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Qiang Guang · Qi-Rui Li · Xu-Jia Wang

The L_p-Minkowski problem with super-critical exponents

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Abstract. The L_p-Minkowski problem deals with the existence of closed convex hypersurfaces in $\mathbb{R}^{n+1}$ with prescribed p-area measures. It extends the classical Minkowski problem and embraces several important geometric and physical applications. Existence of solutions has been obtained in the sub-critical case $p > -n - 1$, but the problem remains open in the super-critical case $p < -n - 1$. In this paper, we introduce new ideas to solve the problem for all the super-critical exponents. A crucial ingredient in our proof is a topological method based on the calculation of the homology of a topological space of ellipsoids. Our results show that the L_p-Minkowski problem admits a solution in both the sub-critical and super-critical cases but does not have a solution in general in the critical case.

Keywords: L_p-Minkowski problem, Monge–Ampère equation.

1. Introduction

A central problem in convex geometry is the characterisation of geometric measures for convex bodies in the Euclidean space $\mathbb{R}^{n+1}$. The best-known example is the classical Minkowski problem, which was a major impetus for the development of fully nonlinear PDEs. In the last three decades, a focus of research in convex geometry is the L_p-Minkowski problem introduced by Lutwak [38]. It includes the classical Minkowski problem ($p = 1$), the logarithmic Minkowski problem ($p = 0$), the centro-affine Minkowski problem and elliptic affine spheres ($p = -n - 1$) as special cases [7, 16]. The L_p-Minkowski problem was derived from the Brunn–Minkowski theory, and research of the problem paved the way for further development of this theory [18, 31, 40]. The L_p-Minkowski problem also plays a significant role in other applications. Of particular interest is that

Qiang Guang: Mathematical Sciences Institute, Australian National University, Hanna Neumann Building #145, Canberra, ACT 2601, Australia; qguang8@gmail.com

Qi-Rui Li: School of Mathematical Sciences, Zhejiang University, 866 Yuhangtang Road, 310058 Hangzhou, P. R. China; qi-rui.li@zju.edu.cn

Xu-Jia Wang: Institute for Theoretical Sciences, Westlake University, 600 Dunyu Road, 310030 Hangzhou, P. R. China; wangxujia@westlake.edu.cn

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it describes self-similar solutions to Gauss curvature flows [3, 4, 9, 17], and its projective invariance in the case $p = -n - 1$ makes it fundamental in image processing [2, 5].

Let $\mathcal{K}_o$ denote the set of closed convex bodies in $\mathbb{R}^{n+1}$ with the origin in the interior. For any $\Omega \in \mathcal{K}_o$ and $p \in \mathbb{R}$, its p-area measure is defined as $dS_p = u^{1-p} dS$ [38], where u is the support function of Ω and S is the classical surface area measure of Ω. Given a finite nonnegative Borel measure μ on the unit sphere $\mathbb{S}^n$, the L_p-Minkowski problem asks for the existence of solutions $\Omega \in \mathcal{K}_o$ such that its p-area measure coincides with the given measure μ. If

$$d\mu = f d\sigma_{\mathbb{S}^n}$$

for a density function f on $\mathbb{S}^n$, then the L_p-Minkowski problem can be formulated as finding solutions to the Monge–Ampère equation

$$\det(\nabla^2 u + uI) = f u^{p-1} \quad \text{on } \mathbb{S}^n, \tag{1.1}$$

where ∇ denotes the covariant derivative with respect to an orthonormal frame on $\mathbb{S}^n$, and I is the identity matrix.

The last three decades have witnessed a great progress in the study of the L_p-Minkowski problem. The problem can be divided into three cases.

- In the sub-critical case $p > -n - 1$ (with respect to the Blaschke–Santaló inequality), the existence of solutions was obtained in [16]. However, there is no uniform estimate for equation (1.1) when $p < 0$ [26], and there may exist infinitely many solutions when $p < -n$ [23, 34]. When $p = 0$, (1.1) is called the logarithm Minkowski problem; necessary and sufficient conditions for the existence of solutions were obtained in [7] when the prescribed measure is an even Borel measure. For $p \geq 1$, the existence and regularity of solutions were obtained in [16, 39].
- In the critical case $p = -n - 1$, equation (1.1) is called the centro-affine Minkowski problem [16]. The quantity $u^{n+2}\det(\nabla^2 u + uI)$ is invariant under projective transforms and plays a key role in affine geometry. For example, when $f = 1$, (1.1) is the equation for affine elliptic spheres [42]. The projective invariance also makes it of great interest to image processing [2]. A Kazdan–Warner type condition [16] implies that (1.1) admits no solutions for a general positive function f. There are many works dealing with the critical case [1, 28, 29, 34, 37]. However, results on the existence and multiplicity of solutions in this case are far from being satisfactory. The main difficulty is that the normalisation of blow-up sequences does not lead to a unique limit model.
- In the super-critical case $p < -n - 1$, there are some results in the one-dimensional case. In [19, 46], the authors obtained the existence of $\frac{\pi}{k}$-periodic (k-fold symmetry, $k \geq 2$) convex solutions. In [4], Andrews proved that when $f = 1$ and $p \in [-7, -2)$, a convex solution to (1.1) must be a circle; when $p < -7$, a convex solution to (1.1) is either the circle, or a curve with k-fold symmetry. In high dimensions, Zhu [47] proved the existence of solutions when f is a discrete measure with no essential subspaces, but we are unaware of any existence results when f is a function.

There are many related research works on the L_p-Minkowski problem [6, 13, 14, 24, 48]. It is interesting to compare equation (1.1) with the semi-linear elliptic equation

$$-\Delta_{g_0} u + c_n R_{g_0} u = f(x) u^{\gamma} \quad \text{on } M, \tag{1.2}$$

where (M, g_0) is an n-dimensional Riemannian manifold, c_n is a constant depending only on n, and R_{g_0} is the scalar curvature. There is a vast body of literature on equation (1.2). In the sub-critical case $1 < \gamma < \frac{n+2}{n-2}$, there is a uniform estimate for solutions to (1.2), and one can obtain the existence of nontrivial solutions under suitable conditions. In the critical case $\gamma = \frac{n+2}{n-2}$, (1.2) is the prescribing scalar curvature equation. In particular, it is Nirenberg's problem when $n = 2$ and $M = \mathbb{S}^2$, and the Yamabe problem when $f = 1$. In this case, there is a very rich phenomenon on the existence and multiplicity of solutions, and one can find many significant results [8, 10, 30, 43]. In the super-critical case $\gamma > \frac{n+2}{n-2}$, numerous attempts have been made for the existence of nontrivial solutions to (1.2) but the solution was obtained only in some special cases.

Comparing with (1.2), we find that equation (1.1) is more complicated. There is no uniform estimate for (1.1) in the sub-critical case. There is no solution in general in the critical case by the Kazdan–Warner type condition, and much less is known about sufficient conditions for the existence of solutions. Therefore, one would not expect a complete resolution for the existence of solutions to (1.1) in the super-critical case. Surprisingly, we find that the L_p-Minkowski problem (1.1) admits a solution for all p in the super-critical range when f is bounded by two positive constants, and thus resolve the existence problem in this case.

Theorem 1.1. *Suppose that $p < -n - 1$. Let f be a positive and $C^{1,1}$-smooth function on $\mathbb{S}^n$. Then there is a uniformly convex, $C^{3,\alpha}$-smooth and positive solution to (1.1), where $\alpha \in (0, 1)$.*

By approximation, we also obtain the existence of solutions when f is a non-smooth function.

Corollary 1.2. *Suppose that $p < -n - 1$ and f is a function on $\mathbb{S}^n$ such that $\frac{1}{c_0} \le f \le c_0$ for some constant $c_0 > 1$. Then there is a strictly convex, $C^{1,\alpha}$-smooth and positive weak solution to (1.1) for some $\alpha \in (0, 1)$.*

We point out that the condition $f > 0$ in Theorem 1.1 and Corollary 1.2 cannot be relaxed to $f \ge 0$. Indeed, there exist functions f which are positive except at the north and south poles, such that equation (1.1) admits no solutions [20, Theorem 1.4].

It is well known that the Monge–Ampère equation is of divergence form and equation (1.1) is the Euler equation of the following functional (for $p \ne 0$) for convex bodies $\Omega \in \mathcal{K}_o$:

$$\mathcal{J}(\Omega) = \mathrm{Vol}(\Omega) - \frac{1}{p} \int_{\mathbb{S}^n} f u^p \, d\sigma_{\mathbb{S}^n}. \tag{1.3}$$

Therefore, a natural approach to the L_p-Minkowski problem is to combine the variational method with the Gauss curvature flow [12, 15, 25, 35].

In this paper, we will employ the following Gauss curvature flow:

$$\frac{\partial X}{\partial t}(x,t) = -f(\nu)K(x,t)\langle X,\nu\rangle^p \nu + X(x,t), \tag{1.4}$$

where $X(\cdot,t): \mathbb{S}^n \to \mathbb{R}^{n+1}$ is a parametrisation of the evolving convex hypersurfaces $\mathcal{M}_t$, ν and K are the unit outward normal and Gauss curvature of $\mathcal{M}_t$, respectively. We will show that functional (1.3) is non-decreasing under flow (1.4) (Lemma 2.1).

The main difficulty is the lack of uniform estimate for the problem. The uniform estimate is the key estimate for many geometric problems such as the Yamabe problem [43] or Calabi's conjecture [45]. The L_p-Minkowski problem has been extensively studied in the past three decades, and various techniques have been developed to establish the uniform estimate for the L_p-Minkowski problem and the associated Gauss curvature flow, but none of them applies to the super-critical case.

To overcome the difficulty, our strategy is to use a topological method to find a special initial condition such that the evolving hypersurfaces $\mathcal{M}_t = \partial\Omega_t$ satisfy

$$B_r(0) \subset \Omega_t \subset B_R(0), \tag{1.5}$$

for positive constants $R \geq r > 0$ independent of t, where $B_r(x)$ denotes a closed ball of radius r centred at x. Once the solution satisfies such a C^0-estimates, one can establish the second derivative estimates, and higher regularity follows from Krylov's regularity theory. Hence, by the monotonicity of functional (1.3), the flow converges to a solution of (1.1).

Therefore, the key point in the argument is to find the special initial hypersurface. A crucial ingredient in achieving this goal is to compute the homology for a class of ellipsoids centred at the origin. Let us outline the main ideas of the proof below.

For any convex body Ω in $\mathbb{R}^{n+1}$, it is well known that there is a unique ellipsoid $E(\Omega)$, called John's minimum ellipsoid [42], which achieves the minimal volume among all ellipsoids containing Ω, such that

$$\frac{1}{n+1}E(\Omega) \subset \Omega \subset E(\Omega),$$

where $\frac{1}{n+1}E(\Omega)$ denotes the dilation of $E(\Omega)$ by the factor $\frac{1}{n+1}$ with respect to its centre. Let $r_1(\Omega) \leq r_2(\Omega) \leq \cdots \leq r_{n+1}(\Omega)$ be the lengths of semi-axes of $E(\Omega)$. Denote

$$e_{\mathcal{M}} = e_\Omega = \frac{r_{n+1}(\Omega)}{r_1(\Omega)}$$

the eccentricity of $\mathcal{M} := \partial\Omega$ (or the eccentricity of Ω). We will first prove the following property:

(P) For any given constant $A > \mathcal{J}(B_1(0))$, if one of the quantities e_Ω, $\mathrm{Vol}(\Omega)$, $[\mathrm{Vol}(\Omega)]^{-1}$, and $[\mathrm{dist}(O,\partial\Omega)]^{-1}$ is sufficiently large, we have $\mathcal{J}(\Omega) \geq A$ (Lemmas 2.2–2.4).

Denote by $\mathcal{A}_I$ the set of ellipsoids E such that the origin $O \in E$, $e_E \in [1,\overline{e}]$, and $\overline{v} \leq \mathrm{Vol}(E) \leq \frac{1}{\overline{v}}$, where $\overline{e}$ is a large constant and $\overline{v}$ is a small constant. $\mathcal{A}_I$ is a metric space under the Hausdorff distance. For any ellipsoid $E \in \mathcal{A}_I$, let $\mathcal{M}_E(t)$ be the solution to flow (1.4) with initial condition E. By the above property (P), $\mathcal{M}_E(t)$ has uniformly bounded eccentricity and volume, and $\operatorname{dist}(O, \mathcal{M}_E(t))$ is uniformly bounded from zero if

$$\mathcal{J}(\mathcal{M}_E(t)) \leq A \quad \forall t \geq 0.$$

Now, our focus is to prove that at any given time $t_0 > 0$, there exists an initial $E_0 \in \mathcal{A}_I$ such that the minimum ellipsoid of $\mathcal{M}_{E_0}(t_0)$ is the unit ball centred at the origin (Lemma 3.8), thus validating condition (1.5) (as a result of Lemma 3.12).

If to the contrary there is no such initial E_0, we will construct a continuous map $T\colon \mathcal{A}_I \to \mathcal{P}$ which is the identity map on $\mathcal{P}$, where $\mathcal{P}$ is the boundary of $\mathcal{A}_I$ in the topological space of all ellipsoids. This implies the existence of an injection from the homology group of $\mathcal{P}$ to that of $\mathcal{A}_I$. As a consequence, $\mathcal{P}$ has trivial homology since $\mathcal{A}_I$ is contractible (Lemma 3.4). By Proposition 3.6, this leads to $H_k(\mathcal{E} \times \mathbb{S}^n) = H_k(\mathcal{E}) \oplus H_k(\mathbb{S}^n)$, where $\mathcal{E}$ is introduced in (3.4). We thus reach a contradiction by the Künneth formula and Theorem 3.7 if we take $k = \frac{n(n+1)}{2} + 2n - 1$. This topological fixed-point argument is the main novelty in this paper. A crucial ingredient in the argument is the computation of the homology of the class $\mathcal{E}$ of ellipsoids.

To complete the proof, we choose a sequence $t_k \to \infty$ and let $E_k \in \mathcal{A}_I$ be the initial condition such that the minimum ellipsoid of $\mathcal{M}_{E_k}(t_k)$ is the unit ball. By the Blaschke selection theorem, E_k sub-converges to $E_* \in \mathcal{A}_I$. It follows by the above property (P) that the Gauss curvature flow (1.4) with initial condition E_* satisfies (1.5). Hence, flow (1.4) starting from E_* converges to a solution of (1.1) as $t \to \infty$.

The paper is organised as follows. In Section 2, we derive some a priori estimates for functional (1.3) and the Gauss curvature flow (1.4). In Section 3, we prove the main results (Theorem 1.1 and Corollary 1.2), assuming Proposition 3.6 and Theorems 3.7 and 2.5 temporarily. The proofs of Proposition 3.6 and Theorem 3.7 will be given in Sections 4, and the proof of Theorem 2.5 will be given in Section 5. For the reader's convenience, in Appendix A, we collect some basic and necessary definitions and results in topology used in our argument.

The topological method introduced in this paper enables us to find an initial condition such that the solution has uniform estimate for all time t. The uniform estimate is the most difficult part for many geometric and analysis problems. This method can be adapted to other geometric problems, such as the L_p dual Minkowski problem [24, 41], and the dual centro-affine Minkowski problem [27] and more general prescribing curvature problems. We will study these problems separately.

2. A priori estimates

Let $\mathcal{M}$ be a smooth, closed, and uniformly convex hypersurface in $\mathbb{R}^{n+1}$. The support function of $\mathcal{M}$ is given by

$$u(x) = \langle x, \nu_{\mathcal{M}}^{-1}(x) \rangle \quad \forall x \in \mathbb{S}^n,$$

where $\nu_{\mathcal{M}}: \mathcal{M} \to \mathbb{S}^n$ is the Gauss map and $\nu_{\mathcal{M}}^{-1}$ is its inverse, namely, $\nu_{\mathcal{M}}^{-1}(x)$ is the point $z(x) \in \mathcal{M}$ such that the unit outer normal of $\mathcal{M}$ at $z(x)$ is equal to x. It is well known that $\nu_{\mathcal{M}}^{-1}(x) = u(x)x + \nabla u(x)$ and the Gauss curvature of $\mathcal{M}$ at $\nu_{\mathcal{M}}^{-1}(x)$ is given by

$$K = \frac{1}{\det(u_{ij} + u\delta_{ij})}, \tag{2.1}$$

where $u_{ij} := \nabla_{ij}^2 u$. This implies that the p-area measure of $\mathcal{M}$ is given by [38],

$$d\mathcal{S}_p = u^{1-p} \det(\nabla^2 u + uI) d\sigma_{\mathbb{S}^n}.$$

Hence, the L_p-Minkowski problem is equivalent to solving equation (1.1).

