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On finite time Type I singularities of the Kähler–Ricci flow on compact Kähler surfaces

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Abstract. We show that the underlying complex manifold of a complete non-compact two-dimensional shrinking gradient Kähler–Ricci soliton (M, g, X) with soliton metric g with bounded scalar curvature R_g whose soliton vector field X has an integral curve along which $\mathrm{R}_g \not\to 0$ is biholomorphic to either $\mathbb{C} \times \mathbb{P}^1$ or to the blowup of this manifold at one point, and that the soliton metric g is toric. We also identify the corresponding soliton vector field X in each case. Given these possibilities, we then prove a strong form of the Feldman–Ilmanen–Knopf conjecture for finite time Type I singularities of the Kähler–Ricci flow on compact Kähler surfaces, leading to a classification of the bubbles of such singularities in this dimension.

Keywords: Kähler–Ricci flow, self-similar solutions, shrinking gradient Kähler–Ricci solitons.

1. Introduction

1.1. Overview

A *Ricci soliton* is a triple (M, g, X), where M is a Riemannian manifold endowed with a complete Riemannian metric g and a complete vector field X, such that

$$\mathrm{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \frac{\lambda}{2} g \tag{1.1}$$

for some $\lambda \in \{-1, 0, 1\}$. If $X = \nabla^g f$ for some smooth real-valued function f on M, then we say that (M, g, X) is *gradient*. In this case, the soliton equation (1.1) becomes

$$\mathrm{Ric}(g) + \frac{\lambda}{2} g = \mathrm{Hess}(f).$$

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If g is complete and Kähler with Kähler form ω, then we say that (M, g, X) is a *Kähler–Ricci soliton* if the vector field X is complete and real holomorphic, and the pair (g, X) satisfies the equation

$$\mathrm{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g \tag{1.2}$$

for λ as above. If g is a Kähler–Ricci soliton and if $X = \nabla^g f$ for some smooth real-valued function f on M, then we say that (M, g, X) is *gradient*. In this case, the soliton equation (1.2) may be rewritten as

$$\rho_\omega + i\partial\bar{\partial} f = \lambda\omega,$$

where ρ_ω is the Ricci form of ω.

For Ricci solitons and Kähler–Ricci solitons (M, g, X), the vector field X is called the *soliton vector field*. Its completeness is guaranteed by the completeness of g [46]. If the soliton is gradient, then the smooth real-valued function f satisfying $X = \nabla^g f$ is called the *soliton potential*. It is unique up to the addition of a constant. Finally, a Ricci soliton and a Kähler–Ricci soliton are called *steady* if $\lambda = 0$, *expanding* if $\lambda = -1$, and *shrinking* if $\lambda = 1$ in (1.1) and (1.2), respectively.

The study of Ricci solitons and their classification is important in the context of Riemannian geometry. For example, they provide a natural generalisation of Einstein manifolds and on certain Fano manifolds, shrinking Kähler–Ricci solitons are known to exist where there are obstructions to the existence of a Kähler–Einstein metric [47]. Also, to each soliton, one may associate a self-similar solution of the Ricci flow [8, Lemma 2.4]. These are candidates for singularity models of the flow. The difference in normalisations between (1.1) and (1.2) reflects the difference between the constants preceding the Ricci term in the Ricci flow and in the Kähler–Ricci flow respectively when one takes this dynamic point of view.

In this article, we are concerned with the classification of complete shrinking gradient Kähler–Ricci solitons with bounded curvature, the motivation being that such a soliton encodes how the Kähler–Ricci flow enters a finite time Type I singularity, that is, a singularity where the curvature of the evolving metric does not blow up faster than $O((T-t)^{-1})$ at the finite singular time $T > 0$. More precisely, non-flat shrinking gradient Kähler–Ricci solitons are known to appear as parabolic rescalings of finite time Type I singularities of the Kähler–Ricci flow on compact Kähler manifolds [13, 31]. We focus on the classification in complex dimension 2, where a bound on the scalar curvature of the soliton suffices to bound the full curvature tensor [30]. Assuming therefore bounded scalar curvature, the soliton is either compact, in which case the underlying manifold is Fano and the resulting soliton is (up to automorphism) Kähler–Einstein or the shrinking gradient Kähler–Ricci soliton given by [44] depending on the Fano manifold in question, or is non-compact. Gradient shrinking Kähler–Ricci solitons are connected at infinity [29] and in this latter case, there is a dichotomy in the sense that the scalar curvature of the soliton either tends to zero along every integral curve of X, or X has an integral curve along which the scalar curvature does not tend to zero. In the former case, it follows that the scalar curvature tends to zero globally (cf. Lemma 2.7) and hence the soliton (up to automorphism) is either that of Feldman–Ilmanen–Knopf [15] on the blowup of $\mathbb{C}^2$ at one point or the flat Gaussian shrinking soliton on $\mathbb{C}^2$ [11]. Here we use a result of [31] to prove, in conjunction with [9], that in the latter case the shrinking soliton is isometric to the cylinder $\mathbb{C} \times \mathbb{P}^1$, or it is toric and its underlying manifold is the blowup of $\mathbb{C} \times \mathbb{P}^1$ at one point. Being the only possibilities, this allows us to prove a strong form of the Feldman–Ilmanen–Knopf conjecture [15] for finite time Type I singularities of the Kähler–Ricci flow on compact Kähler surfaces, and in doing so, identify the possible parabolic rescalings that may appear at such singularities.

1.2. Main results

The simplest examples of complete shrinking gradient Kähler–Ricci solitons include any Kähler–Einstein manifold with soliton vector field $X = 0$ and the flat Gaussian shrinking soliton on $\mathbb{C}$ endowed with soliton vector field $2 \cdot \mathrm{Re}(z\partial_z)$, where z is the holomorphic coordinate on $\mathbb{C}$. Taking Cartesian products also provides examples. With this in mind, our first main result can be stated as follows. The statement should be read in the context of the dichotomy explained above.

Theorem A (Holomorphic classification). *Let (M, g, X) be a two-dimensional complete non-compact shrinking gradient Kähler–Ricci soliton with complex structure J and with bounded scalar curvature R_g whose soliton vector field X has an integral curve along which $\mathrm{R}_g \not\to 0$. Then*

(i) *M is biholomorphic to either $\mathbb{C} \times \mathbb{P}^1$ or to $\mathrm{Bl}_p(\mathbb{C} \times \mathbb{P}^1)$, that is, the blowup of $\mathbb{C} \times \mathbb{P}^1$ at a fixed point p of the standard torus action on $\mathbb{C} \times \mathbb{P}^1$.*

(ii) *There exists a biholomorphism $\gamma\colon M \to M$ such that $\gamma_*^{-1}(JX)$ lies in the Lie algebra of the real torus $\mathbb{T}$ acting on these spaces in the standard way and $\gamma^* g$ is $\mathbb{T}$-invariant.*

(iii) *$\gamma_*^{-1}(JX)$ is determined and its flow generates a holomorphic isometric S^1-action of $(M, J, \gamma^* g)$.*

Conclusions (ii)–(iii) for $M = \mathbb{C} \times \mathbb{P}^1$ have already been established in [9] where it was shown that any complete shrinking gradient Kähler–Ricci soliton with bounded scalar curvature on this manifold is isometric to the Cartesian product of the flat Gaussian soliton $\omega_{\mathbb{C}}$ on $\mathbb{C}$ and twice the Fubini–Study metric $\omega_{\mathbb{P}^1}$ on $\mathbb{P}^1$. The new possibility arising is when M is the blowup of $\mathbb{C} \times \mathbb{P}^1$ at one point, in which case $\gamma_*^{-1}(JX)$ is given by (2.16). In [1], it is shown that this manifold admits a unique complete shrinking gradient Kähler–Ricci soliton with bounded scalar curvature. Combined with Theorem A, [9, Corollary C], and [11, Theorem E (3)], this completes the classification of complete shrinking gradient Kähler–Ricci solitons with bounded scalar curvature in complex dimension 2 (cf. [1, Theorem B] for a precise statement). This new example models a finite time Type I collapsing singularity of a Kähler–Ricci flow with diameter bounded uniformly from below on the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at one point in the vicinity of the exceptional curve. Indeed, this is how the soliton is constructed.

The proof of Theorem A is specifically catered to complex dimension 2, making heavy use of the theory of J-holomorphic curves in this dimension. The outline is as follows. We assume that the shrinking soliton (M, g, X) is simply connected as this turns out to suffice. The bounded scalar curvature assumption implies that the curvature is bounded [30] and so by results in [11], the flow of JX will generate the holomorphic isometric action of a real torus on the soliton. Next, we are able to deduce from a result of Naber [31] that on large balls sufficiently far away from the zero set of the soliton vector field X centred along the integral curve of X along which $\mathrm{R}_g \not\to 0$, (M, g) is C^∞-close to the model cylinder $\mathbb{C} \times \mathbb{P}^1$. As the complex structures will consequently also be close, we use the perturbation theory of J-holomorphic curves to perturb a holomorphic $\mathbb{P}^1$ in the cylinder to a holomorphic $\mathbb{P}^1$ with zero self-intersection in M itself. Taking an S^1 inside the aforementioned real torus generated by JX, we can then move this $\mathbb{P}^1$ around and identify M with $\mathbb{C}^* \times \mathbb{P}^1$ at infinity. Complete shrinking solitons with bounded scalar curvature have finite topological type [14], therefore we may blow down all of the (-1)-curves in M and obtain its minimal model $M_{\min}$. A continuity argument using Gromov's compactness theorem for J-holomorphic curves then allows us to extend the $\mathbb{P}^1$-foliation of M at infinity into the interior of $M_{\min}$ and in doing so, identify $M_{\min}$ as a $\mathbb{P}^1$-bundle over a non-compact Riemann surface S. Compactifying this picture, the assumption of simple connectedness allows us to ascertain that S compactifies to an S^2, leaving us with the diffeomorphism type of $M_{\min}$ as $\mathbb{R}^2 \times S^2$. After analysing the structure of the zero set of X, we may then use the flow of the vector fields X and JX to construct a complex torus equivariant biholomorphism between $M_{\min}$ and $\mathbb{C} \times \mathbb{P}^1$. The manifold M is therefore biholomorphic to either $M_{\min}$ or to the blowup of $M_{\min}$ at finitely many points. The blowup points of $M_{\min}$ must be contained in the zero set of the vector field that X induces on $M_{\min}$, which itself is contained in a $\mathbb{P}^1$. Furthermore, the sign of $-K_M$ dictated by the shrinking soliton equation allows M to contain only (-1)-curves, ruling out iterative blowups of $M_{\min}$ at a point. These two properties limit the number of blowup points to one, leading to the statement of Theorem A (i). The biholomorphism constructed between M and the manifolds of part (i) is torus-equivariant and uses the flow of X and JX, hence naturally has the property regarding the vector field stated in (ii). The toricity of the soliton metric follows from an application of the version of Matsushima's theorem for shrinking gradient Kähler–Ricci solitons proved in [11]. For this step, the assumption of bounded scalar curvature is crucial. Finally, knowing that JX lies in the Lie algebra of the ambient torus means that it can be identified as it has the property that it minimises a certain functional, known as the weighted volume functional [11,41]. In fact, knowing the two possibilities for M allows us to compute this vector field explicitly in each case. This yields (iii).

1.2.1. Application to the Kähler–Ricci flow. For a complete shrinking gradient Kähler–Ricci soliton (M, g, X) with $X = \nabla^g f$ for $f\colon M \to \mathbb{R}$ smooth, one can define an ancient solution $g(t)$, $t < 0$, of the Kähler–Ricci flow

$$\frac{\partial g(t)}{\partial t} = -\operatorname{Ric}(g(t))$$

with $g(-1) = g$ by defining $g(t) := -t\varphi_t^* g$, $t < 0$, where φ_t is a family of diffeomorphisms generated by the gradient vector field $-\frac{1}{t}X$ with $\varphi_{-1} = \mathrm{id}$, i.e.,

$$\frac{\partial \varphi_t}{\partial t}(x) = -\frac{\nabla^g f(\varphi_t(x))}{2t}, \quad \varphi_{-1} = \mathrm{id}.$$

These Kähler–Ricci flows model the formation of finite time Type I singularities of the flow [31] which we now define. We recall the following from [13] in the context of the Kähler–Ricci flow.

A family $(M, g(t))$ of smooth complete Kähler manifolds satisfying the Kähler–Ricci flow

$$\frac{\partial g(t)}{\partial t} = -\operatorname{Ric}(g(t))$$

on a finite time interval $[0, T)$, $T < +\infty$, is called a *Type I Kähler–Ricci flow* if there exists a constant $C > 0$ such that for all $t \in [0, T)$,

$$\sup_M |\mathrm{Rm}_{g(t)}|_{g(t)} \leq \frac{C}{T-t}.$$

Such a solution is said to develop a *Type I singularity* at time T (and T is called a *Type I singular time*) if it cannot be smoothly extended past time T. It is well known that this is the case if and only if

$$\limsup_{t \to T^-} \sup_M |\mathrm{Rm}_{g(t)}|_{g(t)} = +\infty; \tag{1.3}$$

see [21] for compact and [36] for complete flows. Here $\mathrm{Rm}_{g(t)}$ denotes the Riemannian curvature tensor of the metric $g(t)$.

Since Type I Kähler–Ricci flows $(M, g(t))$ have bounded curvature for each $t \in [0, T)$, the parabolic maximum principle, applied to the evolution equation satisfied by $|\mathrm{Rm}|^2_{g(t)}$, shows that (1.3) is equivalent to

$$\sup_M |\mathrm{Rm}_{g(t)}|_{g(t)} \geq \frac{1}{8(T-t)} \quad \text{for all } t \in [0, T).$$

This motivates the following definition.

Definition 1.1 ([13, Definition 1.2]). Let $(M, g(t))$, $t \in [0, T)$, $T < +\infty$, be a Kähler–Ricci flow. A space-time sequence (p_i, t_i) with $p_i \in M$ and $t_i \to T^-$ is called an *essential blowup sequence* if there exists a constant $c > 0$ such that

$$|\mathrm{Rm}_{g(t_i)}|_{g(t_i)}(p_i) \geq \frac{c}{T-t_i}.$$

A point $p \in M$ in a Type I Kähler–Ricci flow is called a *Type I singular point* if there exists an essential blowup sequence with $p_i \to p$ on M. We denote the set of all Type I singular points by Σ_I.

The set Σ_I has been characterised in [13, Theorem 1.2]. As already noted, in general it is known that a suitable blowup limit of a complete Kähler–Ricci flow at a point of Σ_I is a non-flat shrinking gradient Kähler–Ricci soliton with bounded curvature [13, 31]. Therefore, assuming the development of a finite time Type I singularity, thanks to the classification given by Theorem A, we are able to obtain as a corollary the following strong form of the Feldman–Ilmanen–Knopf conjecture for such singularities on compact Kähler surfaces [15, Example 2.2 (3)].

Theorem B (Non-collapsing). *Let $(M, g(t))$ be a Type I Kähler–Ricci flow on $[0, T)$, $T < +\infty$, on a compact Kähler surface M and suppose that $x \in \Sigma_I$ is a Type I singular point as defined in Definition 1.1. Then for every sequence $\lambda_j \to +\infty$, the rescaled Kähler–Ricci flows $(M, g_j(t), x)$ defined on $[-\lambda_j T, 0)$ by $g_j(t) := \lambda_j g(T + \frac{t}{\lambda_j})$ subconverge in the smooth pointed Cheeger–Gromov topology to the unique shrinking gradient $U(2)$-invariant Kähler–Ricci soliton of Feldman–Ilmanen–Knopf* [15] *on the blowup of $\mathbb{C}^2$ at one point if and only if* $\lim_{t\to T^-} \mathrm{vol}_{g(t)}(M) > 0$.

This theorem characterises the Feldman–Ilmanen–Knopf shrinking Kähler–Ricci soliton as the unique shrinking soliton that models finite time Type I non-collapsed singularities of the Kähler–Ricci flow on compact Kähler surfaces. The "if" direction of Theorem B is known to hold true for $U(n)$-invariant Kähler–Ricci flows on the blowup of $\mathbb{P}^n$ at one point [20]. Moreover, on this manifold, it is known that any $U(n)$-invariant solution of the Kähler–Ricci flow developing a finite time singularity is a singularity of Type I [39]. Similar results were obtained by Máximo [26] for $n = 2$. However, contrary to a folklore conjecture, not every finite time singularity of the Kähler–Ricci flow is of Type I [25], although this is expected to be the case for Kähler–Ricci flows on compact Kähler surfaces.

The proof of Theorem B is by contradiction. Assuming that the volume is non-collapsing, we consider the volume evolution of the unique (-1)-curve in the Feldman–Ilmanen–Knopf shrinking soliton under the Kähler–Ricci flow to rule out other possible shrinking solitons appearing as the rescaled limit. For the other direction, we assume volume collapsing and the appearance of the Feldman–Ilmanen–Knopf shrinking soliton to derive a nonsensical lower bound on the volume of a (-1)-curve in the original manifold. This direction crucially relies on the structure of collapsing singularities of the Kähler–Ricci flow in complex dimension 2 given by [43] and the asymptotics and symmetry of the aforementioned soliton.

Given Theorem B, we can now classify the finite time Type I rescaled limits of the Kähler–Ricci flow on a compact Kähler surface M. To this end, let $(M, g(t))_{t\in[0,T)}$ be a Kähler–Ricci flow developing a finite Type I singularity when $t = T > 0$. Take the blowup limit as is done in Theorem B. If $\lim_{t\to T^-} \mathrm{vol}_{g(t)}(M) > 0$, then Theorem B asserts that the blowup limit is the Feldman–Ilmanen–Knopf shrinking soliton on the blowup of $\mathbb{C}^2$ at one point. This picture is consistent with finite time singularities of the Kähler–Ricci flow on compact Kähler surfaces being of Type I. Indeed, under the assumption of non-collapsing, it is known that the flow contracts finitely many disjoint (-1)-curves on M [3, Theorem 3.8.3]. On the other hand, if there is finite time collapsing at $t = T > 0$, i.e., if $\lim_{t\to T^-} \mathrm{vol}_{g(t)}(M) = 0$, then either $\lim_{t\to T^-} \mathrm{diam}\,(M, g(t)) = 0$, which is a “finite time extinction”, or $\lim_{t\to T^-} \mathrm{diam}(M, g(t)) > 0$. In the former case, [43] asserts that M is Fano and the Kähler class of $g(0)$ lies in $c_1(M)$. The work of Perelman (see [35]) gives us the upper bound $\mathrm{diam}(M, g(t)) \leq C(T-t)^{\frac{1}{2}}$, which, for the rescaled limit $g_j(t)$, $t < 0$, translates to $\mathrm{diam}(M, g_j(t)) \leq C|t|$. This latter bound implies that the rescaled limit is compact, hence being a shrinking soliton, is Fano with its (up to automorphism) unique shrinking soliton structure. In the latter case, the blowup limit cannot be Fano as the compactness of such a manifold implies that $\lim_{t\to T^-} \mathrm{diam}\,(M, g(t)) = 0$, a contradiction. By Theorem B, the blowup limit cannot be the shrinking soliton of Feldman–Ilmanen–Knopf. Hence the only possibility is that the blowup limit is the cylinder $\mathbb{C} \times \mathbb{P}^1$ or the shrinking soliton of [1]. The precise soliton that appears would depend upon the proximity of the blowup point to a (-1)-curve. This collapsing picture is also consistent with finite time singularities of the Kähler–Ricci flow on compact Kähler surfaces being of Type I as under the assumption of finite time collapsing, it is known that the underlying complex manifold is birational to a ruled surface [3, Proposition 3.8.4].

