W),\mathrm{M}(V)}$ is the half-braiding provided by the trivial lift $\mathit{Vect} \to Z(\mathscr{A})$ of the unit $\mathit{Vect} \to \mathscr{A}$. (More precisely, τ is the half-braiding provided by the unit structure $- \otimes \mathbf{1} \cong \mathrm{id} \cong \mathbf{1} \otimes -$.) A very practical way to observe the above diagram (15) is the following.

Lemma 9.1. *Given a braided tensor functor* $\mathrm{M} : \mathrm{rep}\,\mathrm{SL}(2)_\zeta \to \mathscr{A}$ *into a braided tensor category* $\mathscr{A}$, M *annihilates* $\mathrm{rep}(\mathrm{PSL}(2))$ *if and only if* $\mathrm{M}(\mathrm{rep}(\mathrm{PSL}(2))) \subset \mathit{Vect}$.

Proof. Compatibility with the braiding implies that the diagram (15) commutes, after we replace τ with the braiding c for $\mathscr{A}$. But now, the hypotheses on any braiding requires $c_{X,Y} = \tau_{X,Y}$ for all trivial X, so that compatibility with the braiding implies the diagram (15). ∎

The following is a general result about de-equivariantization, which we state only in the case of $(\operatorname{rep} \mathrm{SL}(2)_\xi)_{\mathrm{PSL}(2)}$. The general situation is discussed in Appendix B.5.

Proposition 9.2. *Suppose a tensor functor* $\mathrm{M} : \operatorname{rep} \mathrm{SL}(2)_\xi \to \mathscr{A}$ *annihilates the central subcategory* $\operatorname{rep}(\mathrm{PSL}(2)) \subset \operatorname{rep} \mathrm{SL}(2)_\xi$. *Then there exists a tensor functor from the de-equivariantization* $\mu : (\operatorname{rep} \mathrm{SL}(2)_\xi)_{\mathrm{PSL}(2)} \to \mathscr{A}$ *which fits into a* (2-)*diagram*



(16)

If $\mathscr{A}$ *is braided and* M *is a map of braided tensor categories, then* μ *can be taken to be a map of braided tensor categories as well. Furthermore, the collection of all such factorizations* $\{\mu$ admitting a diagram (16)$\}$ *admits a natural* PSL(2)*-action.*

As is made clear in the appendix, a more careful statement of Proposition 9.2 would specify not only the functor M, but also a choice of trivialization of $\mathrm{M}|_{\operatorname{rep}(\mathrm{PSL}(2))}$.

The factorization μ can be written explicitly as $\mu = \mathbf{1} \otimes_{\mathrm{M}\mathscr{O}} \mathrm{M}(-)$, where $\mathscr{O} = \mathrm{Fr}\,\mathscr{O}(\mathrm{PSL}(2))$. The ambiguity appears in the choice of the point (algebra map) $\mathrm{M}\mathscr{O} \to \mathbf{1}$ at which we take the fiber to produce μ. The collection of such points for $\mathrm{M}\mathscr{O}$ has the structure of a PSL(2)-torsor.

Proof of Proposition 9.2. Suppose we have such a map $\mathrm{M} : \operatorname{rep} \mathrm{SL}(2)_\xi \to \mathscr{A}$ which annihilates $\operatorname{rep}(\mathrm{PSL}(2))$. Then the algebra object $\mathscr{O} = \mathrm{Fr}\,\mathscr{O}(\mathrm{PSL}(2))$ in $\operatorname{Rep} \mathrm{SL}(2)_\xi$ has image $B = \mathrm{M}\mathscr{O}$, an algebra object in $\operatorname{Ind} \mathit{Vect} \subset \operatorname{Ind} \mathscr{A}$. Thus we may consider the exact monoidal category $\mathrm{mod}_{\mathscr{A}}(B)$ of finitely presented B-modules in $\mathscr{A}$. Given any augmentation (algebra map) $B \to \mathbf{1}$, we have the associated monoidal functor $\mathbf{1} \otimes_B - : \mathrm{mod}_{\mathscr{A}}(B) \to \mathscr{A}$ [57, proof of Theorem 1.6(3), at $\mathscr{C} = \mathrm{mod}_{\mathscr{A}}(B)$ and $A = \mathbf{1}$] for which the composite

$$\mathscr{A} \xrightarrow{\text{Free}} \mathrm{mod}_{\mathscr{A}}(B) \xrightarrow{\mathbf{1}\otimes_B -} \mathscr{A}$$

is isomorphic to the identity [57, Theorem 1.6 (4) at $\mathscr{C} = \mathscr{A}$, $A = B$, $X = \mathbf{1}$]. (Note that we use the canonical central structure on the embedding $\mathit{Vect} \to \mathscr{A}$ in place of the braidings employed in [57, Theorems 1.5, 1.6].)

The tensor functor M induces an exact $\mathbb{C}$-linear functor between categories of modules

$$\mathrm{M}_{\mathscr{O}} : (\operatorname{rep} \mathrm{SL}(2)_\xi)_{\mathrm{PSL}(2)} \to \mathrm{mod}_{\mathscr{A}}(B),$$

which just sends an object N in $\operatorname{Rep} \mathrm{SL}(2)_\xi$ with a given $\mathscr{O}$-action to the object $\mathrm{M}(N)$ in $\operatorname{Ind} \mathscr{A}$ with its induced B-action, $B \otimes \mathrm{M}(N) \cong \mathrm{M}(\mathscr{O} \otimes N) \to \mathrm{M}(N)$. We compose with

the fiber $\mathbf{1} \otimes_B -$ at any (fixed) augmentation $B \to \mathbf{1}$ to obtain the desired factoring of M through the de-equivariantization $\mu = (\mathbf{1} \otimes_B -) \circ \mathrm{M}_{\mathscr{O}}$, as a $\mathbb{C}$-linear functor.

(We note that since $\mathscr{O}$ is a finitely generated commutative algebra object in $\mathrm{Rep}\,\mathrm{PSL}(2)$, B is a finitely generated commutative algebra object in *Vect*, i.e. a finitely generated commutative algebra over $\mathbb{C}$. Hence B admits a plethora of augmentations which are parametrized by $\mathrm{Spec}(B)$.)

We now claim that $\mathrm{M}_{\mathscr{O}}$ is a monoidal functor, or rather admits the structure of a monoidal functor, and hence μ is a monoidal functor. We also claim that μ is exact. Let us argue these points.

Since M annihilates $\mathrm{rep}(\mathrm{PSL}(2)) \subset \mathrm{rep}\,\mathrm{SL}(2)_\zeta$, the two right B-actions on $\mathrm{M}_{\mathscr{O}}(N)$ induced by the braiding on $\mathrm{rep}\,\mathrm{SL}(2)_\zeta$ and the central structure for the *Vect*-action on $\mathscr{A}$, agree. Additionally, exactness of M implies that the image of the equalizer diagram

$$\mathrm{M}\big(N \otimes \mathscr{O} \otimes N' \rightrightarrows N \otimes N' \to N \otimes_{\mathscr{O}} N'\big)$$

for the action maps on $\mathscr{O}$-modules N and N' provides an equalizer diagram

$$\mathrm{M}(N) \otimes B \otimes \mathrm{M}(N') \rightrightarrows \mathrm{M}(N) \otimes \mathrm{M}(N') \to \mathrm{M}(N \otimes_{\mathscr{O}} N') \tag{17}$$

for the corresponding B-modules in $\mathrm{Ind}\,\mathscr{A}$. Since $\mathrm{M}(N) \otimes_B \mathrm{M}(N')$ is the equalizer of the diagram (17), by definition, the monoidal structure on M induces a unique binatural isomorphism $\mathrm{M}(N) \otimes_B \mathrm{M}(N') \cong \mathrm{M}(N \otimes_{\mathscr{O}} N')$ which fits into a diagram

$$\begin{array}{ccc} \mathrm{M}(N) \otimes \mathrm{M}(N') & \xrightarrow[\text{structural}]{\cong} & \mathrm{M}(N \otimes N') \\ \big\downarrow & & \big\downarrow \\ \mathrm{M}(N) \otimes_B \mathrm{M}(N') & \xrightarrow[\text{induced}]{\cong} & \mathrm{M}(N \otimes_{\mathscr{O}} N') \end{array}$$

These isomorphisms give $\mathrm{M}_{\mathscr{O}}$ the structure of a monoidal functor. The functor μ therefore inherits a monoidal structure, as it is a composition of monoidal functors.

For exactness of μ, note that the reduction $\mathbf{1} \otimes_B - : \mathrm{mod}_{\mathscr{A}}(B) \to \mathscr{A}$ admits a right adjoint $\mathscr{A} \to \mathrm{mod}_{\mathscr{A}}(B)$ which sends an object V in $\mathscr{A}$ to the B-module in $\mathscr{A}$ which is simply V, as an object in $\mathscr{A}$, with B acting through the augmentation $B \to \mathbf{1}$. As any functor which has a right adjoint is right exact, we see that the reduction functor is right exact, and hence μ is right exact as a composition of right exact functors. Now, compatibility with duality [34, Exercise 2.10.6] implies that μ is left exact as well.

As for the promised identification $\mu \circ dE \xrightarrow{\cong} \mathrm{M}$, we have the explicit sequence of natural isomorphisms

$$\begin{aligned} \mu \circ dE(V) &= \mu(\mathscr{O} \otimes V) = \mathbf{1} \otimes_B \mathrm{M}(\mathscr{O} \otimes V) \\ &\cong \mathbf{1} \otimes_B (\mathrm{M}(\mathscr{O}) \otimes \mathrm{M}(V)) = \mathbf{1} \otimes_B (B \otimes \mathrm{M}(V)) \xrightarrow{\cong} \mathbf{1} \otimes_B (\mathbf{1} \otimes \mathrm{M}(V)) \cong \mathrm{M}(V). \end{aligned}$$

The inverse is provided by the natural map $\mathrm{M}(V) \to \mathrm{M}(\mathscr{O} \otimes V) \to \mathbf{1} \otimes_B \mathrm{M}(\mathscr{O} \otimes V)$.

In the braided context, $\mathrm{mod}_{\mathscr{A}}(B)$ inherits a braiding from that of $\mathscr{A}$, and the braiding caries through all of the given arguments. The PSL(2)-action comes from the fact that the PSL(2)-action on $\mathscr{O}(\mathrm{PSL}(2))$, by translation, induces a PSL(2)-action on the algebra B. This PSL(2)-action permutes the points $B \to \mathbf{1}$ at which we define the functor $\mu = \mathbf{1} \otimes_B \mathrm{M}_{\mathscr{O}}(-)$. ∎

Remark 9.3. Note that restriction gives us a functor from $\mathrm{rep}\,\mathrm{SL}(2)_\xi$ to the representation category of the small quantum group (not cocycle corrected), which sends rep(PSL(2)) to *Vect*. However, it does not annihilate rep(PSL(2)) in our sense (it does not satisfy (15)), and will not factorize as in (16). So Proposition 9.2 cannot be used to establish an equivalence between representations of the non-cocycle-corrected small quantum group and $(\mathrm{rep}\,\mathrm{SL}(2)_\xi)_{\mathrm{PSL}(2)}$. Likewise, the forgetful functor $\mathrm{rep}\,\mathrm{SL}(2)_\xi \to \mathit{Vect}$ also does not annihilate rep(PSL(2)) in our sense, and so we do not get an induced fiber functor for the de-equivariantization.

9.2. A braided version of Ostrik's theorem

Consider a self-dual object W in a tensor category $\mathscr{A}$. For such W we have an associated endomorphism

$$f_W : W \otimes W \to \mathbf{1} \to W \otimes W, \quad f_W = \mathrm{coev} \circ \mathrm{ev},$$

of the second tensor power of W. When the endomorphism algebra $\mathrm{End}_{\mathscr{A}}(W)$ is 1-dimensional, the morphism f_W is independent of the choice of structure maps coev and ev. This occurs, for example, when W is simple.

Theorem 9.4. *Suppose W is a self-dual object in a braided tensor category $\mathscr{A}$ with $d(W) = -(\zeta + \zeta^{-1})$. Suppose also that* $\mathrm{ord}(\zeta) < \infty$*, W is non-reduced, and the braiding endomorphism $c_{W,W} \in \mathrm{End}_{\mathscr{A}}(W \otimes W)$ is in the subspace spanned by f_W and the identity.*

Then, under one of the four braided structures on $\mathrm{rep}\,\mathrm{SL}(2)_\xi$ *provided in Proposition 6.2, the tensor functor $F_W : \mathrm{rep}\,\mathrm{SL}(2)_\xi \to \mathscr{A}$ promised in Theorem 5.3 is a* braided *tensor functor.*

Proof. Since the tensor functor F_W is necessarily faithful, its restriction $F_W|_{\mathcal{T}_\xi}$ to the subcategory of tilting modules is faithful as well. As in the proof of Proposition 7.1, faithfulness and the fact that the braiding endomorphism $c_{W,W}$ is in the image of the map

$$F_W : \mathrm{End}_{\mathcal{T}_\xi}(\mathbb{V} \otimes \mathbb{V}) \to \mathrm{End}_{\mathscr{A}}(W \otimes W)$$

implies that $F_W|_{\mathcal{T}_\xi}$ respects the braidings on $\mathcal{T}_\xi$ and $\mathscr{A}$, under one of the four braidings on $\mathcal{T}_\xi$ considered in Proposition 6.3. Hence the corresponding functor of triangulated categories

$$R(F_W|_{\mathcal{T}_\xi}) : K^b(\mathcal{T}_\xi) \to D^b(\mathscr{A})$$

is also a map of braided monoidal categories.

We recall that the inclusion $\mathcal{T}_\xi \to \operatorname{rep} \mathrm{SL}(2)_q$ induces an equivalence of tensor triangulated categories $K^b(\mathcal{T}_\xi) \xrightarrow{\sim} D^b(\mathrm{SL}(2)_\xi)$ [11], and note the diagram

$$\begin{array}{ccc} & K^b(\mathcal{T}_\xi) & \\ \overset{\sim}{\swarrow} & & \searrow^{R(F_W|_{\mathcal{T}_\xi})} \\ D^b(\mathrm{SL}(2)_\xi) & \xrightarrow[RF_W]{} & D^b(\mathscr{A}) \end{array}$$

The above diagram implies that the functor RF_W is braided monoidal, since both maps out of $K^b(\mathcal{T}_\xi)$ are braided. It follows that the original map F_W, which one recovers by restricting RF_W to the subcategory of complexes concentrated in degree 0, is braided monoidal as well. ∎

9.3. A modular tensor equivalence

Theorem 9.5. *There is an equivalence of modular tensor categories*

$$\Theta : \operatorname{rep}(u_q(\mathfrak{sl}_2))^{\mathrm{rev}} \xrightarrow{\sim} \operatorname{rep}(\mathcal{W}_p).$$

The functor Θ *is such that* $\Theta(\mathbb{V}) \cong X_2^+$.

The superscript $(-)^{\mathrm{rev}}$ indicates that we consider the representation category for $u_q(\mathfrak{sl}_2)$ with inverse braiding, and inverse twist, relative to that of Section 3.1.

Proof of Theorem 9.5. Throughout the proof we consider $\operatorname{rep} \mathrm{SL}(2)_q$ with its inverse ribbon structure. Since we have the ribbon equivalence $(\operatorname{rep} \mathrm{SL}(2)_q)_{\mathrm{PSL}(2)} \cong \operatorname{rep}(u_q(\mathfrak{sl}_2))$ of Theorem 3.4, it suffices to produce such an equivalence Θ from the de-equivariantization $(\operatorname{rep} \mathrm{SL}(2)_q)_{\mathrm{PSL}(2)}$.