Denote by $\mathrm{Cl}(\mathcal{M})$ the convex body enclosed by $\mathcal{M}$. When no confusion arises, we may abuse the notation $\mathcal{M}$ for $\mathrm{Cl}(\mathcal{M})$, such as writing the functional $\mathcal{J}(\mathrm{Cl}(\mathcal{M}))$ as $\mathcal{J}(\mathcal{M})$. Assume that $\mathrm{Cl}(\mathcal{M}) \in \mathcal{K}_o$. Let r be the radial function of $\mathcal{M}$, which is given by

$$r(\xi) = \max\{\lambda : \lambda\xi \in \mathrm{Cl}(\mathcal{M})\} \quad \forall \xi \in \mathbb{S}^n. \tag{2.2}$$

Then

$$\mathrm{Vol}(\mathrm{Cl}(\mathcal{M})) = \frac{1}{n+1} \int_{\mathbb{S}^n} r^{n+1} d\sigma_{\mathbb{S}^n}. \tag{2.3}$$

Denote $\vec{r}(\xi) = r(\xi)\xi$. We also define the radial Gauss mapping by

$$\mathcal{A}_{\mathcal{M}}(\xi) = \nu_{\mathcal{M}}(\vec{r}(\xi)) \quad \forall \xi \in \mathbb{S}^n.$$

Let $\mathcal{M}_t$ be a solution to flow (1.4) and $X(\cdot, t): \mathbb{S}^n \to \mathbb{R}^{n+1}$ be its parametrisation. Consider the new parametrisation

$$\bar{X}(x, t) = X(\varphi(x, t), t),$$

where $\varphi(\cdot, t): \mathbb{S}^n \to \mathbb{S}^n$ is the mapping such that

$$X(\varphi(x, t), t) = \nu_{\mathcal{M}_t}^{-1}(x) \quad \forall x \in \mathbb{S}^n.$$

Namely, $\varphi(\cdot, t) = X^{-1}(\cdot, t) \circ \nu_{\mathcal{M}_t}^{-1}$. Denote $\varphi = (\varphi^1, \ldots, \varphi^{n+1})$. By extending X to be homogeneous of degree zero on $\mathbb{R}^{n+1} \setminus \{0\}$, we have

$$\frac{\partial \bar{X}}{\partial t} = \sum_i \frac{\partial X}{\partial z^i} \frac{\partial \varphi^i}{\partial t} + \frac{\partial X}{\partial t}.$$

Since the first term on the right-hand side is tangential, taking inner product with the unit outer normal of $\mathcal{M}_t$ gives that

$$\partial_t u(x, t) = \langle x, \partial_t \bar{X}(x, t)\rangle = \langle x, \partial_t X(\varphi(x, t), t)\rangle.$$

Hence, by (2.1), flow (1.4) can be expressed as

$$\partial_t u(x, t) = -\frac{f(x) u^p(x, t)}{\det(\nabla^2 u + uI)} + u(x, t). \tag{2.4}$$

We next show the monotonicity of functional (1.3) under flow (1.4).

Lemma 2.1. *Suppose $\mathcal{M}_t$, $t \in [0, T)$, is a solution to flow* (1.4) *in $\mathcal{K}_o$. Then*

$$\frac{d}{dt}\mathcal{J}(\Omega_t) \geq 0,$$

where $\Omega_t = \mathrm{Cl}(\mathcal{M}_t)$. Moreover, the equality holds if and only if $\mathcal{M}_t$ satisfies (1.1).

Proof. The following formulas can be found in [35]:

$$\frac{\partial_t r}{r}(\xi, t) = \frac{\partial_t u}{u}(\mathcal{A}_{\mathcal{M}_t}(\xi), t), \quad |\mathrm{Jac}\,\mathcal{A}|(\xi) = \frac{r^{n+1}K(\vec{r}(\xi, t))}{u(\mathcal{A}_{\mathcal{M}_t}(\xi))}, \tag{2.5}$$

where $\mathrm{Jac}\,\mathcal{A}$ is the Jacobian of the radial Gauss mapping.

By virtue of (2.1)–(2.5), we obtain

$$\begin{aligned}
\frac{d}{dt}\mathcal{J}(\Omega_t) &= -\int_{\mathbb{S}^n} f u^{p-1}\partial_t u(x) d\sigma_{\mathbb{S}^n}(x) + \int_{\mathbb{S}^n} r^n \partial_t r(\xi) d\sigma_{\mathbb{S}^n}(\xi) \\
&= \int_{\mathbb{S}^n} \Big(\frac{1}{K} - f u^{p-1}\Big)\partial_t u(x) d\sigma_{\mathbb{S}^n}(x) \\
&= \int_{\mathbb{S}^n} \Big(\frac{1}{K} - f u^{p-1}\Big)^2 uK d\sigma_{\mathbb{S}^n} \geq 0.
\end{aligned}$$

Clearly, the equality $\frac{d}{dt}\mathcal{J}(\Omega_t) = 0$ holds if and only if $u(\cdot, t)$ satisfies (1.1). ■

The proof of Lemma 2.1 also verifies that (1.1) is the Euler–Lagrange equation of functional (1.3).

2.1. Properties of functional (1.3)

Next, we prove property (P) stated in the introduction.

Lemma 2.2. *Suppose that $p < -n - 1$ and $\frac{1}{c_0} \leq f \leq c_0$ for some $c_0 \geq 1$. For any given constant $A > 0$, there exists a small constant $d_0 > 0$ depending only on n, p, c_0 and A such that if $\Omega \in \mathcal{K}_o$ satisfies $\mathrm{dist}(O, \partial\Omega) \in (0, d_0)$, then $\mathcal{J}(\Omega) > A$.*

Proof. Denote by $d = \mathrm{dist}(O, \partial\Omega) > 0$. Take $x_0 \in \mathbb{S}^n$ such that

$$u(x_0) = \min_{\mathbb{S}^n} u = d,$$

where u is the support function of Ω. Let E be the minimum ellipsoid of Ω. We choose the coordinates such that

$$E - \zeta_E = \Big\{z \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} \frac{z_i^2}{a_i^2} \leq 1\Big\},$$

$$x_0 \cdot \mathbf{e}_{n+1} = \max\{|x_0 \cdot \mathbf{e}_i| : 1 \leq i \leq n + 1\},$$

where ζ_E is the centre of E. This implies that $x_0 \cdot \mathbf{e}_{n+1} \geq c_n$. We use c_n to denote a constant which depends only on n, but it may change from line to line.

Let $w(x) = u(x) + u(-x)$, $x \in \mathbb{S}^n$, be the width function of Ω. Since the ball $B_d(0)$ is contained in Ω and $\frac{1}{n+1}E \subset \Omega \subset E$, we have

$$d \le \min_{\mathbb{S}^n} w \le c_n a_{n+1} \quad \text{and} \quad w(\mathbf{e}_i) \le c_n a_i.$$

This yields that

$$\mathcal{J}(\Omega) > \mathrm{Vol}(\Omega) \ge c_n \prod_{i=1}^{n+1} a_i \ge c_n d \prod_{i=1}^{n} w(\mathbf{e}_i) \ge c_n d \prod_{i=1}^{n} u(\mathbf{e}_i). \tag{2.6}$$

Next, we consider the set $\Omega^*_{n+1} = \Omega^* \cap L$, where Ω^* is the polar dual of Ω and $L = \{z \in \mathbb{R}^{n+1} : z \cdot \mathbf{e}_{n+1} = 0\}$. Let r^* be the radial function of Ω^*. Since the origin O and points $r^*(\mathbf{e}_i)\mathbf{e}_i$, $i = 1, \ldots, n$, are contained in Ω^*_{n+1}, their convex hull is an n-dimensional convex set in Ω^*_{n+1}, namely,

$$\mathcal{C} =: \text{convex hull of } \{O, r^*(\mathbf{e}_1)\mathbf{e}_1, \ldots, r^*(\mathbf{e}_n)\mathbf{e}_n\} \subset \Omega^*_{n+1}.$$

Let V be the cone in $\mathbb{R}^{n+1}$ with base $\mathcal{C}$ and vertex $p_0 = r^*(x_0)x_0$. Since $r^*(x) = \frac{1}{u(x)}$ and $u(x_0) = d$, the height of the cone V (in the direction of $\mathbf{e}_{n+1}$) satisfies

$$r^*(x_0)x_0 \cdot \mathbf{e}_{n+1} = \frac{x_0 \cdot \mathbf{e}_{n+1}}{d} \ge \frac{c_n}{d}. \tag{2.7}$$

Consider the following subset of V:

$$V' = \Big\{z \in V : z_{n+1} \ge \frac{r^*(x_0)}{2} x_0 \cdot \mathbf{e}_{n+1}\Big\}.$$

By (2.7), we have

$$|z| \ge \frac{c_n}{2d} \quad \forall z \in V'. \tag{2.8}$$

In view of $V' \subset \Omega^*$, $f \ge \frac{1}{c_0}$ and (2.8), we have

$$\mathcal{J}(\Omega) \ge -\frac{1}{p}\int_{\mathbb{S}^n} u^p f \ge \frac{1}{c_0}\int_{\Omega^*} |z|^{-p-n-1} dz \ge \frac{1}{c_0}\int_{V'} |z|^{-p-n-1} dz \ge \frac{c_n \mathrm{Vol}(V')}{c_0 d^{-p-n-1}}.$$

Since

$$\mathrm{Vol}(V') \ge \frac{c_n}{d}\mathrm{Area}(\mathcal{C}), \quad \mathrm{Area}(\mathcal{C}) \ge c_n \prod_{i=1}^{n} r^*(\mathbf{e}_i) = c_n \Big[\prod_{i=1}^{n} u(\mathbf{e}_i)\Big]^{-1},$$

where $\mathrm{Area}(\mathcal{C})$ is the n-dimensional Hausdorff measure of $\mathcal{C}$, we further obtain that

$$\mathcal{J}(\Omega) \ge \frac{c_n}{c_0 d^{-p-n}}\Big[\prod_{i=1}^{n} u(\mathbf{e}_i)\Big]^{-1}. \tag{2.9}$$

Combining (2.6) and (2.9), we have

$$[\mathcal{J}(\Omega)]^2 \ge \frac{c_n}{c_0 d^{-p-n-1}}.$$

Since $-p-n-1 > 0$, we see that $\mathcal{J}(\Omega) > A$ if d is sufficiently small. ∎

Lemma 2.3. *Suppose that $p < -n-1$ and $\frac{1}{c_0} \le f \le c_0$ for some $c_0 \ge 1$. For any given constant $A > 1$, there exists a small constant $v > 0$ depending only on n, p, c_0, and A, such that if $\Omega \in \mathcal{K}_o$ satisfies either $\mathrm{Vol}(\Omega) \le v$ or $\mathrm{Vol}(\Omega) \ge v^{-1}$, then $\mathcal{J}(\Omega) > A$.*

Proof. If $\mathrm{Vol}(\Omega) \ge v^{-1}$, by definition we have $\mathcal{J}(\Omega) > \mathrm{Vol}(\Omega) \ge v^{-1} > A$ by taking v small.

If $\mathrm{Vol}(\Omega) \le v$, we denote $d = \mathrm{dist}(O, \partial\Omega)$. Since the ball $B_d(0) \subset \Omega$, we have

$$v \ge \mathrm{Vol}(\Omega) \ge \mathrm{Vol}(B_d) = \mathrm{Vol}(B_1) d^{n+1}.$$

Hence, if v is sufficiently small, then $d < d_0$, where $d_0 > 0$ is the constant given by Lemma 2.2. Therefore, we have $\mathcal{J}(\Omega) > A$ by Lemma 2.2. ∎

Lemma 2.4. *Suppose that $p < -n-1$ and $\frac{1}{c_0} \le f \le c_0$ for some $c_0 \ge 1$. For any given constant $A > 0$, there exists a large constant $e > 1$ depending only on n, p, c_0 and A, such that if $\Omega \in \mathcal{K}_o$ satisfies $e_\Omega \ge e$, we have $\mathcal{J}(\Omega) > A$.*

Proof. For the given constant A, let d_0 be the constant determined by Lemma 2.2. We assume that $d = \mathrm{dist}(O, \partial\Omega) \ge d_0$; otherwise, we are done by Lemma 2.2.

Let E be the minimum ellipsoid of Ω with semi-axes $a_1 \le \cdots \le a_{n+1}$. Note that $\frac{a_{n+1}}{a_1} = e_\Omega$. Since $B_{d_0} \subset \Omega$ and $\Omega \subset E$, we obtain that

$$d_0 \le d \le a_1.$$

Noting also that $\frac{1}{n+1} E \subset \Omega$, we have

$$\mathcal{J}(\Omega) > \mathrm{Vol}(\Omega) \ge c_n \mathrm{Vol}(E) = c_n \prod_{i=1}^{n+1} a_i \ge c_n e_\Omega {d_0}^{n+1}.$$

Clearly, we have $\mathcal{J}(\Omega) > A$ if e_Ω is sufficiently large. ∎

2.2. *A priori estimates for the parabolic equation* (2.4)

In this subsection, we state the a priori estimates for the solution u, assuming the uniform estimate for u.

Theorem 2.5. *Let f be a positive and $C^{1,1}$-smooth function on $\mathbb{S}^n$. Let $u(\cdot, t)$ be a positive, smooth and uniformly convex solution to (2.4), $t \in [0, T)$. Assume that*

$$\frac{1}{C_0} \le u(x,t) \le C_0, \quad |\nabla u|(x,t) \le C_0 \tag{2.10}$$

for all $(x,t) \in \mathbb{S}^n \times [0,T)$. Then

$$C^{-1} I \le (\nabla^2 u + uI)(x,t) \le CI \quad \forall (x,t) \in \mathbb{S}^n \times [0,T), \tag{2.11}$$

where C is a positive constant depending only on n, p, C_0, $\min_{\mathbb{S}^n} f$, $\|f\|_{C^{1,1}(\mathbb{S}^n)}$, and the initial condition $u(\cdot, 0)$.

The proof of Theorem 2.5 is based on proper choice of auxiliary functions and will be given in Section 5.

By the second derivative estimate (2.11), equation (2.4) becomes uniformly parabolic. We find by differentiating (2.4) with respect to the time variable that $\partial_t u$ satisfies a uniformly parabolic equation. Applying the Krylov–Safonov's Hölder regularity [32] to the linearised equation satisfied by $\partial_t u$, we obtain the space-time Hölder estimates for $\partial_t u$. We then apply the Evans–Krylov theorem (see, e.g., [21, Section 17.4]) to the uniformly elliptic equation satisfied by $u(\cdot, t)$ (fixing a t)

$$\det(\nabla^2 u + uI) = \frac{f u^p}{u - \partial_t u}$$

and so obtain a space Hölder estimate for $\nabla^2 u(\cdot, t)$ for each t. The Hölder estimate for $\nabla^2 u$ in t can be obtained as in [44]. Differentiating (2.4) with respect to the spatial variable, we see that ∇u satisfies a uniformly parabolic equation with space-time Hölder continuous coefficients. By the Schauder estimates (see, e.g., [36, Section 4.3]), we have the space-time $C^{2,\alpha}$-estimates for ∇u, which yields

$$\|u(\cdot, t)\|_{C^{3,\alpha}(\mathbb{S}^n)} \le C \quad \forall t \in [0, T), \tag{2.12}$$

for any given $\alpha \in (0, 1)$, where the constant C depends only on α, n, p, $\min_{\mathbb{S}^n} f$, $\|f\|_{C^{1,1}(\mathbb{S}^n)}$, and the initial condition $u(\cdot, 0)$. By the a priori estimates (2.12), we have the long-time existence of solutions to flow (1.4), provided that u satisfies (2.10).

Theorem 2.6. *Let f be a positive and $C^{1,1}$-smooth function on $\mathbb{S}^n$. Let $T_{\max}$ be the maximal time such that $u(\cdot, t)$ is a positive, $C^{3,\alpha}$-smooth, and uniformly convex solution to (2.4) on $[0, T_{\max})$. If (2.10) holds for all the time $t \in [0, T_{\max})$, then $T_{\max} = \infty$ and u satisfies estimates (2.11) and (2.12).*

Remark 2.7. Let $\mathcal{M}_t|_{t\in[0,T_{\max})}$ be a $C^{3,\alpha}$-smooth and uniformly convex solution to (1.4) with $\mathrm{Cl}(\mathcal{M}_t) \in \mathcal{K}_o$. By Lemmas 2.2, 2.3 and 2.4, if $\mathcal{J}(\mathcal{M}_t) < A$ for some constant A independent of t, then there exist positive constants e, v, d depending on A, but independent of t, such that

$$e_{\mathcal{M}_t} \le e, \quad v \le \mathrm{Vol}(\Omega_t) \le v^{-1}, \quad \text{and} \quad B_d(0) \subset \Omega_t, \tag{2.13}$$

where Ω_t is the convex body enclosed by $\mathcal{M}_t$.

From (2.13), one infers that (2.10) holds. Hence, the a priori estimates (2.11) and (2.12) hold, and one has the long-time existence of solution (Theorem 2.6). Therefore, for the a priori estimates (2.11), (2.12) and the long-time existence of solution, all we need is that the condition $\mathcal{J}(\mathcal{M}_t) < A$ holds for some constant A.

3. Proof of Theorem 1.1

In this section, we show how to select an initial hypersurface $\mathcal{N}_0$, such that flow (1.4) deforms $\mathcal{N}_0$ to a solution of (1.1).

The initial hypersurface $\mathcal{N}_0$ is an ellipsoid and will be chosen by a topological method. In the proof of the existence of $\mathcal{N}_0$, a key step is the computation of the homology groups of a special class of ellipsoids. The homology groups will be given in Proposition 3.6 and Theorem 3.7, whose proofs are postponed to the next section.

Denote

$$A_0 = 2\Big(-\frac{\|f\|_{L^1(\mathbb{S}^n)}}{p[2(n+1)]^p} + 2^{n+1}\,\mathrm{Vol}(B_1)\Big), \tag{3.1}$$

where $B_1 = B_1(0)$ is the unit ball in $\mathbb{R}^{n+1}$ centred at the origin. Note that we always use $B_1 = B_1(0)$ for short in our paper. Recall that $p < -n-1$. Hence, for any $\Omega \in \mathcal{K}_o$ with $\frac{1}{2(n+1)}B_1 \subseteq \Omega \subseteq 2B_1$, we have

$$\mathcal{J}(\Omega) \le \frac{1}{2}A_0. \tag{3.2}$$

In particular, if the minimum ellipsoid of Ω is $B_1 = B_1(0)$, then $\frac{1}{n+1}B_1 \subset \Omega \subset B_1$ and hence

$$\mathcal{J}(\Omega) \le -\frac{1}{p(n+1)^p}\int_{\mathbb{S}^n} f + \mathrm{Vol}(B_1) \le \frac{1}{2}A_0. \tag{3.3}$$

3.1. A modified flow of (1.4)

We introduce a modified flow of (1.4) such that for any initial condition, the solution exists for all time $t \ge 0$. The purpose of introducing this modified flow is for the convenience of later discussion.