1.3. Outline of paper

We begin Section 2.1 by presenting the background material on J-holomorphic curves that we need to prove Theorem A. We then recall in Section 2.2 the basics of shrinking Ricci and Kähler–Ricci solitons. In Section 2.3, we digress and mention some basics on polyhedrons and polyhedral cones that we need before moving on to some relevant information concerning Hamiltonian actions in Section 2.4. Section 2.5 then comprises the background material on toric geometry that we need. In particular, we recall the definition of the weighted volume functional and discuss its properties in Section 2.5.4. Moreover, in this section, we determine explicitly the unique holomorphic vector field on the manifolds of Theorem A (i) that could be the soliton vector field of a shrinking gradient Kähler–Ricci soliton with bounded scalar curvature.

In Section 3, we prove Theorem A. We first prove in Proposition 3.1 a smooth classification of the underlying manifold, a precursor to the holomorphic classification given by Proposition 3.9. This section concludes by completing the proof of Theorem A.

In the final section, namely Section 4, we prove Theorem B.

2. Preliminaries

2.1. J-holomorphic curves

In this section, we summarise the tools from the theory of J-holomorphic curves that we need in the context of Kähler manifolds. The source for this material is [27, 28].

Let (M, J) be an n-dimensional complex manifold, and let (Σ, j) be a compact Riemann surface with complex structures J and j, respectively. We say that a smooth map $u: \Sigma \to M$ is a *J-holomorphic curve* if the differential du is a complex linear map with respect to j and J, i.e.,

$$J \circ du = du \circ j.$$

A smooth map $u\colon (\Sigma, j) \to (M, J)$ is J-holomorphic if and only if

$$\bar{\partial}_J u = 0, \tag{2.1}$$

where

$$\bar{\partial}_J u := \frac{1}{2}(du + J \circ du \circ j).$$

By definition, a J-holomorphic curve is always parametrised. A J-holomorphic curve $u\colon (\Sigma, j) \to M$ is said to be *multiply covered* if there exist a J-holomorphic curve $u'\colon (\Sigma', j') \to M$ and a branched covering $\phi\colon \Sigma \to \Sigma'$ of degree strictly greater than 1 such that u factors as $u = u' \circ \phi$. The curve u is called *simple* if it is not multiply covered. If u is a multiply covered J-holomorphic curve from $\mathbb{P}^1$, then by the Riemann–Hurwitz formula, $\Sigma' = \mathbb{P}^1$.

We henceforth restrict ourselves to J-holomorphic spheres, that is, when $\Sigma = \mathbb{P}^1$. For a given homology class $A \in H_2(M, \mathbb{Z})$, we denote for such curves the moduli space of solutions to (2.1) by

$$\mathcal{M}(A; J) := \{u \in C^\infty(\mathbb{P}^1, M) \mid J \circ du = du \circ j,\ [u(\mathbb{P}^1)] = A\}$$

and the subspace of simple solutions by

$$\mathcal{M}^*(A; J) := \{u \in \mathcal{M}(A; J) \mid u \text{ is simple}\}.$$

For a compact Riemannian manifold N, let $\Omega^0(N, E)$ denote the space of smooth sections of the bundle $E \to N$. Moreover, let $\Lambda^{0,1} := \Lambda^{0,1} T^*\mathbb{P}^1$ denote the bundle of 1-forms on $\mathbb{P}^1$ of type $(0, 1)$. Assume now that (M, J) is Kähler with a given Kähler form ω and for a given smooth (not necessarily J-holomorphic) curve $u\colon \mathbb{P}^1 \to M$, we define a map

$$\mathcal{F}_u\colon \Omega^0(\mathbb{P}^1, u^*TM) \to \Omega^0(\mathbb{P}^1, \Lambda^{0,1} \otimes_J u^*TM)$$

as follows. Given $\xi \in \Omega^0(\mathbb{P}^1, u^*TM)$, let

$$\Phi_u(\xi)\colon u^*TM \to \exp_u(\xi)^*TM$$

denote the complex bundle isomorphism given by parallel transport along the geodesics $s \mapsto \exp_{u(z)}(s\xi(z))$ with respect to the Levi-Civita connection ∇ induced by ω. Then define

$$\mathcal{F}_u(\xi) := \Phi_u(\xi)^{-1}\bar{\partial}_J(\exp_u(\xi)). \tag{2.2}$$

Write $\Omega_J^{0,1}(\mathbb{P}^1, u^*TM) := \Omega^0(\mathbb{P}^1, \Lambda^{0,1} \otimes_J u^*TM)$, where we drop the subscript J when there is no ambiguity, and let D_u denote the linearisation $d\mathcal{F}_u(0)$ of $\mathcal{F}_u$ at 0. Then D_u defines an operator

$$D_u\colon \Omega^0(\mathbb{P}^1, u^*TM) \to \Omega_J^{0,1}(\mathbb{P}^1, u^*TM),$$

which in our situation with J a complex structure is given by

$$D_u\xi := \frac{1}{2}(\nabla\xi + J(u)\nabla\xi \circ j) \tag{2.3}$$

for every $\xi \in \Omega^0(\mathbb{P}^1, u^*TM)$ [28, Proposition 3.1.1], i.e., $D_u\xi$ is the projection of $\nabla\xi$ onto $\Omega^{0,1}(\mathbb{P}^1, u^*TM)$. This is a real linear "Cauchy–Riemann" operator (cf. [28, Appendix C]), hence is Fredholm [28, Theorem C.1.10], meaning that it has closed range and finite-dimensional kernel and cokernel. The Riemann–Roch theorem asserts that its Fredholm index is

$$\operatorname{index} D_u = 2n + 2c_1(u^*TM),$$

where $n = \dim_{\mathbb{C}} M$. In the case that $u\colon \mathbb{P}^1 \to M$ is actually a J-holomorphic curve, D_u is precisely the Dolbeault $\overline{\partial}$-operator

$$\overline{\partial}\colon \Omega^0(u^*TM) \to \Omega^{0,1}(u^*TM).$$

If in addition D_u is surjective, then $\mathcal{M}^*([u(\mathbb{P}^1)]; J)$ is a smooth oriented manifold near u of real dimension $2n + 2c_1(u^*TM)$ [28, Theorem 3.1.5].

The following is well known.

Proposition 2.1 (Local deformations). *Let M be a two-dimensional complex manifold with complex structure J, let C be a simple embedded J-holomorphic sphere with $C.C = 0$, and let D denote the open ball of radius* 1 *in $\mathbb{C}$. Then there exists an open neighbourhood U of C that is diffeomorphic to $D \times \mathbb{P}^1$ with $\{t\} \times \mathbb{P}^1$ a J-holomorphic sphere in M for each $t \in D$ and $\{0\} \times \mathbb{P}^1 = C$.*

Proof. Fix a parametrisation $u\colon \mathbb{P}^1 \to C \subset M$. As u is J-holomorphic, we know that $\overline{\partial}_J u = 0$. The linearisation D_u of $\mathcal{F}_u$ at 0 is then Fredholm and is precisely the Dolbeault $\overline{\partial}$-operator with respect to J, namely

$$D_u = \overline{\partial}\colon \Omega^0(u^*TM) \to \Omega^{0,1}(u^*TM).$$

Moreover, as C has a trivial holomorphic normal bundle, we have the direct sum decomposition $u^*TM = \mathcal{O} \oplus \mathcal{O}(2)$, a splitting that is respected by $\overline{\partial}$. Therefore, recalling the proof of [28, Lemma 3.3.1], we can consider the action of $\overline{\partial}$ on each factor separately. For any holomorphic line bundle $L \to \mathbb{P}^1$, the cokernel of $\overline{\partial}\colon \Omega^0(\mathbb{P}^1, L) \to \Omega^{0,1}(\mathbb{P}^1, L)$ is precisely the Dolbeault cohomology group $H^{0,1}_{\overline{\partial}}(\mathbb{P}^1, L)$. Now, we have an isomorphism

$$H^{0,1}_{\overline{\partial}}(\mathbb{P}^1, L) \cong (H^{1,0}_{\overline{\partial}}(\mathbb{P}^1, L^*))^*,$$

where $H^{1,0}_{\overline{\partial}}(\mathbb{P}^1, L^*)$ is the space of holomorphic one-forms with values in the dual bundle L^* and which itself is isomorphic to $H^0(\mathbb{P}^1, L^* \otimes K_{\mathbb{P}^1})$, the space of holomorphic sections of the bundle $L^* \otimes K_{\mathbb{P}^1}$ by Kodaira–Serre duality. Hence

$$H^{0,1}_{\overline{\partial}}(\mathbb{P}^1, \mathcal{O}) = H^{0,1}_{\overline{\partial}}(\mathbb{P}^1, \mathcal{O}(2)) = 0.$$

In particular, D_u is surjective of Fredholm index 8, so that $\mathcal{M}^*([C]; J)$ is a smooth oriented manifold of real dimension 8 near u. Indeed, it follows from [28, Corollary 3.3.4] that D_v is surjective for every $v \in \mathcal{M}^*([C]; J)$, hence $\mathcal{M}^*([C]; J)$ itself is a smooth oriented manifold of real dimension 8.

Recall that $\mathcal{M}^*([C]; J)$ comprises parametrised J-holomorphic curves. The six-dimensional real Lie group $\mathrm{PSL}(2,\mathbb{C})$, which we henceforth denote by G, acts freely on $\mathcal{M}^*([C]; J)$ via reparametrisation

$$g \cdot v = v \circ g^{-1} \quad \text{for all } g \in G \text{ and } v \in \mathcal{M}^*([C]; J).$$

We consider the quotient space

$$\widetilde{\mathcal{M}}^*([C]; J) := \mathcal{M}^*([C]; J)/G.$$

This is precisely the space of J-holomorphic spheres in M in the same homology class as C and is a smooth oriented manifold of real dimension $8-6=2$. As C is simple and embedded, McDuff's adjunction formula [28, Corollary E.1.7] implies that every sphere in $\widetilde{\mathcal{M}}^*([C]; J)$ is embedded in M. In addition, the fact that $C.C = 0$ implies that any two distinct $\mathbb{P}^1$'s in $\widetilde{\mathcal{M}}^*([C]; J)$ are disjoint in M.

Set $\mathcal{M}^*([C]; J) \times_G \mathbb{P}^1 \equiv (\mathcal{M}^*([C]; J) \times \mathbb{P}^1)/G$, where G acts on $\mathcal{M}^*([C]; J) \times \mathbb{P}^1$ by $g \cdot (v, z) \mapsto (v \circ g^{-1}, g \cdot z)$. Then $\mathcal{M}^*([C]; J) \times_G \mathbb{P}^1$ is a smooth manifold of real dimension 4 which is a $\mathbb{P}^1$-bundle over $\widetilde{\mathcal{M}}^*([C]; J)$. We define an evaluation map ev by

$$\mathrm{ev}\colon \mathcal{M}^*([C]; J) \times_G \mathbb{P}^1 \mapsto M, \quad [(v, z)] \mapsto v(z).$$

This is a smooth map between two oriented smooth manifolds of the same dimension that maps every fibre $\{[(v, z)] \mid z \in \mathbb{P}^1\}$ biholomorphically onto an embedded J-holomorphic sphere in M, with distinct fibres being mapped to distinct J-holomorphic spheres in M with $\{[(u, z)] \mid z \in \mathbb{P}^1\}$ being mapped to C. In particular, ev is an immersion between two manifolds of the same dimension, hence is a local diffeomorphism. Choosing a trivialisation of the $\mathbb{P}^1$-bundle in a neighbourhood of the fibre $\{[(u, z)] \mid z \in \mathbb{P}^1\}$ now yields the result. ∎

Next, for a compact Riemannian manifold N, for an integer $k \geq 1$ and a real number $p > 2$, let $W^{k,p}(N, E)$ denote the completion of the space $\Omega^0(N, E)$ of smooth sections of the bundle $E \to N$ with respect to the Sobolev $W^{k,p}$-norm. Again, assume that (M, J) is Kähler with Kähler form ω and endow $(\mathbb{P}^1, j)$ with the Fubini–Study form $\omega_{\mathbb{P}^1}$ compatible with j. For a given smooth curve $u\colon \mathbb{P}^1 \to M$ and real number $p > 2$, let

$$\mathcal{X}_u^p := W^{1,p}(\mathbb{P}^1, u^*TM), \quad \mathcal{Y}_u^p := L^p(\mathbb{P}^1, \Lambda^{0,1} \otimes_J u^*TM), \tag{2.4}$$

where all relevant norms are understood to be with respect to ω and $\omega_{\mathbb{P}^1}$ and the Levi-Civita connection ∇ determined by ω. Then the maps $\mathcal{F}_u$ and D_u defined above for smooth sections extend in a natural way to maps $\mathcal{F}_u\colon \mathcal{X}_u^p \to \mathcal{Y}_u^p$.

One can prove that if u is an *approximate* J-holomorphic curve with *sufficiently surjective* operator D_u, then there are J-holomorphic curves near u, and the moduli space can be modelled on a neighbourhood of zero in the kernel of D_u. More precisely, we have the following theorem.

Theorem 2.2 ([27, Theorem 3.3.4] with $\Sigma = \mathbb{P}^1$ and $u\colon \Sigma \to M$ smooth). *Let $p > 2$ and let $\|\cdot\|$ denote the operator norm. Then for every constant $c_0 > 0$, there exist constants $\delta > 0$ and $c > 0$ such that the following holds. Let $u\colon \mathbb{P}^1 \to M$ be a smooth map and $Q_u\colon \mathcal{Y}_u^p \to \mathcal{X}_u^p$ be a right inverse of D_u such that*

$$\|Q_u\| \leq c_0, \quad \|du\|_{L^p} \leq c_0, \quad \|\overline{\partial}_J u\|_{L^p} \leq \delta,$$

with respect to a metric on $\mathbb{P}^1$ such that $\mathrm{vol}(\mathbb{P}^1) \leq c_0$. Then for every $\xi \in \ker(D_u)$ with $\|\xi\|_{L^p} \leq \delta$, there exists a section $\tilde{\xi} = Q_u\eta \in \mathcal{X}_u^p$ such that

$$\overline{\partial}_J(\exp_u(\xi + Q_u\eta)) = 0, \quad \|Q_u\eta\|_{W^{1,p}} \leq c\|\overline{\partial}_J(\exp_u(\xi))\|_{L^p}.$$

This theorem is proved using the implicit function theorem. Given a surjective operator D_u, one technique for constructing a right inverse Q_u is to reduce the domain of D_u by imposing pointwise conditions on ξ so that the resulting operator is bijective, and then taking Q_u to be the inverse of this restricted operator. We will use this to prove the following two corollaries of this theorem.

Corollary 2.3 (Deformation of trivially-embedded curves). *Let M be a manifold of real dimension 4, let (g, J) and $(\tilde{g}, \tilde{J})$ be two Kähler structures on M, and let $u\colon (\mathbb{P}^1, j) \to (M, \tilde{J})$ be a smooth $\tilde{J}$-holomorphic curve with trivial self-intersection. Denote the Levi-Civita connection of $\tilde{g}$ by $\tilde{\nabla}$. Then for all $x \in u(\mathbb{P}^1)$, there exists $\varepsilon > 0$ such that if*

$$|g - \tilde{g}|_{\tilde{g}} + |\tilde{\nabla}(g - \tilde{g})|_{\tilde{g}} + |J - \tilde{J}|_{\tilde{g}} < \varepsilon \tag{2.5}$$

on some sufficiently large compact subset $K \subset M$ containing $u(\mathbb{P}^1)$, then there exists a unique smooth section $\tilde{\xi} \in \Gamma(u^*TM)$ with $\tilde{\xi}(x) = 0$ and $\|\tilde{\xi}\|_{C^0} \leq C\|J - \tilde{J}\|_{C^0(\mathbb{P}^1,\tilde{g})}$ such that

$$v := \exp_u(\tilde{\xi})\colon (\mathbb{P}^1, j) \to (M, J)$$

is a smooth J-holomorphic curve (in the same homology class as $u(\mathbb{P}^1)$ with $x \in v(\mathbb{P}^1)$).

Proof. Let $\tilde{F}_u$ denote map (2.2) corresponding to the data $(u, \tilde{g}, \tilde{J})$ and recall from the proof of Proposition 2.1 that the linearisation $\tilde{D}_u$ of $\tilde{F}_u$ at 0 with respect to $\tilde{J}$ is Fredholm of index 8 and is precisely the Dolbeault $\overline{\partial}$-operator with respect to $\tilde{J}$, namely

$$\tilde{D}_u = \overline{\partial}\colon \Omega^0(u^*TM) \to \Omega^{0,1}_{\tilde{J}}(u^*TM).$$

Via the direct sum decomposition $u^*TM = \mathcal{O} \oplus \mathcal{O}(2)$, the kernel of $\tilde{D}_u$ is spanned by $\{1, z_1^2, z_1z_2, z_2^2\}$ with $[z_1 : z_2]$ homogeneous coordinates on $\mathbb{P}^1$. Identifying x with its pre-image under u, restrict $\tilde{D}_u$ to the subspace $\Omega^0(u^*TM)_{(0)}$ of $\Omega^0(u^*TM)$ of smooth sections that vanish in the tangential directions at the points x, $z_1 = 0$, and $z_2 = 0$ on $\mathbb{P}^1$, and vanish in the normal direction at x. (If $z_i(x) = 0$ for some $i = 1, 2$, then just choose an arbitrary point on $\mathbb{P}^1$ distinct from $z_1 = 0$ and $z_2 = 0$ for the sections to vanish.) Then the restriction

$$\tilde{D}_u^{(0)}\colon \Omega^0(u^*TM)_{(0)} \to \Omega^{0,1}_{\tilde{J}}(u^*TM)$$

is an isomorphism. Fix $p > 2$, and let $(\widetilde{\mathcal{X}}_u^p)_{(0)}$ and $\widetilde{\mathcal{Y}}_u^p$ denote the Sobolev completion of $\Omega^0(u^*TM)_{(0)}$ with respect to the $W^{1,p}$-norm and the completion of $\Omega^0(\mathbb{P}^1, \Lambda^{0,1} \otimes_{\widetilde{J}} u^*TM)$ with respect to the L^p-norm induced by $\widetilde{g}$ and the choice of Kähler metric on $\mathbb{P}^1$, respectively. Then $\widetilde{D}_u^{(0)}$ defines an isomorphism $\widetilde{D}_u^{(0)} \colon (\widetilde{\mathcal{X}}_u^p)_{(0)} \to \widetilde{\mathcal{Y}}_u^p$.