By Theorem 9.4, Proposition 7.1, and Proposition 8.1 we have a braided tensor functor

$$F_W : \operatorname{rep} \mathrm{SL}(2)_q \to \operatorname{rep}(\mathcal{W}_p)$$

with $F_W(\mathbb{V}) = X_2^+$. Since $\operatorname{rep}(\mathcal{W}_p)$ is generated by X_2^+, the functor F_W is surjective. Consequently, the Müger center in $\operatorname{rep} \mathrm{SL}(2)_q$ maps to a Müger central subcategory of $\operatorname{rep}(\mathcal{W}_p)$. But by Theorem 4.7, the category of triplet modules is non-degenerate, so that the only central subcategory in $\operatorname{rep}(\mathcal{W}_p)$ is *Vect*. It follows that $F_W(\operatorname{rep}(\mathrm{PSL}(2))) \subset \mathit{Vect}$. By Proposition 9.2 and Lemma 9.1, the map F_W therefore factors to provide a braided tensor functor

$$\Theta : (\operatorname{rep} \mathrm{SL}(2)_q)_{\mathrm{PSL}(2)} \to \operatorname{rep}(\mathcal{W}_p)$$

from the de-equivariantization.

Since $\Theta \circ dE \cong F_W$ we have $\Theta(\mathbb{V}) \cong X_2^+$, and since $\operatorname{rep}(\mathcal{W}_p)$ is generated by X_2^+ we conclude that Θ is surjective. Finally, since

$$\mathrm{FPdim}\big((\operatorname{rep} \mathrm{SL}(2)_q)_{\mathrm{PSL}(2)}\big) = \mathrm{FPdim}(\operatorname{rep}(u_q(\mathfrak{sl}_2))) = \mathrm{FPdim}(\operatorname{rep}(\mathcal{W}_p)),$$

by Lemma 4.5, surjectivity of Θ implies that Θ is an equivalence, by Theorem 2.5.

Finally, for the ribbon structure, let θ denote the twist of Section 3.1. The category $(\operatorname{rep} \mathrm{SL}(2)_q)_{\mathrm{PSL}(2)}$, with its reversed braiding, admits precisely two twists, θ^{-1} and $K^p\theta^{-1}$. This is because $u_q(\mathfrak{sl}_2)$ has precisely two central grouplike elements, 1 and K^p. These twists are such that $\theta_{\mathbb{V}}^{-1} = -q^{3/2}$ and $K^p\theta_{\mathbb{V}}^{-1} = q^{3/2}$. Since the value of the twist for $\operatorname{rep}(\mathcal{W}_p)$ at X_2^+ is $-q^{3/2}$, we see that Θ is an equivalence of modular tensor categories when we provide $(\operatorname{rep} \mathrm{SL}(2)_q)_{\mathrm{PSL}(2)}$ with the inverse ribbon structure θ^{-1}. ∎

10. Results for logarithmic minimal models

Consider $\mathcal{V}ir_c$, the Virasoro vertex operator algebra at our fixed central charge $c = 1 - 6(p-1)^2/p$. This algebra is not C_2-cofinite (hence not strongly finite). In [66] McRae and Yang verify the existence of a vertex tensor structure on a distinguished "affine" category $\operatorname{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ of Virasoro modules, which we recall below. Some important features of the category $\operatorname{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ are that it (a) admits a tensor generator, and (b) contains all of the integral simples $\mathcal{L}_{r,s}$.

We prove the following theorem, which was first conjectured in [12] (cf. [69, Section 11.2]).

Theorem 10.1. *There is a ribbon tensor equivalence* $\mathrm{K} : \operatorname{rep} \mathrm{SL}(2)_q^{\mathrm{rev}} \xrightarrow{\sim} \operatorname{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ *which fits into a* (2-)*diagram*

$$\begin{array}{ccc} \operatorname{rep} \mathrm{SL}(2)_q^{\mathrm{rev}} & \xrightarrow{\quad \mathrm{K} \quad} & \operatorname{rep}(\mathcal{V}ir_c)_{\mathrm{aff}} \\ \Big\downarrow{\scriptstyle \mathrm{res}^{\omega}} & & \Big\downarrow{\scriptstyle \mathcal{W}_p \otimes -} \\ \operatorname{rep}(u_q(\mathfrak{sl}_2))^{\mathrm{rev}} & \xrightarrow{\quad \Theta \quad} & \operatorname{rep}(\mathcal{W}_p) \end{array} \tag{18}$$

It was argued by Bushlanov, Feigin, Gainutdinov, and Tipunin [12] that the fusion rings for $\operatorname{rep} \mathrm{SL}(2)_q$ and $\operatorname{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ are isomorphic. The equivalence of Theorem 10.1 categorifies the isomorphism of fusion rings proposed in [12].

Throughout the section we ignore (notationally) the inversion of the ribbon structure on $\operatorname{rep} \mathrm{SL}(2)_q$, and take for granted that we are considering the category of $\mathrm{SL}(2)_q$-representations along with its reverse braiding and inverted twist, relative to that of Section 3.1.

10.1. Big categories of VOA modules

We begin with an aside on categories of "big" modules for a given VOA. Let $\mathcal{V}$ be a vertex operator algebra with some category $\operatorname{rep}(\mathcal{V})_{\mathrm{dist}}$ of preferred, finite length modules. We will require that the inclusion $\operatorname{rep}(\mathcal{V})_{\mathrm{dist}} \subset \operatorname{rep}(\mathcal{V})$ into the ambient category of finite length $\mathcal{V}$-modules is fully faithful, and that the class of preferred modules is closed under taking subquotients.

Given such $\operatorname{rep}(\mathcal{V})_{\mathrm{dist}}$, we find ourselves in various situations which require a larger category $\operatorname{Rep}(\mathcal{V})_{\mathrm{dist}}$ which contains $\operatorname{rep}(\mathcal{V})_{\mathrm{dist}}$ and in which we can, say, take infinite direct

sums of modules. Formally speaking, we construct such a big category $\mathrm{Rep}(\mathcal{V})_{\mathrm{dist}}$ by taking the Ind-category of $\mathrm{rep}(\mathcal{V})_{\mathrm{dist}}$ [54, Section 6.1]. In our setting, one can understand this category rather concretely ([54, Corollary 6.3.5], [22, Section 4]): We first consider the category $\mathrm{Rep}(\mathcal{V})$ of arbitrary $\mathcal{V}$-modules, with no finiteness assumptions, then take

$$\mathrm{Rep}(\mathcal{V})_{\mathrm{dist}} := \begin{cases} \text{the full subcategory of } \mathcal{V}\text{-modules } M \text{ in } \mathrm{Rep}(\mathcal{V}) \\ \text{for which } M \text{ is the union } \bigcup_\alpha M_\alpha \text{ of its} \\ \text{(finite length) submodules } M_\alpha \subset M \text{ in } \mathrm{rep}(\mathcal{V})_{\mathrm{dist}}. \end{cases}$$

The category $\mathrm{Rep}(\mathcal{V})_{\mathrm{dist}}$ inherits an abelian structure from the inclusion $\mathrm{Rep}(\mathcal{V})_{\mathrm{dist}} \subset \mathrm{Rep}(\mathcal{V})$, and it is closed under taking arbitrary filtered colimits. We refer to the category $\mathrm{Rep}(\mathcal{V})_{\mathrm{dist}}$ simply as the Ind-category of $\mathrm{rep}(\mathcal{V})_{\mathrm{dist}}$, or informally as the "big" category of preferred $\mathcal{V}$-modules.

We note that if $\mathrm{rep}(\mathcal{V})_{\mathrm{dist}}$ admits a (braided) monoidal structure for which the product is right exact and commutes with sums, then $\mathrm{Rep}(\mathcal{V})_{\mathrm{dist}}$ inherits a unique (braided) monoidal structure which extends that of $\mathrm{rep}(\mathcal{V})_{\mathrm{dist}}$ and commutes with colimits. In particular, when $\mathrm{rep}(\mathcal{V})_{\mathrm{dist}}$ admits a vertex tensor structure then the big category $\mathrm{Rep}(\mathcal{V})_{\mathrm{dist}}$ is canonically braided monoidal. One can see [22] for an explicit, vertex-algebraic, analysis of this extended monoidal structure.

10.2. Categories of Virasoro modules

We recall some results from [17, 66]. We adopt a calligraphic notation for Virasoro modules $\mathcal{L}$, to distinguish such objects from quantum group representations, which we generally denote via an italic L.

We recall that there is a unique simple $\mathcal{V}ir_c$-module of conformal weight h for any $h \in \mathbb{C}$. Consider first the category $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{int}}$ of finite length $\mathcal{V}ir_c$-modules with composition factors among the "integral weighted" simples $\mathcal{L}_{r,s}$. The $\mathcal{L}_{r,s}$ are specifically the simple $\mathcal{V}ir_c$-modules of conformal weight $h_{r,s} = \frac{1}{4p}((rp-s)^2 - (p-1)^2)$, where r is a positive integer and $1 \leq s \leq p$. We consider all morphisms, so that $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{int}}$ is a full subcategory in the ambient category of all $\mathcal{V}ir_c$-modules.

It is shown in work of Creutzig, Jiang, Hunziker, Ridout, and Yang [17], and also McRae and Yang [66], that the category $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{int}}$ satisfies the necessary conditions to admit a vertex tensor structure, and that the corresponding braided monoidal category is rigid. This category furthermore admits a ribbon structure provided by the (standard) exponential $\theta = e^{2\pi i L(0)}$. The authors subsequently define the subcategory

$$\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}} := \text{the full subcategory in } \mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{int}} \text{ (tensor) generated by the simple } \mathcal{L}_{1,2}.$$

The subcategory $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ in fact contains *all* of the simples in $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{int}}$, and so is the subcategory in $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{int}}$ generated by all of the simple objects $\mathcal{L}_{r,s}$. One can alternatively define $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ as the Müger centralizer of the triplet algebra $\mathcal{W}_p = \bigoplus_{n\geq 0}(2n-1)\mathcal{L}_{2n+1,1}$ in $\mathrm{rep}(Vir)_{\mathrm{int}}$ [66, Theorems 4.4, 5.2, 5.4]. The category $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ is *affine* in the sense that it admits a tensor generator (cf. [29, Section II.5, Corollaire 5.2]).

Remark 10.2. The categories $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{int}}$ and $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ are denoted by $\mathcal{O}_c$ and $\mathcal{O}_c^0$ in [66], respectively.

By a general theory of VOA extensions [19, 47], which we recall in Appendix A, we have an induction functor

$$F := I_{\mathcal{V}ir_c}^{\mathcal{W}_p} : \mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}} \to \mathrm{Rep}(\mathcal{W}_p),$$

from the affine category of Virasoro modules to the Ind-category $\mathrm{Rep}(\mathcal{W}_p)$ of $\mathcal{W}_p$-modules. McRae and Yang provide the following calculation.

Proposition 10.3 ([66, Proposition 7.4]). *For the simple objects $\mathcal{L}_{r,s}$ in $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ one has $F(\mathcal{L}_{r,s}) = rX_s^{\varepsilon(r)}$ with $\varepsilon(r) = (-1)^{r+1}$.*

The above proposition implies in particular that the induction of each simple object in $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ lies in the usual category $\mathrm{rep}(\mathcal{W}_p)$ of finite length $\mathcal{W}_p$-modules. We apply Lemma A.3 to find the following.

Lemma 10.4. *Induction provides a surjective ribbon tensor functor $F : \mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}} \to \mathrm{rep}(\mathcal{W}_p)$. In particular, F is faithful and exact.*

Proof. Any such induction functor is braided monoidal [57, Theorems 1.6, 1.10], and exactness of F follows from exactness of the tensor product on $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$, or more directly from exactness of the tensor product and Lemma A.3. Faithfulness holds because any tensor functor between tensor categories is faithful [28, Proposition 1.19]. Since $F(\mathcal{L}_{1,2}) = X_2^+$, and since X_2^+ generates the tensor category $\mathrm{rep}(\mathcal{W}_p)$ (Corollary 4.4), we also see that F is surjective. Compatibility with the ribbon structure follows from the fact that $\mathcal{W}_p$ is central in $\mathrm{Rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ and is a fixed point for the twist, so that

$$e^{2\pi i L(0)}|_{F(\mathcal{L})} = \theta_{\mathcal{W}_p,\mathcal{L}} = (\theta_{\mathcal{W}_p} \otimes \theta_{\mathcal{L}}) c^2_{\mathcal{W}_p,\mathcal{L}} = \theta_{\mathcal{L}} = e^{2\pi i L(0)}|_{\mathcal{L}}$$

at arbitrary $\mathcal{L}$ in $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$. ∎

10.3. At the level of Grothendieck rings

Lemma 10.5. *There is a ribbon tensor functor $\mathrm{K} : \mathrm{rep}\,\mathrm{SL}(2)_q \to \mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ which fits into the diagram (18). In terms of Ostrik's theorem (Theorems 5.3 and 9.4), K is specified by the self-dual simple object $\mathcal{L}_{1,2}$ in $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$.*

Proof. We have the self-dual object $\mathcal{L}_{1,2}$ in $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ [66, Theorem 4.1] which is of intrinsic dimension

$$d = d(\mathcal{L}_{1,2}) = d(F(\mathcal{L}_{1,2})) = d(X_2^+) = -(q + q^{-1}).$$

So we obtain, via the universal property of the Temperley–Lieb category, a linear monoidal functor $F_{\mathcal{L}_{1,2}} : \mathcal{TL}(d) \to \mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$. The composition

$$\mathcal{TL}(d) \to \mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}} \xrightarrow{F} \mathrm{rep}(\mathcal{W}_p)$$

sends $\mathcal{L}_{1,2}$ to X_2^+, so that the composite is the universal map specified by the self-dual object X_2^+ in $\mathrm{rep}(\mathcal{W}_p)$. We have already seen that this map to $\mathrm{rep}(\mathcal{W}_p)$ is faithful, via Proposition 7.1, Theorem 5.3, and Proposition 2.1, so that the original map $F_{\mathcal{L}_{1,2}}$ to $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ must be faithful as well.

By Ostrik's theorem we now have a unique extension $\mathrm{K} : \mathrm{rep}\,\mathrm{SL}(2)_q \to \mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ of $F_{\mathcal{L}_{1,2}}$ to the category of all quantum group representations. Furthermore, by Theorem 9.4, and the fact that the induction functor $F : \mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}} \to \mathrm{rep}(\mathcal{W}_p)$ is a faithful ribbon tensor functor, we find that K is a map of ribbon tensor categories. The diagram (18) commutes simply because the functor $\Theta : \mathrm{rep}(u_q(\mathfrak{sl}_2)) \to \mathrm{rep}(\mathcal{W}_p)$ is constructed by de-equivariantizing the universal map $\mathrm{rep}\,\mathrm{SL}(2)_q \to \mathrm{rep}(\mathcal{W}_p)$ specified by the self-dual object X_2^+, which is (up to natural isomorphism) just the composite $F \circ \mathrm{K}$. ∎

From this point on, $\mathrm{K} : \mathrm{rep}\,\mathrm{SL}(2)_q \to \mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ *denotes the braided tensor functor specified by the self-dual object* $\mathcal{L}_{1,2}$ *in* $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$.

We recall that the collection of all simples in $\mathrm{rep}\,\mathrm{SL}(2)_q$ is precisely the collection of products $\{L(\mu) \otimes V_s : \mu \in \mathbb{Z}_{\geq 0}\frac{p\alpha}{2}, 1 \leq s \leq p\}$. The following describes the behavior of the functor K on simple objects.

Lemma 10.6. *Take* $\mu_r = rp\alpha/2$ *for any non-negative integer* r. *For the simple objects in* $\mathrm{rep}\,\mathrm{SL}(2)_q$ *we have* $\mathrm{K}(L(\mu_r) \otimes V_s) \cong \mathcal{L}_{r+1,s}$.