For a closed, smooth and uniformly convex hypersurface $\mathcal{N}$ such that

$$\Omega_0 = \mathrm{Cl}(\mathcal{N}) \in \mathcal{K}_o,$$

we define a family of time-depending hypersurfaces $\bar{\mathcal{M}}_{\mathcal{N}}(t)$ with initial condition $\mathcal{N}$ as follows:

- If $\mathcal{J}(\mathcal{M}_{\mathcal{N}}(t)) < \frac{9}{10}A_0$ for all time $t \ge 0$, let $\bar{\mathcal{M}}_{\mathcal{N}}(t) = \mathcal{M}_{\mathcal{N}}(t)$ for all $t \ge 0$, where $\mathcal{M}_{\mathcal{N}}(t)$ is the solution to (1.4). We point out that, by Remark 2.7, the solution $\mathcal{M}_{\mathcal{N}}(t)$ exists as long as $\mathcal{J}(\mathcal{M}_{\mathcal{N}}(t))$ is finite.
- If $\mathcal{J}(\mathcal{N}) < \frac{9}{10}A_0$, and $\mathcal{J}(\mathcal{M}_{\mathcal{N}}(t))$ reaches $\frac{9}{10}A_0$ at the first time $t_0 > 0$, we define

$$\bar{\mathcal{M}}_{\mathcal{N}}(t) = \begin{cases} \mathcal{M}_{\mathcal{N}}(t) & \text{if } 0 \le t < t_0, \\ \mathcal{M}_{\mathcal{N}}(t_0) & \text{if } t \ge t_0. \end{cases}$$

- If $\mathcal{J}(\mathcal{N}) \ge \frac{9}{10}A_0$, we let $\bar{\mathcal{M}}_{\mathcal{N}}(t) \equiv \mathcal{N}$ for all $t \ge 0$. That is, the solution is stationary.

For convenience, we call $\bar{\mathcal{M}}_{\mathcal{N}}(t)$ as a modified flow of (1.4).

Remark 3.1. Apparently, $\mathrm{Cl}(\bar{\mathcal{M}}_{\mathcal{N}}(t)) \in \mathcal{K}_o$, and it is easy to verify the following properties:

- $\bar{\mathcal{M}}_{\mathcal{N}}(t)$ is defined for all time $t \ge 0$, and by Lemma 2.1, $\mathcal{J}(\bar{\mathcal{M}}_{\mathcal{N}}(t))$ is non-decreasing.
- By Lemma 2.2, if $\mathrm{dist}(O, \mathcal{N})$ is very small, then $\bar{\mathcal{M}}_{\mathcal{N}}(t) \equiv \mathcal{N}$ for all $t \ge 0$.

- By Lemma 2.3, if $\mathrm{Vol}(\Omega_0)$ is sufficiently large or small, then $\bar{\mathcal{M}}_{\mathcal{N}}(t) \equiv \mathcal{N}$ for all $t \geq 0$.
- By Lemma 2.4, if e_{Ω_0} is sufficiently large, then $\bar{\mathcal{M}}_{\mathcal{N}}(t) \equiv \mathcal{N}$ for all $t \geq 0$.
- We have $\mathcal{J}(\bar{\mathcal{M}}_{\mathcal{N}}(t)) \leq \max\{A_0, \mathcal{J}(\mathcal{N})\}$ for all $t \geq 0$.
- By the a priori estimates, $\bar{\mathcal{M}}_{\mathcal{N}}(t)$ is smooth for any fixed time t, and Lipschitz continuous in time t.

3.2. A special class of ellipsoids $\mathcal{A}_I$

Lemma 3.2. *For the constant A_0 given by (3.1), there exist sufficiently small constants $\bar{v}$ and $\bar{d}$, and a sufficiently large constant $\bar{e}$, such that for any $\Omega \in \mathcal{K}_o$,*

(i) *if* $\mathrm{dist}(O, \partial\Omega) \leq \bar{d}$, *then* $\mathcal{J}(\Omega) > A_0$;

(ii) *if* $e_\Omega \geq \bar{e}$, *then* $\mathcal{J}(\Omega) > A_0$;

(iii) *if* $\mathrm{Vol}(\Omega) \leq \bar{v}$ *or* $\mathrm{Vol}(\Omega) \geq [(n+1)^{n+1}\bar{v}]^{-1}$, *then* $\mathcal{J}(\Omega) > A_0$.

Proof. This is an immediate consequence of Lemmas 2.2, 2.3 and 2.4. See also Remark 2.7. ■

Fix the constants $\bar{d}$, $\bar{v}$ and $\bar{e}$ as in Lemma 3.2. We introduce the following notations:

- $\mathcal{K}$ is the collection of all non-empty, compact and convex sets in $\mathbb{R}^{n+1}$ equipped with the Hausdorff distance, such that $\mathcal{K}$ is a metric space.
- $\bar{\mathcal{K}}_o$ is the closure of $\mathcal{K}_o$ in $\mathcal{K}$.
- $\mathcal{K}_e$ is the subset of $\mathcal{K}_o$ which consists of all origin-symmetric convex bodies.
- Denote

$$\mathcal{A}_I = \left\{E \in \bar{\mathcal{K}}_o \text{ is an ellipsoid in } \mathbb{R}^{n+1} : \ \bar{v} \leq \mathrm{Vol}(E) \leq \frac{1}{\bar{v}} \text{ and } e_E \leq \bar{e}\right\},$$
$$\hat{\mathcal{A}} = \{E \in \mathcal{A}_I : \mathrm{Vol}(E) = \omega_n \text{ and } e_E \leq \bar{e}\}, \quad \text{where } \omega_n = \mathrm{Vol}(B_1),$$
$$\mathcal{A} = \{E \in \hat{\mathcal{A}} : \text{either } e_E = \bar{e} \text{ or } \mathrm{dist}(O, \partial E) = 0\}.$$

- To calculate the homology of $\mathcal{A}_I$, we also introduce

$$\mathcal{E}_I = \mathcal{A}_I \cap \mathcal{K}_e, \quad \hat{\mathcal{E}} = \hat{\mathcal{A}} \cap \mathcal{K}_e, \quad \mathcal{E} = \mathcal{A} \cap \mathcal{K}_e. \tag{3.4}$$

The sets $\mathcal{A}_I$, $\hat{\mathcal{A}}$, $\mathcal{A}$ and $\mathcal{E}_I$, $\hat{\mathcal{E}}$, $\mathcal{E}$ are all closed. In particular, $\mathcal{E}$ (resp. $\mathcal{A}$) is the boundary of $\hat{\mathcal{E}}$ (resp. $\hat{\mathcal{A}}$) in the space of all ellipsoids in $\mathcal{K}_e$ (resp. $\mathcal{K}$) with unit ball volume.

For any $E \in \mathcal{E}$, let $\mathcal{R}$ be a rotation such that

$$\mathcal{R}(E) = \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} \frac{x_i^2}{a_i^2} \leq 1\right\}.$$

Since $\mathrm{Vol}(E) = \omega_n$, we have $\prod_{i=1}^{n+1} a_i = 1$. For $s \in [0, 1]$, denote

$$b_i(s) = (1-s) + s a_i \quad \text{and} \quad \hat{a}_i(s) = \frac{b_i(s)}{[\prod_{j=1}^{n+1} b_j(s)]^{1/(n+1)}}.$$

We then obtain an ellipsoid $\widehat{E}(s)$ such that

$$\mathcal{R}(\widehat{E}(s)) = \left\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} \frac{x_i^2}{(\widehat{a}_i(s))^2} \le 1\right\}.$$

In particular, $\widehat{E}(0) = B_1$, $\widehat{E}(1) = E$. The set $\{\widehat{E}(s)\}_{s\in[0,1]}$ is a path in $\mathcal{K}_e$ connecting B_1 to E, and satisfies $\mathrm{Vol}(\widehat{E}(s)) = \omega_n$ for all $s \in [0, 1]$. We assert

$$\widehat{\mathcal{E}} = \{\widehat{E}(s) : E \in \mathcal{E},\ s \in [0, 1]\}. \tag{3.5}$$

To see (3.5), it suffices to show that, for any $\widehat{E} \in \widehat{\mathcal{E}}$, there is a path $\{\widehat{E}(s)\}_{s\in[0,1]}$ with the above properties goes through $\widehat{E}$. Assume without loss of generality (up to a rotation as above) that

$$\widehat{E} = \left\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} \frac{x_i^2}{a_i'^2} \le 1\right\}$$

with $0 < a_1' \le \cdots \le a_{n+1}'$, $\frac{a_{n+1}'}{a_1'} \le \overline{e}$ and $\prod_{i=1}^{n+1} a_i' = 1$. If

$$\frac{a_{n+1}'}{a_1'} = \overline{e} \quad \text{or} \quad \frac{a_{n+1}'}{a_1'} = 1,$$

then the result clearly holds. Let

$$\beta_0 = \frac{a_{n+1}'/a_1' - 1}{\overline{e} - 1}, \quad b_i'(s) = (1 - s) + \frac{a_i'/a_1' - 1 + \beta_0}{\beta_0} s,$$

and

$$\widehat{a}_i'(s) = \frac{b_i'(s)}{[\prod_{j=1}^{n+1} b_j'(s)]^{1/(n+1)}}.$$

For $s \in [0, 1]$, we consider the ellipsoid

$$\widehat{E}(s) = \left\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} \frac{x_i^2}{\widehat{a}_i'(s)^2} \le 1\right\}.$$

It is not hard to verify that for $\beta_0 \in [0, 1]$, we have

$$\widehat{E}(\beta_0) = \widehat{E}, \quad \widehat{E}(1) \in \mathcal{E}, \quad \text{and} \quad \widehat{E}(s) \in \widehat{\mathcal{E}} \ \ \forall s \in [0, 1].$$

Hence, (3.5) holds. As a result,

$$\mathcal{E}_I = \left\{\left[\frac{\tau}{\omega_n}\right]^{1/(n+1)} \widehat{E} : \widehat{E} \in \widehat{\mathcal{E}},\ \tau \in \left[\overline{v}, \frac{1}{\overline{v}}\right]\right\}. \tag{3.6}$$

Let us compute the homology of the spaces $\mathcal{E}_I$ and $\widehat{\mathcal{E}}$. We refer the readers to Appendix A for the notions and results in homology theory used in our argument.

Lemma 3.3. *Both $\mathcal{E}_I$ and $\widehat{\mathcal{E}}$ are contractible. Hence, the homology*

$$H_k(\mathcal{E}_I) = H_k(\widehat{\mathcal{E}}) = 0 \quad \forall k \ge 1. \tag{3.7}$$

Proof. To see that $\widehat{\mathcal{E}}$ is contractible, by (3.5) it suffices to notice that

$$(\widehat{E}(s), t) \to \widehat{E}((1-t)s),$$

where $t \in [0, 1]$, is a deformation retraction from $\widehat{\mathcal{E}} \to \{B_1\}$. Hence, $\widehat{\mathcal{E}}$ is contractible.

Similarly, by (3.6),

$$\left(\left[\frac{\tau}{\omega_n}\right]^{1/(n+1)} \widehat{E}, t\right) \mapsto \left[\frac{(1-t)\tau + t\omega_n}{\omega_n}\right]^{1/(n+1)} \widehat{E}$$

is a deformation retraction from $\mathcal{E}_I$ to $\widehat{\mathcal{E}}$. As $\widehat{\mathcal{E}}$ is contractible, one sees that $\mathcal{E}_I$ is also contractible. By Lemma A.4, (3.7) follows. ∎

We can combine the two deformation retractions in the above proof and obtain a new one from $\mathcal{E}_I$ to $\{B_1\}$ as follows:

$$\eta\colon \left(\left[\frac{\tau}{\omega_n}\right]^{1/(n+1)} \widehat{E}(s), t\right) \mapsto \left[\frac{(1-t)\tau + t\omega_n}{\omega_n}\right]^{1/(n+1)} \widehat{E}((1-t)s), \tag{3.8}$$

where $s \in [0, 1]$, $\tau \in [\overline{v}, \frac{1}{\overline{v}}]$.

In the following, the notation $\mathcal{H} \simeq \mathcal{H}'$ means two metric spaces $\mathcal{H}$, $\mathcal{H}'$ are homeomorphic.

Lemma 3.4. *We have $\mathcal{A}_I \simeq \mathcal{E}_I \times B_1$ and $\widehat{\mathcal{A}} \simeq \widehat{\mathcal{E}} \times B_1$ (with the product topology). It follows that both $\mathcal{A}_I$ and $\widehat{\mathcal{A}}$ are contractible, and hence the homology*

$$H_k(\mathcal{A}_I) = H_k(\widehat{\mathcal{A}}) = 0 \quad \forall k \geq 1. \tag{3.9}$$

Proof. For any given ellipsoid $E \in \mathcal{K}_e$, the map

$$\vec{\varrho}_E(y) = \begin{cases} \dfrac{y}{r_E(y/|y|)} & \text{if } y \in E \setminus \{0\}, \\ 0 & \text{if } y = 0 \end{cases}$$

defines a homeomorphism between E and B_1, where r_E is the radial function of E (see (2.2)). Denote by $\vec{\varrho}^{\,*}_E$ the inverse of $\vec{\varrho}_E$.

We now show $\mathcal{A}_I \simeq \mathcal{E}_I \times B_1$. For $E \in \mathcal{A}_I$, let ζ_E be its centre. Then

$$E_o =: E - \zeta_E \in \mathcal{E}_I.$$

We define a map $\phi\colon \mathcal{A}_I \to \mathcal{E}_I \times B_1$ by

$$\phi(E) = (E - \zeta_E, \vec{\varrho}_{E_o}(\zeta_E)). \tag{3.10}$$

Its inverse $\phi^*\colon \mathcal{E}_I \times B_1 \to \mathcal{A}_I$ is given by

$$\phi^*(E, y) = E + \vec{\varrho}^{\,*}_E(y). \tag{3.11}$$

It is easy to verify that $\phi^* \circ \phi = \mathrm{id}_{\mathcal{A}_I}$, $\phi \circ \phi^* = \mathrm{id}_{\mathcal{E}_I \times B_1}$ and both ϕ and ϕ^* are continuous.

Restricting ϕ and ϕ^* to $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{E}} \times B_1$, respectively, we see that $\widehat{\mathcal{A}} \simeq \widehat{\mathcal{E}} \times B_1$. ∎

Denote

$$\mathcal{P} = \Big\{E \in \mathcal{A}_I : \text{either } \operatorname{Vol}(E) = \overline{v}, \text{ or } \operatorname{Vol}(E) = \frac{1}{\overline{v}}, \text{ or } e_E = \overline{e}, \text{ or } O \in \partial E\Big\}.$$

Note that $\mathcal{P}$ is the boundary of $\mathcal{A}_I$ if we regard $\mathcal{A}_I$ as a set in the topological space of all ellipsoids.

Lemma 3.5. *There is a retraction Ψ from $\mathcal{A}_I \setminus \{B_1\}$ to $\mathcal{P}$. Namely, $\Psi\colon \mathcal{A}_I \setminus \{B_1\} \to \mathcal{P}$ is continuous and $\Psi|_{\mathcal{P}} = \mathrm{id}$.*

Proof. The retraction Ψ can be constructed as follows.

(1) By the map ϕ in (3.10), we have $\mathcal{A}_I \setminus \{B_1\} \simeq \mathcal{E}_I \times B_1 \setminus (\{B_1\} \times \{0\})$.

(2) Let $\partial\mathcal{E}_I$ be the topological boundary of $\mathcal{E}_I$, regarding $\mathcal{E}_I$ as a set in the space of all ellipsoids in $\mathcal{K}_e$. Then $\partial\mathcal{E}_I = \{E \in \mathcal{E}_I : \text{either } \operatorname{Vol}(E) = \overline{v}, \text{ or } \operatorname{Vol}(E) = \frac{1}{\overline{v}}, \text{ or } e_E = \overline{e}\}$. In this step, we define a continuous map $\psi\colon (\mathcal{E}_I \times B_1) \setminus (\{B_1\} \times \{0\}) \to (\partial\mathcal{E}_I \times B_1) \cup (\mathcal{E}_I \times \partial B_1)$.

Recall the deformation retraction η from $\mathcal{E}_I$ to $\{B_1\}$ in (3.8). For any $t \neq 1$, $\eta(\cdot, t)$ defines a homeomorphism between $\mathcal{E}_I$ and $\eta(\mathcal{E}_I, t)$. For any $E \in \mathcal{E}_I \setminus \{B_1\}$, let

$$t_E = \sup\{t \in [0,1] : \text{there exists } E' \in \mathcal{E}_I \text{ such that } \eta(E', t) = E\}.$$

We have $t_E < 1$. Since $\mathcal{E}_I$ is closed, there exists $\widetilde{E} \in \partial\mathcal{E}_I \subset \mathcal{E}_I$ such that $\eta(\widetilde{E}, t_E) = E$. For any given $t \in [0,1)$, η also defines a homeomorphism between ∂E_I and $\eta(\partial E_I, t)$, and $\eta(\cdot, 0)$ is the identity map on ∂E_I. Define a map $\psi_1\colon \mathcal{E}_I \setminus \{B_1\} \to \partial\mathcal{E}_I$ by letting $\psi_1(E) = \widetilde{E}$, where $\widetilde{E}$ satisfies $\eta(\widetilde{E}, t_E) = E$. Then ψ_1 is a retraction from $\mathcal{E}_I \setminus \{B_1\}$ to $\partial\mathcal{E}_I$ such that $\psi_1 = \mathrm{id}$ on $\partial\mathcal{E}_I$.

Let $\psi_2(x) = \frac{x}{|x|}$. Then ψ_2 is a retraction $B_1 \setminus \{0\}$ to ∂B_1 such that $\psi_2 = \mathrm{id}$ on ∂B_1.

Combining ψ_1 and ψ_2, we can define the map ψ by letting

$$\psi\colon (E, x) \to \begin{cases} \left(\psi_1(E), \dfrac{x}{1 - t_E}\right) & \text{if } 1 - t_E \geq |x|, \\ \left(\eta\left(\psi_1(E), 1 - \dfrac{1 - t_E}{|x|}\right), \psi_2(x)\right) & \text{if } 1 - t_E < |x|. \end{cases}$$

(3) By the map ϕ^* in (3.11), we have $\partial(\mathcal{E}_I \times B_1) \simeq \mathcal{P}$.