Next consider $\overline{\partial}_J u$. Let $\mathcal{X}_u^p$ and $\mathcal{Y}_u^p$ be as in (2.4) defined with respect to ω and the choice of Kähler metric on $\mathbb{P}^1$, and let $(\mathcal{X}_u^p)_{(0)}$ denote the Sobolev completion of $\Omega^0(u^*TM)_{(0)}$ with respect to the $W^{1,p}$-norm induced by the aforementioned metrics. Then the linearisation D_u defines a map

$$D_u \colon \mathcal{X}_u^p \to \mathcal{Y}_u^p$$

which we can restrict to $(\mathcal{X}_u^p)_{(0)}$ and compose with the projection $\mathrm{pr} \colon \mathcal{Y}_u^p \to \widetilde{\mathcal{Y}}_u^p$ to obtain a map

$$(\mathrm{pr} \circ D_u)^{(0)} \colon (\mathcal{X}_u^p)_{(0)} \to \widetilde{\mathcal{Y}}_u^p.$$

Explicitly, the composition $\mathrm{pr} \circ D_u$ is given by

$$(\mathrm{pr} \circ D_u)(\xi) = \frac{1}{2}(D_u\xi + \widetilde{J} D_u \xi \circ j) = \frac{1}{4}((\nabla\xi - \widetilde{J} J \nabla\xi) + (\widetilde{J} + J)\nabla\xi \circ j). \quad (2.6)$$

As clearly $(\mathcal{X}_u^p)_{(0)} = (\widetilde{\mathcal{X}}_u^p)_{(0)}$, we also have an isomorphism $\widetilde{D}_u^{(0)} \colon (\mathcal{X}_u^p)_{(0)} \to \widetilde{\mathcal{Y}}_u^p$. Thus, from the openness of the invertibility of bounded linear operators, we know that there exists $\delta > 0$ such that $\|\widetilde{D}_u^{(0)} - (\mathrm{pr} \circ D_u)^{(0)}\| < \delta$ implies the invertibility of $(\mathrm{pr} \circ D_u)^{(0)}$. In light of (2.3) and (2.6), we estimate that

$$\|\widetilde{D}_u - \mathrm{pr} \circ D_u\| \leq C(\|\nabla - \widetilde{\nabla}\|_{C^0(\mathbb{P}^1,\widetilde{g})} + \|J - \widetilde{J}\|_{C^0(\mathbb{P}^1,\widetilde{g})}),$$

and so $(\mathrm{pr} \circ D_u)^{(0)}$ is invertible if (2.5) holds true for $\varepsilon > 0$ sufficiently small. Moreover, if $\|J - \widetilde{J}\|_{C^0(\mathbb{P}^1,\widetilde{g})}$ is sufficiently small, then pr is an isomorphism. Hence, by shrinking $\varepsilon > 0$ further if necessary, we can assert that the restricted map

$$D_u^{(0)} \colon (\mathcal{X}_u^p)_{(0)} \to \mathcal{Y}_u^p$$

is itself an isomorphism. As

$$\|\overline{\partial}_J u\|_{L^p} \leq C(\|\overline{\partial}_J u - \overline{\partial}_{\widetilde{J}} u\|_{L^p} + \underbrace{\|\overline{\partial}_{\widetilde{J}} u\|_{L^p}}_{=0}) \leq C\|J - \widetilde{J}\|_{C^0(\mathbb{P}^1,\widetilde{g})},$$

control on $\|J - \widetilde{J}\|_{C^0(\mathbb{P}^1,\widetilde{g})}$ allows us to assume that $\|\overline{\partial}_J u\|_{L^p}$ is as small as we please. Therefore, applying Theorem 2.2 with $\xi = 0$, we deduce that for all $\varepsilon > 0$ sufficiently small, there exists a unique section $\widetilde{\xi} \in (\mathcal{X}_u^p)_{(0)}$ such that the map $v := \exp_u(\widetilde{\xi})$ is J-holomorphic and $\|\widetilde{\xi}\|_{W^{1,p}} \leq C\|\overline{\partial}_J u\|_{L^p}$. Thus,

$$\|\widetilde{\xi}\|_{W^{1,p}} \leq C\|\overline{\partial}_J u\|_{L^p} \leq C\|J - \widetilde{J}\|_{C^0(\mathbb{P}^1,\widetilde{g})}.$$

The desired estimate on $\widetilde{\xi}$ now follows from Sobolev embedding. The fact that v is smooth follows from elliptic regularity and the smoothness of J [28, Proposition 3.1.9].

By construction, $\widetilde{\xi}(x) = 0$ so that $x \in v(\mathbb{P}^1)$, and $v(\mathbb{P}^1)$ lies in the same homology class as $u(\mathbb{P}^1)$, hence v has the required properties. Finally, the uniqueness of v is a consequence of the triviality of the normal bundle of $v(\mathbb{P}^1)$ and the positivity of intersections of complex subvarieties in a complex surface [28, Theorem 2.6.3]. ∎

The next corollary is reminiscent of [24, Theorem 5].

Corollary 2.4 (Deformation of (-1)-curves). *Let M be a manifold of real dimension 4, let (g, J) and $(\widetilde{g}, \widetilde{J})$ be two Kähler structures on M, and let $u\colon (\mathbb{P}^1, j) \to (M, \widetilde{J})$ be a smooth $\widetilde{J}$-holomorphic (-1)-curve. Denote the Levi-Civita connection of $\widetilde{g}$ by $\widetilde{\nabla}$. Then there exists $\varepsilon > 0$ such that if*

$$|g - \widetilde{g}|_{\widetilde{g}} + |\widetilde{\nabla}(g - \widetilde{g})|_{\widetilde{g}} + |J - \widetilde{J}|_{\widetilde{g}} < \varepsilon$$

on some sufficiently large compact subset $K \subset M$ containing $u(\mathbb{P}^1)$, then there exists a unique smooth section $\widetilde{\xi} \in \Gamma(u^*TM)$ with $\|\widetilde{\xi}\|_{C^0} \leq C\|J - \widetilde{J}\|_{C^0(\mathbb{P}^1, \widetilde{g})}$ such that $v := \exp_u(\widetilde{\xi})\colon (\mathbb{P}^1, j) \to (M, J)$ is a smooth J-holomorphic (-1)-curve (in the same homology class as $u(\mathbb{P}^1)$).

Proof. Let $\widetilde{F}_u$ denote map (2.2) corresponding to the data $(u, \widetilde{g}, \widetilde{J})$ and recall from the proof of Proposition 2.1 that the linearisation $\widetilde{D}_u$ of $\widetilde{F}_u$ at 0 with respect to $\widetilde{J}$ is precisely the Dolbeault $\bar{\partial}$-operator with respect to $\widetilde{J}$, namely

$$\widetilde{D}_u = \bar{\partial}\colon \Omega^0(u^*TM) \to \Omega^{0,1}_{\widetilde{J}}(u^*TM).$$

This is Fredholm of index 6 (cf. the proof of Proposition 2.1) and via the direct sum decomposition $u^*TM = \mathcal{O}(-1) \oplus \mathcal{O}(2)$, the kernel of $\widetilde{D}_u$ is spanned by $\{z_1^2, z_1 z_2, z_2^2\}$ with $[z_1 : z_2]$ homogeneous coordinates on $\mathbb{P}^1$. Restrict $\widetilde{D}_u$ to the subspace $\Omega^0(u^*TM)_{(1)}$ of $\Omega^0(u^*TM)$ of smooth sections that vanish in the tangential directions at the points $z_1 = 0$, $z_2 = 0$, and at an arbitrary point of $\mathbb{P}^1$ distinct from $z_1 = 0$ and $z_2 = 0$. Then the restriction

$$\widetilde{D}_u^{(1)}\colon \Omega^0(u^*TM)_{(1)} \to \Omega^{0,1}_{\widetilde{J}}(u^*TM)$$

defines an isomorphism. The proof now proceeds verbatim as that of Corollary 2.3 without the last sentence. ∎

2.2. *Shrinking Ricci solitons*

The metrics we are interested in are the following.

Definition 2.5. A *shrinking Ricci soliton* is a triple (M, g, X), where M is a Riemannian manifold endowed with a complete Riemannian metric g and a vector field X satisfying the equation

$$\operatorname{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \frac{1}{2}g. \tag{2.7}$$

We call X the *soliton vector field* and say that (M, g, X) is a *gradient* Ricci soliton if $X = \nabla^g f$ for some real-valued smooth function f on M. In this latter case, equation (2.7) reduces to

$$\mathrm{Ric}(g) + \mathrm{Hess}_g(f) = \frac{1}{2}g,$$

where Hess_g denotes the Hessian with respect to g.

If g is complete and Kähler with Kähler form ω, then we say that (M, g, X) is a *shrinking gradient Kähler–Ricci soliton* if $X = \nabla^g f$ for some real-valued smooth function f on M, X is complete and real holomorphic, and

$$\rho_\omega + i\partial\bar{\partial} f = \omega, \tag{2.8}$$

where ρ_ω is the Ricci form of ω. For gradient Ricci solitons and gradient Kähler–Ricci solitons, the function f satisfying $X = \nabla^g f$ is called the *soliton potential*.

As the next result shows, the soliton potential of a complete non-compact shrinking gradient Ricci soliton grows quadratically with respect to the distance.

Theorem 2.6 ([5, Theorem 1.1]). *Let (M, g, X) be a complete non-compact shrinking gradient Ricci soliton with soliton vector field $X = \nabla^g f$ for a smooth real-valued function $f : M \to \mathbb{R}$. Then for $x \in M$, f satisfies the estimates*

$$\frac{1}{4}(d_g(p, x) - c_1)^2 - C \leq f(x) \leq \frac{1}{4}(d_g(p, x) + c_2)^2$$

for some $C > 0$, where $d_g(p, \cdot)$ denotes the distance to a fixed point $p \in M$ with respect to g. Here, c_1 and c_2 are positive constants depending only on the real dimension of M and the geometry of g on the unit ball $B_p(1)$ based at p.

In particular, f is proper.

We also know the following regarding the asymptotics of four-dimensional shrinking gradient Ricci solitons.

Lemma 2.7. *Let (M, g, X) be a complete non-compact shrinking gradient Ricci soliton of real dimension 4 with soliton vector field $X = \nabla^g f$ for a smooth real-valued function $f : M \to \mathbb{R}$ and with bounded scalar curvature R_g such that $\mathrm{R}_g \to 0$ along every integral curve of X. Then $\mathrm{R}_g \to 0$. Moreover, there exists a constant $C > 0$ such that $0 \leq \mathrm{R}_g \leq C f^{-1}$ outside a sufficiently large compact subset of M.*

Proof. On a shrinking gradient Ricci soliton of real dimension 4 with bounded scalar curvature R_g, we see from [30, Theorem 1.3] that the bounds [30, (3.4)] hold true so that [30, Theorem 3.1] applies. The Harnack estimate from [30, (3.73)] then implies that if R_g is strictly smaller than the constant in this Harnack estimate at some point x in the level set $\{f = t_1\}$ for $t_1 \in \mathbb{R}$ with $\{X = 0\} \subset f^{-1}((-\infty, t_1])$, then R_g decays like $C f^{-1}$ along the integral curve passing through x for some constant $C > 0$ independent of x. Thus, for the first assertion, it suffices to show that there exists $t_1 \in \mathbb{R}$ with $\{X = 0\} \subset f^{-1}((-\infty, t_1])$ so that R_g is as small as we please on $\{f = t_1\}$.

To this end, note that since R_g is bounded, the zero set of X is compact (cf. [11, proof of Lemma 2.26]), hence by properness of f (cf. Theorem 2.6), there exists $t_0 > 0$ so that $\{X = 0\} \subset f^{-1}((-\infty, \frac{t_0}{2}])$. Through the gradient flow of f, the level sets $\{f = t\}$ are therefore diffeomorphic to $\{f = t_0\}$ for all $t > t_0$. In particular, all integral curves of X may be parametrised by $\{f = t_0\}$. Let $x_0 \in \{f = t_0\}$ and choose $\varepsilon > 0$. Then since $\mathrm{R}_g \to 0$ along each integral curve of X by assumption and $\mathrm{R}_g \geq 0$ [46], there exists x_0' lying along the integral curve of X passing through x_0 with $f(x_0') := t_0' > t_0$ such that $0 \leq \mathrm{R}_g(x_0') < \varepsilon$. We can then find an open neighbourhood of x_0' in $\{f = t_0'\}$ such that $0 \leq \mathrm{R}_g < 2\varepsilon$. Flowing this neighbourhood back to x_0 along $-X$, we obtain an open neighbourhood U_0 of x_0 in $\{f = t_0\}$. By properness of f, the level set $\{f = t_0\}$ is compact and so can be covered by finitely many such neighbourhoods U_i, $i = 0, \ldots, N$. Letting t_1 denote the maximum of the corresponding t_i', $i = 0, \ldots, N$, we find that $0 \leq \mathrm{R}_g < 2\varepsilon$ on $\{f = t_1\}$ and $\{X = 0\} \subset f^{-1}((-\infty, t_1])$, as required. By [30, (3.73)], it now follows that R_g decays globally like Cf^{-1}. ■

Complex two-dimensional complete non-compact shrinking gradient Kähler–Ricci solitons with scalar curvature tending to zero at infinity were classified in [11, Theorem E (3)]. They comprise the flat Gaussian soliton on $\mathbb{C}^2$ and the example of Feldman–Ilmanen–Knopf [15] on the blowup of $\mathbb{C}^2$ at one point, up to the action of $\mathrm{GL}(2, \mathbb{C})$.

2.3. Polyhedrons and polyhedral cones

We take the following from [12] and [33, Appendix A].

Let E be a real vector space of dimension n and let E^* denote the dual. Write $\langle \cdot, \cdot \rangle$ for the evaluation $E^* \times E \to \mathbb{R}$. Furthermore, assume that we are given a *lattice* $\Gamma \subset E$, that is, an additive subgroup $\Gamma \simeq \mathbb{Z}^n$. This gives rise to a dual lattice $\Gamma^* \subset E^*$. For any $\nu \in E^*$, $c \in \mathbb{R}$, let $K(\nu, c)$ be the (closed) half space $\{x \in E \mid \langle \nu, x \rangle \geq c\}$ in E. Then we have the following.

Definition 2.8. A *polyhedron* P in E is a finite intersection of half spaces, i.e.,

$$P = \bigcap_{i=1}^{r} K(\nu_i, c_i) \quad \text{for } \nu_i \in E^*,\ c_i \in \mathbb{R}.$$

It is called a *polyhedral cone* if all $c_i = 0$, and moreover a *rational polyhedral cone* if all $\nu_i \in \Gamma^*$ and $c_i = 0$. In addition, a polyhedron is called *strongly convex* if it does not contain any affine subspace of E.

The following definition will be useful.

Definition 2.9. A polyhedron $P \subset E^*$ is called *Delzant* if its set of vertices is non-empty and each vertex $v \in P$ has the property that there are precisely n edges $\{e_1, \ldots, e_n\}$ (one-dimensional faces) emanating from v, and there exists a basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$ of Γ^* such that ε_i lies along the ray $\mathbb{R}(e_i - v)$.

Note that any such P is necessarily strongly convex.

The asymptotic cone of a polyhedron contains all the directions going off to infinity in the polyhedron.

Definition 2.10. Let P be a polyhedron in E. Its *asymptotic cone*, denoted by $\mathcal{C}(P)$, is the set of vectors $\alpha \in E$ with the property that there exists $\alpha^0 \in E$ such that $\alpha^0 + t\alpha \in P$ for sufficiently large $t > 0$.

The asymptotic cone may be identified as follows.

Lemma 2.11 ([33, Lemma A.3]). *If* $P = \bigcap_{i=1}^r K(\nu_i, c_i)$, *then* $\mathcal{C}(P) = \bigcap_{i=1}^r K(\nu_i, 0)$.

In particular, the asymptotic cone of a polyhedron is a polyhedral cone. In addition, we see that for two polyhedrons P, Q, in E,

$$Q \subseteq P \Rightarrow \mathcal{C}(P) \subseteq \mathcal{C}(Q).$$

Compact polyhedrons can be characterised by their asymptotic cone.

Lemma 2.12 ([33, Corollary A.9]). *A polyhedron* P *is compact if and only if* $\mathcal{C}(P) = \{0\}$.

We also have the following.

Definition 2.13. The *dual* of a polyhedral cone C is the set $C^\vee = \{x \in E^* \mid \langle x, C\rangle \geq 0\}$.

It is clear that for two polyhedrons P, Q, in E,

$$Q \subseteq P \Rightarrow P^\vee \subseteq Q^\vee.$$

2.4. *Hamiltonian actions*

Recall what it means for an action to be Hamiltonian.

Definition 2.14. Let (M, ω) be a symplectic manifold, and let T be a real torus acting by symplectomorphisms on (M, ω). Denote by $\mathfrak{t}$ the Lie algebra of T and by $\mathfrak{t}^*$ its dual. Then we say that the action of T is *Hamiltonian* if there exists a smooth map $\mu_\omega \colon M \to \mathfrak{t}^*$ such that for all $\zeta \in \mathfrak{t}$,

$$-\omega \lrcorner \zeta = du_\zeta,$$

where $u_\zeta(x) = \langle \mu_\omega(x), \zeta\rangle$ for all $\zeta \in \mathfrak{t}$ and $x \in M$ and $\langle \cdot, \cdot \rangle$ denotes the dual pairing between $\mathfrak{t}$ and $\mathfrak{t}^*$. We call μ_ω the *moment map* of the T-action, and we call u_ζ the *Hamiltonian (potential)* of ζ.

Define

$$\Lambda_\omega := \{Y \in \mathfrak{t} \mid \mu_\omega(Y) \text{ is proper and bounded below}\} \subseteq \mathfrak{t}.$$

By Theorem 2.6, this set is non-empty for ω a complete non-compact shrinking gradient Kähler–Ricci soliton. In addition, it can be identified through the image of μ_ω in the following way.

Proposition 2.15 ([33, Proposition 1.4]). *Let (M,ω) be a (possibly non-compact) symplectic manifold of real dimension $2n$ with symplectic form ω on which there is a Hamiltonian action of a real torus T with moment map $\mu_\omega\colon M \to \mathfrak{t}^*$, where $\mathfrak{t}$ is the Lie algebra of T and $\mathfrak{t}^*$ its dual. Assume that the fixed point set of T is compact and that $\Lambda_\omega \neq \emptyset$. Then $\Lambda_\omega = \operatorname{int}(\mathcal{C}(\mu_\omega(M))^\vee)$.*

2.5. Toric geometry

In this section, we collect together some standard facts from toric geometry as well as recall those results from [9] that we require. We begin with the following definition.

Definition 2.16. A *toric manifold* is an n-dimensional complex manifold M endowed with an effective holomorphic action of the algebraic torus $(\mathbb{C}^*)^n$ such that the following hold true:

- The fixed point set of the $(\mathbb{C}^*)^n$-action is compact.
- There exists a point $p \in M$ with the property that the orbit $(\mathbb{C}^*)^n \cdot p \subset M$ forms a dense open subset of M.

We will often denote the dense orbit simply by $(\mathbb{C}^*)^n \subset M$ in what follows. The $(\mathbb{C}^*)^n$-action of course determines the action of the real torus $T^n \subset (\mathbb{C}^*)^n$.