Proof. By the descriptions of the fusion rules for $\mathrm{rep}\,\mathrm{SL}(2)_q$ and $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$, given for example in [70, Section 2.5] and [66, Theorem 4.11], we observe that the lengths of the powers $\mathbb{V}^{\otimes n}$ and $\mathcal{L}_{1,2}^{\otimes n}$ are the same, and that each such power has precisely one simple composition factor which does not appear in a lower tensor power. Specifically, the simples $L(\mu_r) \otimes V_s$ and $\mathcal{L}_{r+1,s}$ appear first in the powers $\mathbb{V}^{\otimes(pr+s)}$ and $\mathcal{L}_{1,2}^{\otimes(pr+s)}$. So we observe by induction, and exactness of K, that the simples in $\mathrm{rep}\,\mathrm{SL}(2)_q$ map to simples in $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$, in the prescribed manner. ∎

The following corollary is immediate.

Corollary 10.7. *The induced map on Grothendieck rings*

$$[\mathrm{K}] : K(\mathrm{rep}\,\mathrm{SL}(2)_q) \to K(\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}})$$

is an isomorphism of $\mathbb{Z}_+$*-rings.*

10.4. Projective representations

Proposition 10.8. *The functor* K *sends indecomposable projectives to indecomposable projectives.*

Proof. Since the induction functor $F : \mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}} \to \mathrm{rep}(\mathcal{W}_p)$ has an exact right adjoint, given by restriction along the algebra inclusion $\mathcal{V}ir_c \to \mathcal{W}_p$, it follows that induction preserves projective objects. The object $\mathbf{1} = \mathcal{L}_{1,1}$ in $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ has a projective cover $\mathcal{Q}_1$

with simple socle and cosocle, both of which are just a copy of $\mathcal{L}_{1,1}$. We have the composition series

$$[\mathcal{Q}_1] = 2[\mathcal{L}_{1,1}] + [\mathcal{L}_{2,p-1}]$$

(see [66, Theorem 5.7]).

Since $F(\mathcal{L}_{2,p-1}) \cong (X_{p-1}^-)^{\oplus 2}$ [66, Proposition 7.4], we see that the induction $F(\mathcal{Q}_1)$ is a length 4 projective in $\mathrm{rep}(\mathcal{W}_p)$ which comes equipped with a surjective map to the unit object. Since the projective cover $\mathscr{P}_1^+$ of the unit in $\mathrm{rep}(\mathcal{W}_p)$ is also of length 4, we have $\mathscr{P}_1^+ \cong F(\mathcal{Q}_1)$.

Let P_1 be the projective cover of the unit $\mathbf{1}$ in $\mathrm{rep}\,\mathrm{SL}(2)_q$. This is just a lift of the corresponding projective cover in $\mathrm{rep}(u_q(\mathfrak{sl}_2))$, and so the image under composition

$$F \circ \mathrm{K}(P_1) \cong \Theta(\mathrm{res}\, P_1) \cong \mathscr{P}_1^+$$

is the projective cover of X_1^+. Now, in $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ projectivity of $\mathcal{Q}_1$ implies the existence of a map $\mathcal{Q}_1 \to \mathrm{K}(P_1)$ which lifts the surjection $\mathrm{K}(P_1) \to \mathrm{K}(\mathbf{1}) = \mathcal{L}_{1,1}$. Applying induction to $\mathrm{rep}(\mathcal{W}_p)$ we find that any such map is an isomorphism. Since induction is exact and faithful, it follows that the lift $\mathcal{Q}_1 \to \mathrm{K}(P_1)$ is an isomorphism in $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$. Hence, K sends the projective cover P_1 of $\mathbf{1}$ to the projective cover $\mathcal{Q}_1$ of the unit in $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$.

One similarly argues that the image $\mathrm{K}(P_s)$ of the projective cover of each simple V_s in $\mathrm{rep}\,\mathrm{SL}(2)_q$ is the projective cover of the corresponding simple $\mathcal{L}_{1,s}$ in $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$, for $1 \leq s < p$. For the projective simples V_p and $\mathcal{L}_{1,p}$, we have already established that $\mathrm{K}(V_p) = \mathcal{L}_{1,p}$ at Lemma 10.6.

Now, for the remaining indecomposable projectives we have $\mathcal{Q}_{r,s} = \mathcal{L}_{r,1} \otimes \mathcal{Q}_{1,s}$ in $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ [66, Theorem 5.9], so that $\mathrm{K}(L(rp\alpha/2) \otimes P_s) \cong \mathcal{Q}_{r,s}$ by the above calculations and Lemma 10.6. We note finally that each product $L(rp\alpha/2) \otimes P_s$ in $\mathrm{rep}\,\mathrm{SL}(2)_q$ is the projective cover of the simple $L(rp\alpha/2) \otimes V_s$ [12, Lemma 4.1] to complete the proof. ∎

As we saw in the proof, Proposition 10.8 implies that K sends the projective cover of a given simple in $\mathrm{rep}\,\mathrm{SL}(2)_q$ to the projective cover of the corresponding simple in $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$.

10.5. The proof of Theorem 10.1

By Proposition 10.8 all projectives in $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ are in the image of K. By considering projective resolutions, we see that K is fully faithful, and also essentially surjective, if and only if its restriction to the additive subcategory of projectives in $\mathrm{rep}\,\mathrm{SL}(2)_q$ is fully faithful. In other words, K is an *equivalence* if and only if its restriction to the subcategory of projectives is fully faithful.

As any tensor functor is faithful [28, Proposition 1.19], we already know that K is faithful. So we need only establish an equality of dimensions,

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}(2)_q}(P, P') \stackrel{?}{=} \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{V}ir_c}(\mathcal{Q}, \mathcal{Q}'), \tag{19}$$

for indecomposable projectives P and P' in $\operatorname{rep}\mathrm{SL}(2)_q$ with images $\mathcal{Q}$ and $\mathcal{Q}'$ in $\operatorname{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$. But now, for the corresponding simple L such that P is the projective cover of L, the dimension of the above Hom space is just the multiplicity of the simple L in the composition series for P',

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathrm{SL}(2)_q}(P, P') = [L, P'].$$

Similarly $\operatorname{Hom}_{\mathcal{V}ir_c}(\mathcal{Q}, \mathcal{Q}') = [\mathcal{L}, \mathcal{Q}']$. Since K induces an isomorphism at the level of Grothendieck rings by Corollary 10.7, we observe that $[L, P] = [\mathcal{L}, \mathcal{Q}]$, as desired. ∎

Remark 10.9. As a consequence of Theorem 10.1 we obtain a calculation of the Müger center for the affine module category:

$$\operatorname{rep}(\mathrm{PSL}(2)) = Z_{\mathrm{Müg}}(\operatorname{rep}\mathrm{SL}(2)_q) \xrightarrow{\sim} Z_{\mathrm{Müg}}(\operatorname{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}). \tag{20}$$

This calculation is also obtainable via the theory of compact group actions on vertex operator algebras provided in [65, Corollary 4.8]. See in particular the proof of [66, Theorem 4.3], where the equivalence (20) is obtained modulo the determination of a certain 3-cocyle. This cocycle can furthermore be shown to vanish since the triplet algebra is a VOA, rather than a super VOA or some more general type of algebra.

11. Categories of modules for the singlet algebra

We describe the (affine) representation category for the singlet vertex operator algebra $\mathcal{M}_p$. As with the Virasoro VOA, the singlet is not C_2-cofinite (hence not strongly finite). The singlet algebra sits between the triplet algebra and the Virasoro $\mathcal{V}ir_c \subset \mathcal{M}_p \subset \mathcal{W}_p$; in particular, $\mathcal{M}_p$ is the invariant subalgebra (orbifold) in $\mathcal{W}_p$ of a maximal torus $\mathbb{C}^\times$ in $\operatorname{Aut}(\mathcal{W}_p) \cong \mathrm{PSL}(2, \mathbb{C})$ (a subgroup isomorphic to S^1 also works, but $\mathbb{C}^\times$ is more natural from our perspective). Going in the other direction, $\mathcal{W}_p$ is obtained from $\mathcal{M}_p$ by extending by an infinite order simple current (invertible simple) which generates $\operatorname{Rep}\mathbb{C}^\times \cong \mathbb{Z}$.

Our presentation is based on work of Creutzig, McRae, and Yang [21], but we provide further elaborations on the behavior of induction, both from the Virasoro to the singlet, and from the singlet to the triplet. We also provide a Tannakian (group-theoretic) description of the Müger center of the affine representation category $\operatorname{rep}(\mathcal{M}_p)_{\mathrm{aff}}$. We discuss subsequent relationships between singlet modules and representations of quantum SL(2) in Section 12.

11.1. Categories of singlet modules

The singlet algebra $\mathcal{M}_p$ has simple modules M_λ labeled by arbitrary functions $\lambda \in \mathfrak{h}^* = \mathbb{C}\alpha$ [1, Section 5]. We are particularly interested in the "integral" (atypical) simples $M_{r,s}$,

which are labeled by an arbitrary integer r and $s \in \{1, \dots, p\}$. Each module $M_{r,s}$ appears as the $\mathcal{M}_p$-socle in the corresponding Fock module $F_{\alpha_{r,s}}$ with Heisenberg weight $\alpha_{r,s} = \frac{1}{2}((1-r) - \frac{1}{p}(1-s))\sqrt{p}\alpha \in \mathfrak{h}$ [25, Section 2].

We consider, first, the category $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{int}}$ of finite length grading-restricted $\mathcal{M}_p$-modules with composition factors among the integral simples $M_{r,s}$. It is shown in [21] that the category $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{int}}$ admits a rigid vertex tensor structure, and we have the tensor subcategory $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$ in $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{int}}$ generated by the simples $\{M_{r,s} : r \in \mathbb{Z}, 1 \leq s \leq p\}$.

Remark 11.1. The categories $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{int}}$ and $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$ above are the categories $\mathcal{C}_{\mathcal{M}(p)}$ and $\mathcal{C}^0_{\mathcal{M}(p)}$ of [21], respectively.

According to the fusion rules of [21, Theorem 3.2.6], the category $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$ is alternatively generated by the distinguished simple $M_{1,2}$. For yet another interpretation, the category $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$ is the centralizer of the algebra object $\mathcal{W}_p = \bigoplus_{n\in\mathbb{Z}} M_{2n+1,1}$ in $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{int}}$ [21, Sections 1.1, 5.1].

Each simple $M_{r,s}$ in $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$ admits a decomposition as a product $M_{r,1} \otimes M_{1,s}$, and the simple modules $M_{r,1}$ are all invertible with tensor inverse $M^*_{r,1} = M_{-r+2,1}$. The simples $\{M_{r,1} : r \in \mathbb{Z}\}$ provide a complete list of invertible objects in $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$, and they satisfy the fusion rule $M_{r,1} \otimes M_{r',1} \cong M_{r+r'-1,1}$ [21, Theorem 5.2.1].

We consider the tensor subcategory $\langle M_{3,1}\rangle$ in $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$ generated by the object $M_{3,1}$, in the sense of Section 2.4. By the above information, the simple objects in $\langle M_{3,1}\rangle$ are precisely those of the form $M_{r,1}$ with r odd.

Lemma 11.2. *There are no extensions* $\mathrm{Ext}^1_{\mathcal{M}_p}(M_{r,1}, M_{r',1}) = 0$ *when r and r' are odd. Furthermore, when $p > 2$, there are no extensions between $M_{r,1}$ and $M_{r',1}$ at arbitrary $r, r' \in \mathbb{Z}$.*

Proof. Suppose $p > 2$. It suffices to show that the extensions $\mathrm{Ext}^1_{\mathcal{M}_p}(\mathbf{1}, M_{r,1})$ vanish, via duality. We have the projective resolution of the unit $P^\bullet = \cdots \to P_{0,p-1} \oplus P_{2,p-1} \to P_1 \to 0$, where the $P_{r,s}$ are the projective covers of the simples $M_{r,s}$, and P_1 is the projective cover of the unit [21, Theorem 5.1.3]. In particular, there are no non-zero maps from the degree -1 portion of the resolution P^{-1} to any $M_{r,1}$. Hence the subquotient $\mathrm{Ext}^1_{\mathcal{M}_p}(\mathbf{1}, M_{r,1})$ of $\mathrm{Hom}_{\mathcal{M}_p}(P^{-1}, M_{r,1})$ vanishes.

At $p = 2$ the above projective resolution of the unit is still valid, but now appears as $\cdots \to P_{0,1} \oplus P_{2,1} \to P_1 \to 0$. So we see that there are no such extensions between $\mathbf{1}$ and $M_{r,1}$ when r is odd. ∎

We note that at $p = 2$ there *are* in fact non-vanishing classes in $\mathrm{Ext}^1_{\mathcal{M}_p}(\mathbf{1}, M_{0,1})$ and $\mathrm{Ext}^1_{\mathcal{M}_p}(\mathbf{1}, M_{2,1})$, provided by the length 2 quotients of the projective cover P_1 of the unit. In any case, the above lemma implies the following.

Corollary 11.3. *The tensor subcategory $\langle M_{3,1}\rangle$ generated by $M_{3,1}$ in $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$ is semi-simple, closed under the formation of extensions in $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$, and has simple objects $\{M_{r,1} : r \text{ odd}\}$.*

11.2. Induction from the Virasoro

As a Virasoro module, the algebra $\mathcal{M}_p$ decomposes as the sum of simples $\bigoplus_{n\geq 0}\mathcal{L}_{2n+1,1}$. Hence $\mathcal{M}_p$ lies in the affine representation category $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$, or rather its Ind-category $\mathrm{Rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ (see Section 10.1). We therefore have the induction functor

$$I = I_{\mathcal{V}ir_c}^{\mathcal{M}_p} : \mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}} \to \mathrm{Rep}(\mathcal{M}_p), \quad \mathcal{L} \mapsto \mathcal{M}_p \otimes \mathcal{L}, \tag{21}$$

which has image in the braided monoidal category of $\mathcal{M}_p$-modules whose restriction to $\mathcal{V}ir_c$ lies in $\mathrm{Rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ [19, Theorem 1.2, Theorem 2.67]. Since $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ is rigid, the functor I is exact (see Appendix A).

Lemma 11.4. *There are isomorphisms*

$$I(\mathcal{L}_{3,1}) \cong M_{3,1} \oplus M_{1,1} \oplus M_{-1,1} \quad \textit{and} \quad I(\mathcal{L}_{1,2}) \cong M_{1,2}.$$

Proof. We have the adjunction

$$\mathrm{Hom}_{\mathcal{M}_p}(I(\mathcal{L}_{3,1}), M_{r,1}) \cong \mathrm{Hom}_{\mathcal{V}ir_c}(\mathcal{L}_{3,1}, M_{r,1})$$

and the Virasoro decompositions $M_{1,1} = \bigoplus_{n\geq 0}\mathcal{L}_{2n+1,1}$ and $M_{3,1} = M_{-1,1} = \bigoplus_{n\geq 0}\mathcal{L}_{2n+3,1}$. In particular, there is a unique-up-to-scaling non-zero map $\mathcal{L}_{3+2n} \to M_{r,1}$ over the Virasoro for $r = 3, 1, -1$. So, by the above adjunction, we have a non-zero map $I(\mathcal{L}_{3,1}) \to M_{r,1}$ for each such r, and this non-zero map must be a surjection since each $M_{r,1}$ is simple. We therefore have the product map $I(\mathcal{L}_{3,1}) \to M_{3,1} \oplus M_{1,1} \oplus M_{-1,1}$, which must also be surjective since each simple in the codomain is distinct.