(4) Let $\Psi = \phi^* \circ \psi \circ \phi\colon \mathcal{A}_I \setminus \{B_1\} \to \mathcal{P}$. Then Ψ is a retraction from $\mathcal{A}_I \setminus \{B_1\}$ to $\mathcal{P}$. ∎

The following two results are crucial for our later argument, whose proofs are postponed to Section 4.

Proposition 3.6. *We have the following results:*

(i) $H_{k+1}(\mathcal{P}) = H_k(\mathcal{A})$ *for all* $k \geq 1$.

(ii) *There is a long exact sequence*

$$\cdots \to H_{k+1}(\mathcal{A}) \to H_k(\mathcal{E} \times \mathbb{S}^n) \to H_k(\mathcal{E}) \oplus H_k(\mathbb{S}^n) \to H_k(\mathcal{A}) \to \cdots.$$

Theorem 3.7. *Let $n^* = \frac{n(n+1)}{2}$. The homology group $H_{n^*+n-1}(\mathcal{E}) = \mathbb{Z}$.*

3.3. Selection of a good initial condition

As mentioned in the introduction, we use a topological method to prove the existence of a special initial condition such that the solution to the Gauss curvature flow (1.4) satisfies the uniform estimate (1.5).

We will employ the modified flow with initial data in $\mathcal{A}_I$. For any ellipsoid $\mathcal{N}$ such that $\mathrm{Cl}(\mathcal{N}) \in \mathcal{A}_I$, let $\bar{\mathcal{M}}_{\mathcal{N}}(t)$ be the solution to the modified flow, with the constant A_0 given in (3.1). We have the following properties:

(1) If $\mathrm{Cl}(\mathcal{N})$ is close to $\mathcal{P}$ in Hausdorff distance or on $\mathcal{P}$, we have $\mathcal{J}(\mathcal{N}) \geq \frac{9}{10} A_0$ and so $\bar{\mathcal{M}}_{\mathcal{N}}(t) \equiv \mathcal{N}$ for all t (see Lemma 3.2).

(2) If $\mathrm{Cl}(\mathcal{N})$ is close to $B_1(0)$ in Hausdorff distance, then $\mathcal{J}(\mathcal{N}) < \frac{9}{10} A_0$ (see (3.2)).

(3) By our definition of the modified flow, if $\mathcal{J}(\bar{\mathcal{M}}_{\mathcal{N}}(t)) < A_0$ for all $t \geq 0$, then by Remark 2.7, we have

$$e_{\bar{\mathcal{M}}_{\mathcal{N}}(t)} \leq \bar{e}, \quad \bar{v} \leq \mathrm{Vol}(\bar{\Omega}_{\mathcal{N}}(t)) \leq \frac{1}{\bar{v}}, \quad \text{and} \quad B_{\bar{d}}(0) \subset \bar{\Omega}_{\mathcal{N}}(t) \quad \forall t \geq 0, \tag{3.12}$$

where $\bar{\Omega}_{\mathcal{N}}(t) = \mathrm{Cl}(\bar{\mathcal{M}}_{\mathcal{N}}(t))$, the convex body enclosed by $\bar{\mathcal{M}}_{\mathcal{N}}(t)$. Here the bar over Ω means that $\bar{\Omega}_{\mathcal{N}}(t)$ is the convex body enclosed by the modified flow $\bar{\mathcal{M}}_{\mathcal{N}}(t)$, not the closure of $\Omega_{\mathcal{N}}(t)$. In our notation, $\Omega_{\mathcal{N}}$ is a closed convex body.

With these properties, we can prove the following key lemma.

Lemma 3.8. *For every $t > 0$, there exists $\mathcal{N} = \mathcal{N}_t$ with $\mathrm{Cl}(\mathcal{N}) \in \mathcal{A}_I$, such that the minimum ellipsoid of $\bar{\mathcal{M}}_{\mathcal{N}}(t)$ is the unit ball $B_1(0)$.*

Proof. Suppose by contradiction that there is $t' > 0$ such that, for any $\Omega \in \mathcal{A}_I$, $E_{\mathcal{N}}(t') \neq B_1(0)$, where $\mathcal{N} = \partial\Omega$ and $E_{\mathcal{N}}(t')$ is the minimum ellipsoid of $\bar{\Omega}_{\mathcal{N}}(t') := \mathrm{Cl}(\bar{\mathcal{M}}_{\mathcal{N}}(t'))$. Then we have

$$E_{\mathcal{N}}(t') \in \mathcal{A}_I. \tag{3.13}$$

This is obvious when $\bar{\Omega}_{\mathcal{N}}(t) \equiv \Omega$ for all t. If $\bar{\Omega}_{\mathcal{N}}(t)$ is not identical to Ω, then $\mathcal{J}(\bar{\mathcal{M}}_{\mathcal{N}}(t)) \leq A_0$. By Lemma 3.2, we have $e_{\Omega_{\mathcal{N}}(t')} \leq \bar{e}$ and

$$\bar{v} \leq \mathrm{Vol}(\bar{\Omega}_{\mathcal{N}}(t')) \leq [(n+1)^{n+1}\bar{v}]^{-1}.$$

Since $\frac{1}{n+1}(E_{\mathcal{N}}(t') - \zeta) + \zeta \subset \bar{\Omega}_{\mathcal{N}}(t') \subset E_{\mathcal{N}}(t')$ (here ζ denotes the centre of $E_{\mathcal{N}}(t')$), we have

$$\mathrm{Vol}(\bar{\Omega}_{\mathcal{N}}(t')) \leq \mathrm{Vol}(E_{\mathcal{N}}(t')) \leq (n+1)^{n+1}\,\mathrm{Vol}(\bar{\Omega}_{\mathcal{N}}(t')).$$

As a result,

$$\bar{v} \leq \mathrm{Vol}(E_{\mathcal{N}}(t')) \leq (n+1)^{n+1}[(n+1)^{n+1}\bar{v}]^{-1} = \bar{v}^{-1}.$$

Therefore, (3.13) holds.

Hence, we can define a continuous map $T\colon \mathcal{A}_I \to \mathcal{P}$ by

$$\Omega \in \mathcal{A}_I \mapsto E_{\mathcal{N}}(t') \in \mathcal{A}_I \setminus \{B_1\} \mapsto \Psi(E_{\mathcal{N}}(t')) \in \mathcal{P},$$

where Ψ is given in Lemma 3.5 and $B_1 = B_1(0)$. Note that when $\Omega \in \mathcal{P}$, we have $\mathcal{J}(\Omega) \geq \frac{9}{10} A_0$ and thus $E_{\mathcal{N}}(t') = E_{\mathcal{N}}(0) = \Omega$. This implies that $T|_{\mathcal{P}} = \mathrm{id}_{\mathcal{P}}$. Hence, T is a retraction from $\mathcal{A}_I$ to $\mathcal{P}$, and so there is an injection from $H_*(\mathcal{P})$ to $H_*(\mathcal{A}_I)$. By (3.9), we then have

$$H_k(\mathcal{P}) = 0 \quad \forall k \geq 1.$$

It follows from Proposition 3.6 (ii) that

$$H_k(\mathcal{E} \times \mathbb{S}^n) = H_k(\mathcal{E}) \oplus H_k(\mathbb{S}^n) \quad \forall k \geq 1.$$

Computing the left-hand side by the Künneth formula (Lemma A.7), we further obtain

$$H_k(\mathcal{E}) \oplus H_{k-n}(\mathcal{E}) = H_k(\mathcal{E}) \oplus H_k(\mathbb{S}^n) \quad \forall k > n. \tag{3.14}$$

However, this contradicts Theorem 3.7 by taking $k = n^* + 2n - 1$ in the above. ■

Remark 3.9. Lemma 3.8 asserts that for any $t > 0$, there is an initial hypersurface $\mathcal{N} = \mathcal{N}_t$ such that the minimum ellipsoid $E_{\mathcal{N}}(t)$ is the unit ball $B_1(0)$, even in the case when the L_p-Minkowski problem (1.1) has a unique solution $\partial B_R(z)$ for $R \neq 1$ and $z \neq 0$. In fact, our topological argument implies that for any ellipsoid $E \in \mathcal{A}_I \setminus \mathcal{P}$, and any given time $t > 0$, there is an initial hypersurface $\mathcal{N} = \mathcal{N}_t$ such that $E_{\mathcal{N}}(t) = E$.

Remark 3.10. To derive a contradiction from (3.14), we only need to show that there exists one nontrivial homology group among $H_k(\mathcal{E})$ for all $k \geq 1$. Theorem 3.7 asserts that $H_{n^*+n-1}(\mathcal{E}) = \mathbb{Z}$, which suffices for the proof of the key Lemma 3.8. By (4.31) below, $k = n^* + n - 1$ is the largest integer such that $H_k(\mathcal{E}) \neq 0$. We will not compute other homology groups of $\mathcal{E}$ in this paper, as they are not needed in our proof.

Remark 3.11. We point out that the modified flow depends continuously on its initial data: for each $\Omega \in \mathcal{A}_I$ and a sequence of $\Omega_k \in \mathcal{A}_I$ with $\lim_{k\to\infty} \Omega_k = \Omega$ in the sense of Hausdorff distance, we have $\bar{\mathcal{M}}_{\mathcal{N}_k}(t_*) \to \bar{\mathcal{M}}_{\mathcal{N}}(t_*)$ in Hausdorff metric for any fixed $t_* > 0$, where $\mathcal{N} = \partial\Omega$ and $\mathcal{N}_k = \partial\Omega_k$. The property holds for $\Omega \in \partial\mathcal{A}_I$ clearly. Let us prove the property for $\Omega \in \operatorname{Int} \mathcal{A}_I$. Denote by $u_k(\cdot, t)$ and $u(\cdot, t)$ the support functions of $\mathcal{M}_{\mathcal{N}_k}(t)$ and $\mathcal{M}_{\mathcal{N}}(t)$. By (2.4), $w_k = u_k - u$ satisfies

$$\partial_t w_k = \sum_{i,j} a_{ij}^{(k)} \nabla_{ij}^2 w_k + b^{(k)} w_k, \tag{3.15}$$

where $a_{ij}^{(k)}$ and $b^{(k)}$ depend on u and u_k up to their second derivatives, and $\{a_{ij}^{(k)}\}$ is positive-definite. Suppose T is a time satisfying $J_T := \mathcal{J}(\mathcal{M}_{\mathcal{N}}(T)) < \infty$. Let $T_k \in [0, T]$ be the first time such that $\mathcal{J}(\mathcal{M}_{\mathcal{N}_k})$ is no less than $2J_T$, and if no such time exists, let $T_k = T$. By the a priori estimates (Lemma 3.2 and Theorem 2.5), PDE (3.15) (restricting on $[0, T_k]$) is uniformly parabolic and its coefficients are uniformly bounded. By the maximum principle (see [36, Theorem 2.10]), there is C_{J_T} (depending on J_T but is independent of k) such that

$$\sup_{\mathbb{S}^n \times [0,T_k]} |u - u_k| \leq C_{J_T} \sup_{\mathbb{S}^n} |u(\cdot, 0) - u_k(\cdot, 0)|.$$

Hence, $|\mathcal{J}(\mathcal{M}_{\mathcal{N}_k}(T_k)) - \mathcal{J}(\mathcal{M}_{\mathcal{N}}(T_k))| \le \frac{1}{2}J_T$. This leads to $\mathcal{J}(\mathcal{M}_{\mathcal{N}_k}(T_k)) \le \frac{3}{2}J_T$. Therefore, $T_k = T$ for all large k, and the above estimate becomes

$$\sup_{\mathbb{S}^n \times [0,T]} |u - u_k| \le C_{J_T} \sup_{\mathbb{S}^n} |u(\cdot,0) - u_k(\cdot,0)|. \tag{3.16}$$

Let $t'_* \in [0, t_*]$ be the first time such that $\bar{\mathcal{M}}_{\mathcal{N}}(t) = \mathcal{M}_{\mathcal{N}}(t'_*)$ is stationary for all $t \ge t'_*$. When no such time exists, we set $t'_* = t_*$. Take a small $\tau_0 > 0$ such that $J_{t'_*+\tau_0}$ is finite. Suppose $\mathcal{J}(\mathcal{M}_{\mathcal{N}}(t'_* + \tau_0)) > \mathcal{J}(\mathcal{M}_{\mathcal{N}}(t'_*))$. Then $\bar{\mathcal{M}}_{\mathcal{N}_k}(t_*) = \mathcal{M}_{\mathcal{N}_k}(t_k)$ for some $t_k \in [0, t'_* + \tau_0]$. Since (3.16) holds for $T = t'_* + \tau_0$, we deduce that $\bar{\mathcal{M}}_{\mathcal{N}_k}(t_*) \to \bar{\mathcal{M}}_{\mathcal{N}}(t_*)$. If $\mathcal{J}(\mathcal{M}_{\mathcal{N}}(t'_* + \tau_0)) = \mathcal{J}(\mathcal{M}_{\mathcal{N}}(t'_*))$, we infer by Lemma 2.1 that $\bar{\mathcal{M}}_{\mathcal{N}}(t) = \mathcal{M}_{\mathcal{N}}(t)$ for all $t \in [t'_*, \infty)$. In particular, J_{t_*} is finite. Hence, (3.16) holds for all $T = t_*$. It follows that $\bar{\mathcal{M}}_{\mathcal{N}_k}(t_*) \to \bar{\mathcal{M}}_{\mathcal{N}}(t_*)$.

In the following, we prove the convergence of flow (1.4) with a specially chosen initial condition. Take a sequence $t_k \to \infty$ and let $\mathcal{N}_k = \mathcal{N}_{t_k}$ be the initial data from Lemma 3.8. By our choice of A_0 (see (3.1) and (3.3)), Lemma 3.8 implies that

$$\mathcal{J}(\bar{\mathcal{M}}_{\mathcal{N}_k}(t_k)) \le \frac{1}{2}A_0. \tag{3.17}$$

Hence, by the monotonicity of the functional $\mathcal{J}$, we have

$$\bar{\mathcal{M}}_{\mathcal{N}_k}(t) = \mathcal{M}_{\mathcal{N}_k}(t) \quad \forall t \le t_k. \tag{3.18}$$

Since $\mathrm{Cl}(\mathcal{N}_k) \in \mathcal{A}_I$ and $B_{\bar{d}}(0) \subset \mathrm{Cl}(\mathcal{N}_k)$, by Blaschke's selection theorem, there is a subsequence of $\mathcal{N}_k$ which converges in Hausdorff distance to a limit $\mathcal{N}_*$ such that $\mathrm{Cl}(\mathcal{N}_*) \in \mathcal{A}_I$ and $B_{\bar{d}}(0) \subset \mathrm{Cl}(\mathcal{N}_*)$.

Next, we show that flow (1.4) starting from $\mathcal{N}_*$ satisfies $\mathcal{J}(\mathcal{M}_{\mathcal{N}_*}(t)) < A_0$ for all t.

Lemma 3.12. *For any $t \ge 0$, we have*

$$\mathcal{J}(\bar{\mathcal{M}}_{\mathcal{N}_*}(t)) \le \frac{3}{4}A_0.$$

Hence,

$$\bar{\mathcal{M}}_{\mathcal{N}_*}(t) = \mathcal{M}_{\mathcal{N}_*}(t) \quad \forall t > 0.$$

Proof. For any given $t > 0$, since $\mathcal{N}_k \to \mathcal{N}_*$ and $t_k \to \infty$, when k is sufficiently large and $t_k > t$, we have

$$\mathcal{J}(\bar{\mathcal{M}}_{\mathcal{N}_*}(t)) - \mathcal{J}(\mathcal{M}_{\mathcal{N}_k}(t)) \le \frac{1}{4}A_0. \tag{3.19}$$

Estimate (3.19) is a consequence of

$$\mathcal{J}(\bar{\mathcal{M}}_{\mathcal{N}_k}(t)) \le \mathcal{J}(\mathcal{M}_{\mathcal{N}_k}(t)) \quad \text{and} \quad \lim_{k\to\infty} \bar{\mathcal{M}}_{\mathcal{N}_k}(t) = \bar{\mathcal{M}}_{\mathcal{N}_*}(t)$$

in Hausdorff metric. See Remark 3.11 for this convergence.

By the monotonicity of the functional $\mathcal{J}$, as $t_k > t$,

$$\mathcal{J}(\mathcal{M}_{\mathcal{N}_k}(t)) \le \mathcal{J}(\mathcal{M}_{\mathcal{N}_k}(t_k)).$$

Combining the above inequality with (3.17), (3.18) and (3.19), we obtain that

$$\begin{aligned}\mathcal{J}(\bar{\mathcal{M}}_{\mathcal{N}_*}(t)) &= \mathcal{J}(\bar{\mathcal{M}}_{\mathcal{N}_*}(t)) - \mathcal{J}(\mathcal{M}_{\mathcal{N}_k}(t)) + \mathcal{J}(\mathcal{M}_{\mathcal{N}_k}(t)) \\ &\leq \mathcal{J}(\bar{\mathcal{M}}_{\mathcal{N}_*}(t)) - \mathcal{J}(\mathcal{M}_{\mathcal{N}_k}(t)) + \mathcal{J}(\mathcal{M}_{\mathcal{N}_k}(t_k)) \leq \frac{1}{4}A_0 + \frac{1}{2}A_0 = \frac{3}{4}A_0.\end{aligned}$$

This completes the proof. ∎

3.4. *Convergence of the flow and existence of solutions to* (1.1)

Let $\Omega_{\mathcal{N}_*}(t) = \mathrm{Cl}(\mathcal{M}_{\mathcal{N}_*}(t))$ and $u(\cdot, t)$ be its support function. By Lemma 3.12, $\mathcal{M}_{\mathcal{N}_*}(t)$ satisfies (3.12). Hence, by (3.12), we infer that

$$\bar{d} \leq u(x,t) \leq C \quad \forall (x,t) \in \mathbb{S}^n \times [0,\infty),$$

where $C = \frac{n+1}{\bar{v}\omega_{n-1}\bar{d}^n}$.