2.5.1. Divisors on toric varieties and fans. Let $T^n \subset (\mathbb{C}^*)^n$ be the real torus with Lie algebra $\mathfrak{t}$ and denote the dual pairing between $\mathfrak{t}$ and the dual space $\mathfrak{t}^*$ by $\langle\cdot\,,\cdot\rangle$. There is a natural integer lattice $\Gamma \simeq \mathbb{Z}^n \subset \mathfrak{t}$ comprising all $\lambda \in \mathfrak{t}$ such that $\exp(\lambda) \in T^n$ is the identity. This then induces a dual lattice $\Gamma^* \subset \mathfrak{t}^*$. We have the following combinatorial definition.

Definition 2.17. A *fan* Σ in $\mathfrak{t}$ is a finite set of rational polyhedral cones σ satisfying

(i) For every $\sigma \in \Sigma$, each face of σ also lies in Σ.

(ii) For every pair $\sigma_1, \sigma_2 \in \Sigma$, $\sigma_1 \cap \sigma_2$ is a face of each.

To each fan Σ in $\mathfrak{t}$, one can associate a toric variety X_Σ. Heuristically, Σ contains all the data necessary to produce a partial equivariant compactification of $(\mathbb{C}^*)^n$, resulting in X_Σ. More concretely, one obtains X_Σ from Σ as follows. For each n-dimensional cone $\sigma \in \Sigma$, one constructs an affine toric variety U_σ which we first explain. We have the dual cone $\sigma^\vee$ of σ. Denote by S_σ the semigroup of those lattice points which lie in $\sigma^\vee$ under addition. Then one defines the semigroup ring, as a set, as all finite sums of the form

$$\mathbb{C}[S_\sigma] = \Big\{ \sum \lambda_s s \;\Big|\; s \in S_\sigma \Big\},$$

with the ring structure defined on monomials by $\lambda_{s_1} s_1 \cdot \lambda_{s_2} s_2 = (\lambda_{s_1}\lambda_{s_2})(s_1 + s_2)$ and extended in the natural way. The affine variety U_σ is then defined to be $\operatorname{Spec}(\mathbb{C}[S_\sigma])$. This automatically comes endowed with a $(\mathbb{C}^*)^n$-action with a dense open orbit. This construction can also be applied to the lower-dimensional cones $\tau \in \Sigma$. If $\sigma_1 \cap \sigma_2 = \tau$, then there is a natural way to map U_τ into U_{σ_1} and U_{σ_2} isomorphically. One constructs X_Σ by declaring the collection of all U_σ to be an open affine cover of X_Σ with transition functions determined by U_τ. This identification is also reversible.

Proposition 2.18 ([12, Corollary 3.1.8]). *Let M be a smooth toric manifold. Then there exists a fan Σ such that $M \simeq X_\Sigma$.*

Proposition 2.19 (Orbit-cone correspondence, [12, Theorem 3.2.6]). *The k-dimensional cones $\sigma \in \Sigma$ are in a natural one-to-one correspondence with the $(n-k)$-dimensional orbits O_σ of the $(\mathbb{C}^*)^n$-action on X_Σ.*

In particular, each ray $\sigma \in \Sigma$ determines a unique torus-invariant divisor D_σ. As a consequence, a torus-invariant Weil divisor D on X_Σ naturally determines a polyhedron $P_D \subset \mathfrak{t}^*$. Indeed, we can decompose D uniquely as $D = \sum_{i=1}^N a_i D_{\sigma_i}$, where $\{\sigma_i\}_i \subset \Sigma$ is the collection of rays. Then by assumption, there exists a unique minimal lattice element $\nu_i \in \sigma_i \cap \Gamma$. The polyhedron P_D is then given by

$$P_D = \{x \in \mathfrak{t}^* \mid \langle \nu_i, x\rangle \geq -a_i\} = \bigcap_{i=1}^N K(\nu_i, -a_i). \tag{2.9}$$

2.5.2. Kähler metrics on toric varieties. For a given toric manifold M endowed with a Riemannian metric g invariant under the action of the real torus $T^n \subset (\mathbb{C}^*)^n$ and Kähler with respect to the underlying complex structure of M, the Kähler form ω of g is also invariant under the T^n-action. We call such a manifold a *toric Kähler manifold*. In what follows, we always work with a fixed complex structure on M.

Hamiltonian Kähler metrics have a useful characterisation due to Guillemin.

Proposition 2.20 ([19, Theorem 4.1]). *Let ω be any T^n-invariant Kähler form on M. Then the T^n-action is Hamiltonian with respect to ω if and only if the restriction of ω to the dense orbit $(\mathbb{C}^*)^n \subset M$ is exact, i.e., there exists a T^n-invariant potential ϕ such that*

$$\omega = 2i\partial\bar{\partial}\phi.$$

Fix once and for all a $\mathbb{Z}$-basis $(X_1, \dots, X_n)$ of $\Gamma \subset \mathfrak{t}$. This in particular induces a background coordinate system $\xi = (\xi^1, \dots, \xi^n)$ on $\mathfrak{t}$. Using the natural inner product on $\mathfrak{t}$ to identify $\mathfrak{t} \cong \mathfrak{t}^*$, we can also identify $\mathfrak{t}^* \cong \mathbb{R}^n$. For clarity, we will denote the induced coordinates on $\mathfrak{t}^*$ by $x = (x^1, \dots, x^n)$. Let $(z_1, \dots, z_n)$ be the natural coordinates on $(\mathbb{C}^*)^n$ as an open subset of $\mathbb{C}^n$. There is a natural diffeomorphism $\mathrm{Log}\colon (\mathbb{C}^*)^n \to \mathfrak{t} \times T^n$ which provides a one-to-one correspondence between T^n-invariant smooth functions on $(\mathbb{C}^*)^n$ and smooth functions on $\mathfrak{t}$. Explicitly,

$$(z_1, \dots, z_n) \overset{\mathrm{Log}}{\longmapsto} (\log(r_1), \dots, \log(r_n), \theta_1, \dots, \theta_n) = (\xi_1, \dots, \xi_n, \theta_1, \dots, \theta_n), \tag{2.10}$$

where $z_j = r_j e^{i\theta_j}$, $r_j > 0$. Given a function $H(\xi)$ on $\mathfrak{t}$, we can extend H trivially to $\mathfrak{t} \times T^n$ and pull back by Log to obtain a T^n-invariant function on $(\mathbb{C}^*)^n$. Clearly, any T^n-invariant function on $(\mathbb{C}^*)^n$ can be written in this form.

Choose any branch of log and write $w = \log(z)$. Then clearly $w = \xi + i\theta$, where $\xi = (\xi^1, \dots, \xi^n)$ are real coordinates on $\mathfrak{t}$ (or, more precisely, there is a corresponding lift of θ to the universal cover with respect to which this equality holds), and so if ϕ is T^n-invariant and $\omega = 2i\partial\bar{\partial}\phi$, then we have that

$$\omega = 2i\frac{\partial^2\phi}{\partial w^i \partial \bar{w}^j} dw_i \wedge d\bar{w}_j = \frac{\partial^2 \phi}{\partial \xi^i \partial \xi^j} d\xi^i \wedge d\theta^j. \tag{2.11}$$

In this setting, the metric g corresponding to ω is given on $\mathfrak{t} \times T^n$ by

$$g = \phi_{ij}(\xi) d\xi^i d\xi^j + \phi_{ij}(\xi) d\theta^i d\theta^j, \tag{2.12}$$

and the moment map μ as a map $\mu\colon \mathfrak{t} \times T^n \to \mathfrak{t}^*$ is defined by the relation

$$\langle \mu(\xi, \theta), b\rangle = \langle \nabla\phi(\xi), b\rangle$$

for all $b \in \mathfrak{t}$, where $\nabla\phi$ is the Euclidean gradient of ϕ. The T^n-invariance of ϕ implies that it depends only on ξ when considered a function on $\mathfrak{t} \times T^n$ via (2.10). Since ω is Kähler, we see from (2.11) that the Hessian of ϕ is positive definite so that ϕ itself is strictly convex. In particular, $\nabla\phi$ is a diffeomorphism onto its image. Using the identifications mentioned above, we view $\nabla\phi$ as a map from $\mathfrak{t}$ into an open subset of $\mathfrak{t}^*$.

2.5.3. Kähler–Ricci solitons on toric manifolds. We define what we mean by a shrinking Kähler–Ricci soliton in the toric category.

Definition 2.21. A complex n-dimensional shrinking Kähler–Ricci soliton (M, g, X) with complex structure J is *toric* if M is a toric manifold as in Definition 2.16, JX lies in the Lie algebra $\mathfrak{t}$ of the underlying real torus T^n that acts on M, and g is T^n-invariant. In particular, the zero set of X is compact.

It follows from [45] that $H^1(M) = 0$, hence the induced real T^n-action is automatically Hamiltonian with respect to ω. Working on the dense orbit $(\mathbb{C}^*)^n \subset M$, the condition that a vector field JY lies in $\mathfrak{t}$ is equivalent to saying that in the coordinate system $(\xi^1, \dots, \xi^n, \theta_1, \dots, \theta_n)$ from (2.10), there is a constant $b_Y = (b_Y^1, \dots, b_Y^n) \in \mathbb{R}^n$ such that

$$JY = b_Y^i \frac{\partial}{\partial\theta^i} \quad \text{or equivalently,} \quad Y = b_Y^i \frac{\partial}{\partial \xi^i}. \tag{2.13}$$

From Proposition 2.20, we know that $\mathcal{L}_X\omega = 2i\partial\bar{\partial}X(\phi)$. In addition, the function $X(\phi)$ on $(\mathbb{C}^*)^n$ can be written as $\langle b_X, \nabla\phi\rangle = b_X^j \frac{\partial\phi}{\partial\xi^j}$, where $b_X \in \mathbb{R}^n$ corresponds to the soliton vector field X via (2.13). These observations allow us to write the shrinking soliton equation (2.8) as a real Monge–Ampère equation for ϕ on $\mathbb{R}^n$.

Proposition 2.22 ([9, Proposition 2.6]). *Let (M, g, X) be a toric shrinking gradient Kähler–Ricci soliton with Kähler form ω. Then there exists a unique smooth convex real-valued function ϕ defined on the dense orbit $(\mathbb{C}^*)^n \subset M$ such that $\omega = 2i\partial\bar{\partial}\phi$ and*

$$\det(\phi_{ij}) = e^{-2\phi + \langle b_X, \nabla\phi\rangle}. \tag{2.14}$$

A priori, the function ϕ is defined only up to addition of a linear function. However, (2.14) provides a normalisation for ϕ which in turn provides a normalisation for $\nabla\phi$, the moment map of the action. The next lemma shows that this normalisation coincides with that for the moment map as defined in [11, Definition 5.16].

Lemma 2.23. *Let (M, g, X) be a toric complete shrinking gradient Kähler–Ricci soliton with complex structure J and Kähler form ω with soliton vector field $X = \nabla^g f$ for a smooth real-valued function $f : M \to \mathbb{R}$. Let ϕ be given by Proposition 2.20 and normalised by (2.14), let $JY \in \mathfrak{t}$, and let $u_Y = \langle \nabla\phi, b_Y \rangle$ be the Hamiltonian potential of JY with b_Y as in (2.13) so that $\nabla^g u_Y = Y$. Then $\mathcal{L}_{JX} u_Y = 0$ and $\Delta_\omega u_Y + u_Y - \frac{1}{2} Y \cdot f = 0$.*

To see the equivalence with [11, Definition 5.16], simply replace Y by JY in this latter definition as here we assume that $JY \in \mathfrak{t}$, contrary to the convention in [11, Definition 5.16] where it is assumed that $Y \in \mathfrak{t}$.

Proof of Lemma 2.23. By definition, we have that

$$d(\mathcal{L}_{JX} u_Y) = \mathcal{L}_{JX}(du_Y) = -\mathcal{L}_{JX}(\omega \lrcorner JY) = 0,$$

where we have used the fact that $\mathcal{L}_{JX}\omega = 0$ and $[JX, JY] = 0$. Therefore, $\mathcal{L}_{JX} u_Y$ is equal to a constant which must be zero as JX has a zero because $X = \nabla^g f$ and f is proper and bounded from below (cf. Theorem 2.6), hence attains a local minimum. This proves the JX-invariance of u_Y.

The final equation follows by differentiating (2.14) with respect to Y. Indeed, from (2.11) and (2.12) we see that on the dense orbit,

$$\begin{aligned} \frac{\omega^n}{n!} &= \mathrm{vol}_g = \det(\phi_{ij})\, d\xi^1 \wedge d\theta^1 \wedge \cdots \wedge d\xi^n \wedge d\theta^n \\ &= \frac{\det(\phi_{ij})}{(-2i)^n}\, dw^1 \wedge d\bar{w}^1 \wedge \cdots \wedge dw^n \wedge d\bar{w}^n. \end{aligned}$$

Recalling that f denotes the Hamiltonian potential of $JX \in \mathfrak{t}$ so that $f = \langle b_X, \nabla\phi \rangle$ on the dense orbit, (2.14) may therefore be rewritten as

$$\log\det\left(\frac{(-2i)^n \omega^n}{n!\, dw^1 \wedge d\bar{w}^1 \wedge \cdots \wedge dw^n \wedge d\bar{w}^n}\right) + 2\phi - f = 0.$$

By differentiating along Y, this yields the relation

$$\begin{aligned} 0 &= Y \cdot \log\det\left(\frac{(-2i)^n \omega^n}{n!\, dw^1 \wedge d\bar{w}^1 \wedge \cdots \wedge dw^n \wedge d\bar{w}^n}\right) + 2Y \cdot \phi - Y \cdot f \\ &= \mathrm{tr}_\omega \mathcal{L}_Y \omega + 2u_Y - Y \cdot f = 2\Delta_\omega u_Y + 2u_Y - Y \cdot f, \end{aligned}$$

where we have made use of [9, Lemma 2.5] in the last line. From this, the result follows. ∎

Given normalisation (2.14), the next lemma identifies the image of the moment map $\mu = \nabla\phi$.

Lemma 2.24 ([9, Lemmas 4.4 and 4.5]). *Let (M, g, X) be a complete toric shrinking gradient Kähler–Ricci soliton, let $\{D_i\}$ be the prime $(\mathbb{C}^*)^n$-invariant divisors in M, and let $\Sigma \subset \mathfrak{t}$ be the fan determined by Proposition 2.18. Let $\sigma_i \in \Sigma$ be the ray corresponding to D_i with minimal generator $\nu_i \in \Gamma$.*

(i) *There is a distinguished Weil divisor representing the anticanonical class $-K_M$ given by $-K_M = \sum_i D_i$ whose associated polyhedron (cf. (2.9)) is given by*

$$P_{-K_M} = \{x \mid \langle \nu_i, x\rangle \geq -1\} \tag{2.15}$$

which is strongly convex and has full dimension in $\mathfrak{t}^*$. In particular, the origin lies in the interior of P_{-K_M}.

(ii) *If μ is the moment map for the induced real T^n-action normalised by (2.14), then the image of μ is precisely P_{-K_M}.*

2.5.4. The weighted volume functional. As a result of Lemma 2.23, we can now define the weighted volume functional.

Definition 2.25 (Weighted volume functional, [11, Definition 5.16]). Let (M, g, X) be a complex n-dimensional toric shrinking gradient Kähler–Ricci soliton with Kähler form $\omega = 2i\partial\bar{\partial}\phi$ on the dense orbit with ϕ strictly convex with moment map $\mu = \nabla\phi$ normalised by (2.14). Assume that the fixed point set of the torus is compact and recall that

$$\Lambda_\omega := \{Y \in \mathfrak{t} \mid \langle \mu, Y\rangle \text{ is proper and bounded below}\} \subseteq \mathfrak{t}.$$

Then the *weighted volume functional* $F\colon \Lambda_\omega \to \mathbb{R}$ is defined by

$$F_\omega(v) = \int_M e^{-\langle \mu, v\rangle}\omega^n.$$

As the fixed point set of the torus is compact by definition, F_ω is well defined by the non-compact version of the Duistermaat–Heckman formula [33] (see also [11, Theorem A.3]). This leads to two important lemmas concerning the weighted volume functional in the toric category, the independence of Λ_ω and F_ω from the choice of shrinking soliton ω.

Lemma 2.26. *The set Λ_ω is independent of the choice of toric shrinking Kähler–Ricci soliton ω in Definition 2.25 and is given by $\Lambda_\omega = \operatorname{int}(C^\vee)$, where $C := \{x \mid \langle \nu_i, x\rangle \geq 0\}$ and $\{\nu_i\}$ are as in Lemma 2.24.*

Proof. Recall from Proposition 2.15 that Λ_ω is given by $\operatorname{int}(\mathcal{C}(\mu_\omega(M))^\vee)$, where the moment map μ_ω with respect to ω, normalised by (2.14), depends on ω. However, no matter the choice of ω in Definition 2.25, normalisation (2.14) implies by Lemma 2.24 (ii) that the image of M under the moment map is always given by P_{-K_M}, a fixed polytope determined solely by the torus action. Therefore, Λ_ω is independent of the choice of ω in Definition 2.25. Finally, the asymptotic cone of this polytope (as a subset of $\mathfrak{t}^*$) is, by Lemma 2.11, given by C. This leads to the desired expression for Λ_ω. ∎

Note that C is always a strongly convex rational polyhedral cone in $\mathfrak{t}$, although not necessarily of full dimension, whereas $C^\vee$ is always full-dimensional, although not necessarily strongly convex.

Lemma 2.27. *The functional F_ω is independent of the choice of toric shrinking Kähler–Ricci soliton ω in Definition 2.25. Moreover, after identifying Λ_ω with a subset of $\mathbb{R}^n$ via (2.13), F_ω is given by $F_\omega(v) = (2\pi)^n \int_{P_{-K_M}} e^{-\langle v,x\rangle}\,dx$, where $x = (x^1,\dots,x^n)$ denotes coordinates on $\mathfrak{t}^*$ dual to the coordinates $(\xi^1,\dots,\xi^n)$ on $\mathfrak{t}$ introduced in Section 2.5.2.*

Proof. We first show that the given integral is finite. To demonstrate this, it suffices to show that $\langle v,x\rangle > 0$ on the complement of a compact subset of P_{-K_M}. To this end, recall that $0 \in \operatorname{int}(P_{-K_M})$ so that the intersection of the hyperplane $\{x \in \mathbb{R}^n \mid \langle v,x\rangle = 0\}$ with P_{-K_M} is non-empty. We claim that the polyhedron $Q := \{x \in P_{-K_M} \mid \langle v,x\rangle \le 0\}$ is compact. Indeed, by Lemma 2.12, Q is compact if and only if $\mathcal{C}(Q) = \{0\}$. To derive a contradiction, assume that there exists a non-zero vector $w \in \mathcal{C}(Q)$. Then from the definition of the asymptotic cone, one can see that Q contains a ray of the form $x_0 + tw$, $t \ge 0$, for some $x_0 \in Q$. Taking the inner product with v, it follows that $\langle v, x_0 + tw\rangle > 0$ for $t \gg 0$ because $\langle v,w\rangle > 0$ by virtue of the fact that

$$Q \subseteq P \Rightarrow \mathcal{C}(Q) \subseteq \mathcal{C}(P) \Rightarrow \mathcal{C}(P)^\vee \subseteq \mathcal{C}(Q)^\vee \Rightarrow \operatorname{int}(\mathcal{C}(P)^\vee) \subseteq \operatorname{int}(\mathcal{C}(Q)^\vee).$$

This yields the desired contradiction.