Note that the above surjection of $\mathcal{M}_p$-modules is also a surjection of modules over the Virasoro VOA. Since the decompositions of $I(\mathcal{L}_{3,1})$ and $M_{3,1} \oplus M_{1,1} \oplus M_{-1,1}$ into simple Virasoro modules agree:

$$I(\mathcal{L}_{3,1}) = \mathcal{M}_p \otimes \mathcal{L}_{3,1} \cong \left(\bigoplus_{n\geq 0}\mathcal{L}_{2n+1,1}\right) \oplus \left(\bigoplus_{n\geq 0} 2\cdot\mathcal{L}_{2n+3,1}\right) = M_{3,1} \oplus M_{1,1} \oplus M_{-1,1},$$

this surjection must be an isomorphism.

Similarly, we consider the decomposition $M_{1,2} = \bigoplus_{n\geq 0}\mathcal{L}_{2n+1,2}$ to see that there is a surjection $I(\mathcal{L}_{1,2}) \to M_{1,2}$. This surjection must again be an isomorphism since these two objects have the same decompositions into simples over the Virasoro:

$$I(\mathcal{L}_{1,2}) = \mathcal{M}_p \otimes \mathcal{L}_{1,2} \cong \bigoplus_{n\geq 0}\mathcal{L}_{2n+1,2} = M_{1,2}. \qquad \blacksquare$$

Of course, the significance of the object $\mathcal{L}_{1,2}$ is that it is the distinguished tensor generator for the affine representation category $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$. Similarly, the object $\mathcal{L}_{3,1}$ generates the Müger center in $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$. This just follows from the fact that it is the image $\mathcal{L}_{3,1} = \mathrm{K}(L(p\alpha))$ of the central generator $L(p\alpha)$ in $\mathrm{rep}(\mathrm{PSL}(2)) \subset \mathrm{rep}\,\mathrm{SL}(2)_q$, under the equivalence of Theorem 10.1. One can see in particular Lemma 10.6. The previous lemma and the aforementioned generating property for $\mathcal{L}_{1,2}$ provide the following.

Corollary 11.5. *The induction functor I has image in the affine subcategory* $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$*, and restricts to a surjective, ribbon, tensor functor*

$$I : \mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}} \to \mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}. \tag{22}$$

Proof. Let $\mathrm{Rep}(\mathcal{M}_p)$ denote the braided monoidal category of $\mathcal{M}_p$-modules which restrict to objects in $\mathrm{Rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$, along the inclusion $\mathcal{V}ir_c \to \mathcal{M}_p$, by an abuse of notation. We note that all of the simples $M_{r,s}$ in $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$ lie in this category $\mathrm{Rep}(\mathcal{M}_p)$ [16, Section 2.3], and that the induction functor I is an *exact* braided monoidal functor, via rigidity of $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$.

We have the (rigid) tensor subcategory $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$ in $\mathrm{Rep}(\mathcal{M}_p)$ which is closed under taking subquotients, and is tensor generated by the simple $M_{1,2}$. Since $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ is tensor generated by the simple $\mathcal{L}_{1,2}$, and $I(\mathcal{L}_{1,2}) = \mathcal{M}_{1,2}$ by Lemma 11.4, we see that I restricts to a surjective, braided, tensor functor as in (22). Compatibility of I with the twist follows by the same arguments as in the proof of Lemma 10.4. ∎

11.3. The Müger center in $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$

As a module over the singlet algebra, we have $\mathcal{W}_p = \bigoplus_{n\in\mathbb{Z}} M_{2n+1,1}$. In particular, $\mathcal{W}_p$ lies in the Ind-category of affine representations for the singlet $\mathcal{M}_p$, and we have the braided monoidal functor $I' : \mathrm{rep}(\mathcal{M}_p) \to \mathrm{Rep}(\mathcal{W}_p)$ provided by induction. The following basic description of induction is deduced from Creutzig, McRae, and Yang [21] (see also Lemma A.3).

Proposition 11.6 ([21, Proposition 3.2.5]). *Induction restricts to a surjective, braided tensor functor*

$$I' : \mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}} \to \mathrm{rep}(\mathcal{W}_p),$$

and furthermore satisfies $I'(M_{r,s}) = X_s^{\epsilon(r)}$ with $\epsilon(r) = (-1)^{r+1}$.

One can use this result to determine the Müger center in $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$.

Lemma 11.7. *The Müger center of* $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$ *is precisely the tensor subcategory $\langle M_{3,1}\rangle$ generated by the invertible simple $M_{3,1}$.*

Proof. Since the induction functor $I : \mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}} \to \mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$ is surjective and braided, it must send the Müger center in $\mathrm{rep}(Vir)_{\mathrm{aff}}$ into the Müger center in $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$. By Lemma 11.4, $M_{3,1}$ is in the surjective image of the Müger center of $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$, and hence the Müger center contains $\langle M_{3,1}\rangle$. By Corollary 11.3, $\langle M_{3,1}\rangle$ is the subcategory of all objects in $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$ with composition factors (only) among the $M_{r,1}$ with r odd. To see that no other simples in $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$ are central, we simply apply the braided monoidal functor $I' : \mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}} \to \mathrm{rep}(\mathcal{W}_p)$ and note that $I'(M_{r,s}) = X_s^{\epsilon(r)}$ centralizes $I'(M_{1,2}) = X_2^+$ if and only if r is odd and $s = 1$, by Theorem 4.7. ∎

We provide a further analysis of the Müger center in $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$, and its behavior under induction both from the Virasoro and to the triplet algebra.

Lemma 11.8. *The induction functor* $I : \mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}} \to \mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$ *restricts to a surjective, symmetric tensor functor*

$$Z_{\mathrm{Müg}}(\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}) \to Z_{\mathrm{Müg}}(\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}).$$

Proof. Surjectivity of I implies that I sends the Müger center in $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ to the Müger center in $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$. So the result follows from the computation $I(\mathcal{L}_{3,1}) = M_{3,1} \oplus M_{1,1} \oplus M_{-1,1}$ of Lemma 11.4, the fact that $M_{3,1}$ generates the Müger center in $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$, and the fact that $\mathcal{L}_{3,1} = \mathrm{K}(L(p\alpha))$ is Müger central in $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$. ∎

Via Theorem 10.1 and Section 3.3, we know that the Müger center in $\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ is equivalent to the representation category of PSL(2). We have a corresponding group-theoretic interpretation of the center in $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$.

Proposition 11.9. *The induction functor* $I' : \mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}} \to \mathrm{rep}(\mathcal{W}_p)$ *restricts to a symmetric fiber functor* $Z_{\mathrm{Müg}}(\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}) \to$ *Vect. Furthermore, there is a symmetric tensor equivalence*

$$\mathrm{Fr}' : \mathrm{rep}(\mathbb{G}_{\mathrm{m}}) \xrightarrow{\sim} Z_{\mathrm{Müg}}(\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}). \tag{23}$$

For the unfamiliar reader, $\mathbb{G}_{\mathrm{m}}$ denotes the multiplicative group $\mathbb{C}^\times$, considered as an affine algebraic group, and the category $\mathrm{rep}(\mathbb{G}_{\mathrm{m}})$ is identified with the symmetric tensor category of $\mathbb{Z}$-graded vector spaces. This is the group $\mathbb{Z}$ of simple currents, with which we extend the VOA $\mathcal{M}_p$ to obtain $\mathcal{W}_p$.

Proof of Proposition 11.9. Take $\mathscr{Z} = Z_{\mathrm{Müg}}(\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}})$. Since I' is a braided, surjective, tensor functor, it sends the Müger center in $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$ to the Müger center in $\mathrm{rep}(\mathcal{W}_p)$. But the Müger center in $\mathrm{rep}(\mathcal{W}_p)$ is trivial by Theorem 4.7, so that I' restricts to a symmetric fiber functor from the Müger center in $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$, as claimed. It follows now, by Tannakian reconstruction [28, Theorem 2.11], that there is an algebraic group $\mathbb{G}$ which admits a symmetric tensor equivalence $\mathrm{rep}(\mathbb{G}) \xrightarrow{\sim} \mathscr{Z}$.

By our understanding of $\mathscr{Z}$ provided by Corollary 11.3 and Lemma 11.7, $\mathbb{G}$ has an invertible representation $\mathbb{C}_\chi$ which generates $\mathrm{rep}(\mathbb{G})$ and admits no extensions from its various tensor powers. This representation therefore specifies a surjective map of algebraic groups

$$\mathbb{G} \to \mathrm{GL}(\mathbb{C}_\chi) = \mathbb{G}_{\mathrm{m}}$$

for which the restriction functor $\mathrm{rep}(\mathbb{G}_{\mathrm{m}}) \to \mathrm{rep}(\mathbb{G}) \cong \mathscr{Z}$ is fully faithful and essentially surjective. This functor is therefore an equivalence, so that $\mathscr{Z} \cong \mathrm{rep}(\mathbb{G}_{\mathrm{m}})$. ∎

Remark 11.10. As was the case with the Virasoro, the calculation of the center (23) can also be deduced from [65].

12. The singlet algebra and torus extended $u_q(\mathfrak{sl}_2)$

In this section we establish a quantum group equivalence for the affine representation category of the singlet vertex operator algebra $\mathcal{M}_p$. We compare the singlet to the "torus

extended" small quantum group $\dot{\mathbf{u}}_q(\mathfrak{sl}_2)$ of [64, Section 36.2.1], at the given parameter $q = \exp(\pi i/p)$. We prove the following analog of Theorems 9.5 and 10.1 for the singlet (see Section 11.1).

Theorem 12.1. *There is an equivalence* $\Psi : \mathrm{rep}(\dot{\mathbf{u}}_q(\mathfrak{sl}_2))^{\mathrm{rev}} \xrightarrow{\sim} \mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$ *of ribbon tensor categories which fits into a* (2-)*diagram*

$$\begin{array}{ccc} \mathrm{rep}(\dot{\mathbf{u}}_q(\mathfrak{sl}_2))^{\mathrm{rev}} & \xrightarrow{\Psi} & \mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}} \\ \Big\downarrow{\scriptstyle \mathrm{res}^{\omega}} & & \Big\downarrow{\scriptstyle \mathcal{W}_p \otimes -} \\ \mathrm{rep}(u_q(\mathfrak{sl}_2))^{\mathrm{rev}} & \xrightarrow{\Theta} & \mathrm{rep}(\mathcal{W}_p) \end{array}$$

The proof of Theorem 12.1 relies in essential ways on notions of equivariantization and de-equivariantization for tensor categories, relative to a given algebraic group action. We recall the relevant constructions in Appendix B.

Remark 12.2. As remarked in the introduction, a quantum group equivalence for the singlet was conjectured in [14, 24]. The works [14, 24] conjecture, specifically, an equivalence between representations of the so-called unrolled quantum group $\mathbf{u}_q^H(\mathfrak{sl}_2)$ and a certain extension $\mathrm{rep}_{\langle s \rangle}(\mathcal{M}_p)$ [23] of $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$ by the category of $\mathbb{C}/\mathbb{Z}$-graded vector spaces. The category of $\mathbf{u}_q^H(\mathfrak{sl}_2)$-representations is just the category of $\mathfrak{h}^*$-graded representations of $u_q(\mathfrak{sl}_2)$, as opposed to those graded by the character lattice $\Lambda \subset \mathfrak{h}^*$. So the above theorem differs, to some degree, from the precise conjectures of [14, 24].

12.1. The category of $\dot{\mathbf{u}}_q(\mathfrak{sl}_2)$*-representations*

We understand the algebra $\dot{\mathbf{u}}_q(\mathfrak{sl}_2)$ directly through its representations. A representation of $\dot{\mathbf{u}}_q(\mathfrak{sl}_2)$ is a finite-dimensional $\Lambda = \frac{1}{2}\mathbb{Z}\alpha$-graded vector space V which comes equipped with p-nilpotent linear endomorphisms $E, F : V \to V$ which (a) shift the degree by α and $-\alpha$, respectively, and (b) satisfy the quantum group relations of [63]. So, the construction of $\mathrm{rep}(\dot{\mathbf{u}}_q(\mathfrak{sl}_2))$ is completely analogous to the construction of $\mathrm{rep}\,\mathrm{SL}(2)_q$ given in Section 3.1, where we simply forget about the additional operators $E^{(p)}$ and $F^{(p)}$.

The expected coproduct

$$\Delta(E) = E \otimes 1 + K \otimes E \quad \text{and} \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F$$

provides the category $\mathrm{rep}(\dot{\mathbf{u}}_q(\mathfrak{sl}_2))$ with a rigid tensor structure. Furthermore, the R-matrix and twist from Section 3.1 define a (unique) ribbon structure on the category $\mathrm{rep}(\dot{\mathbf{u}}_q(\mathfrak{sl}_2))$ so that the forgetful functor $\mathrm{rep}\,\mathrm{SL}(2)_q \to \mathrm{rep}(\dot{\mathbf{u}}_q(\mathfrak{sl}_2))$ is a ribbon tensor functor. This is the *standard* ribbon structure on the category of $\dot{\mathbf{u}}_q(\mathfrak{sl}_2)$-representations, and in the statement of Theorem 12.1 we consider $\mathrm{rep}(\dot{\mathbf{u}}_q(\mathfrak{sl}_2))$ with its reversed braiding and inverted twist, relative to this standard structure.

As in the previous sections, we omit the superscript $(-)^{\mathrm{rev}}$ from our analysis, and take for granted that we are considering the category of $\dot{\mathbf{u}}_q(\mathfrak{sl}_2)$-representations with the reversed braiding and inverted twist.

12.2. Triplet modules via (de-)equivariantization

We let $\mathrm{Rep}(\mathcal{M}_p)_{\mathrm{aff}}$ denote the Ind-category of $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$, which we define as in Section 10.1.

The equivalence Fr' from $\mathrm{rep}(\mathbb{G}_{\mathrm{m}})$ to the Müger center of $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$ sends the algebra object $\mathscr{O}(\mathbb{G}_{\mathrm{m}})$ to a commutative algebra object A in the Müger center of $\mathrm{Rep}(\mathcal{M}_p)_{\mathrm{aff}}$ which, as an $\mathcal{M}_p$-module, is of the form

$$A = \bigoplus_{n\in\mathbb{Z}} M_{2n+1,1} \cong \mathcal{W}_p. \tag{24}$$

We have the following little lemma.

Lemma 12.3. *There is a unique commutative algebra structure on the $\mathcal{M}_p$-module $\bigoplus_{n\in\mathbb{Z}} M_{2n+1,1}$ under which it becomes a simple module over itself, i.e. has no non-trivial ideals.*

Proof. Via the equivalence of Proposition 11.9, it suffices to show that the vector space $\mathscr{O} = \mathbb{C}[t, t^{-1}] = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}t^n$ admits a unique commutative, graded algebra structure for which $\deg(t^n) = n$ and for which $\mathscr{O}$ admits no graded ideals. Under any such algebra structure on $\mathscr{O}$, the grading forces t to act as a map $t\cdot -|_{\mathscr{O}_n} : \mathscr{O}_n \to \mathscr{O}_{n+1}$ on each homogeneous degree. Now, the image of the action map $t\cdot - : \mathscr{O} \to \mathscr{O}$ is a graded ideal which contains t, and so must be all of $\mathscr{O}$ by simplicity. Since each graded component $\mathscr{O}_n$ is of dimension 1, it follows that $t\cdot -$ must be an isomorphism. After rescaling the basis elements t^n if necessary, we then have $t\cdot t^n = t^{n+1}$, and in particular $t\cdot t^{-1} = 1$. Associativity forces the general calculation $t^n \cdot t^m = t^{n+m}$. So $\mathscr{O}$ is identified with the expected localization of the polynomial ring, $\mathscr{O} \cong \mathscr{O}(\mathbb{G}_{\mathrm{m}})$, as a graded algebra. ∎

Since both $\mathscr{O}(\mathbb{G}_{\mathrm{m}})$ and $\mathcal{W}_p$ are simple modules over themselves, and the functor Fr' is an equivalence, the above lemma tells us that the identification of $\mathcal{M}_p$-modules (24) implies an identification of algebras $A = \mathrm{Fr}'\,\mathscr{O}(\mathbb{G}_{\mathrm{m}}) \cong \mathcal{W}_p$ in $\mathrm{Rep}(\mathcal{M}_p)_{\mathrm{aff}}$.