By the convexity,

$$|\nabla u(x,t)| \leq \max_{\mathbb{S}^n} u(\cdot,t) \leq C \quad \forall (x,t) \in \mathbb{S}^n \times [0,\infty).$$

Namely, condition (2.10) holds. By Section 2.2, we obtain the existence of solutions to (1.1) as follows.

Proof of Theorem 1.1. Denote $\mathcal{M}(t) = \mathcal{M}_{\mathcal{N}_*}(t)$ and $\mathcal{J}(t) = \mathcal{J}(\mathcal{M}(t))$. By Lemmas 2.1 and 3.12,

$$\mathcal{J}(t) < A_0 \quad \text{and} \quad \mathcal{J}'(t) \geq 0 \quad \forall t \geq 0.$$

Therefore,

$$\int_0^\infty \mathcal{J}'(t)dt \leq \limsup_{T\to\infty} \mathcal{J}(T) - \mathcal{J}(0) \leq A_0.$$

This implies that there exists a sequence $t_i \to \infty$ such that

$$\mathcal{J}'(t_i) = \int_{\mathbb{S}^n} \left[\left(\frac{1}{K} - f u^{p-1}\right)^2 uK\right]\Big|_{t=t_i} d\sigma_{\mathbb{S}^n} \to 0.$$

Passing to a subsequence, we obtain by the a priori estimate (2.12) that $u(\cdot, t_i) \to u_\infty$ in $C^{3,\alpha}(\mathbb{S}^n)$-topology and u_∞ satisfies (1.1). ∎

Corollary 1.2 follows from Theorem 1.1 by an approximation argument. To prove Corollary 1.2, we first point out that all arguments in Sections 2.1 and 3.1–3.3 depend on $\inf f$ and $\sup f$ but are independent of the smoothness of f. Therefore, the constants $\bar{e}$, $\bar{v}$, $\bar{d}$ in (3.12) are independent of the smoothness of f.

Proof of Corollary 1.2. Choose a sequence of functions $f_j \in C^\infty(\mathbb{S}^n)$ such that $\inf_{\mathbb{S}^n} f \leq f_j \leq \sup_{\mathbb{S}^n} f$ and $f_j \to f$ a.e. (such as the mollifications of f). By our proof of Theorem 1.1, there exists an initial convex hypersurface $\mathcal{N}_j \in \mathcal{A}_I$ such that the solution $\mathcal{M}_j(t) := \mathcal{M}_{\mathcal{N}_j}(t)$ to (1.4) converges to a solution $\mathcal{M}_j$ of (1.1) with $f = f_j$, and $\mathcal{M}_j(t)$

satisfies (3.12), uniformly for all j and t. Hence, $\mathcal{M}_j$ satisfies the estimates in (3.12). Passing to a subsequence, we may assume that $\mathcal{M}_j$ converges in Hausdorff distance to a limit $\mathcal{M}$. Then $\mathcal{M}$ satisfies (3.12). By the weak convergence of the Monge–Ampère equation, $\mathcal{M}$ is a weak solution of (1.1). By the regularity theory of the Monge–Ampère equation, $\mathcal{M}$ is strictly convex and $C^{1,\alpha}$ smooth for some $\alpha \in (0, 1)$. ∎

4. Proofs of Proposition 3.6 and Theorem 3.7

In this section, we prove Proposition 3.6 and Theorem 3.7. Our proof uses some results from the homology theory of topology, which are collected in Appendix A.

4.1. *Proof of Proposition* 3.6

Proof of part (i). Note that for any $\tau \in [\overline{v}, \overline{v}^{-1}]$, we have

$$\{E \in \mathcal{A}_I : \mathrm{Vol}(E) = \tau\} \simeq \widehat{\mathcal{A}},$$
$$\{E \in \mathcal{A}_I : \mathrm{Vol}(E) = \tau;\ e_E = \overline{e}, \text{ or } O \in \partial E\} \simeq \mathcal{A}.$$

Hence, $\mathcal{P}$ consists of three components (up to homeomorphism)

$$\mathcal{A} \times [\overline{v}, \overline{v}^{-1}], \quad \widehat{\mathcal{A}} \times \{\overline{v}\}, \quad \text{and} \quad \widehat{\mathcal{A}} \times \{\overline{v}^{-1}\}.$$

Since $\mathcal{A} \times [\overline{v}, \overline{v}^{-1}]$ is topologically a cylinder with base $\mathcal{A}$, and $\mathcal{A}$ can be viewed as the boundary of $\widehat{\mathcal{A}}$ in the space $\{E \in \mathcal{K}$ is an ellipsoid in $\mathbb{R}^{n+1} : \mathrm{Vol}(E) = \omega_n\}$ (equipped with the Hausdorff distance), we see that $\mathcal{P}$ can be viewed as attaching two copies of $\widehat{\mathcal{A}}$ along the boundary of $\mathcal{A} \times [\overline{v}, \overline{v}^{-1}]$. As $\widehat{\mathcal{A}}$ is contractible, we conclude that $\mathcal{P}$ is homotopy-equivalent to $S\mathcal{A}$ (the suspension of $\mathcal{A}$), which is the quotient of $\mathcal{A} \times [\overline{v}, \overline{v}^{-1}]$ obtained by collapsing $\mathcal{A} \times \{\overline{v}\}$ to one point and $\mathcal{A} \times \{\overline{v}^{-1}\}$ to another point. Hence, the two spaces have the same homology $H_*(\mathcal{P}) = H_*(S\mathcal{A})$. By Lemma A.6,

$$H_{k+1}(S\mathcal{A}) = H_k(\mathcal{A}) \quad \forall k \geq 1.$$

This completes the proof. ∎

Proof of part (ii). Let $\mathcal{B}$ be the set of unit balls such that the origin lies on the boundary of the ball. Consider the following subspaces of $\mathcal{A}$:

$$\mathcal{A}_1 = \mathcal{A} \setminus \mathcal{B}, \quad \mathcal{A}_2 = \mathcal{A} \setminus \mathcal{E}, \quad \text{and} \quad \mathcal{A}_3 = \mathcal{A}_1 \cap \mathcal{A}_2.$$

We have the Mayer–Vietoris sequence (Lemma A.5) for the decomposition $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$:

$$\cdots \to H_{k+1}(\mathcal{A}) \to H_k(\mathcal{A}_3) \to H_k(\mathcal{A}_1) \oplus H_k(\mathcal{A}_2) \to H_k(\mathcal{A}) \to \cdots. \tag{4.1}$$

Let $L = ([0, 1] \times \{1\}) \cup (\{1\} \times [0, 1]) \subset \mathbb{R}^2$. Denote

$$L_1 = \{(s, \rho) \in L : s > 0\}, \quad L_2 = \{(s, \rho) \in L : \rho > 0\}, \quad \text{and} \quad L_3 = L_1 \cap L_2.$$

Let $G_1: L_1 \times [0,1] \to L_1$ be a strong deformation retraction from L_1 onto the point $(1,0)$; $G_2: L_2 \times [0,1] \to L_2$ be a strong deformation retraction from L_2 onto $(0,1)$; and $G_3: L_3 \times [0,1] \to L_3$ be a strong deformation retraction from L_3 onto $(1,1)$. Denote by $G_{i,1}$ and $G_{i,2}$ the components of the map G_i such that $G_i(s,\rho,t) = (G_{i,1}, G_{i,2})(s,\rho,t)$, where $1 \le i \le 3$.

Given $E \in \mathcal{A}$, we take $(E', \xi, s, \rho) \in \mathcal{E} \times \mathbb{S}^n \times L$ such that $\phi(E) = (\hat{E}'(s), \rho\xi)$ with ϕ being map (3.10). Define $\mathcal{G}_i: \mathcal{A}_i \times [0,1] \to \mathcal{A}_i$ by

$$\mathcal{G}_i(E,t) = \phi^*(\hat{E}'(G_{i,1}(s,\rho,t)), G_{i,2}(s,\rho,t)\xi).$$

Then $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ are deformation retractions from $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ onto $\mathcal{E}$, $\mathcal{B}$ and $\mathcal{A}'$, respectively, where

$$\mathcal{A}' = \{E \in \mathcal{A} : e_E = \bar{e} \text{ and } O \in \partial E\}.$$

Therefore,

$$H_*(\mathcal{A}_1) = H_*(\mathcal{E}), \quad H_*(\mathcal{A}_2) = H_*(\mathcal{B}), \quad \text{and} \quad H_*(\mathcal{A}_3) = H_*(\mathcal{A}').$$

Inserting these identities into (4.1), we obtain the following long exact sequence:

$$\cdots \to H_{k+1}(\mathcal{A}) \to H_k(\mathcal{A}') \to H_k(\mathcal{E}) \oplus H_k(\mathcal{B}) \to H_k(\mathcal{A}) \to \cdots . \tag{4.2}$$

On the other hand, the maps ϕ and ϕ^* (see (3.10) and (3.11)) yield the homeomorphisms

$$\mathcal{B} \simeq \mathbb{S}^n \quad \text{and} \quad \mathcal{A}' \simeq \mathcal{E} \times \mathbb{S}^n. \tag{4.3}$$

Our conclusion follows by combining (4.2) and (4.3). ∎

The remaining of this section is devoted to proving Theorem 3.7.

4.2. *Proof of Theorem* 3.7

In the following, we always take $n^* = \frac{n(n+1)}{2}$, which is the dimension of the Lie group $\mathrm{SO}(n+1)$.

For $n = 1$, the proof of Theorem 3.7 is straightforward. In this case, the lengths of semi-axes $r_1 = r_1(E)$ and $r_2 = r_2(E)$ of $E \in \mathcal{E}$ satisfy

$$r_1 r_2 = 1 \quad \text{and} \quad r_2 = \bar{e} r_1.$$

Hence,

$$r_1(E) = \frac{1}{\sqrt{\bar{e}}} \quad \text{and} \quad r_2(E) = \sqrt{\bar{e}} \quad \forall E \in \mathcal{E}.$$

Therefore, each element $E \in \mathcal{E}$ is determined by the major axis of E. It follows that $\mathcal{E}$ is homeomorphic to RP^1. As a result,

$$H_1(\mathcal{E}) = H_1(RP^1) = \mathbb{Z}.$$

This proves Theorem 3.7 when $n = 1$, since $n^* = 1$.

In the following, we deal with the case $n \geq 2$. First we introduce some notations. Let

$$\begin{aligned} H_1 &= \{(x_2, \dots, x_n) \in \mathbb{R}^{n-1} : x_2 \geq 1\}, \\ H_i &= \{(x_2, \dots, x_n) \in \mathbb{R}^{n-1} : x_{i+1} \geq x_i\}, \quad i = 2, \dots, n-1, \\ H_n &= \{(x_2, \dots, x_n) \in \mathbb{R}^{n-1} : e \geq x_n\}. \end{aligned}$$

Then H_i are closed half-spaces of $\mathbb{R}^{n-1}$, $i = 1, 2, \dots, n$. Denote

$$\Delta_{n-1} = \bigcap_{i=1}^{n} H_i. \tag{4.4}$$

If $n = 2$, then $\Delta_{n-1} = \{1 \leq x_2 \leq \overline{e}\} \subset \mathbb{R}$ is an interval. If $n = 3$, then $\Delta_{n-1} = \{(x_2, x_3) \in \mathbb{R}^2 : 1 \leq x_2 \leq x_3 \leq \overline{e}\}$ is a triangle. For $n \geq 3$, Δ_{n-1} is an $(n-1)$-simplex in $\mathbb{R}^{n-1}$.

Denote by

$$F_i = \partial H_i \cap \Delta_{n-1},$$

a face of Δ_{n-1}, $i = 1, \dots, n$. We also denote by $\Delta_{n-1}^{(i)}$ the subset of Δ_{n-1}, obtained by removing the face F_i from Δ_{n-1}, namely

$$\Delta_{n-1}^{(i)} = \Delta_{n-1} \setminus F_i. \tag{4.5}$$

There is a natural projection $\pi\colon \mathcal{E} \to \Delta_{n-1}$, given by

$$E \mapsto \pi(E) = (\widetilde{r}_2(E), \dots, \widetilde{r}_n(E)),$$

where $r_i(E)$ are lengths of semi-axes of E satisfying $r_1(E) \leq r_2(E) \leq \cdots \leq r_{n+1}(E)$ and

$$\widetilde{r}_i(E) = \frac{r_i(E)}{r_1(E)}, \quad i = 2, \dots, n.$$

Note that $\frac{r_{n+1}(E)}{r_1(E)} = \overline{e}$ is a fixed constant for all $E \in \mathcal{E}$.

The mapping π can be written as a composition of two mappings π_1 and π_2. Namely,

$$\begin{aligned} &\pi_1\colon \mathcal{E} \to \mathbb{L}_{n+1}, \quad \text{given by } E \mapsto (r_1(E), \dots, r_{n+1}(E)), \\ &\pi_2\colon \mathbb{L}_{n+1} \to \Delta_{n-1}, \quad \text{given by } (x_1, \dots, x_{n+1}) \mapsto \Big(\frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}\Big), \end{aligned}$$

where

$$\mathbb{L}_{n+1} = \Big\{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : 0 < x_1 \leq \cdots \leq x_{n+1}, \prod_{i=1}^{n+1} x_i = 1,\ x_{n+1} = \overline{e} x_1\Big\}.$$

It is readily seen that π_2 is a bijection and $\pi = \pi_2 \circ \pi_1$ is surjective.

Remark 4.1. For any given ellipsoid $E \in \mathcal{E}$, there is a unique positive definite, unimodular matrix A such that $E = \{x \in \mathbb{R}^{n+1} : x' \cdot Ax = 1\}$. Let $\lambda_1 \geq \cdots \geq \lambda_{n+1}$ be the eigenvalues of A. Then $\pi_1(E) = (\lambda_1^{-1/2}, \dots, \lambda_{n+1}^{-1/2})$. Hence, for any point (vector) $\vec{r} = (r_1, \dots, r_{n+1}) \in \mathbb{R}^{n+1}$ such that $0 < r_1 \leq \cdots \leq r_{n+1}$ and $\prod_{i=1}^{n+1} r_i = 1$, we can define $\pi_1^{-1}(\vec{r})$ as the set of ellipsoids $E \in \mathcal{E}$ such that the lengths of the semi-axes of E are equal to $r_1, \dots, r_{n+1}$, namely, the eigenvalues of the ellipsoid matrix A are equal to $r_1^{-2}, \dots, r_{n+1}^{-2}$. For convenience, we say that a component r_i of the vector $\vec{r}$ is single if $r_{i-1} < r_i < r_{i+1}$, and that a component r_i has multiplicity k ($k \geq 2$) if $r_{i-1} < r_i = \cdots = r_{i+k-1} < r_{i+k}$.

For $1 \leq i \leq n$, consider the following subsets of $\mathcal{E}$:

$$\mathcal{E}_i = \pi^{-1}(\Delta_{n-1}^{(i)}). \tag{4.6}$$

By our notation $\Delta_{n-1}^{(i)}$ in (4.5), $\mathcal{E}_i$ is a subset of $\mathcal{E}$ and can be written as

$$\mathcal{E}_i = \{E \in \mathcal{E} : r_i(E) \neq r_{i+1}(E)\}.$$

For any $1 \leq j_1 < j_2 < \cdots < j_l \leq n$, we denote

$$\mathcal{E}_{j_1;j_2;\dots;j_l} = \bigcup_{s=1}^{l} \mathcal{E}_{j_s} \quad \text{and} \quad \mathcal{W}_{j_1;j_2;\dots;j_l} = \bigcap_{s=1}^{l} \mathcal{E}_{j_s}. \tag{4.7}$$

For brevity, we write $\mathcal{E}_{j_1 j_2 \dots j_l} = \mathcal{E}_{j_1;j_2;\dots;j_l}$ and $\mathcal{W}_{j_1 j_2 \dots j_l} = \mathcal{W}_{j_1;j_2;\dots;j_l}$. We see that

$$\begin{aligned}\mathcal{E}_{j_1 j_2 \dots j_l} &= \{E \in \mathcal{E} : r_i(E) \neq r_{i+1}(E) \text{ for some } i = j_1, \dots, j_l\},\\ \mathcal{W}_{j_1 j_2 \dots j_l} &= \{E \in \mathcal{E} : r_i(E) \neq r_{i+1}(E) \text{ for all } i = j_1, \dots, j_l\}.\end{aligned}$$

For convenience of the reader, we first prove Theorem 3.7 for lower dimensions $n = 2$ and $n = 3$, and then for higher dimensions. One may also skip Sections 4.3 and 4.4, and go through Section 4.5 for general case directly. Our method is based on dividing $\mathcal{E}$ into suitable parts and employing the Mayer–Vietoris sequences (Lemma A.5).

4.3. *Dimension $n = 2$*

For $n = 2$, simplex (4.4) is $\Delta_1 = [1, \bar{e}]$. Recall that

$$\Delta_1^{(1)} = (1, \bar{e}], \quad \Delta_1^{(2)} = [1, \bar{e}), \quad F_1 = \{1\}, \quad F_2 = \{\bar{e}\}.$$

The subsets $\mathcal{E}_1 = \pi^{-1}(\Delta_1^{(1)})$ and $\mathcal{E}_2 = \pi^{-1}(\Delta_1^{(2)})$ of $\mathcal{E}$ are given by

$$\mathcal{E}_1 = \{E \in \mathcal{E} : r_1(E) < r_2(E)\}, \quad \mathcal{E}_2 = \{E \in \mathcal{E} : r_2(E) < r_3(E)\}.$$

We have the Mayer–Vietoris sequence (Lemma A.5) for the decomposition $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$:

$$\begin{aligned}\cdots \to H_k(\mathcal{E}_1) \oplus H_k(\mathcal{E}_2) \to H_k(\mathcal{E}) \to H_{k-1}(\mathcal{E}_1 \cap \mathcal{E}_2)\\ \to H_{k-1}(\mathcal{E}_1) \oplus H_{k-1}(\mathcal{E}_2) \to \cdots .\end{aligned} \tag{4.8}$$

In order to prove Theorem 3.7 by using (4.8), we compute the homology groups of $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_1 \cap \mathcal{E}_2$.