Now, no matter the choice of shrinking soliton, the map $\nabla\phi\colon \mathfrak{t} \to P_{-K_M}$ defines a diffeomorphism with image the fixed polytope P_{-K_M} thanks to normalisation (2.14). The independence of F_ω from ω and the given expression then follows from the following computation, where $\nabla\phi\colon \mathfrak{t} \to P_{-K_M}$ is used as a change of coordinates:

$$\begin{aligned} F_\omega(v) &= \int_M e^{-\langle\mu,v\rangle}\omega^n = \int_{\mathfrak{t}\times T^n} e^{-\langle\nabla\phi(\xi),v\rangle}\det(\phi_{ij}(\xi))\,d\xi d\theta \\ &= (2\pi)^n \int_{\mathfrak{t}} e^{-\langle\nabla\phi(\xi),v\rangle}\det(\phi_{ij}(\xi))\,d\xi = (2\pi)^n \int_{P_{-K_M}} e^{-\langle x,v\rangle}\,dx. \end{aligned} \qquad \blacksquare$$

Thus, we henceforth drop the subscript ω from F_ω and Λ_ω when working in the toric category. The functional $F\colon \Lambda \to \mathbb{R}$ is proper in this category, hence attains a critical point in Λ.

Proposition 2.28 ([9, proof of Proposition 3.1]). *The functional*

$$F(v) = (2\pi)^n \int_{P_{-K_M}} e^{-\langle v,x\rangle}\,dx$$

is proper on Λ.

In general, such a critical point turns out to be unique and characterises the soliton vector field of a complete shrinking gradient Kähler–Ricci soliton.

Theorem 2.29 ([11, Lemma 5.17], [5, Theorem 1.1]). *Let (M, g, X) be a complete shrinking gradient Kähler–Ricci soliton with complex structure J, Kähler form ω, and bounded Ricci curvature. Then $JX \in \Lambda_\omega$, F_ω is strictly convex on Λ_ω, and JX is the unique critical point of F_ω in Λ_ω.*

Having established in Lemmas 2.26 and 2.27 that in the toric category the weighted volume functional F and its domain Λ are determined solely by the polytope P_{-K_M} which itself, by Lemma 2.24, depends only on the torus action on M (i.e., is independent of the choice of shrinking soliton), and having an explicit expression for F given by Lemma 2.27, after using the torus action to identify P_{-K_M} via (2.15), we can determine explicitly the soliton vector field of a toric shrinking gradient Kähler–Ricci soliton on M. We illustrate how to do this in the following examples.

Example 2.30. Consider $\mathbb{P}^1$ with the $\mathbb{C}^*$-action given by $\lambda \cdot [Z_0 : Z_1] = [\lambda Z_0 : Z_1]$. Then its torus-invariant divisors are $D_0 = [0 : 1]$ and $D_\infty = [1 : 0]$. The corresponding fan in $\mathbb{R}$ is given by $\Sigma_{\mathbb{P}^1} = \{0, [0, \infty), (-\infty, 0]\}$ and $-K_{\mathbb{P}^1} = D_0 + D_1$, the associated polyhedron $P_{-K_{\mathbb{P}^1}}$ of which can naturally be identified with the interval $[-1, 1] \subset \mathbb{R}$. The Fubini–Study metric $\omega_{\mathbb{P}^1}$ is Kähler–Einstein and, in particular, $2\omega_{\mathbb{P}^1}$ is a shrinking gradient Kähler–Ricci soliton on $\mathbb{P}^1$ with soliton vector field $X = 0$. Working with $2\omega_{\mathbb{P}^1} \in 2\pi c_1(-K_{\mathbb{P}^1})$, on the dense orbit $\mathbb{C}^* \subset \mathbb{P}^1$, $2\omega_{\mathbb{P}^1}$ has Kähler potential

$$\phi_{2\omega_{\mathbb{P}^1}} := \log(1 + |z|^2) - \frac{1}{2}\log(4|z|^2) = \log(e^{2\xi} + 1) - \xi - \log(2),$$

so that $\omega_{\mathbb{P}^1} = 2i\partial\bar{\partial}\phi_{2\omega_{\mathbb{P}^1}}$. It is then straightforward to verify that $\phi_{2\omega_{\mathbb{P}^1}}$ satisfies (2.14) with $b_X = 0$ and that the image of $\frac{\partial\phi_{2\omega_{\mathbb{P}^1}}}{\partial\xi}$ is the interval $[-1, 1]$. The weighted volume functional is then given by

$$F_{\mathbb{P}^1}(v) = 2\pi \int_{-1}^{1} e^{-vx}\, dx.$$

This is defined for all $v \in \mathbb{R}$ and indeed, the asymptotic cone of the compact polytope $[-1, 1]$ is just the point 0 so that $C^\vee = \mathbb{R}$. Clearly, $F'(v) = 0$ if and only if $v = 0$, as expected.

Example 2.31. Consider $\mathbb{C}$ endowed with the standard $\mathbb{C}^*$-action. Then there is only one torus-invariant divisor, namely $D = \{0\}$. The fan in $\mathbb{R}$ is simply $\Sigma_{\mathbb{C}} = \{0, [0, \infty)\}$ and $-K_{\mathbb{C}} = D$ with the corresponding polyhedron given by $P_{-K_{\mathbb{C}}} = [-1, \infty)$. On the dense orbit $\mathbb{C}^* \subset \mathbb{C}$, the Euclidean metric $\omega_{\mathbb{C}}$ has Kähler potential

$$\phi_{\omega_{\mathbb{C}}} = \frac{1}{4}|z|^2 - \log|z| - \frac{1}{2} = \frac{1}{4}e^{2\xi} - \xi - \frac{1}{2}.$$

This satisfies (2.14) with $b_X = 1$ and the image of $\frac{\partial\phi_{\omega_{\mathbb{C}}}}{\partial\xi}$ is $[-1, \infty)$. The asymptotic cone C of $[-1, \infty)$ is given by $[0, \infty)$ and accordingly, the weighted volume functional

$$F_{\mathbb{C}}(v) = 2\pi \int_{-1}^{\infty} e^{-vx}\, dx$$

is only defined on the interior of the dual cone $C^\vee$, namely $(0, \infty)$. We compute

$$F'(v) = -2\pi \int_{-1}^{\infty} x e^{-vx}\, dx = \frac{e^v}{v^2}(1 - v).$$

Hence, as expected, $F_{\mathbb{C}}$ has a unique critical point at $v = 1$, with the corresponding soliton vector field on $\mathbb{C}$ given by

$$X = \frac{\partial}{\partial \xi} = r\frac{\partial}{\partial r},$$

where $z = re^{i\theta} \in \mathbb{C}$ for $r > 0$ and $\xi = \log(r)$.

Example 2.32. Next we consider the Cartesian product $\mathbb{C} \times \mathbb{P}^1$ of the previous two examples. We equip $\mathbb{C} \times \mathbb{P}^1$ with the product $(\mathbb{C}^*)^2$-action and denote by $\mathfrak{t}_1$ and $\mathfrak{t}_2$ the Lie algebras of the real S^1's that act on $\mathbb{C}$ and $\mathbb{P}^1$, respectively. Then we have an obvious solution to (2.14) given by the product metric $\omega_{\mathbb{C}} + 2\omega_{\mathbb{P}^1}$ together with the soliton vector field $X_{\mathbb{C}\times\mathbb{P}^1} = X_{\mathbb{C}} + X_{\mathbb{P}^1} = r\frac{\partial}{\partial r}$ with $r = |z|$, where z is the complex coordinate on the $\mathbb{C}$-factor. Explicitly, the fan $\Sigma_{\mathbb{C}\times\mathbb{P}^1}$ comprises products $\sigma_1 \times \sigma_2 \subset \mathfrak{t}_1 \oplus \mathfrak{t}_2$, where $\sigma_1 \in \Sigma_{\mathbb{C}}$ and $\sigma_2 \in \Sigma_{\mathbb{P}^1}$. The polyhedron $P_{-K_{\mathbb{C}\times\mathbb{P}^1}} \subset \mathfrak{t} = \mathfrak{t}_1 \oplus \mathfrak{t}_2$ can be identified with the subset of $\mathbb{R}^2$ defined by the inequalities

$$P_{-K_{\mathbb{C}\times\mathbb{P}^1}} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq -1,\ -1 \leq x_2 \leq 1\}.$$

From this, one can easily see that if $v = v_1 + v_2$ with $v_1 \in \mathfrak{t}_1$ and $v_2 \in \mathfrak{t}_2$, then

$$F_{\mathbb{C}\times\mathbb{P}^1}(v) = F_{\mathbb{C}}(v_1)F_{\mathbb{P}^1}(v_2).$$

The fact that $F_{\mathbb{C}}$ and $F_{\mathbb{P}^1}$ are convex and positive implies that $F'_{\mathbb{C}\times\mathbb{P}^1}(v) = 0$ if and only if $F'_{\mathbb{C}}(v_1) = F'_{\mathbb{P}^1}(v_2) = 0$, as expected.

Example 2.33. Let $M = \mathrm{Bl}_p(\mathbb{C} \times \mathbb{P}^1)$ denote the blowup of a fixed point p of the $(\mathbb{C}^*)^2$-action on $\mathbb{C} \times \mathbb{P}^1$ and write J for the complex structure on M. Then M inherits a natural $(\mathbb{C}^*)^2$-action with respect to which the blowdown map $\pi\colon M \to \mathbb{C} \times \mathbb{P}^1$ is $(\mathbb{C}^*)^2$-equivariant. In terms of the toric data, the exceptional divisor E of π defines an additional invariant divisor and the polyhedron $P_{\mathbb{C}\times\mathbb{P}^1}$ is modified accordingly,

$$P_{-K_M} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq -1,\ -1 \leq x_2 \leq 1,\ x_1 + x_2 \geq -1\}.$$

Here, the new face with inner normal $\nu_E = (1, 1)$ corresponds to E. Define two auxiliary functions F_1 and F_2 of a real variable $t > 0$ by

$$F_1(t) = \int_{P_{-K_M}} x_1 e^{-t(2x_1+x_2)}\, dx_1 dx_2,$$

$$F_2(t) = \int_{P_{-K_M}} x_2 e^{-t(2x_1+x_2)}\, dx_1 dx_2.$$

These functions are, up to a scaling factor of $-(2\pi)^{-2}$, the components of the gradient of the weighted volume functional $F_M\colon \Lambda \to \mathbb{R}$ of M restricted to the ray generated by

$(2,1) \in \mathfrak{t}$. Thus, if there exists some $\lambda > 0$ such that $F_1(\lambda) = F_2(\lambda) = 0$, then the point $(2\lambda, \lambda) \in \mathbb{R}^2$ would be a critical point of F_M. First, we claim that $F_2(t) \equiv 0$. Indeed, computing directly, we see that

$$\begin{aligned} F_2(t) &= \int_0^1 \int_{-1}^{\infty} x_2 e^{-t(2x_1+x_2)}\, dx_1 dx_2 + \int_{-1}^{0} \int_{-(x_2+1)}^{\infty} x_2 e^{-t(2x_1+x_2)}\, dx_1 dx_2 \\ &= \frac{t^{-3}}{2} e^t (e^t - 1) - \frac{t^{-2}}{2} e^t - \frac{t^{-3}}{2} e^t (e^t - 1) + \frac{t^{-2}}{2} e^t = 0. \end{aligned}$$

Since $F_M(t(2,1))$ is proper and convex as a function of $t > 0$ (cf. Proposition 2.28 and Theorem 2.29), this implies that there is a $\lambda > 0$ such that both $F_1(\lambda)$ and $F_2(\lambda)$ vanish simultaneously. We next determine the value of λ. Since

$$\begin{aligned} F_1(t) &= -\int_{-1}^{0} \int_{-(x_1+1)}^{1} x_1 e^{-t(2x_1+x_2)} dx_2 dx_1 - \int_0^{\infty} \int_{-1}^{1} x_1 e^{-t(2x_1+x_2)} dx_2 dx_1 \\ &= \frac{t^{-2}}{2} e^t + \frac{t^{-3}}{2} \sinh(t) - e^t (t^{-2} e^t + t^{-3}(e^t - 1)) - \frac{t^{-3}}{2} \sinh(t) \\ &= \frac{t^{-3}}{2} e^t (2e^t(1-t) - (2-t)), \end{aligned}$$

we see that $F_1(\lambda) = 0$ for λ the x-coordinate of the unique non-zero point of intersection of the graphs of $G_1(t) = 2e^t(1-t)$ and $G_2(t) = 2 - t$. In particular, λ cannot be equal to $1, 2, \frac{1}{2}$, or indeed any algebraic number. Numerical approximations in fact give $\lambda \approx 0.64$.

Let (z_1, z_2) be complex coordinates on the dense orbit $(\mathbb{C}^*)^2 \subset \mathbb{C}^* \times \mathbb{P}^1 \subset M$. Writing $z_j = r_j e^{i\theta_j}$ with $r_j > 0$, set $\xi_j = \log(r_j)$ as before. Then the soliton vector field X on M may be written as

$$X = \lambda\Big(2\frac{\partial}{\partial \xi^1} + \frac{\partial}{\partial \xi^2}\Big) = \lambda\Big(2r_1 \frac{\partial}{\partial r_1} + r_2 \frac{\partial}{\partial r_2}\Big). \tag{2.16}$$

3. Proof of Theorem A

Consider a complete non-compact shrinking gradient Kähler–Ricci soliton (M, g, X) with bounded scalar curvature, with complex structure J and with soliton vector field $X = \nabla^g f$ for a smooth real-valued function $f : M \to \mathbb{R}$. Then f is proper and bounded from below (cf. Theorem 2.6), hence attains a minimum, and g complete implies that X is complete [46]. Let G_0^X denote the connected component of the identity of the holomorphic isometries of (M, J, g) that commute with the flow of X. Since g has bounded Ricci curvature, G_0^X is a compact Lie group by [11, Lemma 5.12] and X being complete implies that JX is complete by [11, Lemma 2.35]. Moreover, JX is Killing by [17, Lemma 2.3.8]. Hence the closure of the flow of JX in G_0^X yields the holomorphic isometric action of a real torus T on (M, J, g) with Lie algebra $\mathfrak{t}$ containing JX. Compactness of the zero set of X [11, Lemma 2.26] and hence JX implies that the fixed point set of T is compact. Moreover, as f attains a minimum, T will have at least one fixed point. Finally, as $H^1(M) = 0$ by [45], T will act on M in a Hamiltonian fashion.

Using results from J-holomorphic curves (cf. Section 2.1), we identify the candidate non-compact complex surfaces that may admit a complete shrinking gradient Kähler–Ricci soliton with bounded scalar curvature whose soliton vector field has an integral curve along which the scalar curvature does not tend to zero. We first work under the assumption of simple connectedness and classify up to diffeomorphism.

Proposition 3.1 (Smooth classification). *Let (M, g, X) be a two-dimensional simply connected complete non-compact shrinking gradient Kähler–Ricci soliton with bounded scalar curvature R_g with $X = \nabla^g f$ for some smooth function $f : M \to \mathbb{R}$. Assume that X has an integral curve along which $\mathrm{R}_g \not\to 0$. Then M is diffeomorphic to either $\mathbb{C} \times \mathbb{P}^1$ or $\mathrm{Bl}_p(\mathbb{C} \times \mathbb{P}^1)$, that is, the blowup of $\mathbb{C} \times \mathbb{P}^1$ at one point p. In the former case, the zero set of X is contained in a unique $\mathbb{P}^1$ and in the latter case, in the pre-image under the blowup map of the $\mathbb{P}^1$-fibre containing the blowup point.*

Proof. Since there exists a point of M where $\mathrm{R}_g \neq 0$, g is non-flat and so globally we know that $\mathrm{R}_g > 0$ [32]. As X has an integral curve along which $\mathrm{R}_g \not\to 0$ by assumption, this means that there exist $\varepsilon > 0$ and a sequence of points $\{x_i\}_i$ lying along this integral curve going off to infinity as $i \to \infty$ such that $\mathrm{R}_g(x_i) > \varepsilon$. By assumption R_g is bounded, thus we read from [30, Theorem 1.3] that the norm of the full curvature tensor $\mathrm{Rm}(g)$ of g is bounded. It subsequently follows from [31, Corollary 4.1] and the classification of real three-dimensional complete shrinking gradient Ricci solitons [30, Theorem 1.2] that the sequence of pointed manifolds (M, g, x_i), after passing to a subsequence if necessary, converges in the smooth pointed Cheeger–Gromov sense to $(\widehat{M}, \widehat{g}, \widehat{p})$, where $\widehat{p} \in \widehat{M}$ is a base point and $(\widehat{M}, \widehat{g})$ is isometric to $\mathbb{R}^4$ endowed with the flat metric, or $\widehat{M}$ is diffeomorphic to $\mathbb{R}^2 \times S^2$ or to the $\mathbb{Z}_2$-quotient $\mathbb{R} \times ((S^2 \times \mathbb{R})/\mathbb{Z}_2)$, where $\mathbb{Z}_2$ flips both S^2 and $\mathbb{R}$, or to a quotient of $\mathbb{R} \times S^3$ by a finite group acting on the S^3-factor, and $\widehat{g}$ is the standard product metric on these spaces. Write $\widehat{\nabla}$ for the Levi-Civita connection of $\widehat{g}$. What we have then is a sequence of relatively compact open subsets $U_i \subset\subset \widehat{M}$ exhausting $\widehat{M}$ and containing $\widehat{p}$, together with a sequence of smooth maps $\phi_i : U_i \to M$ that are diffeomorphisms onto their image, such that $\phi_i(\widehat{p}) = x_i$ and

$$|\widehat{\nabla}^k(\phi_i^* g - \widehat{g})|_{\widehat{g}} \to 0 \quad \text{for all } k \geq 0, \tag{3.1}$$

smoothly locally on $\widehat{M}$ as $i \to \infty$. Now, the aforementioned boundedness of $|\mathrm{Rm}(g)|_g$ implies that all of the covariant derivatives with respect to g of $\mathrm{Rm}(g)$ are bounded by Shi's derivative estimates. Furthermore, by [31], (M, g) has a lower bound on its injectivity radius. The conditions of [7, Theorem 3.22] are therefore satisfied and consequently we can assert that $(\widehat{M}, \widehat{g})$ is Kähler. Since $\mathrm{R}_g(x_i) > \varepsilon$ for all i, $(\widehat{M}, \widehat{g})$ is clearly not flat and so the limit $\mathbb{R}^4$ can be discarded. Lifting to the universal cover of the remaining candidates for $\widehat{M}$, we obtain a Kähler structure on $\mathbb{R}^2 \times S^2$ or on $\mathbb{R} \times S^3$ with Kähler metric we still denote by $\widehat{g}$. The following claim allows us to discount the case $\mathbb{R} \times S^3$ next.