The following is obtained almost immediately from [19, Theorem 3.65].

Lemma 12.4. *Restriction $\mathrm{rep}(\mathcal{W}_p) \to \mathrm{Rep}(\mathcal{M}_p)_{\mathrm{aff}}$, and any choice of algebra isomorphism $\mathcal{W}_p \cong A$, identifies $\mathrm{rep}(\mathcal{W}_p)$ with the (braided tensor) category of finitely generated A-modules in $\mathrm{Rep}(\mathcal{M}_p)_{\mathrm{aff}}$. Furthermore, any finitely generated A-module is finitely presented.*

To be clear, by a *finitely generated* A-module X we mean one which admits an A-module surjection $A\otimes V \to X$ from a free module, with V in $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$. By a *finitely presented* module we mean one which admits an exact sequence $A\otimes W \to A\otimes V \to X \to 0$, where W and V are in $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$.

Proof of Lemma 12.4. We understand from [19, Theorem 3.65] that the forgetful functor $\mathrm{Rep}(\mathcal{W}_p)_{\mathrm{aff}} \to \mathrm{Rep}(\mathcal{M}_p)_{\mathrm{aff}}$ provides an identification between $\mathrm{Rep}(\mathcal{W}_p)_{\mathrm{aff}}$ and the category of (arbitrary) $\mathcal{W}_p$-modules in $\mathrm{Rep}(\mathcal{M}_p)_{\mathrm{aff}}$, as braided monoidal categories. By restricting along any algebra isomorphism $A \xrightarrow{\cong} \mathcal{W}_p$ we then get an identification between

$\mathrm{Rep}(\mathcal{W}_p)_{\mathrm{aff}}$ and the category of arbitrary A-modules in $\mathrm{Rep}(\mathcal{M}_p)_{\mathrm{aff}}$. So the claim here is that the *finite length* property for objects in $\mathrm{Rep}(\mathcal{W}_p)_{\mathrm{aff}}$ corresponds to the *finitely presented* property for $\mathcal{W}_p$-modules in $\mathrm{Rep}(\mathcal{M}_p)_{\mathrm{aff}}$.

Now, Lemma 11.6 implies that any free module $I'(V) = \mathcal{W}_p \otimes V$ is, in particular, a finite length $\mathcal{W}_p$-module in $\mathrm{Rep}(\mathcal{M}_p)_{\mathrm{aff}}$. Hence any quotient $\mathcal{W}_p \otimes V \to X$ of a free module is a finite length $\mathcal{W}_p$-module. So we see that any finitely generated $\mathcal{W}_p$-module is of finite length, and so lies in $\mathrm{rep}(\mathcal{W}_p)$. Conversely, since the category $\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}$ has enough projectives, surjectivity of the induction functor I' (Lemma 11.6) is equivalent to the claim that any finite length $\mathcal{W}_p$-module X admits a surjection from a free module $\mathcal{W}_p \otimes V \to X$, and so is finitely generated. Finally, since the kernel of such a surjection $\mathcal{W}_p \otimes V \to X$ is also of finite length, we observe that any finitely generated module is also finitely presented. This shows that $\mathrm{rep}(\mathcal{W}_p)$ is identified with the category of finitely presented $\mathcal{W}_p$-modules in $\mathrm{Rep}(\mathcal{M}_p)_{\mathrm{aff}}$ under restriction, as claimed. ∎

In the language of Section 3.5 and Appendix B.4, we have an identification of braided tensor categories

$$\mathrm{rep}(\mathcal{W}_p) = (\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}})_{\mathbb{G}_{\mathrm{m}}} := \left\{ \begin{array}{c} \text{finitely presented } A = \mathrm{Fr}'\,\mathscr{O}(\mathbb{G}_{\mathrm{m}}) \\ \text{modules in } \mathrm{Rep}(\mathcal{M}_p)_{\mathrm{aff}} \end{array} \right\}$$

between finite length $\mathcal{W}_p$-modules and the de-equivariantization of $\mathrm{rep}(\mathcal{M}_p)$ along the central functor $\mathrm{rep}(\mathbb{G}_{\mathrm{m}}) \to \mathrm{rep}(\mathcal{M}_p)$. By a general result [9, 69], it follows that there is a categorical action of $\mathbb{G}_{\mathrm{m}}$ on $\mathrm{rep}(\mathcal{W}_p)$ for which we have an equivalence

$$\mathrm{rep}(\mathcal{W}_p)^{\mathbb{G}_{\mathrm{m}}} \cong \mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}} \tag{25}$$

between the category of $\mathbb{G}_{\mathrm{m}}$-equivariant objects in $\mathrm{rep}(\mathcal{W}_p)$ and the affine representation category of $\mathcal{M}_p$ [69, Proposition A.2].

Remark 12.5. The equivalence (25) was first observed in [66, Section 7.2], where the authors proceed via the theory of Lie group actions on vertex operator algebras.

Let us explain the equivalence (25) in more tangible terms. As explained above, the restriction functor $\mathrm{rep}(\mathcal{W}_p) \to \mathrm{Rep}(\mathcal{M}_p)_{\mathrm{aff}}$ identifies finite length modules over the triplet algebra with finitely generated A-modules in $\mathrm{Rep}(\mathcal{M}_p)_{\mathrm{aff}}$. The algebra A can be written as $\mathcal{W}_p \cong A = \mathcal{M}_p[t, t^{-1}]$ where t is the invertible object $M_{3,1}$ and $t^{-1} = M_{3,1}^* = M_{-1,1}$. Furthermore, the translation action of $\mathbb{G}_{\mathrm{m}}$ on $A = \mathrm{Fr}'\,\mathscr{O}(\mathbb{G}_{\mathrm{m}})$ corresponds precisely to the $\mathbb{Z}$-grading on A – and also $\mathcal{W}_p$ – specified by taking $\deg(t) = 1$, $\deg(t^{-1}) = -1$.

With the above framing in mind, a $\mathbb{G}_{\mathrm{m}}$-equivariant object in $\mathrm{rep}(\mathcal{W}_p)$ is just a $\mathcal{W}_p$-module X in $\mathrm{Rep}(\mathcal{M}_p)_{\mathrm{aff}}$ with a compatible $\mathbb{Z}$-grading so that $t^{\pm 1} \cdot X_k = X_{k \pm 1}$. We are claiming at (25) that the map

$$\mathrm{rep}(\mathcal{W}_p)^{\mathbb{G}_{\mathrm{m}}} = \mathcal{W}_p\text{-mod}^{\mathbb{Z}} \to \mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}, \quad X \mapsto X_0, \tag{26}$$

is an equivalence of categories, and has inverse provided by induction. One can see directly (or abstractly as above) that this is indeed the case. The functor (26) is furthermore an equivalence of ribbon tensor categories, as its inverse (induction) is compatible with the ribbon tensor structure.

12.3. Principles for Theorem 12.1

As recalled in Appendix B, the equivalence $(\operatorname{rep} \mathrm{SL}(2)_q)_{\mathrm{PSL}(2)} \cong \operatorname{rep}(u_q(\mathfrak{sl}_2))$ of Theorem 3.4 induces a PSL(2)-action on the category of $u_q(\mathfrak{sl}_2)$-representations. We have the standard torus $\mathbb{G}_{\mathrm{m}} \cong T \subset \mathrm{PSL}(2)$ of diagonal matrices, and we can restrict the action of PSL(2) to an action of the torus on $\operatorname{rep}(u_q(\mathfrak{sl}_2))$.

Given this torus action on $\operatorname{rep}(u_q(\mathfrak{sl}_2))$, we can consider the non-full tensor subcategory $\operatorname{rep}(u_q(\mathfrak{sl}_2))^T \subset \operatorname{rep}(u_q(\mathfrak{sl}_2))$ of T-equivariant representations. One should view objects in $\operatorname{rep}(u_q(\mathfrak{sl}_2))^T$ as $u_q(\mathfrak{sl}_2)$-representations equipped with a compatible rational T-action, although this is an oversimplification. This subcategory is identified with the category of $\dot{\mathbf{u}}_q(\mathfrak{sl}_2)$-representations via restriction.

Proposition 12.6 ([69, Proposition 9.1], [9]). *There is an equivalence* $\operatorname{rep}(u_q(\mathfrak{sl}_2))^T \cong \operatorname{rep}(\dot{\mathbf{u}}_q(\mathfrak{sl}_2))$ *of ribbon tensor categories.*

We consider the above theorem in parallel with the equivalence (26) for the singlet. We claim at this point that (after some error correction if necessary) the equivalence $\Theta : \operatorname{rep}(u_q(\mathfrak{sl}_2)) \xrightarrow{\sim} \operatorname{rep}(\mathcal{W}_p)$ of Theorem 9.5 commutes with the $\mathbb{G}_{\mathrm{m}}$-actions on these categories, where we act on $\operatorname{rep}(u_q(\mathfrak{sl}_2))$ via the torus $\mathbb{G}_{\mathrm{m}} \cong T$ and we act on $\operatorname{rep}(\mathcal{W}_p)$ in the manner prescribed in Section 12.2. One then obtains an induced equivalence on the subcategories of $\mathbb{G}_{\mathrm{m}}$-equivariant objects

$$\operatorname{rep}(\dot{\mathbf{u}}_q(\mathfrak{sl}_2)) \cong \operatorname{rep}(u_q(\mathfrak{sl}_2))^T \xrightarrow[\Psi := \Theta^{\mathbb{G}_{\mathrm{m}}}]{\sim} \operatorname{rep}(\mathcal{W}_p)^{\mathbb{G}_{\mathrm{m}}} \cong \operatorname{rep}(\mathcal{M}_p),$$

which provides the claimed result.

12.4. The proof of Theorem 12.1

We use the calculus of equivariantization and de-equivariantization in the proof. One can see the original texts [9, 26, 69], and/or Appendix B for a review of the topic. (See also [32].) When needed, we reference specific results from the appendix in the proof of Theorem 12.1.

Proof of Theorem 12.1. We have the (2-)diagram of braided tensor functors

$$\begin{array}{ccccc}
\operatorname{rep} \mathrm{SL}(2)_q \cong \operatorname{rep}(\mathcal{V}ir_c)_{\mathrm{aff}} & \xrightarrow{I} & \operatorname{rep}(\mathcal{M}_p)_{\mathrm{aff}} & \xrightarrow{I'} & \operatorname{rep}(\mathcal{W}_p) \\
\uparrow{\scriptstyle \mathrm{Fr}} & & \uparrow{\scriptstyle \mathrm{Fr}'} & & \uparrow{\scriptstyle \mathrm{unit}} \\
\operatorname{rep}(\mathrm{PSL}(2)) & \xrightarrow[\text{surject}]{\text{Lem. 11.8}} & \operatorname{rep}(\mathbb{G}_{\mathrm{m}}) & \xrightarrow[\text{fiber}]{\text{Prop. 11.9}} & \mathit{Vect}
\end{array}$$

where I and I' are the appropriate induction functors. By Tannakian reconstruction, and surjectivity of the map

$$I|_{\operatorname{rep}(\mathrm{PSL}(2))} : \operatorname{rep}(\mathrm{PSL}(2)) \to \operatorname{rep}(\mathbb{G}_{\mathrm{m}}),$$

we find that $I|_{\mathrm{rep}(\mathrm{PSL}(2))}$ is isomorphic to restriction $\mathrm{res}_\alpha : \mathrm{rep}(\mathrm{PSL}(2)) \to \mathrm{rep}(\mathbb{G}_\mathrm{m})$ along a group embedding $\alpha : \mathbb{G}_\mathrm{m} \to \mathrm{PSL}(2)$ ([74, Theorem 2.3.2], [28, Proposition 2.21]). So the above diagram can be replaced with a (2-)diagram

$$\begin{array}{ccccc} \mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}} & \xrightarrow{I} & \mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}} & \xrightarrow{I'} & \mathrm{rep}(\mathcal{W}_p) \\ \uparrow{\scriptstyle \mathrm{Fr}} & & \uparrow{\scriptstyle \mathrm{Fr}'} & & \uparrow{\scriptstyle \mathrm{unit}} \\ \mathrm{rep}(\mathrm{PSL}(2)) & \xrightarrow{\mathrm{res}_\alpha} & \mathrm{rep}(\mathbb{G}_\mathrm{m}) & \xrightarrow{\mathrm{fiber}} & \mathit{Vect} \end{array} \tag{27}$$

Furthermore, the induction functors provide equivalences

$$(\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}})_{\mathrm{PSL}(2)} \cong \mathrm{rep}(\mathcal{W}_p) \quad \text{and} \quad (\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}})_{\mathbb{G}_\mathrm{m}} \cong \mathrm{rep}(\mathcal{W}_p),$$

by Theorems 9.5 and 10.1, and Lemma 12.4 (see also Lemma B.6). These equivalences provide rational actions of PSL(2) and $\mathbb{G}_\mathrm{m}$ on $\mathrm{rep}(\mathcal{W}_p)$ ([9], [69, Section A.1]), respectively, and there is a unique equivalence of ribbon tensor categories

$$f : (\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}})_{\mathrm{PSL}(2)} \xrightarrow{\sim} (\mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}})_{\mathbb{G}_\mathrm{m}}$$

which fits into a diagram over $\mathrm{rep}(\mathcal{W}_p)$. The above diagram (27) implies that f is $\mathbb{G}_\mathrm{m}$-equivariant (Lemma B.9), where $\mathbb{G}_\mathrm{m}$ acts on $(\mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}})_{\mathrm{PSL}(2)}$ by restricting the natural PSL(2)-action along the group embedding $\alpha : \mathbb{G}_\mathrm{m} \to \mathrm{PSL}(2)$. That is, the $\mathbb{G}_\mathrm{m}$-action on $\mathrm{rep}(\mathcal{W}_p)$ deduced from singlet de-equivariantization agrees with the $\mathbb{G}_\mathrm{m}$-action deduced from Virasoro de-equivariantization and the map α.

Via the diagram of Theorem 10.1, we understand that the equivalence

$$\Theta : \mathrm{rep}(u_q(\mathfrak{sl}_2)) \to \mathrm{rep}(\mathcal{W}_p)$$

of Theorem 9.5 is PSL(2)-equivariant (Lemma B.5), where PSL(2) acts on $\mathrm{rep}(\mathcal{W}_p)$ as above. We can therefore restrict these PSL(2)-actions along $\alpha : \mathbb{G}_\mathrm{m} \to \mathrm{PSL}(2)$, and equivariantize, to obtain an equivalence of braided monoidal categories

$$\Theta^{\mathbb{G}_\mathrm{m}} : \mathrm{rep}(u_q(\mathfrak{sl}_2))^{\mathbb{G}_\mathrm{m}} \xrightarrow{\sim} \mathrm{rep}(\mathcal{W}_p)^{\mathbb{G}_\mathrm{m}} \tag{28}$$

(Lemma B.1).