The following lemma helps us to simplify the computation of the homology groups of $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{W}_{12} = \mathcal{E}_1 \cap \mathcal{E}_2$.

Lemma 4.2. *The following statements hold:*

(1) *For $i = 1, 2$, $\pi^{-1}(F_{3-i})$ is a deformation retract of $\mathcal{E}_i$.*

(2) *For any given point $P \in \Delta_1^{(1)} \cap \Delta_1^{(2)}$, $\pi^{-1}(P)$ is a deformation retract of $\mathcal{E}_1 \cap \mathcal{E}_2$.*

As a direct consequence, we have for all $k \geq 1$,

$$H_k(\mathcal{E}_i) = H_k(\pi^{-1}(F_{3-i})), \quad i = 1, 2,$$
$$H_k(\mathcal{E}_1 \cap \mathcal{E}_2) = H_k(\pi^{-1}(P)), \quad P \in \Delta_1^{(1)} \cap \Delta_1^{(2)}.$$

Proof. For part (1), it suffices to consider the case when $i = 1$. Recall that $\Delta_1^{(1)} = (1, \overline{e}]$ and $F_2 = \{\overline{e}\}$. Let $G\colon \Delta_1^{(1)} \times [0, 1] \to \Delta_1^{(1)}$ be a strong deformation retraction of $\Delta_1^{(1)}$ onto F_2 (such deformation clearly exists). We then define $\mathcal{G}\colon \mathcal{E}_1 \times [0, 1] \to \mathcal{E}_1$ as follows. For any $E \in \mathcal{E}_1$, let $\mathcal{G}(E, t)$ be the ellipsoid such that its axial directions are all the same with E, and its axial lengths are determined by

$$(r_1(t), r_2(t), r_3(t)) = \pi_2^{-1} \circ G(\pi(E), t).$$

Namely, we continuously deform the axial lengths of E while keeping the directions of its axes so that the resulting ellipsoid belongs to $\pi^{-1}(F_2)$. It is easy to check that $\mathcal{G}$ is a deformation retraction from $\mathcal{E}_1$ onto $\pi^{-1}(F_2)$.

For part (2), the argument is similar. ∎

Since $\pi^{-1}(F_1) = \{E \in \mathcal{E} : r_1(E) = r_2(E)\}$ and $\pi^{-1}(F_2) = \{E \in \mathcal{E} : r_2(E) = r_3(E)\}$, we see that $\pi^{-1}(F_1)$ and $\pi^{-1}(F_2)$ are both homeomorphic to RP^2. Using Lemma 4.2 and the homology of RP^2, we have

$$H_k(\mathcal{E}_i) = 0 \quad \forall k \geq 3. \tag{4.9}$$

Next, we study the homology groups of $\mathcal{W}_{12} = \mathcal{E}_1 \cap \mathcal{E}_2$.

Lemma 4.3. *For any given $\vec{r} = (r_1, r_2, r_3) \in \mathbb{L}_3$ with $r_1 < r_2 < r_3$, denote $\mathcal{E}_{\vec{r}} = \pi_1^{-1}(\vec{r})$, that is,*

$$\mathcal{E}_{\vec{r}} = \{E \in \mathcal{E} : r_i(E) = r_i,\ 1 \leq i \leq 3\}.$$

Then $\mathcal{E}_{\vec{r}} \simeq \mathrm{SO}(3)/\Gamma$, where $\Gamma = \{A \in \mathrm{SO}(3) : A = \mathrm{diag}\{\pm 1, \pm 1, \pm 1\}\}$ is a finite discrete subgroup of SO(3). *It follows that $\mathcal{E}_{\vec{r}}$ is orientable and so*

$$H_3(\mathcal{E}_{\vec{r}}) = \mathbb{Z}.$$

Proof. We consider the following Lie group action on $\mathcal{E}_{\vec{r}}$:

$$\mathrm{SO}(3) \times \mathcal{E}_{\vec{r}} \to \mathcal{E}_{\vec{r}}, \quad (g, E) \mapsto T_g(E),$$

where T_g is the linear transformation of $\mathbb{R}^3$ associated to g. This groups action is clearly transitive and the stabiliser of

$$E_{\vec{r}} = \left\{x \in \mathbb{R}^3 : \sum_{i=1}^{3} \frac{x_i^2}{r_i^2} \leq 1\right\}$$

is given by Γ. It follows that $\mathcal{E}_{\vec{r}}$ is homeomorphic to $\mathrm{SO}(3)/\Gamma$ [33, Section 21.3]. As the orientation of SO(3) is preserved by all the diffeomorphisms of Γ, we find that $\mathrm{SO}(3)/\Gamma$ is an orientable manifold. This together with $\dim(\mathrm{SO}(3)/\Gamma) = 3$ yields

$$H_3(\mathcal{E}_{\vec{r}}) = H_3(\mathrm{SO}(3)/\Gamma) = \mathbb{Z}$$

(Lemma A.8). ∎

For any $P \in \Delta_1^{(1)} \cap \Delta_1^{(2)}$, let $\pi_2^{-1}(P) = (r_1, r_2, r_3) \in \mathbb{L}_3$ such that $r_1 < r_2 < r_3$. By Lemmas 4.2 and 4.3, we obtain that

$$H_3(\mathcal{E}_1 \cap \mathcal{E}_2) = \mathbb{Z}. \tag{4.10}$$

Now, we are ready to prove Theorem 3.7 for $n = 2$.

Taking $k = 4$ in (4.8) and using (4.9), we obtain the following short exact sequence:

$$0 \to H_4(\mathcal{E}) \to H_3(\mathcal{E}_1 \cap \mathcal{E}_2) \to 0.$$

As a result,

$$H_4(\mathcal{E}) = H_3(\mathcal{E}_1 \cap \mathcal{E}_2).$$

This together with (4.10) gives $H_4(\mathcal{E}) = \mathbb{Z}$, and thus proves Theorem 3.7 for $n = 2$.

4.4. *Dimension $n = 3$*

When $n = 3$, the simplex $\Delta_2 = \{(x_2, x_3) \in \mathbb{R}^2 : 1 \le x_2 \le x_3 \le \bar{e}\}$ is a triangle, and F_i, $1 \le i \le 3$, are the sides of this triangle Δ_2. Recall that $\mathcal{E}_i = \pi^{-1}(\Delta_2^{(i)})$, $1 \le i \le 3$, are given by

$$\mathcal{E}_i = \{E \in \mathcal{E} : r_i(E) < r_{i+1}(E)\}.$$

Arguing similarly as in Lemma 4.2, we have the following result.

Lemma 4.4. *The following statements hold:*

(1) *For $i = 1, 2, 3$, $\pi^{-1}(P)$ is a deformation retract of $\mathcal{E}_i$, where $P = F_{j_1} \cap F_{j_2}$ and j_1 and j_2 are distinct such that $\{j_1, j_2\} = \{1, 2, 3\} \setminus \{i\}$.*

(2) *For any point $P \in \bigcap_{k=1}^3 \Delta_2^{(k)}$, $\pi^{-1}(P)$ is a deformation retract of $\mathcal{W}_{123} = \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3$.*

Suppose that $P = F_1 \cap F_2$ and $\pi_2^{-1}(P) = (r_1, r_2, r_3, r_4) \in \mathbb{L}_4$. We then have $r_1 = r_2 = r_3$. Hence, $\pi^{-1}(P)$ is homeomorphic to RP^3. By Lemma 4.4, we conclude that $\mathcal{E}_3$ is homotopy equivalent to RP^3. Similarly, we obtain that $\mathcal{E}_1$ is homotopy equivalent to RP^3 and $\mathcal{E}_2$ is homotopy equivalent to the Grassmannian $G(2, 4)$, the space of all real two-dimensional planes in $\mathbb{R}^4$ that pass through the origin. Using the homology of RP^3 and $G(2, 4)$ (Remark A.9), we have for $1 \le i \le 3$,

$$H_k(\mathcal{E}_i) = 0 \quad \text{if } k \ge 5. \tag{4.11}$$

By part (2) of Lemma 4.4 and an analogue of Lemma 4.3 for $n = 3$ (see Lemma 4.7 for the general case), we conclude that $\mathcal{W}_{123}$ is homotopy equivalent to $\mathrm{SO}(4)/\Gamma$, where $\Gamma = \{A \in \mathrm{SO}(4) : A = \mathrm{diag}\{\pm 1, \pm 1, \pm 1, \pm 1\}\}$ is a finite subgroup of SO(4). Since $\mathrm{SO}(4)/\Gamma$ is orientable and has dimension 6, we have by Lemma A.8 that

$$H_6(\mathcal{W}_{123}) = H_6(\mathrm{SO}(4)/\Gamma) = \mathbb{Z}. \tag{4.12}$$

Next, we consider the homology groups of $\mathcal{W}_{12} = \mathcal{E}_1 \cap \mathcal{E}_2$, $\mathcal{W}_{13} = \mathcal{E}_1 \cap \mathcal{E}_3$ and $\mathcal{W}_{23} = \mathcal{E}_2 \cap \mathcal{E}_3$. Recall that

$$\mathcal{W}_{12} = \{E \in \mathcal{E} : r_1(E) < r_2(E) < r_3(E)\}.$$

Take any point $P \in \Delta_2^{(1)} \cap \Delta_2^{(2)} \cap F_3$. Clearly, $\pi^{-1}(P) \in \mathcal{W}_{12}$. Arguing as in Lemma 4.2, we see that $\pi^{-1}(P)$ is a deformation retract of $\mathcal{W}_{12}$. Assume that

$$\pi_2^{-1}(P) = (r_1, r_2, r_3, r_4) := \vec{r}.$$

We then have $r_1 < r_2 < r_3 = r_4$. Denote $\mathcal{E}_{\vec{r}} = \pi^{-1}(P)$. We consider the Lie group action on $\mathcal{E}_{\vec{r}}$ as in Lemma 4.3: $\mathrm{SO}(4) \times \mathcal{E}_{\vec{r}} \to \mathcal{E}_{\vec{r}}$. The stabiliser of this group action is given by $\mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(1) \times \mathrm{O}(2))$, i.e., the set of matrices in $\mathrm{O}(1) \times \mathrm{O}(1) \times \mathrm{O}(2)$ with determinant 1. Therefore, we conclude that

$$H_k(\mathcal{W}_{12}) = H_k(\mathrm{SO}(4)/\mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(1) \times \mathrm{O}(2))) \quad \forall k.$$

As the space $\mathrm{SO}(4)/\mathrm{S}(\mathrm{O}(1) \times \mathrm{O}(1) \times \mathrm{O}(2))$ has dimension 5, we get

$$H_k(\mathcal{W}_{12}) = 0 \quad \forall k \geq 6.$$

The above discussion yields the following result.

Lemma 4.5. *Let $1 \leq j_1 < j_2 \leq 3$. Then*

(1) *$\pi^{-1}(P)$ is a deformation retract of $\mathcal{W}_{j_1 j_2}$, where $P \in \Delta_2^{(j_1)} \cap \Delta_2^{(j_2)} \cap F_i$ with $\{i\} = \{1, 2, 3\} \setminus \{j_1, j_2\}$.*

(2) *$H_k(\mathcal{W}_{j_1 j_2}) = 0$ for all $k \geq 6$.*

The Mayer–Vietoris sequence (Lemma A.5) for the decomposition $\mathcal{E} = \mathcal{E}_1 \cup (\mathcal{E}_2 \cup \mathcal{E}_3)$ gives

$$\begin{aligned} \cdots \to H_k(\mathcal{E}_1) \oplus H_k(\mathcal{E}_2 \cup \mathcal{E}_3) \to H_k(\mathcal{E}) \to H_{k-1}(\mathcal{W}_{12} \cup \mathcal{W}_{13}) \\ \to H_{k-1}(\mathcal{E}_1) \oplus H_{k-1}(\mathcal{E}_2 \cup \mathcal{E}_3) \to \cdots . \end{aligned} \tag{4.13}$$

For the decomposition $\mathcal{E}_{23} = \mathcal{E}_2 \cup \mathcal{E}_3$, we have

$$\begin{aligned} \cdots \to H_k(\mathcal{E}_2) \oplus H_k(\mathcal{E}_3) \to H_k(\mathcal{E}_2 \cup \mathcal{E}_3) \to H_{k-1}(\mathcal{W}_{23}) \\ \to H_{k-1}(\mathcal{E}_2) \oplus H_{k-1}(\mathcal{E}_3) \to \cdots . \end{aligned} \tag{4.14}$$

Taking $k \geq 7$ in (4.14) and using (4.11) and part (2) of Lemma 4.5, we see that

$$H_k(\mathcal{E}_2 \cup \mathcal{E}_3) = 0 \quad \forall k \geq 7. \tag{4.15}$$

Letting $k = 8$ in (4.13) and inserting (4.11) and (4.15) in the exact sequence, we obtain that

$$H_8(\mathcal{E}) = H_7(\mathcal{W}_{12} \cup \mathcal{W}_{13}). \tag{4.16}$$

The proof reduces to the computation of the right-hand side of (4.16), i.e., $H_7(\mathcal{W}_{12} \cup \mathcal{W}_{13})$. The Mayer–Vietoris sequence

$$\begin{aligned}\cdots \to H_7(\mathcal{W}_{12}) \oplus H_7(\mathcal{W}_{13}) \to H_7(\mathcal{W}_{12} \cup \mathcal{W}_{13}) \to H_6(\mathcal{W}_{123})\\ \to H_6(\mathcal{W}_{12}) \oplus H_6(\mathcal{W}_{13}) \to \cdots\end{aligned}$$

together with part (2) in Lemma 4.5 yields that

$$H_7(\mathcal{W}_{12} \cup \mathcal{W}_{13}) = H_6(\mathcal{W}_{123}). \tag{4.17}$$

Combining (4.16), (4.17) and (4.12), we complete the proof of Theorem 3.7 for $n = 3$.

4.5. *General dimensions*

Now, we consider the general dimensions. Before we use the Mayer–Vietoris sequence and the induction arguments, we first prove several lemmas concerning the homology groups of $\mathcal{E}_i$, $\mathcal{E}_{j_1 j_2 \dots j_l}$ and $\mathcal{W}_{j_1 j_2 \dots j_l}$ (see notations in (4.6) and (4.7)). Recall the following notations:

(a) If $P \in \operatorname{Int} \Delta_{n-1}$, then

$$\pi^{-1}(P) \in \mathcal{W}_{12\dots n} = \{E \in \mathcal{E} : r_i(E) \neq r_j(E) \text{ whenever } i \neq j\}.$$

(b) If $P \in (\bigcap_{s=1}^{l} \Delta_{n-1}^{(j_s)}) \cap (\bigcap_{i \neq j_1, j_2, \dots, j_l} F_i)$, where $1 \le l < n$ and $1 \le j_1 < \cdots < j_l \le n$, then

$$\pi^{-1}(P) \in \mathcal{W}_{j_1 j_2 \dots j_l} \cap \{E \in \mathcal{E} : r_i(E) = r_{i+1}(E) \text{ for all } i \neq j_s,\ s = 1, \dots, l\}.$$

The following lemma shows that $\pi^{-1}(P)$ in cases (a) and (b) above are deformation retracts of $\mathcal{W}_{12\dots n}$ and $\mathcal{W}_{j_1 j_2 \dots j_l}$, respectively. It is the generalisation of Lemmas 4.2, 4.4 and 4.5 for high dimensions.

Lemma 4.6. *The two statements below hold:*

(i) *For any given $P \in \operatorname{Int} \Delta_{n-1}$, $\pi^{-1}(P)$ is a deformation retract of $\mathcal{W}_{12\dots n}$.*

(ii) *For $1 \le l < n$ and $1 \le j_1 < \cdots < j_l \le n$, if $P \in (\bigcap_{s=1}^{l} \Delta_{n-1}^{(j_s)}) \cap (\bigcap_{i \neq j_1, j_2, \dots, j_l} F_i)$. Then $\pi^{-1}(P)$ is a deformation retract of $\mathcal{W}_{j_1 j_2 \dots j_l}$.*

Proof. For (i), let

$$G\colon \operatorname{Int} \Delta_{n-1} \times [0,1] \to \operatorname{Int} \Delta_{n-1}$$

be a deformation retraction of $\operatorname{Int} \Delta_{n-1}$ onto P. Define $\mathcal{G}\colon \mathcal{W}_{12\dots n} \times [0,1] \to \mathcal{W}_{12\dots n}$ as follows. For any $E \in \mathcal{W}_{12\dots n}$, let $\mathcal{G}(E,t)$ be the ellipsoid such that its axial directions are all the same as E, and its axis lengths $r_i(t)$ are determined by

$$(r_1(t), \dots, r_{n+1}(t)) = \pi_2^{-1} \circ G(\pi(E), t). \tag{4.18}$$

It can be verified that $\mathcal{G}$ is a deformation retraction from $\mathcal{W}_{12\dots n}$ onto $\pi^{-1}(P)$ that we want.

For (ii), the argument is similar. Denote $W = \bigcap_{s=1}^{l} \Delta_{n-1}^{(j_s)}$. Now let $G\colon W \times [0,1] \to W$ be a deformation retraction form W onto P such that

- $G(W \cap (\bigcap_{i \neq j_1,\dots,j_l} F_i), t) \subset W \cap (\bigcap_{i \neq j_1,\dots,j_l} F_i)$ for all $t \in [0,1]$;
- $G(Q,t) \in W \setminus \bigcap_{i \neq j_1,\dots,j_l} F_i$ for all $t \in [0,1)$ and $Q \in W \setminus \bigcap_{i \neq j_1,\dots,j_l} F_i$.

We then define the deformation retraction $\mathcal{G}\colon \mathcal{W}_{j_1 j_2 \dots j_l} \times [0,1] \to \mathcal{W}_{j_1 j_2 \dots j_l}$ as follows: $\mathcal{G}(E,t)$ keeps all the axis-directions of E unchanged but its axis-lengths $r_i(t)$ are again given by (4.18). ∎

The next lemma gives the general case of Lemma 4.3.