Claim 3.2. $\mathbb{R} \times S^3$ *does not admit a complex structure with respect to which the product metric $\widehat{g}$ is Kähler.*

Proof. Suppose to the contrary that $\mathbb{R} \times S^3$ admits a complex structure J with respect to which the product metric $\hat{g}$ is Kähler. Let $q = (x, y) \in \mathbb{R} \times S^3$. Then we have the decomposition $T_q\hat{M} = T_x\mathbb{R} \oplus T_yS^3$. Let $Y \in T_x\mathbb{R}$ be a unit vector. Then J compatible with $\hat{g}$ implies that $\hat{g}(Y, JY) = 0$ which in turn implies that $JY \in T_yS^3$. Let H be an arbitrary holonomy transformation with respect to $\hat{g}$ of $T_q\hat{M}$ and act by H on JY. Then since the holonomy of $\hat{g}$ is trivial on the $\mathbb{R}$-factor and J is parallel, we see that $H(JY) = J(HY) = JY$, i.e., H fixes JY, forcing $JY = 0$. This is a contradiction. ∎

We therefore arrive at the fact that $\hat{M}$ is covered by $\mathbb{R}^2 \times S^2$. We next identify the $\hat{g}$-compatible complex structure on this space.

Claim 3.3. *Up to a sign on each factor, the only complex structure on $\mathbb{R}^2 \times S^2$ with respect to which $\hat{g}$ is Kähler is the standard complex structure $\hat{J}$ on $\mathbb{C} \times \mathbb{P}^1$.*

Proof. Suppose that $q = (x, y) \in \mathbb{R}^2 \times S^2 \approx \hat{M}$. Then we have the decomposition $T_q\hat{M} = T_x\mathbb{R}^2 \oplus T_yS^2$. Suppose that there exists another complex structure $\tilde{J}$ on $\mathbb{R}^2 \times S^2$ with respect to which $\hat{g}$ is Kähler, and let $Y \in T_yS^2$ be a unit vector. Then with respect to the aforementioned decomposition, we can write $\tilde{J}Y = a\hat{J}Y \oplus U$ for some $a \in \mathbb{R}$, $|a| \leq 1$, and $U \in T_x\mathbb{R}^2$. We parallel transport the quadruple $\{Y, \tilde{J}Y, \hat{J}Y, U\}$ around a non-trivial closed loop in the S^2-fibre of $\hat{M}$ containing q using the connection $\hat{\nabla}$. As $\hat{g}$ is flat in the $\mathbb{R}^2$-direction, U will remain unchanged under this action. Moreover, $\tilde{J}$ and $\hat{J}$ are parallel with respect to $\hat{g}$. Thus, as the holonomy of S^2 is SO(2), we find that for every unit vector $Z \in T_yS^2$, $\tilde{J}Z = a\hat{J}Z \oplus U$, leaving us with $U = 0$ and $|a| = 1$.

We next consider a unit vector $Y \in T_x\mathbb{R}^2$. Then with respect to the splitting $T_q\hat{M} = T_x\mathbb{R}^2 \oplus T_yS^2$, we have that $\tilde{J}Y = U \oplus b\hat{J}Y$ for some $b \in \mathbb{R}$, $|b| \leq 1$, and $U \in T_yS^2$. Arguing as before, parallel transport in the S^2-fibre of $\hat{M}$ containing q using $\hat{\nabla}$ demonstrates that $\tilde{J}Y = V \oplus b\hat{J}Y$ for all $V \in T_yS^2$, forcing $V = 0$ and $|b| = 1$. From this, the assertion follows. ∎

Thus, without loss of generality, we may assume that the aforementioned Kähler structure on $\mathbb{R}^2 \times S^2$ is standard, i.e., simply $(\hat{g}, \hat{J})$. It then follows that $\hat{M}$ is biholomorphic to $\mathbb{C} \times \mathbb{P}^1$ as the $\mathbb{Z}_2$-quotient thereof, acting freely and holomorphically, would introduce an $\mathbb{R}P^2$ as a complex submanifold yielding a contradiction. Returning to (3.1), set $J_i := \phi_i^*J$ and $g_i := \phi_i^*g$. Arguing as in [7, proof of Theorem 3.22] (see also [34, pp. 16–18]), we see that J_i converges smoothly locally to a $\hat{g}$-parallel complex structure J_∞ on $\hat{M}$ which by Claim 3.3, we can without loss of generality take to be equal to $\hat{J}$.

Fix a large ball $B_R(\hat{p}, \hat{g}) \subset \hat{M}$ of radius $R > 0$ centred at $\hat{p}$ with respect to $\hat{g}$, and let $\hat{u}\colon \mathbb{P}^1 \to \mathbb{C}$ denote the unique $\hat{J}$-holomorphic sphere passing through $\hat{p}$. Since $|J_i - \hat{J}|_{\hat{g}} \to 0$ as $i \to \infty$, by Corollary 2.3, for i sufficiently large, $\hat{u}$ may be deformed to a J_i-holomorphic sphere $u\colon \mathbb{P}^1 \to \hat{M}$ with zero self-intersection. By the estimate given in Corollary 2.3, the image of $u\colon \mathbb{P}^1 \to \hat{M}$ will eventually be contained in $B_R(\hat{p}, \hat{g})$. Thus, outside any fixed compact subset K of M, $v := \phi_i^{-1} \circ u\colon \mathbb{P}^1 \to M$ will define a J-holomorphic curve in M with trivial normal bundle and zero self-intersection lying in $M \setminus K$ for i sufficiently large.

Henceforth, we write $C := \upsilon(\mathbb{P}^1)$. Then $C.C = 0$. Recall the real torus T acting on M introduced at the beginning of this section, and let ω denote the Kähler form of g. The function f, the Hamiltonian potential of JX, is, as the soliton potential, proper and bounded from below (cf. Theorem 2.6). Consequently, Proposition 2.15 allows us to find an element $JY \in \Lambda_\omega \subseteq \mathfrak{t}$ whose flow generates an S^1-action and that admits a real Hamiltonian potential u_Y that is proper and bounded from below. Since the fixed point set of T is non-empty and contained in the zero set of X, a compact subset, [33, Proposition 1.2] implies that the zero set of Y is non-empty and compact. Moreover, by [11, Lemma 2.34], we also know that Y and JY are complete. Hence we can define for all time the holomorphic flow of the vector fields Y and JY which we denote by ϕ_t^Y and ϕ_t^{JY}, respectively, for $t \in \mathbb{R}$. As the next claim shows, the image of a holomorphic sphere under the flow of Y and JY is determined by the image of one point on the sphere.

Claim 3.4. *For $x \in M$, let $L_x \in [C]$ be a holomorphic sphere in M with $x \in L_x$. Then $\phi_t^Y(L_x)$ (resp. $\phi_t^{JY}(L_x)$) is the unique holomorphic sphere in M lying in $[C]$ passing through $\phi_t^Y(x)$ (resp. $\phi_t^{JY}(L_x)$).*

Proof. It is clear that the image of L_x under the flow of Y and JY is a holomorphic sphere in M lying in $[C]$ passing through $\phi_t^Y(x)$ and $\phi_t^{JY}(x)$, respectively. No other holomorphic sphere in $[C]$ can pass through these points since $C.C = 0$. ∎

Holomorphic spheres containing a zero of Y are fixed by the flow of Y and JY.

Claim 3.5. *Let $L \in [C]$ be a holomorphic sphere in M. Then the following are equivalent:*

(i) *Y vanishes at some point $x \in L$.*

(ii) *$\phi_t^Y(L) = \phi_t^{JY}(L) = L$ for all t.*

(iii) *Y is tangent to L.*

Proof. (i) ⇒ (ii) By Claim 3.4, $\phi_t^Y(L)$ is the unique holomorphic curve in $[C]$ passing through $\phi_t^Y(x)$. Since $\phi_t^Y(x) = x$, we deduce that $\phi_t^Y(L) = L$.

(ii) ⇒ (iii) This is clear.

(iii) ⇒ (i) A holomorphic vector field tangent to $\mathbb{P}^1$ has at least one zero. ∎

If Y is nowhere vanishing along the holomorphic sphere, then the image sphere is disjoint from the original.

Claim 3.6. *Let $L \in [C]$ be a holomorphic sphere in M. Then Y is nowhere vanishing on L if and only if there exists $\varepsilon > 0$ such that $\phi_t^Y(L) \cap L = \emptyset$ and $\phi_t^{JY}(L) \cap L = \emptyset$ for all $0 < |t| < \varepsilon$.*

Proof. If Y is nowhere vanishing on L, then Y cannot be tangent to L for otherwise it would have a zero along L. Thus, Y has a normal component at some point $x \in L$ so that $\phi_t^Y(x) \notin L$ for $0 < |t| < \varepsilon$ for some $\varepsilon > 0$. By Claim 3.4, for such values of t, $\phi_t^Y(L)$ will be the unique holomorphic sphere in $[C]$ passing through $\phi_t^Y(x)$, hence will be disjoint from L. A similar argument applies to JY. The converse follows from the implication (i) $\Rightarrow$ (ii) of Claim 3.5. ■

As the zero set of Y is compact, by choosing i sufficiently large, we can guarantee that Y is nowhere vanishing along C so that Claim 3.6 applies with $L = C$. Henceforth, working with the $\mathbb{C}^*$-action generated by Y and JY, in light of Claim 3.5, we then see that Y and JY will be nowhere vanishing on the $\mathbb{C}^*$-orbit of C in M. Define $\mathrm{Orb}_{\mathbb{C}^*}(C) := \{g \cdot C \mid g \in \mathbb{C}^*\} \subseteq M$.

Claim 3.7. *There exists a finite cyclic group $\mathbb{Z}_k \subset S^1 \subset \mathbb{C}^*$, $k \geq 1$, such that the induced action of $\mathbb{C}^*/\mathbb{Z}_k$ on $\mathrm{Orb}_{\mathbb{C}^*}(C)$ is free.*

Proof. Claim 3.4 implies that the $\mathbb{C}^*$-action on this orbit descends to a (transitive) $\mathbb{C}^*$-action on the holomorphic $\mathbb{P}^1$'s in $[C]$ contained in $\mathrm{Orb}_{\mathbb{C}^*}(C)$. Claim 3.6 then implies that this action on the holomorphic $\mathbb{P}^1$'s is locally free. Compactness of S^1 implies that the stabiliser group in $S^1 \subset \mathbb{C}^*$ of C under this action is a finite subgroup of S^1, hence is a cyclic group of the form $\mathbb{Z}_k$ for some $k \geq 1$. The induced action of $\mathbb{C}^*/\mathbb{Z}_k$ on the holomorphic $\mathbb{P}^1$'s in $[C]$ contained in $\mathrm{Orb}_{\mathbb{C}^*}(C)$ will therefore be free. Claim 3.4 then tells us that the induced action of $\mathbb{C}^*/\mathbb{Z}_k$ on $\mathrm{Orb}_{\mathbb{C}^*}(C)$ will be free. ■

As $\mathbb{C}^*/\mathbb{Z}_k \cong \mathbb{C}^*$, we may therefore assume without loss of generality that the $\mathbb{C}^*$-action generated by Y and JY on $\mathrm{Orb}_{\mathbb{C}^*}(C)$ is free. We define a map

$$\Phi\colon \mathbb{C}^* \times \mathbb{P}^1 \to M, \quad (g, y) \mapsto g \cdot (v(y)). \tag{3.2}$$

Since the $\mathbb{C}^*$-action is free, this defines a biholomorphism onto its image, holomorphic along the $\mathbb{P}^1$-direction, and for dimensional reasons demonstrates that for some compact subset K of M containing the zero set of Y, $M \setminus K$ is biholomorphic to $\mathbb{P}^1 \times \mathbb{C}^*$. Indeed, recall that the Hamiltonian potential $u_Y\colon M \to \mathbb{R}$ of Y is proper and bounded from below and that the zero set of Y is compact so that the level sets $u_Y^{-1}(\{y\})$ of u_Y are compact and, through the gradient flow of u_Y, diffeomorphic for all $y > R$ for some R sufficiently large and positive. Hence with M having only one end [29, Theorem 0.1], we obtain a decomposition of the unique end of M as $\bigcup_{y \in (R, +\infty)} u_Y^{-1}(\{y\})$. In this picture, one can see that the positive gradient flow of u_Y, that is, the positive flow of Y, moves out to infinity along the unique end of M and from [11, Proposition 2.28], we also read that the negative gradient flow of u_Y, i.e., the negative flow of Y, accumulates in the zero set of Y, a non-empty compact analytic subset of M. Thus, the image of Φ is precisely the complement in M of the zero set of Y, and so M fibres as a trivial $\mathbb{P}^1$-bundle on the complement of this compact analytic subset. Notice that all the $\mathbb{P}^1$-fibres of the fibration are homologous to C.

Next, being complete and having bounded scalar curvature, M has finite topological type [14, Theorem 1.2], hence K contains only finitely many (-1)-curves. There therefore exists a sequence of blowdown maps, each contracting at least one (-1)-curve, which give rise to the minimal model $\varpi\colon M \to M_{\min}$ of M whose complex structure we still denote by J. As M is simply connected, $M_{\min}$ will also be simply connected. Furthermore, the

(-1)-curves in K are necessarily fixed by the $\mathbb{C}^*$-action on M induced by the flow of Y and JY and so the $\mathbb{C}^*$-action will extend to a $\mathbb{C}^*$-action on M_{min} in such a way that the map $\varpi\colon M \to M_{\mathrm{min}}$ is equivariant with respect to these two actions. The holomorphic vector field Y on M therefore descends to a holomorphic vector field Y on M_{min} with compact zero set, vanishing at least at the points of M_{min} that are blown-up to obtain M. It is also clear that Φ induces a biholomorphism from $M_{\mathrm{min}} \setminus \varpi(K)$ to $\mathbb{P}^1 \times \mathbb{C}^*$. We claim that this $\mathbb{P}^1$-fibration at infinity extends in a smooth manner to the interior of M_{min}. This we prove via a continuity argument.

To this end, consider the set

$$A := \{x \in \varpi(K) \mid x \text{ is contained in a holomorphic } \mathbb{P}^1 \text{ representing } [C]\},$$

where we enlarge K if necessary so that a tubular neighbourhood of its boundary is foliated by $\mathbb{P}^1$'s representing $[C]$. Then we have the following.

Claim 3.8. *The set A is equal to $\varpi(K)$.*

Proof. First note that A is non-empty and that the openness of A is immediate from Proposition 2.1. As for closedness, let x_i be a sequence of points in A with $x_i \to x$ for some $x \in A$. Then for each i, there exists a J-holomorphic curve $u_i\colon \mathbb{P}^1 \to M$ passing through x_i representing $[C]$. Being contained in the same homology class, these curves all have uniformly bounded area. Therefore, by the Gromov compactness theorem [18], there exists a subsequence converging to a tree of k holomorphic $\mathbb{P}^1$'s with multiplicity in $[C]$. This limit may be written as $[C] = \sum_{i=1}^{k} a_i[C_i]$, $a_i > 0$. Then

$$0 = [C].[C] = \sum_{i \neq j} a_i a_j [C_i].[C_j] + \sum_i a_i^2 [C_i].[C_i].$$

Now, from the equation defining a shrinking Kähler–Ricci soliton, we know that for any J-holomorphic curve $\hat{C}$ in M, $-K_M.[\hat{C}] > 0$ so that $[\hat{C}].[\hat{C}] \geq -1$ by adjunction. As we are working on M_{min}, this implies that $[C_i].[C_i] \geq 0$ and so $k = 1$ and accordingly, the limit is a smooth $\mathbb{P}^1$ with multiplicity one. This gives closedness and the claim now follows. ∎

Hence we conclude that M_{min} exhibits the global smooth structure of a $\mathbb{P}^1$-fibration over a real surface S, with each fibre lying in the homology class $[C]$.

We next holomorphically compactify M_{min} by adjoining a $\mathbb{P}^1$ at infinity using the Φ from (3.2) to obtain a closed compact real manifold $\bar{M}_{\mathrm{min}}$ that admits the structure of a smooth S^2-bundle over a closed compact real surface $\bar{S}$ that itself is obtained from S by adding a point at infinity. By construction, this additional fibre will be preserved by the induced $\mathbb{C}^*$-action on $\bar{M}_{\mathrm{min}}$. As M_{min} is simply connected, $\bar{M}_{\mathrm{min}}$ will be simply connected by the Seifert–Van Kampen theorem. It then follows from a long exact sequence [2, (17.4)] that $\bar{S}$ is simply connected, hence is diffeomorphic to S^2. Consequently, $\bar{M}_{\mathrm{min}}$ is diffeomorphic to either $S^2 \times S^2$ or to the blowup of $\mathbb{P}^2$ at one point, the only two S^2-bundles over S^2 [40]. In either case, removing an S^2-fibre shows that S is diffeomorphic to $\mathbb{R}^2$ and that M_{min} is diffeomorphic to $S^2 \times \mathbb{R}^2$ with the S^2-fibres defining J-holomorphic spheres in M_{min}.

Being a compact analytic subvariety of $M_{\min}$, the zero set of Y must comprise a finite union of isolated points and $\mathbb{P}^1$-fibres of $M_{\min}$. Now, those fibres containing a zero of Y are fixed by the $\mathbb{C}^*$-action induced by Y and JY by Claim 3.5. Otherwise, by Claim 3.6, the image of a fibre is disjoint from the original. What we deduce therefore is that the S^1-action defined by the flow of JY on $M_{\min}$ induces an S^1-action on $S \approx \mathbb{R}^2$ with finitely many zeroes, and in turn via Φ an S^1-action on $\overline{S} \approx S^2$ with finitely many zeroes, one of which is at infinity. Averaging the round metric on S^2 over this action, we may assume that the S^1-action is isometric. Then [23, Theorem (4)] tells us that the S^1-action on $\overline{S}$ has precisely two zeroes. As one of these zeroes occurs at infinity, we conclude that the $\mathbb{C}^*$-action on $M_{\min}$ fixes precisely one $\mathbb{P}^1$-fibre. Denote this fibre by L_0. By Claim 3.5, Y is then tangent to L_0, and by Claim 3.6, the zero set of Y is contained in L_0. As the flow of JY induces an S^1-action on L_0, we again see from [23] that the zero set of Y comprises the whole of L_0 (if the S^1-action is trivial) or precisely two points. As M is obtained from $M_{\min}$ by blowing up finitely many points of $M_{\min}$ at which the vector field Y vanishes, we see that M is obtained from $M_{\min}$ by blowing up finitely many points of L_0. Blowing up more than one point would introduce at least one holomorphic sphere in M with self-intersection $(-k)$ for some $k \geq 2$. This is not possible because using adjunction, the restriction of $-K_M$ to every holomorphic curve in M must be positive by the shrinking soliton condition. Hence $\varpi\colon M \to M_{\min}$ is the identity or the blowup of $M_{\min}$ at one point of L_0. Set $E := \varpi^{-1}(L_0)$. Then E contains the zero set of Y on M and hence also the fixed point set of T. But the flow of JX, being dense in T, implies that this latter set coincides with the zero set of X. This completes the proof of the proposition. ∎

We now consider $\widehat{M} := \mathbb{C} \times \mathbb{P}^1$ endowed with the standard holomorphic action of the real two-dimensional torus $\widehat{T}$ with Lie algebra $\hat{\mathfrak{t}}$ and $\widetilde{M} := \mathrm{Bl}_p(\mathbb{C} \times \mathbb{P}^1)$, the blowup of $\widehat{M}$ at a fixed point p of the $\widehat{T}$-action on $\widehat{M}$. The torus action on $\widehat{M}$ induces in a natural way the holomorphic action of a real two-dimensional torus $\widetilde{T}$ on $\widetilde{M}$ with Lie algebra $\tilde{\mathfrak{t}}$ such that the blowdown map $\sigma\colon \widetilde{M} \to \widehat{M}$ is equivariant with respect to the action of $\widetilde{T}$ and $\widehat{T}$. Recall the real torus T generated by the flow of JX with Lie algebra $\mathfrak{t}$ containing JX acting on (M, J, g) in a holomorphic isometric fashion with a compact fixed point set introduced at the beginning of this section. Theorem A (i) will follow from the next proposition, an improvement from the smooth category of the previous proposition to the complex category.