Now, the map $\alpha : \mathbb{G}_\mathrm{m} \to \mathrm{PSL}(2)$ identifies $\mathbb{G}_\mathrm{m}$ with a maximal torus in PSL(2), and all maximal tori are conjugate. So there is an element (closed point) $x \in \mathrm{PSL}(2)$ under which conjugation

$$\mathrm{Ad}_x : \mathrm{PSL}(2) \to \mathrm{PSL}(2)$$

sends $\alpha(\mathbb{G}_\mathrm{m})$ to the standard torus $T = \{\mathrm{diag}\{a, a^{-1}\} : a \in \mathbb{C}^\times\} \subset \mathrm{PSL}(2)$. It follows that the action of x on $\mathrm{rep}(u_q(\mathfrak{sl}_2))$ provides a braided monoidal equivalence

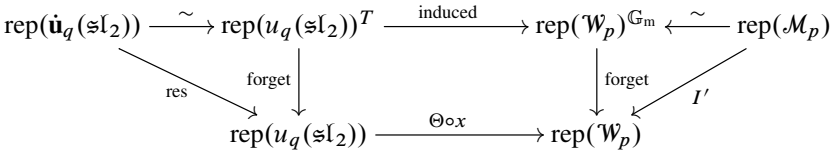
$$x \cdot - : \mathrm{rep}(u_q(\mathfrak{sl}_2))^T \xrightarrow{\sim} \mathrm{rep}(u_q(\mathfrak{sl}_2))^{\mathbb{G}_\mathrm{m}},$$

and we find from (28) an equivalence $\mathrm{rep}(u_q(\mathfrak{sl}_2))^T \cong \mathrm{rep}(\mathcal{W}_p)^{\mathbb{G}_\mathrm{m}}$. We recall that the equivariantization of $\mathrm{rep}(u_q(\mathfrak{sl}_2))$ by the torus of diagonal matrices is the representation

category $\mathrm{rep}(\dot{\mathbf{u}}_q(\mathfrak{sl}_2))$ [69, Proposition 9.1] to observe finally

$$\mathrm{rep}(\dot{\mathbf{u}}_q(\mathfrak{sl}_2)) \cong \mathrm{rep}(u_q(\mathfrak{sl}_2))^T \cong \mathrm{rep}(\mathcal{W}_p)^{\mathbb{G}_m} \cong \mathrm{rep}(\mathcal{M}_p)_{\mathrm{aff}}.$$

Compatibility with the ribbon structure follows from the diagram



and the fact that all of the functors present, save for possibly the induced map in question, are (faithful) ribbon tensor functors. ∎

Remark 12.7. In the proof we have employed a PSL(2)-action on $\mathrm{rep}(\mathcal{W}_p)$ deduced from the central embedding $\mathrm{rep}(\mathrm{PSL}(2)) \to \mathrm{rep}(\mathcal{V}ir_c)_{\mathrm{aff}}$ and the induction functor (see Remark 10.9). We do not claim that this PSL(2)-action is identified with the categorical action induced by the known PSL(2)-action on $\mathcal{W}_p$ by VOA automorphisms ([38], [3, Section 7], [2, Section 2]). However, we certainly expect that this is the case.

Appendix A. Induction for VOA extensions

We recall some information regarding extensions of vertex operator algebras and induction. The original references for the following material are [19, 47, 57].

A.1. Vertex algebra extensions and induction

Let $\mathcal{V}$ be a vertex operator algebra with $\mathrm{rep}(\mathcal{V})_{\mathrm{dist}}$ a full subcategory of distinguished objects in the category of finite length $\mathcal{V}$-modules. We suppose additionally that the subcategory $\mathrm{rep}(\mathcal{V})_{\mathrm{dist}}$ is closed under taking subquotients and admits a vertex tensor structure as described in Section 2.6. We let $\mathrm{Rep}(\mathcal{V})_{\mathrm{dist}}$ denote the associated Ind-category of distinguished modules, which one can describe as in Section 10.1. In this setting, the category $\mathrm{Rep}(\mathcal{V})_{\mathrm{dist}}$ inherits a unique braided monoidal structure for which the product $\otimes$ commutes with colimits. Furthermore, this monoidal structure is specified by intertwining operators [22, Section 6].

Consider $\mathrm{rep}(\mathcal{V})_{\mathrm{dist}}$ as above, and suppose we have a vertex operator algebra extension $\mathcal{V} \to \mathcal{W}$ with $\mathcal{W}$ lying in $\mathrm{Rep}(\mathcal{V})_{\mathrm{dist}}$, as a $\mathcal{V}$-module. Then the vertex algebra structure on $\mathcal{W}$ gives it the structure of a commutative algebra object in $\mathrm{Rep}(\mathcal{V})_{\mathrm{dist}}$ ([47, Theorem 3.2], [22, Theorem 7.5]). In such a setting we let $\mathrm{Rep}(\mathcal{W})_{\mathrm{dist}}$ denote the abelian category of $\mathcal{W}$-modules which restrict to $\mathcal{V}$-modules in $\mathrm{Rep}(\mathcal{V})_{\mathrm{dist}}$ along the extension $\mathcal{V} \to \mathcal{W}$. We let $\mathrm{rep}(\mathcal{W})_{\mathrm{dist}}$ denote the full, exact subcategory of finite length $\mathcal{W}$-modules in $\mathrm{Rep}(\mathcal{W})_{\mathrm{dist}}$.

Consider now a general commutative algebra object A in $\mathrm{Rep}(\mathcal{V})_{\mathrm{dist}}$. Recall that an A-module M is called *local* if the action map $\mathrm{act} : A \otimes M \to M$ is such that

act $\circ\, c_{A,M}^2 = \mathrm{act}$, where $c_{A,M}$ is the braiding for $\mathrm{Rep}(\mathcal{V})_{\mathrm{dist}}$. In this way any local A-module admits an unambiguous A-bimodule structure, and the category of local A-modules inherits a braided monoidal structure under the product $\otimes_A$ [71, Theorem 2.5] (see also [57]). We let $\mathrm{Rep}^0(A)$ denote the category of all local A-modules in $\mathrm{Rep}(\mathcal{V})_{\mathrm{dist}}$, and let $\mathrm{rep}^0(A)$ denote the exact subcategory of finite length modules.

Remark A.1. We note that objects in the categories $\mathrm{Rep}(\mathcal{W})_{\mathrm{dist}}$ and $\mathrm{Rep}^0(A)$ are not, a priori, unions of their finite length subobjects. So our use of "Rep" here deviates slightly from its earlier uses in Section 10.1. However, in all cases which are of interest to us, all objects in $\mathrm{Rep}(\mathcal{W})_{\mathrm{dist}}$ and $\mathrm{Rep}^0(A)_{\mathrm{dist}}$ will in fact be unions of their finite length submodules, and these categories will be identified with the Ind-categories of their small counterparts (cf. Lemma A.3 below).

We have the following essential results of Creutzig, Kanade, McRae, and Yang, and also Huang, Kirillov, and Lepowsky.

Theorem A.2 ([47, Theorem 3.4], [19, Theorem 3.65]). *Consider* $\mathrm{rep}(\mathcal{V})_{\mathrm{dist}}$ *as above, and an extension of vertex operator algebras* $\iota : \mathcal{V} \to \mathcal{W}$ *with* $\mathcal{W}$ *lying in* $\mathrm{Rep}(\mathcal{V})_{\mathrm{dist}}$. *Let* A *denote* $\mathcal{W}$, *considered as an algebra object in* $\mathrm{Rep}(\mathcal{V})_{\mathrm{dist}}$. *Then the category* $\mathrm{Rep}(\mathcal{W})_{\mathrm{dist}}$ *admits a vertex tensor structure, and restriction along* ι *provides an identification of braided monoidal categories*

$$\mathrm{Rep}(\mathcal{W})_{\mathrm{dist}} = \mathrm{Rep}^0(A).$$

Having fixed a commutative algebra object A in $\mathrm{Rep}(\mathcal{V})_{\mathrm{dist}}$, we let $\mathrm{Rep}^0(\mathcal{V})_{\mathrm{dist}}$ denote the Müger centralizer of A in $\mathrm{Rep}(\mathcal{V})_{\mathrm{dist}}$. We then have the free module functor $A \otimes - : \mathrm{Rep}^0(\mathcal{V})_{\mathrm{dist}} \to \mathrm{Rep}^0(A)$. This functor is braided monoidal, and is left adjoint to the restriction functor $\mathrm{Rep}^0(A) \to \mathrm{Rep}(\mathcal{V})_{\mathrm{dist}}$ [57, Theorems 1.6]. Taking this fact and Theorem A.2 into account, we observe a braided monoidal functor

$$I_{\mathcal{V}}^{\mathcal{W}} = \mathcal{W} \otimes - : \mathrm{Rep}^0(\mathcal{V})_{\mathrm{dist}} \to \mathrm{Rep}(\mathcal{W})_{\mathrm{dist}} \tag{29}$$

to the category of $\mathcal{W}$-modules, for any VOA extension $\mathcal{V} \to \mathcal{W}$ with $\mathcal{W}$ in $\mathrm{Rep}(\mathcal{V})_{\mathrm{dist}}$.

Note that when the category $\mathrm{rep}(\mathcal{V})_{\mathrm{dist}}$ is rigid, the tensor product $\otimes$ is necessarily biexact and commutes with colimits [10, Proposition 2.1.8]. It follows that the induction functor $I_{\mathcal{V}}^{\mathcal{W}}$ is exact in this case.

Lemma A.3. *Consider* $\mathcal{V}$ *and* $\mathcal{W}$ *as above. Suppose that* $\mathcal{W}$ *lies in the Müger center of* $\mathrm{Rep}(\mathcal{V})_{\mathrm{dist}}$, *the category* $\mathrm{rep}(\mathcal{V})_{\mathrm{dist}}$ *is rigid, and the free modules* $\mathcal{W} \otimes L$ *are of finite length over* $\mathcal{W}$ *for each simple module* L *in* $\mathrm{rep}(\mathcal{V})_{\mathrm{dist}}$. *Then*

(1) *all modules in* $\mathrm{Rep}(\mathcal{W})_{\mathrm{dist}}$ *are unions of their finite length submodules;*

(2) *induction restricts to an exact braided monoidal functor*

$$I_{\mathcal{V}}^{\mathcal{W}} : \mathrm{rep}(\mathcal{V})_{\mathrm{dist}} \to \mathrm{rep}(\mathcal{W})_{\mathrm{dist}}, \tag{30}$$

which is left adjoint to restriction.

Proof. We first prove (2) and then return to (1). Given that $I_{\mathcal{V}}^{\mathcal{W}}$ is exact, and $I_{\mathcal{V}}^{\mathcal{W}}(L)$ is of finite length for each simple L, we see by induction on the lengths of objects that $I_{\mathcal{V}}^{\mathcal{W}}$ restricts to a functor (30) between the subcategories of finite length objects.

(1) Any module M in $\mathrm{Rep}(\mathcal{W})_{\mathrm{dist}}$ admits a surjection from its corresponding free module $I_{\mathcal{V}}^{\mathcal{W}}(M) = \mathcal{W} \otimes M \to M$. Here $\otimes$ denotes the product on $\mathrm{Rep}(\mathcal{V})_{\mathrm{dist}}$ and the aforementioned surjection is the action map for M. It therefore suffices to show that any free module $\mathcal{W} \otimes M$, with M in $\mathrm{Rep}(\mathcal{V})_{\mathrm{dist}}$, is a colimit of finite length modules over $\mathcal{W}$. We therefore write M as a colimit $M = \varinjlim_{\alpha} M_\alpha$ of finite length modules over $\mathcal{V}$ and consider the formula

$$\mathcal{W} \otimes M = \mathcal{W} \otimes \varinjlim_{\alpha} M_\alpha \cong \varinjlim_{\alpha} (\mathcal{W} \otimes M_\alpha) = \varinjlim_{\alpha} I_{\mathcal{V}}^{\mathcal{W}}(M_\alpha)$$

to observe that $\mathcal{W} \otimes M$ is in fact a colimit of finite length $\mathcal{W}$-modules. ∎

This should be contrasted with the more familiar case where both $\mathcal{V}$ and $\mathcal{W}$ are strongly finite. In this case, $\mathcal{W}$ will not lie in the Müger center in $\mathrm{Rep}(\mathcal{V})$ (unless $\mathcal{W} = \mathcal{V}$), and induction lands in $\mathrm{Rep}(A)$ rather than in its subcategory $\mathrm{Rep}^0(A)$.

Appendix B. Rational (de-)equivariantization, again

We elaborate on the presentation of [69, Appendix A], which itself is an elaboration on the presentations of [9, 26], in order to clarify some of the mechanics involved in the proof of Theorem 12.1. We are generally concerned with equivariantization and de-equivariantization operations under algebraic group actions, and the stability of these operations under isomorphisms and restriction. For the VOA theorist, equivariantization and de-equivariantization are the two steps which, when combined, capture tensor-categorically the orbifold construction of VOAs.

B.1. Equivariantization and equivariant tensor functors

Throughout this section we understand that a linear category is, by definition if one likes, an additive category with a specified action of *Vect*. Here *Vect* denotes the symmetric monoidal category of finite-dimensional vector spaces, and Ind *Vect* denotes its Ind-category of generally infinite-dimensional vector spaces. For a linear category $\mathscr{C}$ we let $\otimes_{\mathbb{C}} : \mathit{Vect} \times \mathscr{C} \to \mathscr{C}$ denote the implicit action map. We note that when $\mathscr{C}$ is furthermore monoidal, this *Vect*-action can be identified with the action induced by the unit structure $\mathit{Vect} \to \mathscr{C}$.

For an arbitrary algebra R in Ind *Vect* and a linear category $\mathscr{C}$ the base change $\mathscr{C}_R$ is the category of finitely presented R-modules in Ind $\mathscr{C}$ (cf. [69, Definition 6.1]). Given an algebra map $t : R \to S$ we have the restriction functor $t^* : \mathscr{C}_S \to \mathrm{Ind}\,\mathscr{C}_R$ and its left adjoint $t_* : \mathscr{C}_R \to \mathscr{C}_S$, which is given by base change $t_* = S \otimes_R -$.

We consider an affine algebraic group $\mathbb{G}$ and write $R = \mathscr{O}(\mathbb{G})$ for the algebra of global functions on $\mathbb{G}$, viewed as a Hopf algebra in Ind *Vect* with comultiplication Δ and counit ϵ.

We recall from [9, 26, 69] that a rational action of an affine algebraic group $\mathbb{G}$ on a tensor category $\mathscr{C}$, or more generally a $\mathbb{C}$-linear monoidal category, is a triple $\phi = (\phi, \nabla, \varepsilon)$ of an exact monoidal functor $\phi : \mathscr{C} \to \mathscr{C}_R$ equipped with natural transformations $\nabla : \phi \to \Delta^* \phi^2$ and $\varepsilon : \phi \to \epsilon^*$ whose adjoint maps provide isomorphisms $\Delta_* \phi \xrightarrow{\cong} \phi^2$ and $\epsilon_* \phi \xrightarrow{\cong} \mathrm{id}_{\mathscr{C}}$ [69, Section 8.1]. We write

$$\phi : \mathbb{G} \to \underline{\mathrm{Aut}}_{\otimes}(\mathscr{C}) \tag{31}$$

for a particular choice of $\mathbb{G}$-action on $\mathscr{C}$. From the above description we see that an action of $\mathbb{G}$ on $\mathscr{C}$ is a choice of a particularly structured comonad ϕ for $\mathscr{C}$, or more precisely for $\mathrm{Ind}\,\mathscr{C}$.

An isomorphism between two actions $\phi, \phi' : \mathbb{G} \rightrightarrows \underline{\mathrm{Aut}}_{\otimes}(\mathscr{C})$ is a natural isomorphism of monoidal functors $\eta : \phi \xrightarrow{\cong} \phi'$ which forms the appropriate diagrams with the structure maps ∇, ∇', ε, and ε'. Also, given two categories $\mathscr{C}$ and $\mathscr{D}$ equipped with actions $\phi : \mathbb{G} \to \underline{\mathrm{Aut}}_{\otimes}(\mathscr{C})$ and $\psi : \mathbb{G} \to \underline{\mathrm{Aut}}_{\otimes}(\mathscr{C})$, a $\mathbb{G}$-equivariant structure on a tensor functor $F : \mathscr{C} \to \mathscr{D}$ is a choice of an isomorphism of monoidal functors $\tau : F\phi \to \psi F$ which again forms the appropriate diagrams with the structure maps for ϕ and ψ.