Lemma 4.7. *Suppose that r_i, $i = 1, 2, \dots, n+1$, are distinct positive constants such that $\vec{r} = (r_1, \dots, r_{n+1}) \in \mathbb{L}_{n+1}$. Denote $\mathcal{E}_{\vec{r}} = \pi_1^{-1}(\vec{r})$. Then $\mathcal{E}_{\vec{r}}$ is homeomorphic to* $\mathrm{SO}(n+1)/\Gamma$*, where $\Gamma = \{A \in \mathrm{SO}(n+1) : A = \mathrm{diag}\{\pm 1, \dots, \pm 1\}\}$ is a finite discrete subgroup of* $\mathrm{SO}(n+1)$. *As a result, $\mathcal{E}_{\vec{r}}$ is orientable and has dimension n^*, and*

$$H_{n^*}(\mathcal{E}_{\vec{r}}) = \mathbb{Z}.$$

Proof. Consider the Lie group action on $\mathcal{E}_{\vec{r}}$,

$$\mathrm{SO}(n+1) \times \mathcal{E}_{\vec{r}} \to \mathcal{E}_{\vec{r}}, \quad (g, E) \mapsto T_g(E), \tag{4.19}$$

where T_g represents the linear transformation of $\mathbb{R}^{n+1}$ associated to g. Then the stabiliser of

$$E_{\vec{r}} = \left\{ x \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} \frac{x_i^2}{r_i^2} \le 1 \right\} \tag{4.20}$$

is given by $\Gamma = \{A \in \mathrm{SO}(n+1) : A = \mathrm{diag}\{\pm 1, \dots, \pm 1\}\}$, the set of diagonal $(n+1) \times (n+1)$ matrices with determinant 1. Since the group action is transitive and Γ is closed in $\mathrm{SO}(n+1)$, we conclude that $\mathcal{E}_{\vec{r}}$ has a smooth manifold structure such that

$$\mathcal{F}\colon \mathrm{SO}(n+1)/\Gamma \to \mathcal{E}_{\vec{r}}, \quad \mathcal{F}(g\Gamma) \mapsto T_g(E_{\vec{r}}),$$

is a diffeomorphism [33, Section 21.3]. As such manifold structure yields the same topology of $\mathcal{E}_{\vec{r}}$ induced by the Hausdorff metric, we see that $\mathcal{E}_{\vec{r}} \simeq \mathrm{SO}(n+1)/\Gamma$ and these two spaces have the same homology.

Since Γ is a finite discrete group and the orientation of $\mathrm{SO}(n+1)$ is preserved by all the diffeomorphisms of Γ, we conclude that $\mathrm{SO}(n+1)/\Gamma$ is an orientable closed manifold. As $\dim(\mathrm{SO}(n+1)/\Gamma) = n^*$, we see by Lemma A.8 that

$$H_{n^*}(\mathrm{SO}(n+1)/\Gamma) = \mathbb{Z}.$$

This completes the proof. Indeed, one can also show that $\mathrm{SO}(n+1)/\Gamma$ is diffeomorphic to the complete flag variety in $\mathbb{R}^{n+1}$ (see [11]). ∎

The following result corresponds to the general case of Lemma 4.5.

Lemma 4.8. *Suppose that* $\vec{r} = (r_1, \dots, r_{n+1}) \in \mathbb{L}_{n+1}$ *such that* r_{i_k+1} *has multiplicity* m_k *for* $k = 1, \dots, l$ *and all other components* r_i *are single, where*

$$0 \le i_1 < i_2 < \cdots < i_l \le n-1, \quad i_j + m_j \le i_{j+1}, \quad i_l + m_l \le n+1, \quad \sum_{j=1}^{l} m_j \le n+1.$$

Let $\mathcal{E}_{\vec{r}} = \pi_1^{-1}(\vec{r})$. *Then*

$$H_k(\mathcal{E}_{\vec{r}}) = 0 \quad \text{if } k \ge n^* + 1 - \sum_{j=1}^{l} \frac{m_j(m_j-1)}{2}. \tag{4.21}$$

Proof. The components of $\vec{r}$ can be divided into two groups:

$$\{r_{s_1}, \dots, r_{s_p}\} \quad \text{and} \quad \{r_{i_1+1}, \dots, r_{i_l+m_l}\}.$$

Components in the first group are single ones and components in the second group are multiple ones (see Remark 4.1). We have $p + \sum_{j=1}^{l} m_j = n+1$.

For simplicity, we may assume that (after a proper permutation) $\vec{r}$ can be written as

$$(r_{s_1}, \dots, r_{s_p} \mid r_{i_1+1}, \dots, r_{i_1+m_1}, \dots, r_{i_l+1}, \dots, r_{i_l+m_l}),$$

where the components before the symbol | are single ones and the components after the symbol | are multiple ones, and the components are in the ascending order $r_{s_1} < \cdots < r_{s_p}$, $r_{i_1+1} < r_{i_2+1} < \cdots < r_{i_l+1}$.

Consider the Lie group action on $\mathcal{E}_{\vec{r}}$ as in (4.19). This group action is transitive and the stabiliser of $E_{\vec{r}}$ (given by (4.20)) is the collection of matrices in the form

$$\mathcal{S} = \operatorname{diag}\{\pm 1, \dots, \pm 1 \mid \mathcal{O}_{m_1}, \dots, \mathcal{O}_{m_l}\}$$

with the property $\det \mathcal{S} = 1$, where $\mathcal{O}_{m_k} \in \mathrm{O}(m_k)$, the set of $m_k \times m_k$ orthogonal matrices. By the same argument as in Lemma 4.7, we obtain

$$\mathcal{E}_{\vec{r}} \simeq \mathrm{SO}(n+1)/\mathrm{S}(\mathrm{O}(1) \times \cdots \times \mathrm{O}(1) \times \mathrm{O}(m_1) \times \cdots \times \mathrm{O}(m_l)).$$

It is known that the space on the right-hand side has dimension $\frac{n^* - \sum_{j=1}^{l} m_j(m_j-1)}{2}$. We then deduce (4.21) as desired. ∎

By using Lemmas 4.6 and 4.7, we have the following conclusion.

Lemma 4.9. *We have* $H_{n^*}(\mathcal{W}_{12\dots n}) = \mathbb{Z}$.

Proof. Let P be a point of $\bigcap_{i=1}^{n} \Delta_{n-1}^{(i)}$. By Lemma 4.6, $H_*(\mathcal{W}_{12\dots n}) = H_*(\pi^{-1}(P))$. Since $\pi_2^{-1}(P) = \vec{r} \in \mathbb{L}_{n+1}$ satisfies $r_1 < r_2 < \cdots < r_{n+1}$, it follows from Lemma 4.7 that

$$H_{n^*}(\pi^{-1}(P)) = \mathbb{Z}.$$

This completes the proof. ∎

Lemma 4.10. *For any* $1 \le j_1 < j_2 < \cdots < j_l \le n$, *we have*

$$H_k(\mathcal{W}_{j_1 j_2 \dots j_l}) = 0, \quad \forall k \ge n^* + 1 - \sum_{s=1}^{l+1} \frac{m_s(m_s - 1)}{2},$$

where $m_s = j_s - j_{s-1}$ *and* $j_0 = 0$, $j_{l+1} = n + 1$.

Proof. Let P be a point in $(\bigcap_{s=1}^{l} \Delta_{n-1}^{(j_s)}) \cap (\bigcap_{i \ne j_1, j_2, \dots, j_l} F_i)$. By Lemma 4.6, we find

$$H_*(\mathcal{W}_{j_1 j_2 \dots j_l}) = H_*(\pi^{-1}(P)). \tag{4.22}$$

As $\pi_2^{-1}(P) = \vec{r} \in \mathbb{L}_{n+1}$ satisfies

$$\begin{aligned} r_1 &= \cdots = r_{j_1} < r_{j_1+1} = \cdots = r_{j_2} < r_{j_2+1} = \cdots = r_{j_{l-1}} < r_{j_{l-1}+1} = \cdots \\ &= r_{j_l} < r_{j_l+1} = \cdots = r_{n+1}, \end{aligned}$$

we obtain the conclusion by Lemma 4.8 and (4.22). ∎

Propositions 4.11 and 4.12 below are consequences of Lemmas 4.9 and 4.10, which can be viewed as a generalisation of these two lemmas.

Proposition 4.11. *Suppose* $1 \le p_1 < \cdots < p_r < j \le n$, *we have*

$$H_k\left(\bigcup_{l=j}^{n} \mathcal{W}_{p_1 \dots p_r l}\right) = 0, \quad \text{if } k \ge n^* + n + 1 - j - \sum_{s=1}^{r+1} \frac{m_s(m_s - 1)}{2}, \tag{4.23}$$

where $m_s = p_s - p_{s-1}$, $p_0 = 0$ *and* $p_{r+1} = j$. *Furthermore, if* $k \ge n^* + n + 1 - j - \frac{j(j-1)}{2}$, *then*

$$H_k(\mathcal{E}_{j(j+1)\dots n}) = 0. \tag{4.24}$$

Proof. For $j = n$, (4.23) follows from Lemma 4.10.

We now verify (4.23) by the induction argument on j. For this purpose, let us assume that (4.23) holds when $j = n - m$ for some $m \ge 0$. We next show that (4.23) holds for $j = n - (m + 1)$. The Mayer–Vietoris sequence (Lemma A.5) for the decomposition

$$\bigcup_{l=n-m-1}^{n} \mathcal{W}_{p_1 \dots p_r l} = \mathcal{W}_{p_1 \dots p_r (n-m-1)} \cup \left(\bigcup_{l=n-m}^{n} \mathcal{W}_{p_1 \dots p_r l}\right)$$

yields

$$\begin{aligned} \cdots &\to H_k\left(\bigcup_{l=n-m}^{n} \mathcal{W}_{p_1 \dots p_r l}\right) \oplus H_k(\mathcal{W}_{p_1 \dots p_r (n-m-1)}) \\ &\to H_k\left(\bigcup_{l=n-m-1}^{n} \mathcal{W}_{p_1 \dots p_r l}\right) \to H_{k-1}\left(\bigcup_{l=n-m}^{n} \mathcal{W}_{p_1 \dots p_r (n-m-1) l}\right) \\ &\to H_{k-1}\left(\bigcup_{l=n-m}^{n} \mathcal{W}_{p_1 \dots p_r l}\right) \oplus H_{k-1}(\mathcal{W}_{p_1 \dots p_r (n-m-1)}) \to \cdots. \end{aligned} \tag{4.25}$$

By our induction assumption, (4.23) holds when $j = n - m$. That is,

$$H_k\Big(\bigcup_{l=n-m}^{n} \mathcal{W}_{p_1\dots p_r l}\Big) = 0 \quad \text{if } k \geq k(m), \tag{4.26}$$

where $k(m)$ is an integer function of m given by

$$k(m) := n^* + m + 1 - \sum_{s=1}^{r} \frac{m_s(m_s-1)}{2} - \frac{(n-m-p_r)(n-m-p_r-1)}{2}.$$

By Lemma 4.10,

$$H_k(\mathcal{W}_{p_1\dots p_r(n-m-1)}) = 0 \quad \text{if } k \geq k'(m), \tag{4.27}$$

where $k'(m)$ is another integer function of m given by

$$k'(m) := n^* + 1 - \sum_{s=1}^{r} \frac{m_s(m_s-1)}{2} - \frac{(n-m-p_r-1)(n-m-p_r-2)}{2} - \frac{(m+2)(m+1)}{2}.$$

It can be verified that $k(m+1) \geq \max\{k(m), k'(m)\} + 1$. Now inserting (4.26) and (4.27) in the long exact sequence (4.25), we obtain then

$$H_k\Big(\bigcup_{l=n-m-1}^{n} \mathcal{W}_{p_1\dots p_r l}\Big) = H_{k-1}\Big(\bigcup_{l=n-m}^{n} \mathcal{W}_{p_1\dots p_r(n-m-1)l}\Big) \quad \text{if } k \geq k(m+1). \tag{4.28}$$

By our induction assumption again, the right-hand side above

$$H_{k-1}\Big(\bigcup_{l=n-m}^{n} \mathcal{W}_{p_1\dots p_r(n-m-1)l}\Big) = 0 \quad \text{if } k \geq k(m+1).$$

Hence, by (4.28), we conclude that (4.23) holds when $j = n - (m+1)$.

Note that

$$\mathcal{W}_i = \mathcal{E}_i \quad \text{and} \quad \mathcal{E}_{j(j+1)\dots n} = \bigcup_{l=j}^{n} \mathcal{W}_l.$$

By the same discussion as above but deleting p_i's, we obtain (4.24). ■

Proposition 4.12. *For any $2 \leq j \leq n$, we have*

$$H_{n^*+n-j}\Big(\bigcup_{l=j}^{n} \mathcal{W}_{12\dots(j-1)l}\Big) = \mathbb{Z}. \tag{4.29}$$

In particular, if $j = 2$, then

$$H_{n^*+n-2}\Big(\bigcup_{l=2}^{n} \mathcal{W}_{1l}\Big) = \mathbb{Z}.$$

Proof. For $j = n$, (4.29) is the conclusion of Lemma 4.9. Suppose by induction argument that (4.29) holds for $j = n - m$ for some $m \geq 0$. Applying Lemma A.5 to the pair $\bigcup_{l=n-m}^{n} \mathcal{W}_{12\ldots(n-m-2)l}$ and $\mathcal{W}_{12\ldots(n-m-2)(n-m-1)}$, we obtain

$$\begin{aligned} \cdots &\to H_k\Big(\bigcup_{l=n-m}^{n} \mathcal{W}_{12\ldots(n-m-2)l}\Big) \oplus H_k(\mathcal{W}_{12\ldots(n-m-2)(n-m-1)}) \\ &\to H_k\Big(\bigcup_{l=n-m-1}^{n} \mathcal{W}_{12\ldots(n-m-2)l}\Big) \to H_{k-1}\Big(\bigcup_{l=n-m}^{n} \mathcal{W}_{12\ldots(n-m-1)l}\Big) \\ &\to H_{k-1}\Big(\bigcup_{l=n-m}^{n} \mathcal{W}_{12\ldots(n-m-2)l}\Big) \oplus H_{k-1}(\mathcal{W}_{12\ldots(n-m-2)(n-m-1)}) \to \cdots . \end{aligned}$$

It follows from (4.23) in Proposition 4.11 and Lemma 4.10 that

$$H_k\Big(\bigcup_{l=n-m}^{n} \mathcal{W}_{12\ldots(n-m-2)l}\Big) = H_k(\mathcal{W}_{12\ldots(n-m-2)(n-m-1)}) = 0 \quad \text{if } k \geq n^* + m,$$

and therefore the long exact sequence above implies that

$$H_{n^*+m+1}\Big(\bigcup_{l=n-m-1}^{n} \mathcal{W}_{12\ldots(n-m-2)l}\Big) = H_{n^*+m}\Big(\bigcup_{l=n-m}^{n} \mathcal{W}_{12\ldots(n-m-1)l}\Big).$$

Hence, (4.29) follows when $j = n - m - 1$ by our induction assumption. This completes the proof. ∎

Now, we are ready to give the proof of Theorem 3.7 for general dimensions.

Proof of Theorem 3.7. Lemma A.5 for the decomposition $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_{23\ldots n}$ implies

$$\begin{aligned} \cdots &\to H_{n^*+n-1}(\mathcal{E}_1) \oplus H_{n^*+n-1}(\mathcal{E}_{23\ldots n}) \to H_{n^*+n-1}(\mathcal{E}) \\ &\to H_{n^*+n-2}(\mathcal{E}_1 \cap \mathcal{E}_{23\ldots n}) \to H_{n^*+n-2}(\mathcal{E}_1) \oplus H_{n^*+n-2}(\mathcal{E}_{23\ldots n}) \to \cdots . \end{aligned} \tag{4.30}$$

By virtue of (4.24) in Proposition 4.11 and Lemma 4.10 (for $l = 1$ and $j_1 = 1$),

$$H_{n^*+n-1}(\mathcal{E}_1) = H_{n^*+n-1}(\mathcal{E}_{23\ldots n}) = H_{n^*+n-2}(\mathcal{E}_1) = H_{n^*+n-2}(\mathcal{E}_{23\ldots n}) = 0.$$

Hence, by (4.30),

$$H_{n^*+n-1}(\mathcal{E}) = H_{n^*+n-2}(\mathcal{E}_1 \cap \mathcal{E}_{23\ldots n}).$$

Since $\mathcal{E}_1 \cap \mathcal{E}_{23\ldots n} = \bigcup_{l=2}^{n} \mathcal{W}_{1l}$, we complete the proof by Proposition 4.12. ∎

Remark 4.13. For any given $k \geq n^* + n$, by Proposition 4.11 and Lemma 4.10, we have

$$H_i(\mathcal{E}_1) = H_i(\mathcal{E}_{23\ldots n}) = 0 \quad \text{for } i = k - 1 \text{ or } k.$$

Using (4.30) with $n^* + n - 1$ replaced by k, we then obtain

$$H_k(\mathcal{E}) = H_{k-1}(\mathcal{E}_1 \cap \mathcal{E}_{23\ldots n}) = H_{k-1}\Big(\bigcup_{l}^{n} \mathcal{W}_{1l}\Big).$$

By (4.23) (with $j = 2$), the right-hand side above $H_{k-1}(\bigcup_l^n \mathcal{W}_{1l}) = 0$. Therefore,

$$H_k(\mathcal{E}) = 0 \quad \text{for all } k \geq n^* + n. \tag{4.31}$$

5. Proof of Theorem 2.5

In this section, we prove Theorem 2.5 by showing

(i) the Gauss curvature of $\mathcal{M}_t$ is bounded from above,

(ii) the principal curvatures of $\mathcal{M}_t$ have a positive lower bound.