Proposition 3.9 (Holomorphic classification). *Let (M, g, X) be a two-dimensional simply connected complete non-compact shrinking gradient Kähler–Ricci soliton with bounded scalar curvature R_g with $X = \nabla^g f$ for some smooth function $f\colon M \to \mathbb{R}$. Assume that X has an integral curve along which $\mathrm{R}_g \not\to 0$. Then there exists an equivariant biholomorphism α from (M, T) to $(\widehat{M}, \widehat{T})$ or $(\widetilde{M}, \widetilde{T})$ with respect to which $\alpha_*(JX)$ lies in $\hat{\mathfrak{t}}$ or $\tilde{\mathfrak{t}}$, respectively. In particular, in the latter case, $\alpha_*(JX)$ is given by (2.16).*

Proof. We have already established in Proposition 3.1 that M is diffeomorphic to either $\mathbb{C} \times \mathbb{P}^1$ or to $\mathrm{Bl}_p(\mathbb{C} \times \mathbb{P}^1)$ and that there is a map $\varpi\colon M \to M_{\min}$ to $M_{\min}$, a manifold diffeomorphic to $S^2 \times \mathbb{R}^2$ with the S^2-fibres defining holomorphic spheres in $M_{\min}$, with ϖ the identity or the blowup of $M_{\min}$ at a point p of a $\mathbb{P}^1$-fibre L_0 of $M_{\min}$, as appropriate. Let $\hat{\pi}\colon M_{\min} \to \mathbb{R}^2$ denote the projection map. Then we obtain a map $\pi := \hat{\pi} \circ \varpi\colon M \to \mathbb{R}^2$. Without loss of generality, we may assume that $L_0 = \hat{\pi}^{-1}(\{0\})$. Proposition 3.1 then tells us that $M_0(X)$, that is, the zero set of X, is a compact analytic subset of $E := \pi^{-1}(\{0\})$, where E is equal to L_0 if ϖ is the identity, or to two holomorphic $\mathbb{P}^1$'s meeting transversely, each of self-intersection (-1), otherwise. In this latter case, we denote these curves by L_1 and L_2. In both cases, the forward flow of $-X$ accumulates in E by [11, Proposition 2.28] and the action of T preserves E. Indeed, this last point follows from Claims 3.4 and 3.5 (which also hold with Y replaced by X) if ϖ is the identity map, and from the following claim otherwise.

Claim 3.10. *The vector field X is tangent to any (-1)-curve in M.*

Proof. A neighbourhood of any (-1)-curve in M is biholomorphic to a neighbourhood of the zero section of $\mathcal{O}_{\mathbb{P}^1}(-1)$. Along this zero section, we have a canonical holomorphic splitting of TM as $T\mathbb{P}^1 \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. The normal component of X in this splitting must therefore vanish which yields the claim. ∎

This claim in fact implies that when ϖ is the blowup map, L_1 and L_2 are both preserved by the action of T. Thus, no matter what ϖ may be, the action of T on M will induce an action of T on $M_{\min}$. In the particular case when ϖ is the blowup map, the point of intersection of L_1 and L_2 will be fixed by T. We denote this point by x so that $x \in M_0(X) \cap E \neq \emptyset$.

Suppose first that ϖ is the blowup map so that M is diffeomorphic to $\mathrm{Bl}_p(\mathbb{C} \times \mathbb{P}^1)$. We begin by noting the following.

Claim 3.11. *If $X|_{L_i}$ is non-trivial for $i = 1$ or $i = 2$, then $|M_0(X) \cap L_i| = 2$.*

Proof. As $X|_{L_i}$ is non-trivial, the restriction of f to L_i is non-constant, hence attains a global maximum and a global minimum on L_i. At these points, $d(f|_{L_i}) = 0$. Then as X is tangent to L_i by Claim 3.10, we actually have that $df = 0$ at these points so that $X|_{L_i}$ has at least two zeroes. But $X|_{L_i}$ is a holomorphic vector field on $\mathbb{P}^1$, hence has at most two zeroes. ∎

Now, by [16, proof of Lemma 1], f is a Morse–Bott function on M. The critical submanifolds of f are precisely the connected components of $M_0(X)$. Since M is Kähler, the Morse indices (i.e., the number of negative eigenvalues of $\mathrm{Hess}(f)$) of the critical submanifolds are all even [16]. Write

$$M_0(X) = M^{(0)} \cup M^{(2)} \cup M^{(4)},$$

where $M^{(j)}$ denotes the disjoint union of the critical submanifolds of $M_0(X)$ of index j. We already know from Proposition 3.1 that $M_0(X) \subseteq E$, and from [11, Claim 2.30] we know that $M^{(0)}$ is a non-empty connected compact complex submanifold of M, hence is equal to either L_1, L_2, or an isolated point of E. We analyse the structure of $M_0(X)$ in each of these cases separately, beginning with the following.

Claim 3.12. *If $M^{(0)}$ comprises a single point, then $|M_0(X) \cap L_i| = 2$ for $i = 1, 2$.*

Proof. Recall that $\{x\} = L_1 \cap L_2$ and assume that $M^{(0)} = \{y\}$ for some point $y \in E$. If $x = y$, then y is an isolated zero of both $X|_{L_1}$ and $X|_{L_2}$. The result then follows by applying Claim 3.11 to both L_1 and L_2. If $x \neq y$, then assume without loss of generality that $y \in L_1$. Then the zero set of $X|_{L_1}$ comprises at least two points, namely x and y. Claim 3.11 implies that in fact $|M_0(X) \cap L_1| = \{x, y\}$. Considering $X|_{L_2}$ next, the zero set of this vector field contains x. If $X|_{L_2}$ is non-trivial, then the result follows from Claim 3.11. Otherwise, assume that $X|_{L_2} \equiv 0$. Then as $M^{(0)}$ is connected, we have that $x \in M^{(2)} \cup M^{(4)}$. Now, if $x \in M^{(4)}$, then by [4, Proposition 6] there exist local holomorphic coordinates (z_1, z_2) centred at x such that the holomorphic vector field $X^{1,0} := \frac{1}{2}(X - iJX)$ takes the form

$$X^{1,0} = a_1 z_1 \frac{\partial}{\partial z_1} + a_2 z_2 \frac{\partial}{\partial z_2}$$

for some $a_1, a_2 \in \mathbb{R}_{<0}$. This implies in particular that x is an isolated zero of X, contradicting the fact that $X|_{L_2} \equiv 0$. Hence necessarily $x \in M^{(2)}$ so that $L_2 \subseteq M^{(2)}$.

For each $z \in L_2$, f is decreasing along the forward flow of $-X$ emanating from z, hence this flow accumulates at $y \in L_1$ by [11, Proposition 2.28]. As in [6, p. 3332], we can use the forward flow of $-X$ to construct a holomorphic sphere $R_z\colon \mathbb{P}^1 \to M$ in M with $R_z(0) = z$ and $R_z(\infty) = y$. Assume that $x \neq z$ and call the resulting holomorphic sphere D. Also recall the holomorphic sphere C with zero self-intersection from the proof of Proposition 3.1. Then for $i \neq j$, $L_j = \varpi^* C - L_i$, $D.L_i > 0$, and $D.L_j > 0$, which leads to the conclusion that $D.\varpi^* C > 0$. This is a contradiction and the claim now follows. ∎

Next, we have the following.

Claim 3.13. *If $M^{(0)} = L_i$, then $|M_0(X) \cap L_j| = 2$ for $j \neq i$.*

Proof. In this case, $X|_{L_j}$ is non-trivial as $M^{(0)} \neq E$ and is connected. The result then follows from an application of Claim 3.11 to L_j. ∎

Thus, the induced action of the real torus T on $M_{\min}$ will fix either two points on L_0 or the whole of L_0 and the forward flow lines of the vector field $-X$ induced on $M_{\min}$ accumulate in L_0. If $\dim_{\mathbb{R}} T = 2$, then by identifying a point off of L_0 and $\{0\} \times \mathbb{P}^1$ in M and $\hat{M}$, respectively, and using the actions, one can construct an equivariant biholomorphism $\alpha\colon (M_{\min}, T) \to (\hat{M}, \hat{T})$.

If $|M^{(0)}| = 1$ and $\dim_{\mathbb{R}} T = 1$, then the flow of X and JX on $M_{\min}$ induces a $\mathbb{C}^*$-action on $M_{\min}$ which by Claim 3.7 we may assume to be free on $M_{\min} \setminus L_0$. In addition, Claim 3.12 implies that the fixed point set of T will comprise precisely two isolated points in L_0, say a and b. As the forward flow lines of $-X$ on $M_{\min}$ accumulate in L_0, the closure of every orbit of this $\mathbb{C}^*$-action on $M_{\min}$ is a copy of $\mathbb{C}$ obtained by adjoining either a or b to the orbit in question. Choose an orbit O_a and O_b passing through a and b, respectively. Then each orbit will intersect every fibre of the $\mathbb{P}^1$-foliation of $M_{\min} \setminus L_0$

at precisely one point. Indeed, if an orbit intersected a $\mathbb{P}^1$-fibre L in $M_{\min} \setminus L_0$ at two points $x_1, x_2 \in L, x_1 \neq x_2$, say, then there would exist a $g \in \mathbb{C}^*$, $g \neq 1$, such that $g \cdot x_1 = x_2$. By Claim 3.4, we would then have that $g \cdot L = L$. The element g would then define an automorphism of $L \cong \mathbb{P}^1$ and would therefore have at least one fixed point. This contradicts the freeness of the $\mathbb{C}^*$-action on $M_{\min} \setminus L_0$. Define a global real holomorphic vector field JV on $M_{\min}$ in the following way. Restricted to a $\mathbb{P}^1$-fibre L, JV will be the unique real holomorphic vector field tangent to L vanishing at $O_a \cap L$ and $O_b \cap L$ that generates a holomorphic S^1-action, the direction of the flow of which relative to the points $O_a \cap L$ and $O_b \cap L$ will be the same as that on L_0 relative to a and b, and the time 2π-flow of which is the identity map. The flow of V and JV will generate a $\mathbb{C}^*$-action on $M_{\min}$ that commutes with the flow of X and JX. To see this last point, it suffices to verify that $[X, V] = 0$ on $M_{\min} \setminus L_0$. To this end, we set up an equivariant biholomorphism $\mathbb{C}^* \times \mathbb{P}^1 \to M_{\min} \setminus L_0$ in the following way. Pick an arbitrary fibre $u\colon \mathbb{P}^1 \to L \subset M_{\min} \setminus L_0$. By pre-composing with a suitable Möbius transformation, we can assume that $u(0) = O_a \cap L$ and $u(\infty) = O_b \cap L$. As in (3.2), we extend u to a biholomorphism $\Psi\colon \mathbb{C}^* \times \mathbb{P}^1 \to M_{\min} \setminus L_0$, equivariant with respect to the standard $\mathbb{C}^*$-action on the first component of the domain and the $\mathbb{C}^*$-action generated by X and JX on the range. By construction, Ψ^{-1} has the property that it pushes forward $\frac{1}{2}(V - iJV)$ to a global holomorphic vector field on $\mathbb{C}^* \times \mathbb{P}^1$ tangent to the $\mathbb{P}^1$-fibres and vanishing along $(\mathbb{C}^* \times \{0\}) \cup (\mathbb{C}^* \times \{\infty\})$. In particular, this holomorphic vector field generates another $\mathbb{C}^*$-action on $\mathbb{C}^* \times \mathbb{P}^1$ and the map Ψ will also be $\mathbb{C}^*$-equivariant with respect to this action on the domain and that generated by $\frac{1}{2}(V + iJV)$ on the range. Observing that the two $\mathbb{C}^*$-actions on the domain of Ψ commute, the desired vanishing of $[X, V]$ is now clear. The result of this is that T is contained in a real two-dimensional torus acting holomorphically on $M_{\min}$ and hence we reduce to the previous case.

If $M^{(0)} = L_i$ for some $i = 1, 2$, then by Claim 3.13, the fixed point set of T will comprise either two isolated points, a case that we have already dealt with (independent of the dimension of T), or a $\mathbb{P}^1$ given by $\varpi(M^{(0)})$. In this latter case, the argument of the proof of [9, Claim 4.15] tells us that $\dim_{\mathbb{R}} T = 1$ (this argument is local). The argument of [9, Claims 4.16 and 4.17] then yields an equivariant biholomorphism $\alpha\colon (M_{\min}, T) \to (\widehat{M}, \widehat{T})$.

Finally, if $M^{(0)} = E$, then the fixed point set of $\widehat{T}$ will comprise a $\mathbb{P}^1$ given by $\varpi(M^{(0)})$, that is, an instance of the previous case. This covers all possibilities for ϖ equal to the blowdown map and so we have an equivariant biholomorphism $\alpha\colon (M_{\min}, T) \to (\widehat{M}, \widehat{T})$. Being equivariant then allows us to lift this to an equivariant biholomorphism $\alpha\colon (M, T) \to (\widetilde{M}, \widetilde{T})$.

Suppose now that ϖ is the identity map so that M is diffeomorphic to $\mathbb{C} \times \mathbb{P}^1$ and $M = M_{\min}$. Then $E = \mathbb{P}^1$ and is preserved by the action of T. As $M_0(X) \subseteq E$, we must therefore have that the fixed point set of T comprises either two points in E or the whole of E. Being connected, it follows that $|M^{(0)}| = 1$ in the former case and that $M^{(0)} = E$ in the latter case. All possibilities thereafter have then been dealt with above and we conclude that there is an equivariant biholomorphism $\alpha\colon (M, T) \to (\widehat{M}, \widehat{T})$.

In both cases, the fact that JX generates T and α is equivariant implies $\alpha_*(JX) \in \tilde{\mathfrak{t}}$ or $\alpha_*(JX) \in \hat{\mathfrak{t}}$, as appropriate. ∎

We now conclude the proof of Theorem A.

Completion of the proof of Theorem A

Given (M, g, X) as in the statement of Theorem A, let M_{univ} denote the universal cover of M. Then since M has finite fundamental group [45, Theorem 1.1], we can write $M = M_{\mathrm{univ}}/\Gamma$, where Γ is a finite group of biholomorphisms of M_{univ} acting freely. Lifting the shrinking soliton structure to M_{univ}, we read from Proposition 3.9 that M_{univ} is biholomorphic to either $\widehat{M}$ or $\widetilde{M}$. Thus, item (i) of Theorem A will follow from Proposition 3.9 if we can establish that $\mathrm{Fix}(\Gamma) \neq \emptyset$. This we prove in the next claim.

Claim 3.14. *Every element of a finite group* Γ *of biholomorphisms acting on* $\widehat{M}$ *or* $\widetilde{M}$ *has a fixed point.*

Proof. Any biholomorphism of $\widetilde{M}$ must preserve the two (-1)-curves, hence it must fix their point of intersection x.

As for $\widehat{M}$, any automorphism $\gamma \in \Gamma$ sends a $\mathbb{P}^1$-fibre to a $\mathbb{P}^1$-fibre, hence γ induces an automorphism of the $\mathbb{C}$-factor of $\widehat{M}$. Every finite automorphism group of $\mathbb{C}$ is a rotational group. In particular, the origin is fixed by the action, and so there exists a $\mathbb{P}^1$-fibre of $\widehat{M}$ fixed by Γ. Every Möbius transformation has a fixed point. This observation completes the proof of the claim. ∎

The biholomorphism $\alpha\colon M \to M$ given by Proposition 3.9 has the property that $\alpha_*(JX)$ lies in $\mathrm{Lie}(\mathbb{T})$, the Lie algebra of the real torus $\mathbb{T}$ from Theorem A (ii). Let $X' := \alpha_*(X)$, $g' := (\alpha^{-1})^* g$, and consider the complete shrinking soliton (M, X', g'). The fact that α is a biholomorphism implies that the background complex structure here is still J. In particular, $JX' \in \mathrm{Lie}(\mathbb{T})$.

Let $G_0^{X'}$ denote the connected component of the identity of the holomorphic isometries of (M, J, g') that commute with the flow of X'. As explained at the beginning of Section 3, the assumption of bounded scalar curvature implies that the closure of the flow of JX' in $G_0^{X'}$ yields the holomorphic isometric action of a real torus T' on (M, J, g') with Lie algebra $\mathfrak{t}'$ containing JX'. Without loss of generality, we may assume that T' is maximal in $G_0^{X'}$. Corollary 5.13 of [11] asserts that $G_0^{X'}$ is a maximal compact Lie subgroup of the Lie group $\mathrm{Aut}_0^{X'}(M)$, the connected component of the identity of the group of automorphisms of (M, J) that commute with the flow of X'; cf. [11, Proposition 5.8] as for why $\mathrm{Aut}_0^{X'}(M)$ is a Lie group. Thus, T' is a maximal real torus in $\mathrm{Aut}_0^{X'}(M)$. For each $v \in \mathrm{Lie}(\mathbb{T})$, $JX' \in \mathrm{Lie}(\mathbb{T})$ implies that $[v, JX'] = 0$ so that $[v, X'] = 0$. Hence each element of $\mathbb{T}$ commutes with the flow of X' and so $\mathbb{T}$ itself is a Lie subgroup of $\mathrm{Aut}_0^{X'}(M)$. For dimensional reasons, $\mathbb{T}$ is maximal in $\mathrm{Aut}_0^{X'}(M)$, therefore by Iwasawa's theorem [22] there exists an element $\beta \in \mathrm{Aut}_0^{X'}(M)$ such that $\beta(T')\beta^{-1} = \mathbb{T}$. Since β commutes with the flow of X', necessarily $d\beta^{-1}(X') = X'$. Moreover, $\beta^*(g')$ is invariant under the action of $\mathbb{T}$. Let $\gamma := \alpha^{-1} \circ \beta \colon M \to M$. Unravelling the definitions, we conclude that $\gamma^* g$ is invariant under the action of $\mathbb{T}$ and $\gamma_*^{-1}(JX) = JX' \in \mathrm{Lie}(\mathbb{T})$. This yields item (ii) of Theorem A. Note that the background complex structure $\gamma^* J$ is still equal to J because γ is a biholomorphism.