Recall, finally, that the equivariantization $\mathscr{C}^{\mathbb{G}} = \mathscr{C}^{\mathbb{G},\phi}$ of a tensor category, relative to some $\mathbb{G}$-action ϕ, is the non-full subcategory of objects V in $\mathscr{C}$ which are equipped with coassociative, counital, coaction $\rho : V \to \phi(V)$. This is the category of comodules over the comonad ϕ. We have the following basic lemma.

Lemma B.1. (a) *If two actions $\phi, \phi' : \mathbb{G} \rightrightarrows \underline{\mathrm{Aut}}_{\otimes}(\mathscr{C})$ on a tensor category $\mathscr{C}$ are isomorphic, via some isomorphism $\eta : \phi \xrightarrow{\cong} \phi'$, then η induces an equivalence of categories $\eta_* : \mathscr{C}^{\mathbb{G},\phi} \to \mathscr{C}^{\mathbb{G},\phi'}$.*

(b) *If $F : \mathscr{C} \to \mathscr{D}$ is a $\mathbb{G}$-equivariant tensor functor, with comparison transformation $\tau : F\phi \xrightarrow{\cong} \psi F$, then there is an induced monoidal functor between equivariantizations*

$$F^{\mathbb{G}} : \mathscr{C}^{\mathbb{G}} \to \mathscr{D}^{\mathbb{G}}, \quad (V, \rho) \mapsto (FV, \tau F\rho).$$

Furthermore, if F is an equivalence then $F^{\mathbb{G}}$ is an equivalence.

Proof. (a) The equivalence η_* sends a ϕ-comodule (V, ρ_V) to the ϕ'-comodule $(V, \eta_V \rho_V)$. One similarly constructs the inverse η_*^{-1} via η^{-1}.

(b) We only speak to the second point, and so assume F is an equivalence. It is clear that fully faithfulness of F implies fully faithfulness of $F^{\mathbb{G}}$. Similarly, essential surjectivity of F implies essential surjectivity of $F^{\mathbb{G}}$. Indeed, if an object W in $\mathscr{D}$ is isomorphic to some $F(V)$, and W admits a ψ-comodule structure ρ_W, then $F(V)$ admits an isomorphic ψ-comodule structure, which then induces a corresponding ϕ-comodule structure on V via τ. So we see that (W, ρ_W) is in the essential image of $F^{\mathbb{G}}$. ∎

B.2. Restricting group actions

Consider a map of algebraic groups $\alpha : \mathbb{T} \to \mathbb{G}$. Let R and S be the algebras in Ind *Vect* associated to $\mathscr{O}(\mathbb{G})$ and $\mathscr{O}(\mathbb{T})$, respectively. Then we have the map of algebras $\bar{\alpha} : R \to S$

determined by α, and base change along $\bar{\alpha}$ provides a monoidal functor $\bar{\alpha}_* = S \otimes_R - : \mathscr{C}_R \to \mathscr{C}_S$. The restriction

$$\phi|_{\mathbb{T}} : \mathbb{T} \to \underline{\mathrm{Aut}}_{\otimes}(\mathscr{C})$$

of a $\mathbb{G}$-action (31) along α is defined as the composite functor $\phi|_{\mathbb{T}} := \bar{\alpha}_*\phi : \mathscr{C} \to \mathscr{C}_S$ with its induced structure maps. For example, the structure map $\nabla|_{\mathbb{T}} : \phi|_{\mathbb{T}} \to \Delta^*(\phi|_{\mathbb{T}})^2$ is identified under (Δ_*, Δ^*)-adjunction by the isomorphism

$$\Delta_*\phi|_{\mathbb{T}} = \Delta_*\bar{\alpha}_*\phi = (\bar{\alpha} \otimes \bar{\alpha})_*\Delta_*\phi \xrightarrow{\cong} (\bar{\alpha} \otimes \bar{\alpha})_*\phi^2 = (\phi|_{\mathbb{T}})^2.$$

We note that any $\mathbb{G}$-equivariant functor $F : \mathscr{C} \to \mathscr{D}$ inherits a $\mathbb{T}$-equivariant structure, under the restricted actions on $\mathscr{C}$ and $\mathscr{D}$, via base change.

B.3. Actions by ribbon and braided automorphisms

Let $\mathscr{C}$ be a ribbon tensor category. Then the braiding and twist on $\mathscr{C}$ induces a unique braiding and twist on the base change $\mathscr{C}_R$ so that the base change functor $R \otimes_{\mathbb{C}} - : \mathscr{C} \to \mathscr{C}_R$ is a braided monoidal functor which commutes with the twist. We say an action $\mathbb{G} \to \underline{\mathrm{Aut}}_{\otimes}(\mathscr{C})$ is an *action by ribbon automorphisms* if the associated monoidal functor $\phi : \mathscr{C} \to \mathscr{C}_R$ is braided and commutes with the twist. Note that preservation of the ribbon structure, for a given action, is a property and not an additional structure.

One can check that when $\mathbb{G}$ acts by ribbon automorphisms, the equivariantization $\mathscr{C}^{\mathbb{G}}$ admits a unique ribbon structure such that the forgetful functor $\mathscr{C}^{\mathbb{G}} \to \mathscr{C}$ is a map of ribbon tensor categories. One also observes that the class of ribbon actions $\mathbb{G} \to \underline{\mathrm{Aut}}_{\otimes}(\mathscr{C})$ is closed under isomorphism.

Of course, we can omit the twist to obtain the appropriate notion of a rational group action by braided automorphisms. If a group acts $\mathbb{G} \to \underline{\mathrm{Aut}}_{\otimes}(\mathscr{C})$ by braided automorphisms then the equivariantization $\mathscr{C}^{\mathbb{G}}$ inherits a unique braiding such that the forgetful functor is a braided tensor functor.

B.4. De-equivariantization

We again follow [9, 26, 69]. Consider a braided tensor category $\mathscr{C}$ with a Müger central tensor functor $i : \mathrm{rep}(\mathbb{G}) \to \mathscr{C}$, where $\mathbb{G}$ is an affine algebraic group. We have the regular (co)representation $\mathscr{O}(\mathbb{G})$ in $\mathrm{Rep}(\mathbb{G}) = \mathrm{Corep}(\mathscr{O}(\mathbb{G}))$ and the corresponding algebra object $\mathscr{O} = i\mathscr{O}(\mathbb{G})$ in $\mathrm{Ind}\,\mathscr{C}$. The de-equivariantization $\mathscr{C}_{\mathbb{G}} = \mathscr{C}_{\mathbb{G},i}$ is, by definition, the category of finitely presented $\mathscr{O}$-modules in $\mathrm{Ind}\,\mathscr{C}$.

The de-equivariantization $\mathscr{C}_{\mathbb{G}}$ is an exact $\mathbb{C}$-linear monoidal category, with product $\otimes_{\mathscr{O}}$, and exact structure induced by the faithful inclusion $\mathscr{C}_{\mathbb{G}} \to \mathrm{Ind}\,\mathscr{C}$. In most cases, the de-equivariantization $\mathscr{C}_{\mathbb{G}}$ is actually an *abelian* subcategory in $\mathrm{Ind}\,\mathscr{C}$. Rigidity of the monoidal structure, however, should only hold when i is an embedding, i.e. when the image of $\mathrm{rep}(\mathbb{G})$ in $\mathscr{C}$ is closed under taking subquotients (see [69, Sections 5.1, 5.2]).

For a Müger central functor $i : \mathrm{rep}(\mathbb{G}) \to \mathscr{C}$ as above we have the de-equivariantization functor, or free module functor $dE : \mathscr{C} \to \mathscr{C}_{\mathbb{G}}$, $V \mapsto \mathscr{O} \otimes V$. The category $\mathscr{C}_{\mathbb{G}}$ admits

a unique braiding such that the de-equivariantization map is a map of braided monoidal categories. If $\mathscr{C}$ is additionally ribbon, and the twist acts trivially on the image of $\mathrm{rep}(\mathbb{G})$, then $\mathscr{C}_{\mathbb{G}}$ also inherits a twist. This twist gives $\mathscr{C}_{\mathbb{G}}$ a ribbon structure, in the event that the de-equivariantization is indeed rigid.

In the statement of the following lemma, by "the" fiber functor for $\mathrm{rep}(\mathbb{G})$ we mean the forgetful functor $\mathrm{rep}(\mathbb{G}) \to \mathit{Vect}$. We note that there is only one fiber functor for $\mathrm{rep}(\mathbb{G})$ up to isomorphism, in any case [28].

Lemma B.2. *Given any Müger central tensor functor* $\mathrm{rep}(\mathbb{G}) \to \mathscr{C}$*, the composite* $\mathrm{rep}(\mathbb{G}) \to \mathscr{C} \to \mathscr{C}_{\mathbb{G}}$ *factors through the fiber functor* $\mathrm{rep}(\mathbb{G}) \to \mathit{Vect} \to \mathscr{C}_{\mathbb{G}}$.

By factoring through the fiber functor, we mean that the two maps to $\mathscr{C}_{\mathbb{G}}$ are isomorphic as monoidal functors.

Proof of Lemma B.2. The unit $\mathit{Vect} \to \mathscr{C}_{\mathbb{G}}$ sends a vector space V to the free module $\mathscr{O} \otimes V$. The composite $\mathrm{rep}(\mathbb{G}) \to \mathscr{C}_{\mathbb{G}}$ factors through the de-equivariantization map $\mathrm{rep}(\mathbb{G}) \to (\mathrm{rep}(\mathbb{G}))_{\mathbb{G}}$ for $\mathrm{rep}(\mathbb{G})$ itself. By the fundamental theorem of Hopf modules [67, Theorem 1.9.4] the de-equivariantization map for $\mathrm{rep}(\mathbb{G})$ is isomorphic to the composite $\mathrm{rep}(\mathbb{G}) \to \mathit{Vect} \to (\mathrm{rep}(\mathbb{G}))_{\mathbb{G}}$ of the forgetful functor and the unit morphism for $(\mathrm{rep}(\mathbb{G}))_{\mathbb{G}}$. ∎

Remark B.3. One could more generally consider de-equivariantization along central tensor functors from $\mathrm{rep}(\mathbb{G})$ to an arbitrary tensor category $\mathscr{C}$, in the sense of [32, Definition 4.15]. We stick to the braided setting for simplicity.

B.5. The rational $\mathbb{G}$-action on $\mathscr{C}_{\mathbb{G}}$, and $\mathrm{rep}(\mathbb{G})$-linear functors

As in the previous section, we consider a Müger central tensor functor $i : \mathrm{rep}(\mathbb{G}) \to \mathscr{C}$ into a braided tensor category $\mathscr{C}$, and fix $\mathscr{O} = i\mathscr{O}(\mathbb{G})$.

Take R to be the image of $\mathscr{O}(\mathbb{G})$ in $\mathrm{Ind}\,\mathit{Vect}$ under the forgetful functor $\mathrm{Rep}(\mathbb{G}) \to \mathrm{Ind}\,\mathit{Vect}$. We have the unit map $\mathrm{Ind}\,\mathit{Vect} \to \mathrm{Rep}(\mathbb{G})$ so that R is also an algebra object in $\mathrm{Rep}(\mathbb{G})$. Comultiplication provides an R-comodule algebra structure $\Delta : \mathscr{O}(\mathbb{G}) \to R \otimes \mathscr{O}(\mathbb{G})$ on the underlying algebra $\mathscr{O}(\mathbb{G})$ in $\mathrm{Rep}\,\mathbb{G}$, and we apply i to obtain an iR-comodule algebra structure on $\mathscr{O}$.

Now, the group $\mathbb{G}$ acts naturally on the de-equivariantization $\mathscr{C}_{\mathbb{G}}$ by braided automorphisms, and ribbon automorphisms when in the ribbon context. This action is specified by the functor $\phi(M) := iR \otimes M$, where M is an $\mathscr{O}$-module in $\mathrm{Ind}\,\mathscr{C}$ and $\mathscr{O}$ acts on $iR \otimes M$ via the coaction $\Delta : i\mathscr{O} \to iR \otimes i\mathscr{O}$. Coassociativity of the R-coaction on $\mathscr{O}$, and the counit, provide the necessary transformations $\phi \to \Delta^*\phi^2$ and $\phi \to \epsilon^*$. The following lemma is straightforward.

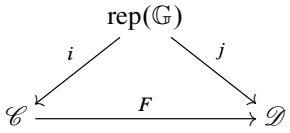
Lemma B.4. *Consider two Müger central embeddings* $i, i' : \mathrm{rep}(\mathbb{G}) \to \mathscr{C}$*, and the associated algebras* $\mathscr{O} = i\mathscr{O}(\mathbb{G})$ *and* $\mathscr{O}' = i'\mathscr{O}(\mathbb{G})$ *in* $\mathrm{Ind}\,\mathscr{C}$*. If* i *and* i' *are isomorphic, with a chosen monoidal isomorphism* $\eta : i \to i'$*, then restricting along the algebra isomorphism*

$\eta : \mathcal{O} \to \mathcal{O}'$ provides a $\mathbb{G}$-equivariant braided monoidal equivalence

$$\mathscr{C}_{\mathbb{G},i'} \to \mathscr{C}_{\mathbb{G},i}, \quad M \mapsto M|_{\mathcal{O}}.$$

$\mathbb{G}$-equivariance of the restriction functor is realized by the natural isomorphism $\eta \otimes \mathrm{id}$: $\phi(M) = iR \otimes M \to i'R \otimes M = \phi'(M)$.

Let us call a braided category $\mathscr{C}$, with a fixed central tensor functor $i : \mathrm{rep}(\mathbb{G}) \to \mathscr{C}$, a tensor category *over* $\mathrm{rep}(\mathbb{G})$. By a map of tensor categories over $\mathrm{rep}(\mathbb{G})$ we mean a tensor functor $F : \mathscr{C} \to \mathscr{D}$ and a specific choice of natural isomorphism of tensor functors $l : j \xrightarrow{\cong} Fi$ which makes the diagram



(2-)commute. In this case we have immediately a $\mathbb{G}$-equivariant functor between the de-equivariantizations

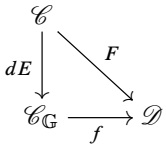
$$\mathscr{C}_{\mathbb{G}} \to \mathscr{D}_{\mathbb{G},Fi}, \quad M \mapsto FM,$$

and Lemma B.4 provides a $\mathbb{G}$-equivariant equivalence $\mathscr{D}_{\mathbb{G},Fi} \xrightarrow{\cong} \mathscr{D}_{\mathbb{G},j}$. Hence we have the following.

Lemma B.5. *Given a map $F = (F, l) : \mathscr{C} \to \mathscr{D}$ of braided tensor categories over* $\mathrm{rep}(\mathbb{G})$*, there is an induced $\mathbb{G}$-equivariant braided monoidal functor between the de-equivarianti-zations $F_{\mathbb{G}} : \mathscr{C}_{\mathbb{G}} \to \mathscr{D}_{\mathbb{G}}$.*

We have the following general, and slightly stronger version of Proposition 9.2.