By approximation, we may assume directly that f is C^2-smooth. The Gauss curvature flow has been extensively studied. The technique and calculation presented here are similar to those in [35].

Let $X(\cdot, t)$ be the solution of flow (1.4). Recall that the Gauss curvature of $X(\cdot, t)$ is given by (2.1), and the principal radii of curvature of $X(\cdot, t)$ are eigenvalues of the matrix $\{b_{ij}\}$, where

$$b_{ij} = u_{ij} + u\delta_{ij},$$

where u is the support function of $X(\cdot, t)$.

First, we derive an upper bound for the Gauss curvature.

Lemma 5.1. *Let $X(\cdot, t)$ be a uniformly convex solution to flow (1.4) for $t \in [0, T)$. Suppose that the support function u satisfies (2.10). Then there exists a constant C depending on n, p, $\min_{\mathbb{S}^n} f$, $\max_{\mathbb{S}^n} f$, the initial condition $\mathcal{M}_0$, and the constant C_0 in (2.10), such that*

$$K(\cdot, t) \leq C \quad \forall t \in [0, T).$$

Proof. We introduce the auxiliary function

$$Q = -\frac{u_t}{u - \varepsilon_0} = \frac{Ku^p f - u}{u - \varepsilon_0},$$

where $\varepsilon_0 = \frac{1}{2} \min_{\mathbb{S}^n \times [0,T)} u > 0$. It suffices to show that $Q(x, t) \leq C$ for all $(x, t) \in \mathbb{S}^n \times [0, T)$.

For any given $T' \in (0, T)$, we assume that Q attains its maximum over $\mathbb{S}^n \times [0, T']$ at (x_0, t_0). If $t_0 = 0$, then $\max_{\mathbb{S}^n \times [0,T']} Q = \max_{\mathbb{S}^n} Q(\cdot, 0)$ and we are through. If $t_0 > 0$, then at the point (x_0, t_0), we have

$$0 = \nabla_i Q = -\frac{u_{ti}}{u - \varepsilon_0} + \frac{u_t u_i}{(u - \varepsilon_0)^2}.$$

Hence, $u_{ti} = -Qu_i$ and we have

$$\begin{aligned} 0 \geq \nabla_{ij}^2 Q &= -\frac{u_{tij}}{u - \varepsilon_0} + \frac{u_{ti}u_j + u_{tj}u_i + u_t u_{ij}}{(u - \varepsilon_0)^2} - \frac{2u_t u_i u_j}{(u - \varepsilon_0)^3} \\ &= -\frac{u_{tij}}{u - \varepsilon_0} + \frac{u_t u_{ij}}{(u - \varepsilon_0)^2}. \end{aligned}$$

It follows that

$$-b_{ijt} = -u_{ijt} - u_t\delta_{ij} \le (b_{ij} - \varepsilon_0\delta_{ij})Q. \tag{5.1}$$

Let $\{h^{ij}\}$ be the inverse matrix of $\{b_{ij}\}$. By a rotation, we assume $\{b_{ij}(x_0, t_0)\}$ is diagonal. Then, at (x_0, t_0),

$$\sum h^{ii} \ge n\Big(\prod h^{ii}\Big)^{1/n} = nK^{1/n}.$$

This together with (5.1) yields, at (x_0, t_0),

$$\partial_t K = -K\sum h^{ij}b_{ijt} \le \Big(n - \varepsilon_0\sum h^{ii}\Big)KQ \le CQ^2 - \frac{\varepsilon_0}{C}Q^{2+1/n}. \tag{5.2}$$

We next compute, at (x_0, t_0),

$$\begin{aligned} 0 \le \partial_t Q &= -\frac{u_{tt}}{u-\varepsilon_0} + Q^2 = \frac{1}{u-\varepsilon_0}\frac{\partial}{\partial t}(Ku^p f) + Q + Q^2 \\ &\le \frac{1}{u-\varepsilon_0}(fu^p\partial_t K) + CQ^2, \end{aligned} \tag{5.3}$$

where we assume without loss of generality that $K \approx Q \gg 1$.

Combining (5.2) and (5.3), we obtain, at (x_0, t_0),

$$0 \le (C - \varepsilon_0 Q^{1/n})Q^2.$$

This implies that $\max_{\mathbb{S}^n\times[0,T']} Q$ is bounded from above. As this bound is independent of T', by sending $T' \to T$, we complete the proof. ∎

Next, we derive a lower bound on the principal curvatures.

Lemma 5.2. *Let $X(\cdot,t)$ be a uniformly convex solution to flow (1.4) for $t \in [0,T)$. Assume the support function u satisfies (2.10). Then there exists a constant $\overline{\kappa}$ depending on n, p, C_0, $\min_{\mathbb{S}^n} f$, $\|f\|_{C^{1,1}(\mathbb{S}^n)}$, and the initial condition $\mathcal{M}_0$, such that*

$$\kappa_i(\cdot,t) \ge \overline{\kappa} \quad \forall t \in [0,T),\ 1 \le i \le n, \tag{5.4}$$

where κ_i's are the principal curvatures of $X(\cdot,t)$.

Proof. Consider the following auxiliary function:

$$\widetilde{w}(x,t) = \log\lambda_{\max}(\{b_{ij}\}) - A\log u + B|\nabla u|^2,$$

where A and B are large constants to be determined, and $\lambda_{\max}(\{b_{ij}\})$ denotes the maximal eigenvalue of $\{b_{ij}\}$. Our purpose is to show that $\widetilde{w}$ is bounded from above.

For any given $T' \in (0,T)$, assume that $\widetilde{w}(x,t)$ achieves its maximum over $\mathbb{S}^n \times [0,T']$ at some point (x_0, t_0). We also suppose $t_0 > 0$, otherwise estimate (5.4) follows from the initial condition. By a proper rotation, we may assume that $\{b_{ij}\}$ is diagonal at (x_0, t_0) and $\lambda_{\max}(\{b_{ij}\})(x_0, t_0) = b_{11}(x_0, t_0)$.

Then the function

$$w(x,t) = \log b_{11} - A\log u + B|\nabla u|^2$$

attains its maximum at (x_0, t_0). We may assume $b_{11} \gg 1$, otherwise we are through. Denote by $\{h^{ij}\}$ the inverse matrix of $\{b_{ij}\}$. At (x_0, t_0), we have

$$\begin{aligned} 0 = \nabla_i w &= h^{11}\nabla_i b_{11} - A\frac{u_i}{u} + 2B\sum_k u_k u_{ki} \\ &= h^{11}(u_{i11} + u_1\delta_{i1}) - A\frac{u_i}{u} + 2Bu_i u_{ii}, \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} 0 \geq \nabla_{ii} w = &h^{11}\nabla^2_{ii}b_{11} - (h^{11})^2(\nabla_i b_{11})^2 - A\Big(\frac{u_{ii}}{u} - \frac{u_i^2}{u^2}\Big) \\ &+ 2B\Big(u_{ii}^2 + \sum_k u_k u_{kii}\Big). \end{aligned} \tag{5.6}$$

In the above, we have used the properties that $\nabla_k b_{ij}$ are symmetric in all indices and that $\nabla_k b^{ij} = -h^{il}h^{jp}\nabla_k b_{lp}$.

We also have

$$\partial_t w = h^{11}(u_{11t} + u_t) - A\frac{u_t}{u} + 2B\sum u_k u_{kt}.$$

Next, we estimate the term $b^{11}u_{11t}$. Recall that

$$\log(u - u_t) = \log K + \log(f u^p). \tag{5.7}$$

Set

$$\phi(x, u) = \log(f u^p).$$

Differentiating (5.7) gives

$$\frac{u_k - u_{kt}}{u - u_t} = -\sum h^{ij}\nabla_k b_{ij} + \nabla_k\phi = -\sum h^{ii}(u_{kii} + u_i\delta_{ik}) + \nabla_k\phi, \tag{5.8}$$

and

$$\frac{u_{11} - u_{11t}}{u - u_t} - \frac{(u_1 - u_{1t})^2}{(u - u_t)^2} = -\sum h^{ii}\nabla^2_{11}b_{ii} + \sum h^{ii}h^{jj}(\nabla_1 b_{ij})^2 + \nabla^2_{11}\phi. \tag{5.9}$$

By (5.9) and the Ricci identity $\nabla^2_{11}b_{ii} = \nabla^2_{ii}b_{11} - b_{11} + b_{ii}$, we have

$$\begin{aligned} \frac{\partial_t w}{u - u_t} &= h^{11}\Big[\frac{u_{11t} - u_{11}}{u - u_t} + \frac{u_{11} + u - u + u_t}{u - u_t}\Big] - \frac{A}{u}\frac{u_t - u + u}{u - u_t} + 2B\frac{\sum u_k u_{kt}}{u - u_t} \\ &\leq h^{11}\Big[\sum h^{ii}\nabla^2_{11}b_{ii} - \sum h^{ii}h^{jj}(\nabla_1 b_{ij})^2 - \nabla^2_{11}\phi\Big] \\ &\quad + \frac{1}{u - u_t} + \frac{A}{u} - \frac{A}{u - u_t} + 2B\frac{\sum u_k u_{kt}}{u - u_t} \\ &\leq h^{11}\Big[\sum h^{ii}(\nabla^2_{ii}b_{11} - b_{11} + b_{ii}) - \sum h^{ii}h^{jj}(\nabla_1 b_{ij})^2\Big] - h^{11}\nabla^2_{11}\phi \\ &\quad + \frac{1 - A}{u - u_t} + \frac{A}{u} + 2B\frac{\sum u_k u_{kt}}{u - u_t}. \end{aligned} \tag{5.10}$$

Inserting (5.5) and (5.6) into (5.10), we obtain, at (x_0, t_0),

$$\begin{aligned}
\frac{\partial_t w}{u-u_t} \le & \sum h^{ii}\Big[(h^{11})^2(\nabla_i b_{11})^2 + A\Big(\frac{u_{ii}}{u} - \frac{u_i^2}{u^2}\Big) - 2B\Big(u_{ii}^2 + \sum u_k u_{kii}\Big)\Big] \\
& + h^{11}\sum h^{ii}(b_{ii} - b_{11}) - h^{11}\sum h^{ii}h^{jj}(\nabla_1 b_{ij})^2 - h^{11}\nabla_{11}^2\phi \\
& + \frac{1-A}{u-u_t} + \frac{A}{u} + 2B\frac{\sum u_k u_{kt}}{u-u_t} \\
\le & \sum h^{ii}\Big[A\Big(\frac{u_{ii}}{u} - \frac{u_i^2}{u^2}\Big) - 2B\Big(u_{ii}^2 + \sum u_k u_{kii}\Big)\Big] - h^{11}\nabla_{11}^2\phi \\
& + 2B\frac{\sum u_k u_{kt}}{u-u_t} + \frac{1-A}{u-u_t} + CA \\
\le & -A\sum h^{ii} - 2B\sum b_{ii} - 2B\sum h^{ii}u_k u_{kii} - h^{11}\nabla_{11}^2\phi \\
& + 2B\frac{\sum u_k u_{kt}}{u-u_t} + \frac{1-A}{u-u_t} + CA + CB,
\end{aligned}$$

where $\sum h^{ii}h^{11}(\nabla_i b_{11})^2 \le \sum h^{ii}h^{jj}(\nabla_1 b_{ij})^2$ is used in the second inequality. By (5.8),

$$\begin{aligned}
\frac{\partial_t w}{u-u_t} \le & (2B|\nabla u|^2 - A)\sum h^{ii} - 2B\sum b_{ii} - h^{11}\nabla_{11}^2\phi \\
& - 2B\sum u_k\nabla_k\phi + \frac{2B|\nabla u|^2 + 1 - A}{u-u_t} + CA + CB \\
\le & (2B|\nabla u|^2 - A)\sum h^{ii} - 2B\sum b_{ii} + Cb_{11} \\
& + \frac{2B|\nabla u|^2 + 1 - A}{u-u_t} + CA + CB. \qquad (5.11)
\end{aligned}$$

Choose B large such that $B\sum b_{ii} \ge Cb_{11}$, and let

$$A = 2B\max_{\mathbb{S}^n\times[0,T)}|\nabla u|^2 + 1.$$

Since $\partial_t w \ge 0$ at (x_0, t_0), we obtain by (5.11) that

$$0 \le \frac{\partial_t w}{u-u_t} \le -B\sum b_{ii} + CA + CB.$$

Hence, $\lambda_{\max}(\{b_{ij}\})(x_0, t_0)$ is bounded and so $\max_{\mathbb{S}^n\times[0,T']}\lambda_{\max}(b_{ij}) \le C$. Since this upper bound is independent of T', we then let $T' \to T$ and finish the proof. ■

Appendix A. Basic results from algebraic topology

In this appendix, we present some preliminary results in algebraic topology which are used in Sections 3.2, 3.3 and 4. For a comprehensive introduction to the theory, we refer the readers to the book [22].

Let X be a topological space and A be a subspace of X. We start by recalling the definitions of retraction and deformation retraction (see [22, Chapter 0]).

Definition A.1. A continuous map $r\colon X \to A$ is called a retraction if the restriction of r to A is the identity of A. If there is a retraction from X to A, we say that A is a retract of X.

Definition A.2. Let $r\colon X \to A$ be a retraction. We say that r is a deformation retraction if $\iota \circ r$ is homotopic to the identity map of X, where $\iota\colon A \hookrightarrow X$ is the inclusion map. In this case, A is said to be a deformation retract of X.

It can be seen that $A \subset X$ is a deformation retract of X if and only if there exists a homotopy $F\colon X \times [0,1] \to X$ which satisfies

$$F(x,0) = x, \quad F(x,1) \in A \quad \forall x \in X, \qquad \text{and} \qquad F(a,1) = a \quad \forall a \in A.$$

We remark that if in addition $F(a,t) = a$ for all $a \in A$ and $t \in [0,1]$, then A is called a strong deformation retract of X.

Let $H_k(X)$ be the k-th (singular) homology group (with integer coefficients) of the topological space X, where k denotes nonnegative integers. The homology group is an important object in the algebraic topology. It was shown that every continuous map $f\colon X \to Y$ induces a homomorphism $f_*\colon H_k(X) \to H_k(Y)$ for each k, and f_* is an isomorphism if f is a homotopy equivalence [22, Section 2.1]. In particular, we have the following facts.

Lemma A.3 (Homology of a retract). (i) *If $A \subset X$ is a retract of X, then for each k, the homology homomorphism $H_k(A) \to H_k(X)$ induced by the inclusion is injective.*

(ii) *If A is a deformation retract of X, then for each k, the homology homomorphism $H_k(A) \to H_k(X)$ induced by the inclusion is an isomorphism.*

Recall that X is contractible if it is homotopy equivalent to a point. As a consequence, we have the following result.

Lemma A.4. *If X is contractible, then $H_k(X) = 0$ for all $k \geq 1$.*

The following Mayer–Vietoris theorem is a main tool for our argument in Section 4. It is quite useful in calculating the homology [22, Section 2.2].

Lemma A.5 (Mayer–Vietoris sequence). *For a pair of subspaces $A, B \subset X$ such that X is the union of the interiors of A and B, there is a long exact sequence as follows:*

$$\begin{aligned}\cdots \to H_{k+1}(X) \to H_k(A \cap B) \to H_k(A) \oplus H_k(B) \to H_k(X)\\ \to H_{k-1}(A \cap B) \to \cdots \to H_0(X) \to 0.\end{aligned}$$

For a space X, the suspension SX is the quotient of $X \times [0,1]$ obtained by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point. We call these two points p_0 and p_1, respectively, and consider two open subsets $A = SX \setminus \{p_1\}$ and $B = SX \setminus \{p_0\}$ to use for the Mayer–Vietoris sequence

$$\cdots \to H_{k+1}(A) \oplus H_{k+1}(B) \to H_{k+1}(SX) \to H_k(A \cap B) \to H_k(A) \oplus H_k(B) \to \cdots.$$

As A and B are contractible, and $A \cap B$ has a deformation retraction to X, we find by Lemmas A.3 and A.4 that

$$H_k(A) = H_k(B) = 0 \quad \text{and} \quad H_k(A \cap B) = H_k(X) \quad \forall k \geq 1,$$

and consequently the exact sequence yields an isomorphism between $H_{k+1}(SX)$ and $H_k(X)$.

Lemma A.6. *We have $H_{k+1}(SX) = H_k(X)$ for all $k \geq 1$.*

The Künneth theorem relates the homology of two topological spaces and their product space. We need the following special case of the Künneth theorem [22, Example 3B.3].

Lemma A.7 (Künneth formula). *We have $H_k(X \times \mathbb{S}^n) = H_k(X) \oplus H_{k-n}(X)$ for all $k > n$.*

The lemma below can be found in [22, Theorem 3.26].

Lemma A.8. *Let M be a closed connected n-manifold. Then $H_k(M) = 0$ for all $k > n$. If M is moreover orientable, then $H_n(M) = \mathbb{Z}$.*

We conclude the appendix by collecting known results on the homology groups of some topological spaces used in Section 4.

Remark A.9. The homology groups of the real projective spaces RP^n, spheres $\mathbb{S}^n$ and the Grassmannian $G(2,4)$ are all known.

- The homology groups of RP^n are given by

$$H_k(RP^n) = \begin{cases} \mathbb{Z}, & k = 0 \text{ or } k = n = \text{odd}, \\ \mathbb{Z}_2, & 0 < k < n \text{ and } k = \text{odd}, \\ 0, & \text{otherwise}. \end{cases}$$

- The homology groups of an n-dimensional sphere $\mathbb{S}^n$ are

$$H_k(\mathbb{S}^n) = \begin{cases} \mathbb{Z}, & k = 0, n, \\ 0, & \text{otherwise}. \end{cases}$$

- The homology groups of Grassmannian $G(2,4)$ of all real two-dimensional planes in $\mathbb{R}^4$ that pass through the origin are

$$H_k(G(2,4)) = \begin{cases} \mathbb{Z}_2, & k = 1, 2, \\ \mathbb{Z}, & k = 0, 4, \\ 0, & \text{otherwise}. \end{cases}$$

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