Finally, the fact that $\gamma_*^{-1}(JX)$ is determined in item (iii) is a result of Proposition 2.28 and Theorem 2.29, as we know for this latter theorem that the Ricci curvature of g, hence that of $\gamma^* g$, is bounded. That its flow generates an S^1-action is clear from the explicit expression of the vector field, given in Examples 2.32 and 2.33 for each respective possibility of M. As explained at the beginning of this section, JX is holomorphic and Killing and so the flow of $\gamma_*^{-1}(JX)$ is holomorphic and isometric for $(J, \gamma^* g)$, as claimed in the same item.

4. Proof of Theorem B

Recall that $(M, g(t))$ is a finite time Type I Kähler–Ricci flow on $[0, T)$, $T < +\infty$, defined on a compact Kähler surface M, $x \in \Sigma_I \subset M$ is a Type I singular point, and $g_j(t) := \lambda_j g(T + \frac{t}{\lambda_j})$, $t \in [-\lambda_j T, 0)$, for a sequence $\lambda_j \to +\infty$. Let J denote the complex structure of M. From [13,31], we know that a subsequence of $(M, g_j(t), x)$ converges in the smooth pointed Cheeger–Gromov sense [42, Definition 7.2.1] to a non-flat complete shrinking gradient Ricci soliton (N, h, p) with bounded curvature and soliton potential f and associated Kähler–Ricci flow $h(t)$, $t \in (-\infty, 0)$, with $h(-1) = h$. Uniformly bounded curvature implies from Shi's derivative estimates that the norms of the derivatives of the curvatures of the metrics $g_j(t)$ are uniformly bounded, hence an application of [7, Theorem 3.23] demonstrates that the limit is in fact Kähler so that (N, h) is a two-dimensional shrinking gradient Kähler–Ricci soliton with bounded scalar curvature. Let $\widetilde{J}$ denote the complex structure of N.

First assume that $\lim_{t \to T^-} \mathrm{vol}_{g(t)}(M) > 0$. Then if N were compact, N would be a del Pezzo surface with h Kähler–Einstein or the shrinking gradient Kähler–Ricci soliton on the blowup of $\mathbb{P}^2$ at one or two points [44]. After unravelling the scaling factors in the definition of smooth pointed Cheeger–Gromov convergence, this would then imply that $\lim_{t \to T^-} \mathrm{vol}_{g(t)}(M) = 0$, a contradiction. Indeed, let $h(t)$, $t \in (-\infty, 0)$, denote the Kähler–Ricci flow associated to (N, h). Then compactness of N implies that for all $0 < \delta < 1$, there exists a diffeomorphism $\phi_k \colon N \to M$ such that $|\phi_k^* g_k(t) - h(t)| < 1$ with derivatives for all $t \in [-1, -\delta]$ for k sufficiently large. In particular,

$$\mathrm{vol}_{g(T + \frac{t}{\lambda_k})}(M) = \frac{\mathrm{vol}_{\phi_k^* g_k(t)}(N)}{\lambda_k^2} \leq \frac{C}{\lambda_k^2} \to 0 \quad \text{as } k \to \infty.$$

Thus, (N, h) is non-compact and according to Lemma 2.7, the scalar curvature R_h of h tends to zero along the unique end of N or there exists an integral curve of the soliton vector field of (N, h) along which $\mathrm{R}_h \not\to 0$. If $\mathrm{R}_h \to 0$, then we would be done by [11, Theorem E (3)]. Therefore, to conclude the proof of this direction of the theorem, it suffices to rule out the latter case.

To this end, recall from Theorem A that if there exists an integral curve of the soliton vector field of (N, h) along which $\mathrm{R}_h \not\to 0$, then up to pullback by biholomorphism, (N, h) is the cylinder $\mathbb{C} \times \mathbb{P}^1$ or a hypothetical shrinking Kähler–Ricci soliton on the blowup of $\mathbb{C} \times \mathbb{P}^1$ at one point. In either case, choose $R > 0$ such that $p \in f^{-1}((-\infty, 3R])$, $A := f^{-1}([2R, 3R])$ is a non-empty annulus in N, and such that any (-1)-curves are contained in the set $f^{-1}((-\infty, R])$. This can be done because f is proper (cf. Theorem 2.6). Then there exist a compact subset $U \subset N$ containing A, $\delta \in (0, 1)$, and diffeomorphisms $\phi_k : U \to M$ with $\phi_k(p) = x$ such that $\phi_k^* g_k(t) \to h(t)$ with derivatives on U as $k \to \infty$ for all $t \in [-1, -\delta]$.

Next, fix a $\widetilde{J}$-holomorphic sphere $u : \mathbb{P}^1 \to A$ in N with trivial self-intersection. Then by Corollary 2.3, there exists a sequence of $\phi_k^* J$-holomorphic spheres $u_k : \mathbb{P}^1 \to N$ with trivial self-intersection converging in C^0 to u as $k \to \infty$. In fact, it follows from standard elliptic bootstrapping arguments that there exists a subsequence, still denoted by u_k, that converges uniformly with all derivatives to u; cf. [27, Proposition 3.3.5 and Appendix B.4]. Set $C := u(\mathbb{P}^1)$, $C_k := u_k(\mathbb{P}^1)$, and let $\tau(t)$, $t \in (-\infty, 0)$, denote the Kähler form associated to $h(t)$ with associated Ricci form $\rho_{\tau(t)}$. Then (cf. [7, (1.13)])

$$[\rho_{\tau(t)}] = \frac{[\tau(t)]}{-t}, \quad t < 0.$$

Consequently, using adjunction, we find that for $t < 0$,

$$\mathrm{vol}_{h(t)}(C) = \int_C [\tau(t)|_C] = -t \int_C [\rho_{\tau(t)}|_C] = -2\pi t \int_C c_1(-K_N|_C) = -4\pi t.$$

Since $\phi_k^* g_k(t) \to h(t)$ and $u_k \to u$ in C^1 as $k \to \infty$, we can assert that for $t \in [-1, -\delta]$, $u_k^*(\mathrm{vol}_{\phi_k^* g_k(t)}) \to u^*(\mathrm{vol}_{h(t)})$ on $\mathbb{P}^1$ as $k \to \infty$, so that for $t \in [-1, -\delta]$,

$$\mathrm{vol}_{\phi_k^* g_k(t)}(C_k) \to \mathrm{vol}_{h(t)}(C) = -4\pi t \quad \text{as } k \to \infty.$$

In other words, for $t \in [-1, -\delta]$,

$$|\mathrm{vol}_{\phi_k^* g_k(t)}(C_k) - (-4\pi t)| \to 0 \quad \text{as } k \to \infty. \tag{4.1}$$

On the other hand, let $\omega(t)$ denote the Kähler form of $g(t)$ and $\rho_{\omega(t)}$ the corresponding Ricci form. Then $\frac{\partial \omega(s)}{\partial s} = -\rho_{\omega(s)}$, $s \in [0, T)$, implies that $[\omega(s)] = [\omega(0)] - s[\rho_{\omega(0)}]$. Using this and the fact that $\phi_k(C_k)$ is J-holomorphic, we compute that

$$\begin{aligned}
\mathrm{vol}_{\phi_k^* g_k(t)}(C_k) &= \mathrm{vol}_{\lambda_k g(T + \frac{t}{\lambda_k})}(\phi_k(C_k)) = \lambda_k \, \mathrm{vol}_{g(T + \frac{t}{\lambda_k})}(\phi_k(C_k)) \\
&= \lambda_k \int_{\phi_k(C_k)} \left[\omega\Big(T + \frac{t}{\lambda_k}\Big)\Big|_{\phi_k(C_k)} \right] \\
&= \lambda_k \int_{\phi_k(C_k)} \Big([\omega(0)] - 2\pi \Big(T + \frac{t}{\lambda_k}\Big) c_1(-K_M|_{\phi_k(C_k)}) \Big) \\
&= \lambda_k \int_{\phi_k(C_k)} ([\omega(0)] - 2\pi T c_1(-K_M|_{\phi_k(C_k)})) - 2\pi t \int_{C_k} c_1(-K_M|_{\phi_k(C_k)}) \\
&= \lambda_k \lim_{s \to T^-} \mathrm{vol}_{g(s)}(\phi_k(C_k)) - 4\pi t.
\end{aligned} \tag{4.2}$$

To derive a contradiction, we need to show that $\lim_{s\to T^-} \mathrm{vol}_{g(s)}(\phi_k(C_k)) > c$ for some positive constant c independent of k. For this, we require

Claim 4.1. *There exists an open subset $U \subset M$ such that for all k,*

$$\phi_k(C_k) \cap (M \setminus U) \neq \emptyset.$$

Proof. Let U denote the union of the open neighbourhoods of each (-1)-curve in M for which there exists a biholomorphism onto a neighbourhood of the zero section of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{P}^1$. Then for every k, $\phi_k(C_k)$ has trivial self-intersection in U and so cannot be contained in U for any k. In other words, $\phi_k(C_k) \cap (M \setminus U) \neq \emptyset$ as claimed. ∎

Let V be an open subset of M containing every (-1)-curve in M with $\bar{V} \subset U$. Since $\lim_{s\to T^-} \mathrm{vol}_{g(s)}(M) > 0$ by assumption, we read from [3, Theorem 3.8.3] (cf. also [3, Definition 3.7.9]) that as $t \to T^-$, $g(t)$ contracts only (-1)-curves and converges smoothly locally to a Kähler metric g_T on the complement of these curves. In particular, $g(t) \to g_T$ smoothly on $M \setminus V$ as $t \to T^-$, so that $\mathrm{inj}_{M\setminus U}\, g(t) \to \mathrm{inj}_{M\setminus U}\, g_T$ and $\mathrm{dist}_{g(t)}(\partial U, \partial V) \to \mathrm{dist}_{g_T}(\partial U, \partial V)$ as $t \to T^-$. Moreover, by the previous claim, for every k there exists a point $x_k \in \phi_k(C_k) \cap (M \setminus U)$. Let $\varepsilon := \min\{\mathrm{dist}_{g_T}(\partial U, \partial V), \mathrm{inj}_{M\setminus U}\, g_T\}$. Then for $s \in (0, T)$ sufficiently close to T, $B_{g(s)}(x_k, \frac{\varepsilon}{2})$ is contained in $M \setminus V$ and an application of [37, p. 178, Comment 1] and [37, Proposition 4.3.1 (ii)] (see also [10, Lemma 5.2]) yields for such values of s the lower bound

$$\mathrm{vol}_{g(s)}(\phi_k(C_k)) \geq \mathrm{vol}_{g(s)}\left(B_{g(s)}\left(x_k, \frac{\varepsilon}{2}\right) \cap \phi_k(C_k)\right) \geq \frac{\pi}{4}\left(\frac{\varepsilon}{2}\right)^2 = \frac{\pi\varepsilon^2}{16}.$$

As a consequence, we obtain the following uniform lower bound on $\mathrm{vol}_{g_T}(\phi_k(C_k))$:

$$\mathrm{vol}_{g_T}(\phi_k(C_k)) = \lim_{s\to T^-} \mathrm{vol}_{g(s)}(\phi_k(C_k)) \geq \frac{\pi\varepsilon^2}{16}. \tag{4.3}$$

To conclude, substitute expression (4.2) into (4.1), then use the lower bound (4.3), and finally let $k \to \infty$. This gives the desired contradiction.

Conversely, suppose that (N, h) is the shrinking gradient Kähler–Ricci soliton of [15] on the blowup of $\mathbb{C}^2$ at the origin and for the sake of a contradiction, assume that

$$\lim_{t\to T^-} \mathrm{vol}_{g(t)}(M) = 0.$$

Then [43] tells us that M exhibits the structure of a Fano fibration $\pi\colon M \to B$ over a base B, where in particular $-K_M$ is π-ample. If B is a point, then M is a del Pezzo surface and [43] (see also [38]) further tells us that the Kähler class of the initial metric $g(0)$ is $c_1(M)$ and that the diameter $\mathrm{diam}(M, g(t))$ of $(M, g(t))$ tends to zero as $t \to T$. In fact, the work of Perelman (see [35]) gives us the upper bound $\mathrm{diam}(M, g(t)) \leq C(T-t)^{\frac{1}{2}}$, which, for the rescaled limit $g_j(t)$, $t < 0$, translates to $\mathrm{diam}(M, g_j(t)) \leq C|t|$. This latter bound implies that (N, h) is compact which yields a contradiction. Hence we conclude that B is one-dimensional. The fact that $-K_M$ is π-ample now tells us that the generic fibre of $\pi: M \to B$ is a holomorphic $\mathbb{P}^1$. Furthermore, by considering the minimal model of M and using the π-ampleness of $-K_M$, Claim 3.8 applies with $\varpi(K)$ replaced by M and $[C]$ replaced by the homology class of a $\mathbb{P}^1$-fibre of the fibration $\pi: M \to B$. The result is that the singular fibres of M comprise a bubble tree of two (-1)-curves.

Now, recalling the setup outlined at the beginning of this section, let $B_R := B_h(p, R)$ denote the ball of radius $R > 0$ centred at p with respect to h. Then for all $R > 0$ and $\delta \in (0, 1)$, there exist diffeomorphisms $\phi_k: \overline{B_R} \to M$ with $\phi_k(p) = x$ such that $\phi_k^* g_k(t) \to h(t)$ with derivatives on $\overline{B_R}$ as $k \to \infty$ for all $t \in [-1, -\delta]$. Let $\widetilde{E}$ denote the exceptional curve in N and choose R sufficiently large, $R = R_1$ say, so that $\widetilde{E} \subset B_{R_1}$. Then since $\phi_k^* J$ converges smoothly locally to $\widetilde{J}$ as $k \to +\infty$, we can, by Corollary 2.4, construct a $\phi_k^* J$-holomorphic curve E_k in B_{R_1} for each k sufficiently large such that $E_k \to \widetilde{E}$ in C^0 as $k \to +\infty$.

Recall from [15] that the soliton $h = h(-1)$ lives on $\mathbb{C}^2$ blown-up at a point, is $U(2)$-invariant, and is asymptotic to a Kähler cone metric on $\mathbb{C}^2$. As such, for all $\lambda > 0$, there exists a compact subset $K_\lambda \subset N$ containing $\widetilde{E}$ in the interior such that for all $x \in N \setminus K_\lambda$, $\mathrm{inj}_h(x) \geq 3\lambda$ and $\sup_{B_h(x, \mathrm{inj}_h(x))} |\mathrm{Rm}(h)|_h \leq \frac{\pi^2}{3\lambda^2}$. Set $\lambda = 4$, take the corresponding K_λ, and choose $x \in N \setminus K_\lambda$ with $|x| = \widehat{R}$ for $\widehat{R} > 0$ to be chosen later. By the $U(2)$-invariance of h, the aforementioned bounds on the injectivity radius and curvature hold at all points on the sphere $\{|z| = \widehat{R}\}$. Choose $\widehat{R}$ sufficiently large so that $B_{R_1} \subset \{|z| \leq \widehat{R}\}$ and such that $\overline{B_h(y, 3\lambda)} \cap \widetilde{E} = \emptyset$ for all $y \in \{|z| = \widehat{R}\}$. Next, choose $R > R_1$ sufficiently large so that $\{|z| \leq \widehat{R}\} \subset B_R$ and so that B_R contains $\overline{B_h(y, 3\lambda)}$ for every $y \in \{|z| = \widehat{R}\}$. Finally, fix k (depending on R) sufficiently large so that $\phi_k^* g_k(-1)$ is sufficiently close to h in derivatives to guarantee that for all $y \in \{|z| = \widehat{R}\}$,

(i) $\mathrm{inj}_{\phi_k^* g_k(-1)}(y) \geq 2\lambda$,

(ii) $B_{\phi_k^* g_k(-1)}(y, 2\lambda) \subset B_h(y, 3\lambda)$,

(iii) $\sup_{B_{\phi_k^* g_k(-1)}(y, 2\lambda)} |\mathrm{Rm}(\phi_k^* g_k(-1))|_{\phi_k^* g_k(-1)} \leq \frac{\pi^2}{2\lambda^2}$.

As a consequence of (ii), by choosing k larger if necessary, we may assume in addition that for all $y \in \{|z| = \widehat{R}\}$,

(vi) $\overline{B_{\phi_k^* g_k(-1)}(y, 2\lambda)} \cap E_k = \emptyset$ and $\overline{B_{\phi_k^* g_k(-1)}(y, 2\lambda)} \cap \partial B_R = \emptyset$.

Now, $\phi_k(E_k)$ will comprise one of the components of the bubble tree of the two (-1)-curves in some exceptional fibre of the fibration $\pi: M \to B$. Write $E_{(1)} := \phi_k(E_k)$ and let $E_{(2)}$ denote the other component. Then $\phi_k^{-1}(E_{(2)} \cap \phi_k(\overline{B_R}))$ defines a real surface in $\overline{B_R}$ intersecting E_k at precisely one point. Let $S_k \subset \overline{B_R}$ denote the unique connected component of this real surface intersecting E_k. Then $S_k \cap \partial B_R \neq \emptyset$, for otherwise S_k would be contained in B_R defining a $\phi_k^* J$-holomorphic $\mathbb{P}^1$ which, using Corollary 2.4, could be perturbed to a $\widetilde{J}$-holomorphic curve in N distinct from $\widetilde{E}$ (after choosing k larger if necessary), thereby leading to a contradiction. In particular, it follows that S_k must intersect the hypersurface $\{|z| = \widehat{R}\}$ at some point q. Take the unique connected component $S_k^q \subset B_{\phi_k^* g_k(-1)}(q, 2\lambda)$ of $S_k \cap B_{\phi_k^* g_k(-1)}(q, 2\lambda)$ passing through q. Clearly, if non-empty, the connected components of the boundary ∂S_k are contained in ∂B_R. Thus, from (iv) above it follows that $\partial S_k^q \subset \partial B_{\phi_k^* g_k(-1)}(q, 2\lambda)$. Next recalling points (i) and (iii) above, after unravelling the definitions and noting that $\phi_k(S_k^q)$ is J-holomorphic, an application of [37, p. 178, Comment 1] and [37, Proposition 4.3.1 (ii)] (see also [10, Lemma 5.2]) allows us to assert that

$$\mathrm{vol}_{g_k(-1)}(\phi_k(S_k^q) \cap B_{g_k(-1)}(\phi_k(q), r)) \geq \frac{\pi r^2}{4}$$

for all $0 < r < 2\lambda$. Set $r = \lambda = 4$. Then we find that

$$\mathrm{vol}_{g_k(-1)}(\phi_k(S_k^q) \cap B_{g_k(-1)}(\phi_k(q), 4)) \geq 4\pi,$$

which, as $\phi_k(S_k^q) \cap B_{g_k(-1)}(\phi_k(q), 4) \subseteq E_{(2)}$, leads to the lower bound

$$\mathrm{vol}_{g_k(-1)}(E_{(2)}) \geq 4\pi.$$

On the other hand, using [43, (1.2)] and computing as in (4.2) with $t = -1$, keeping in mind the fact that $(E_{(2)})^2 = -1$, we derive that

$$\mathrm{vol}_{g_k(-1)}(E_{(2)}) = 2\pi \int_{E_{(2)}} c_1(-K_M|_{E_2}) = 2\pi.$$

This is a contradiction. We therefore conclude that $\lim_{t\to T^-} \mathrm{vol}_{g(t)}(M) > 0$, as desired.

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