Lemma B.6. *Suppose a tensor category $\mathscr{D}$ is equipped with the trivial* $\mathrm{rep}(\mathbb{G})$ *structure,* $\mathrm{rep}(\mathbb{G}) \to Vect \to \mathscr{D}$*, and $i : \mathrm{rep}(\mathbb{G}) \to \mathscr{C}$ is an arbitrary tensor category over* $\mathrm{rep}(\mathbb{G})$*. Then for any map $F : \mathscr{C} \to \mathscr{D}$ of tensor categories over* $\mathrm{rep}(\mathbb{G})$*, there is a right exact monoidal functor $f : \mathscr{C}_{\mathbb{G}} \to \mathscr{D}$ which fits into a* (2-)*diagram*



(32)

The functor f is unique up to isomorphism, is braided when F is braided, and is exact when $\mathscr{C}_{\mathbb{G}}$ is a tensor category.

Proof. The existence of f, and its conditional exactness and/or braidedness, are established as in Proposition 9.2. Below we let R denote the algebra of functions $\mathcal{O}(\mathbb{G})$, considered as an algebra object in Ind *Vect*, and $\mathcal{O} = i\mathcal{O}(\mathbb{G})$.

To recall, we have $F\mathcal{O} \cong \mathrm{unit}(R)$ in $\mathrm{Ind}\,\mathscr{D}$ and the counit for R provides an algebra map $F\mathcal{O} \to \mathbf{1}$. We then take $f = \mathbf{1} \otimes_{F\mathcal{O}} F$, where F denotes the restriction of the given functor $F : \mathscr{C} \to \mathscr{D}$ to a functor between the non-full subcategories of modules

$$\mathscr{C}_{\mathbb{G}} = \{\text{fin pres } \mathcal{O}\text{-modules in } \mathrm{Ind}\,\mathscr{C}\} \to \{\text{fin pres } F\mathcal{O}\text{-modules in } \mathrm{Ind}\,\mathscr{D}\},$$

by abuse of notation.

As for uniqueness, we first claim that the monoidal functor $\mathbf{1} \otimes_{dE\mathcal{O}} dE : \mathscr{C}_{\mathbb{G}} \to \mathscr{C}_{\mathbb{G}}$ is isomorphic to the identity. Explicitly, $dE\mathcal{O} = \mathcal{O} \otimes \mathcal{O}$ and the counit for R corresponds to the algebra map $dE\mathcal{O} = \mathcal{O} \otimes \mathcal{O} \to \mathcal{O} = \mathbf{1}$ given by multiplication. For any M in $\mathscr{C}_{\mathbb{G}}$, $dE(M) = \mathcal{O} \otimes M$ is the $\mathcal{O} \otimes \mathcal{O}$-module in $\mathrm{Ind}\,\mathscr{C}$, with each copy of $\mathcal{O}$ acting on its respective factor, and the module structure map $dE(M) = \mathcal{O} \otimes M \to M$ is a natural morphism of $dE\mathcal{O}$-modules, where $dE\mathcal{O}$ acts on M via the counit/multiplication map $dE\mathcal{O} \to \mathcal{O} = \mathbf{1}$. One can check that this map reduces to a natural isomorphism

$$\mathbf{1} \otimes_{dE\mathcal{O}} dE(M) \xrightarrow{\cong} M.$$

The above isomorphism provides the claimed natural isomorphism of monoidal functors $\mathbf{1} \otimes_R dE \cong \mathrm{id}$.

Consider again $F : \mathscr{C} \to \mathscr{D}$ as above, and take $f = \mathbf{1} \otimes_R F$. Then for any right exact monoidal functor $f' : \mathscr{C}_G \to \mathscr{D}$ which fits into a diagram (32) we have

$$f' \cong f' \circ (\mathbf{1} \otimes_R dE) \cong \mathbf{1} \otimes_R (f' \circ dE) \cong 1 \otimes_R F = f. \qquad \blacksquare$$

Remark B.7. If we identify *Vect* with the category of $\mathcal{O}(\mathbb{G})$-modules in $\mathrm{Rep}\,\mathbb{G}$, then Lemma B.6 is in line with various categorical base change formulas in the literature. One can compare with [31] and [61, Theorem 4.8.4.6], for example.

B.6. Some relative results

Lemma B.8. *Consider a map of algebraic groups $\alpha : \mathbb{T} \to \mathbb{G}$, and a 2-diagram of braided tensor functors*

$$\begin{array}{ccc} \mathrm{rep}(\mathbb{G}) & \xrightarrow{\mathrm{res}_\alpha} & \mathrm{rep}\,\mathbb{T} \\ & {\scriptstyle i=j\,\mathrm{res}_\alpha} \searrow \quad \swarrow {\scriptstyle j} & \\ & \mathscr{D} & \end{array} \tag{33}$$

Suppose that i and j have Müger central image. Then, for $\mathcal{O}' = j\mathcal{O}(\mathbb{T})$ and $\mathcal{O} = i\mathcal{O}(\mathbb{G})$, the braided monoidal functor $\mathcal{O}' \otimes_{\mathcal{O}} - : \mathscr{D}_{\mathbb{G}} \to \mathscr{D}_{\mathbb{T}}$ admits a natural $\mathbb{T}$-equivariant structure.

Let us clarify a few points here. First, the de-equivariantizations $\mathscr{D}_{\mathbb{G}}$ and $\mathscr{D}_{\mathbb{T}}$ are defined via the central maps i and j respectively. Second, the base change operation $\mathcal{O}' \otimes_{\mathcal{O}} -$ is defined via the algebra map

$$\mathcal{O} = i\mathcal{O}(\mathbb{G}) \cong j\,\mathrm{res}_\alpha\,\mathcal{O}(\mathbb{G}) \xrightarrow{j\tilde{\alpha}} j\mathcal{O}(\mathbb{T}) = \mathcal{O}',$$

where the first isomorphism is provided by the natural isomorphism implicit in the diagram (33) and $\bar{\alpha} : \mathrm{res}_\alpha \mathscr{O}(\mathbb{G}) \to \mathscr{O}(\mathbb{T})$ is the algebra map dual to the map of affine group schemes α. Third, the $\mathbb{T}$-action on $\mathscr{D}_\mathbb{G}$ is defined via restriction along α, as described in Section B.2 above. We continue with the proof.

Proof of Lemma B.8. Take $\mathrm{res} = \mathrm{res}_\alpha$. By Lemma B.4 we may replace i with $j \circ \mathrm{res}$ to assume that the 2-diagram (33) strictly commutes, i.e. $i = j \circ \mathrm{res}$. So we consider such a strict diagram, take $\mathscr{O} = i\mathscr{O}(\mathbb{G})$ and $\mathscr{O}' = j\mathscr{O}(\mathbb{T})$, and let R and S be the corresponding Hopf algebras in Ind *Vect*.

The algebraic group map $\alpha : \mathbb{T} \to \mathbb{G}$ specifies a dual Hopf algebra map $\bar{\alpha} : R \to S$ and, at the level of $\mathbb{T}$-representations, $\bar{\alpha} : \mathrm{res}\, \mathscr{O}(\mathbb{G}) \to \mathscr{O}(\mathbb{T})$ is a map of S-comodule algebras in $\mathrm{Rep}\,\mathbb{T}$. Here the S-coaction on $\mathscr{O}(\mathbb{T})$ is provided by comultiplication, and the S-coaction on $\mathscr{O}(\mathbb{G})$ is provided by corestricting the R-coaction along $\bar{\alpha}$. (Geometrically, we are restricting the translation action of $\mathbb{G}$ on itself along α.)

We apply j to recover the morphism $j\bar{\alpha} : \mathscr{O} \to \mathscr{O}'$ along which we have changed base to define the functor $\mathscr{O}' \otimes_\mathscr{O} - : \mathscr{D}_\mathbb{G} \to \mathscr{D}_\mathbb{T}$. The map $j\bar{\alpha}$ is a map of jS-comodule algebras in $\mathscr{D}$. As we explain below, compatibility of $j\bar{\alpha}$ with the given jS-comodule structures provides the functor $\mathscr{O}' \otimes_\mathscr{O} -$ with its natural $\mathbb{T}$-equivariant structure.

To say things precisely, we are in search of a natural isomorphism

$$\bar{\eta} : \mathscr{O}' \otimes_\mathscr{O} (jS \otimes M) \xrightarrow{\cong} jS \otimes (\mathscr{O}' \otimes_\mathscr{O} M)$$

of $jS \otimes \mathscr{O}'$-modules, where $\mathscr{O}'$ acts on each product $jS \otimes X$ via the coaction $\mathscr{O}' \to jS \otimes \mathscr{O}'$, and $\mathscr{O}$ acts on the product $jS \otimes M$ similarly. We consider the two natural maps in $\mathscr{D}_S$,

$$\eta : \mathscr{O}' \otimes (jS \otimes M) \to jS \otimes (\mathscr{O}' \otimes M) \quad \text{and} \quad \gamma : jS \otimes (\mathscr{O}' \otimes M) \to \mathscr{O}' \otimes (jS \otimes M),$$

which are obtained by applying $j(-) \otimes M$ to the isomorphism $\mathscr{O}(\mathbb{T}) \otimes S \to S \otimes \mathscr{O}(\mathbb{T})$, $f \otimes s \mapsto f_1 s \otimes f_2$, and its inverse $S \otimes \mathscr{O}(\mathbb{T}) \to \mathscr{O}(\mathbb{T}) \otimes S$, $s \otimes f \mapsto f_2 \otimes S(f_1)s$, respectively. The maps η and γ are mutually inverse, and the fact that $\bar{\alpha} : \mathscr{O} \to \mathscr{O}'$ is jS-colinear implies that the composites with the projections

$$\begin{array}{ccc} \mathscr{O}' \otimes (jS \otimes M) & \xrightarrow{\eta} & jS \otimes (\mathscr{O}' \otimes M) \\ \vdots & & \downarrow \mathrm{proj} \\ \mathscr{O}' \otimes_\mathscr{O} (jS \otimes M) & \xrightarrow{\exists!} & jS \otimes (\mathscr{O}' \otimes_\mathscr{O} M) \end{array}$$

and

$$\begin{array}{ccc} jS \otimes (\mathscr{O}' \otimes M) & \xrightarrow{\gamma} & \mathscr{O}' \otimes (jS \otimes M) \\ \vdots & & \downarrow \mathrm{proj} \\ jS \otimes (\mathscr{O}' \otimes_\mathscr{O} M) & \xrightarrow{\exists!} & \mathscr{O}' \otimes_\mathscr{O} (jS \otimes M) \end{array}$$

are appropriately $\mathscr{O}$-bilinear, and hence induce mutually inverse natural isomorphisms on the quotients

$$\bar{\eta} : \mathscr{O}' \otimes_{\mathscr{O}} (jS \otimes M) \xrightarrow{\cong} jS \otimes (\mathscr{O}' \otimes_{\mathscr{O}} M), \quad \bar{\gamma} : jS \otimes (\mathscr{O}' \otimes_{\mathscr{O}} M) \xrightarrow{\cong} \mathscr{O}' \otimes_{\mathscr{O}} (jS \otimes M).$$

The fact that $\bar{\eta}$ commutes with the structure maps $\Delta \otimes - : jS \otimes - \to jS \otimes jS \otimes -$ for the $\mathbb{T}$-actions on $\mathscr{D}_{\mathbb{G}}$ and $\mathscr{D}_{\mathbb{T}}$ follows from the comodule algebra axioms for $\mathscr{O}'$. Specifically, this compatibility follows from the equality of the two maps

$$\mathscr{O}(\mathbb{T}) \otimes S \to S \otimes S \otimes \mathscr{O}(\mathbb{T}), \quad f \otimes a \mapsto \Delta(f_1 a) \otimes f_2 = f_1 a_1 \otimes f_2 a_2 \otimes f_3.$$

The natural isomorphism $\bar{\eta}$ hence realizes $\mathbb{T}$-equivariance of the base change functor, as a functor between non-monoidal $\mathbb{T}$-linear categories.

Having established the existence of the above natural isomorphism, we note that $\bar{\eta}$ is in fact the unique $jS \otimes \mathscr{O}'$-linear map which fits into a diagram

$$\begin{array}{ccc} & M & \\ \swarrow & & \searrow \\ \mathscr{O}' \otimes_{\mathscr{O}} (jS \otimes M) & \xrightarrow{\cong} & jS \otimes (\mathscr{O}' \otimes_{\mathscr{O}} M) \end{array}$$

Similarly, the isomorphisms providing the monoidal structures on the functors $M \mapsto jS \otimes M$ and $M \mapsto \mathscr{O}' \otimes_{\mathscr{O}} M$ are the unique jS-linear and $\mathscr{O}'$-linear maps fitting into diagrams

$$\begin{array}{ccc} & M \otimes N & \\ \swarrow & & \searrow \\ (jS \otimes M) \otimes_{jS \otimes \mathscr{O}} (jS \otimes N) & \xrightarrow{\cong} & jS \otimes (M \otimes_{\mathscr{O}} N) \end{array}$$

and

$$\begin{array}{ccc} & M \otimes N & \\ \swarrow & & \searrow \\ (\mathscr{O}' \otimes_{\mathscr{O}} M) \otimes_{\mathscr{O}'} (\mathscr{O}' \otimes_{\mathscr{O}} N) & \xrightarrow{\cong} & \mathscr{O}' \otimes_{\mathscr{O}} (M \otimes_{\mathscr{O}} N) \end{array}$$

respectively. One can use these uniqueness properties to check that $\bar{\eta}$ is indeed compatible with the monoidal structure on the functor $\mathscr{O}' \otimes_{\mathscr{O}} -$. ∎

We now consider a 2-diagram of braided tensor functors

$$\begin{array}{ccc} \mathrm{rep}(\mathbb{G}) & \xrightarrow{\mathrm{res}_\alpha} & \mathrm{rep}(\mathbb{T}) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ \mathscr{C} & \xrightarrow{F} & \mathscr{D} \end{array} \tag{34}$$

where the functor res_α is restriction along an algebraic group map $\alpha : \mathbb{T} \to \mathbb{G}$. We assume that i and j are Müger central and that the de-equivariantization $\mathscr{D}_{\mathbb{T}}$ is a tensor category.[2] The composition $\mathscr{C} \to \mathscr{D} \to \mathscr{D}_{\mathbb{T}}$ then annihilates $\mathrm{rep}(\mathbb{G})$, and so by Lemma B.6 there is a unique right exact monoidal functor $f : \mathscr{C}_{\mathbb{G}} \to \mathscr{D}_{\mathbb{T}}$ which fits into a 2-diagram

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{F} & \mathscr{D} \\ \downarrow & & \downarrow \\ \mathscr{C}_{\mathbb{G}} & \xrightarrow{f} & \mathscr{D}_{\mathbb{T}} \end{array} \tag{35}$$

Lemma B.9. *In the above setting, the induced map $f : \mathscr{C}_{\mathbb{G}} \to \mathscr{D}_{\mathbb{T}}$ inherits a natural $\mathbb{T}$-equivariant structure.*

Proof. Take $\mathscr{O} = i\mathscr{O}(\mathbb{G})$ and $\mathscr{O}' = j\mathscr{O}(\mathbb{T})$. The map $f : \mathscr{C}_{\mathbb{G}} \to \mathscr{D}_{\mathbb{T}}$ is, by definition, the composition

$$f := \Big[\mathscr{C}_{\mathbb{G}} \xrightarrow{F_{\mathbb{G}}} \{\text{fin pres } F\mathscr{O}\text{-modules in Ind}\,\mathscr{D}\} = \mathscr{D}_{\mathbb{G}} \xrightarrow{\mathscr{O}' \otimes_{F\mathscr{O}} -} \{\text{fin pres } \mathscr{O}'\text{-modules in Ind}\,\mathscr{D}\} = \mathscr{D}_{\mathbb{T}}\Big].$$

The first map $F_{\mathbb{G}}$ in this composite is known to be $\mathbb{G}$-equivariant and hence $\mathbb{T}$-equivariant, by Lemma B.5 above. The second map is $\mathbb{T}$-equivariant by Lemma B.8. We conclude that the composite f is $\mathbb{T}$-equivariant as well. ∎

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